Mixed Volumes, Mixed Ehrhart Theory

and Applications to

Tropical Geometry and Linkage Configurations

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Abstract

The aim of this thesis is the discussion of mixed volumes, their interplay with algebraic geometry, discrete geometry and tropical geometry and their use in applications such as linkage configuration problems. Namely we present new technical tools for mixed volume computation, a novel approach to Ehrhart theory that links mixed volumes with counting integer points in Minkowski sums, new expressions in terms of mixed volumes of combinatorial quantities in tropical geometry and furthermore we employ mixed volume techniques to obtain bounds in certain graph embedding problems.

Mixed volumes. Mixed volumes arise naturally as the combination of the fundamental concepts of Minkowski addition and volume. Minkowski showed that for polytopes $P_1, \ldots, P_n$ and non-negative real parameters $\lambda_1, \ldots, \lambda_n$ the volume of the scaled Minkowski sum $\mathrm{vol}_n(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ depends polynomially on the parameters λ_i. The coefficient of $\lambda_1 \cdots \lambda_n$ is called the *mixed volume* of $P_1, \ldots, P_n$.

In addition to their geometric significance mixed volumes can contain information about algebraic-geometric objects. Let $f_1, \ldots, f_n$ be Laurent polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ and denote by $\mathrm{P}(f_1), \ldots, \mathrm{P}(f_n)$ their Newton polytopes, i.e. the convex hulls of their support sets. Then Bernstein's Theorem [**Ber75**] states that the number of common isolated zeroes in the algebraic torus $(\mathbb{C}^*)^n$ of the system $f_i = 0$ $(i = 1, \ldots, n)$ is bounded above by the mixed volume of $\mathrm{P}(f_1), \ldots, \mathrm{P}(f_n)$. For generic coefficients in $f_1, \ldots, f_n$ this quantity gives the exact number of common isolated solutions counting multiplicities.

Bernstein's Theorem is a generalization of Bézout's Theorem which bounds the number of common solutions to $f_i = 0$ $(i = 1, \ldots, n)$ by the product of the degrees of the f_i. For sparse systems Bernstein's bound is significantly better. Therefore mixed volumes provide an interesting technique to study sparse systems of polynomial equations.

Mixed volumes have been studied in several contexts before. The following choice of literature references provides a discussion of those characteristics of mixed volumes which are important for this work. Schneider [**Sch93**] outlines geometric properties, Ewald's book [**Ewa96**] describes the connection between algebraic geometry and convex geometric objects, Emiris and Canny [**EC95**] as well as Huber and Sturmfels [**HS95**] provide algorithmic tools for mixed volume computation, the appearance of mixed volumes as intersection numbers in tropical geometry is characterized in the articles by Bertrand and Bihan [**BB07**] and by Sturmfels, Tevelev and Yu [**STY07**] and the survey [**MS83**] of McMullen and Schneider discusses mixed volumes as valuations. Even though mixed volumes have been studied already for many decades they still provide a variety of open questions.

Mixed volume computation for large classes of polynomial systems. Sparse systems of polynomial equations appear in a variety of applications. These systems often inherit a special structure from the context that they were obtained from. In general, solving systems of polynomial equations is an active area of research, cf. [**DE05**]. For applications it often suffices to use numerical methods, e.g. homotopy continuation (see [**Li97**, **Ver99**]), to approximate the common solutions to a polynomial system. For the running time of numerical solvers it is of crucial importance to have a good estimate on the number of solutions which are to be found. The mixed volume gives the best bound for this quantity and it is therefore of fundamental interest to develop methods for its efficient computation.

While for concrete systems of equations, the mixed volume can be computed algorithmically, studying the mixed volume for *classes of polytopes* is connected with a variety of issues in convex geometry (such as understanding the Minkowski sum of the polytopes). Determining the mixed volume is computationally very hard (#P-hard), cf. [**GK94**], and hence it is furthermore desirable to exploit the special structure of specific systems of polynomial equations to simplify the computation.

Using results of Betke [**Bet92**] on dissections of Minkowski sums, we provide a method to decouple the computation of mixed volumes in the case when several polytopes lie in a lower dimensional subspace (Lemma 2.6). Polynomial systems to which this statement applies allow therefore a significantly easier approach as we demonstrate by applying Lemma 2.6 on systems obtained from a simple class of minimally rigid graphs (Theorem 5.1).

The most efficient method to compute the mixed volume of polytopes $P_1, \dots, P_n \subset \mathbb{R}^n$ known so far is to construct a *mixed subdivision* of $P_1 + \cdots + P_n$ and add up the volumes of the *mixed cells* in this subdivision. Mixed subdivisions are constructed by lifting the polytopes P_i to $(n+1)$-dimensional space, building the Minkowski sum of the lifted polytopes and then projecting the lower hull of this sum back to $\mathbb{R}^n$. Those cells arising in this process that can be described as a sum of edges from the polytopes P_i are called mixed cells.

Employing the algorithmic methods of Canny and Emiris [**EC95**] we use linear programming duality to show another result that is applicable to compute the mixed volume of a system of polynomial equations. Namely, we give explicit conditions on sets of linear lifting vectors that induce subdivisions of Minkowski sums that contain a given cell as a mixed cell (Lemma 2.9). This enables us to pick large cells and compute liftings that induce these as mixed cells. Repeating this provides a method to approximate the mixed volume from below. Furthermore we specify Lemma 2.9 in 2-dimensional space (Corollary 2.10) which allows a nice geometric interpretation.

The tools described above are employed later in an actual application, namely they help to establish bounds on the number of embeddings of minimally rigid graphs.

Mixed Ehrhart theory. Let P be a polytope with vertices in the integer lattice $\mathbb{Z}^n$ and denote by $\mathrm{L}(P)$ the number of integer lattice points that lie in P. Ehrhart showed [**Ehr67**] that for natural numbers t, the function $\mathrm{L}(t \cdot P)$ is a polynomial in t of degree n, called the *Ehrhart polynomial* of P. Furthermore he found that some coefficients of this polynomial have a nice geometric interpretation. Namely the leading coefficient of $\mathrm{L}(t \cdot P)$ equals the volume of P, the second highest coefficient is the sum of the relative volumes of all facets of P and the constant term is always 1. In particular Ehrhart's results on the coefficients of the Ehrhart polynomial establish a fruitful connection between a continuous quantity, $\mathrm{vol}(P)$, and discrete quantities like the number of integer points in P (cf. [**BR07**]). Studying the lattice point enumerator L, its generating function and related questions is known as Ehrhart theory (cf. [**Bar08**, **Gru07**]).

In this work we introduce a mixed version of Ehrhart theory. Namely, for polytopes $P_1, \ldots, P_k$ with integer vertices we study the function

$$\mathrm{ME}_{P_1,\ldots,P_k}(t) := \sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \; \mathrm{L}\left(t \cdot \sum_{j \in J} P_j\right) .$$

Since this resembles the way mixed volumes are obtained from volumes we call this function the *mixed Ehrhart polynomial* of $P_1, \ldots, P_k$.

Surprisingly it turns out that mixed Ehrhart polynomials have a very simple structure. We show that the coefficient of t^r in $\mathrm{ME}_{P_1,\ldots,P_k}(t)$ vanishes whenever $1 \leq r < k$ (Lemma 3.4) and prove furthermore that the highest two coefficients can be expressed in terms of mixed volumes (Lemma 3.7 and Lemma 3.8). This allows in particular to explicitly state the mixed Ehrhart polynomial in the cases $k = n$ and $k = n - 1$ (Theorem 3.9 and Theorem 3.12). Since it is an open problem to provide a geometric interpretation of the intermediate coefficients of the classical Ehrhart polynomial these results were rather unexpected.

As corollaries to the explicit description of $\mathrm{ME}_{P_1,\ldots,P_n}(t)$ and $\mathrm{ME}_{P_1,\ldots,P_{n-1}}(t)$ we obtain formulas (Corollary 3.10 and Corollary 3.13) that compare alternating sums of integer points in Minkowski sums and expressions in mixed volumes. The statement in Corollary 3.10 was already conjectured by Kušnirenko [**Kuš76**] and then shown by Bernstein [**Ber76**] who used essentially different methods for his proof. On the other hand Corollary 3.13 gives a novel formula which turns out to be the crucial ingredient in our proof that the tropical and toric genus of an intersection curve with the same underlying Newton polytopes coincide.

Combinatorics of tropical intersections. Tropical geometry allows to express certain algebraic-geometric problems in terms of discrete geometric problems using *correspondence theorems*. One general aim is to establish new tropical methods to study the original algebraic problem (see, e.g. [**DFS07**, **Dra08**, **EKL06**, **Mik06**]). A prominent example is the work of Mikhalkin [**Mik05**] who gave a tropical formula for the number of plane curves of given degree and genus passing through a given number of points; see also [**GM07**, **IKS03**, **NS06**] for related theorems. Providing methods to establish these correspondence statements is an important task of current research. Another important objective in tropical geometry is to understand the combinatorial structure of the tropical varieties which can be regarded as polyhedral complexes in n-dimensional space. See, e.g. [**Spe08**, **SS04**].

In this work, we consider intersections of tropical hypersurfaces given by polynomials $g_1, \ldots, g_k$ in $\mathbb{R}^n$ with Newton polytopes $P_1, \ldots, P_k$. For the special case $k = n - 1$ and all

P_i standard simplices, Vigeland studied the number of vertices and unbounded edges as well as the genus of this intersection [**Vig07**]. His methods strongly rely on the special structure of the Newton polytopes.

Our contributions can be stated as follows. Firstly, we provide a uniform and systematic treatment of the whole f-vector (i.e. the vector of face numbers) of tropical transversal and non-transversal intersections. In particular, we show how to reduce these counts to well-established tropical intersection theorems. Generalizing the results in [**Vig07**], our approach also covers the general mixed case, where we start from polynomials $g_1, \ldots, g_k$ with arbitrary Newton polytopes $P_1, \ldots, P_k$. We obtain formulas expressing the number of faces (Theorems 4.4 and 4.9) and the genus (Theorem 4.15) in terms of mixed volumes.

Secondly, we establish a combinatorial connection from the tropical genus of a curve to the genus of a toric curve corresponding to the same Newton polytopes. In [**Kho78**], Khovanskiĭ gave a characterization of the genus of a toric variety in terms of integer points in Minkowski sums of polytopes. We show that in the case of a curve this toric genus coincides with the tropical genus (Theorem 4.20). In particular we think that the methods to establish this result are of particular interest. Khovanskiĭ's formula is stated using numbers of integer points in Minkowski sums of polytopes, whereas the mentioned formula for the tropical genus is given in terms of mixed volumes. For the special case $n = 2$ the connection boils down to the classical Theorem of Pick relating the number of integer points in a polygon to its area. We develop a Pick-type formula for the surface volume of a lattice complex in terms of integer points (Theorem 4.21) to show that in the generalized *unmixed* case (n arbitrary, all P_i coincide) the connection reduces to certain n-dimensional generalizations of Pick's theorem (Macdonald [**Mac63**]). To approach the general mixed case we employ the new aspects of mixed Ehrhart theory (Theorem 3.12) that we presented earlier in this work.

Linkage configuration problems. A series of bars connected with joints that form a closed chain is called a *linkage*. The joints are interpreted to be mobile such that they allow motion between the bars. Linkages arise in various applications in engineering and have as well been studied by mathematicians for over two centuries (cf. [**ES97**]). Our focus is on linkage structures which have no degrees of freedom, i.e. linkages which are designed such that no motion is possible. Linkages of this kind, as well as the graphs that model them, will be called *rigid*. A graph is called *minimally rigid* if it is rigid and becomes flexible if any edge is removed. In 2-dimensional space, minimally rigid graphs are also called *Laman graphs*. Given generic positive lengths for the edges of a minimally rigid graph $G = (V, E)$, we are interested in counting the number of ways in which G can be drawn in the plane or in higher dimensional spaces where we do not count drawings separately if they differ only by rigid motions, i.e. translations and rotations.

Determining the maximal number of embeddings (modulo rigid motions) for a given minimally rigid graph is an open problem. The best upper bounds are due to Borcea and Streinu (see [**Bor02**, **BS04**]) who show that the number of embeddings in 2-dimensional space is bounded by $\binom{2N-4}{N-2} \approx \frac{4^{N-2}}{\sqrt{N-2}}$ where N denotes the number of vertices. Their bounds are based on degree results of determinantal varieties.

As seen above, a general method to study the number of (complex) solutions of systems of polynomial equations is to use Bernstein's Theorem [**Ber75**] for sparse polynomial systems. Since the systems of polynomial equations describing the embedding problem for minimally rigid graphs are sparse, the question arose how good these Bernstein bounds are for the embedding problem.

We study the quality of the Bernstein bound on the minimally rigid graph embedding problem using mixed volume techniques to handle the resulting convex geometric problems. In most cases, our bounds are worse than the bounds in [**BS04**], see Theorem 5.3 and Corollary 5.4. However, we think that the general methodology of studying Bernstein bounds nevertheless provides an interesting technique. It is particularly interesting that for some classes of graphs, the mixed volume bound is tight, see Theorem 5.1 and Corollary 5.2.

Thesis Overview. The thesis is structured as follows. Chapter 1 introduces the basic concepts employed in this work. This includes polytopes, Minkowski sums, volume, mixed volume, mixed subdivisions, tropical geometry and Bernstein's Theorem. Readers with a sound discrete geometric background may skip most of the material here. The methods of Paragraph 1.2.5 on exploiting symmetries in mixed volume computation are less known in the community and might be of interest to all readers.

In Chapter 2 we state and prove some technical tools for explicit mixed volume computation. The main results are a lemma which allows to decouple the computation of mixed volumes in certain situations and a lemma that states explicit conditions on lifting vectors to induce certain cells in mixed subdivisions. The methods established in this chapter are the crucial tools in dealing with systems of polynomial equations that arise in linkage configuration problems.

Chapter 3 describes a new flavor of Ehrhart theory which we call *mixed Ehrhart theory* since it resembles the way mixed volumes are obtained from volumes. We introduce the mixed Ehrhart polynomial and show that coefficients of low order vanish while coefficients of high order can be expressed in terms of mixed volumes. These results imply new identities for alternating sums of integer points in Minkowski sums which play an important role in our proof for the equality of the toric and tropical genus of an intersection curve presented in Chapter 4.

Chapter 4 studies the combinatorics of tropical intersections. In particular we determine the number of bounded and unbounded faces of a tropical intersection in terms of mixed volumes. This leads as well to a new formula for the genus of a tropical intersection curve. Using methods from Chapter 3 we show that the tropical genus coincides with the toric genus defined by polynomials with the same Newton polytopes.

In Chapter 5 we discuss the problem of determining the number of embeddings of minimally rigid graphs with generic edge lengths. Our focus is on the use of discrete geometric techniques, in particular Bernstein's Theorem, to provide upper bounds for the number of embeddings.

Content published in advance. Some results of this work are published in the articles [**ST10**, **ST09**] and the extended conference abstract [**ST08a**]. In addition this thesis contains generalizations of the results in these papers and enhances the presentation of the statements by providing more examples and, where appropriate, graphical illustrations of important ideas that sustain a geometric intuition.

Zusammenfassung

Ziel dieser Arbeit ist die Diskussion gemischter Volumina, ihres Zusammenspiels mit der algebraischen Geometrie, der diskreten Geometrie und der tropischen Geometrie sowie deren Anwendungen im Bereich von Gestänge-Konfigurationsproblemen. Wir präsentieren insbesondere neue Methoden zur Berechnung gemischter Volumina, einen neuen Zugang zur Ehrhart Theorie, welcher gemischte Volumina mit der Enumeration ganzzahliger Punkte in Minkowski-Summen verbindet, neue Formeln, die kombinatorische Größen der tropischen Geometrie mithilfe gemischter Volumina beschreiben, und einen neuen Ansatz zur Verwendung gemischter Volumina zur Lösung eines Einbettungsproblems der Graphentheorie.

Gemischte Volumina. Gemischte Volumina treten in natürlicher Weise als Kombination der fundamentalen Konzepte Volumen und Minkowski-Summation auf. Minkowski zeigte, dass für Polytope $P_1, \ldots, P_n$ und nicht-negative reelle Parameter $\lambda_1, \ldots, \lambda_n$ das Volumen der skalierten Minkowski-Summe $\mathrm{vol}_n(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ polynomiell von den Parametern λ_i abhängt. Den Koeffizienten des Monoms $\lambda_1 \cdots \lambda_n$ nennt man das *gemischte Volumen* von $P_1, \ldots, P_n$.

Zusätzlich zu ihrer geometrischen Bedeutung beinhalten gemischte Volumina Informationen über Objekte der algebraischen Geometrie. Seien $f_1, \ldots, f_n$ Laurent-Polynome in $\mathbb{C}[x_1, \ldots, x_n]$ und bezeichne mit $\mathrm{P}(f_i)$ das Newton-Polytop von f_i, d.h. die konvexe Hülle der Exponenten der auftretenden Monome in f_i. Dann gilt nach dem Satz von Bernstein [**Ber75**], dass die Anzahl der isolierten gemeinsamen Nullstellen im algebraischen Torus $(\mathbb{C}^*)^n$ des Systems $f_i = 0$ $(i = 1, \ldots, n)$ durch das gemischte Volumen von $\mathrm{P}(f_1), \ldots, \mathrm{P}(f_n)$ nach oben beschränkt ist. Falls die Koeffizienten der Polynome $f_1, \ldots, f_n$ generisch gewählt sind, so gibt das gemischte Volumen sogar die exakte Anzahl isolierter Nullstellen, unter Berücksichtigung von Vielfachheiten, an.

Der Satz von Bernstein stellt eine Verallgemeinerung des Satzes von Bézout dar, der die Anzahl gemeinsamer Lösungen durch das Produkt der Grade der Polynome f_i beschränkt. Für dünnbesetzte Polynomgleichungssysteme gibt Bernsteins Satz eine deutlich bessere Schranke an. Daher bieten gemischte Volumina eine interessante Technik zum Studium dünnbesetzter Polynomgleichungssysteme.

Gemischte Volumina wurden bereits in zahlreichen Zusammenhängen studiert. Die folgenden Literaturreferenzen geben einen guten Überblick über diejenigen Eigenschaften gemischter Volumina, die für diese Arbeit relevant sind. Schneider [**Sch93**] diskutiert geometrische Eigenschaften, Ewald's Buch [**Ewa96**] beschreibt die Verbindungen zwischen algebraischer Geometrie und Objekten der konvexen Geometrie, sowohl Emiris und Canny [**EC95**] als auch Huber und Sturmfels [**HS95**] stellen algorithmische Methoden

zur Berechnung gemischter Volumina bereit, die Interpretation gemischter Volumina als Schnittmultiplizitäten tropischer Hyperflächen findet sich in den Arbeiten von Bertrand und Bihan [**BB07**] und von Sturmfels, Tevelev und Yu [**STY07**] und der Übersichtsartikel [**MS83**] von McMullen und Schneider diskutiert die Eigenschaften gemischter Volumina im Kontext von Bewertungen. Obwohl gemischte Volumina schon seit vielen Jahrzehnten studiert werden, sind dennoch viele Fragen offen und bieten daher ein weites Feld zukünftiger Forschung.

Berechnung gemischter Volumina für große Klassen von Polynomgleichungssystemen. Dünnbesetzte Systeme polynomieller Gleichungen tauchen in einer Vielzahl von Anwendungsproblemen auf. Solche Systeme beinhalten oft eine spezielle Struktur die durch den Kontext bestimmt wird, welchen sie modellieren. Das Lösen von Polynomgleichungssystemen ist daher ein wichtiges Feld aktueller Forschung (vgl. [**DE05**]). In Anwendungsproblemen ist es oft ausreichend mithilfe numerischer Verfahren, wie z.B. *homotopy continuation* (siehe [**Li97**, **Ver99**]), die gemeinsamen Lösungen eines Polynomgleichungssystems zu approximieren. Für die Laufzeit solcher Verfahren ist es von entscheidender Bedeutung, gute Schätzungen der Anzahl von Lösungen zu haben, die berechnet werden sollen. Das gemischte Volumen gibt die beste Schranke für diese Anzahl und es ist daher von fundamentalem Interesse, Methoden für dessen effektive Berechnung zu entwickeln.

Im Allgemeinen ist die Komplexität der Berechnung des gemischten Volumens sehr hoch (#P-hart), vgl. [**GK94**]. Daher ist es erstrebenswert die spezielle Struktur mancher Polynomgleichungssysteme auszunutzen, um die Berechnung zu vereinfachen.

Wir verwenden Betkes [**Bet92**] Resultate bezüglich Zerlegungen von Minkowski-Summen um eine Methode zu beschreiben, die Berechnung gemischter Volumina zu entkoppeln, für den Fall, dass einige der Polytope in einem niederdimensionalen Unterraum liegen (Lemma 2.6). Polynomgleichungssysteme, auf die dieses Resultat anwendbar ist, erlauben daher eine wesentlich vereinfachte Herangehensweise. Dies demonstrieren wir durch die Anwendung von Lemma 2.6 auf Systeme, die einer bestimmten Klasse von minimal starren Graphen zugrunde liegen (Satz 5.1).

Die zur Zeit effizienteste Methode zur Berechnung des gemischten Volumens der Polytope $P_1, \ldots, P_n \subset \mathbb{R}^n$ ist es, eine *gemischte Unterteilung* der Minkowski-Summe $P_1 + \cdots + P_n$ zu berechnen und dann das Volumen der *gemischten Zellen* dieser Unterteilung aufzuaddieren. Gemischte Unterteilungen wiederum werden konstruiert, indem man die Polytope P_i in den $\mathbb{R}^{n+1}$ anhebt, die Minkowski-Summe dieser angehobenen Polytope bildet und die unteren Facetten dieser Summe zurück in den $\mathbb{R}^n$ projiziert. Diejenigen Zellen, die in diesem Prozess entstehen und sich als Summen von Kanten der Polytope P_i darstellen lassen, nennt man gemischte Zellen.

Wir verwenden die algorithmischen Methoden von Canny und Emiris [**EC95**] und die Dualität linearer Programme, um ein weiteres Resultat zu erhalten, dass bei der Berechnung des gemischten Volumens großer Polynomgleichungssysteme hilfreich ist. Genauer gesagt benennen wir explizite Bedingungen an Mengen von Lifting-Vektoren, die garantieren, dass eine vorgegebene Zelle als gemischte Zelle der von den Vektoren induzierten gemischten Unterteilung vorkommt (Lemma 2.9). Dieses Werkzeug ermöglicht es Zellen mit großem Volumen auszuwählen und Lifting-Vektoren zu berechnen, die diese Zellen als gemischte Zellen induzieren. Durch Wiederholung dieses Prozesses erhalten wir eine Methode gemischte Volumina von unten zu approximieren. Des Weiteren zeigen wir, dass Lemma 2.9 im 2-dimensionalen Raum (Korollar 2.10) eine schöne geometrische Interpretation zulässt.

Die Verwendung der hier entwickelten Methoden wird später in echten Anwendungen demonstriert. Insbesondere werden mithilfe dieser Ergebnisse Schranken für die Anzahl von Einbettungen minimal starrer Graphen bestimmt.

Gemischte Ehrhart Theorie. Sei P ein Polytop mit Knoten im Gitter $\mathbb{Z}^n$ und sei $\mathrm{L}(P)$ die Anzahl der ganzzahligen Punkte in P. Ehrhart zeigte [**Ehr67**], dass die Funktion $\mathrm{L}(t \cdot P)$ für natürliche Zahlen t ein Polynom in t vom Grad n ist. Dieses Polynom bezeichnen wir als das *Ehrhart-Polynom* von P. Desweiteren haben einige Koeffizienten dieses Polynoms eine interessante geometrische Bedeutung. Genauer gesagt ist der Leitkoeffizient von $\mathrm{L}(t \cdot P)$ gleich dem Volumen von P, der zweithöchste Koeffizient beschreibt die Summe der Volumina der Facetten von P und der Absolutkoeffizient ist immer 1. Insbesondere beschreiben Ehrharts Resultate bezüglich der Koeffizienten von $\mathrm{L}(t \cdot P)$ einen schönen Zusammenhang zwischen einer stetigen Größe, $\mathrm{vol}_n(P)$, und diskreten Größen wie der Anzahl der ganzzahligen Punkte in P (vgl. [**BR07**]). Das Studium des Gitterpunktzählers L, seiner Erzeugendenfunktion und verwandter Fragestellungen wird als *Ehrhart Theorie* bezeichnet (vgl. [**Bar08**, **Gru07**]).

In dieser Arbeit wird eine gemischte Version der Ehrhart Theorie eingeführt. Das heißt, wir betrachten für Polytope $P_1, \ldots, P_k$ mit ganzzahligen Knoten die Funktion

$$\mathrm{ME}_{P_1,\ldots,P_k}(t) := \sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \ \mathrm{L}\left(t \cdot \sum_{j \in J} P_j\right) .$$

Da dies die Art widerspiegelt, in der gemischte Volumina aus normalen Volumina gebildet werden, bezeichnen wir diese Funktion als *gemischtes Ehrhart-Polynom* von $P_1, \ldots, P_k$.

Überraschenderweise stellt sich heraus, dass gemischte Ehrhart-Polynome häufig eine sehr einfache Struktur besitzen. Wir zeigen, dass der Koeffizient von t^r in $\mathrm{ME}_{P_1,\ldots,P_k}(t)$ verschwindet falls $1 \leq r < k$ (Lemma 3.4) und beweisen weiterhin, dass die höchsten Koeffizienten durch gemischte Volumina ausgedrückt werden können (Lemma 3.7 und Lemma 3.8). Insbesondere kann man durch diese Ergebnisse das gemischte Ehrhart-Polynom in den Fällen $k = n$ und $k = n - 1$ vollständig beschreiben (Theorem 3.9 und Theorem 3.12). Da eine geometrische Interpretation der mittleren Koeffizienten des klassischen Ehrhart-Polynoms immer noch ein offenes Problem ist, waren Resultate dieser Art eher unerwartet.

Als Folgerungen der expliziten Beschreibung von $\mathrm{ME}_{P_1,\ldots,P_n}(t)$ und $\mathrm{ME}_{P_1,\ldots,P_{n-1}}(t)$ erhalten wir Formeln (Korollar 3.10 und Korollar 3.13), die alternierende Summen von Gitterpunktanzahlen in Minkowski-Summen und Ausdrücke in gemischten Volumina vergleichen. Die Aussage von Korollar 3.10 wurde bereits von Kušnirenko [**Kuš76**] vermutet und später von Bernstein [**Ber76**] bewiesen, der für seinen Beweis essentiell andere Methoden verwendete. Auf der anderen Seite ist Korollar 3.13 eine neue Formel, die sich als wesentliches Werkzeug beim Beweis der Aussage herausstellt, dass das tropische und das torische Geschlecht von Kurven, denen die gleichen Newton-Polytope zugrunde liegen, übereinstimmt.

Die Kombinatorik tropischer Schnitte. Tropische Geometrie ermöglicht es, mithilfe sogenannter *Korrespondenz-Sätze*, Probleme der algebraischen Geometrie in Probleme der diskreten Geometrie zu übersetzen. Dabei ist es ein generelles Ziel, neue tropische Methoden zu erschließen, um die zugrunde liegenden Probleme der algebraischen Geometrie zu studieren (siehe z.B. [**DFS07**, **Dra08**, **EKL06**, **Mik06**]). Ein bekanntes Beispiel dafür ist die Arbeit von Mikhalkin [**Mik05**], in der eine tropische Formel für die Anzahl der ebenen Kurven eines bestimmten Geschlechts und Grades, auf der eine gegebene Anzahl von Punkten liegt, beschrieben wird; siehe auch [**GM07**, **IKS03**, **NS06**] für ähnliche Resultate.

Es ist eine wichtige Aufgabe zukünftiger Forschung, Methoden bereitzustellen, um Korrespondenz-Sätze dieser Art zu ergründen. Tropischer Varietäten kann man als polyedrische Komplexe im n-dimensionalen Raum auffassen und es ist ein weiteres wesentliches Ziel innerhalb der tropischen Geometrie, die kombinatorische Struktur tropischer Varietäten zu untersuchen (siehe z.B. [**Spe08**, **SS04**].

In dieser Arbeit betrachten wir Schnitte tropischer Hyperflächen im $\mathbb{R}^n$, die durch Polynome $g_1, \ldots, g_k$ mit Newton-Polytopen $P_1, \ldots, P_k$ beschrieben sind. Für den speziellen Fall, dass $k = n - 1$ ist und alle P_i Standardsimplexe, hat Vigeland sowohl die Anzahl der Knoten und unbeschränkten Kanten als auch das Geschlecht dieses Schnittes studiert [**Vig07**]. Seine Methoden beruhen dabei stark auf der speziellen Struktur der Newton-Polytope.

Unser Beitrag kann wie folgt beschrieben werden. Zum einen bieten wir ein einheitliches und systematisches Studium des gesamten f-Vektors (d.h. des Vektors der Seitenanzahlen) von tropischen transversalen und nicht-transversalen Schnitten. Insbesondere zeigen wir, wie das Zählen von Seitenzahlen auf wohlbekannte tropische Sätze über Schnittanzahlen zurückgeführt werden kann. Unsere Resultate verallgemeinern die Ergebnisse in [**Vig07**] und decken weiterhin den allgemeinen gemischten Fall ab, in dem wir von Polynomen $g_1, \ldots, g_k$ mit beliebigen Newton-Polytopen $P_1, \ldots, P_k$ ausgehen. Dabei erhalten wir Formeln, die die Anzahl der Seiten (Satz 4.4 und 4.9) sowie das Geschlecht (Satz 4.15) durch Ausdrücke in gemischten Volumina beschreiben.

Zum anderen beschreiben wir einen kombinatorischen Zusammenhang zwischen dem tropischen Geschlecht einer Kurve und dem torischen Geschlecht einer Kurve, der die selben Newton-Polytope zugrunde liegen. Khovanskiĭ gibt in [**Kho78**] eine Charakterisierung des Geschlechts einer torischen Varietät in Termen von Gitterpunktanzahlen in Minkowski-Summen der Newton-Polytope an. Wir zeigen, dass für den Fall von Kurven das torische und das tropische Geschlecht übereinstimmen (Satz 4.20). Dabei sind insbesondere die Methoden, die für dieses Resultat verwendet werden, von besonderem Interesse. Während Khovanskiĭs Formel durch Ausdrücke in Gitterpunktanzahlen von Minkowski-Summen gegeben ist, ist die Formel für das tropische Geschlecht durch gemischte Volumina beschrieben. Im Spezialfall $n = 2$ ist dieser Zusammenhang durch den klassischen Satz von Pick gegeben, der die Anzahl der Gitterpunkte in einem Polygon mit dessen Fläche verbindet. Wir entwickeln eine neue Pick-artige Formel für das

Oberflächenvolumen eines Gitterkomplexes (Satz 4.21) um zu zeigen, dass sich im allgemeinen *ungemischten Fall* (n beliebig, alle P_i identisch) dieser Zusammenhang auf eine bestimmte n-dimensionale Verallgemeinerung des Satzes von Pick (Macdonald [**Mac63**]) zurückführen lässt. Um den allgemeinen gemischten Fall zu behandeln verwenden wir Resultate der gemischten Ehrhart Theorie (Satz 3.12), die wir an einer früherer Stelle dieser Arbeit präsentiert haben.

Konfigurationen von Gestängen. Als ein *Gestänge* bezeichnet man eine durch Gelenke verbundene Reihe von Stäben, die eine geschlossene Struktur bilden. Dabei sind die Gelenke als beweglich anzusehen, so dass sie Bewegungen der Stäbe relativ zueinander zulassen. Gestänge werden in diversen Anwendungen der Ingenieurwissenschaften benötigt und wurden ebenfalls von Mathematikern seit mehr als zwei Jahrhunderten studiert (vgl. [**ES97**]). Wir beschäftigen uns vornehmlich mit Gestängestrukturen, die keine Freiheitsgrade haben, d.h. dass sie so gestaltet sind, dass Bewegungen der Stäbe relativ zueinander verhindert werden. Sowohl Gestänge dieser Art, als auch Graphen die solche Gestänge modellieren, nennt man *starr*. Ein Graph wird *minimal starr* genannt, wenn er starr ist und durch Hinwegnahme eines Stabes beweglich wird. Minimal starre Graphen im 2-dimensionalen Raum werden auch *Laman Graphen* genannt. Für gegebene positive Kantenlängen eines minimal starren Graphen $G = (V, E)$ interessieren wir uns für die Anzahl von Möglichkeiten den Graphen G in der Ebene oder in höher-dimensionalen Räumen zu zeichnen. Hierbei zählen wir Einbettungen, die sich lediglich durch starre Bewegungen (d.h. Rotationen und Translationen) unterscheiden, nicht mehrfach.

