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Investment, Income, and Incompleteness

Abstract: The utility-maximizing consumption and investment strategy of an individual investor receiving an unspanned labor income stream seems impossible to find in closed form and very difficult to find using numerical solution techniques. We suggest an easy procedure for finding a specific, simple, and admissible consumption and investment strategy, which is near-optimal in the sense that the wealth-equivalent loss compared to the unknown optimal strategy is very small. We first explain and implement the strategy in a simple setting with constant interest rates, a single risky asset, and an exogenously given income stream, but we also show that the success of the strategy is robust to changes in parameter values, to the introduction of stochastic interest rates, and to endogenous labor supply decisions.

Keywords: Optimal consumption and investment, labor income, incomplete markets, artificially completed markets, welfare loss

JEL-Classification: G11

Investment, Income, and Incompleteness

"However, the largest component of wealth for most households is human capital, which is nontradable." (*John Campbell on Household Finance in his Presidential Address to the American Finance Association on January 7, 2006.*)

1 Introduction

Human wealth is a dominant asset of most individuals and households and is known to have potentially large effects on the optimal consumption and investment decisions over the life-cycle. However, since labor income is typically not spanned by financial assets and the income insurance contracts offered by governments and insurance companies are far from perfect, human wealth is a non-traded asset. Due to this fact, it seems impossible to find closed-form expressions for the dynamic consumption and investment strategies maximizing the life-time utility of an individual consumer-investor. In fact, most of the portfolio choice literature disregards labor income completely (e.g. Samuelson (1969), Merton (1969), Kim and Omberg (1996), Sørensen (1999), Campbell and Viceira (2001), Brennan and Xia (2002), and Liu (2007)) or assumes that labor income is deterministic or spanned by traded assets (e.g. Hakansson (1970) and Bodie, Merton, and Samuelson (1992)). Some recent papers do allow for unspanned labor income but they have to resort to coarse and computationally intensive numerical solution techniques that can handle only low-dimensional problems, have an unknown precision, and do not provide much understanding of the economic forces driving consumption and portfolio decisions (e.g. Cocco, Gomes, and Maenhout (2005), Van Hemert (2009), and Koijen, Nijman, and Werker (2009)).[4] In this paper we suggest an easy procedure for finding a simple consumption and investment strategy, which is near-optimal in the sense that the wealth-equivalent loss compared to the unknown optimal strategy is very small.

Throughout the paper we take a continuous-time framework where uncertainty is generated by a number of standard Brownian motions. The labor income is spanned when the standard Brownian motions driving income changes contemporaneously affect the returns of sufficiently many traded financial assets. In that case the entire labor income stream can be seen as the dividend stream from a trading strategy in those assets so that the human wealth, i.e. the present value of all future

[4] Explicit solutions have been found for special and unrealistic cases involving negative exponential utility, a normally distributed income stream, and very simple asset price dynamics, cf. Svensson and Werner (1993) and Henderson (2005). Duffie and Jackson (1990) and Teplá (2000) derive similar solutions for investors receiving an unspanned income only at the terminal date.

labor income, is uniquely valued by the no-arbitrage principle. The optimal consumption and portfolio decisions of an investor will then follow from the (often well-known) solution to the same problem without labor income basically by replacing financial wealth by the sum of financial and human wealth. In the more realistic case of unspanned labor income, the dynamics of the income rate is affected by a standard Brownian motion unrelated to the returns on traded financial assets. Since the market price of risk, λ_I, associated with that Brownian motion cannot be read off the prices of financial markets, the market is incomplete so that the no-arbitrage valuation of human wealth breaks down, and it is no longer possible to derive the optimal decisions with income from the optimal decisions without income.

The specific consumption and investment strategy we propose to follow with unspanned income is motivated by the optimal decisions in a set of *artificially completed* markets, a concept originally introduced by Karatzas, Lehoczky, Shreve, and Xu (1991) and Cvitanić and Karatzas (1992). For any given market price of risk λ_I (which may in general be a stochastic process), we define an artificially completed market where the individual can invest in the same assets as in the original incomplete market and a hypothetical asset completing the market. The risk-return tradeoff of the hypothetical asset is governed by λ_I. When the price dynamics of the traded assets is sufficiently simple, i.e. interest rates and risk premia have affine or quadratic dynamics (see e.g. Liu (2007)), and λ_I is a deterministic function of time, we can derive a simple, closed-form expression for the optimal strategy of a power-utility maximizer in the artificially completed market. We transform this strategy into an admissible strategy in the true, incomplete market by disregarding the investment in the hypothetical asset and modifying the remaining strategy slightly to ensure non-negative wealth. Each specification of λ_I leads to one specific strategy. We then optimize over λ_I to find the best of those strategies. In the optimization we compute the expected utility generated by a given strategy using straightforward Monte Carlo simulation. In order to evaluate the strategy we propose, we would like to compare the expected utility it generates to the maximum expected utility, but the whole problem is that the latter and the associated strategies are unknown. However, we can easily compute an upper bound on the maximum expected utility in the incomplete market by taking a minimum of the expected utilities obtainable in the artificially completed markets considered. Comparing the expected utility of the specific strategy with this upper bound on the maximum expected utility, we derive an upper bound on the wealth-equivalent loss associated with the specific strategy.

Although our approach is not restricted to low-dimensional problems, we explain and test our strategy in a simple setting with constant interest rates, a single risky asset, and an exogenously given income stream. First, we consider only the artificial markets corresponding to different constant values of λ_I. With our benchmark parameter values we find that a long-term, moderately risk-averse investor following our proposed strategy will suffer a loss less than 2.3% of total wealth for a zero correlation between shocks to labor income and stock returns. When the correlation is increased, the upper bound on the loss becomes even smaller, e.g. roughly 0.9% for a income-stock correlation of 0.6. Second, we generalize to the case where λ_I is a deterministic, affine function of time. This leads to a significant reduction of the upper bound on the loss, e.g. 1.04% for a zero correlation and 0.04% for a correlation of 0.6. These results are robust to changes in key parameter values. We generalize the idea and the procedure to the case, where the investor endogenously determines his labor supply at a stochastic, unspanned wage rate. We find that the bound on the welfare loss is slightly bigger than in the exogenous income case, but still only 1% or lower when the wage-stock correlation is 0.4 or higher. Finally, we generalize our approach to a setting with stochastic interest rates where the individual can invest in a long-term bond in addition to the stock and short-term deposits. We find that the wealth-equivalent losses are also very small in this case. In sum, our numerical results demonstrate that the simple consumption and investment strategy we propose is near-optimal.

As mentioned above, a number of related papers assume that labor income is spanned by traded assets in order to obtain closed-form solutions for the optimal consumption and investment strategies or to reduce the dimension of the numerical solution scheme. If the labor income is really unspanned, the misspecified strategy derived assuming spanning is no longer optimal. We evaluate the performance of this particular strategy in the same way as explained above for our near-optimal strategy. We find that an investor following this misspecified strategy will suffer a significant loss when the true income-asset correlation is low, but minor losses if the true correlation is higher. For example, in our benchmark case the loss is approximately 14% of total wealth if the true income-stock correlation is zero and approximately 3.2% if the correlation is 0.6. Empirical estimates of the correlation between individual household income and returns on broad stock indices are typically close to zero (see, e.g, Cocco, Gomes, and Maenhout 2005) so a strategy derived from a complete market model will perform quite badly.[5]

The remainder of the paper is structured as follows. Section 2 describes the consumption and portfolio choice problem of the investor and summarizes the solution for the case where labor income is spanned by traded assets. Section 3 describes the artificially completed markets and derives the optimal consumption and investment strategies in such markets. Section 4 explains how we transform the optimal strategies in the artificial markets into admissible strategies in the real market, how we find the best of such strategies, and how we evaluate the performance of these strategies. Section 5 discusses numerical results from an implementation of our procedure. Section 6 shows that our ideas and strong numerical results extend to the case of endogenous labor supply, while Section 7 covers the case of stochastic interest rates. Finally, Section 8 concludes. All proofs can be found in the Appendix.

[5]House prices are more highly correlated with labor income so in a setting where investors are allowed to invest in houses in addition to stocks, a complete market assumption will be less harmful, cf., e.g., Cocco (2005) and Kraft and Munk (2008).

2 The problem

We are going to analyze the life-cycle consumption and portfolio problem of a utility-maximizing investor receiving uncertain labor income until retirement. For simplicity we assume for now that the individual can only invest in a bank account offering a constant risk-free rate of r (with continuous compounding) and a single stock (e.g. representing the stock market index). The time t price of the stock is denoted by S_t and the price dynamics is assumed to be

$$dS_t = S_t \left[(r + \sigma_S \lambda_S)\, dt + \sigma_S \, dW_t \right], \tag{1}$$

where $W = (W_t)$ is a standard Brownian motion. Hence, σ_S is the volatility of the stock and λ_S is the Sharpe ratio of the stock, both assumed constant.

We assume in our main analysis that the individual earns an exogenously given labor income until a predetermined retirement date $\tilde{T}$ after which the individual lives on until time $T > \tilde{T}$. The labor income rate at time t is denoted by Y_t and we assume that

$$dY_t = Y_t \left[\alpha \, dt + \beta \left(\rho \, dW_t + \sqrt{1-\rho^2}\, d\tilde{W}_t \right) \right], \quad 0 \leq t \leq \tilde{T}, \tag{2}$$

where $\tilde{W} = (\tilde{W}_t)$ is another standard Brownian motion, independent of W. For $t > \tilde{T}$, $y_t = 0$. The parameter α is the expected growth rate of labor income, β is the income volatility, and ρ is the instantaneous correlation between stock returns and income growth. We assume that α, β, and ρ are all constants, but our analysis goes through with deterministic age-related variations in α and β, as documented by e.g. Cocco, Gomes, and Maenhout (2005). Note that, except for $|\rho| = 1$, the investor faces an incomplete market, since he is not able to fully hedge against unfavorable income shocks.

