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Generalized Gram–Hadamard Inequality

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We generalize the classical Gram determinant inequality. Our generalization follows from the boundedness of the antisymmetric tensor product operator. We use fermionic Fock space methods.

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Let $\mathcal{H}$ be a Hilbert space with scalar product $(\cdot,\cdot)$, $f_{ij}\in\mathcal{H}$ for $i=1,\dots,r$, $j=1,\dots,n$ and $h_{kl}\in\mathcal{H}$ for $k=1,\dots,s$, $l=1,\dots,m$. We introduce the notations

$$\begin{aligned} f_i &= (f_{i1},\dots,f_{in}), \quad & i&=1,\dots,r,\\ h_k &= (h_{k1},\dots,h_{km}), \quad & k&=1,\dots,s \end{aligned}$$

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and

$$F=\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} \equiv \begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & & \\ f_{r1} & \dots & f_{rn} \end{pmatrix}$$

$$H=\begin{pmatrix} h_1 \\ \vdots \\ h_s \end{pmatrix} \equiv \begin{pmatrix} h_{11} & \dots & h_{1m} \\ \vdots & & \\ h_{s1} & \dots & h_{sm} \end{pmatrix}.$$

Further denote

$$G_p(u;v) = \det \begin{pmatrix} (u_1, v_1) & \dots & (u_1, v_p) \\ \vdots & & \\ (u_p, v_1) & \dots & (u_p, v_p) \end{pmatrix}$$

for $u=(u_1,\dots,u_p)$; $v=(v_1,\dots,v_p)$, $u_i, v_i \in \mathcal{H}$, $i=1,\dots,p$. Let $\alpha=(\alpha_1,\dots,\alpha_r)$, $\beta=(\beta_1,\dots,\beta_s)$, $\alpha_i, \beta_j \in \mathbf{C}$,

$$G_n(F;\alpha) = \sum_{i,i'\leq r} \alpha_i^* \alpha_{i'} G_n(f_i; f_{i'}),$$
$$G_m(H;\beta) = \sum_{j,j'\leq s} \beta_j^* \beta_{j'} G_m(h_j; h_{j'}).$$

where $*$ denotes the complex conjugation in $\mathbf{C}$. For $u=v$, $G_p(u;v)=G_p(u;u)\equiv G_p(u)$ is the classical Gram determinant. Besides $G_p(u)\geq 0$, the Gram determinant satisfies the Gram (or Gram–Hadamard) inequality [1]:

$$G_p(u_1,\dots,u_p) \leq G_{p_1}(u_1,\dots,u_{p_1}) G_{p_2}(u_{p_1+1},\dots,u_{p_2}), \quad p=p_1+p_2. \tag{1}$$

In the case where u_i, v_i are L^2-functions, the generalization $G_p(u, v)$ of the Gram determinant $G_p(u)$ appeared in the theory of integral equations. It can be related to the Slater determinants over u and v by the Landsberg integral formula [2,3] but this will not interest us here.

In order to get our generalized Gram inequality in a compact form we will write $F\otimes H$ and $\alpha\otimes\beta$ for tensor products of matrices and vectors, respectively. We prove

THEOREM *Suppose that the dimension of the linear span generated by $f_{ij}, i = 1, \ldots, r, j = 1, \ldots, n$ is not higher than n or the dimension of the linear span generated by $h_{ij}, i = 1, \ldots, s, j = 1, \ldots, m$ is not higher than m. Then the following generalized Gram–Hadamard inequalities hold:*

$$G_n(F; \alpha) \geq 0, \tag{2a}$$

$$G_{n+m}(F \otimes H; \alpha \otimes \beta) \leq G_n(F; \alpha) G_m(H; \beta). \tag{2b}$$

Before going into the proof of the theorem we write (2b) in the explicit form for the convenience of the reader

$$\sum_{\substack{1 \leq i, i' \leq r \\ 1 \leq k, k' \leq s}} \alpha_i^* \beta_k^* \alpha_{i'} \beta_{k'} G_{n+m}(f_i, g_k; f_{i'}, g_{k'})$$
$$\leq \sum_{1 \leq i, i' \leq r} \alpha_i^* \alpha_{i'} G_n(f_i; f_{i'}) \sum_{1 \leq k, k' \leq s} \beta_k^* \beta_{k'} G_m(g_k; g_{k'}). \tag{3}$$

Note that (1) can be obtained as a special case of (3) by taking $r = s = 1$ and cancelling the α, β-constants. In the proof to follow we use fermionic Fock space methods borrowed from physics [4]. Going through the proof the mathematically oriented reader will find out that in fact we simply exploit boundedness of the antisymmetric tensor product operator, writing out this property in a form convenient for our purposes.