Die Bestimmung der maximalen Anzahl von Einbettungen (modulo starrer Bewegungen) eines gegebenen minimal starren Graphen ist ein offenes Problem. Die besten bekannten oberen Schranken gehen zurück auf Borcea und Streinu (vgl. [**Bor02**, **BS04**]), die beweisen, dass die Anzahl der Einbettungen durch $\binom{2N-4}{N-2} \approx \frac{4^{N-2}}{\sqrt{N-2}}$ beschränkt ist, wobei N die Anzahl der Knoten von G bezeichnet. Die Resultate von Borcea und Streinu beruhen auf Grad-Berechnungen geeigneter Determinanten-Varietäten.

Wie bereits oben beschrieben ist der Satz von Bernstein [**Ber75**] eine Methode die Anzahl komplexer Lösungen von dünnbesetzten Polynomgleichungssystemen zu studieren. Da die polynomiellen Gleichungen, die die Einbettungen von minimal starren Graphen beschreiben, dünnbesetzt sind, stellte sich die Frage, wie gut die Bernstein-Schranken für dieses Problem sind. Während das gemischte Volumen konkreter Gleichungssysteme algorithmisch behandelt werden kann, ist das Studium gemischter Volumina für ganze Klassen von Polytopen mit einer Vielzahl von Fragestellungen der diskreten Geometrie verknüpft (wie beispielsweise der Untersuchung von Minkowski-Summen).

Wir studieren die Qualität der Bernstein-Schranken für das Einbettungsproblem minimal starrer Graphen unter Verwendung von Techniken zur Berechnung gemischter Volumina der Polytope, die die Problemstellung beschreiben. In den meisten Fällen sind die daraus resultierenden Schranken schwächer als die in [**BS04**], siehe Satz 5.3 und Korollar 5.4. Allerdings denken wir, dass die generelle Methode des Studiums der Bernstein-Schranken eine interessante Technik bietet. Es ist insbesondere hervorzuheben, dass die Schranken, die durch das gemischte Volumen beschrieben werden, in einigen Fällen scharf sind (vgl. Satz 5.1 und Korollar 5.2).

Gliederung der Dissertation. Diese Arbeit ist wie folgt gegliedert. Kapitel 1 führt in die Konzepte ein, die dieser Arbeit zugrunde liegen. Dies beinhaltet: Polytope, Minkowski-Summen, Volumina, gemischte Volumina, gemischte Unterteilungen, tropische Geometrie und den Satz von Bernstein. Leser mit einem guten Hintergrundwissen in diskreter Geometrie können diesen Teil getrost überspringen. Lediglich die Methoden in Absatz 1.2.5, die das Ausnützen von Symmetrien in der Berechnung von gemischten Volumina beschreiben, sind weniger bekannt und sind daher auch für erfahrene Leser interessant.

In Kapitel 2 werden einige technische Methoden beschrieben, die bei der expliziten Berechnung gemischter Volumina hilfreich sind. Die wesentlichen Resultate sind hierbei ein Lemma zur Entkopplung der Berechnung gemischter Volumina in speziellen Fällen und ein Lemma, dass explizite Bedingungen an Lifting-Vektoren beschreibt, die gegebene Zellen als gemischte Zellen einer gemischten Unterteilung induzieren. Die Methoden dieses Kapitels sind die entscheidenden Werkzeuge beim Studium der Polynomgleichungssysteme, die in Gestänge-Konfigurationsproblemen auftauchen.

Kapitel 3 beschreibt eine neue Variante der Ehrhart Theorie, welche wir als gemischte Ehrhart Theorie bezeichnen, da sie die Art wie gemischte Volumina aus Volumina gebildet werden widerspiegelt. Wir definieren das sogenannte gemischte Ehrhart-Polynom und zeigen, dass Koeffizienten niedriger Ordnung dieses Polynoms verschwinden und Koeffizienten hoher Ordnung durch Ausdrücke in gemischten Volumina darstellbar sind. Diese Resultate implizieren neue Formeln für die Anzahl ganzzahliger Punkte in Minkowski-Summen. In unserem Beweis in Kapitel 4, der zeigt, dass das torische und das tropische Geschlecht von Kurven, denen die selben Newton-Polytope zugrunde liegen, übereinstimmen, spielen diese Formeln eine entscheidende Rolle.

Kapitel 4 widmet sich dem Studium der Kombinatorik tropischer Schnitte. Insbesondere drücken wir die Anzahl der beschränkten und unbeschränkten Seiten eines tropischen Schnittes durch Terme in gemischten Volumina aus. Dies führt ebenfalls zu einer neuen Formel für das Geschlecht einer tropischen Schnittkurve. Mit den Methoden aus Kapitel 3 zeigen wir darauf, dass das tropische und das torische Geschlecht von Kurven, die durch Polynome mit den gleichen Newton-Polytopen beschrieben werden, identisch sind.

In Kapitel 5 diskutieren wir die Bestimmung der Anzahl der Einbettung minimal starrer Graphen mit generischen Kantenlängen. Dabei liegt das Hauptaugenmerk auf der Verwendung diskret geometrischer Methoden, insbesondere des Satzes von Bernstein, um Schranken für die Anzahl der Einbettungen bereitzustellen.

Bereits veröffentlichte Inhalte. Einige Ergebnisse dieser Arbeit wurden bereits in den Artikeln [**ST10**, **ST09**] und dem Konferenzbeitrag [**ST08a**] veröffentlicht. Diese Dissertation enthält zusätzlich einige Verallgemeinerungen der Resultate dieser Arbeiten. Außerdem wurde die Präsentation der Aussagen durch zusätzliche Beispiele und grafische Darstellungen wichtiger Ideen verbessert um dem Leser die Entwicklung einer geometrischen Intuition zu erleichtern.

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Furthermore I would like to thank the following people for helpful discussions, comments and proof-reading: Hardwig Bosse, Eric Böse-Wolf, Ioannis Emiris, Kerstin Hept, Martin Hutzenthaler, Christian Kohl, Benjamin Mühlbauer, Cordian Riener, Raman Sanyal, Henning Sulzbach, Thorsten Theobald, and Timo de Wolff.

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Contents

CHAPTER 1

Preliminaries

This chapter gives a short introduction to the basic concepts of this thesis. We begin with an introduction to the language of polytopes and convex bodies, including basic facts on volumes, Minkowski sums and polyhedral complexes. With these definitions mixed volumes are introduced, followed by a discussion of those properties of mixed volumes which are crucial for this thesis. Also we review a way to compute the mixed volume using mixed subdivisions.

Furthermore the reader is familiarized with tropical geometry from two different viewpoints. Here we stress in particular the duality between tropical hypersurfaces and polyhedral complexes which is of significant importance for later results. The chapter ends with a discussion of Bernstein's first and second theorem.

1.1. Polytopes

Most methods applied in this work are discrete geometric which implies that polytopes will play a crucial role in everything we do. We give a brief introduction here and refer readers with less background to [**Grü03**, **Zie95**] for polytopes in general and to [**Ewa96**] concerning the interplay of discrete geometry and algebraic geometry.

In the following paragraphs we define the most important objects of this work and clarify the notation that is used throughout this thesis.

1.1.1. Basic definitions and notation. A set $A \subset \mathbb{R}^n$ is called *convex* if with any two points $p, q \in A$ it also contains the straight line segment $[p, q] := \{\lambda p + (1-\lambda)q \,|\, 0 \leq \lambda \leq 1\}$ between p and q. We say that p is a *convex combination* of $p_1, \ldots, p_r \subset \mathbb{R}^n$ if there are $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ such that

(1) $p = \lambda_1 p_1 + \cdots + \lambda_r p_r$
(2) $\sum_{i=1}^{r} \lambda_i = 1$
(3) $\lambda_i \geq 0$.

If condition (3) is dropped, p is called an *affine combination* of $p_1, \ldots, p_r$, if condition (2) is dropped p is called a *positive combination* of $p_1, \ldots, p_r$ and if both conditions (2) and (3) are dropped p is called a *linear combination* of $p_1, \ldots, p_r$. For a set A the set of all convex combinations of points in A is the *convex hull* of A and denoted by $\mathrm{conv}(A)$. In the same way we define the *affine hull* of A: $\mathrm{aff}(A)$, the *positive hull* of A: $\mathrm{pos}(A)$ (which is also sometimes called the *cone* of A: $\mathrm{cone}(A)$) and the *linear hull* of A: $\mathrm{lin}(A)$.

Each affine hull is the translate of a linear hull and the *dimension* of an affine hull is defined as the dimension of the corresponding linear hull. Affine subspaces of dimensions

$0, 1, \ldots, k, \ldots, n-1$ in $\mathbb{R}^n$ will be called *points*, *lines*, *k-planes* and *hyperplanes*, respectively. Each hyperplane H separates the space $\mathbb{R}^n$ into two halfspaces denoted by H^+ and H^-. The intersection of a finite number of halfspaces is called a *polyhedron*.

Compact convex sets $K \subset \mathbb{R}^n$ are called *convex bodies* and a convex body P that is the convex hull of a finite point set $v_1, \ldots, v_r \in \mathbb{R}^n$ is a *polytope*. The space of all polytopes in $\mathbb{R}^n$ is denoted by $\mathcal{P}_n$. We say that a hyperplane H *supports* a closed convex set A if $H \cap A \neq \emptyset$ and $A \subset H^+$ or $A \subset H^-$. This intersection $H \cap A$ is called a (*proper*) *face* of A. We make the convention to call $\emptyset$ and A itself faces of A as well but refer to them as *improper*. Faces of dimensions $0, 1, \ldots, k, \ldots, n-1$ will be called *vertex*, *edge*, *k-face* and *facet*. The convex hull of points $v_1, \ldots, v_r$ which are *affinely independent*, i.e. none of the

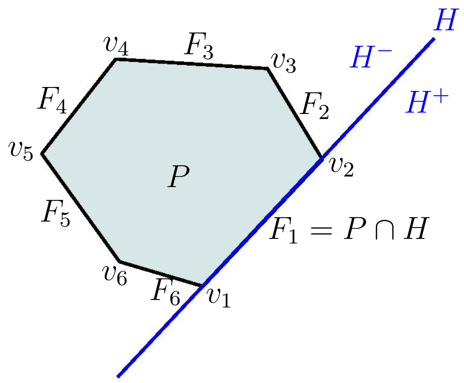

Figure 1.1. A polytope P with a supporting hyperplane H.

points is an affine combination of the others, is called a *simplex*.

We chose to define polytopes as the convex hull of its vertices, i.e. $P = \operatorname{conv}\{v_1, \ldots, v_r\}$ but the following proposition states that another description is equivalent.

Proposition 1.1. *A subset $P \subset \mathbb{R}^n$ is the convex hull of a finite point set (a $\mathcal{V}$-polytope) if and only if it is a bounded intersection of a finite number of halfspaces (an $\mathcal{H}$-polytope).*

To every proper face F of a closed convex set A corresponds a cone $N(F)$ of linear functions $v \in (\mathbb{R}^n)^*$ which are maximized in F on A. We identify $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ and call such a function v an (*outer*) *normal vector* of F on A. In the following the face that is maximal with respect to v will be denoted by $(A)^v$. The cone N_F is called the *normal cone* of F and the normal cones of all faces of a polytope P form a complete fan, the *normal fan*, $\mathcal{N}_P$, of P. I.e. every non-empty face of a normal cone is also a normal cone of some face of P, the intersection of two normal cones is a face of both and the union of all cones covers $\mathbb{R}^n$.

1.1.2. Volume. From basic calculus we know that every n-dimensional convex body $K \subset \mathbb{R}^n$ has an n-dimensional Euclidean volume $\operatorname{vol}_n(K)$. In this work the volume is normalized by assuming the volume of the unit cube in $\mathbb{R}^n$ to be 1.

Most convex bodies which are considered in this thesis are *lattice polytopes*, i.e. polytopes with vertices in a lattice $\Lambda \subset \mathbb{R}^n$, i.e. a discrete subgroup of $\mathbb{R}^n$, and often this

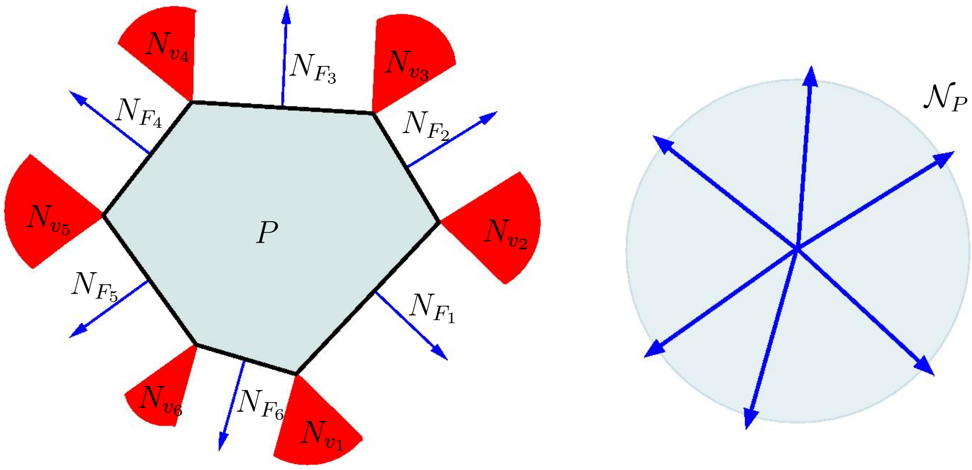

Figure 1.2. Left: The normal cones of vertices and edges of P. Right: The normal fan of P.

lattice is $\mathbb{Z}^n$ itself. We denote by $\mathcal{P}_n(\Lambda)$ the space of lattice polytopes in $\mathbb{R}^n$. If Λ is k-dimensional, then $\Lambda \simeq \mathbb{Z}^k$ holds and Λ lies in a k-dimensional supspace $\Lambda_{\mathbb{R}}$ of $\mathbb{R}^n$. A basis of Λ induces an isomorphism between Λ and $\mathbb{Z}^k$ and also between $\Lambda_{\mathbb{R}}$ and $\mathbb{R}^k$. The volume vol_Λ on $\Lambda_{\mathbb{R}}$ is defined as the pull-back of the usual Euclidean volume on $\mathbb{R}^k$ under this isomorphism. As the lattice will usually be clear from the context this volume will often be denoted as just vol'_k. Note that the definition of vol_Λ is independent of the choice of the basis B of Λ since any other basis can be obtained from B by a volume preserving linear map.

The parallelotope $\mathfrak{P}$ that is generated by the basis of a k-dimensional lattice Λ is called the *fundamental lattice parallelotope of* Λ . This notation allows to state the relation of the volume with respect to Λ and the usual Euclidean volume in $\mathbb{R}^k$ as follows:

$$\mathrm{vol}'_k(K) = \frac{\mathrm{vol}_k(K)}{\mathrm{vol}_k(\mathfrak{P})} \ . \tag{1.1}$$

Example 1.2. Let Λ be the lattice spanned by $v_1 = (1,3)^T$ and $v_2 = (2,1)^T$ and let Q be the polytope with vertices $0, v_1, 2v_2$ (see Figure 1.3). For this choice we have that the volume of Q with respect to Λ is $\mathrm{vol}'_2(Q) = \frac{2\cdot 1}{2} = 1$, the volume of Q with respect to the lattice $\mathbb{Z}^2$ is $\mathrm{vol}_2(Q) = \frac{\sqrt{20}\sqrt{5}}{2} = 5$ and the volume of the fundamental lattice parallelotope $\mathfrak{P}$ with respect to $\mathbb{Z}^2$ is $\mathrm{vol}_2(\mathfrak{P}) = \sqrt{5}\sqrt{5} = 5$.

1.1.3. Polyhedral complexes. A *polyhedral complex* Γ is a finite collection of polyhedra such that the empty set is in Γ, if $P \in \Gamma$ then all faces of P are in Γ as well, and the intersection $P \cap Q$ of two polyhedra $P, Q \in \Gamma$ is a face of both. The largest dimension of a polyhedron in Γ is set as the *dimension* of Γ and the k-dimensional complex Γ is called *pure* if all inclusion maximal elements have dimension k. We already came across a polyhedral complex in Paragraph 1.1.1 when we defined the normal fan $\mathcal{N}_P$ of a polytope P.

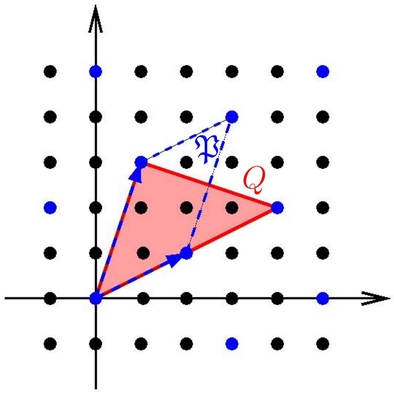

Figure 1.3. The lattice Λ from Example 1.2 with the fundamental lattice parallelotope $\mathfrak{P}$ and the polytope Q.

Let f_k denote the number of k-dimensional elements of an n-dimensional complex Γ. The vector $(f_0, \ldots, f_n)$ is then called the *f-vector* of Γ. If all elements of Γ are polytopes or simplices then we have a *polytopal complex* or *simplicial complex*, respectively. Furthermore, if all the vertices of a polyhedral complex Γ lie in a lattice Λ, e.g. in $\mathbb{Z}^n$, then Γ is called a *lattice complex*.

1.1.4. Minkowski sums. The *Minkowski sum* of two sets $A_1, A_2 \subset \mathbb{R}^n$ is defined as

$$A_1 + A_2 = \{a_1 + a_2 \,|\, a_1 \in A_1, a_2 \in A_2\} \enspace .$$

The set $\mathcal{K}_n$ of convex bodies in $\mathbb{R}^n$ (as well as the set $\mathcal{P}_n$ of polytopes in $\mathbb{R}^n$) together with the Minkowski addition forms a commutative semi group with the set containing the origin as the neutral element. It is possible to define a *Minkowski difference* (cf. Remark 1.3) as

$$A_1 - A_2 = \{p \in \mathbb{R}^n \,|\, p + A_2 \subset A_1\}$$

but this is not the inverse to Minkowski addition. In general we have $A_2 + (A_1 - A_2) \subsetneq A_1$. Only if A_1 is itself a Minkowski sum $A_2 + A_3$, then $A_1 - A_3 = A_2$.

Since Minkowski addition is commutative and associative it generalizes naturally to more than two polytopes. If $\lambda \in \mathbb{R}$ and $A \subset \mathbb{R}^n$

$$\lambda\, A = \{\lambda \cdot p \,|\, p \in A\}$$

is called a *multiple of* A and $\lambda_1 A_1 + \cdots + \lambda_r A_r$ is called a *linear combination* of $A_1, \ldots, A_r$. If $A_1, \ldots, A_r$ are convex then all their linear combinations are as well. Note that the combinatorics of a linear combination $\lambda_1 A_1 + \cdots + \lambda_r A_r$ only depends on which λ_i are zero, which are negative and which are positive (see e.g. [**HRS00**]).

λ might be negative, but again this can not be interpreted as the inverse to Minkowski addition. For $\lambda \in \mathbb{N}$, λA can be pictured geometrically as either the scaling of A by a factor of λ or as $\underbrace{A + \cdots + A}_{\lambda-\text{times}}$.

Remark 1.3. Note that $A_1 - A_2$ and $A_1 + (-A_2)$ are two different things. Unless stated otherwise we will always deal with the object $A_1 + (-A_2)$. Some authors (e.g. [**Sch93**]) refer to the latter as *Minkowski subtraction*.

If P and Q are (lattice) polytopes, then $P + Q$ is again a (lattice) polytope. Also it holds that

$$(P+Q)^v = (P)^v + (Q)^v \ , \tag{1.2}$$

which means in particular that each vertex of $P + Q$ is the sum of vertices of P and Q. Furthermore it can be shown (see e.g. [**Zie95**]) that the normal fan of $P + Q$ is the common refinement of the normal fans of P and of Q. I.e. we have

$$\mathcal{N}_{P+Q} = \{C_P \cap C_Q \,|\, C_P \in \mathcal{N}_P,\ C_Q \in \mathcal{N}_Q\} \ . \tag{1.3}$$

Example 1.4. Let

$$P = \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}\right\}, \quad Q = \operatorname{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\\frac{3}{2}\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}\right\} .$$

Figure 1.4 depicts P, Q, their Minkowski sum $P + Q$ and the sum $P + (-Q)$. Note that $P - Q$ is empty for these polytopes. In Figure 1.5 the normal fans of these polytopes are shown

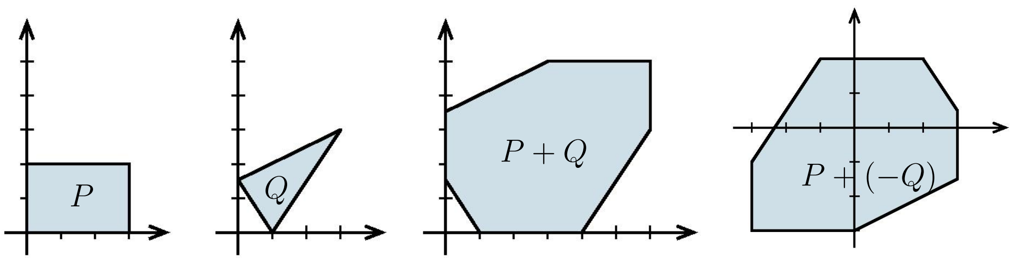

Figure 1.4. From left to right: P, Q, $P + Q$ and $P + (-Q)$.

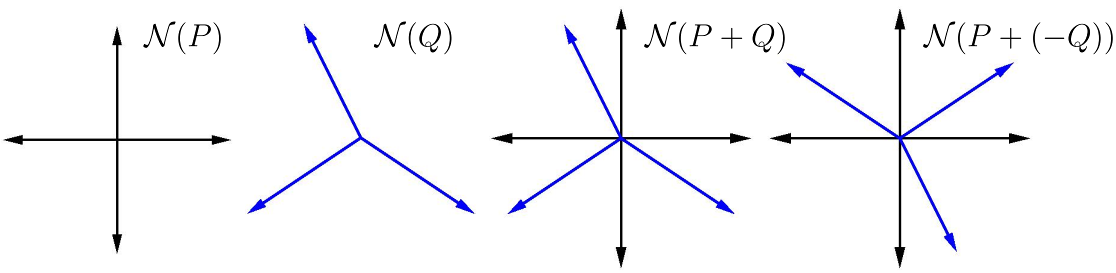

Figure 1.5. From left to right: $\mathcal{N}(P)$, $\mathcal{N}(Q)$, $\mathcal{N}(P + Q)$ and $\mathcal{N}(P + (-Q))$.

Remark 1.5. We chose a geometric approach to Minkowski summation to encourage geometric intuition and to keep the notation pleasant. For precise combinatorial statements it is often more convenient to keep track of information that is lost in the geometric picture. Namely one gives each element $p_1 \in A_1$ and $p_2 \in A_2$ a label and assigns the sum $p = p_1 + p_2$ the tuple of the labels of the summands. This way points having the same geometric coordinates but arising as a sum of different combinations of points are distinguished. For further background and examples see [**LRS**, Section 9.2].

A discussion on the computational complexity of Minkowski summation can be found in [**GS93**] and in [**FW09**].

1.1.5. Hausdorff metric. Let $\mathcal{B}$ be the unit ball in $\mathbb{R}^n$ and let $\lambda \geq 0$. The *Hausdorff distance* of the convex bodies K_1 and K_2 is defined by

$$\delta(K_1, K_2) := \inf\{\lambda \,|\, K_1 \subset K_2 + \lambda \cdot \mathcal{B} \text{ and } K_2 \subset K_1 + \lambda \cdot \mathcal{B}\} \ .$$

Note that the Hausdorff distance is a metric on $\mathcal{K}_n$.

Proposition 1.6 (see [**Ewa96**]). *For every convex body K there exists a sequence of polytopes $(P_j)_{j\in\mathbb{N}}$ that converges to K with respect to the Hausdorff metric.*

With respect to δ the volume $\mathrm{vol}_n(K)$ and the Minkowski sum $K_1 + K_2$ depend continuously on the convex bodies K, K_1 and K_2.

1.2. Mixed Volumes

Mixed volumes have been studied for several decades, nevertheless many questions about them remain open problems. The introduction here is far from being complete and is intended to introduce the most important definitions and properties used in this work. To obtain a solid background we refer the reader to [**BZ88**, **Sch93**] for a thorough geometric discussion, to [**CLO05**] for an easy accessible introduction and to [**Ewa96**, **Ful93**] for a treatment of mixed volumes in the context of algebraic geometry.

1.2.1. Definition and basic properties. Let $K_1, \dots, K_n$ be n convex bodies in $\mathbb{R}^n$ and let $\lambda_1, \dots, \lambda_n$ be non-negative real parameters.

Proposition 1.7 (Minkowski, see e.g. [**Sch93**]). *The function $\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_n K_n)$ is a homogeneous polynomial of degree n in $\lambda_1, \dots, \lambda_n$.*

The coefficient of the mixed monomial $\lambda_1 \cdots \lambda_n$ is called the *mixed volume* of $K_1, \dots, K_n$ and is denoted by $\mathrm{MV}_n(K_1, \dots, K_n)$. The mixed volume can be explicitly computed as

$$\mathrm{MV}_n(K_1, \dots, K_n) = \sum_{j=1}^{n} (-1)^j \sum_{I \subset \{1,\dots,n\},\, |I|=j} \mathrm{vol}_n\left(\sum_{i\in I} K_i\right) \ , \tag{1.4}$$

but for most practical purposes this is not a very useful expression. We will introduce a more convenient method using mixed subdivisions in the next paragraph.

Example 1.8. Take the polytopes P and Q from Example 1.4. Then (1.4) states that $\mathrm{MV}_2(P, Q) = \mathrm{vol}_2(P + Q) - \mathrm{vol}_2(P) - \mathrm{vol}_2(Q) = 24 - 6 - 3 = 15$. [1]

[1] The volume computations were carried out using the polytope software `polymake`, see [**GJ00**].

Remark 1.9. Note that some authors prefer to factor out $n!$ in the definition of the mixed volume. We choose to keep that factor since this scaling guarantees that the mixed volume of polytopes with integer vertices is an integer.

We write $\mathrm{MV}_n(K_1, d_1; \ldots; K_k, d_k)$ to denote the mixed volume where K_i is taken d_i times and $\sum_{i=1}^k d_i = n$. Actually all coefficients of $\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_k K_k)$ can be written as mixed volumes using this notation. Namely we have (see [**Sch93**, Section 5.1])

(1.5)
$$\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_k K_k) = \frac{1}{n!} \sum_{d_1,\ldots,d_k=0}^{n} \binom{n}{d_1 \ldots d_k} \lambda_1^{d_1} \cdots \lambda_k^{d_k} \, \mathrm{MV}(K_1, d_1; \ldots; K_k, d_k) \ ,$$

where the *multinomial coefficient* is defined by

$$\binom{n}{d_1 \ldots d_k} = \begin{cases} \frac{n!}{d_1! \cdots d_k!} & \text{if } d_i \geq 0 \text{ and } \sum_i d_i = n \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Remark 1.10. Since the volume and the Minkowski addition both depend continuously on the convex bodies with respect to the Hausdorff metric, the mixed volume does as well. Due to Proposition 1.6 it will hence often be enough to prove properties of the mixed volume for polytopes and then use the continuity to extend the statement for general convex bodies.

Mixed volumes are always non-negative (see [**Ful93**, Section 5.4]) and they are monotone with respect to inclusion, i.e.

$$\mathrm{MV}(K_1, \ldots, K_n) \geq \mathrm{MV}(K_1', \ldots, K_n') \quad \text{if } K_i \supset K_i' \text{ for all } i \ . \tag{1.7}$$

Furthermore $\mathrm{MV}(K_1, \ldots, K_n)$ is strictly positive if and only if there exist segments $S_i \subset K_i$ $(i = 1, \ldots, n)$ whose directions are linearly independent.

The mixed volume is invariant under permutation of its arguments, i.e.

$$\mathrm{MV}(K_1, \ldots, K_n) = \mathrm{MV}(K_{\sigma(1)}, \ldots, K_{\sigma(n)}) \quad \text{for any permutation } \sigma \tag{1.8}$$

and is linear in each argument, i.e.

$$\mathrm{MV}_n(\ldots, \alpha K_i + \beta K_i', \ldots) = \alpha \, \mathrm{MV}_n(\ldots, K_i, \ldots) + \beta \, \mathrm{MV}_n(\ldots, K_i', \ldots) \ . \tag{1.9}$$

Also it generalizes the usual volume in the sense that

$$\mathrm{MV}_n(K, \ldots, K) = n! \, \mathrm{vol}_n(K) \tag{1.10}$$

holds (cf. [**Sch93**]).

It is possible to express the n-dimensional mixed volume in terms of $(n-1)$-dimensional mixed volumes as stated in the next proposition. Here, we have to take care again that the volume is taken with respect to the underlying lattice. Namely we set

$$\mathrm{MV}_{n-1}'((P_1)^v, \ldots, (P_{n-1})^v) := \frac{\mathrm{MV}_{n-1}((P_1)^v, \ldots, (P_{n-1})^v)}{\mathrm{vol}_{n-1}(\mathfrak{P})} \tag{1.11}$$

where $\mathfrak{P}$ denotes a fundamental lattice parallelotope in the hyperplane orthogonal to v.

Proposition 1.11 (see e.g. [**CLO05**, **EK08**]). *Let K be a convex, full-dimensional body in $\mathbb{R}^n$ and let $P_1, \dots, P_{n-1} \subset \mathbb{R}^n$ be integer polytopes. Then*

$$\mathrm{MV}_n(P_1, \dots, P_{n-1}, K) = \sum_v \max_{a \in K} \langle a, v \rangle \cdot \mathrm{MV}'_{n-1}((P_1)^v, \dots, (P_{n-1})^v)$$

where the sum is taken over all primitive outer normals $v \in \mathbb{Z}^n$, i.e. $\gcd(v_1, \dots, v_n) = 1$, *of facets F of $P_1 + \dots + P_{n-1}$.*

1.2.2. Mixed subdivisions. Let $\mathcal{S} = (S^{(1)}, \dots, S^{(m)})$ be a sequence of finite point sets in $\mathbb{R}^n$ that affinely spans the full space. A sequence $C = (C^{(1)}, \dots, C^{(m)})$ of subsets $C^{(i)} \subset S^{(i)}$ is called a *cell* of $\mathcal{S}$. A *subdivision* of $\mathcal{S}$ is a collection $\Gamma = (C_1, \dots, C_k)$ of cells such that

(i) $\dim(\mathrm{conv}(C_i)) = n$ for all cells C_i,
(ii) $\mathrm{conv}(C_i) \cap \mathrm{conv}(C_j)$ is a face of both convex hulls and
(iii) $\bigcup_{i=1}^k \mathrm{conv}(C_i) = \mathrm{conv}(\mathcal{S})$

where $\mathrm{conv}(A) := \mathrm{conv}(A^{(1)} + \dots + A^{(m)})$ for a sequence A of point sets. A subdivision is called *mixed* if additionally

(iv) $\sum_{i=1}^m \dim(\mathrm{conv}(C_j^{(i)})) = n$ for all cells C_j in Γ

and it is called *fine mixed* if furthermore

(v) $\sum_{i=1}^m (|C_j^{(i)}| - 1)) = n$ for all cells C_j in Γ

where $|A|$ denotes the number of points in a finite set $A \subset \mathbb{R}^n$. The *type* of a cell is defined as

$$\mathrm{type}(C) = \big(\dim(\mathrm{conv}(C^{(1)})), \dots, \dim(\mathrm{conv}(C^{(m)}))\big)$$

and cells of type $(d_1, \dots, d_m)$ with $D_i \geq 1$ for all i will be called *mixed cells*.