The individual has to choose a consumption strategy represented by a stochastic process $c = (c_t)$ and an investment strategy represented by a stochastic process $\pi_S = (\pi_{St})$, where π_{St} is the fraction of financial wealth invested in the stock at time t with the remaining financial wealth being invested in the bank account. Let X_t denote the financial wealth at time t. For a given consumption and portfolio strategy (c, π_S), the wealth dynamics is given by

$$dX_t = X_t \left[(r + \pi_{St} \sigma_S \lambda_S)\, dt + \pi_{St} \sigma_S \, dW_t \right] + (Y_t - c_t)\; dt. \tag{3}$$

We will say that a strategy (c, π) is admissible, if it is adapted and $X_T \geq 0$ (almost surely). We denote the set of admissible strategies from time t and onwards by $\mathcal{A}_t$.

The individual has preferences consistent with time-additive expected utility of consumption and terminal wealth. An admissible consumption and investment strategy (c, π_S) generates the expected utility

$$J(t, x, y; c, \pi_S) = \mathrm{E}_t \left[\int_t^T e^{-\delta(s-t)} U(c_s)\, ds + \varepsilon e^{-\delta(T-t)} U(X_T) \right], \tag{4}$$

where the expectation is conditional on $X_t = x$ and $Y_t = y$, δ is the subjective time preference rate, and ε models the relative weight of terminal wealth (bequests) and intermediate consumption. The indirect utility function is given by

$$J(t,x,y) = \max_{(c,\pi_S)\in\mathcal{A}_t} J(t,x,y;c,\pi_S). \tag{5}$$

We assume throughout that the utility function exhibits a constant relative risk aversion $\gamma > 1$, i.e. $U(c) = c^{1-\gamma}/(1-\gamma)$.

If the market is indeed complete, that is $|\rho| = 1$, the problem has the following simple solution:

Theorem 1 (Solution in a truly complete market) *Assume $|\rho| = 1$. Then the indirect utility function is given by*

$$J^{\text{com}}(t,x,y) = \frac{1}{1-\gamma}(g^{\text{com}}(t))^{\gamma}(x + yF^{\text{com}}(t))^{1-\gamma}, \tag{6}$$

where

$$g^{\text{com}}(t) = \frac{1}{r_g}\left(1 - e^{-r_g(T-t)}\right) + \varepsilon^{1/\gamma}e^{-r_g(T-t)}, \tag{7}$$

$$F^{\text{com}}(t) = \mathbf{1}_{\{t\le\tilde{T}\}}\frac{1}{r_F}\left(1 - e^{-r_F(\tilde{T}-t)}\right), \tag{8}$$

and we have introduced the constants[6]

$$r_g = \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}r + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\lambda_S^2, \tag{9}$$

$$r_F = r - \alpha + \rho\beta\lambda_S. \tag{10}$$

The optimal consumption and investment strategy is given by

$$c_t = \frac{X_t + Y_tF^{\text{com}}(t)}{g^{\text{com}}(t)}, \tag{11}$$

$$\pi_{St} = \frac{\lambda_S}{\gamma\sigma_S}\frac{X_t + Y_tF^{\text{com}}(t)}{X_t} - \frac{\beta\rho}{\sigma_S}\frac{Y_tF^{\text{com}}(t)}{X(t)}. \tag{12}$$

In the complete market, the labor income can be uniquely valued as a stream of dividends. Due to the assumptions about the dynamics of labor income and asset prices, the time t value of all future income will be given by $Y_tF^{\text{com}}(t)$. The function g^{com} captures the non-wealth dependent parts of the individual's indirect utility. Compared to a problem without labor income, the initial financial wealth is simply adjusted by adding the initial value of human wealth yF^{com}. The optimal consumption strategy is to consume the fraction $1/g^{\text{com}}(t)$ out of total wealth at any date. The

[6]In the case $r_g = 0$, the term $\frac{1}{r_g}(1 - e^{-r_g(T-t)})$ is interpreted as its limit as $r_g \to 0$, which is simply $T - t$. Similarly for r_F.

optimal investment strategy can be deduced in the following way. First, determine the optimal riskiness of *total* wealth with respect to the exogenous shock, which was originally determined by Merton (1969, 1971). Then subtract the risk exposure of *human* wealth in order to find the optimal exposure of *financial* wealth, which pinpoints the investment strategy. The same intuitive approach holds with more general asset price dynamics, as long as the income is spanned by the traded assets, cf. e.g. Munk and Sørensen (2008).

For the more reasonable situation of unspanned labor income risk, i.e. $|\rho| < 1$, it is impossible to value the human wealth as a traded asset so that the separation (6) and the associated intuitive derivation of the optimal strategy break down. This is demonstrated in the following theorem.

Theorem 2 (Expected Utility in the Incomplete Market) *Assume* $|\rho| < 1$*. Then, for any admissible consumption and investment strategy* (c, π_S) *for which the consumption and portfolio at any time* t *depends at most on* t*,* X_t*, and* Y_t*, the associated expected utility function* $J(t, x, y; c, \pi_S)$ *will not satisfy the separation (6) for any functions* $g(t)$ *and* $F(t)$*.*

In particular, this theorem implies that a separation like (6) does not hold for the *optimal* consumption and investment strategy in the incomplete market case.

To summarize, a closed-form solution for the optimal consumption and investment strategy and the investor's indirect utility does not seem to be available when labor income risk is not fully spanned. Consequently, one has to resort to numerical methods to find an optimal strategy. The numerical methods appropriate for problems of this type are quite intricate and, by the nature of numerical techniques, can only produce an approximation to the optimal strategy. See Cocco, Gomes, and Maenhout (2005), Munk and Sørensen (2008), Koijen, Nijman, and Werker (2009), and Van Hemert (2009) for examples of numerical approaches to consumption/investment choice problems with labor income. Note that little is known about the precision of such methods and, since the methods are based on finite difference lattice techniques, they suffer from the curse of dimensionality. Below we introduce a specific consumption and investment strategy, which is very simple to compute and implement, and we demonstrate that this strategy is close to optimal in a certain, very reasonable metric. The consumption and investment strategy we suggest for the incomplete market will be motivated from the optimal solution in an artificially completed market to which we turn now.

3 The artificially completed markets

Now make the realistic assumption that labor income shocks are not fully hedgeable by traded financial assets, i.e. the income-asset correlation is less than perfect, $|\rho| < 1$. Following an idea originally introduced by Karatzas, Lehoczky, Shreve, and Xu (1991) and Cvitanić and Karatzas

(1992), we will consider an artificially completed market, which consists of the original risk-free bank account and the stock, augmented by an asset making the market complete. Clearly, the individual can do at least as well in any artificially completed market as in the original incomplete market. Karatzas, Lehoczky, Shreve, and Xu (1991) and Cvitanić and Karatzas (1992) show that the solution to the incomplete market problem is identical to the least favorable of solutions in artificially completed markets, but this does not facilitate the actual computation of the optimal solution. We take the following approach. We look at a subset of artificially completed markets in which fairly simple closed-form expressions for the optimal consumption and investment strategies exist. By ignoring the investment in the hypothetical asset which these strategies involve, we obtain a family of consumption and investment strategies admissible in the true incomplete market. We then perform a utility maximization over this family of strategies. That will define a specific consumption and investment strategy in the incomplete market. While this strategy is presumably different from the unknown optimal strategy, we show that it provides almost as high a utility level as the optimal one. The utility generated by the optimal incomplete market strategy is unknown, but certainly lower than the utility obtained in any of the artificially completed markets. We can therefore derive an upper bound on the maximum obtainable utility in the incomplete market by minimizing expected utility over our family of artificially completed markets. We show that the difference between the expected utility induced by our specific strategy and this upper bound on the maximum expected utility is very small (in certainty equivalent terms), implying that our strategy is near-optimal.

More specifically, until retirement we will let the individual trade in a hypothetical asset with time t price I_t having dynamics

$$dI_t = I_t \left[(r + \lambda_I)\, dt + d\tilde{W}_t \right]. \tag{13}$$

Note that, for simplicity and without loss of generality, we assume that this asset only depends on the income-specific motion and has a unit volatility. We can interpret λ_I as a market price of risk associated with the unspanned income shock represented by $d\tilde{W}_t$. We focus for now on a constant λ_I, but we discuss generalizations later. After retirement, the labor income is assumed to be zero so that the market is already complete. Shiller (1993) suggested to establish so-called macro markets where, for instance, claims on (aggregate) income are traded. While Shiller's suggestion has been implemented in the housing market, claims on labor income remain hypothetical. In the following, we will refer to the above hypothetical asset as a *Shiller contract*. The fraction of wealth invested in the Shiller contract will be denoted by π_{It}.

In the artificially completed market, the investor's wealth dynamics for a given consumption-investment strategy (c, π_S, π_I) is given by

$$dX_t = X_t \left[\left(r + \pi_{St}\sigma_S\lambda_S + \mathbf{1}_{\{t\le\tilde{T}\}}\pi_{It}\lambda_I \right) dt + \pi_{St}\sigma_S\, dW_t + \mathbf{1}_{\{t\le\tilde{T}\}}\pi_{It}\, d\tilde{W}_t \right] + \left(\mathbf{1}_{\{t\le\tilde{T}\}} Y_t - c_t \right) dt.$$

For a given market price of risk λ_I, the indirect utility in the artificially completed market is

$$J^{\text{art}}(t,x,y;\lambda_I) = \max_{(c,\pi_S,\pi_I)} \left\{ \int_t^T \mathrm{E}_t \left[e^{-\delta(s-t)} U(c_s) \right] ds + \varepsilon e^{-\delta(T-t)} \mathrm{E}_t[U(X_T)] \right\}, \tag{14}$$

where U is still the power utility function. The indirect utility and the corresponding optimal strategy can be derived in closed form as summarized by the following theorem.