Proof We concentrate on fermionic Fock space methods (see for instance [4]). Consider (smeared) fermionic annihilation and creation operators $a(f)$, $a^+(f)$ satisfying

$$\begin{gathered} \{a(f), a(g)\} = 0 = \{a^+(f), a^+(g)\}, \\ \{a(f), a^+(g)\} = (f, g), \end{gathered} \tag{4}$$

where $a^+(f)$ is the adjoint of $a(f)$ and $\{\cdot, \cdot\}$ is the anticommutator. In the L^2-realization the action in the fermionic Fock–Hilbert space is as usual [4] given by

$$(a(f)\psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int \mathrm{d}x f(x)^* \psi^{(n+1)}(x, x_1, \ldots, x_n),$$

$$(a^+(f)\psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^{i-1} f(x_i) \psi^{(n-1)}(x_1, \ldots, \hat{x}_i, \ldots x_n),$$

where $\hat{x}_i$ indicates that the ith variable is to be omitted and $\psi^{(n)}(x_1, \ldots, x_n)$ is totally antisymmetric. $\psi^{(0)} = c\Omega$ where $c \in \mathbf{C}$ and Ω is the normalized vacuum.

Let

$$\Psi = \sum_{j=1}^{r} \alpha_j a^+(f_{j1}) \cdots a^+(f_{jn}), \tag{5}$$

$$\Phi = \sum_{k=1}^{s} \beta_k a^+(h_{k1}) \cdots a^+(h_{km}). \tag{6}$$

Then we have on the vacuum Ω

$$\|\Psi\Omega\|^2 = G(F; \alpha), \tag{7}$$

$$\|\Phi\Omega\|^2 = G(H; \beta) \tag{8}$$

and

$$\|\Psi\Phi\Omega\|^2 = \sum_{j,j',k,k'} \alpha_j^* \beta_k^* \alpha_{j'} \beta_{k'} G_{n+m}(f_j, g_k; f_{j'}, g_{k'}). \tag{9}$$

Suppose that the dimension of the linear span generated by $f_{ij}, j = 1, \ldots, n$ is not higher than n. The generalized Gram inequalities (2a) and (2b) are proved if we can show that the operator norm $\|\Psi\|^2$ is equal to

$$\|\Psi\|^2 = \sum_{j,j'} \alpha_j^* \alpha_{j'} G_n(f_j; f_{j'}) = G_n(F; \alpha). \tag{10}$$

It then follows from

$$\|\Psi\Phi\Omega\| \leq \|\Psi\| \, \|\Phi\Omega\|$$

or from

$$\|\Psi\Phi\Omega\| \leq \|\Psi\| \, \|\Phi\|.$$

Equation (10) is a consequence of Wick's theorem about normal ordering of operator products which we write down with the following simplified notation:

$$a(f_n)\cdots a(f_1)a^+(g_1)\cdots a^+(g_n) = :a(f_n)\cdots a^+(g_n):$$
$$+ :\overline{a(f_n)\cdots a^+(g_j)}\cdots a^+(g_n): + \cdots + G(f_1,\ldots,f_n;g_1,\ldots,g_n). \tag{11}$$

The r.h.s. is obtained from the l.h.s. by normal ordering which is denoted by double dots, that means by anticommuting all creation operators a^+ to the left. The "contractions" (indicated by the bracket overhead) represent the anticomutators (f_n, g_j) (4) which appear in this process. The last term has all operators contracted in pairs in all possible ways and this gives just the Gram determinant.

Now let us consider the square

$$\begin{aligned}(\Psi\Psi^+)^2 &= \left[\sum_{j,j'}\alpha_j\alpha_{j'}a^+(f_{j1})\cdots a^+(f_{jn})a(f_{j'n})\cdots a(f_{j'1})\right]^2\\ &= \sum_{jj'll'}\alpha_j\alpha^*_{j'}\alpha_l\alpha^*_{l'}a^+(f_{j1})\cdots a^+(f_{jn})\\ &\quad\times a(f_{j'n})\cdots a(f_{j'1})a^+(f_{l1})\cdots a^+(f_{ln})a(f_{l'n})\cdots a(f_{l'1}).\end{aligned}$$

In the last line we substitute Wick's theorem (11). Then only the last term with Gram's determinant contributes because all other terms contain at least two equal Fermi operators. This gives

$$\begin{aligned}(\Psi\Psi^+)^2 &= \sum_{j',l}\alpha^*_{j'}\alpha_l G(f_{j'1},\ldots,f_{j'n};f_{l1},\ldots,f_{ln})\\ &\quad\times\sum_{j,l'}\alpha_j\alpha^*_{l'}a^+(f_{j1})\cdots a^+(f_{jn})a(f_{l'n})\cdots a(f_{l'1})\\ &= \left(\sum\alpha^*\alpha G\right)(\Psi\Psi^+)\end{aligned}$$

with obvious short-hand notation. Since $\Psi\Psi^+$ is self-adjoint this implies

$$\|\Psi\Psi^+\| = \left|\sum\alpha^*\alpha G\right| = \|\Psi\|^2 = \|\Psi^+\|^2$$

which is the desired result (10). Applications of the present inequality in physics are given in [5].

Remark We caution the reader that the generalized Gram inequality (2b) is generally not true without the restriction on the dimension of the linear span of f_{ij} or h_{ij}. Nevertheless, it seems that the violation of the inequality without the linear span condition is not very stringent and some variants of it still hold. We realized that this problem has interesting implications in two-dimensional physics and the theory of vertex operator algebras [5].

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