Example 1.12. Let $\mathcal{S} = (\{(0,0)^T, (3,0)^T, (0,2)^T, (3,2)^T\}, \{(1,0)^T, (0, \frac{3}{2})^T, (3,3)^T\})$. Then $\Gamma = (C_1, \dots, C_6)$ where

$$\begin{aligned}
C_1 &= (\{(0,2)^T, (3,2)^T\}, \{(0, \tfrac{3}{2})^T, (3,3)^T\}), \\
C_2 &= (\{(3,0)^T, (3,2)^T\}, \{(0, \tfrac{3}{2})^T, (1,0)^T\}), \\
C_3 &= (\{(3,0)^T, (3,2)^T\}, \{(1,0)^T, (3,3)^T\}), \\
C_4 &= (\{(0,0)^T, (3,0)^T\}, \{(1,0)^T, (0, \tfrac{3}{2})^T\}), \\
C_5 &= (\{(0,0)^T, (3,0)^T, (0,2)^T, (3,2)^T\}, \{(0, \tfrac{3}{2})^T\}), \\
C_6 &= (\{(3,2)^T\}, \{(1,0)^T, (0, \tfrac{3}{2})^T, (3,3)^T\})
\end{aligned}$$

is a mixed subdivision of $\mathcal{S}$. $C_1, \dots, C_4$ are cells of type $(1,1)$, C_5 is of type $(2,0)$ and C_6 is of type $(0,2)$. The mixed subdivision Γ is not fine mixed since C_5 violates condition (v).

Remark 1.13. For technical reasons we prefer here to define mixed subdivisions on point sets rather then on polytopes. These definitions extend naturally to sequences of polytopes P_i by considering their *vertex sets* $\mathrm{vert}(P_i)$ as the point sets above. By abuse of notation we speak then of a mixed subdivision of $P := P_1 + \cdots + P_m$ meaning a mixed subdivision

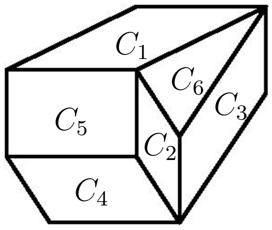

Figure 1.6. A mixed subdivision Γ of $P+Q$.

of $(\mathrm{vert}(P_1),\ldots,\mathrm{vert}(P_m))$. As cells of such a subdivision we always consider sums of faces $F_1+\cdots+F_m$ where F_i is a face of P_i. If all cells of a subdivision Γ of $P_1+\cdots+P_m$ are simplices then Γ is called a *triangulation*.

With this terminology an explicit formula to calculate the mixed volume can be stated (cf. [**HS95**]):

$$\mathrm{MV}_n(P_1,d_1;\ldots;P_r,d_r) = \sum_{\substack{C \text{ cell type } (d_1,\ldots,d_r) \\ \text{of a mixed subdivision} \\ \text{of } (P_1,\ldots,P_r)}} d_1!\cdots d_r!\,\mathrm{vol}_n(C)\ . \tag{1.12}$$

For a cell $C=(C^{(1)},\ldots,C^{(r)})$ of type $(d_1,\ldots,d_r)$ in a mixed subdivision with $C^{(i)} = \{p_0^{(i)},\ldots,p_{d_i}^{(i)}\}$ we define the matrix $M(C)$ to be the $n\times n$ matrix whose rows are $p_j^{(i)}-p_0^{(i)}$ for $1\le i\le r$ and $1\le j\le d_i$. We have that

$$|\det(M(C))| = d_1!\cdots d_r!\cdot\mathrm{vol}(C) \tag{1.13}$$

which simplifies the computation of (1.12).

Example 1.14. Consider again the polytopes P and Q from Example 1.4. Figure 1.6 shows a mixed subdivision of $P+Q$ (which is of course the subdivision from Example 1.12). By (1.12) we have that $\mathrm{MV}_2(P,Q)=\mathrm{vol}_2(C_1)+\mathrm{vol}_2(C_2)+\mathrm{vol}_2(C_3)+\mathrm{vol}_2(C_4)=\frac{9}{2}+2+4+\frac{9}{2}=15$.

To construct mixed subdivisions we proceed as in [**HS95**]. Not every subdivision can be constructed in this way but for our purposes this construction suffices. For each of the point sets $S^{(i)}$ from $\mathcal{S}$ choose a lifting function $\mu_i : S^{(i)}\to\mathbb{R}$ and denote by $\hat{A}$ the lifted point set $\{(q,\mu_i(q)) : q\in A\}\subset\mathbb{R}^{n+1}$.

The set of those facets of $\mathrm{conv}(\hat{S}^{(1)}+\ldots+\hat{S}^{(m)})$ which have an inward pointing normal with a positive last coordinate is called the *lower hull* of the Minkowski sum. If we project down this lower hull back to $\mathbb{R}^n$ by forgetting the last coordinate we get a subdivision of $(S^{(1)},\ldots,S^{(m)})$. We call such a subdivision *coherent* (or *regular*) and will say it is *induced* by $\mu=(\mu_1,\ldots,\mu_m)$.

Example 1.15. Once more we consider the polytopes P and Q from Example 1.4. Now the following lifting functions μ_1,μ_2 are chosen:

$$\mu_1(p) := \left\langle p, \begin{pmatrix}1/5\\2/5\end{pmatrix}\right\rangle,\quad \mu_2(p) := \left\langle p, \begin{pmatrix}1/2\\1/2\end{pmatrix}\right\rangle\ .$$

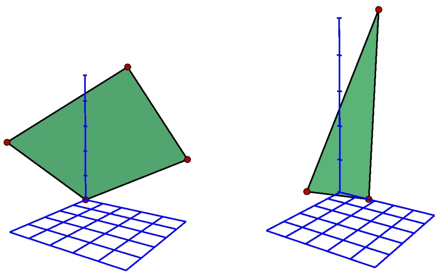

Figure 1.7. The lifted polytopes $\hat{P}$ and $\hat{Q}$.

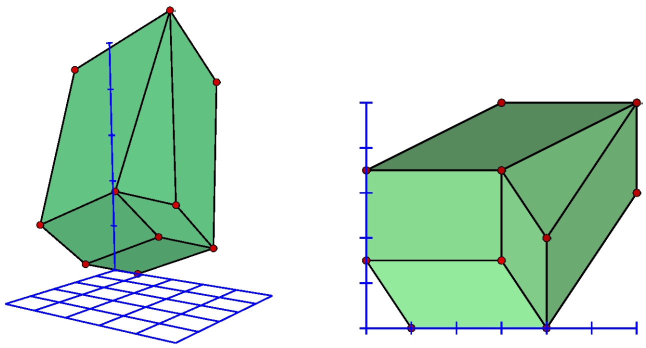

Figure 1.8. The sum $\hat{P} + \hat{Q}$ of the lifted polytopes and the projection of the lower hull to $\mathbb{R}^2$.

Figure 1.7 shows the lifted polytopes $\hat{P}$ and $\hat{Q}$ and Figure 1.8 illustrates the sum and its projection to the first two coordinates.

Not all coherent subdivisions are mixed but there are conditions on liftings which guarantee that the induced subdivision is mixed.

Proposition 1.16 (See [**HS95**]). *If for each n-dimensional cell C in the subdivision of $(S^{(1)}, \dots, S^{(m)})$ induced by μ we have that $M(\hat{C})$ has maximal rank then the subdivision is fine mixed.*

A lifting μ that satisfies the condition of Proposition 1.16 is called *sufficiently generic*. The maximal minors of $M(\hat{C})$ give linear conditions on the values $\mu(q)$ for $q \in S^{(i)}$.

To achieve this sufficient genericity it is enough that every vertex of the lower envelope can be expressed uniquely as a Minkowski sum and this can be achieved by considering linear lifting functions $\mu_i : \mathbb{R}^n \to \mathbb{R}$ (see [**HS95**, **EC95**]).

1.2.3. The Cayley-Trick and fiber polytopes. The *Cayley-Trick* relates mixed subdivisions of a sequence of point sets $\mathcal{S} = (S^{(1)}, \dots, S^{(m)})$ to subdivisions of a single point set that is constructed from $\mathcal{S}$ (see [**GKZ94**, **HRS00**, **Stu94**]). We sketch here the basic ideas and refer to [**LRS**, Chapter 9] for a precise combinatorial treatment as well as some nice graphical illustrations.

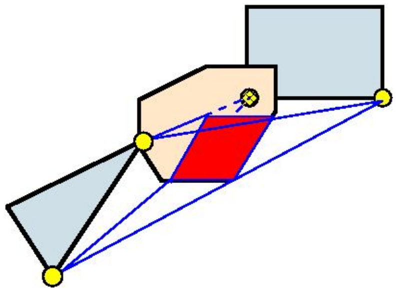

Figure 1.9. A correspondence between cells in the Cayley-Trick.

The *Cayley embedding* $\mathcal{C}(\mathcal{S}) \subset \mathbb{R}^n \times \mathbb{R}^m$ of the sequence of point sets $\mathcal{S} = (S^{(1)}, \dots, S^{(m)})$ in $\mathbb{R}^n$ is defined as

$$\mathcal{C}(S^{(1)}, \dots, S^{(m)}) := \bigcup_{i=1}^{m} (S^{(i)} \times \xi_i) \tag{1.14}$$

where ξ_i denotes the i^{th} unit vector in $\mathbb{R}^m$. The Cayley-Trick states now that there is a one-to-one correspondence between subdivisions of $\mathcal{C}(S^{(1)}, \dots, S^{(m)})$ and mixed subdivisions of $(S^{(1)}, \dots, S^{(m)})$.

Figure 1.9 shows an example of how a cell of $\mathcal{C}(\mathrm{vert}(P), \mathrm{vert}(Q))$ corresponds to a mixed cell of $P + Q$ for the polytopes P and Q from Example 1.4.

Let $P \subset \mathbb{R}^N$ be a polytope and let $\pi : \mathbb{R}^N \to \mathbb{R}^n$ be a linear function that projects P to the polytope $Q \subset \mathbb{R}^n$. For any point $x \in Q$ its fiber $\pi^{-1}(x) \cap P$ is a $(N-n)$-dimensional polytope and the *fiber polytope* $\Sigma_\pi(P) \subset \mathbb{R}^{N-n}$ is defined as the following Minkowski integral:

$$\Sigma_\pi(P) := \frac{1}{\mathrm{vol}_n(Q)} \int_Q (\pi^{-1}(x) \cap P)\, dx. \tag{1.15}$$

The combinatorics of a fiber polytope contains a nice surprise. Namely the faces of $\Sigma_\pi(P)$ are in bijection with the coherent polyhedral subdivisions of Q which are induced by the boundary of P (cf. [**BS92**, **Zie95**]). Hence in particular, if P is a simplex then the vertices of $\Sigma_\pi(P)$ correspond to triangulations of Q and therefore fiber polytopes generalize the secondary polytopes from Gel'fand, Kapranov and Zelevinsky [**GKZ94**].

Of course the question arises whether there is a similar combinatorial structure, a *mixed fiber polytope*, which describes the mixed subdivisions of a set of polytopes. McDonald [**McD02**] as well as Michiels and Cools [**MC00**] predicted the existence of such a structure and McMullen [**McM04**] and independently Esterov and Khovanskii [**EK08**] were able to give a construction. Namely, for polytopes $P_1, \dots, P_r$ and positive real parameters $\lambda_1, \dots, \lambda_r$ the fiber polytope

$$\Sigma_\pi(\lambda_1 P_1 + \cdots + \lambda_r P_r)$$

depends polynomially on $\lambda_1, \dots, \lambda_r$ and this polynomial is homogeneous of degree $n+1$. The mixed fiber polytope is defined as the coefficient of $\lambda_1 \cdots \lambda_r$ in $\Sigma_\pi(\lambda_1 P_1 + \cdots + \lambda_r P_r)$.

To compute fiber polytopes and mixed fiber polytopes Sturmfels and Yu provide the software package `TrIM`, see [**SY08**].

1.2.4. The lift-prune algorithm. In this section a state of the art algorithm from Emiris and Canny [**EC95**] to compute the mixed volume is sketched.[2]

Assume that we already have a sufficiently generic linear lifting μ_i for each polytope P_i $(i = 1, \dots, n)$ in the sense of Paragraph 1.2.2. The lifted polytopes will be denoted by $\hat{P}_i$ and the Minkowski sum of the P_i is denoted by P. The idea for the computation of $\mathrm{MV}(P_1, \dots, P_n)$ is then the following. For each combination of n edges from the given polytopes it is tested whether their lifted Minkowski sum lies on the lower envelope of $\hat{P}$. If so, compute the volume of the corresponding mixed cell and add it to the mixed volume. To make this naive algorithm efficient we employ the fact (see [**EC95**]) that $\sum_{j \in J} \hat{e}_j$ lies on the lower envelope of $\sum_{j \in J} \hat{P}_j$ only if $\sum_{t \in T} \hat{e}_t$ lies on the lower envelope of $\sum_{t \in T} \hat{P}_t$ for every subset $T \subset J$.

So instead of performing a few expensive tests on the sum of n edges, many small tests are done to build up valid sums of edges step by step. Each test for a k-tuple of edges $e_1, \dots, e_k$ is implemented as a linear program (LP) as follows. Let $\hat{m}_i \in \mathbb{R}^{k+1}$ denote the midpoint of the lifted edge $\hat{e}_i$ of $\hat{P}_i$ such that $\hat{m} = \hat{m}_1 + \cdots + \hat{m}_k$ is an interior point of the Minkowski sum $\hat{e}_1 + \cdots + \hat{e}_k$. Consider the linear program

$$\begin{aligned} &\text{maximize } s \in \mathbb{R}_{\geq 0} \\ &\text{s.t. } \hat{m} - (0, \dots, 0, s) \in \hat{P}_1 + \cdots + \hat{P}_k \ . \end{aligned} \tag{1.16}$$

If we denote the vertices of P_i by $v_1^{(i)}, \dots, v_{r_i}^{(i)}$ this can be written as

$$\begin{aligned} &\text{maximize } s \in \mathbb{R}_{\geq 0} \\ &\text{s.t. } \hat{m} - (0, \dots, 0, s) = \sum_{i=1}^{k} \sum_{j=1}^{r_i} \lambda_j^{(i)} \hat{v}_j^{(i)} \\ &\sum_{j=1}^{r_i} \lambda_j^{(i)} = 1 \quad \forall\, i = 1, \dots, n \\ &\lambda_j^{(i)} \geq 0 \quad \forall\, i, j \ . \end{aligned} \tag{1.17}$$

s measures the vertical distance of $\hat{m}$ to the lower envelope of the Minkowski sum. Hence $\hat{m}$ lies on the lower envelope of $\hat{P}_1 + \cdots + \hat{P}_k$ if and only if the optimal value of (1.16) is zero.

See Algorithm 1 for a pseudo-code description. An implementation of this algorithm can be found at: `http://www-sop.inria.fr/galaad/logiciels/emiris/soft_geo.html` or in the `PHCpack` by Jan Verschelde, see [**Ver99**].

The worst case complexity of the algorithm arising from these ideas is in $r^{O(n)}$ where r denotes the maximal number of vertices of the P_i (cf. [**Emi96**]). Computing the volume of the convex hull of a point set is #P-hard (cf. [**Kha93**]). Since the mixed volume is a

[2] There are heuristic improvements of this algorithm, see [**ZE05**].

Input: The vertex sets of polytopes $P_1, \dots, P_n \subset \mathbb{R}^n$
Output: $\mathrm{MV}_n(P_1, \dots, P_n)$
begin
  Enumerate the edges of all polytopes $P_1, \dots, P_n$ respectively into sets $E_1, \dots, E_n$;
  Compute random lifting vectors $\mu_1 \dots, \mu_n \in \mathbb{Q}^n$;
  for $i \in \{1, \dots, n\}$ and $e_i \in E_i$ **do**
    Compute the lifted edge $\hat{e}_i$;
  end
  $\mathrm{MV}_n(P_1, \dots, P_n) \leftarrow 0$;
* **if** $E_1 = \emptyset$ **then** terminate;
  else Pick any $e_1 \in E_1$;
  $E_1 \leftarrow E_1 \setminus \{e_1\}$;
  Create current tuple (e_1);
** **for** $j \in \{2, \dots, n\}$ **do**
    $E'_j \leftarrow E_j$;
  end
  $k \leftarrow 1$;
  for $i \in \{k+1, \dots, n\}$ **do**
    for $e_i \in E'_i$ **do**
      if $\sum_{j=1}^{k} \hat{e_j} + \hat{e_i}$ does not lie on the lower envelope of $\sum_{j=1}^{k} \hat{P}_j + \hat{P}_i$ **then**
        $E'_i \leftarrow E'_i \setminus \{e_i\}$;
      end
    end
  end
  $k \leftarrow k + 1$;
  if $k > n$ **then**
    $\mathrm{MV}_n(P_1, \dots, P_n) \leftarrow \mathrm{MV}_n(P_1, \dots, P_n) + \mathrm{vol}_n(e_1 + \dots + e_n)$;
    Continue from line $*$;
  end
  if $k \leq n$ **then**
    if $E'_k = \emptyset$ **then** Continue from line $*$;
    else Add some edge $e_k \in E'_k$ to the current tuple $(e_1, \dots, e_{k-1})$;
    $E'_k \leftarrow E'_k \setminus \{e_k\}$;
    Continue from line $**$;
  end
end

Algorithm 1: The Lift-Prune Algorithm from Emiris and Canny [**EC95**].

generalization of the volume (see (1.10)) this gives a lower bound on the complexity. For further discussions of computational aspects see [**GK94**].

1.2.5. Exploiting symmetries. Let $\mathcal{S} = (S^{(1)}, \dots, S^{(n)})$ be a sequence of point sets $S^{(i)} \subset \mathbb{R}^n$ and let G be a finite group, e.g. a subgroup of the symmetric group $\mathbb{S}_n$. We say that $\mathcal{S}$ is *G-symmetric* if

$$g \,.\, \mathcal{S} = \mathcal{S} \circ g \quad \forall g \in G \tag{1.18}$$

where the operation on the left means a permutation of the support sets $S^{(i)}$ while the operation on the right means a permutation of the vector components in all points of all $S^{(i)}$.

Furthermore we call a lifting $\mu = (\mu_1, \ldots, \mu_n)$ *G-symmetric* if

$$g \,.\, \hat{\mathcal{S}} = \hat{\mathcal{S}} \circ \hat{g} \quad \forall g \in G \tag{1.19}$$

where $\hat{g}$ acts like g on the first n coordinates and leaves the $(n+1)$ coordinate fixed. If a support set or a lifting is G-symmetric then it is as well G'-symmetric for every subgroup G' of G.

The problem of finding a symmetric lifting that is still generic in the sense of Proposition 1.16 is not fully understood. We investigate now conditions on lifting values that arise from the symmetries. For a point $q \in S^{(i)}$ we will define the *point orbit* of the tuple $(q, S^{(i)})$ as

$$\mathcal{O}_{q,i} := \{(q \circ g, g(i)) \mid g \in G\} \ .$$

A lifting μ for a G-symmetric support set is G-symmetric if and only if in each point orbit $\mathcal{O}_{q,i}$ every point has the same lifting value. The symmetries of a support set and a lifting function imply the following properties of the cell structure.

Proposition 1.17 (See [**VG95**]). *Let $\mathcal{S}$ and μ be G-symmetric and let C be a cell of the μ-induced subdivision of $\mathcal{S}$ such that $\hat{C}$ has inner normal $(\gamma, 1)$. Then we have for all $g \in G$ that*

$$D := g^{-1} \,.\, C \circ g$$

is a cell of the μ-induced subdivision as well and $\hat{D}$ has inner normal $(\gamma \circ g, 1)$.

For a cell C we define the *cell orbit* of C under G by

$$\mathcal{O}_C := \{g^{-1} \,.\, C \circ g \mid g \in G\} \ .$$

Then the following statement holds which simplifies the calculation of mixed volumes for symmetric Newton polytopes.

Proposition 1.18 (See [**VG95**]). *Let $\mathcal{S} = (S^{(1)}, \ldots, S^{(r)})$ and μ be G-symmetric such that μ induces a fine mixed subdivision on $\mathcal{S}$. Then*

$$\mathrm{MV}_n(\mathrm{conv}(S^{(1)}), d_1; \ldots; \mathrm{conv}(S^{(r)}), d_r) = \sum_{\mathcal{O}_C} d_1! \cdots d_r! \cdot \#\mathcal{O}_c \cdot \mathrm{vol}_n(C) \tag{1.20}$$

where C is a cell in the μ induced subdivision of type $(k_1, \ldots, k_r)$ that generates the orbit $\mathcal{O}_C$.

1.3. Tropical Geometry

There are several approaches to tropical geometry and each has its advantages. Tropical hypersurfaces can be defined as the image under a (non-archemedian) valuation map of varieties over an algebraically closed field (see e.g. [**SS04**]), as the corner locus of piecewise linear functions (see e.g. [**Vig07**]), as limits of amoebas (see e.g. [**EKL06**]) or

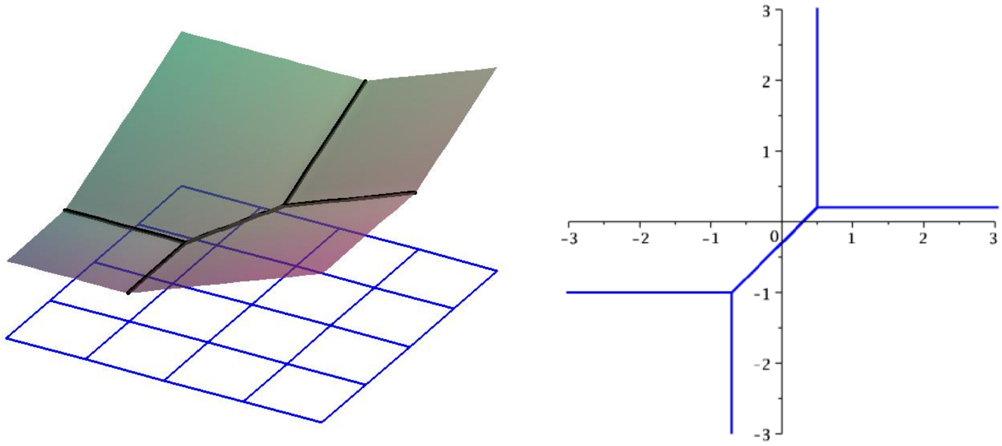

Figure 1.10. The function f and its non-linear locus.

as polyhedral complexes that satisfy certain balancing conditions (see e.g. [**Mik04**]). A comparison of these approaches in 2-dimensional space and a discussion of the difficulties or chances the different viewpoints inherit is given in [**Gat06**] (see also [**Mik06**]). The focus of this work is on the first two approaches since they are the most suitable for the techniques employed here.

1.3.1. Tropical hypersurfaces as corner loci. Let $\mathbb{R}_{\text{trop}} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ denote the *tropical semiring*. The arithmetic operations of *tropical addition* $\oplus$ and *tropical multiplication* $\odot$ are

$$x \oplus y = \max\{x, y\} \quad \text{and} \quad x \odot y = x + y\ . \tag{1.21}$$

Equivalently tropical addition can be defined as $\min\{x, y\}$ (e.g. [**RGST05**]) but results in either preferred notation can easily be translated into each other. A *tropical Laurent polynomial* f in n variables $x_1, \ldots, x_n$ is an expression of the form

$$f = \bigoplus_{\alpha \in \mathcal{S}(f)} c_\alpha \odot x_1^{\alpha_1} \odot \cdots \odot x_n^{\alpha_n} = \max_{\alpha \in \mathcal{S}(f)} (c_\alpha + \alpha_1 x_1 + \cdots + \alpha_n x_n) \tag{1.22}$$

with real numbers c_α. The support set $\mathcal{S}(f)$ is always assumed to be a finite subset of $\mathbb{Z}^n$, and its convex hull $\mathrm{P}(f) \subset \mathbb{R}^n$ is called the *Newton polytope of* f. A tropical polynomial $f(x_1, \ldots, x_n)$ defines a convex, piecewise linear function $f : \mathbb{R}^n \to \mathbb{R}$ and we define the *tropical hypersurface* $X(f)$ as the non-linear locus of f (see Figure 1.10). These are those points $x \in \mathbb{R}^n$ such that $\max_{\alpha \in \mathcal{S}(f)}(c_\alpha + \alpha_1 x_1 + \cdots + \alpha_n x_n)$ is attained at least twice.

Example 1.19. Let $f = 4 \oplus 4.7 \odot x \oplus 5 \odot y \oplus 4.5 \odot x \odot y$. Figure 1.10 shows the function f and its nonlinear locus.

1.3.2. Tropical hypersurfaces via Puiseux series. A *Puiseux series* is a formal power series

$$g = \sum_{\beta \in \mathbb{Q}} c_\beta t^\beta \tag{1.23}$$

in the variable t with coefficients in $\mathbb{C}$ and such that the subset of those $\beta \in \mathbb{Q}$ with $c_\beta \neq 0$ is bounded below and has a finite set of denominators. The Puiseux series form an algebraically closed field (see e.g. [**Wal50**]) which we denote here by $\mathbb{K}$. For a non-zero element $g \in \mathbb{K}$ the minimum of all $\beta \in \mathbb{Q}$ with $c_\beta \neq 0$ is called the *order* of g and is denoted by $\operatorname{ord}(g)$. Note that the order defines a so called *valuation* [3] on $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$, i.e.

$$\operatorname{ord}(g_1 + g_2) \geq \min\{\operatorname{ord}(g_1), \operatorname{ord}(g_2)\} \quad \text{and} \quad \operatorname{ord}(g_1 \cdot g_2) = \operatorname{ord}(g_1) + \operatorname{ord}(g_2) \ . \tag{1.24}$$

For a polynomial $h \in \mathbb{K}[x_1, \ldots, x_n]$ we denote by $\operatorname{trop}(h)$ the *tropicalization* of h which is the tropical Laurent polynomial obtained from h by replacing the usual multiplication and addition by their tropical counterparts and by replacing the coefficients $g \in \mathbb{K}$ of h by the negative value of their orders $-\operatorname{ord} g$. Namely we have:

$$h = \sum_{\alpha \in \mathcal{S}(h)} g_\alpha x^\alpha \quad \Rightarrow \quad \operatorname{trop}(h) = \bigoplus_{\alpha \in \mathcal{S}(h)} -\operatorname{ord}(g_\alpha) \odot x_1^{\alpha_1} \odot \cdots \odot x_n^{\alpha_n} \ .$$

Furthermore we denote by $X_{\mathbb{K}}(h)$ the subset of $(\mathbb{K}^*)^n$ on which h vanishes and $\mathcal{V}$ denotes the map

$$\begin{aligned} \mathcal{V} : (\mathbb{K}^*)^n \quad &\longrightarrow \quad \mathbb{Q}^n \\ (g_1, \ldots, g_n) \quad &\mapsto \quad (-\operatorname{ord}(g_1), \ldots, -\operatorname{ord}(g_n)) \ . \end{aligned} \tag{1.25}$$

Proposition 1.20 (Kapranov). *With the notation from above we have*

$$X(\operatorname{trop}(h)) \cap \mathbb{Q}^n = \mathcal{V}(X_{\mathbb{K}}(h)) \ .$$

This implies that we could have equivalently defined tropical hypersurfaces as the closure in $\mathbb{R}^n$ of the image under the valuation map of a codimension 1 variety defined in $(\mathbb{K}^*)^n$. Note that the minus sign in the definition of the valuation map (1.25) resembles our choice of "max" over "min" in (1.21).

Remark 1.21. Note that instead of the field $\mathbb{K}$ of Puiseux series we could have done this construction using any field with a non-archemedian valuation, e.g. the p-adic numbers (cf. [**JSY07**]).

1.3.3. Tropical varieties. The correspondence from Proposition 1.20 becomes more complicated when we deal with intersections of hypersurfaces. Let $I = \langle h_1, \ldots, h_k \rangle$ be an ideal in $\mathbb{K}[x_1, \ldots, x_n]$, then the *tropical variety* of I, denoted by X_I, can be defined as the closure in $\mathbb{R}^n$ of $\mathcal{V}(X_{\mathbb{K}}(I))$ just like above. Unfortunately it is not guaranteed that the intersection of the tropical hypersurfaces $\mathcal{I} := X(\operatorname{trop}(h_1)) \cap \cdots \cap X(\operatorname{trop}(h_k))$ equals X_I, and even worse, there does not even need to exist an ideal J such that $\mathcal{I} = X_J$. Hence an intersection $\mathcal{I}$ of tropical varieties, which is called a *tropical prevariety*, does not need to be a tropical variety itself. It has been shown however, that every ideal I in

[3] The term "valuation" is used in a different way in Chapter 3.

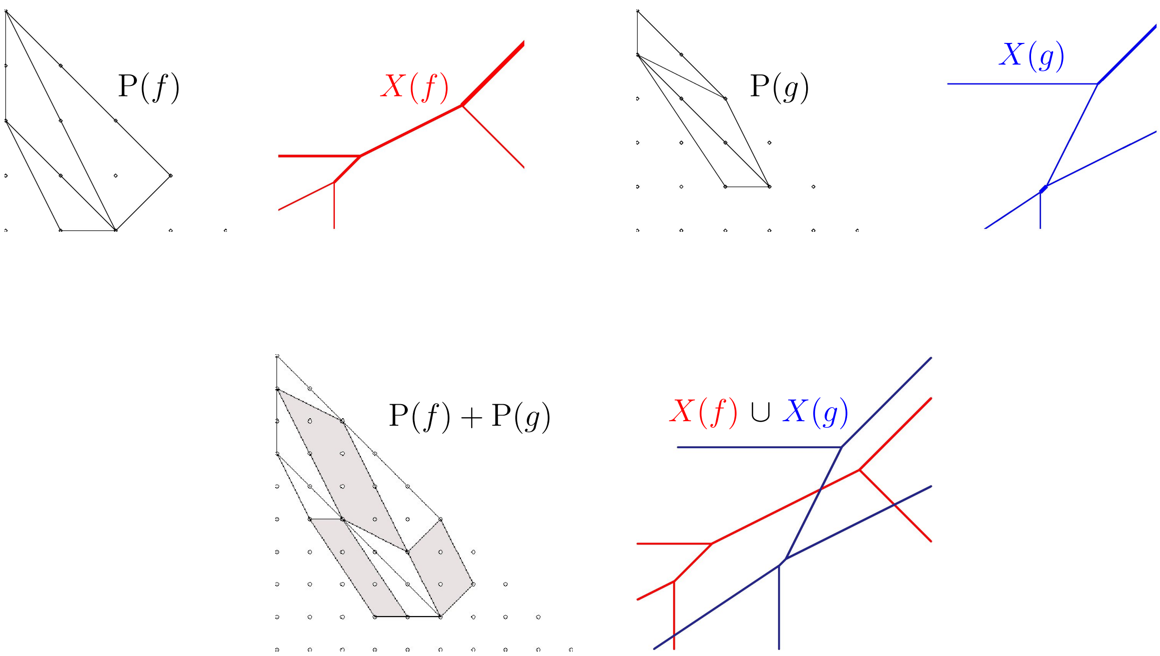

Figure 1.11. Top: $\mathrm{P}(f)$ and $\mathrm{P}(g)$ with the privileged subdivision and the tropical curves $X(f)$ and $X(g)$. (Here bold edges indicate higher multiplicities.) Bottom: $\mathrm{P}(f \odot g)$ with the privileged subdivision and the tropical curve $X(f \odot g)$. (The shaded regions are the mixed cells of the privileged subdivision.)

$\mathbb{K}[x_1, \dots, x_n]$ has a finite set of generators $h_1, \dots, h_r$, called a *tropical basis* of I, such that $X_I = X(\operatorname{trop}(h_1)) \cap \dots \cap X(\operatorname{trop}(h_r))$. Concerning the computation of tropical bases for a given ideal we refer to [**BJS$^+$07**, **HT08**].