Theorem 3 (Solution with Shiller Contracts) *If the investor has access to Shiller contracts with constant λ_I until retirement, then his indirect utility is given by*

$$J^{\text{art}}(t,x,y;\lambda_I) = \frac{1}{1-\gamma} g^{\text{art}}(t;\lambda_I)^{\gamma} (x + yF^{\text{art}}(t;\lambda_I))^{1-\gamma}, \tag{15}$$

where

$$g^{\text{art}}(t;\lambda_I) = \begin{cases} \frac{1}{r_g^{\text{art}}}(1 - e^{-r_g^{\text{art}}(\tilde{T}-t)}) + g^{\text{com}}(\tilde{T}) e^{-r_g^{\text{art}}(\tilde{T}-t)}, & t < \tilde{T}, \\ g^{\text{com}}(t), & t \geq \tilde{T}, \end{cases} \tag{16}$$

$$F^{\text{art}}(t;\lambda_I) = \mathbf{1}_{\{t \leq \tilde{T}\}} \frac{1}{r_F^{\text{art}}} (1 - e^{-r_F^{\text{art}}(\tilde{T}-t)}), \tag{17}$$

with

$$r_F^{\text{art}} = r_F + \beta\lambda_I\sqrt{1-\rho^2} = r - \alpha + \beta(\lambda_S\rho + \lambda_I\sqrt{1-\rho^2}),$$

$$r_g^{\text{art}} = r_g + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\lambda_I = \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma} r + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\lambda_S^2 + \lambda_I^2).$$

The optimal consumption and investment strategies are

$$c_t^{\text{art}}(\lambda_I) = \frac{X_t + Y_t F^{\text{art}}(t;\lambda_I)}{g^{\text{art}}(t;\lambda_I)}, \tag{18}$$

$$\pi_{St}^{\text{art}}(\lambda_I) = \frac{\lambda_S}{\gamma\sigma_S} \frac{X_t + Y_t F^{\text{art}}(t;\lambda_I)}{X_t} - \frac{\beta\rho}{\sigma_S} \frac{Y_t F^{\text{art}}(t;\lambda_I)}{X_t}, \tag{19}$$

$$\pi_{It}^{\text{art}}(\lambda_I) = \mathbf{1}_{\{t \leq \tilde{T}\}} \left\{ \frac{\lambda_I}{\gamma} \frac{X_t + Y_t F^{\text{art}}(t;\lambda_I)}{X_t} - \beta\sqrt{1-\rho^2} \frac{Y_t F^{\text{art}}(t;\lambda_I)}{X_t} \right\}. \tag{20}$$

Note that after retirement, $t \geq \tilde{T}$, the portfolio problem collapses into a problem without labor income and without Shiller contracts. In particular, this implies that the solutions for the complete and incomplete case coincide after retirement.

For any choice of λ_I, the solution in the artificially completed market will be at least as good as the unknown solution in the truly incomplete market. Given Theorem 3, it is easy to find $\bar{\lambda}_I = \arg\min_{\lambda_I} J^{\text{art}}(t,x,y;\lambda_I)$, which defines an upper bound for the truly incomplete market, i.e.

$$J(t,x,y) \leq \bar{J}(t,x,y) \equiv J^{\text{art}}(t,x,y;\bar{\lambda}_I). \tag{21}$$

Although we only minimize over constant market prices of risk associated with the unspanned income risk, it follows from our numerical results below that this upper bound will be very tight. We will also discuss an extension to the class of deterministic market prices of risk that are affine in time. Even for that class, we can compute the upper bound $\bar{J}(t,x,y)$ explicitly. In principle, the ideas could be extended to stochastic market prices of risk, but then it will be very difficult to find closed-form solutions, and given the excellent results with simpler specifications the extra trouble is not worthwhile.

4 A simple, near-optimal strategy with unspanned income risk

While we are not able to derive the optimal consumption and investment strategy in the truly incomplete market, we can evaluate the performance of any admissible consumption and investment strategy (c,π_S) in the following way. We compare the expected utility generated by the strategy, $J(t,x,y;c,\pi_S)$, to the upper bound $\bar{J}(t,x,y)$ on the maximum utility. If the distance is close, the strategy is near-optimal. More precisely, we can compute an upper bound on the welfare loss $L = L(t,x,y;c,\pi_S)$ suffered when following the specific strategy (c,π_S) by solving the equation

$$J(t,x,y;c,\pi_S) = \bar{J}(t,x[1-L],y[1-L]). \tag{22}$$

$L(t,x,y;c,\pi_S)$ is interpreted as an upper bound on the fraction of total wealth (current wealth and future income) that the individual would be willing to throw away to get access to the unknown optimal strategy, instead of following the strategy (c,π_S). Given Theorem 3,

$$\bar{J}(t,x[1-L],y[1-L]) = J^{\text{art}}(t,x[1-L],y[1-L];\bar{\lambda}_I) = (1-L)^{1-\gamma} J^{\text{art}}(t,x,y;\bar{\lambda}_I),$$

so the upper bound on the welfare loss becomes

$$L(t,x,y;c,\pi_S) = 1 - \left(\frac{J(t,x,y;c,\pi_S)}{J^{\text{art}}(t,x,y;\bar{\lambda}_I)} \right)^{\frac{1}{1-\gamma}}. \tag{23}$$

Our basic idea for finding good strategies is the following. For any given λ_I, we have found the optimal consumption and investment strategy in the artificially completed market in the preceding section. Disregarding the investment in the hypothetical Shiller contract leaves us with a specific strategy for consumption and investments in the stock and the bank account, namely the strategy

$$c_t = \frac{X_t + Y_t F^{\text{art}}(t;\lambda_I)}{g^{\text{art}}(t;\lambda_I)}, \quad \pi_{St} = \frac{\lambda_S}{\gamma\sigma_S} \frac{X_t + Y_t F^{\text{art}}(t;\lambda_I)}{X_t} - \frac{\beta\rho}{\sigma_S} \frac{Y_t F^{\text{art}}(t;\lambda_I)}{X_t}. \tag{24}$$

After retirement, the strategy is identical to the known optimal strategy without income, cf. Theorem 1. Since this specific strategy is derived from the optimal strategy in a closely related market, it seems reasonable to conjecture that it will perform well.

However, we have to modify the suggested strategy (24) slightly to ensure that it is admissible, i.e. that it generates a non-negative terminal wealth, $X_T \geq 0$ (almost surely). With unspanned income risk, this requires non-negative financial wealth at any date, $X_t \geq 0$, as future income may dry out due to negative shocks to $\tilde{W}_t$ and the investor cannot hedge that by financial investments. Hence, there is no way to ensure that a negative financial wealth is made up by future labor income. In the artificial complete market, the strategy stated in Theorem 3 is admissible exactly because of the hedge term. The strategy (c, π_S) stated above is not admissible in the true, incomplete market as X_t can become negative. In fact, $X_t + Y_t F^{\text{art}}(t;\lambda_I)$ can become negative. To see this, substitute the strategy (24) into (3) and apply Itô's lemma to find that

$$d(X_t + Y_t F^{\text{art}}(t;\lambda_I)) = \underbrace{(X_t + Y_t F^{\text{art}}(t;\lambda_I))\left[\left(r + \frac{\lambda_S^2}{\gamma} - \frac{1}{g^{\text{art}}(t;\lambda_I)}\right) dt + \frac{\lambda_S}{\gamma}\, dW_t\right]}_{(i)} + \underbrace{Y_t F^{\text{art}}(t;\lambda_I)\lambda_I\sqrt{1-\rho^2}\, dt}_{(ii)} + \underbrace{Y_t F^{\text{art}}(t;\lambda_I)\beta\sqrt{1-\rho^2}\, d\tilde{W}_t}_{(iii)}.$$

The term (i) alone would be a geometric Brownian motion (with deterministic drift) and thus stays positive. The term (ii) has a sign determined by λ_I. The term (iii) is normally distributed and can thus become negative enough to pull $X_t + Y_t F^{\text{art}}(t;\lambda_I)$ to a negative value. Since $Y_t F^{\text{art}}(t;\lambda_I) \geq 0$, the financial wealth will be negative in that case. We modify the strategy as follows. As long as $X_t > k$ for some small positive k, we follow the strategy (24). Whenever $X_t \leq k$, we replace $F^{\text{art}}(t;\lambda_I)$ by zero in the expression for the stock investment, and if c_t from (24) exceeds Y_t, we set consumption equal to some fraction $\zeta \in (0,1]$ of current income, i.e. $c_t = \zeta Y_t$. The full strategy is therefore

$$\begin{aligned} c_t(\lambda_I) &= \begin{cases} \frac{X_t + Y_t F^{\text{art}}(t;\lambda_I)}{g^{\text{art}}(t;\lambda_I)}, & \text{if } \frac{X_t + Y_t F^{\text{art}}(t;\lambda_I)}{g^{\text{art}}(t;\lambda_I)} < Y_t \text{ or } X_t > k, \\ \zeta Y_t, & \text{otherwise,} \end{cases} \\ \pi_{St}(\lambda_I) &= \frac{\lambda_S}{\gamma\sigma_S}\frac{X_t + \mathbf{1}_{\{X_t > k\}} Y_t F^{\text{art}}(t;\lambda_I)}{X_t} - \mathbf{1}_{\{X_t > k\}}\frac{\beta\rho}{\sigma_S}\frac{Y_t F^{\text{art}}(t;\lambda_I)}{X_t}. \end{aligned} \tag{25}$$

Whenever X_t is below k, the dynamics becomes

$$dX_t = X_t\left[\left(r + \frac{\lambda_S^2}{\gamma}\right) dt + \frac{\lambda_S}{\gamma}\, dW_t\right] + (\mathbf{1}_{\{t \leq \tilde{T}\}} Y_t - c_t)\, dt, \quad X_t < k,$$

that is a geometric Brownian motion plus a non-negative net income, and therefore X_t stays non-negative. In the following numerical implementation, the boundary $X_t \geq k$ was rarely violated for our choice of k.[7]

[7]Note that when consumption and investments are really adjusted continuously in time, we can put $k = 0$ in the

For any given λ_I, we can compute the expected utility $J(t,x,y;c(\lambda_I),\pi_S(\lambda_I))$ generated by the strategy (25) by Monte Carlo simulation of the processes $X=(X_t)$ and $Y=(Y_t)$. Since the market is complete in the retirement phase, the dynamic programming principle and Theorem 1 imply that

$$J(t,x,y;c(\lambda_I),\pi_S(\lambda_I)) = \frac{1}{1-\gamma}\mathrm{E}_t\Big[\int_t^{\tilde{T}} e^{-\delta(s-t)}(c_s(\lambda_I))^{1-\gamma}\,ds + e^{-\delta(\tilde{T}-t)}(g^{\mathrm{com}}(\tilde{T}))^{\gamma}X_{\tilde{T}}^{1-\gamma}\Big],$$

where $X_{\tilde{T}}$ is the time $\tilde{T}$ wealth generated by the strategy $(c(\lambda_I),\pi_S(\lambda_I))$. Consequently, we only need to simulate until the retirement date $\tilde{T}$. In our implementation we use 10,000 paths and along each path the consumption and investment strategy is reset with a frequency of $\Delta = 0.004$, i.e. 250 times a year (roughly corresponding to the number of trading days), unless mentioned otherwise.[8] We maximize over λ_I to find the best strategy in this family of strategies parameterized by λ_I. Define $\hat{\lambda}_I = \arg\max_{\lambda_I} J(t,x,y;c(\lambda_I),\pi_S(\lambda_I))$. This defines a specific admissible strategy

$$(\hat{c},\hat{\pi}_S) = \Big(c(\hat{\lambda}_I),\pi_S(\hat{\lambda}_I)\Big), \tag{26}$$

with associated expected utility $\hat{J}(t,x,y)\equiv J(t,x,y;\hat{c},\hat{\pi}_S)$. The unknown optimal expected utility is now bounded from below and above by

$$\hat{J}(t,x,y)\le J(t,x,y)\le \bar{J}(t,x,y).$$

An upper bound on the welfare loss $\hat{L}=\hat{L}(t,x,y)$ associated with the strategy $(\hat{c},\hat{\pi}_S)$ follows from (23) as

$$\hat{L}(t,x,y) = 1-\left(\frac{\hat{J}(t,x,y)}{J^{\mathrm{art}}(t,x,y;\bar{\lambda}_I)}\right)^{\frac{1}{1-\gamma}}. \tag{27}$$

In order to reduce any simulation bias in the loss, we also compute $J^{\mathrm{art}}(t,x,y;\bar{\lambda}_I)$ by Monte Carlo simulation using the same set of random numbers as used in the computation of $\hat{J}(t,x,y)$.

5 Numerical results

This section contains a quantitative study of the consumption and investment strategy suggested in (26) above. Our benchmark values for the parameters describing the characteristics of the indi-

modified strategy defined above. However, we will have to evaluate the performance of that strategy by a simulation study with non-continuous decisions, hence a strictly positive k is needed to avoid that simulated wealth drops below zero. The value of k can be lowered, if the frequency of decisions is increased, but that will be at the expense of increased computation time.

[8]We find that rebalancing the portfolio 250 times per year leads to indirect utilities that are virtually indistinguishable from the optimal indirect utilities that we can calculate explicitly and that obtain when the individuum rebalances his holdings continuously. For a portfolio problem with stocks only, a similar pattern was observed by Rogers (2001). We thus conclude that the bounds resulting from the Monte Carlo simulations are very close to the explicit ones, which is also supported by very low standard errors.

Investor Characteristics						Financial Market			Labor Income		
δ	γ	t	$\tilde{T}$	T	x	r	λ_S	σ_S	α	β	y
0.03	4	0	20	40	2	0.02	0.25	0.2	0.02	0.1	2

Table 1: **Benchmark parameter values.** The table shows the values of the model parameters used in the numerical computations unless mentioned otherwise. Time is measured in years. The initial wealth $x = 2$ and income $y = 2$ are interpreted as USD 20,000.

vidual, the income process, and the financial market are summarized in Table 1. The benchmark values are similar to those used in the existing literature, cf. Munk and Sørensen (2008) and the references therein. Whenever we need to use levels of current wealth, labor income etc., we use a unit of USD 10,000 scaled by one plus the inflation rate in the perishable consumption good. As the benchmark we put $x = 2$ and $y = 2$, which represents the initial endowment of an investor having USD 20,000 in financial wealth and an annual income of USD 20,000. We will study the sensitivity of our results with respect to various parameter values below. Note that we consider an individual with a relative risk aversion of 4 who receives income for the next 20 years and subsequently lives for another 20 years.

5.1 Basic results

Table 2 reports the upper bounds on the welfare losses for different correlations ρ between stock market and labor income as well as for three different weights ε of terminal wealth. For all combinations of ε and ρ, the welfare loss from implementing the simple strategy $(\hat{c}, \hat{\pi}_S)$ is very small and at most 2.3%. As can be seen in Table 2, the effect from changing the weight ε of bequest is negligible. The impact of the correlation ρ between income and stock market shocks is more pronounced, and the welfare loss increases with increasing incompleteness (decreasing ρ). This is not surprising because the investor implements a strategy in the incomplete market that was derived from a complete market setting.[9]

Figure 1 provide additional information on the small welfare loss. The graphs show how the various expected utilities depend on the parameter λ_I for the case where $\rho = 0.4$ and $\varepsilon = 1$.[10] The

[9] Part of the loss is due to the introduction of the strictly positive wealth level k at which we force the investor to switch to a more prudent strategy. As mentioned earlier, with truly continuous decision making we could let $k = 0$. To gauge the importance of k for the magnitude of the loss, we have also performed simulations with a lower k, namely $k = 0.15$. In that case, we increased the number of time steps to 1000 per year. For an income-stock correlation of zero and $\varepsilon = 0$, the upper bound on the welfare loss was reduced to 2.05% (from 2.27%) and the losses for positive correlation were slightly reduced.

[10] The curves are similar for $\varepsilon = 0.1$ and $\varepsilon = 10$. The figures depict the utility functions multiplied by $\delta = 0.03$, but this is without loss of generality.

	Income-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\varepsilon = 0.1$	2.18%	1.53%	1.19%	0.86%	0.46%
$\varepsilon = 1$	2.27%	1.55%	1.20%	0.86%	0.48%
$\varepsilon = 10$	2.22%	1.56%	1.22%	0.88%	0.48%

Table 2: **Welfare loss for the near-optimal strategy with constant λ_I.** The table shows the upper bound $\hat{L}$ on the welfare loss associated with the strategy $(\hat{c}, \hat{\pi}_S)$ defined in (26) for different values of the income-stock correlation ρ and the parameter ε capturing the relative weight of terminal weight in the preferences. We use $k = 0.3$, $\zeta = 0.5$, and the benchmark parameter values of Table 1. The expected utility from the near-optimal strategy is computed by Monte Carlo simulations involving 10,000 paths and 250 time steps per year.

dark-blue curve is the graph of $J^{\text{art}}(t, x, y; \lambda_I)$ as a function of λ_I using Theorem 3. The yellow curve depicts the same expected utility computed by Monte Carlo simulation, and the fact that the two curves are almost coinciding indicates that the simulation procedure is correctly implemented. The diamond on the dark-blue curve marks the minimum value $\bar{J}(t, x, y)$, which defines the upper bound for the obtainable utility in the incomplete market. The red curve shows how the expected utility $J(t, x, y; c(\lambda_I), \pi_S(\lambda_I))$ of our simple strategy varies with λ_I. The diamond on the red curve marks the maximum value $\hat{J}(t, x, y)$ obtained for the best of the simple strategies. The light-blue curve shows the upper bound on the welfare loss associated with implementing the given strategy $(c(\lambda_I), \pi_S(\lambda_I))$, compared to the smallest upper bound on the obtainable expected utility, $\bar{J}(t, x, y)$. The welfare loss is measured on the vertical axis on the right-hand side of the diagram. Although the red curve seems to be very flat around its maximum, the welfare loss does vary somewhat with λ_I and, by definition, achieves its minimum exactly where the red curve has its maximum. Still the loss curve is quite flat around its minimum, which indicates that the success of the suggested strategy does not require that the best λ_I is determined very precisely.

To check the robustness of our results, we now vary the parameters of our benchmark case. The results are reported in Table 3. First, consider the relative risk aversion γ. For low [high] income-stock correlations the welfare loss is decreasing [increasing] in γ. Our procedure implicitly involves approximations of both the hedge portfolio and the valuation of future income and, consequently, approximations of both components of the optimal portfolio. The quality of these approximations depend on the degree of incompleteness and on investor-specific parameters and variables. For high risk aversion the intertemporal hedge term has a higher weight and it is thus more important to use the right hedge. On the other hand, for low risk aversion the speculative part of the portfolio

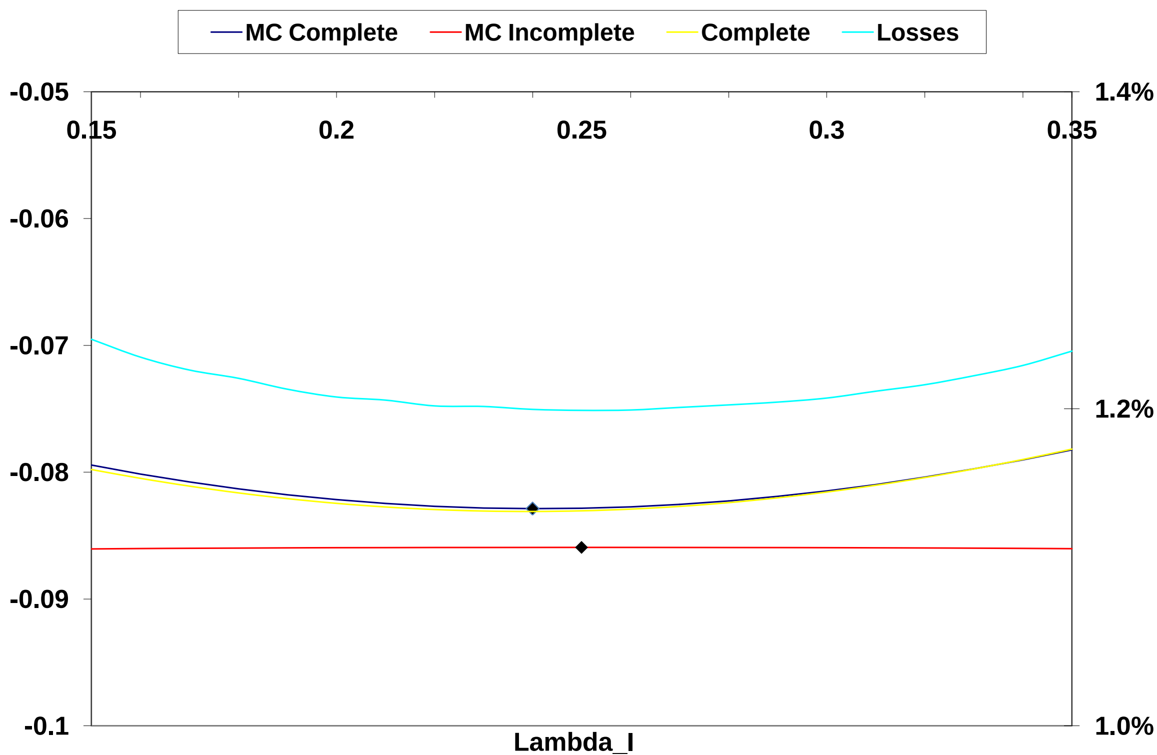

Figure 1: **Expected utilities and the welfare loss for a correlation of $\rho = 0.4$.** The dark-blue curve depicts the optimal expected utility in the artificially completed market, $J^{\text{art}}(t, x, y; \lambda_I)$ stated in (15), as a function of λ_I. The yellow curve (almost coinciding with the dark-blue curve) depicts the same expected utility computed by Monte Carlo simulation using 10,000 paths and 250 time steps per year. The diamond on the dark-blue curve marks the minimum value $\bar{J}(t, x, y)$ defining the upper bound on the obtainable utility in the incomplete market. The red curve shows the expected utility $J(t, x, y; c(\lambda_I), \pi_S(\lambda_I))$ of the simple strategy (25) as a function of λ_I. The diamond on the red curve marks the maximum value $\hat{J}(t, x, y)$ obtained for the best of the simple strategies. The expected utilities have been multiplied by $\delta = 0.03$ and can be read off the vertical axis to the left. The light-blue curve shows the upper bound on the welfare loss associated with the strategy $(c(\lambda_I), \pi_S(\lambda_I))$ and is read off the vertical axis to the right. All graphs are generated assuming $k = 0.3$, $\zeta = 0.5$, the benchmark parameters in Table 1, an income-stock correlation of $\rho = 0.4$, and $\varepsilon = 1$.