1.3.4. Privileged subdivisions and duality. Any tropical hypersurface $X(f)$ is a pure polyhedral complex of codimension 1 in $\mathbb{R}^n$ which has bounded and unbounded cells. The set of m-dimensional cells of a polyhedral complex X will be denoted by $X^{(m)}$. For tropical polynomials f_1, f_2 we have $\mathrm{P}(f_1 \odot f_2) = \mathrm{P}(f_1) + \mathrm{P}(f_2)$ and $X(f_1 \odot f_2) = X(f_1) \cup X(f_2)$, see [**Vig07**, Lemma 1.2]

Example 1.22. Consider the two tropical polynomials

$$
\begin{aligned}
f &= -62 \odot x \oplus 97 \odot x^2 \oplus -73 \odot y^2 \oplus -4 \odot x^3 \odot y \oplus -83 \odot x^2 \odot y^2 \oplus -10 \odot y^4 \\
g &= -10 \odot x^2 \odot y \oplus 31 \odot x^3 \odot y \oplus -51 \odot x \odot y^3 \oplus 77 \odot y^4 \oplus 95 \odot x^2 \odot y^3 \oplus y^5 \ .
\end{aligned}
$$

Figure 1.11 shows their curves and their Newton polytopes as well as the Newton polytope of the product $f \odot g$ and the union $X(f) \cup X(g)$.[4]

[4] All 2-dimensional pictures of tropical hypersurfaces were made with the tropical maple package of N. Grigg: `math.byu.edu/tropical/maple`

The Newton polytope $\mathrm{P}(f)$ of a tropical polynomial f comes with a *privileged subdivision* $\Gamma(f)$. Namely we lift the points $\alpha \in \mathcal{S}(f)$ into $\mathbb{R}^{n+1}$ using the coefficients c_α as lifting values. The set of those facets of $\hat{\mathrm{P}}(f) := \mathrm{conv}\{(\alpha, c_\alpha) \mid \alpha \in \mathcal{S}(f)\}$ which have an inward pointing normal with a negative last coordinate is called the *upper hull*. If we project down this upper hull back to $\mathbb{R}^n$ by forgetting the last coordinate we get a subdivision of $\mathrm{P}(f)$ (see Figure 1.11 for examples). On a set of k tropical polynomials $f_1, \ldots, f_k$ the coefficients induce a *privileged subdivision* $\Gamma(f_1, \ldots, f_k)$ of $\mathrm{P}(f_1) + \cdots + \mathrm{P}(f_k)$ by projecting down the upper hull of $\hat{\mathrm{P}}(f_1) + \cdots + \hat{\mathrm{P}}(f_k)$. For a generic choice of coefficients in the system $f_1, \ldots, f_k$ this subdivision will be mixed (cf. [**HS95**]).

Remark 1.23. This is of course similar to the construction of a mixed subdivision as described in Paragraph 1.2.2. That we use here the upper hull of the lifted Minkowski sum instead of the lower hull is due to our choice of "max" over "min" in the definition of the tropical addition (1.21).

The subdivision $\Gamma(f_1, \ldots, f_k)$ and the union $X(f_1) \cup \cdots \cup X(f_k)$ of tropical hypersurfaces are polyhedral complexes which are dual in the sense that there is a one-to-one correspondence between their cells which reverses the inclusion relations (see [**BB07**,**Mik04**]). Each cell C in $\Gamma(f_1, \ldots, f_k)$ corresponds to a cell A in $X(f_1) \cup \cdots \cup X(f_k)$ such that $\dim(C) + \dim(A) = n$, C and A span orthogonal real affine spaces and A is unbounded if and only if C lies on the boundary of $\mathrm{P}(f_1) + \cdots + \mathrm{P}(f_k)$. Furthermore we have that a cell A of $X(f_1) \cup \cdots \cup X(f_k)$ is in the intersection $\mathcal{I} = X(f_1) \cap \cdots \cap X(f_k)$ if and only if the corresponding dual cell C in $\Gamma(f_1, \ldots, f_k)$ is mixed.

A cell A in $\mathcal{I}$ can be written as $A = \bigcap_{i=1}^k A_i$ where $A_i \in X_i$. If we require that A lies in the relative interior of each A_i then this representation is unique. The dual cell C of A has then a unique decomposition into a Minkowski sum $C = F_1 + \cdots + F_k$ where each F_i is dual to A_i. We will always refer to this decomposition if not stated otherwise.

1.4. Bernstein's Theorem

One of the most important tools in this work is Bernstein's Theorem. This result provides a method to study the solutions of systems of polynomial equations by discrete geometric methods. We state here Bernstein's original work and discuss some generalizations. Furthermore a proof of Bernstein's Theorem using Puiseux series is sketched (cf. [**HS95**]).

1.4.1. The BKK bound.

For a Laurent polynomial $f = \sum_{\alpha \in S(f)} c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ the Newton polytope $\mathrm{P}(f) \subset \mathbb{R}^k$ is the convex hull of the monomial exponent vectors, i.e. $\mathrm{P}(f) = \mathrm{conv}\, S(f)$. Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Proposition 1.24 (Bernstein's Theorem [**Ber75**])**.** *Given Laurent polynomials* $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ *with finitely many common zeroes in* $(\mathbb{C}^*)^n$ *and let* $\mathrm{P}(f_i)$ *denote the Newton polytope of* f_i. *Then the number of common zeros of* $f_i = 0$ *in* $(\mathbb{C}^*)^n$ *is bounded above by the mixed volume* $\mathrm{MV}_n(\mathrm{P}(f_1), \ldots, \mathrm{P}(f_n))$. *Moreover for generic choices of coefficients in the* f_i, *the number of common solutions is exactly* $\mathrm{MV}_n(\mathrm{P}(f_1), \ldots, \mathrm{P}(f_n))$.

Remark 1.25. Here, and throughout this work *generic* is interpreted as follows. A subset A of $\mathbb{C}^m$ is called *Zariski open* if there is an algebraic variety V, i.e. a solution set to a system of algebraic equations, such that $A = \mathbb{C}^m \setminus V$. We say that a statement is true for a *generic choice* in $\mathbb{C}^m$ if it is true for a non-empty Zariski open subset of $\mathbb{C}^m$. This implies that the statement is true "almost everywhere" in a measure theoretic sense.

The statement can be formulated even more general. Fulton [**Ful93**] showed that the cardinality of the common isolated zeros of the system $f_i = 0$ $(i = 1, \ldots, n)$ in $(\mathbb{C}^*)^n$ is bounded by $\mathrm{MV}_n(\mathrm{P}(f_1), \ldots, \mathrm{P}(f_n))$, regardless of the dimension of the variety. Canny and Rojas [**CR91**, **Roj94**] showed furthermore that equality holds if a certain subset of the coefficients corresponding to the vertices of the $\mathrm{P}(f_i)$ is generic.

On a first view the statement of Bernstein's Theorem is very surprising. The possibility to obtain algebraic information from a discrete geometric object might be unexpected. To give an intuition for this correspondence and since Bernstein's Theorem is crucial for our work we sketch the proof for the case of generic coefficients. The focus is on describing the method of toric deformation which provides a nice way to understand the interplay of discrete geometry and algebra here. The proof presented is not Bernstein's original proof from [**Ber75**] but an independent version of Huber and Sturmfels [**HS95**].

Proof. Assume the system

$$f_i(x) = \sum_{\alpha \in S(f_i)} c_\alpha x^\alpha, \quad i = 1, \ldots, n \tag{1.26}$$

has finitely many common zeroes in $(\mathbb{C}^*)^n$ and choose a generic lifting $\mu = (\mu_1, \ldots, \mu_n)$ in the sense of Paragraph 1.2.2 to obtain a fine mixed subdivision of $\mathrm{P}(f_1) + \cdots + \mathrm{P}(f_n)$.

We perform a *toric deformation* of the system (1.26) by introducing a new complex variable t and setting

$$\hat{f}_i(x, t) = \sum_{\alpha \in S(f_i)} c_\alpha x^\alpha t^{\mu_i(\alpha)}, \quad i = 1, \ldots, n\ . \tag{1.27}$$

The roots of (1.27) are algebraic functions $x(t) = (x_1(t), \ldots, x_n(t))$ in t (cf. [**Wal50**]) whose branches can be expressed as Puiseux series

$$x(t) = \bar{x} \cdot t^v + \text{ higher-order terms in } t \tag{1.28}$$

with $v \in \mathbb{Q}^n$ and $\bar{x} \in (\mathbb{C}^*)^n$ and where $\bar{x} \cdot t^v$ is interpreted as $(\bar{x}_1 t^{v_1}, \ldots, \bar{x}_n t^{v_n})$. The idea is to insert this expression into (1.27) and study the terms of lowest order in t. We denote by $\hat{P}^{(v)}$ and $P^{(v)}$ the face of $\hat{P} := \hat{\mathrm{P}}(f_1) + \cdots + \hat{\mathrm{P}}(f_n)$ on which $(v, 1)^T$ is minimized and its projection to a cell of $\mathrm{P}(f_1) + \cdots + \mathrm{P}(f_n)$, respectively. Plugging (1.28) into (1.27) yields

$$\underbrace{\Big(\sum_{\alpha \in \mathrm{P}(f_i)^{(v)}} c_\alpha \bar{x}^\alpha \Big)}_{=:\, \mathrm{init}_v f_i} \cdot t^{\langle v, \alpha \rangle + \mu_i(\alpha)} + \text{h.o.t.}(t), \quad i = 1, \ldots, n \tag{1.29}$$

where $P^{(v)} = \mathrm{P}(f_1)^{(v)} + \cdots + \mathrm{P}(f_n)^{(v)}$ is the decomposition of the cell $P^{(v)}$ into its Minkowski summands.

Let $(d_1, \ldots, d_n)$ by the type of the cell $P^{(v)}$. Suppose one of the d_i equals 0, then the equation $\operatorname{init}_v f_i = 0$ has no solution $\bar{x}$ in $(\mathbb{C}^*)^n$. Since the subdivision induced by μ was assumed to be fine mixed, all d_i must be equal to 1. Hence the system (1.27) has branches of the form (1.28) if and only if $P^{(v)}$ has type $(1, \ldots, 1)$.

So pick v with corresponding cell $P^{(v)}$ of type $(1, \ldots, 1)$, i.e. $P^{(v)} = \mathrm{P}(f_1)^{(v)} + \cdots + \mathrm{P}(f_n)^{(v)}$ where all $\mathrm{P}(f_i)^{(v)}$ are edges. We claim that the binomial face system with respect to v,

$$\operatorname{init}_v f_i = 0 \quad i = 1, \ldots, n \ , \tag{1.30}$$

has $\operatorname{vol}_n(P^{(v)})$ solutions in $(\mathbb{C}^*)^n$. We sketch briefly how this can be shown.

Without loss of generality assume that each edge contains the origin such that the binomial system (1.30) is of the form

$$c_{\beta_1} x^{\beta_1} = \cdots = c_{\beta_n} x^{\beta_n} = 1 \ . \tag{1.31}$$

Set $B := \begin{pmatrix} \beta_1 & \cdots & \beta_n \end{pmatrix}$ and compute the Smith normal form (see [**DF04**, Chapter 12])

$$U \cdot B \cdot V = \begin{pmatrix} k_1 & 0 & \ldots & 0 \\ 0 & k_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & k_n \end{pmatrix}$$

where U and V are invertible integer matrices with determinant 1 and $k_i \in \mathbb{Z}_{>0}$.Now the matrices U and V are used to change the coordinate system such that the system (1.31) becomes

$$c'_1 x_1^{k_1} = \cdots = c'_n x_n^{k_n} = 1 \ . \tag{1.32}$$

This system has $k_1 \cdots k_n = \det(B) = \operatorname{vol}_n(P^{(v)})$ solutions in $(\mathbb{C}^*)^n$ and this proves the claim.

Together with the previous considerations this shows that (1.27) has

$$\sum_{\substack{C \text{ mixed cell} \\ \text{of } \mathrm{P}(f_1)+\cdots+\mathrm{P}(f_n)}} \operatorname{vol}_n(C)$$

many solutions in $(\mathbb{C}^*)^n$ and this equals $\mathrm{MV}_n(\mathrm{P}(f_1), \ldots, \mathrm{P}(f_n))$ by formula (1.12). □

Remark 1.26. The proof by Huber and Sturmfels that is sketched here contains the idea for a construction of Puiseux series solutions $x(t)$ to systems of the form

$$f_i(x_1, \ldots, x_n, t) = \sum_{\alpha \in S(f_i)} c_\alpha \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot t^{\alpha_{n+1}} = 0, \quad i = 1, \ldots, n \ .$$

McDonald [**McD95**, **McD02**] gives a detailed description of this construction in even more general cases. Note that the construction for the case $n = 1$ goes back to Newton (cf. [**Wal50**]) which gave rise to the term "Newton polytope".

The bound on the number of solutions of a polynomial system arising from Bernstein's Theorem is also often referred to as the *BKK bound* due to the work of Bernstein [**Ber75**], Khovanskiĭ [**Kho77**] and Kušnirenko [**Kuš75**, **Kuš76**]. The 2-dimensional case of this statement was already known to Minding [**Min03**, English translation] in 1841. For a description of the BKK bound in the context of toric varieties see [**Dan78**, **Ful93**, **GKZ94**].

1.4.2. Bernstein vs. Bézout. The BKK bound generalizes the Bézout bound (cf. [**CLO05**, Chapter 7]) and for sparse polynomial systems it is often significantly better. We will demonstrate this in an example below. For a discussion of the BKK bound in comparison with the Bézout bound and multihomogenous Bézout bounds see [**MSW95**].

Example 1.27. We want to determine the number of unit length eigenvectors in $\mathbb{C}^n$ of an $n \times n$ matrix $A = (a_{ij})$ (with generic entries a_{ij}) using Bernstein's Theorem. The following system of $n+1$ equations in the variables $(x_1, \ldots, x_n, \lambda)$ describes the setting.

$$\sum_{j=1}^{n} a_{ij}x_j \quad - \lambda x_i \quad = \quad 0 \quad \text{for } i = 1, \ldots, n \tag{1.33}$$

$$\sum_{i=1}^{n} x_i^2 \quad - 1 = 0 \tag{1.34}$$

Since each polynomial has total degree 2, the Bézout bound on the number of solutions is 2^{n+1}. The Newton polytopes of (1.33) and (1.34) are

$$P_i = \operatorname{conv}\{\xi_1, \ldots, \xi_n, \xi_i + \xi_{n+1}\} \quad \text{and} \quad P_{n+1} = \operatorname{conv}\{2\xi_1, \ldots 2\xi_n, 0\} \tag{1.35}$$

where ξ_i denotes the i-th unit vector in $\mathbb{R}^{n+1}$. Since each eigenspace intersects the unit sphere in two points the system (1.33), (1.34) has $2n$ solutions and therefore we have by Proposition 1.24 that $\mathrm{MV}_{n+1}(P_1, \ldots, P_{n+1}) = 2n$.

1.4.3. A bound on the number of solutions in $\mathbb{C}^n$. There are various works which generalize Bernstein's results to count all common roots in the affine space $\mathbb{C}^k$ (see e.g. [**EV99**, **HS97**, **LW96**, **Roj99**]). We state here the result of Li and Wang [**LW96**] which is not the tightest bound in every case but which is the most suitable for our purposes.

Proposition 1.28 (see [**LW96**]). *For a polynomial system $f_1(x) = \cdots = f_n(x) = 0$, the quantity*

$$\mathrm{MV}_n(\operatorname{conv}(\mathrm{P}(f_1) \cup 0), \ldots, \operatorname{conv}(\mathrm{P}(f_n) \cup 0))$$

is an upper bound for the number of isolated solutions in $\mathbb{C}^n$ counting multiplicities.

1.4.4. Bernstein's Second Theorem. Bernstein also gives an explicit algebraic condition that characterizes when a choice of coefficients is generic. Let v be a non-zero vector and let $(P)^v$ denote as before the face of a polytope P which is maximal with respect to the direction v. For a given $f = \sum_{\alpha \in S(f)} c_\alpha x^\alpha$ we set $\operatorname{init}_v f = \sum_\alpha c_\alpha x^\alpha = 0$ to be the *face equation* with respect to v, where the sum is over all integer points $\alpha \in (\mathrm{P}(f))^v$.

Proposition 1.29 (Bernstein's Second Theorem [**Ber75**]). *If for all $v \neq 0$, the face system $\operatorname{init}_v f_1 = 0, \ldots, \operatorname{init}_v f_n = 0$ has no solution in $(\mathbb{C}^*)^n$, then the mixed volume of the Newton polytopes of the f_i gives the exact number of common zeros in $(\mathbb{C}^*)^n$ and all solutions are isolated. Otherwise it is a strict upper bound.*

The system $\operatorname{init}_v f_i = 0$, $(i = 1, \ldots, n)$ has a solution in $(\mathbb{C}^*)^n$ only if none of the polynomials $\operatorname{init}_v f_i$ is a monomial. Hence it is necessary for a direction v to be a witness of the degeneracy that for each i, $\max_{\alpha \in \mathrm{P}(f_i)}(v_1\alpha_1 + \cdots + v_n\alpha_n)$ is attained at least twice. So in the language of Section 1.3 this implies that v must be in a certain tropical prevariety, namely

$$v \in \bigcap_{i=1}^{n} X\left(\operatorname{trop}\left(f_i\right)\right) \ .$$

It is furthermore interesting (see [**Roj97**]) that the BKK bound is a strict upper bound at most on a codimension 1 subset of the coefficient space. Rojas and Canny [**CR91**, **Roj99**] give explicit combinatorial criteria for a system of polynomials to be non degenerate in the sense of Proposition 1.29 but we will not make further use of these.

CHAPTER 2

Techniques for Explicit Mixed Volume Computation

This chapter introduces some new techniques for explicit mixed volume computation. The motivation for these results came from their application in the study of embedding numbers of minimally rigid graphs that will be presented in Chapter 5.

The goal of the first section is to present a tool to decouple the mixed volume computation of larger systems with a special structure. Namely Lemma 2.6 gives a method to compute the mixed volume of a large system in two smaller steps if some of the polytopes are contained in a lower dimensional subspace.

The second section describes how to obtain explicit conditions on a set of lifting vectors that ensure that a chosen cell appears in the induced mixed subdivision (see Lemma 2.9). We break down this result to the 2-dimensional case (Corollary 2.10) to give a better geometric intuition and carefully study the implications in an example.

2.1. Separation Lemma

Let $P \subset \mathbb{R}^n$ be a polytope and denote faces of P by F_P. The outer normal cone of F_P, will be denoted as $N(F_P)$ (see Paragraph 1.1.1). We call v *generic with respect to* P *and* Q if there is no face F_P of P and no face F_Q of Q such that

$$v \in N(F_P) - N(F_Q) \quad \text{and} \quad \dim(F_P) + \dim(F_Q) > n$$

where $N(F_P) - N(F_Q) := \{w_1 - w_2 \,|\, w_1 \in N(F_P),\ w_2 \in N(F_Q)\}$ (see Remark 1.3). In particular a generic v can not be a point on the boundary of $N(F_P) - N(F_Q)$.

Proposition 2.1 (Betke [**Bet92**]). *Let* $P, Q \subset \mathbb{R}^n$ *be polytopes and let* v *be generic with respect to* P *and* Q*. Then*

$$\text{vol}_n(P+Q) = \sum_{d=0}^{n} \sum_{\substack{type(F_P,F_Q)=(d,n-d) \\ v \in N(F_P)-N(F_Q)}} \text{vol}'_d(F_P) \cdot \text{vol}'_{n-d}(F_Q) \cdot \text{vol}_n(\mathfrak{P}) \tag{2.1}$$

where $\mathfrak{P}$ *is the parallelotope spanned by the unit cubes in* $\text{aff}(F_P)$ *and* $\text{aff}(F_Q)$*.*

We give a rough sketch of Betke's proof here since his approach was of significant importance for the work of Huber and Sturmfels [**HS95**] in which they obtain the methods to compute the mixed volume described in Paragraph 1.2.2, which are crucial for this work.

Proof. Let μ_1 and $\mu_2 \in \mathbb{R}^n$ be linear lifting functions for P and Q respectively such that $\mu_2 - \mu_1 = v$. Clearly the volumes of the cells in the mixed subdivision induced by μ_1 and μ_2 add up to $\text{vol}_n(P+Q)$. Betke showed that the cells $C = F_P + F_Q$ of this mixed subdivision correspond to those tuples (F_P, F_Q) of type $(d, n-d)$ for which $v \in N(F_P) -$

$N(F_Q)$. Since C is full-dimensional the faces F_P and F_Q lie in complementary subspaces and hence the volume of C can be computed as $\mathrm{vol}'_d(F_P) \cdot \mathrm{vol}'_{n-d}(F_Q) \cdot \mathrm{vol}_n(\mathfrak{P})$. □

Identity (1.5) from Paragraph 1.2.1 states for the case of two polytopes P and Q:

$$\mathrm{vol}_n(\lambda_1 Q + \lambda_2 P) = \sum_{d=0}^{n} \frac{\lambda_1^d \lambda_2^{n-d}}{d!\,(n-d)!} \mathrm{MV}_n(Q, d; P, n-d) \ . \tag{2.2}$$

Comparing coefficients in (2.1) and (2.2) yields the following statement.

Proposition 2.2 (Betke [**Bet92**]). *Let $P, Q \subset \mathbb{R}^n$ be polytopes and let v be generic with respect to P and Q. Then*

$$\mathrm{MV}_n(P, d; Q, n-d) = d!\,(n-d)! \sum_{\substack{type(F_P, F_Q) = (d, n-d) \\ v \in N(F_P) - N(F_Q)}} \mathrm{vol}'_d(F_P) \cdot \mathrm{vol}'_{n-d}(F_Q) \cdot \mathrm{vol}_n(\mathfrak{P})$$

where $\mathfrak{P}$ is defined as above in Lemma 2.1.

With these results we can formulate and prove the first tool to decouple mixed volume computation.

Lemma 2.3. *Let $P \subset \mathbb{R}^{m+k}$ and $Q \subset \mathbb{R}^m \subset \mathbb{R}^{m+k}$ be polytopes. Then*

$$\mathrm{MV}_{m+k}(Q, m; P, k) = \mathrm{MV}_m(Q, m) \cdot \mathrm{MV}_k(\pi(P), k) \tag{2.3}$$

where $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ denotes the projection on the last k coordinates.

Proof. By Proposition 2.2 and equation (1.10) it remains to show that

$$\sum_{\substack{type(F_Q, F_P) = (m, k) \\ v \in N(F_P) - N(F_Q)}} \mathrm{vol}'_m(F_Q) \cdot \mathrm{vol}'_k(F_P) \cdot \mathrm{vol}_n(\mathfrak{P}) = \mathrm{vol}_m(Q)\ \mathrm{vol}_k(\pi(P)) \tag{2.4}$$

for a v that is generic with respect to P and Q. Since Q is m-dimensional it follows that $F_Q = Q$ and $N(Q) = \mathbb{R}^k$. Since $Q \subset \mathbb{R}^m$ we have that $\mathrm{vol}'_m(Q) \cdot \mathrm{vol}'_k(F_P) \cdot \mathrm{vol}_n(\mathfrak{P})$ is simply $\mathrm{vol}_m(Q) \cdot \mathrm{vol}_k(\pi(F_P))$ where π is defined as above. Hence (2.4) is equivalent to

$$\sum_{\substack{F_P\ k\text{-dim. face of } P \\ v \in N(F_P) - \mathbb{R}^k}} \mathrm{vol}_k(\pi(F_P)) = \mathrm{vol}_k(\pi(P)) \ . \tag{2.5}$$

So let v be generic, i.e. if $v \in N(F_P) - \mathbb{R}^k$ then F_P is at most k-dimensional. Denote by $\mathcal{F}_v^{(k)}$ the set of k-dimensional faces F_P of P that satisfy $v \in N(F_P) - \mathbb{R}^k$. With this notation (2.5) is equivalent to

$$\bigcup_{F \in \mathcal{F}_v^{(k)}} \pi(F) = \pi(P) \quad \text{and } \dim\left(\pi(F_i) \cap \pi(F_j)\right) < k \ \text{ for all } F_i, F_j \in \mathcal{F}_v^{(k)} \ . \tag{2.6}$$

So pick a point $r \in \pi(P)$. Then $S := \pi^{-1}(r) \cap P$ is a polytope in an m-dimensional subspace which is parallel to $\mathbb{R}^m$. The normal cone of a vertex of S is of the form $N(F) + \mathbb{R}^k$ where F is a face of P with $\dim(F) \leq k$. v lies in exactly one of these normal cones since the projection of the normal fan of S to $\mathbb{R}^m$ is a complete fan. We denote by F^* a face of P such that $\pi^{-1}(r) \cap F^*$ is the vertex of S for which $v \in N(F^*) + \mathbb{R}^k$. Then either F^*

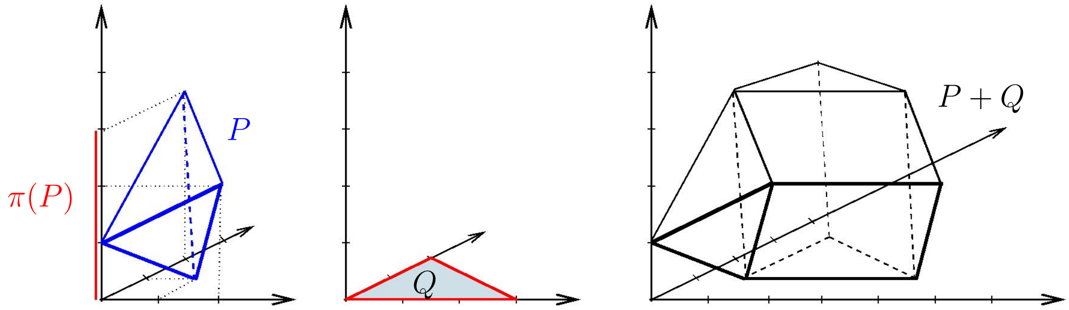

Figure 2.1. The polytopes P, Q and $P+Q$ from Example 2.4

is k-dimensional and hence $F^* \in \mathcal{F}_v^{(k)}$ or F^* is the proper face of some element of $\mathcal{F}_v^{(k)}$. In either case we have $r \in \pi(\mathcal{F}_v^{(k)})$. This shows the first statement in (2.6). Figure 2.2 depicts these ideas fro the polytopes from Example 2.4.

Assume now that $dim\,(\pi(F_i) \cap \pi(F_j)) \geq k$ for some $F_i, F_j \in \mathcal{F}_v^{(k)}$. Then we can choose a point r in this intersection which lies in the relative interior of both $\pi(F_i)$ and $\pi(F_j)$. $S := \pi^{-1}(r) \cap P$ is again an m-dimensional polytope which is parallel to $\mathbb{R}^m$. Since $\pi(F_i)$ and $\pi(F_j)$ are k-dimensional, $\pi^{-1}(r) \cap F_i$ and $\pi^{-1}(r) \cap F_j$ are vertices of S. These vertices are also distinct since r was chosen to lie in the relative interior of the projections $\pi(F_i)$ and $\pi(F_j)$. But as seen above v lies in only one of the normal cones of the vertices of S and this is a contradiction to $F_i, F_j \in \mathcal{F}_v^{(k)}$. □

Example 2.4. Let $P := \operatorname{conv}\left\{(1,1,0)^T, (2,0,2)^T, (0,0,1)^T, (0,2,3)^T\right\}$, $Q := \left\{(0,0,0)^T, (3,0,0)^T, (0,2,0)^T\right\}$. See Figure 2.1 where P, Q and their Minkowski sum $P+Q$ is depicted. The mixed volume $\mathrm{MV}_3(Q,2;P,1)$ equals $2\cdot \mathrm{vol}_2(Q)\cdot \mathrm{vol}_1(\pi(P)) = 2\cdot 3\cdot 3 = 18$ according to Lemma 2.3.

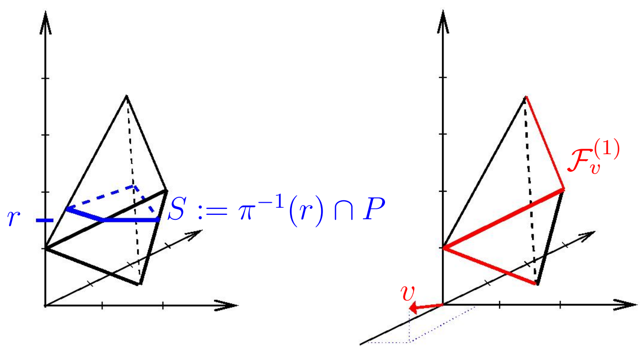

Figure 2.2. Illustration of some notations of the proof to Lemma 2.3 using the polytopes from Example 2.4

Remark 2.5. Lemma 2.3 can also be obtained as a special case of [**Ewa96**, Chapter IV, Lemma 4.9] where essentially different methods are used.

Exploiting the properties (1.8) and (1.9) of mixed volumes allows now to expand the statement of Lemma 2.3 to the case where all polytopes are different.

Lemma 2.6 (Separation Lemma). *Let $P_1, \dots, P_k$ be polytopes in $\mathbb{R}^{m+k}$ and $Q_1, \dots, Q_m$ be polytopes in $\mathbb{R}^m \subset \mathbb{R}^{m+k}$. Then*

$$\mathrm{MV}_{m+k}(Q_1, \dots, Q_m, P_1, \dots, P_k) = \mathrm{MV}_m(Q_1, \dots, Q_m) \cdot \mathrm{MV}_k(\pi(P_1), \dots, \pi(P_k)) \tag{2.7}$$

where $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ denotes the projection on the last k coordinates.

Proof. We show that both sides of the desired equation define a symmetric multilinear function and then use combinatorial identities for symmetric multilinear functions and Lemma 2.3 to show the full result.

Let $\mathcal{P}^m$ (resp. $\mathcal{P}^{m+k}$) be the set of all m-dimensional (resp. $(m+k)$-dimensional) polytopes and define two functions g_1 and g_2 on $(\mathcal{P}^m)^m \times (\mathcal{P}^{m+k})^k$ via

$$\begin{aligned} g_1(Q_1, \dots, Q_m, P_1, \dots, P_k) &:= \mathrm{MV}_{m+k}(Q_1, \dots, Q_m, P_1, \dots, P_k) \\ g_2(Q_1, \dots, Q_m, P_1, \dots, P_k) &:= \mathrm{MV}_m(Q_1, \dots, Q_m) \cdot \mathrm{MV}_k(\pi(P_1), \dots, \pi(P_k)) \ . \end{aligned}$$

Due to the properties of mixed volumes (see Paragraph 1.2) it is easy to see that g_1 and g_2 are invariant under changing the order of the Q_i and under changing the order of the P_j. Furthermore it follows from (1.9) that both functions are linear in each argument.

Hence, for fixed $P_1, \dots, P_k$ the induced mappings

$$\tilde{g}_i^{(P_1,\dots,P_k)}(Q_1, \dots, Q_m) := g_i(Q_1, \dots, Q_m, P_1, \dots, P_k) \qquad (i = 1, 2)$$

are symmetric and multilinear, and analogously, for fixed Q, the mappings

$$\bar{g}_i^{(Q)}(P_1, \dots, P_k) := g_i(Q, \dots, Q, P_1, \dots, P_k) \qquad (i = 1, 2)$$

are symmetric and multilinear. For any semigroups A, B and any symmetric multilinear function $f : A^n \to B$, it follows from an inclusion-exclusion argument (see [**Ewa96**, Theorem 3.7]) that

$$f(a_1, \dots, a_n) = \frac{1}{n!} \sum_{1 \le i_1 < \dots < i_q \le n} (-1)^{n-q} f(a_{i_1} + \dots + a_{i_q}, \dots, a_{i_1} + \dots + a_{i_q}) \ . \tag{2.8}$$

Hence we have for $i = 1, 2$ that

$$\begin{aligned} & g_i(Q_1, \dots, Q_m, P_1, \dots, P_k) \\ &= \tilde{g}_i^{(P_1,\dots,P_k)}(Q_1, \dots, Q_m) \\ &= \frac{1}{m!} \sum_{1 \le i_1 < \dots < i_q \le m} (-1)^{m-q} \ \tilde{g}_i^{(P_1,\dots,P_k)}(Q_{i_1} + \dots + Q_{i_q}, \dots, Q_{i_1} + \dots + Q_{i_q}) \\ &= \frac{1}{m!} \sum_{1 \le i_1 < \dots < i_q \le m} (-1)^{m-q} \ \bar{g}_i^{(Q_{i_1} + \dots + Q_{i_q})}(P_1, \dots, P_k) \ . \end{aligned}$$

Since we can expand $\bar{g}_i^{(Q_{i_1} + \dots + Q_{i_q})}(P_1, \dots, P_k)$ by using (2.8) as well, we see that both functions g_1 and g_2 are fully determined by their images of tuples of polytopes where $Q_1 = \dots = Q_m = Q$ and $P_1 = \dots = P_k = P$. Hence the statement reduces to Lemma 2.3. □

In the special case where all polytopes Q_i, P_j are lattice polytopes, i.e. their vertices have integer coordinates, Lemma 2.6 can be shown independently of the results of this section by using Bernstein's Theorem (see Section 1.4).