	Income-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\gamma = 2$	5.71%	3.46%	1.89%	0.89%	0.35%
$\gamma = 6$	2.43%	2.14%	1.77%	1.32%	0.77%
$y = 1$	1.74%	1.32%	1.06%	0.76%	0.41%
$y = 3$	2.40%	1.64%	1.25%	0.90%	0.49%
$\beta = 0.05$	1.52%	0.87%	0.49%	0.24%	0.11%
$\beta = 0.15$	4.00%	3.25%	2.67%	2.02%	1.31%
$\tilde{T} = 30, T = 50$	3.79%	2.16%	1.50%	1.13%	0.71%

Table 3: **Robustness of the welfare loss for the near-optimal strategy with constant λ_I.** The table shows the upper bound $\hat{L}$ on the welfare loss associated with the strategy $(\hat{c}, \hat{\pi}_S)$ defined in (26) with $k = 0.3$ and $\zeta = 0.5$, when key input variables are varied one by one. Other parameter values are taken from Table 1 and we put $\varepsilon = 1$. The expected utility from the near-optimal strategy is computed by Monte Carlo simulations involving 10,000 paths and 250 time steps per year.

has a higher weight and this term is highly sensitive to the valuation of future income. The total dependence of the welfare loss on the risk aversion coefficient for the different correlations is a mix of the varying quality of the approximations of the two portfolio components and the relative weights of those components. From the table we also see that the welfare loss increases with the initial level of income y, the riskiness of the income stream measured by its volatility β, and the length of the life-cycle measured by $\tilde{T}$ and T. The effects of variations in y and $\tilde{T}$ stem from the fact that increasing these parameters leads to a higher value of the individual's labor income and thus makes labor income relatively more important. Consequently, the welfare loss of strategies that are derived from simplifying assumptions about the income stream becomes more significant. Finally, the volatility of the labor income stream also increases the sensitivity of the individual's life-cycle problem towards suboptimally specified strategies. To summarize, the welfare losses remain very small even for fairly extreme parameter values.

5.2 An improvement

Although the welfare losses for strategies based on constant market prices of risk λ_I are already small, we now analyze whether these results can be further improved if we work with a time-dependent market prices of risk of the affine form

$$\lambda_I(t) = \Lambda_1 t + \Lambda_0, \quad \Lambda_1, \Lambda_0 \in \mathbb{R}. \tag{28}$$

The closed-form solution of Theorem 3 carries over to this case with a slight modification of $g^{\text{art}}(t)$ and $F^{\text{art}}(t)$:

$$g^{\text{art}}(t;\lambda_I) = \begin{cases} \int_t^{\tilde{T}} e^{-r_g^{\text{art}}(s-t)+h(t,s)}\,ds + g^{\text{com}}(\tilde{T})e^{-r_g^{\text{art}}(\tilde{T}-t)+h(t,\tilde{T})}, & t<\tilde{T}, \\ g^{\text{com}}(t), & t \geq \tilde{T}, \end{cases} \tag{29}$$

$$F^{\text{art}}(t;\lambda_I) = \mathbf{1}_{\{t\leq \tilde{T}\}} \int_t^{\tilde{T}} e^{-r_F^{\text{art}}(s-t)-\frac{1}{2}\beta\sqrt{1-\rho^2}\Lambda_1(s^2-t^2)}\,ds, \tag{30}$$

with

$$\begin{aligned} h(t,s) &= \frac{1-\gamma}{2\gamma^2}\Lambda_0\Lambda_1(s^2-t^2) + \frac{1-\gamma}{6\gamma^2}\Lambda_1^2(s^3-t^3), \\ r_F^{\text{art}} &= r_F + \beta\Lambda_0\sqrt{1-\rho^2} = r-\alpha+\beta(\lambda_S\rho+\Lambda_0\sqrt{1-\rho^2}), \\ r_g^{\text{art}} &= r_g + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\Lambda_0 = \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}r + \frac{1}{2}\frac{\gamma-1}{\gamma^2}(\lambda_S^2+\Lambda_0^2). \end{aligned}$$

Of course, we can do at least as well with the affine specification as with the constant market price of risk considered above. Intuitively, when the investor is young and has a long working life ahead, he should be more concerned with the market incompleteness caused by labor income than when he is close to retirement. Therefore, it seems relevant to let the market price of risk and thus the consumption and investment strategy depend on time.

We can find an upper bound on the obtainable utility by minimizing the closed-form indirect utility in the artificially completed market over (Λ_0, Λ_1). Let $\bar{\Lambda}_0, \bar{\Lambda}_1$ denote the minimizing coefficients. On the other hand, for any constants (Λ_0, Λ_1), we can therefore define a strategy $c(\Lambda_0, \Lambda_1), \pi_S(\Lambda_0, \Lambda_1)$ very similar to (25) and evaluate that strategy by Monte Carlo simulation and compute (an upper bound on) the associated welfare loss. In principle, by maximizing the expected utility over Λ_0, Λ_1, we could find the best of these simple strategies. However, this is quite time-consuming due to the Monte Carlo procedure. For simplicity, we thus take the strategy defined by the coefficients $\bar{\Lambda}_0, \bar{\Lambda}_1$ defining the lowest upper bound on expected utility. The resulting upper bounds L on the welfare losses are reported in Table 4. Losses are significantly reduced compared to the case of a constant λ_I and for moderate and high values of the correlation the loss is virtually zero. This demonstrates how close we can get to the optimum by allowing for time-dependent market prices of risk λ_I. Since assumption (28) already leads to very small welfare losses, we have not tried to improve the results further.

	Income-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\bar{\Lambda}_1$	-0.0165	-0.0163	-0.0154	-0.0135	-0.0102
$\bar{\Lambda}_0$	0.4059	0.3947	0.3675	0.3207	0.2415
L	1.04%	0.36%	0.12%	0.04%	0.01%

Table 4: **Welfare loss for the near-optimal strategy with affine $\lambda_I(t)$.** The table shows, for different values of the income-stock correlation ρ, the coefficients $\bar{\Lambda}_1, \bar{\Lambda}_0$ defining the lowest upper bound on expected utility obtainable when $\lambda_I(t)$ has the affine form (28) and the upper bound L on the welfare loss associated with the specific strategy $c(\bar{\Lambda}_0, \bar{\Lambda}_1), \pi_S(\bar{\Lambda}_0, \bar{\Lambda}_1)$. The benchmark parameter values of Table 1 and $\varepsilon = 1$ are used. The expected utility from the specific strategy is computed by Monte Carlo simulations involving 10,000 paths and 250 time steps per year.

5.3 The welfare loss from assuming market completeness

In the literature on optimal consumption and investment strategies with labor income some papers assume that the labor income is spanned by traded assets so that markets are complete and a closed-form solution can often be found, cf., e.g., Bodie, Merton, and Samuelson (1992) and Kraft and Munk (2008). We now evaluate the welfare loss from using the consumption and investment strategy derived under a complete market assumption, when the labor income is really unspanned so that the true market is incomplete. It follows from Theorem 1 that an investor believing in a complete market with perfect income-stock correlation, $\rho = 1$, would follow the strategy

$$c_t = \frac{X_t + Y_t F^{\text{com}}(t)}{g^{\text{com}}(t)}, \qquad \pi_{St} = \frac{\lambda_S}{\gamma \sigma_S} \frac{X_t + Y_t F^{\text{com}}(t)}{X_t} - \frac{\beta}{\sigma_S} \frac{Y_t F^{\text{com}}(t)}{X(t)},$$

where g^{com} and F^{com} are given by (7) and (8) and where ρ is replaced by 1 in the expression for r_F. Again, such a strategy is not admissible in an incomplete market as it may lead to bankruptcy. We modify the strategy just as in Section 4 to ensure admissibility. Note that the modified strategy is identical to the strategy $c_t(\lambda_I), \pi_S(\lambda_I)$ defined in (25) if we put $\lambda_I = 0$ *and* use $\rho = 1$ in the coefficient r_F^{art} entering the function $F^{\text{art}}(t)$. With our parametrization, this modification becomes active only very rarely. We will refer to the strategy $(\tilde{c}, \tilde{\pi}_S)$ defined this way as the *misspecified* strategy. We compute the expected utility generated by this strategy using Monte Carlo simulation and let $\tilde{L}$ denote the upper bound on the associated welfare loss.