PROOF. Let $f_1 = \cdots = f_m = g_1 = \cdots = g_k = 0$ be a polynomial system of equations with Newton polytopes $Q_1, \ldots, Q_m, P_1, \ldots, P_k$ and generic coefficients. We will count the number of solutions in $(\mathbb{C}^*)^{m+k}$ to this system in two ways using Bernstein's Theorem. On the one hand $\mathrm{MV}_{m+k}(Q_1, \ldots, Q_m, P_1, \ldots, P_k)$ gives this quantity according to Bernstein's Theorem. On the other hand, since $Q_1, \ldots, Q_m \subset \mathbb{R}^m$, the m-dimensional polynomial system $f_1 = \cdots = f_m = 0$ has $\mathrm{MV}_m(Q_1, \ldots, Q_m)$ solutions in $(\mathbb{C}^*)^m$. Each solution to this smaller system can be plugged into the remaining polynomials g_i to obtain the system $g_1^* = \cdots = g_k^* = 0$ having Newton polytopes $\pi(P_1), \ldots, \pi(P_k)$. Each of these new systems has $\mathrm{MV}_k(\pi(P_1), \ldots, \pi(P_k))$ solutions in $(\mathbb{C}^*)^k$. Hence the number of solutions to $f_1 = \cdots = f_m = g_1 = \cdots = g_k = 0$ is $\mathrm{MV}_m(Q_1, \ldots, Q_m) \cdot \mathrm{MV}_k(\pi(P_1), \ldots, \pi(P_k))$ which proves the desired identity. □

Corollary 2.7. *Let $K_1, \ldots, K_n$ be convex bodies in $\mathbb{R}^n$ such that the first m of them lie in an m-dimensional subspace V of $\mathbb{R}^n$. Then*

$$\mathrm{MV}_n(K_1, \ldots, K_n) = \mathrm{MV}'_m(K_1, \ldots, K_m) \cdot \mathrm{MV}'_{n-m}(\pi_{\bar{V}}(K_{m+1}), \ldots, \pi_{\bar{V}}(K_n))$$

where $\pi_{\bar{V}}$ denotes the projection to the orthogonal complement $\bar{V}$ of V respectively.

Remark 2.8. This result was already mentioned in [**BZ88**] in which the authors refer to [**Fed78**] (in Russian) for the proof which unfortunately we were unable to obtain and therefore unable to check.

PROOF. Note first that the mixed volume does not change if all arguments are mapped under the same volume preserving function. So it suffices that m arguments lie in an m dimensional subspace of $\mathbb{R}^n$. To generalize Lemma 2.6 to the case where the arguments are general convex bodies one can use the fact that for every convex body K there exists a sequence of polytopes which converges to K (see Proposition 1.6) and that the mixed volume is continuous with respect to the Hausdorff metric (see Remark 1.10). □

2.2. Lifting Lemma

In this section we take a closer look at the idea of Emiris and Canny [**EC95**] as seen in Paragraph 1.2.4 to use linear programming and the formula (1.12) to compute the mixed volume.[1] This section's main result is a technical lemma that describes explicit conditions on linear lifting vectors to induce a certain cell as a mixed cell in a subdivision. To make the statement more comprehensible we formulate it in the 2-dimensional case and study it in a longer example.

Lemma 2.9. *Given polytopes $P_1, \ldots, P_k \subset \mathbb{R}^k$ and lifting vectors $\mu_1, \ldots, \mu_k \in \mathbb{R}^k_{\geq 0}$. Denote the vertices of P_i by $v_1^{(i)}, \ldots, v_{r_i}^{(i)}$ and choose one edge $e_i = [v_{t_i}^{(i)}, v_{l_i}^{(i)}]$ from each P_i. Then $C := e_1 + \cdots + e_k$ is a mixed cell of the mixed subdivision induced by the liftings μ_i if and only if*

[1] As pointed out by the second referee, some ideas of this section aprallel results by Emiris and Verschelde [**EV99**] and Verschelde and Gatermann [**VGC96**].

i) The edge matrix $E := V_b - V_a$ *is non-singular (where* $V_a := (v_{t_1}^{(1)}, \ldots, v_{t_k}^{(k)})$ *and* $V_b := (v_{l_1}^{(1)}, \ldots, v_{l_k}^{(k)})$*) and*

ii) For all polytopes P_i *and all vertices* $v_s^{(i)}$ *of* P_i *which are not in* e_i *we have:*

$$(\langle \mu_1 - \mu_i, \vec{e_1} \rangle, \ldots, \langle \mu_k - \mu_i, \vec{e_k} \rangle) \cdot E^{-1} \cdot \left(v_{t_i}^{(i)} - v_s^{(i)} \right) \geq 0 \tag{2.9}$$

where $\vec{e_i} = v_{l_i}^{(i)} - v_{t_i}^{(i)}$.

Before beginning with the proof we start with some auxiliary considerations about how to apply linear programming here. Recall from Paragraph 1.2.4 that the test whether $\hat{e}_1 + \cdots + \hat{e}_k$ lies on the lower envelope of $\hat{P}_1 + \cdots + \hat{P}_k$ can be formulated as the linear program (1.17). Setting $x^T = (\lambda_1^{(1)}, \ldots, \lambda_{r_1}^{(1)}, \ldots\ldots, \lambda_1^{(k)}, \ldots, \lambda_{r_k}^{(k)}, s) \in \mathbb{R}^{r_1 + \cdots + r_k + 1}$, the linear program (1.17) can be written in standard matrix form $\max\{c^T x : Ax = b, x \geq 0\}$ with

$$A = \begin{pmatrix} v_1^{(1)} & \ldots & v_{r_1}^{(1)} & \ldots & \ldots & v_1^{(k)} & \ldots & v_{r_k}^{(k)} & \mathbf{0}_k \\ \langle \mu_1, v_1^{(1)} \rangle & \ldots & \langle \mu_1, v_{r_1}^{(1)} \rangle & \ldots & \ldots & \langle \mu_k, v_1^{(k)} \rangle & \ldots & \langle \mu_k, v_{r_k}^{(k)} \rangle & 1 \\ & \mathbf{1}_{r_1}^T & & \mathbf{0}_{r_2}^T & \ldots & & \mathbf{0}_{r_k}^T & & 0 \\ & \mathbf{0}_{r_1}^T & & \mathbf{1}_{r_2}^T & \ldots & & \mathbf{0}_{r_k}^T & & 0 \\ & \vdots & & & \ddots & & \vdots & & \vdots \\ & \mathbf{0}_{r_1}^T & & \mathbf{0}_{r_2}^T & \ldots & & \mathbf{1}_{r_k}^T & & 0 \end{pmatrix},$$

$$b^T = (\hat{m}, \mathbf{1}_k^T) \in \mathbb{R}^{2k+1},$$

$$c^T = (\mathbf{0}_{r_1 + \cdots + r_k}^T, 1) \in \mathbb{R}^{r_1 + \cdots + r_k + 1}.$$

Here $\mathbf{0}_k$ and $\mathbf{1}_k$ denote the all-0-vector and the all-1-vector in $\mathbb{R}^k$, respectively. In this notation the point $\hat{m}$ from (1.16) corresponds to $\bar{x} = (\lambda_1^{(1)}, \ldots, \lambda_{r_k}^{(k)}, s)$ where $s = 0$ and $\lambda_j^{(i)} = \frac{1}{2}$ if the edge $\hat{e}_i$ contains the vertex $\hat{v}_j^{(i)}$ and $\lambda_j^{(i)} = 0$ otherwise.

Assume a feasible vertex $\bar{x} \geq 0$ of the linear program (1.17) is given. For a subset $S \subset \{1, \ldots, r_1 + \cdots + r_k + 1\}$ let A_S be the submatrix of A that consists of the columns with indices in S. If v is a vector, then v_S is understood as the vector where all entries with indices which are not in S are deleted. Now let B be a (not necessarily unique) choice of $2k+1$ indices such that $A_B^{-1} \cdot b = \bar{x}_B$ and denote by N those indices which are not in B. By linear programming duality (see, e.g. [**GLS93**]) $\bar{x}$ is optimal if and only if

$$c_N^T - c_B^T \cdot A_B^{-1} \cdot A_N \leq 0, \tag{2.10}$$

where the equation is understood componentwise, i.e. each component of the vector on the left hand side is non-positive.

To prove Lemma 2.9 we assume that $\bar{x}$ is optimal and deduce conditions on the lifting vectors μ_i by using the inequality (2.10).

Proof of Lemma 2.9. Note that C is full-dimensional if and only if E is non-singular. In the following only this full-dimensional case will be considered. To simplify the notation write $\mu(V)$ to denote $(\langle \mu_1, v_1 \rangle, \ldots, \langle \mu_k, v_k \rangle)$.

We know that C is a mixed cell if and only if the following $\bar{x}$ is the optimal solution to the linear program defined above:

$$\bar{x} = (\lambda_1^{(1)}, \ldots, \lambda_{r_k}^{(k)}, 0) \text{ where } \lambda_j^{(i)} = \begin{cases} \frac{1}{2}, & j \in \{t_i, l_i\} \\ 0, & \text{else} \end{cases} .$$

The submatrices of A corresponding to $\bar{x}$ are

$$A_B = \begin{pmatrix} V_a & V_b & \mathbf{0}_k \\ \mu(V_a) & \mu(V_b) & 1 \\ \mathrm{Id}_k & \mathrm{Id}_k & \mathbf{0}_k \end{pmatrix} \quad \text{and} \quad A_N = \begin{pmatrix} v_s^{(i)} \\ \langle \mu_i, v_s^{(i)} \rangle \\ \xi_i \end{pmatrix}_{\substack{1 \le i \le k \\ 1 \le s \le r_i \\ s \ne t_i, l_i}}$$

where ξ_i denotes the i^{th} unit vector. Since

$$A_B^{-1} = \begin{pmatrix} -E^{-1} & \mathbf{0}_k & E^{-1} \cdot V_b \\ E^{-1} & \mathbf{0}_k & -E^{-1} \cdot V_a \\ -\mu(E) \cdot E^{-1} & 1 & \mu(E) \cdot E^{-1} \cdot V_a - \mu(V_a) \end{pmatrix}$$

and $c_N = (0, \ldots, 0)$ the criterion (2.10) implies that $\bar{x}$ is optimal if and only if

$$(0, \ldots, 0, 1) \cdot A_B^{-1} \cdot A_N \ge 0 \quad \text{(componentwise)} .$$

But the i^{th} component of the vector on the left can be explicitly computed as

$$-\big(\mu(E) \cdot E^{-1}\big) \cdot v_s^{(i)} + \langle \mu_r, v_s^{(i)} \rangle + \big(\mu(E) \cdot E^{-1} \cdot V_b - \mu(V_b)\big) \cdot \xi_i$$

which equals the left hand side of (2.9) since $\langle \mu_i, v_s^{(i)} \rangle = (\langle \mu_i, \vec{e}_1 \rangle, \ldots, \langle \mu_i, \vec{e}_n \rangle) \cdot E^{-1} \cdot v_s^{(i)}$ and $\mu(V_b) \cdot \xi_i = \langle \mu_i, v_{l_i}^{(i)} \rangle$. ☐

Note that (2.9) is linear in the μ_j. Hence, for a given a choice of edges this condition defines a cone of lifting vectors which induce a mixed subdivision that contains our chosen cell as a mixed cell.

To get a better comprehension of Lemma 2.9 we consider the case $n = 2$. So let P and Q be 2-dimensional polytopes and let $e_P = v_2^{(P)} - v_1^{(P)}$ and $e_Q = v_2^{(Q)} - v_1^{(Q)}$ be the edges that sum up to the cell C. The first condition of Lemma 2.9 states that the edge matrix $E = (e_P, e_Q)$ has to be non-singular which is the case if and only if e_P and e_Q are not parallel.

Then condition (2.9) states

$$(0, \langle \mu_2 - \mu_1, e_Q \rangle) \cdot E^{-1} \cdot (v_1^{(P)} - v^{(P)}) \ \ge \ 0 \tag{2.11}$$

$$(\langle \mu_1 - \mu_2, e_P \rangle, 0) \cdot E^{-1} \cdot (v_1^{(Q)} - v^{(Q)}) \ \ge \ 0 \tag{2.12}$$

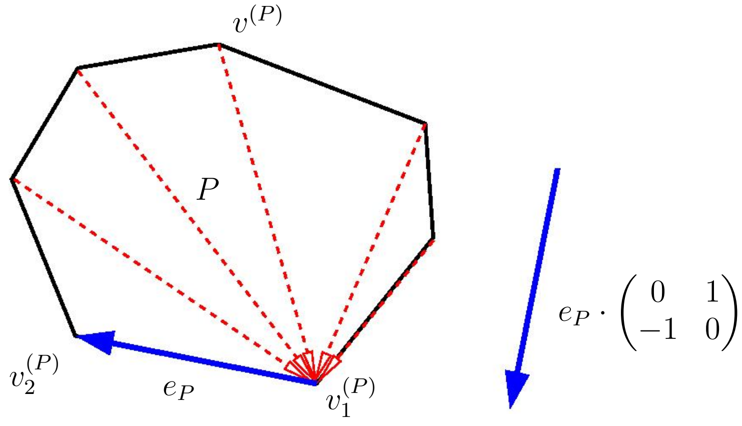

Figure 2.3

for all vertices $v^{(P)}$ of P and $v^{(Q)}$ of Q. In this case the matrix $E^{-1} = (e_P, e_Q)^{-1}$ can be explicitly described as

$$E^{-1} = \frac{1}{e_P^T \cdot \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \cdot e_Q} \begin{pmatrix} \langle e_Q, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle & \langle e_Q, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \rangle \\ \langle e_P, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \rangle & \langle e_P, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \end{pmatrix}$$

$$(2.13) \qquad = \frac{1}{e_P^T \cdot \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \cdot e_Q} \begin{pmatrix} -e_Q^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ e_P^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} .$$

Note that $e_P^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is an outer or inner normal to P depending on the orientation of the edge e_P. Hence $e_P^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot (v_1^{(P)} - v^{(P)})$ keeps the same sign when $v^{(P)}$ runs over all vertices of P except those in the edge e_P (see Figure 2.3). Exactly the same argumentation works for Q such that we can define

$$\alpha_P := \operatorname{sign}\left(\frac{e_P^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot (v_1^{(P)} - v^{(P)})}{e_P^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot e_Q} \right)$$

$$\alpha_Q := \operatorname{sign}\left(\frac{e_Q^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot (v_1^{(Q)} - v^{(Q)})}{e_P^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot e_Q} \right) .$$

Hence using (2.13) in (2.11) and (2.12) shows that the following holds.

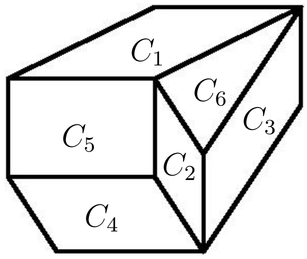

Figure 2.4. A mixed subdivision Γ of $P+Q$.

Corollary 2.10. *Let $P, Q \subset \mathbb{R}^2$ be polytopes and $\mu_1, \mu_2 \in \mathbb{R}^2$ be the corresponding lifting vectors. If e_P and e_Q are edges of P and Q respectively which are not parallel, then $C = e_P + e_Q$ is a mixed cell of the subdivision induced by μ_1, μ_2 if an only if*

$$\begin{aligned} \langle \mu_2 - \mu_1, e_Q \rangle \cdot \alpha_P &\geq 0 \\ \text{and } \langle \mu_2 - \mu_1, e_P \rangle \cdot \alpha_Q &\geq 0 \end{aligned}$$

where α_P and α_Q are defined as above.

Example 2.11. We demonstrate the previous results in a 2-dimensional example. Take once more the polytopes P and Q from Example 1.4. For the convenience of the reader we repeat their definition here.

$$P = \operatorname{conv}\left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix} \right\} \quad Q = \operatorname{conv}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\\frac{3}{2}\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix} \right\}$$

The Minkowski sum of P and Q is depicted in Figure 1.4 and Figure 2.4 shows one of the possible coherent mixed subdivisions.

The mixed cells $C_1, \ldots, C_4$ of the mixed subdivision Γ shown in Figure 2.4 will now be studied using Lemma 2.9. Fixing a cell C_i Lemma 2.9 describes a cone in $\mathbb{R}^2$ that contains the difference $\mu_2 - \mu_1$ for all linear lifting functions μ_1, μ_2 that induce a subdivision containing C_i.

Denote the vertices of P by $v_1, \ldots, v_4$ and the vertices of Q by w_1, w_2, w_3 respectively. Then the cell C_1 is the sum of the edges $\{v_3, v_4\}$ and $\{w_2, w_3\}$. Plugging these values into Lemma 2.9 we obtain two conditions, namely $\left\langle \begin{pmatrix}2\\1\end{pmatrix}, \mu_2 - \mu_1 \right\rangle \geq 0$ and $\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \mu_2 - \mu_1 \right\rangle \geq 0$. Similarly $C_2 = \{v_2, v_4\} + \{w_1, w_2\}$ leads to the cone described by the vectors $\begin{pmatrix}2\\-3\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, $C_3 = \{v_2, v_4\} + \{w_1, w_3\}$ yields $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ and finally $C_4 = \{v_1, v_2\} + \{w_1, w_2\}$ results in $\begin{pmatrix}-2\\3\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$. The cells and their corresponding cones are shown in Figure 2.5.

We can also answer the question which linear liftings μ_1, μ_2 induce the whole subdivision Γ. Since all cells $C_1, \ldots, C_4$ have to be induced it is necessary that $\mu_2 - \mu_1$ lies in the intersection of all four cones corresponding to $C_1, \ldots, C_4$.

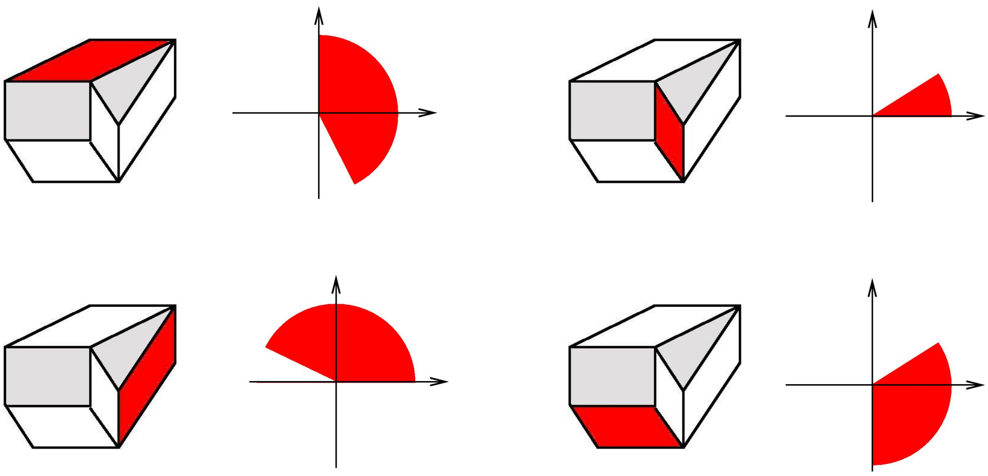

Figure 2.5. The mixed cells $C_1, \ldots, C_4$ of the subdivision Γ with their corresponding lifting cones.

The same process can be applied to all coherent mixed subdivisions of $P + Q$ which leads to a *fan of mixed subdivisions* depicted in Figure 2.6.

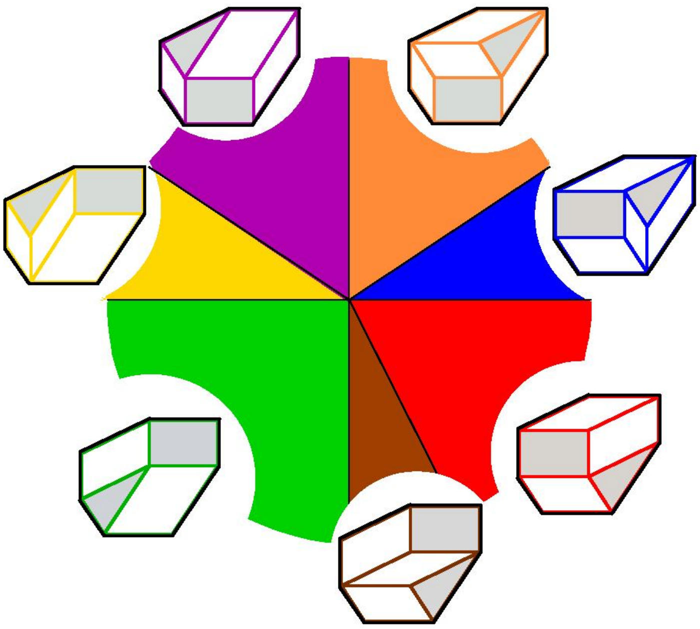

Figure 2.6. The fan of coherent mixed subdivisions of $P + Q$.

CHAPTER 3

Mixed Ehrhart Theory

The motivation for the work in this chapter came from the desire to prove a theorem that compares the toric and tropical genus of an intersection curve (see Section 4.4). Due to the formula for the tropical genus (Theorem 4.15) and Khovanskii's formula for the toric genus (Proposition 4.19) this boils down to show that a certain alternating sum of lattice points of polytopes equals an expression in mixed volumes of these polytopes. We are able to show this result and even generalize it by studying a mixed version of the Ehrhart polynomial $E_P(t)$.

This chapter begins with a short introduction to classical Ehrhart theory and the theory of valuations (see the survey articles [**GW93**,**MS83**,**McM93**], the books [**Bar08**, **BR07**,**EGH89**] or the collection [**BBC$^+$08**] for more details). With these tools we define and study the *mixed Ehrhart polynomial* $\mathrm{ME}_{P_1,\dots,P_k}(t)$ which turns out to have a much simpler structure then expected. In particular many coefficients vanish and the coefficients of highest order allow an interpretation in terms of mixed volumes. As corollaries we get formulas that compare the mixed volume to an alternating sum of integer points in a set of polytopes. In each case we provide a graphic example to strengthen the geometric intuition. To conclude the multivariate case is discussed briefly.

3.1. Ehrhart Theory and Valuations

3.1.1. Classical results. Let $\Lambda \subset \mathbb{R}^n$ be a lattice and let $\mathrm{L}(P)$ and $\mathrm{L}^{\circ}(P)$ denote the number of lattice points and the number of interior lattice points of a lattice polytope P, respectively. First, the case $\Lambda = \mathbb{Z}^n$ is treated.

Ehrhart showed (see [**Bar08**, **Ehr67**]) that the number of integer points in $t \cdot P$ for $t \in \mathbb{N}$ is a polynomial in t of degree n, i.e.

$$\mathrm{L}(t \cdot P) = E_P(t) \quad \text{for some polynomial } E_P(x) = \sum_{i=0}^{n} e_i(P) \cdot x^i \ . \tag{3.1}$$

The polynomial $E_P(t)$ is called the *Ehrhart polynomial* of P and its coefficients $e_i(P)$ are called *Ehrhart coefficients*. The following identities hold for the coefficients:

$$e_n(P) = \mathrm{vol}_n(P), \quad e_{n-1}(P) = \frac{1}{2} \sum_{F \text{ facet of } P} \mathrm{vol}'_{n-1}(F), \quad e_0(P) = 1 \tag{3.2}$$

For the remaining coefficients we do not have explicit expressions but the results in [**McM93**] show that the coefficient $e_k(P)$ can be expressed in terms of the faces of P.

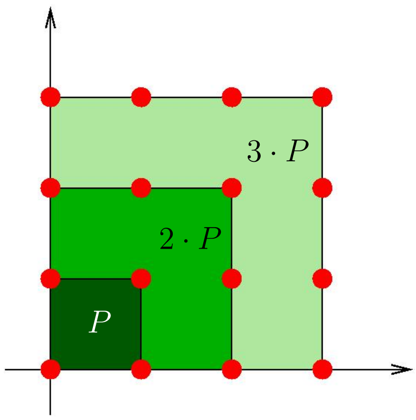

Figure 3.1. Integer points in P, $2 \cdot P$ and $3 \cdot P$.

Concerning the number of interior integer points, similar results can be stated using the *reciprocity law* (see [**Ehr67**, **Sta80**])

$$\mathrm{L}^{\circ}(t \cdot P) = (-1)^n E_P(-t) = (-1)^n \sum_{i=0}^{n} (-1)^i e_i(P) \cdot t^i \ . \tag{3.3}$$

Note furthermore that it is equivalent to ask for the number of $\mathbb{Z}^n$-points in $t \cdot P$ or to ask for the number of $\frac{1}{t}\mathbb{Z}^n$-points in P.

Example 3.1. Let P be the unit cube in $\mathbb{R}^n$. Then $\mathrm{L}(t \cdot P) = (t+1)^n$ and $\mathrm{L}^{\circ}(t \cdot P) = (t-1)^n$. See Figure 3.1 for a 2-dimensional example.

3.1.2. Valuations. Suppose K_1, K_2 are convex bodies in $\mathbb{R}^n$. φ is called a *valuation*[1] if

$$\varphi(K_1 \cup K_2) + \varphi(K_1 \cap K_2) = \varphi(K_1) + \varphi(K_2) \tag{3.4}$$

holds whenever $K_1 \cup K_2$ is convex. If it holds furthermore that for any convex body K

$$\varphi(t \cdot K) = t^r \varphi(K) \ ,$$

then the valuation φ will be called *homogeneous of degree* r. Let Δ be an additive subgroup of $\mathbb{R}^n$ such that $\mathrm{aff}(\Delta) = \mathbb{R}^n$. Most of the forthcoming results just use the special case $\Delta = \mathbb{Z}^n$, however a more general treatment might be of independent interest to the reader. φ is called a Δ-*valuation* if (3.4) holds for any elements of $\mathcal{P}_n(\Delta)$ and $\varphi(P+a) = \varphi(P)$ for all $a \in \Delta$. Furthermore, in the case that the additive subgroup is a lattice Λ, McMullen [**McM09**] showed that Λ-valuations satisfy the inclusion exclusion principle.

Example 3.2. (1) The n-dimensional volume vol_n is a valuation which is homogeneous of degree n (see e.g. [**McM93**]).

[1]Note that the term "valuation" is used differently here then in Section 1.3.

(2) For any $1 \leq r \leq n$ and any convex bodies $K_1, \ldots, K_{n-r}$ the function

$$\begin{array}{rcl} \mathrm{MV}_n^{(r)} : \mathcal{K}_n & \rightarrow & \mathbb{R} \\ K & \mapsto & \mathrm{MV}_n(\underbrace{K, \ldots, K}_{r-\text{times}}, K_1, \ldots, K_{n-r}) \end{array}$$

is a valuation which is homogeneous of degree r (see e.g. [**McM77**]).

(3) For $\Delta = \mathbb{Z}^n$ the number of lattice points $\mathrm{L}(P)$ and the number of interior lattice points $\mathrm{L}^{\circ}(P)$ is a $\mathbb{Z}^n$-valuation (see e.g. [**McM93**]).

(4) For $\Delta = \mathbb{Z}^n$ the Ehrhart coefficients $e_r(P)$ are $\mathbb{Z}^n$-valuations which are homogeneous of degree r (see [**BK85, McM77**]).

Many of the previous results can now be formulated more general with this new language (see [**McM77**]). For example Ehrhart's result (3.1) holds for any Δ-valuation φ. Namely for $P \in \mathcal{P}(\Delta)$ we have that $\varphi(tP) = \sum_{i=0}^{n} \varphi_i(P)\, t^i$ is a polynomial in t of degree at most n, whose coefficients φ_i are homogeneous Δ-valuations of degree i.

It is even possible to generalize Minkowski's Proposition 1.7 from Section 1.2. Namely for a valuation φ which is continuous with respect to the Hausdorff metric (see Paragraph 1.1.5) and monotone with respect to inclusion, $\varphi(\lambda_1 K_1 + \cdots + \lambda_r K_r)$ with $\lambda_1, \ldots, \lambda_r \geq 0$ is a polynomial in the λ_i which is homogeneous of degree n.

In the following section it is of significant importance that we can decompose a homogeneous Δ-valuation. The key ingredient is the following lemma by McMullen (see [**McM77**]).

Proposition 3.3 (McMullen [**McM77**]). *Let φ_r be a homogeneous Δ-valuation of degree r and let $t_1, \ldots, t_k$ be integers. Then for any polytopes $P_1, \ldots, P_k \in \mathcal{P}(\Delta)$ we have*

$$\varphi_r(t_1 \cdot P_1 + \cdots + t_k \cdot P_k) = \sum_{r_1,..,r_k} \binom{r}{r_1 \ldots r_k} \varphi'_r(P_1, r_1; \ldots; P_k, r_k)\ t_1^{r_1} \cdots t_k^{r_k}\ .$$

The coefficients $\varphi'_r(P_1, r_1; \ldots; P_k, r_k)$ are called *mixed Δ-valuations*. One can show that

$$\varphi_r(P) = \varphi'_r(P, r) = \varphi'_r(\underbrace{P, \ldots, P}_{r-\text{times}})$$

and that φ'_r is independent of P_i if $r_i = 0$ but we will not need this or any other explicit expression of these coefficients.

3.2. The mixed Ehrhart polynomial

For lattice polytopes $P_1, \ldots, P_k \subset \mathbb{R}^n$ [2] in the integer lattice $\mathbb{Z}^n$ and $t \in \mathbb{N}$ consider the following version of a *mixed Ehrhart polynomial* [3] in one variable t:

$$\mathrm{ME}_{P_1,\ldots,P_k}(t) := \sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|}\ \mathrm{L}\left(t \cdot \sum_{j \in J} P_j\right),$$

[2] k and n are independent throughout this chapter.

[3] We chose to call this the mixed Ehrhart polynomial since it resembles the way that mixed volumes are obtained from volumes, see (1.4). The mixed Ehrhart polynomial equals the Ehrhart polynomial for the case $k = 1$, but note that this is not true though in the case that $k > 1$ and where all P_i coincide.

where we used the notation $[k] := \{1, \dots, k\}$. This alternating sum of Ehrhart polynomials turns out to have a very simple structure as will be seen below. Namely all coefficients of t^r for $1 \le r < k$ vanish and in the case $k = n$ and $k = n-1$ the remaining coefficients have a nice interpretation in terms of mixed volumes.

Clearly $\mathrm{ME}_{P_1,\dots,P_k}(t)$ is a polynomial in t of degree at most n since it is the alternating sum of Ehrhart polynomials:

$$\begin{aligned}\mathrm{ME}_{P_1,\dots,P_k}(t) &= \sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \; E_{\sum_J P_j}(t) \\ &= \sum_{r=0}^{n} t^r \left(\sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \; e_r(\textstyle\sum_J P_j) \right) .\end{aligned}$$

We denote the coefficients of this polynomial by $\mathrm{me}_r(P_1, \dots, P_k)$.

If we have to consider the alternating sum of numbers of interior integer points L^{o} instead of just integer points L, the Ehrhart reciprocity (3.3) allows to translate each result. Namely we have that

$$\sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \; \mathrm{L}^{\mathrm{o}}(t \cdot \sum_{j \in J} P_j) = \sum_{r=0}^{n} t^r \; (-1)^{n+r} \; \mathrm{me}_r(P_1, \dots, P_k) \; . \tag{3.5}$$

3.2.1. Coefficients of low order. Though the main focus of this work is on the case $\Delta = \mathbb{Z}^n$, we state the first result for an arbitrary additive subgroup Δ of $\mathbb{R}^n$ since it might be of independent interest in other contexts.

Lemma 3.4. *For any polytopes $P_1, \dots, P_k \in \mathcal{P}_n(\Delta)$ and any Δ-valuation φ_r which is homogeneous of degree $r < k$ we have that*

$$\sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \varphi_r(\sum_J P_j) = 0 \; .$$

Remark 3.5. In particular this implies that $\mathrm{me}_r(P_1, \dots, P_k) = 0$ for $1 \le r < k$.

Proof. By McMullen's result on homogeneous valuations (see Proposition 3.3) we obtain

$$\sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \varphi_r(\sum_J P_j) = \sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \sum_{r_1, \dots, r_{|J|}} \binom{r}{r_1 \dots r_{|J|}} \varphi'_r(P_{j_1}, r_1; \dots; P_{j_{|J|}}, r_{|J|}) \; .$$

Here the $\varphi'_r(P_{j_1}, r_1; \dots; P_{j_{|J|}}, r_{|J|})$ are mixed valuations which we do not need to state more explicitly. We write the right hand side of the previous equation slightly different as

$$(-1)^k \sum_{\emptyset \neq J \subset [k]} (-1)^{|J|} \sum_{\substack{s_1, \dots, s_k \ge 0 \\ \sum s_i = r \\ s_i = 0 \text{ if } i \in [k] \setminus J}} \binom{r}{s_1 \dots s_{n-1}} \varphi'_r(P_1, s_1; \dots; P_k, s_k) \; . \tag{3.6}$$

Now fix $s_1, \dots, s_k \ge 0$ and ask for which sets J does $\varphi'_r(P_1, s_1; \dots; P_k, s_k)$ appear in the inner sum of (3.6). Denote by J_s the set of indices i for which $s_i \neq 0$ then $\varphi'_r(P_1, s_1; \dots; P_k, s_k)$

appears whenever $J \supset J_s$. Whenever this term appears it has the same multinomial coefficient but possibly different sign depending on the number of elements in J.

Let α_s be the number of elements in $[k]$ which are not in J_s. Then we can write

$$\sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \varphi_r(\sum_J P_j) = (-1)^k \sum_{\substack{s_1,..,s_k \geq 0 \\ \sum s_i = r}} A(s) \cdot \varphi'_r(P_1, s_1; \ldots; P_k, s_k) \tag{3.7}$$

where

$$A(s) = (-1)^{|J_s|} \binom{r}{s_1 \ldots s_k} \sum_{i=0}^{\alpha_s} (-1)^i \binom{\alpha_s}{i} .$$

Now $\sum_{i=0}^{\alpha_s} (-1)^i \binom{\alpha_s}{i}$ equals 0 if $\alpha_s > 0$ and equals 1 if $\alpha_s = 0$. Since $r < k$ the case $\alpha_s = 0$ can not occur and hence (3.7) vanishes for $1 \leq r < k$. □

Consider now again the specific case of Ehrhart coefficients and not the case of a general homogeneous valuation.

Lemma 3.6. *The absolute coefficient of* $\mathrm{ME}_{P_1,\ldots,P_k}(t)$ *equals* $(-1)^{k+1}$.

Proof. By (3.2) it is known that $e_0(P) = 1$ for any polytope P. Hence

$$\begin{aligned} \mathrm{me}_0(P_1, \ldots, P_k) &= \sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|} \cdot e_0\big(\sum_{j \in J} P_j\big) \\ &= (-1)^k \sum_{\emptyset \neq J \subset [k]} (-1)^{|J|} = (-1)^{k+1} . \end{aligned}$$

□

3.2.2. Leading coefficients. With the specific identities for Ehrhart coefficients (3.2) and some knowledge about mixed volumes it is possible to determine the two leading coefficients of $\mathrm{ME}_{P_1,\ldots,P_k}(t)$ by using a similar combinatorial methods as in the proof of Lemma 3.4.

Lemma 3.7. *The leading coefficient of the mixed Ehrhart polynomial* $\mathrm{ME}_{P_1,\ldots,P_k}(t)$ *equals*

$$\mathrm{me}_n(P_1, \ldots, P_k) = \sum_{\substack{s_1+\cdots+s_k=n \\ s_i \geq 1}} \frac{\mathrm{MV}_n(P_1, s_1; \ldots, P_k, s_k)}{s_1! \cdots s_k!} .$$

Proof. Considering $e_n(P)$, (3.2) states that the coefficient of t^n in the Ehrhart polynomial equals $\mathrm{vol}_n(P)$. Hence by definition

$$\mathrm{me}_n(P_1, \ldots, P_k) = (-1)^k \sum_{1 \leq i_1 < \cdots < i_u \leq k} (-1)^u \,\mathrm{vol}_n(P_{i_1} + \cdots + P_{i_u}) . \tag{3.8}$$

Using identity (1.5) from Section 1.2 shows

$$\mathrm{vol}_n(P_{i_1} + \cdots + P_{i_u}) = \sum_{\substack{j_1+\cdots+j_u=n \\ j_s \geq 0}} \frac{1}{j_1! \cdots j_u!} \mathrm{MV}_n(P_{i_1}, j_1; \ldots; P_{i_u}, j_u) ;$$

thus the right hand side of (3.8) can be written as

$$
\begin{aligned}
&(-1)^k \sum_{1\le i_1<\cdots<i_u\le k} (-1)^u \sum_{\substack{j_1+\cdots+j_u=n\\ j_s\ge 0}} \frac{1}{j_1!\cdots j_u!}\,\mathrm{MV}_n(P_{i_1},j_1;\ldots;P_{i_u},j_u)\\
&= (-1)^k \sum_{\emptyset\ne J\subset[k]} (-1)^{|J|} \sum_{\substack{s_i\ge 0,\ \sum s_i=n\\ s_i=0 \text{ if } i\in[k]\setminus J}} \frac{\mathrm{MV}_n(P_1,s_1;\ldots;P_k,s_k)}{s_1!\cdots s_k!} \qquad (3.9)
\end{aligned}
$$

With the same notation J_s and α_s as in the proof of Lemma 3.4 we see that $\mathrm{MV}_n(P_1,s_1;\ldots;P_k,s_k)$ appears in the inner sum of (3.9) whenever $J_s\subset J$. Using this in (3.9) we get

$$
\mathrm{me}_n(P_1,\ldots,P_k) = (-1)^k \sum_{s_i\ge 0,\ \sum s_i=n} A'(s)\cdot \mathrm{MV}_n(P_1,s_1;\ldots;P_k,s_k) \qquad (3.10)
$$

where $A'(s)=\frac{(-1)^{|J_s|}}{s_1!\cdots s_k!}\sum_{i=0}^{\alpha_s}(-1)^i\binom{\alpha_s}{i}$. As seen before $A'(s)=0$ for $\alpha_s\ne 0$. Hence only terms with $\alpha_s=0$ (i.e. $J_s=[k]$) remain in which case $\sum_{i=0}^{\alpha_s}(-1)^i\binom{\alpha_s}{i}=1$ and we obtain

$$
\mathrm{me}_n(P_1,\ldots,P_k) = (-1)^k \sum_{s_i\ge 1,\ \sum s_i=n} (-1)^k\, \frac{\mathrm{MV}_n(P_1,s_1;\ldots;P_k,s_k)}{s_1!\cdots s_k!}\ .
$$

□

Lemma 3.8. *The coefficient of t^{n-1} in $\mathrm{ME}_{P_1,\ldots,P_k}(t)$ equals*

$$
\mathrm{me}_{n-1}(P_1,\ldots,P_k) = \frac{1}{2}\sum_{v\in\mathbb{S}^n} \sum_{\substack{s_i\ge 1\\ s_1+\cdots+s_k=n-1}} \frac{\mathrm{MV}'_{n-1}((P_1)^v,s_1;\ldots;(P_k)^v,s_k)}{s_1!\cdots s_k!}\ .
$$

Proof. The coefficient of t^{n-1} can be computed using the same combinatorial trick as in the proof of Lemma 3.7. The only difference is that we start here with the identity $e_{n-1}(P)=\frac{1}{2}\sum_{F \text{ facet of } P}\mathrm{vol}'_{n-1}(F)$ from (3.2).

$$
\begin{aligned}
\mathrm{me}_{n-1}(P_1,\ldots,P_k) &= (-1)^k \sum_{1\le i_1<\cdots<i_u\le k} (-1)^u\,\frac{1}{2} \sum_{\substack{F \text{ facet of}\\ P_{i_1}+\cdots+P_{i_u}}} \mathrm{vol}'_{n-1}(F)\\
&= (-1)^k \sum_{1\le i_1<\cdots<i_u\le k} (-1)^u\,\frac{1}{2}\sum_{v\in\mathbb{S}^n} \mathrm{vol}'_{n-1}\left((P_{i_1}+\cdots+P_{i_u})^v\right)\ , \qquad (3.11)
\end{aligned}
$$

where the last equation holds since $\mathrm{vol}'_{n-1}\left((P_{i_1}+\cdots+P_{i_u})^v\right)$ vanishes whenever v is not a facet normal of $P_{i_1}+\cdots+P_{i_u}$. Since $(P_{i_1}+\cdots+P_{i_u})^v=(P_{i_1})^v+\cdots+(P_{i_u})^v$ holds, the term in (3.11) can be written as

$$
\frac{1}{2}\sum_{v\in\mathbb{S}^n}\left[(-1)^k \sum_{1\le i_1<\cdots<i_u\le k} (-1)^u\,\mathrm{vol}'_{n-1}\left((P_{i_1})^v+\cdots+(P_{i_u})^v\right)\right]\ . \qquad (3.12)
$$

With the same method as before (starting from equation (3.8)) we can show that the term in the large brackets in (3.12) equals

$$\sum_{\substack{s_i \geq 1 \\ s_1+\cdots+s_k=n-1}} \frac{\mathrm{MV}'_{n-1}((P_1)^v, s_1; \ldots ; (P_k)^v, s_k)}{s_1! \cdots s_k!} \ .$$

Now finally using this in (3.12) yields

$$\mathrm{me}_{n-1}(P_1, \ldots, P_k) = \frac{1}{2} \sum_{v \in \mathbb{S}^n} \sum_{\substack{s_i \geq 1 \\ s_1+\cdots+s_k=n-1}} \frac{\mathrm{MV}'_{n-1}((P_1)^v, s_1; \ldots ; (P_k)^v, s_k)}{s_1! \cdots s_k!} \ .$$

□

3.2.3. The cases $\mathbf{k = n}$ and $\mathbf{k = n - 1}$. For $k = n$, Lemma 3.4 states that the coefficient $\mathrm{me}_r(P_1, \ldots, P_n)$ vanishes for $1 \leq r < n$. Since we consider the case $k = n$, Lemma 3.6 and Lemma 3.7 determine the remaining coefficients.

Theorem 3.9. $\mathrm{ME}_{P_1,\ldots,P_n}(t)$ *is a polynomial in* t *of degree* n *and we have*

$$\mathrm{ME}_{P_1,\ldots,P_n}(t) = t^n \cdot \mathrm{MV}_n(P_1, \ldots, P_n) \ + (-1)^{n+1} \ .$$

This theorem has a straight forward corollary by setting $t = 1$, which allows to express an alternating sum of integer point cardinalities of Minkowski sums by a mixed volume. Note that this statement appears already in [**Kuš76**] as a conjecture and is proven by Bernstein in [**Ber76**] using significantly different methods.

Corollary 3.10. *With the notation from above we have*

$$\mathrm{MV}_n(P_1, \ldots, P_n) = \sum_{\emptyset \neq J \subset [n]} (-1)^{n-|J|} \, \mathrm{L}\Big(\sum_{j \in J} P_j\Big) + (-1)^n \ .$$

Example 3.11. Take the following two polytopes in $\mathcal{P}(\mathbb{Z}^2)$:

$$\begin{aligned} P &:= \mathrm{conv}\{(0,0)^T, (2,0)^T, (0,1)^T, (1,2)^T, (2,1)^T\} \\ Q &:= \mathrm{conv}\{(1,0)^T, (0,2)^T, (3,2)^T, (2,3)^T\} \ . \end{aligned}$$

See Figure 3.2 for an illustration of P, Q and their Minkowski sum $P + Q$. The number of integer points in P, Q and $P + Q$ is $\mathrm{L}(P) = 7$, $\mathrm{L}(Q) = 8$ and $\mathrm{L}(P + Q) = 24$, respectively. Furthermore we have that $\mathrm{MV}_2(P, Q) = 10$ which agrees with Corollary 3.10 since $10 = 24 - 8 - 7 + 1$.

Similarly we can explicitly state the univariate mixed Ehrhart polynomial in the case $k = n - 1$.

Theorem 3.12. $\mathrm{ME}_{P_1,\ldots,P_{n-1}}(t)$ *is a polynomial in* t *of degree* n *and we have*

$$\begin{aligned} \mathrm{ME}_{P_1,\ldots,P_{n-1}}(t) = t^n \cdot \frac{1}{2} \mathrm{MV}_n &\left(P_1, \ldots, P_{n-1}, \sum_{i=1}^{n-1} P_i\right) \\ &+ t^{n-1} \cdot \frac{1}{2} \sum_{v \in \mathbb{S}^n} \mathrm{MV}'_{n-1}((P_1)^v, \ldots, (P_{n-1})^v) \ + \ (-1)^n \ . \end{aligned}$$

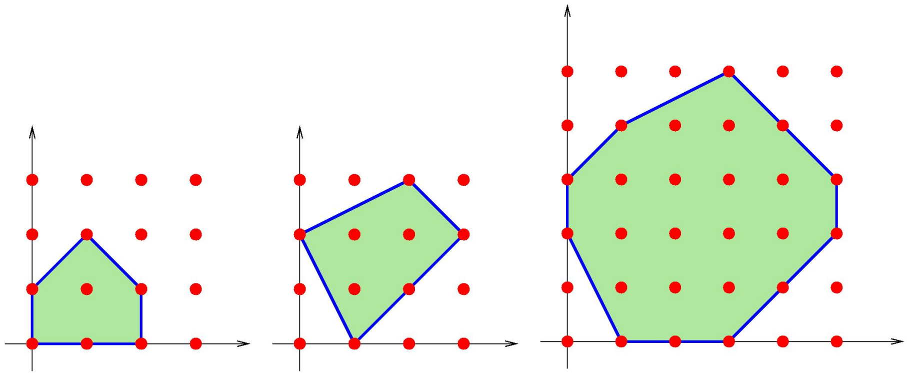

Figure 3.2. Integer points of P, Q and $P+Q$.

Proof. By Lemma 3.4 the coefficients of t^k vanish for $1 \le k < n-1$. The absolute coefficient equals $(-1)^n$ by Lemma 3.6. Considering the highest coefficient Lemma 3.7 yields

$$\begin{aligned}\mathrm{me}_n(P_1,\dots,P_{n-1}) &= \sum_{s_i\ge 1,\ \sum s_i=n} \frac{\mathrm{MV}_n(P_1,s_1;\dots;P_{n-1},s_{n-1})}{s_1!\cdots s_{n-1}!} \\ &= \frac{1}{2}\mathrm{MV}_n\left(P_1,\dots,P_{n-1},\sum_{i=1}^{n-1}P_i\right) .\end{aligned}$$

And finally Lemma 3.8 can be employed to determine the coefficient of t^{n-1}:

$$\begin{aligned}\mathrm{me}_{n-1}(P_1,\dots,P_{n-1}) &= \frac{1}{2}\sum_{v\in\mathbb{S}^n}\ \sum_{\substack{s_1+\dots+s_n-1=n-1\\ s_i\ge 1}} \frac{\mathrm{MV}'_{n-1}((P_1)^v,s_1;\dots;(P_{n-1})^v,s_{n-1})}{s_1!\cdots s_{n-1}!} \\ &= \frac{1}{2}\sum_{v\in\mathbb{S}^n}\mathrm{MV}'_{n-1}((P_1)^v,\dots,(P_{n-1})^v) .\end{aligned}$$

□

Of course this Theorem has as well a straight forward corollary. Surprisingly this statement plays a crucial role in Chapter 4 where it is employed to show that the tropical genus equals the toric genus of a curve depending on the same Newton polytopes.

Corollary 3.13. *With the notation from above we have*

$$\begin{aligned}&\sum_{\emptyset\ne J\subset[n-1]}(-1)^{n-1-|J|}\,\mathrm{L}(\sum_J P_j) \\ &= \frac{1}{2}\,\mathrm{MV}_n(P_1,\dots,P_{n-1},\sum_{i=1}^{n-1}P_i) + \frac{1}{2}\sum_{v\in\mathbb{S}^n}\mathrm{MV}'_{n-1}((P_1)^v,\dots,(P_{n-1})^v) + (-1)^n .\end{aligned}$$

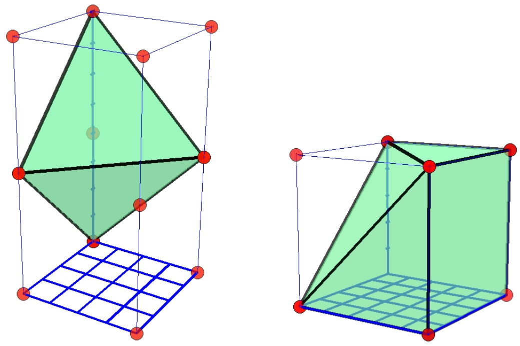

Figure 3.3. The polytopes P and Q in the integer lattice $\mathbb{Z}^3$.

Example 3.14. To illustrate the use of Corollary 3.13 consider the following two polytopes in $\mathcal{P}(\mathbb{Z}^3)$:

$$\begin{aligned} P &:= \operatorname{conv}\left\{(0,0,0)^T,(1,0,1)^T,(0,1,1)^T,(2,0,0)^T\right\} \\ Q &:= \operatorname{conv}\left\{(0,0,0)^T,(0,0,1)^T,(1,0,0)^T,(0,1,0)^T,(1,1,0)^T,(0,1,1)^T,(1,1,1)^T\right\} \end{aligned}$$

Figure 3.3 shows the two polytopes and the integer lattice. Here we see that $\mathrm{L}(P) = 5$ and $\mathrm{L}(Q) = 7$. The Minkowski sum of P and Q is depicted in Figure 3.4 from several perspectives to simplify counting the integer points in it. After careful counting we see that $\mathrm{L}(P+Q) = 22$.

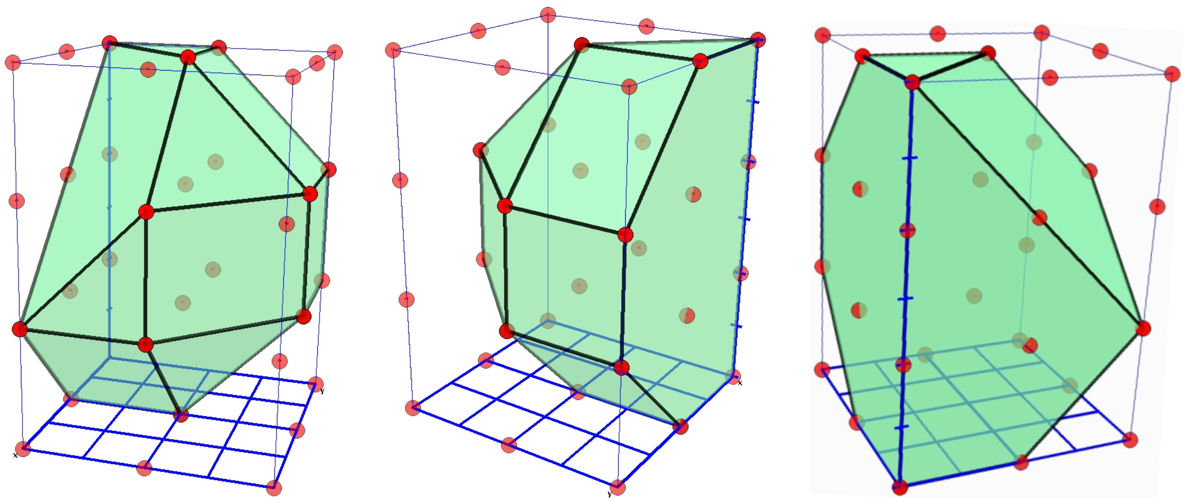

Figure 3.4. The sum $P+Q$ shown from 3 different viewpoints.

Furthermore $\mathrm{MV}_3(P, Q, P+Q) = 12$ as well as $\sum_{v \in \mathbb{S}^3} \mathrm{MV}'_2((P)^v, (Q)^v) = 1 + 1 + 1 + 1 + 1 + 3 + 2 = 10$. Hence we have

$$\frac{1}{2}\,\mathrm{MV}_3(P, Q, P+Q) + \frac{1}{2}\sum_{v \in \mathbb{S}^n} \mathrm{MV}'_2((P)^v, (Q)^v) + (-1)^3 = \frac{12}{2} + \frac{10}{2} - 1 = 10$$

$$\text{and} \quad \mathrm{L}(P+Q) - \mathrm{L}(P) - \mathrm{L}(Q) = 22 - 5 - 7 = 10$$

as predicted by Corollary 3.13.

3.2.4. The multivariate mixed Ehrhart polynomial. We conclude by discussing briefly what is known about the multivariate version of the mixed Ehrhart polynomial

$$\mathrm{ME}_{P_1,\dots,P_k}(t_1, \dots, t_k) := \sum_{\emptyset \neq J \subset [k]} (-1)^{k-|J|}\, \mathrm{L}(\textstyle\sum_{j \in J} t_j \cdot P_j) \tag{3.13}$$

for lattice polytopes $P_1, \dots, P_k$ and integers $t_1, \dots, t_k$. To study this multivariate function it is crucial to understand the number of lattice points in scaled Minkowski sums. The following statement is known concerning this aspect.

Proposition 3.15 (McMullen [**McM77**] and Bernstein [**Ber76**])**.** *Let φ be a Δ-valuation and let $P_1, \dots, P_k \in \mathcal{P}_n(\Delta)$. Then for integers $t_1, \dots, t_k \geq 0$, $\varphi(t_1 P_1 + \cdots + t_k P_k)$ is a polynomial in $t_1, \dots, t_k$ of total degree at most n. Moreover the coefficient of $t_1^{r_1} \cdots t_k^{r_k}$ is a homogeneous Δ-valuation of degree r_i in P_i.*

So in particular we have that $\mathrm{ME}_{P_1,\dots,P_k}(t_1, \dots, t_k)$ is a polynomial in $t_1, \dots, t_k$ of total degree at most n whose coefficients are alternating sums of homogeneous valuations. It is possible to compute the absolute coefficient just like in the univariate case but obtaining more information on the remaining coefficients is an open problem that deserves further research.

CHAPTER 4

Combinatorics and Genus of Tropical Intersections

Let $g_1, \ldots, g_k$ be tropical polynomials in n variables $x_1, \ldots, x_n$ with Newton polytopes $P_1, \ldots, P_k$ and let $X_i := X(g_i)$ denote their tropical hypersurfaces in $\mathbb{R}^n$ (see Section 1.3). In this chapter we study combinatorial questions on the intersection of the tropical hypersurfaces $X_1, \ldots, X_k$, such as the f-vector, the number of unbounded faces and (in case of a curve) the genus. Our point of departure is Vigeland's work [**Vig07**] who considered the special case $k = n-1$ and where all Newton polytopes are standard simplices. We generalize these results to arbitrary k and arbitrary Newton polytopes $P_1, \ldots, P_k$. This provides new formulas for the number of faces and the genus in terms of mixed volumes. Furthermore using the results on mixed Ehrhart polynomials from Chapter 3 we show that the genus of a tropical intersection curve equals the genus of a toric intersection curve corresponding to the same Newton polytopes.

4.1. Intersection Multiplicities

An intersection $\mathcal{I} = X_1 \cap \cdots \cap X_k$ is called *proper* if $\dim(\mathcal{I}) = n - k$. $\mathcal{I}$ is *transversal along a cell* A of this complex if the dual cell $C = F_1 + \cdots + F_k$ in the privileged subdivision of $P_1 + \cdots + P_k$ satisfies

$$\dim(C) = \dim(F_1) + \cdots + \dim(F_k) \ .$$

We call the intersection *transversal* if for each subset $J \subset \{1, \ldots, k\}$ the intersection is proper and transversal along each cell of the complex. In the dual picture a transversal intersection implies that the privileged subdivision of $P_1 + \cdots + P_k$ is mixed. Note that in a transversal intersection each cell A of $\mathcal{I}$ lies in the relative interior of each cell A_i from X_i that is involved in the intersection.

In the case of a non-transversal intersection $\mathcal{I}$ we can perturb the hypersurfaces by a small parameter ε to obtain again a transversal intersection $\mathcal{I}_\varepsilon$. The *stable intersection* $\mathcal{I}_{\mathrm{st}}$ is defined as the limit of these transversal intersections when ε goes to 0,

$$\mathcal{I}_{\mathrm{st}} := X_1 \cap_{\mathrm{st}} \cdots \cap_{\mathrm{st}} X_k := \lim_{\varepsilon \to 0} X_1^{(\varepsilon_1)} \cap \cdots \cap X_k^{(\varepsilon_k)}$$

(cf. [**RGST05**]). Stable intersections are always proper and they have some more comfortable features. As mentioned above a tropical hypersurface $X(g) \subset \mathbb{R}^n$ is a pure polyhedral complex of dimension $n-1$. The stable intersection of $X(g)$ with itself gives the $(n-2)$-skeleton of $X(g)$. In particular we can isolate the vertices of $X(g)$ by intersecting $X(g)$ n-times with itself.

Every face of a tropical intersection $\mathcal{I}$ naturally comes with a multiplicity. We follow the notation of Bertrand and Bihan [**BB07**], whose approach is consistent with those in [**Kat09**, **Mik06**].

Definition 4.1 (Intersection multiplicity)**.** Each cell A in an intersection $\mathcal{I}$ can be assigned a *multiplicity* (or *weight*) as follows. Let $C = F_1 + \cdots + F_k$ be its dual cell in $P_1 + \cdots + P_k$. If A is of dimension j then C is of dimension $n - j$ and we denote its type by $(d_1, \ldots, d_k)$. For a transversal intersection define

$$\begin{aligned} m_A &:= \left(\prod_{i=i}^{k} d_i! \cdot \mathrm{vol}'_{d_i}(F_i) \right) \cdot \mathrm{vol}_{n-j}(\mathfrak{P}) \\ &= \mathrm{MV}'_{n-j}(F_1, d_1; \ldots; F_k, d_k) \end{aligned} \tag{4.1}$$

where $\mathfrak{P}$ is a fundamental lattice polytope in the $(n-j)$-dimensional sublattice $\mathbb{Z}(F_1) + \cdots + \mathbb{Z}(F_k)$ and where vol'_{d_i} denotes the volume in the lattice $\mathbb{Z}(F_i)$ spanned by the integer vectors of F_i. (For more background on these relative volume forms and the proof that equality holds in (4.1) see [**BB07**].)

In the non-transversal case we have that $n - j \leq d_1 + \cdots + d_k$ and we define,

$$m_A := \sum_{\substack{(e_1,\ldots,e_k) \text{ s.t.} \\ \sum e_i = n-j;\ e_i \leq d_i}} \mathrm{MV}'_{n-j}(F_1, e_1; \ldots; F_k, e_k) \ .$$

Example 4.2. Take the tropical hypersurfaces defined by the following tropical polynomials which do not intersect transversally.

$$f = 4 \odot x^2 \oplus 6 \odot x \oplus 9 \odot y^2 \oplus 2, \quad g = 2.6 \odot y \oplus 2.1 \odot x \oplus 0.1$$

Small perturbations of the second hypersurface result in transversal intersections (see Figure 4.1 above).

To obtain the multiplicity of the intersection point of f and g we study the cell $C = \mathrm{conv}\left\{(1,0)^T, (2,0)^T, (0,2)^T, (0,3)^T, (1,2)^T\right\}$ in the subdivision of $P(f) + P(g)$. Note that this is the union of the cells which correspond to the intersection in the perturbed situations. C is of type $(1,2)$ and has the unique privileged decomposition $C = F_1 + F_2$ where $F_1 = \mathrm{conv}\left\{(1,0)^T, (0,2)^T\right\}$ and $F_2 = \mathrm{conv}\left\{(0,0)^T, (1,0)^T, (0,1)^T\right\}$. According to Definition 4.1, the multiplicity of this intersection point is $\mathrm{MV}_2(F_1, 1; F_2, 1) + \mathrm{MV}_2(F_1, 0; F_2, 2) = 2 + 0 = 2$. This agrees with the multiplicity that would be assigned to this point as a stable intersection since it is the limit of either two intersection points of multiplicity 1 or one intersection point of multiplicity 2.

Proposition 4.3 (Tropical Bernstein, see [**BB07**, **RGST05**])**.** *Suppose the tropical hypersurfaces $X_1, \ldots, X_n \subset \mathbb{R}^n$ with Newton polytopes $P_1, \ldots, P_n$ intersect in finitely many points. Then the number of intersection points counted with multiplicity is* $\mathrm{MV}_n(P_1, \ldots, P_n)$*. Furthermore the stable intersection of n tropical hypersurfaces $X_1, \ldots, X_n$ always consists of* $\mathrm{MV}_n(P_1, \ldots, P_n)$ *points counted with multiplicities.*

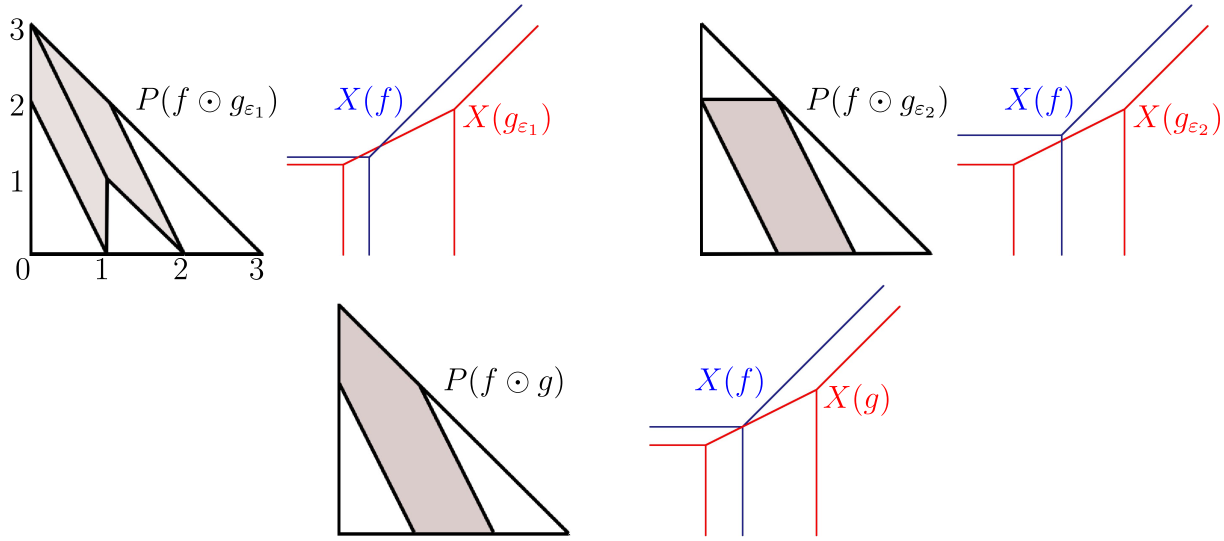

Figure 4.1. Above: The Newton polytope $P(f \odot g_\varepsilon) = P(f) + P(g_\varepsilon)$ with the privileged subdivision and the hypersurface $X(f \odot g_\varepsilon) = X(f) \cup X(g_\varepsilon)$ for two choices of ε. Below: The Newton polytope $P(f \odot g_\varepsilon) =$ with the privileged subdivision and the tropical hypersurface $X(f \odot g)$.

4.2. The number of j-faces in $\mathcal{I}$

Let $\mathcal{I} = X_1 \cap \cdots \cap X_k$ be a transversal intersection. Hence the intersection is proper which implies that the number of j-dimensional faces in $\mathcal{I}$ is 0 if $j \geq n-k$. By using the duality approach described in Section 1.3 the number of j-faces can be expressed in terms of mixed volumes.