Table 5 shows the welfare loss from the misspecified strategy for different combinations of the true income-stock correlation ρ and the terminal wealth coefficient ε. If the true correlation is small, there are significant welfare losses of up to 14.4%, which is much higher than for the near-optimal strategy defined in (26). On the other hand, the welfare loss from the misspecified strategy

	Income-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\varepsilon = 0.1$	14.41%	9.95%	6.21%	3.25%	1.15%
$\varepsilon = 1$	14.43%	9.93%	6.21%	3.24%	1.14%
$\varepsilon = 10$	14.39%	9.94%	6.20%	3.24%	1.15%

Table 5: **Welfare loss for the misspecified strategy.** The table shows, for different values of the true income-stock correlation ρ and the parameter ε, the upper bound $\tilde{L}$ on the welfare loss associated with the misspecified strategy, i.e. the strategy followed by an investor believing that labor income is perfectly correlated with stock returns. We use $k = 0.3$, $\zeta = 0.5$, and the benchmark parameter values of Table 1. The expected utility from the misspecified strategy is computed by Monte Carlo simulations involving 10,000 paths and 250 time steps per year.

is closer to that of the near-optimal strategy and closer to zero if the correlation approaches unity (see $\rho = 0.8$). The latter result follows from two facts: Firstly, for $\rho = 0.8$ the assumption of having a perfect correlation is less problematic. Secondly, the indirect utility function interpreted as a function of λ_I becomes flatter if ρ increases. Therefore, the error from setting λ_I equal to zero, which is what the investor implementing $\tilde{\pi}_S$ is doing, also becomes less pronounced. Consequently, both effects go in the same direction bringing down the differences in the welfare losses of the two strategies as ρ approaches 1. Additional numerical experiments have shown that the welfare loss associated with the misspecified strategy increases significantly with the initial labor income rate, the income volatility, and the time until retirement.

6 Extension: flexible labor supply

In the previous sections we have assumed that labor supply is exogenously fixed. In this section we relax this assumption by allowing the individual to decide on how much time he wishes to work. Let w_t denote the wage rate and assume that

$$dw_t = w_t[\alpha \, dt + \beta(\rho \, dW_t + \sqrt{1-\rho^2} \, d\tilde{W}_t)], \tag{31}$$

where the Brownian motions W and $\tilde{W}$ are uncorrelated. If l_t denotes the fraction of time that the individual chooses to work over a short time period $[t, t+dt]$, the total labor income earned in that period is $l_t w_t \, dt$. We continue to assume that the individual retires from the labor market at the predetermined date $\tilde{T}$ so that $l_t \equiv 0$ for $t > \tilde{T}$. As before, the dynamics of the stock price is given by (1) and the risk-free bank account offers a constant rate of return of r. Given a consumption-labor-investment strategy (c, l, π_S), the dynamics of financial wealth is

$$dX_t = X_t \left[(r + \pi_{St}\sigma_S\lambda_S)\, dt + \pi_{St}\sigma_S\, dW_t \right] + (l_t w_t - c_t)\, dt, \tag{32}$$

and the expected utility is

$$J(t, x, w; c, l, \pi_S) = \mathrm{E}_t \left[\int_t^T e^{-\delta(s-t)} U\big(c_s^{\xi}[1 - l_s]^{1-\xi}\big)\, ds + \varepsilon e^{-\delta(T-t)} U(X_T) \right],$$

where $\xi \in (0, 1)$ defines the relative weight of consumption and leisure, and $U(x) = x^{1-\gamma}/(1-\gamma)$ as before. We assume $\xi = 1$ after retirement. Again, it seems impossible to find a closed-form solution for the strategy (c, l, π_S) maximizing the expected utility, and numerical solution techniques will be complicated and of unknown precision.[11] However, as before, we can find a closed-form solution in an artificially completed market, where the individual can invest in Shiller contracts with price dynamics (13).

Theorem 4 (Solution With Shiller Contracts and Endogenous Labor Supply) *If, until retirement, the investor can endogenously control his labor supply and invest in Shiller contracts with a constant λ_I, his indirect utility is given by*

$$J^{\text{art}}(t, x, w; \lambda_I) = \frac{1}{1-\gamma} g^{\text{art}}(t, w; \lambda_I)^{\gamma} (x + wF^{\text{art}}(t; \lambda_I))^{1-\gamma}, \tag{33}$$

where F^{art} is given by (17), and

$$g^{\text{art}}(t, w; \lambda_I) = \begin{cases} \xi^{-\xi(\gamma-1)/\gamma}(1-\xi)^{-k} w^k \frac{1}{R_g}\left(1 - e^{-R_g(\tilde{T}-t)}\right) + g^{\text{com}}(\tilde{T}) e^{-r_g^{\text{art}}(\tilde{T}-t)}, & t < \tilde{T}, \\ g^{\text{com}}(t), & t \geq \tilde{T}, \end{cases} \tag{34}$$

with $k = \frac{(\gamma-1)(1-\xi)}{\gamma}$ and $R_g = r_g^{\text{art}} + \frac{1}{2}\beta^2 k(1-k) - k\left(\alpha - \frac{\gamma-1}{\gamma}\beta\left[\lambda_S\rho + \lambda_I\sqrt{1-\rho^2}\right]\right)$. The optimal consumption, labor supply, and investment strategy is given by

$$c_t^{\text{art}} = \begin{cases} \xi^{1-\xi(\gamma-1)/\gamma}(1-\xi)^{-k} w_t^k \frac{X_t + w_t F^{\text{art}}(t;\lambda_I)}{g^{\text{art}}(t, w_t; \lambda_I)}, & t < \tilde{T} \\ \frac{X_t}{g(t)}, & t \geq \tilde{T} \end{cases} \tag{35}$$

$$l_t^{\text{art}} = \mathbf{1}_{\{t<\tilde{T}\}} \left\{ 1 - \xi^{-\xi(\gamma-1)/\gamma}(1-\xi)^{1-k} w_t^{k-1} \frac{X_t + w_t F^{\text{art}}(t; \lambda_I)}{g^{\text{art}}(t, w; \lambda_I)} \right\}, \tag{36}$$

$$\pi_{St}^{\text{art}} = \frac{\lambda_S}{\gamma\sigma_S} \frac{X_t + w_t F^{\text{art}}(t; \lambda_I)}{X_t} + \frac{\beta\rho}{\sigma_S} \left[\frac{w_t g_w^{\text{art}}(t, w_t; \lambda_I)}{g^{\text{art}}(t, w_t; \lambda_I)} \frac{X_t + w_t F^{\text{art}}(t; \lambda_I)}{X_t} - \frac{w_t F^{\text{art}}(t; \lambda_I)}{X_t} \right], \tag{37}$$

$$\pi_{It}^{\text{art}} = \mathbf{1}_{\{t<\tilde{T}\}} \left\{ \frac{\lambda_I}{\gamma} \frac{X_t + w_t F^{\text{art}}(t; \lambda_I)}{X_t} + \beta\sqrt{1-\rho^2} \left[\frac{w_t g_w^{\text{art}}(t, w_t; \lambda_I)}{g^{\text{art}}(t, w_t; \lambda_I)} \frac{X_t + w_t F^{\text{art}}(t; \lambda_I)}{X_t} - \frac{w_t F^{\text{art}}(t; \lambda_I)}{X_t} \right] \right\}. \tag{38}$$

[11] Only few papers have solved dynamic utility maximization problems with endogenous labor supply. Bodie, Merton, and Samuelson (1992) derive a closed-form solution for the case of perfect wage-stock correlation, while Cvitanić, Goukasian, and Zapatero (2007) consider a slightly different setting with a fixed wage rate.

Note that, in contrast to the case of exogenous income, the function g^{art} now depends on the income stream via the wage level, which is stochastic. Also note $g_w^{\text{art}} = 0$ after retirement, i.e. for $t \geq \tilde{T}$, so that the entire solution collapses to the solution in the truly complete market summarized in Theorem 1.

From the above solution in the artificially completed market, we can define a consumption, labor supply, and investment strategy $c(\lambda_I), l(\lambda_I), \pi_S(\lambda_I)$ in the true, incomplete market. As before, we do that by ignoring the investment in the hypothetical Shiller contract and by modifying the remaining elements of the strategy to ensure admissibility. The expected utility generated by such a strategy is computed with Monte Carlo simulation and an optimization over λ_I gives us our candidate for a near-optimal strategy. We can also obtain an upper bound on the expected utility by computing the minimum of $J^{\text{art}}(t, x, w; \lambda_I)$ over λ_I, i.e. $\bar{J}(t, x, w) = \min_{\lambda_I} J^{\text{art}}(t, x, w; \lambda_I) \equiv J^{\text{art}}(t, x, w; \bar{\lambda}_I)$. Analogously to the exogenous income case, we define an upper bound L on the welfare loss associated with any given strategy (c, l, π_S) as the solution to

$$
\begin{aligned}
J(t, x, w; c, l, \pi_S) &= \bar{J}(t, x[1-L], w[1-L]) \\
&= \frac{1}{1-\gamma} g^{\text{art}}(t, w[1-L]; \bar{\lambda}_I)^{\gamma} \left(x + wF^{\text{art}}(t; \bar{\lambda}_I)\right)^{1-\gamma} (1-L)^{1-\gamma}.
\end{aligned}
$$

Note that, in contrast to the situation with exogenous income, the function g^{art} depends on the wage rate, and thus we cannot completely separate out L on the right-hand side but have to solve the equation numerically for L.

We perform a numerical analysis along the lines of the case with exogenous income using the same benchmark parameter values as in Table 1 together with $\xi = 0.5$. We assume an initial wage rate of $w = 6$ (instead of an exogenous income starting at $y = 2$) so that, with the optimal initial labor supply, the initial income rates will be about the same as in the exogenous income case. Table 6 shows the upper bound on the welfare loss for various combinations of the wage-stock correlation ρ and the preference coefficient ε on terminal wealth. The welfare losses are still very small for high correlations. However, for zero or low correlations, the welfare loss is now bigger, although still not dramatically high. This increase in the welfare loss may in part be due to the discretization needed to evaluate the suggested strategy. To avoid negative wealth in the simulation we need to set the critical wealth level k to 0.5, which is higher than in the case with exogenous income. This is due to the fact that the wealth dynamics (32) involves the term $l_t w_t$ instead of y_t in the case with endogenous income so when wealth is low, the individual wants to work harder (increase l_t) but is then also hit more severely by adverse, non-hedgeable shocks to the wage rate. In Table 7 we generalize to an affine market price of risk (28). Here we maximize over Λ_0, Λ_1 to find the best of our simple strategies and minimize over Λ_0, Λ_1 to find the lowest upper bound from the artificially completed markets. As in the case with exogenous income, we see a significant reduction in the loss, in particular for high correlations.

	Wage-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\varepsilon = 0.1$	5.72%	2.63%	1.11%	0.41%	0.17%
$\varepsilon = 1$	5.56%	2.59%	1.09%	0.41%	0.17%
$\varepsilon = 10$	5.47%	2.50%	1.03%	0.41%	0.16%

Table 6: **Welfare loss for the near-optimal strategy with endogenous income and constant λ_I.** The table shows the upper bound on the welfare loss associated with the near-optimal consumption-labor-investment strategy for different values of the wage-stock correlation ρ and the parameter ε capturing the relative weight of terminal weight in the preferences. Only constant market prices of risk λ_I are considered. The benchmark parameter values of Table 1 are used together with a consumption-leisure weight of $\xi = 0.5$. The expected utility from the near-optimal strategy is computed by Monte Carlo simulations involving 10,000 paths and 700 time steps per year, and the wealth level at which the investor switches to a more prudent strategy is set to $k = 0.5$.

	Wage-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\Lambda_1(AC)$	-0.0075	-0.0072	-0.0069	-0.0065	-0.0049
$\Lambda_0(AC)$	0.2893	0.2821	0.2632	0.2294	0.1722
$\Lambda_1(I)$	-0.01634	-0.01612	-0.01591	-0.01504	-0.01408
$\Lambda_0(I)$	0.5523	0.5021	0.3945	0.3658	0.3156
L	5.18%	2.12%	0.73%	0.12%	0.03%

Table 7: **Welfare loss for the near-optimal strategy with endogenous income and affine $\lambda_I(t)$.** The two upper rows show the values of the coefficients Λ_1, Λ_0 that defines the lowest upper bound on expected utility obtainable when labor supply is endogenous and $\lambda_I(t)$ has the affine form (28). The third and fourth rows show the values of the coefficients Λ_1, Λ_0 that produce the highest utility of the simple, admissible strategies. The lower row shows the upper bound on the corresponding welfare loss. The benchmark parameter values of Table 1 are used together with a consumption-leisure weight of $\xi = 0.5$ and $\varepsilon = 1$. The expected utility from the near-optimal strategy is computed by Monte Carlo simulations involving 10,000 paths and 700 time steps per year, and the wealth level at which the investor switches to a more prudent strategy is set to $k = 0.5$.

	Income-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\varepsilon = 0.1$	12.33%	8.20%	4.86%	2.34%	0.70%
$\varepsilon = 1$	12.33%	8.20%	4.85%	2.34%	0.70%
$\varepsilon = 10$	12.33%	8.20%	4.85%	2.34%	0.70%

Table 8: **Welfare loss for the misspecified strategy with endogenous income.** The table shows, for different values of the true income-stock correlation ρ and the parameter ε, the upper bound on the welfare loss associated with the misspecified consumption-labor-investment strategy, i.e. the strategy followed by an investor believing that the wage rate is perfectly correlated with stock returns. The benchmark parameter values of Table 1 are used together with a consumption-leisure weight of $\xi = 0.5$ and $\varepsilon = 1$. The expected utility from the misspecified strategy is computed by Monte Carlo simulations involving 10,000 paths and 700 time steps per year, and the wealth level at which the investor switches to a more prudent strategy is set to $k = 0.5$.

In our formulation of the problem and our solution for the artificially completed market, we did not impose the very natural constraint $l_t \leq 1$ on the labor supply of the individual, but in our numerical experiments this constraint was never violated.

Finally, we have again considered the misspecified strategy that an investor assuming perfect wage-stock correlation would follow, modified to ensure admissibility. This strategy corresponds to the strategy $c(\lambda_I), l(\lambda_I), \pi_S(\lambda_I)$ with $\lambda_I = 0$ and $\rho = 1$. Table 8 illustrates that the welfare loss induced by the misspecified strategy is much larger than for our near-optimal strategy, but even the misspecified strategy performs quite well for fairly high wage-stock correlations.

7 Extension: stochastic interest rates

Until now we have assumed a simple Black-Scholes type financial market, but our approach applies to more general settings. As an example we consider now the case where interest rates are stochastic as described by the Vasicek (1977) model so that the short-term interest rate r_t has dynamics

$$dr_t = (\vartheta - \kappa r_t)\, dt - \sigma_r\, dW_{rt},$$

where ϑ, κ, and σ_r are constants, and W_r is a standard Brownian motion. The price B_t of any bond has dynamics of the form

$$dB_t = B_t \left[\left(r_t + \lambda_B \sigma_B(r_t, t) \right) dt + \sigma_B(r_t, t)\, dW_{rt} \right],$$

where λ_B is a constant market price of interest rate risk. For a zero-coupon bond with a time-to-maturity of τ, the price is of the form $B_t = \exp\{-\mathcal{A}(\tau) - \mathcal{B}(\tau) r_t\}$, where $\mathcal{B}(\tau) = (1 - e^{-\kappa\tau})/\kappa$ and $\mathcal{A}$ is another deterministic function of minor importance for what follows, so that $\sigma_B(r_t, t) = \sigma_r \mathcal{B}(\tau)$. The dynamics of the stock price and the labor income is now assumed to be

$$dS_t = S_t \left[(r_t + \lambda_S \sigma_S)\, dt + \sigma_S(\rho_{SB}\, dW_{rt} + \hat{\rho}_S\, dW_{St})\right],$$
$$dY_t = Y_t \left[\alpha\, dt + \beta(\rho_{YB}\, dW_{rt} + \hat{\rho}_{YS}\, dW_{St} + \hat{\rho}_Y\, d\tilde{W}_t)\right], \quad t < \tilde{T},$$

where $W_r, W_S, \tilde{W}$ are independent standard Brownian motions and

$$\hat{\rho}_S = \sqrt{1 - \rho_{SB}^2}, \quad \hat{\rho}_{YS} = \frac{\rho_{YS} - \rho_{SB}\rho_{YB}}{\sqrt{1 - \rho_{SB}^2}}, \quad \hat{\rho}_Y = \sqrt{1 - \rho_{YB}^2 - \hat{\rho}_{YS}^2}$$

where ρ_{SB}, ρ_{YB}, and ρ_{YS} are the pairwise stock-bond, income-bond, and income-stock correlations. Income is zero after the retirement date $\tilde{T}$. Before retirement the market is incomplete unless $\hat{\rho}_Y = 0$. Again we will artificially complete the market by introducing a Shiller contract on the unspanned income component, i.e. an asset with price dynamics

$$dI_t = I_t \left[(r_t + \lambda_I)\, dt + d\tilde{W}_t\right].$$

Let π_{Bt}, π_{St}, and π_{It} denote the fractions of financial wealth invested in the bond, the stock, and the Shiller contract, respectively. As before, c_t denotes the consumption rate at time t. The dynamics of financial wealth X_t is now

$$\begin{aligned} dX_t = X_t \Big[&(r_t + \pi_{Bt}\sigma_B(r_t,t)\lambda_B + \pi_{St}\sigma_S\lambda_S + \pi_{It}\lambda_I)\ dt + (\pi_{Bt}\sigma_B(r_t,t) + \pi_{St}\rho_{SB}\sigma_S)\ dW_{rt} \\ &+ \pi_{St}\hat{\rho}_S\sigma_S\, dW_{St} + \mathbf{1}_{\{t \le \tilde{T}\}}\pi_{It}\, d\tilde{W}_t \Big] + (Y_t - c_t)\, dt, \end{aligned} \tag{39}$$

while the indirect utility function reads

$$J^{\text{art}}(t, x, y, r; \lambda_I) = \max_{(c, \pi_S, \pi_B, \pi_I)} \left\{ \int_t^T \mathrm{E}_t \left[e^{-\delta(s-t)} U(c_s) \right] ds + \varepsilon e^{-\delta(T-t)} \mathrm{E}_t[U(X_T)] \right\}.$$

The indirect utility and the corresponding optimal strategy in the artificially completed market can be derived in closed form as summarized by the following theorem.

Theorem 5 (Solution with Shiller Contracts and Stochastic Interest Rates) *If the investor has access to Shiller contracts on his labor income until retirement, his indirect utility is given by*

$$J^{\text{art}}(t, x, y, r; \lambda_I) = \frac{1}{1-\gamma} g^{\text{art}}(t, r; \lambda_I)^{\gamma} \big(x + y F^{\text{art}}(t, r; \lambda_I)\big)^{1-\gamma}, \tag{40}$$

where

$$F^{\text{art}}(t,r;\lambda_I) = 1_{\{t\le\tilde{T}\}} \int_t^{\tilde{T}} e^{-A(s-t)-\mathcal{B}(s-t)r}\,ds \tag{41}$$

$$g^{\text{art}}(t,r;\lambda_I) = \begin{cases} \int_t^{\tilde{T}} e^{-D(s-t)-\frac{\gamma-1}{\gamma}\mathcal{B}(s-t)r}\,ds + g^{\text{art}}(\tilde{T},r;\lambda_I)e^{-D(\tilde{T}-t)-\frac{\gamma-1}{\gamma}\mathcal{B}(\tilde{T}-t)r}, & t<\tilde{T} \\ \int_t^{T} e^{-\tilde{D}(s-t)-\frac{\gamma-1}{\gamma}\mathcal{B}(s-t)r}\,ds + \varepsilon^{\frac{1}{\gamma}} e^{-\tilde{D}(T-t)-\frac{\gamma-1}{\gamma}\mathcal{B}(T-t)r}, & t\ge\tilde{T} \end{cases} \tag{42}$$

and A, D, and $\tilde{D}$ are deterministic functions stated in Equations (69), (70), and (61) in the Appendix.