Theorem 4.4. *The number of j-faces in $\mathcal{I}$ counting multiplicities is*

$$\sum_{A \in \mathcal{I}^{(j)}} m_A = \sum_{\substack{(d_1,\ldots,d_k) \text{ s.t.} \\ d_i \geq 1;\, \sum_i d_i = n-j}} \mathrm{MV}'_{n-j}(P_1, d_1; \ldots ; P_k, d_k) \ , \tag{4.2}$$

where $\mathrm{MV}'_{n-j}(P_1, d_1; \ldots ; P_k, d_k)$ *is interpreted as the sum over the relative volume of all* $(n-j)$*-dimensional cells of type* $(d_1, \ldots, d_k)$ *in a mixed subdivision of* $P_1 + \cdots + P_k$.

Note that this implies the tropical version of Bernstein's Theorem (see Proposition 4.3) for $k = n$ and $j = 0$.

Proof. Each j-dimensional cell C in the mixed subdivision of $P_1 + \cdots + P_k$ is dual to an $(n-j)$-dimensional cell A of $X_1 \cup \cdots \cup X_k$. If C is a mixed cell, i.e. $d_i \geq 1$ for all i, its dual A is contained in every X_i. Hence, by Definition 4.1

$$\sum_{A \in \mathcal{I}^{(j)}} m_A = \sum_{\substack{(d_1,\ldots,d_k) \text{ s.t.} \\ d_i \geq 1;\, \sum_i d_i = n-j}} \ \sum_{C = F_1 + \cdots + F_k} \mathrm{MV}'_{n-j}(F_1, d_1; \ldots ; F_k, d_k)$$

where the second sum runs over all cells C of type $(d_1, \dots, d_k)$. If we denote by $\mathrm{vol}'_{d_i}(F_i)$ the volume of F_i in the lattice spanned by the integer points of F_i and furthermore denote by $\mathfrak{P}$ the fundamental lattice parallelotope in $\mathbb{Z}^{n-j}$ defined by $F_1, \dots, F_k$ then (4.1) implies

$$\begin{aligned}\mathrm{MV}'_{n-j}(F_1, d_1; \dots; F_k, d_k) &= d_1! \cdots d_k! \ \mathrm{vol}'_{d_1}(F_1) \cdots \mathrm{vol}'_{d_k}(F_k) \ \mathrm{vol}_{n-j}(\mathfrak{P}) \\ &= d_1! \cdots d_k! \ \mathrm{vol}_{n-j}(C) \ .\end{aligned}$$

Hence we have

$$\begin{aligned}\sum_{A \in \mathcal{I}^{(j)}} m_A &= \sum_{\substack{(d_1,\dots,d_k) \text{ s.t.} \\ d_i \geq 1; \sum_i d_i = n-j}} \ \sum_{\substack{C \text{ of type} \\ (d_1,\dots,d_k)}} d_1! \cdots d_k! \ \mathrm{vol}_{n-j}(C) \\ &= \sum_{\substack{(d_1,\dots,d_k) \text{ s.t.} \\ d_i \geq 1; \sum_i d_i = n-j}} \mathrm{MV}'_{n-j}(P_1, d_1; \dots; P_k, d_k)\end{aligned}$$

where we used (1.12) for the last identity. □

In Section 4.4 we focus on the number of vertices in tropical intersection curves. Hence we state Theorem 4.4 for $k = n-1$ and $j = 0$ again which gives a much nicer expression.

Corollary 4.5. *Let $\mathcal{I} = X(f_1) \cap \cdots \cap X(f_{n-1})$ be a transversal intersection curve in $\mathbb{R}^n$ of $n-1$ tropical hypersurfaces with corresponding Newton polytopes $P_1, \dots, P_{n-1}$. Then the number of vertices in $\mathcal{I}$ counting multiplicities is*

$$\sum_{A \in \mathcal{I}^{(0)}} m_A = \mathrm{MV}_n(P_1, \dots, P_{n-1}, P_1 + \cdots + P_{n-1}) \ . \tag{4.3}$$

Remark 4.6. Corollary 4.5 generalizes [**Vig07**, Theorem 3.3] where each P_i is a standard simplex of the form $\mathrm{conv}\{s_i \cdot \xi^{(i)} \cup \{0\} \, : \, 1 \leq i \leq n\}$ where $\xi^{(i)}$ denotes the i-th unit vector and $s_i \in \mathbb{Z}_{>0}$. In this case (4.3) gives $s_1 \cdots s_{n-1} \cdot (s_1 + \cdots + s_{n-1})$ as the number of vertices counting multiplicities.

Proof. For $k = n-1$ the sum in (4.2) runs over all cells of type $(2, 1, \dots, 1)$, $(1, 2, 1, \dots, 1)$, $\dots$, $(1, \dots, 1, 2)$. Using the linearity of the mixed volume (1.9) we get

$$\begin{aligned}\sum_{A \in \mathcal{I}^{(0)}} m_A &= \mathrm{MV}_n(P_1, 2; P_2, 1; \dots, P_{n-1}, 1) + \cdots + \mathrm{MV}_n(P_1, 1; P_2, 1; \dots, P_{n-1}, 2) \\ &= \mathrm{MV}_n(P_1, \dots, P_{n-1}, P_1 + \cdots + P_{n-1}) \ .\end{aligned}$$

□

We can also prove Corollary 4.5 independently of the dual approach by using stable intersections.

Proof. Define $\mathcal{J} := X(f_1 \odot \cdots \odot f_{n-1}) = X(f_1) \cup \cdots \cup X(f_{n-1})$. We know that $\underbrace{\mathcal{J} \cap_{\mathrm{st}} \cdots \cap_{\mathrm{st}} \mathcal{J}}_{n\text{-times}} = \mathcal{J}^{(0)}$. Since $\mathcal{I} \subset \mathcal{J}^{(1)}$ holds, this implies that $\mathcal{I} \cap_{\mathrm{st}} \mathcal{J} \subset \mathcal{J}^{(0)}$. Furthermore we have $\mathcal{I} \cap_{\mathrm{st}} \mathcal{J} \subset \mathcal{I} \cap \mathcal{J} = \mathcal{I}$ and $\mathcal{J}^{(0)} \cap \mathcal{I} = \mathcal{I}^{(0)}$ such that

$$\mathcal{I}^{(0)} = \mathcal{I} \cap_{\mathrm{st}} \mathcal{J} \ .$$

The Newton polytope of $f_1 \odot \cdots \odot f_{n-1}$ is $P_1 + \cdots + P_{n-1}$. Now using the tropical Bernstein Theorem for stable intersections (Proposition 4.3) we have that the number of points in $\mathcal{I}^{(0)}$ counted with multiplicities is $\mathrm{MV}_n(P_1, \ldots, P_{n-1}, P_1 + \cdots + P_{n-1})$. $\square$

Example 4.7. We illustrate Corollary 4.5 in a 3-dimensional example. Let

$$f := 7 \odot x \oplus 6 \odot y \oplus 8 \odot z \oplus 5 \odot x \odot z \oplus -7 \odot x \odot y \oplus -2 \odot y \odot z$$
$$g := 9 \oplus 9 \odot x \oplus 7 \odot y \oplus -7 \odot x \odot z \oplus -17 \odot y \odot z \oplus -5 \odot x \odot y \odot z$$

be two tropical polynomials. Figure 4.2 shows the intersection $\mathcal{I} = X(f) \cap X(g)$ of the hypersurfaces defined by f and g. We have that $\mathrm{MV}_3(\mathrm{P}(f), \mathrm{P}(g), \mathrm{P}(f) + \mathrm{P}(g)) = 12$ which equals the number of vertices in $\mathcal{I}$ as can be seen in Figure 4.2 when counted carefully.

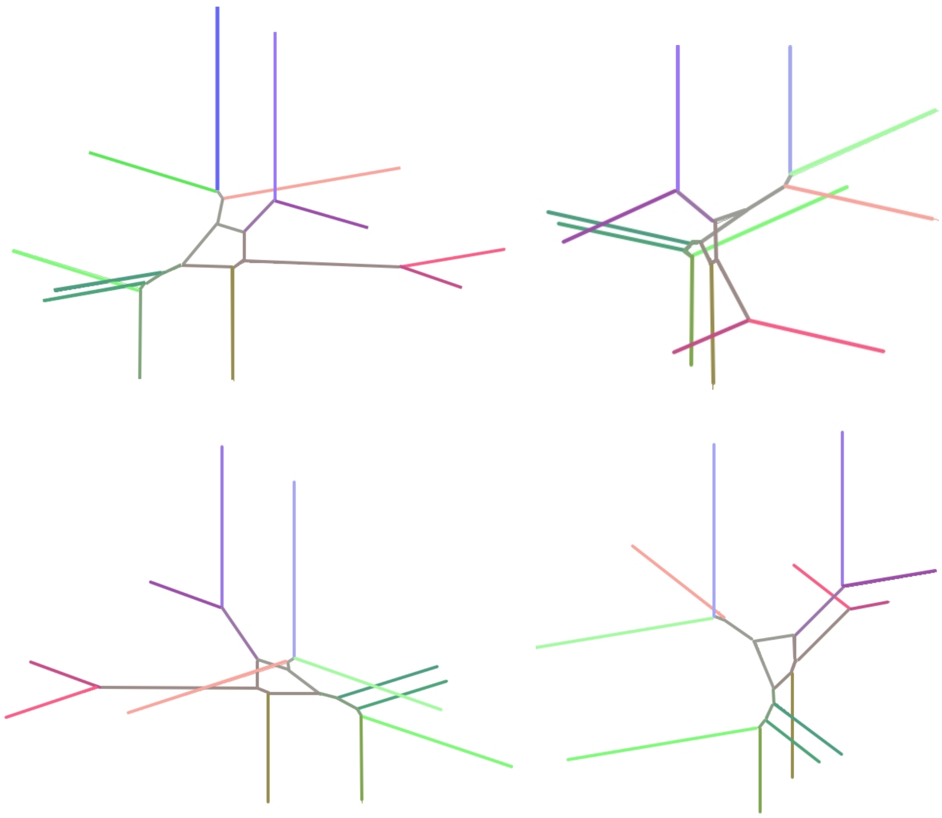

Figure 4.2. The intersection curve $\mathcal{I} = X(f) \cap X(g)$ of Example 4.7 from four different viewpoints.

For a non-transversal intersection $\mathcal{I}$ the same argumentation as in the proof of Theorem 4.4 leads to the following statement expressing the number of faces via mixed volumes. Note that for non-transversal intersections the privileged subdivision is in general not a mixed subdivision.

Theorem 4.8. *Let $\mathcal{I} = X_1 \cap \cdots \cap X_k$ be an intersection in $\mathbb{R}^n$ (where $n \geq k$) of k tropical hypersurfaces with corresponding Newton polytopes $P_1, \ldots, P_k$. Then the number of j-faces in $\mathcal{I}$ counting multiplicities is*

$$\sum_{A \in \mathcal{I}^{(j)}} m_A = \sum_{C = F_1 + \cdots + F_k} \sum_{\substack{t_1 + \cdots + t_k = n-j \\ \text{and } t_i \geq 0}} \mathrm{MV}'_{n-j}(F_1, t_1; \ldots ; F_k, t_k) \ . \tag{4.4}$$

where the first sum goes over all $(n-j)$-dimensional cells $C = F_1 + \cdots + F_k$ of the privileged subdivision of $P_1 + \cdots + P_k$ such that $\dim(F_i) \geq 1$ *for all i.*

4.3. The number of unbounded j-faces in $\mathcal{I}$

With similar techniques we count the number of unbounded faces in $\mathcal{I} = X_1 \cap \cdots \cap X_k$. Again, we formulate the result in a general manner though our main interest will later be the case $k = n-1$ and $j = 1$, i.e. the number of unbounded edges in a tropical intersection curve.

Theorem 4.9. *The number of unbounded j-faces in $\mathcal{I}$ is*

$$\sum_{F = (P_1)^v + \cdots + (P_k)^v} \mathrm{MV}'_{n-j}((P_1)^v, \ldots, (P_k)^v) \ . \tag{4.5}$$

Here the sum is taken over all $(n-j)$-faces F of $P := P_1 + \cdots + P_k$, $v \in \mathbb{S}^n$ is the outer unit normal vector of F and MV'_{n-j} denotes the $(n-j)$-dimensional mixed volume taken with respect to the lattice defined by the face F.

Proof. As seen in Section 1.3 the unbounded j-faces of the union $X_1 \cup \cdots \cup X_k$ correspond to $(n-j)$-dimensional cells in the boundary of $P = P_1 + \cdots + P_k$. So to count the unbounded j-faces in the intersection $\mathcal{I}$ we count mixed cells in all $(n-j)$-faces of P. Each face F of P has an outer unit normal vector v and $F = (P_1)^v + \cdots + (P_k)^v$ where $(P_i)^v$ denotes the face of P_i which is maximal with respect to v. So the number of unbounded j-faces counted with multiplicity (see Definition 4.1) which are dual to cells in F is $\mathrm{MV}'_{n-j}((P_1)^v, \ldots, (P_k)^v)$ and the result follows. $\square$

Example 4.10. Take the tropical polynomials f and g from Example 4.7. We would like to count the unbounded rays of $\mathcal{I} = X(f) \cap X(g)$. Careful counting in Figure 4.2 yields 12 rays. Theorem 4.9 states that this number can be obtained by computing the mixed volume on the facets of $\mathrm{P}(f) + \mathrm{P}(g)$. To simplify this for the reader we depict in Figure 4.3 the Newton polytopes of f and g as well as their Minkowski sum. The sum $\mathrm{P}(f) + \mathrm{P}(g)$ has 10 facets. 4 of those arise as the sum of a point and a facet and have therefore mixed volume 0. The remaining 6 facets have relative mixed volume 2 and hence we have $\sum_{F=(\mathrm{P}(f))^v + (\mathrm{P}(g))^v} \mathrm{MV}'_2((\mathrm{P}(f))^v, (\mathrm{P}(g))^v) = 12$ as predicted by Theorem 4.9.

The focus of Section 4.4 is on unbounded rays of an intersection $\mathcal{I}$. So let $j = 1$. To simplify the terms in the formulas we obtain it would be desirable to express the term in (4.5) as a single mixed volume. A tool to achieve this is Proposition 1.11 which states that for a full-dimensional convex body K

$$\mathrm{MV}_n(P_1, \ldots, P_{n-1}, K) = \sum_v \max_{a \in K} \langle a, v \rangle \cdot \mathrm{MV}'_{n-1}((P_1)^v, \ldots, (P_{n-1})^v)$$

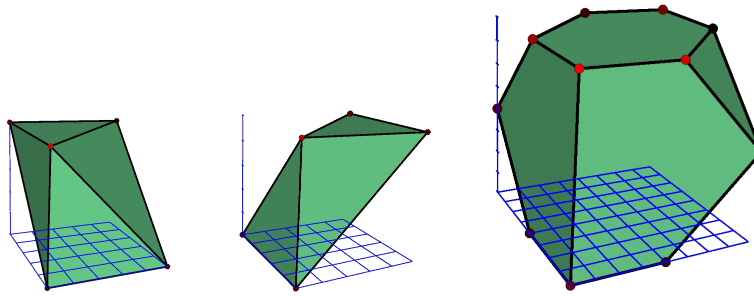

Figure 4.3. The Newton polytopes P(f) and P(g) and their Minkowski sum.

where the sum is taken over all primitive outer normals $v \in \mathbb{Z}^n$ of facets F of $P_1 + \ldots + P_{n-1}$.

The goal is now of course to find a convex body K such that $\max_{a\in K}\langle a, v\rangle = 1$ for all primitive outer facet normals v. Unfortunately such a body does not exist in general, see Example 4.13.

Corollary 4.11. *Denote by $v_1, \ldots, v_s$ the primitive outer facet normals to $P := P_1 + \ldots + P_{n-1}$. If none of the v_i lies in the convex hull of the remaining $s-1$ primitive normals then the number of unbounded rays in $\mathcal{I} = X_1 \cap \ldots \cap X_{n-1}$ is*

$$\mathrm{MV}_n(P_1, \ldots, P_{n-1}, Q) \tag{4.6}$$

where Q is the polar polytope of $\operatorname{conv}\{v_1, \ldots, v_k\}$, *i.e.*

$$Q = \bigcap_{i=1}^{s} \{x \in \mathbb{R}^n \,:\, \langle x, v_i\rangle \le 1\} \ .$$

Remark 4.12. In the situation that at least one of the primitive outer normals, say v_i, is in the convex hull of the remaining v_j, then (4.6) still gives a lower bound on the number of unbounded edges, since then $\max_{a\in Q}\langle a, v_i\rangle < 1$.

Example 4.13. The tropical polynomial f defined below is a 2-dimensional example that shows that the conditions of Corollary 4.11 do not always apply. Let

$$f := 2 \oplus 7 \odot x^7 \oplus 2 \odot x^3 \odot y \oplus 3 \odot x^4 \odot y$$

then $P(f) = \operatorname{conv}\left\{(0,0)^T, (3,1)^T, (4,1)^T, (7,0)^T\right\}$. The tropical line $X(f)$ has

$$\sum_{F \text{ facet of } P(f)} \mathrm{MV}_1'(F) = 1 + 1 + 1 + 8 = 11$$

unbounded rays counted with multiplicities. The primitive outer normals of $P(f)$ are $(0,-1)^T, (0,1)^T, (1,3)^T$, and $(-1,3)^T$ such that $Q = \operatorname{conv}\left\{(4,-1)^T, (-4,-1)^T, (0,\frac{1}{3})^T\right\}$ and therefore $\mathrm{MV}_2(P(f), Q) = \frac{28}{3} < 11$. The situation is illustrated in Figure 4.4

4.4. The Genus of Tropical Intersection Curves

Suppose we are given the intersection curve of $n-1$ smooth tropical hypersurfaces in $\mathbb{R}^n$, where a tropical hypersurface X is called *smooth* if the maximal cells of its privileged

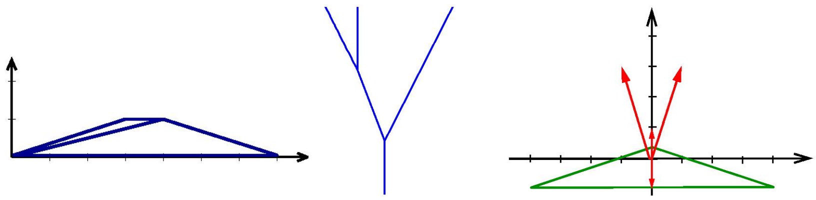

Figure 4.4. From left to right: $P(f)$ with privileged subdivision. The hypersurface $X(f)$. The fan of primitive outer normals of $P(f)$ and Q.

subdivision are simplices of volume $\frac{1}{n!}$. We apply the results of the last section to express its tropical genus g (as defined below) in terms of mixed volumes of the Newton polytopes corresponding to the defining hypersurfaces. Our goal is to prove that this genus coincides with the genus $\bar{g}$ of a toric variety X that was obtained using the same Newton polytopes. Due to a result by Khovanskiĭ [**Kho78**] the toric genus can be expressed via alternating sums of interior integer point numbers. To show that the combinatorial expressions for g and $\bar{g}$ are equal we employ our results on mixed Ehrhart theory from Chapter 3.

4.4.1. The genus via mixed volumes. Assume in the following that the intersection curve $\mathcal{I}$ is connected and was obtained by a transversal intersection of $n-1$ hypersurfaces $X_1 \cap \cdots \cap X_{n-1}$ with Newton polytopes $P_1, \ldots, P_{n-1}$. For a transversal intersection curve $\mathcal{I}$ in $\mathbb{R}^n$ define the *genus* $g = g(\mathcal{I})$ as the number of independent cycles of $\mathcal{I}$, i.e. its first Betti number.

Since $\mathcal{I}$ is a transversal intersection each vertex A in $\mathcal{I}$ is dual to a cell C of type $(1, \ldots, 1, 2, 1, \ldots, 1)$ in the privileged subdivision of $P_1 + \cdots + P_{n-1}$. So C is a sum of $n-1$ edges and one 2-dimensional face F_i of P_i. The *degree* (or *valence*) of A is the number of outgoing edges (bounded and unbounded) in $\mathcal{I}$. Each such outgoing edge A' is dual to an $(n-1)$-dimensional mixed cell C' which is a facet of C. Hence the degree of A equals the number of edges of the 2-dimensional face F_i.

Vigeland gave in [**Vig07**] an expression for the genus of a 3-valent curve in terms of inner vertices and outgoing edges. The proof does not apply tropical properties of $\mathcal{I}$ and works for any 3-valent graph with unbounded edges. Note that the vertices and edges are not counted with multiplicities in this statement.

Proposition 4.14 (see [**Vig07**])**.** *For a 3-valent tropical intersection curve $\mathcal{I}$ we have*

$$2g - 2 = \#\{\textit{vertices in } \mathcal{I}\} - \#\{\textit{unbounded edges in } \mathcal{I}\} .$$

If $\mathcal{I}$ is obtained as an intersection of smooth hypersurfaces, then $\mathcal{I}$ is 3-valent and each vertex and unbounded edge has multiplicity 1.

Theorem 4.15. *Let $\mathcal{I}$ be a connected transversal intersection of $n-1$ smooth tropical hypersurfaces in $\mathbb{R}^n$ with Newton polytopes $P_1, \ldots, P_{n-1}$. Then the genus g of $\mathcal{I}$ is given* *by*

$$2g-2 = \mathrm{MV}_n\left(P_1,\dots,P_{n-1},\sum_{i=1}^{n-1} P_i\right) - \sum_v \mathrm{MV}'_{n-1}((P_1)^v,\dots,(P_{n-1})^v) \tag{4.7}$$

where v runs over all outer unit normal vectors of $P_1+\cdots+P_{n-1}$.

Remark 4.16. If the smoothness condition of the hypersurfaces X_i is dropped, the right hand side of (4.7) still gives an upper bound for $2g-2$.

Proof. Using Corollary 4.5, Theorem 4.9 and Proposition 4.14 we immediately get the result. □

In particular we see that under the conditions of Theorem 4.15 the genus only depends on the Newton polytopes $P_1,\dots,P_{n-1}$ and we will write $g(P_1,\dots,P_{n-1})$ to denote this value.

Example 4.17. Consider this theorem in the case $n=2$. Here we just have one smooth tropical hypersurface X with corresponding Newton polytope P. The genus g of this curve equals the number of interior integer points of P, see e.g. [**RGST05**]. So Theorem 4.15 states that

$$\begin{aligned} 2\cdot\#\left\{\begin{smallmatrix}\text{interior integer}\\ \text{points of } P\end{smallmatrix}\right\} - 2 &= \mathrm{MV}_2(P,P) - \sum_{v\in\mathbb{S}^2}\mathrm{MV}'_1((P)^v) \\ &= 2\cdot\mathrm{vol}_2(P) - \#\left\{\begin{smallmatrix}\text{integer points on}\\ \text{the facets of } P\end{smallmatrix}\right\} . \end{aligned}$$

Hence Theorem 4.15 implies that

$$\mathrm{vol}_2(P) = \#\left\{\begin{smallmatrix}\text{interior integer}\\ \text{points of } P\end{smallmatrix}\right\} + \frac{1}{2}\cdot\#\left\{\begin{smallmatrix}\text{integer points on}\\ \text{the facets of } P\end{smallmatrix}\right\} - 1$$

which is known as Pick's theorem for convex polygons (see [**AZ04**]).

Example 4.18. Take once more the two tropical polynomials f and g from Example 4.7. As can be seen in Figure 4.2 the intersection curve $\mathcal{I} = X(f)\cap X(g)$ has genus 1. In Example 4.7 it was shown that $\mathrm{MV}_3(\mathrm{P}(f),\mathrm{P}(g),\mathrm{P}(f)+\mathrm{P}(g)) = 12$ and in Example 4.10 we saw that $\sum_v \mathrm{MV}'_2((\mathrm{P}(f))^v,(\mathrm{P}(g))^v) = 12$. Hence we have that $2\cdot g-2 = 2\cdot 1-2 = 0$ equals $\mathrm{MV}_3(\mathrm{P}(f),\mathrm{P}(g),\mathrm{P}(f)+\mathrm{P}(g)) - \sum_v \mathrm{MV}'_2((\mathrm{P}(f))^v,(\mathrm{P}(g))^v = 12-12 = 0$ as predicted by Theorem 4.15.

4.4.2. Khovanskiĭ's toric genus. An introduction to toric varieties is beyond the scope of this work and we refer the reader to [**Ful93**]. In [**Kho78**], Khovanskiĭ gave a formula for the genus of a complete intersection in a toric variety. Let the variety X in $(\mathbb{C}^*)^n$ be defined by a non-degenerate system of equations $f_1=\cdots=f_k=0$ with Newton polyhedra $P_1,\dots,P_k$ where each has full dimension n. Let $\bar{X}$ be the closure of X in a sufficiently complete projective toric compactification.

Proposition 4.19 (Khovanskiĭ [**Kho78**])**.** *If $\bar{X}$ is connected and has no holomorphic forms of intermediate dimension, then the geometric genus $\bar{g}$ of X can be calculated by the formula*

$$\bar{g} = \sum_{\emptyset\neq J\subset[k]} (-1)^{k-|J|}\,\mathrm{L}^{\circ}(\sum_{j\in J} P_j) \tag{4.8}$$

where $\mathrm{L}^{\circ}(P)$ *denotes the number of interior integer points of the lattice polytope* P *and* $[k] := \{1, \ldots, k\}$.

Thus for any variety satisfying the conditions of Proposition 4.19, the genus only depends on $P_1, \ldots, P_k$. We call this value $\bar{g}(P_1, \ldots, P_k)$.

4.4.3. Toric genus equals tropical genus. We are ready now to state and proof our theorem comparing the genus of tropical and toric intersection curves.

Theorem 4.20. *Let* $P_1, \ldots, P_{n-1} \subset \mathbb{R}^n$ *be full-dimensional lattice polytopes. Then the tropical and the toric genus with respect to* $P_1, \ldots, P_{n-1}$ *coincide, i.e.*

$$\bar{g}(P_1, \ldots, P_{n-1}) = g(P_1, \ldots, P_{n-1}) \ .$$

We prove this theorem by showing that the combinatorial quantities of Proposition 4.19 and Theorem 4.15 are the same, i.e.

$$\begin{aligned} &\frac{1}{2}\,\mathrm{MV}_n\big(P_1, \ldots, P_{n-1}, \sum_{i=1}^{n-1} P_i\big) - \frac{1}{2}\sum_{v \in \mathbb{S}^n} \mathrm{MV}'_{n-1}((P_1)^v, \ldots, (P_{n-1})^v) + 1 \\ &= \sum_{\emptyset \neq J \subset [n-1]} (-1)^{n-1-|J|}\,\mathrm{L}^{\circ}\big(\sum_J P_j\big) \ . \end{aligned} \tag{4.9}$$

That (4.9) holds for $n = 2$ can be seen in Example 4.17 when we take Pick's theorem as given.

Proof. Corollary 3.13 almost states the desired equation. Namely we have that

$$\begin{aligned} &\frac{1}{2}\,\mathrm{MV}_n\big(P_1, \ldots, P_{n-1}, \sum_{i=1}^{n-1} P_i\big) + \frac{1}{2}\sum_{v \in \mathbb{S}^n} \mathrm{MV}'_{n-1}((P_1)^v, \ldots, (P_{n-1})^v) + (-1)^n \\ &= \sum_{\emptyset \neq J \subset [n-1]} (-1)^{n-1-|J|}\,\mathrm{L}\big(\sum_J P_j\big) \ . \end{aligned}$$

Using the Ehrhart reciprocity (3.3) in the same way as in (3.5) yields the result. □

Even though Theorem 4.20 is already proved we would like to give another independent proof for the *unmixed case* $P_1 = \cdots = P_{n-1}$ of (4.9) that does not need the heavy machinery of mixed Ehrhart theory. The results obtained here on the surface volume and the number of integer points of a *lattice complex*, i.e. a bounded polyhedral complexes with vertices in $\mathbb{Z}^n$, might be of general interest in other contexts.

Let $\chi(Q)$ denote the Euler-Poincaré characteristic of a polyhedral complex Q. For simplicity we set $\mathrm{L}(0 \cdot Q) := \chi(Q)$ and by ∂Q we denote the boundary complex of Q.

Theorem 4.21. *Let* Q *be a pure* n*-dimensional lattice complex. Then*

$$\sum_{F \text{ facet of } Q} (n-1)!\,\mathrm{vol}'_{n-1}(F) = (-1)^{n-1} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k}\,\mathrm{L}(k \cdot \partial Q) \ . \tag{4.10}$$

Proof. We subdivide the facets of $k \cdot Q$ into fundamental lattice simplices [1] (i.e. simplices Δ of volume $\frac{1}{(\dim \Delta)!}$) with respect to the lattices defined by the facets. Let f_i be the number of i-dimensional faces of this simplicial complex. Note that the left hand side of (4.10) counts the number of $(n-1)$-dimensional faces, i.e.

$$f_{n-1} = \sum_{F \text{ facet of } Q} (n-1)! \operatorname{vol}'_{n-1}(F) \ .$$

Each i-dimensional face of our complex is a fundamental lattice simplex. The number of interior integer points of a fundamental lattice simplex Δ of dimension i stretched by a factor of $k \geq 1$ is equal to

$$\# \left\{ x \in \mathbb{N}^i \ : \ x_j \geq 1 \text{ and } \sum_j x_j \leq k-1 \right\} = \binom{k-1}{i} \ .$$

Hence we have for $k \geq 1$ that

$$\mathrm{L}(k \cdot \partial Q) = \sum_{i=0}^{k-1} \binom{k-1}{i} f_i \ .$$

Up to the term for $k = 0$ the sum on the right hand side of (4.10) evaluates to

$$\begin{aligned} &\sum_{k=1}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \sum_{i=0}^{k-1} \binom{k-1}{i} f_i \\ &\qquad = (-1)^{n-1} \sum_{i=0}^{n-2} f_i \sum_{k=1+i}^{n-1} (-1)^k \binom{n-1}{k} \binom{k-1}{i} \\ &\qquad = \sum_{i=0}^{n-2} f_i \sum_{r=0}^{n-2-i} (-1)^r \binom{n-1}{r} \binom{n-2-r}{n-2-i-r} \end{aligned} \tag{4.11}$$

where we substituted $r = n - k - 1$ to obtain the last equation. Using the following binomial identity (see e.g. [**Grü03**, p. 149])

$$\text{For } 0 \leq c \leq a : \quad \sum_{i=0}^{c} (-1)^i \binom{b}{i} \binom{a-i}{c-i} = \binom{a-b}{c}$$

yields that the right hand side in (4.10) equals

$$(-1)^{n-1} \chi(\partial Q) + \sum_{i=0}^{n-2} f_i \binom{-1}{n-2-i} = (-1)^{n-1} \chi(\partial Q) + \sum_{i=0}^{n-2} (-1)^{n-2-i} f_i \ .$$

By the Euler-Poincaré formula $\chi(\partial Q) = \sum_{i=0}^{n-1} (-1)^i f_i$ (see [**Bre93**]) this expression simplifies to f_{n-1} which proves the theorem. ☐

[1] We assume here that Q allows such a subdivision. If not choose an $N \in \mathbb{N}$ such that $N \cdot Q$ allows a subdivision into fundamental lattice simplices. Then our proof shows that (4.10) holds for $N \cdot Q$. Since both sides of (4.10) are polynomials in N the equation holds as well for Q. We thank Benjamin Nill for helpful remarks on this point.

By combining Theorem 4.21 and the generalization of Pick's theorem by Macdonald (see Reeve [**Ree57**] for the 3-dimensional case) we get the unmixed version of (4.9).

Proposition 4.22 (Macdonald [**Mac63**]). *Let P be a pure n-dimensional lattice complex, let $\mathrm{L}(0 \cdot P) := \chi(P)$ be the Euler-Poincaré characteristic of P and denote by ∂P the boundary complex of P. Then we have*

$$\frac{n-1}{2} n! \operatorname{vol}_n(P) = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \left[\mathrm{L}(k \cdot P) - \frac{1}{2} \mathrm{L}(k \cdot \partial P) \right] .$$

Corollary 4.23. *For n-dimensional lattice polytopes P we have*

$$\frac{n-1}{2} n! \operatorname{vol}_n(P) - \frac{1}{2} \sum_{F \text{ facet of } P} (n-1)! \operatorname{vol}'_{n-1}(F) \; + 1 = \sum_{k=1}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \mathrm{L}^{\circ}(k \cdot P) .$$

Proof. Knowing Proposition 4.22 and the fact that $\mathrm{L}(P) - \frac{1}{2}\mathrm{L}(\partial P) = \mathrm{L}^{\circ}(P) + \frac{1}{2}\mathrm{L}(\partial P)$ we still have to show that

$$\begin{aligned} &(-1)^{n-1} \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k} \mathrm{L}(k \cdot \partial P) + 2(-1)^{n-1}(\chi(P) - \chi(\partial P)) \\ &= \sum_{F \text{ facet of } P} (n-1)! \operatorname{vol}'_{n-1}(F) \; - 2 . \end{aligned}$$

Since the Euler-Poincaré formula implies $\chi(P) = \chi(\partial P) + (-1)^n$, the last equation reduces to the statement of Theorem 4.21. ☐

CHAPTER 5

The Number of Embeddings of Minimally Rigid Graphs

Determining the number of embeddings of minimally rigid graph frameworks is an open problem which corresponds to understanding the solutions of the resulting systems of equations. In this chapter we investigate the bounds which can be obtained from the viewpoint of Bernstein's Theorem (see Section 1.4). To do this, the techniques to study the mixed volume of systems of polynomial equations described in Chapter 2 are employed. While in most cases the resulting bounds are weaker than the best known bounds on the number of embeddings, for some classes of graphs the bounds are tight.

The focus here is on the 2-dimensional case. With respect to 3 and higher dimensions we give a brief discussion at the end of this chapter.

5.1. Laman graphs

Let $G = (V, E)$ be an undirected and loop-free graph with no multiple edges on N vertices with $2N - 3$ edges. If each subset of k vertices spans at most $2k - 3$ edges, we say that G has the *Laman property* and call it a *Laman graph* (see [**Lam70**]). A *framework* is a tuple (G, L) where $G = (V, E)$ is a graph and $L = \{l_{i,j} : [v_i, v_j] \in E\}$ is a set of $|E|$ positive numbers interpreted as edge lengths. For generic edge lengths, Laman graph frameworks are *minimally rigid* (see [**Con93**]), i.e. they are rigid and they become flexible if any edge is removed. Note that some authors call such frameworks *isostatic*.

A *Henneberg sequence* (cf. [**Hen11**]) for a graph G is a sequence $(G_i)_{3 \le i \le r}$ of Laman graphs such that G_3 is a triangle, $G_r = G$ and each G_i is obtained by G_{i-1} via one of the following two types of steps: A *Henneberg I step* adds one new vertex v_{i+1} and two new edges, connecting v_{i+1} to two arbitrary vertices of G_i. A *Henneberg II step* adds one new vertex v_{i+1} and three new edges, connecting v_{i+1} to three vertices of G_i such that at least two of these vertices are connected via an edge e of G_i and this certain edge e is removed (see Figure 5.1).

Any Laman graph G can be constructed via a Henneberg sequence and any graph constructed via a Henneberg sequence has the Laman property (see [**ST08b**,**TW85**]). We call G a *Henneberg I graph* if it is constructable using only Henneberg I steps. Otherwise we call it *Henneberg II*.

Given a Laman graph framework we want to know how many embeddings, i.e. maps $\alpha : V \to \mathbb{R}^2$, exist such that the Euclidean distance between two points in the image is exactly $l_{i,j}$ for all $[v_i, v_j] \in E$. Since every rotation or translation of an embedding gives another one, we ask how many embeddings exist *modulo rigid motions*. Due to the minimal rigidity property, questions about embeddings of Laman graphs arise naturally in

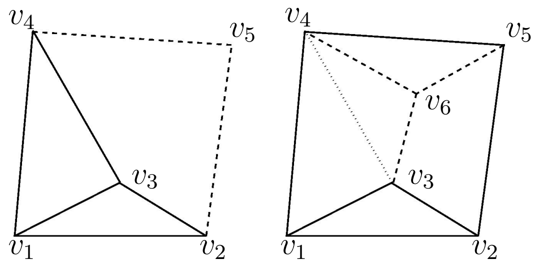

Figure 5.1. A Henneberg I and a Henneberg II step. New edges are dashed and the deleted edge is pointed.

rigidity and linkage problems (see [**HOR**$^+$**05**, **TD99**]). Graphs with less edges will have zero or infinitely many embeddings modulo rigid motions, and graphs with more edges do not have any embeddings for a generic choice of edge lengths.

Please note that we do allow edges to cross in the embeddings. For methods to enumerate non-crossing minimally rigid frameworks see [**AKO**$^+$**07**, **AKO**$^+$**08**].

To use these algebraic tools for the embedding problem we formulate that problem as a system of polynomial equations in the $2N$ unknowns $(x_1, y_1, \ldots, x_N, y_N)$ where (x_i, y_i) denote the coordinates of the embedding of the vertex v_i. Each prescribed edge length translates into a polynomial equation. I.e. if $e_k := [v_i, v_j] \in E$ with length $l_{i,j}$, we require $h_k(x) := (x_i - x_j)^2 + (y_i - y_j)^2 - l_{i,j}^2 = 0$. Thus we obtain a system of $|E|$ quadratic equations whose solutions represent the embeddings of our framework. To get rid of translations and rotations we fix the points $(x_1, y_1) = (c_1, c_2)$ and $(x_2, y_2) = (c_3, c_4)$ (Here we assume without loss of generality that there is an edge between v_1 and v_2.) For practical reasons we choose $c_i \neq 0$ and c_3, c_4 are chosen such that the embedded points (x_1, y_1) and (x_2, y_2) have distance $l_{1,2}$. Hence we want to study the solutions to the following system of $2N$ equations.

$$
(5.1) \qquad \left\{
\begin{array}{l}
h_1(x) := x_1 - c_1 = 0 \\
h_2(x) := y_1 - c_2 = 0 \\
h_3(x) := x_2 - c_3 = 0 \\
h_4(x) := y_2 - c_4 = 0 \\
h_k(x) := (x_i - x_j)^2 + (y_i - y_j)^2 - l_{i,j}^2 = 0 \quad \forall e_k = [v_i, v_j] \in E - \{[v_1, v_2]\}
\end{array}
\right\}
$$

5.2. Application of the BKK theory on the graph embedding problem

Our goal is to apply Bernstein's results to give bounds on the number of embeddings of Laman graphs. A first observation shows that for the formulation (5.1) the Bernstein bound is not tight. Namely, the system (5.1) allows to choose a direction v that satisfies the conditions of Bernstein's Second Theorem (Proposition 1.29). The choice

$v = (0, 0, 0, 0, -1, -1, \ldots, -1)$ yields the face system

$$\begin{cases} \mathrm{init}_v\, h_1 = x_1 - c_1 = 0 \\ \mathrm{init}_v\, h_2 = y_1 - c_2 = 0 \\ \mathrm{init}_v\, h_3 = x_2 - c_3 = 0 \\ \mathrm{init}_v\, h_4 = y_2 - c_4 = 0 \\ \mathrm{init}_v\, h_k = x_i^2 + y_i^2 = 0 \quad \forall e_k = [v_1, v_i], [v_2, v_i] \in E \\ \mathrm{init}_v\, h_k = (x_i - x_j)^2 + (y_i - y_j)^2 = 0 \quad \forall e_k = [v_i, v_j] \in E \text{ with } i, j \neq 1, 2 \end{cases}$$

which has $(x_1, y_1, \ldots, x_N, y_N) = (c_1, c_2, c_3, c_4, 1, i, 1, i, \ldots, 1, i)$ as a solution with non-zero complex entries. So the mixed volume of the system in (5.1) is a strict upper bound on the number of graph embeddings.

To decrease this degeneracy we apply an idea of Ioannis Emiris[1] (see [**Emi94**]). Surprisingly the introduction of new variables for common subexpressions, which increases the Bézout bound, can decrease the BKK bound. To the best of our knowledge it is an open problem to characterize in general when substitutions can be applied to remove degeneracies and reduce the mixed volume.

Here we introduce for every $i = 1, \ldots, N$ the variable s_i together with the new equation $s_i = x_i^2 + y_i^2$. This leads to the following system of equations.

$$\begin{cases} x_1 - c_1 = 0 \\ y_1 - c_2 = 0 \\ x_2 - c_3 = 0 \\ y_2 - c_4 = 0 \\ s_i + s_j - 2x_i x_j - 2y_i y_j - l_{i,j}^2 = 0 \quad \forall [v_i, v_j] \in E - \{[v_1, v_2]\} \\ s_i - x_i^2 - y_i^2 = 0 \quad \forall i = 1, \ldots, N \end{cases} \tag{5.2}$$

Experiments show that the system (5.2) is still not generic in the sense of Proposition 1.29 for every underlying minimally rigid graph. Hence the upper bound on the number of embeddings given by the mixed volume might not be tight in every case.

5.2.1. Henneberg I graphs. For this simple class of Laman graphs the mixed volume bound is tight as we will demonstrate below. Our proof exploits the inductive structure of Henneberg I graphs which is why it cannot be used for Henneberg II graphs.

Theorem 5.1. *For a Henneberg I graph on N vertices, the mixed volume of system* (5.2) *equals* 2^{N-2}.

PROOF. Each Henneberg sequence starts with a triangle for which system (5.2) has mixed volume 2. Starting from the triangle we consider a sequence of Henneberg I steps and show that the mixed volume doubles in each of these steps.

In a Henneberg I step we add one vertex v_{N+1} and two edges $[v_r, v_{N+1}]$, $[v_q, v_{N+1}]$ with lengths $l_{r,N+1}$ and $l_{q,N+1}$. So our system of equations (5.2) gets three new equations,

[1] Personal communication at EuroCG 2008, Nancy.

namely

$$s_{N+1} - x_{N+1}^2 - y_{N+1}^2 = 0 \tag{5.3}$$

$$s_r + s_{N+1} - 2x_r x_{N+1} - 2y_r y_{N+1} - l_{r,N+1}^2 = 0 \tag{5.4}$$

$$s_q + s_{N+1} - 2x_q x_{N+1} - 2y_q y_{N+1} - l_{q,N+1}^2 = 0. \tag{5.5}$$

In the new system of equations these three are the only polynomials involving x_{N+1}, y_{N+1} and s_{N+1}, so Lemma 2.6 can be used to calculate the mixed volume separately. The projections of the Newton polytopes of equations (5.3), (5.4) and (5.5) to the coordinates x_{N+1}, y_{N+1} and s_{N+1} are

$$\operatorname{conv}\left\{\begin{pmatrix}2 & 0 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 2 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 0 & 1\end{pmatrix}^T\right\}$$

and twice

$$\operatorname{conv}\left\{\begin{pmatrix}1 & 0 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 1 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 0 & 1\end{pmatrix}^T, \begin{pmatrix}0 & 0 & 0\end{pmatrix}^T\right\} .$$

The mixed volume of these equals 2. So by Lemma 2.6 the mixed volume of the new system is twice the mixed volume of the system before the Henneberg I step. □

To get two new embeddings in every Henneberg I step we choose the new edge lengths to be almost equal to each other and much larger then all previous edge lengths (larger then the sum of all previous is certainly enough).

Corollary 5.2 (Borcea and Streinu [**BS04**]). *The number of embeddings of Henneberg I graph frameworks is less than or equal to* 2^{N-2} *and this bound is sharp.*

Of course the elementary proof described in [**BS04**] of this statement does not need such heavy machinery as Bernstein's Theorem. The purpose of Theorem 5.1 is to show that the techniques described in this work apply here and that the BKK bound is tight in this case.

5.2.2. Laman graphs on 6 vertices. The first Laman graphs which are not constructable using only Henneberg I steps arise on 6 vertices. A simple case analysis shows that up to isomorphisms there are only two such graphs, the Desargues graph and $K_{3,3}$ (see Figure 5.2).

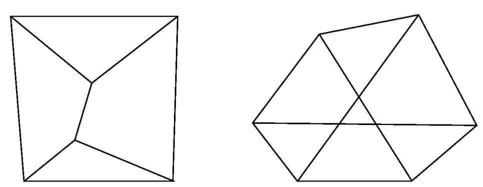

Figure 5.2. Left: Desargues graph. Right: $K_{3,3}$.

The number of embeddings of both graphs has been studied in detail. The Desargues graph is studied in [**BS04**] where the authors show that there can only be 24 embeddings and that there exists a choice of edge lengths giving 24 different embeddings. This is obtained by investigating the curve that is traced out by one of the vertices after one

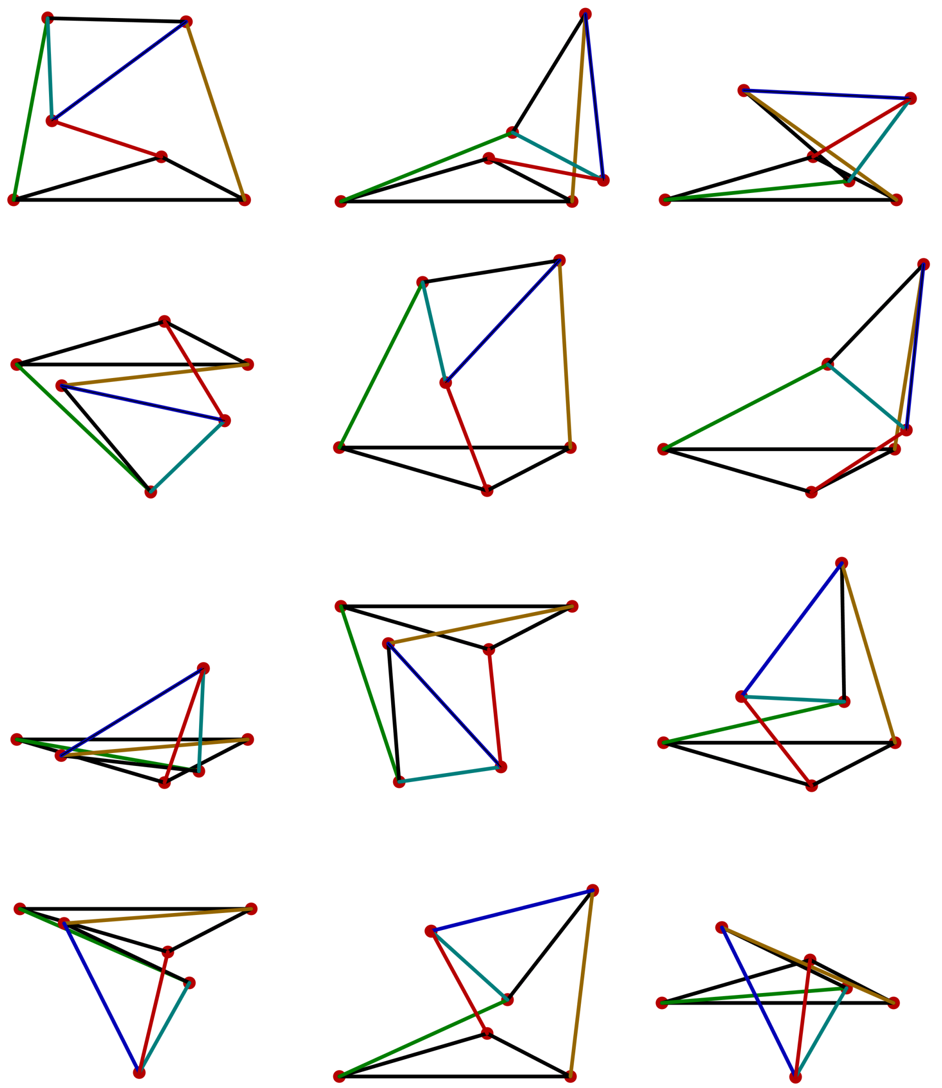

Figure 5.3. 12 embeddings of the Desargues graph. (Edge legths used in this example: $l_{1,2} = 4, l_{1,3} = 2.667, l_{2,3} = 1.622, l_{1,4} = 3.2, l_{2,5} = 3.244, l_{3,6} = 2, l_{4,5} = 2.4, l_{4,6} = 1.778, l_{5,6} = 2.889$.)

incident edge is removed. Figure 5.3 shows a situation with 24 embeddings. 12 of them are shown here and the remaining 12 are obtained by reflecting each embedding at the horizontal axis.

Husty and Walter [**HW07**] apply resultants to show that $K_{3,3}$ can have up to 16 embeddings and give as well specific edge lengths leading to 16 different embeddings.

Both approaches rely on the special combinatorial structure of the specific graphs. The general bound in [**BS04**] for the number of embeddings of a graph with 6 vertices yields

$\binom{2\cdot(6-2)}{6-2} = 70$. In this case the BKK bound gives a closer estimate. Namely the mixed volume of the system (5.2) (which uses the substitution trick to remove degeneracies) can be shown to be 32 for both graphs.

5.2.3. General case. For the classes discussed above (Henneberg I, graphs on six vertices) as well as some other special cases, the BKK bound on the number of embeddings resembles or even improves the general bound of $\binom{2N-4}{N-2}$. For the general case, the mixed volume approach for the system (5.1) without the substitutions suggested by Emiris provides a simple, but very weak bound. However, it may be of independent interest that the mixed volume can be exactly determined as a function of N and that in particular the value is independent of the structure of the Laman graph.

Theorem 5.3. *For any Laman graph on N vertices, the mixed volume of the initial system* (5.1) *is exactly* 4^{N-2}.

PROOF. The mixed volume of (5.1) is at most the product of the degrees 2^{2N-4} of the polynomial equations because it is less than or equal to the Bézout bound (see [**Stu02**]). To show that the mixed volume is at least this number we will use Lemma 2.9 to give a lifting that induces a mixed cell of volume 4^{N-2}.

For $i \in \{1, \ldots, 4\}$ the Newton polytope $\mathrm{P}(h_i)$ is a segment. We claim that the polynomials h_i can be ordered in a way such that for $i \geq 5$, $\mathrm{P}(h_i)$ contains the edge $[0, 2\xi_i]$ where ξ_i denotes the i^{th} unit vector. To see this, note first that every polynomial h_j $(1 \leq j \leq 2N)$ has a non-vanishing constant term and therefore $0 \in \mathrm{P}(h_j)$. For $i \in \{1, \ldots, N\}$, each of the monomials x_i^2 and y_i^2 occurs in h_j (for $j \geq 5$) if and only if the edge which is modeled by h_j is incident to v_i.

Let $E' := E \setminus \{[v_1, v_2]\}$. The Henneberg construction of a Laman graph allows to orient the edges such that in the graph (V, E') each vertex in $V \setminus \{v_1, v_2\}$ has exactly two incoming edges (see [**BJ03, LS08**]). Namely, in a Henneberg I step the two new edges point to the new vertex. For a Henneberg II step we remember the direction of the deleted edge $\overrightarrow{[v_r, v_s]}$ and let the new edge, which connects the new vertex to v_s, point to v_s. The other two new edges point to the new vertex. (Figure 5.4 depicts this in an example where $v_r = v_3$ and $v_s = v_4$.)

This orientation shows how to order the polynomials $h_5, \ldots, h_{2N}$ in such a way that the polynomials h_{2i-1} and h_{2i} model edges which are incoming edges of the vertex v_i within the directed graph. Remembering that the order of the variables was $(x_1, y_1, \ldots, x_N, y_N)$ this implies that $2\xi_{2i-1} \in \mathrm{P}(h_{2i-1})$ and $2\xi_{2i} \in \mathrm{P}(h_{2i})$.

Now Lemma 2.9 can be used to describe a lifting that induces a subdivision that has

$$[\xi_1, 0] + \cdots + [\xi_4, 0] + [2\xi_5, 0] + \cdots + [2\xi_{2N}, 0] \tag{5.6}$$

as a mixed cell. In the notation of Lemma 2.9 the chosen edges give rise to the edge matrix $\begin{pmatrix} \mathrm{Id}_4 & \mathbf{0} \\ \mathbf{0} & 2\mathrm{Id}_{2N-4} \end{pmatrix}$, where Id_k denotes the $k \times k$ identity matrix. Substituting this into the second condition (2.9) of Lemma 2.9 we get that for each Newton polytope $\mathrm{P}(h_i)$

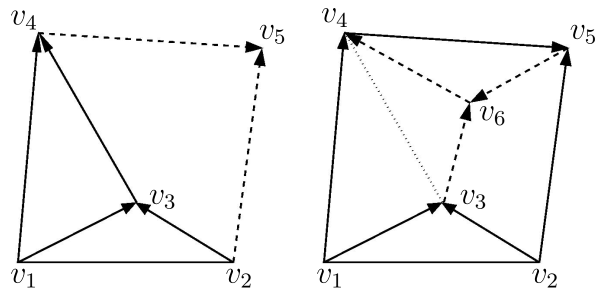

Figure 5.4. A Henneberg I and a Henneberg II step with directed edges.

all vertices $v_s^{(i)}$ of $\mathrm{P}(h_i)$ which are not 0 or $2\xi_i$ have to satisfy

$$(\mu_{1_1} - \mu_{i_1}, \ldots, \mu_{2N_{2N}} - \mu_{i_{2N}}) \cdot v_s^{(i)} \leq 0\,,$$

where we denote by $\mu_j = (\mu_{j_1}, \ldots, \mu_{j_{2N}}) \in \mathbb{Q}^{2N}$ the lifting vector for $\mathrm{P}(h_j)$. Since all the entries of each $v_s^{(i)}$ are non-negative this can easily be done by choosing the vectors μ_j such that their j^{th} entry is sufficiently small and all other entries are sufficiently large. Note that for $i < 5$ the Newton polytope $\mathrm{P}(h_i)$ is an edge and therefore is part of any full-dimensional cell.

Since the cell (5.6) has volume $2^{2N-4} = 4^{N-2}$, this proves the theorem. □

Since the Newton polytopes of system (5.1) all contain the point 0 as a vertex, the mixed volume of (5.1) yields, according to Proposition 1.28, a bound on the number of solutions in $\mathbb{C}$ rather than only on those in $\mathbb{C}^*$.

Corollary 5.4. *The number of embeddings of a Laman graph framework with generic edge lengths is strictly less then* 4^{N-2}*.*

Examples like the case study of Laman graph frameworks on 6 vertices in Section 5.2.2 suggest that the mixed volume of the system (5.2) gives a significantly better bound on the number of embeddings than the one analyzed in Theorem 5.3. However it remains open to compute the mixed volume of the system (5.2) as a function of N like it was done for the system (5.1) in Theorem 5.3.

5.3. Minimally rigid graphs in higher dimensions

We discuss briefly what the BKK techniques yield in the 3-dimensional and higher dimensional cases. Borcea and Streinu [**BS04**] as well as Emiris and Varvitsiotis [**EV09**] gave bounds for embeddings into 3-dimensional and general n-dimensional spaces. Since for these 3- and n-dimensional problems the resulting polynomial equations are sparse as well, the BKK techniques are also applicable. With regard to the Bernstein bounds there are straightforward analogs of Theorem 5.1 and Theorem 5.3 to higher dimensions as we sketch below.

Unfortunately the combinatorics of minimally rigid frameworks in dimensions higher than 2 is not fully understood (cf. [**Hen92**,**TW85**]). Namely, so far no general Henneberg-type construction for an arbitrary n-dimensional minimally rigid graph is known. Nevertheless there is an n-dimensional generalization of the Henneberg I steps that leads to minimally rigid graphs (cf. [**Whi96**]). We start with the 1-skeleton of an n-dimensional simplex and add in each step one new vertex and n new edges connecting the new vertex to arbitrary old vertices of the graph. Graphs obtained in this fashion will be called *Henneberg I graphs*. In the special case $n = 3$ we have furthermore that 1-skeleta of simplicial polyhedra are minimally rigid (see [**Glu75**]).

Clearly the 1-skeleton of an n-dimensional simplex has 2 embeddings in $\mathbb{R}^n$ up to rigid motions differing by a reflection. So let $G = (V, E)$ be an n-dimensional Henneberg I framework on N vertices with generic edge lengths. We employ the same techniques as in the proof of Theorem 5.1 to show that an n-dimensional Henneberg I step at most doubles the mixed volume of the underlying system of polynomial equations. Let v_{N+1} be the new vertex and let $v_{k_1}, \ldots, v_{k_n}$ be the vertices which are connected to the new vertex. Then a Henneberg I step adds the following equations to the polynomial system, describing the embeddings:

$$s_{N+1} - \sum_{i=1}^{n} (x_i^{(N+1)})^2 = 0 \tag{5.7}$$

$$s_{k_j} + s_{N+1} - 2\sum_{i=1}^{n} x_i^{(k_j)} x_i^{(N+1)} = 0 \quad \text{for } j = 1, \ldots, n \tag{5.8}$$

where $(x_1^{(j)}, \ldots, x_n^{(j)})$ denote the coordinates of the embedded vertex v_j. These are again the only $n+1$ equations which involve the $n+1$ variables $s_{N+1}, x_1^{(N+1)}, \ldots, x_n^{(N+1)}$ and hence Lemma 2.6 can be used to decouple the mixed volume computation.

The projection of the Newton polytope of the polynomials in (5.7) and (5.8) to the coordinates $s_{N+1}, x_1^{(N+1)}, \ldots, x_n^{(N+1)}$ yields $Q := \operatorname{conv}\{\xi_1, 2\xi_2, \ldots, 2\xi_{n+1}\}$ and the $(n+1)$-dimensional standard simplex Δ_{n+1}, respectively. It holds that $Q \subset 2 \cdot \Delta_{n+1}$ and therefore the monotonicity of the mixed volume implies

$$\begin{aligned} \mathrm{MV}_{n+1}(Q, \Delta_{n+1}, \ldots, \Delta_{n+1}) &\leq \mathrm{MV}_{n+1}(2 \cdot \Delta_{n+1}, \Delta_{n+1}, \ldots, \Delta_{n+1}) \\ &= 2 \cdot \mathrm{MV}_{n+1}(\Delta_{n+1}, \Delta_{n+1}, \ldots, \Delta_{n+1}) \\ &= 2 \cdot (n+1)! \ \mathrm{vol}_{n+1}(\Delta) = 2 \ . \end{aligned}$$

Hence a Henneberg I step at most doubles the number of embeddings. Again, it is possible to pick edge lengths for which 2 new embeddings occur. Namely we let the lengths of all new edges be almost equal and sufficiently larger then the lengths of the old edges.

Corollary 5.5. *An n-dimensional Henneberg I framework with generic edge lengths on N vertices has at most 2^{N-n} embeddings and this bound is sharp.*

To conclude we discuss now how Theorem 5.3 can be generalized to 1-skeleta of simplicial 3-polytopes. In [**BF67**] Bowen and Fisk show that each such graph can be constructed starting with the 1-skeleton of a tetrahedron and then adding vertices in three sorts of steps. The first possible step is a Henneberg I step that connects the new vertex to the three vertices of a facet of the simplicial 3-polytope (see Figure 5.5, left). Taking two adjacent facets and replacing the bordering diagonal by a 4-valent vertex (see Figure 5.5, middle) will be called a 3-dimensional Henneberg II step and replacing the two bordering diagonals of three neighboring facets by a 5-valent vertex (see Figure 5.5, right) is a 3-dimensional Henneberg III step.

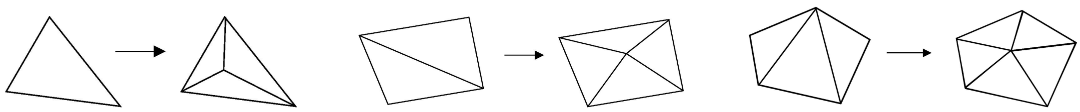

Figure 5.5. From left to right: A 3-dimensional Henneberg I, II and III step.

Theorem 5.6. *The mixed volume of the initial system modelling the* 1*-skeleton of a simplicial* 3*-polytope with* N *vertices is exactly* 2^{3N-3}.

Proof. The proof is a straight forward analog of the proof of Theorem 5.3 except that we have to show that each vertex, that was not fixed to avoid translations and rotations, has exactly three incoming edges. We employ the Henneberg construction to show this. It is obvious that in a Henneberg I step all new edges have to be oriented such that they point to the new vertex. In a Henneberg II step we remember the direction of the deleted edge and let one of the new edges point to the vertex that lost an incoming edge due to the removal of the diagonal.

Finally in a Henneberg III step we have to separate two cases. If the two deleted diagonals do not point to the same vertex then two of the new edges can be oriented such that each vertex that lost an incoming edge gets a new one (see Figure 5.6, above). In the case that both deleted edges point to the same vertex v we have to let one new edge point to this vertex, one new edge point to a vertex w that had a deleted diagonal as an outgoing edge and then we have to reverse the orientation of all edges on a simple path between v and w (see Figure 5.6, below). □

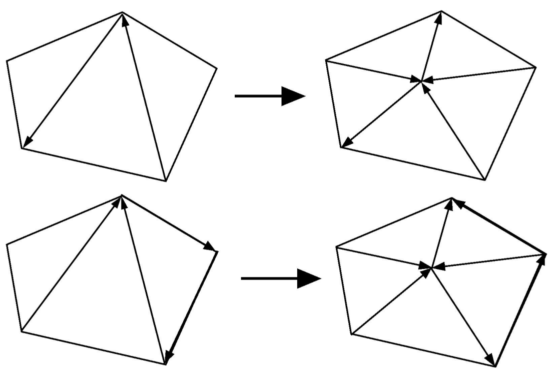

Figure 5.6. Edge orientation in a Henneberg III step.

CHAPTER 6

Open Problems

In the following we present some open questions that arose during the preparation of this thesis.

Chapter 2: Techniques for Explicit Mixed Volume Computation. We briefly described in Section 1.2.3 the correspondence of faces of fiber polytopes and subdivisions. It is an open problem to establish a correspondence like that for mixed subdivisions (compare Example 2.11). Some work in this direction has been done by Michiels and Cools, see [**MC00**]. However it remains unclear how the recent results on mixed fiber polytopes by McMullen [**McM04**] and Esterov and Khovanskiĭ [**EK08**] relate to this question.

Chapter 3: Mixed Ehrhart Theory. It remains open to describe the coefficients of the multivariate mixed Ehrhart polynomial in terms of mixed volumes and to study the mixed Ehrhart (quasi-)polynomials for polytopes with rational coordinates.

In Section 3.2.3 we gave a full description of the mixed Ehrhart polynomial in the cases $k = n$ and $k = n-1$. What can be said about different valuse of k? For $k > n$ there might be a straight forward extension. For $k \leq n-2$ we do not see a way to approach this problem at the moment without knowing more about the classical Ehrhart coefficients. Hence obtaining results for arbitrary $k \leq n-2$ might be a very hard problem.

One general direction of future research is to study the generating functions of mixed Ehrhart polynomials which has been a very fruitful techniques in classical Ehrhart theory. In particular it would be an interesting project to achieve a result paralleling Stanley's work [**Sta92**] who established a connection between the generating function of the classical Ehrhart polynomial of a polyhedral complex and the h-vector of this complex. It is an open problem if there is a similar connection between the h-vector of a mixed subdivision of the underlying polytopes and the generating function of the mixed Ehrhart polynomial.

Chapter 4: Combinatorics and Genus of Tropical Intersections. We studied the genus of a tropical intersection curve in Section 4.4 under rather restrictive conditions. It is open to study the genus of general tropical intersections of arbitrary dimension. In particular it would be interesting to have an interpretation of this genus in terms of mixed volumes or in terms of integer point cardinalities of Minkowski sums. This could also lead to a generalization of Theorem 4.20 that states the equality of the tropical and toric genus for a larger class of intersections.

It also remains open to provide a proof of Theorem 4.20 which is independent of our results about mixed Ehrhart polynomials by studying the amoebas of toric varieties.

Chapter 5: The Number of Embeddings of Minimally Rigid Graphs. We have presented techniques to study the embedding problem of Laman graph frameworks using the BKK theory. As already mentioned in Section 5.2 it is an open question whether the Bernstein bounds can be improved by applying suitable transformations (such as substitutions) on the system of equations. Examples like the case study of Laman graph frameworks on 6 vertices in Section 5.2.2 suggest that the mixed volume of the system (5.2) gives a significantly better bound on the number of embeddings than the one analyzed in Theorem 5.3. However it also remains open to compute the mixed volume of the system (5.2) as a function of n like it was done for the system (5.1) in Theorem 5.3.

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Curiculum Vitae

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Positions

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