The optimal consumption and investment strategies are

$$c_t^{\text{art}}(\lambda_I) = \frac{X_t + Y_t F^{\text{art}}(t,r;\lambda_I)}{g^{\text{art}}(t,r;\lambda_I)}, \tag{43}$$

$$\pi_{St}^{\text{art}}(\lambda_I) = \frac{\lambda_S}{\gamma\sigma_S(1-\rho_{SB}^2)} \frac{X_t + Y_t F^{\text{art}}(t,r;\lambda_I)}{X_t} - \frac{\beta(\rho_{YS}-\rho_{SB}\rho_{YB})}{\sigma_S(1-\rho_{SB}^2)} \frac{Y_t F^{\text{art}}(t,r;\lambda_I)}{X_t}, \tag{44}$$

$$\begin{aligned}\pi_{Bt}^{\text{art}}(\lambda_I) = {} & \frac{\lambda_B - \frac{\rho_{SB}}{(1-\rho_{SB}^2)}\lambda_S}{\gamma\sigma_B} \frac{X_t + Y_t F^{\text{art}}(t,r;\lambda_I)}{X_t} - \frac{\beta(\rho_{YB}-\rho_{SB}\rho_{YS})}{\sigma_B(1-\rho_{SB}^2)} \frac{Y_t F^{\text{art}}(t,r;\lambda_I)}{X_t} \\ & + \frac{\sigma_r}{\sigma_B}\left(\frac{Y_t F_r^{\text{art}}(t,r;\lambda_I)}{X_t} - \frac{g_r^{\text{art}}(t,r;\lambda_I)}{g^{\text{art}}(t,r;\lambda_I)} \frac{X_t + Y_t F^{\text{art}}(t,r;\lambda_I)}{X_t}\right),\end{aligned} \tag{45}$$

$$\pi_{It}^{\text{art}}(\lambda_I) = \mathbf{1}_{\{t\le\tilde{T}\}} \left\{ \frac{\lambda_I}{\gamma} \frac{X_t + Y_t F^{\text{art}}(t,r;\lambda_I)}{X_t} - \beta\hat{\rho}_Y \frac{Y_t F^{\text{art}}(t,r;\lambda_I)}{X_t} \right\}. \tag{46}$$

After retirement, the solution collapses to the well-known solution for the corresponding no-income case, cf. Sørensen (1999).

Disregarding the investment in the Shiller contract and modifying the remaining strategy to ensure admissibility, we can again define a strategy $c(\lambda_I), \pi_S(\lambda_I), \pi_B(\lambda_I)$ for any constant λ_I. Each strategy can be evaluated by Monte Carlo simulation and maximizing over λ_I produces our candidate strategy $\hat{c}, \hat{\pi}_S, \hat{\pi}_B$ in the incomplete market. The expected utility generated by this strategy is again compared with the minimum of $J^{\text{art}}(t,x,y,r;\lambda_I)$ over λ_I and the utility difference is transformed in to a wealth-equivalent loss. For those parameters that were also included in our basic, constant interest rate case, we use the same values, cf. Table 1. The values of the parameters in the interest rate dynamics we use $\kappa = 0.2$, $\vartheta = 0.004$, and $\sigma_r = 0.01$, and the initial short rate is set to its long-term average of $\vartheta/\kappa = 0.02$. The market price of interest rate risk is $\lambda_B = 0.1$, the stock-bond correlation is fixed at $\rho_{SB} = 0.25$, and the income-bond correlation at $\rho_{YB} = 0.1$. In the implementation, we assume that the bond in which the investor trades is a 50-year zero-coupon bond. The utility weight on terminal wealth is assumed to be $\varepsilon = 1$. Table 9 tabulates the upper bound on the welfare loss for different values of the income-stock correlation ρ_{YS}. Clearly, the losses are also very small in the stochastic interest rate setting.

	Income-stock correlation ρ_{YS}				
	0	0.2	0.4	0.6	0.8
Loss	1.56%	1.19%	0.90%	0.61%	0.26%

Table 9: **Welfare loss for the near-optimal strategy with exogenous income, stochastic interest rates, and constant λ_I.** The table shows the upper bound $\hat{L}$ on the welfare loss associated with the strategy $(\hat{c}, \hat{\pi}_S, \hat{\pi}_B)$ derived from (43)–(45) as explained in the text. We use $k = 0.3$, $\zeta = 0.5$, and the benchmark parameter values of Table 1 together with $\varepsilon = 1$, $\kappa = 0.2$, $\vartheta = 0.004$, $\sigma_r = 0.01$, $\lambda_B = 0.1$, $\rho_{SB} = 0.25$, and $\rho_{YB} = 0.1$. The expected utility from the near-optimal strategy is computed by Monte Carlo simulations involving 1000 paths and 1000 time steps per year.

8 Conclusion

This paper has suggested and tested an easy procedure for finding a simple, near-optimal consumption and investment strategy of an investor receiving an unspanned labor income stream. This procedure is valuable since it appears to be impossible to find the truly optimal solution in closed form and very difficult to approximate it precisely using numerical solution techniques. For illustrative purposes we have focused on fairly simple models of the price dynamics of traded assets. However, we emphasize that the procedure can be generalized to models of the affine or quadratic classes considered in many recent papers on portfolio choice in the absence of labor income, since in those settings (i) we would still be able to find explicit solutions in the artificially completed markets and (ii) we can still evaluate the performance of a specific strategy by Monte Carlo simulations. Therefore, our approach shows how to include a realistic (unspanned) specification of the highly important human wealth in the recent literature finding closed-form solutions to optimal consumption and investment problems.

Our ideas should be applicable to other portfolio problems with incomplete markets, e.g. problems with stochastic volatility or stochastic market prices of risk not spanned by traded assets. Some papers find closed-form solutions in such settings (Chacko and Viceira 2005; Kim and Omberg 1996), but only for utility of terminal wealth only, whereas it seems impossible to find optimal strategies in closed form when intermediate consumption is introduced. We conjecture that our approach would lead to near-optimal strategies for investors with utility of intermediate consumption in those models.

A Proofs

Proof of Theorem 1. In the retirement phase, the problem is identical to the problem solved by Merton (1969) with the well-known solution

$$J(t,x,y) = \frac{1}{1-\gamma} g(t)^\gamma x^{1-\gamma}, \quad t \in [\tilde{T}, T],$$

where $g(t)$ is given by (7), and with $c_t = X_t/g(t)$ and $\pi_S = \lambda_S/(\gamma\sigma_S)$. By dynamic programming, we can write the indirect utility before retirement as

$$J(t,x,y) = \max_{(c,\pi_S)} \mathrm{E}_t \left[\int_t^{\tilde{T}} e^{-\delta(s-t)} \frac{1}{1-\gamma} c_s^{1-\gamma}\, ds + \frac{1}{1-\gamma} g(\tilde{T})^\gamma X_{\tilde{T}}^{1-\gamma} \right].$$

The Hamilton-Jacobi-Bellman equation associated with this problem is

$$\delta J = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J, \tag{47}$$

where

$$\mathcal{L}_1 J = \frac{\partial J}{\partial t} + J_x r([x + yF] - yF) + J_x y + J_y y \alpha + \frac{1}{2} J_{yy} y^2 \beta^2, \tag{48}$$

$$\mathcal{L}_2 J = \max_c \left\{ \frac{1}{1-\gamma} c^{1-\gamma} - c J_x \right\}, \tag{49}$$

$$\mathcal{L}_3 J = \max_{\pi_S} \left\{ J_x x \pi_S \sigma_S \lambda_S + \frac{1}{2} x^2 J_{xx} \pi_S^2 \sigma_S^2 + J_{xy} x y \beta \rho \sigma_S \pi_S \right\}, \tag{50}$$

and ρ is either $+1$ or -1. We handle each of these terms separately and then combine them afterwards. We conjecture that the indirect utility function is of the form $J(t,x,y) = \frac{1}{1-\gamma} g(t)^\gamma (x + yF(t))^{1-\gamma}$.

With the conjecture for J, we get

$$\begin{aligned} \mathcal{L}_1 J =& g^\gamma (x+yF)^{1-\gamma} \left(r + \frac{\gamma}{1-\gamma} \frac{g'}{g} \right) + g^\gamma (x+yF)^{-\gamma} y \left(F' - (r-\alpha) F + 1 \right) \\ & - \frac{\gamma}{2} g^\gamma (x+yF)^{-\gamma-1} y^2 F^2 \beta^2. \end{aligned} \tag{51}$$

The first-order condition for c implies that $c = J_x^{-1/\gamma}$, which leads to (11) and

$$\mathcal{L}_2 J = \frac{\gamma}{1-\gamma} g^{\gamma-1} (x+yF)^{1-\gamma}. \tag{52}$$

The first-order condition for π_S implies that

$$\pi_S = -\frac{\lambda_S}{\sigma_S} \frac{J_x}{x J_{xx}} - \frac{\beta\rho}{\sigma_S} \frac{y J_{xy}}{x J_{xx}}, \tag{53}$$

which with the conjecture leads to (12) and

$$\mathcal{L}_3 J = g^{\gamma}(x+yF)^{1-\gamma}\frac{\lambda_S^2}{2\gamma} - g^{\gamma}(x+yF)^{-\gamma} yF\rho\beta\lambda_S + \frac{\gamma}{2} g^{\gamma}(x+yF)^{-\gamma-1} y^2 F^2 \beta^2 \rho^2. \tag{54}$$

Substituting the above expressions back into the HJB-equation (47), we see that the terms involving $(x+yF)^{-\gamma-1}$ clearly cancel out (since $\rho^2 = 1$). Moreover, the terms involving $(x+yF)^{\gamma}$ disappear due to the fact the function $F = F^{\mathrm{com}}$ defined in (8) satisfies the ordinary differential equation $F' - r_F F + 1 = 0$ and $F(\tilde{T}) = 0$. All the remaining terms involve $(x+yF)^{1-\gamma}$. For these terms to cancel out, we need the function g to satisfy the ordinary differential equation $g' - r_g g + 1 = 0$, which is satisfied by the same function $g(t)$ as in the retirement phase, namely the function stated in (7), and thus clearly has the appropriate value at time $\tilde{T}$. □

Proof of Theorem 2. For any admissible strategy (c, π_S), the expected utility function $J(t, x, y; c, \pi_S)$ will, under mild technical conditions, satisfy the partial differential equation[12]

$$\begin{aligned} 0 = U(c) - \delta J + \frac{\partial J}{\partial t} + x(r + \pi_S\sigma_S\lambda_S)J_x + (y-c)J_x + \frac{1}{2}x^2\sigma_S^2\pi_S^2 J_{xx} + y\alpha J_y \\ + \frac{1}{2}y^2\beta^2 J_{yy} + y\beta x\rho\sigma_S\pi_S J_{xy} \end{aligned} \tag{55}$$

for $t < \tilde{T}$. Without loss of generality, the proportion invested in stock can be written as

$$\pi_S = -\frac{\lambda_S}{\sigma_S}\frac{J_x}{xJ_{xx}} - \frac{h}{x\sigma_S}, \tag{56}$$

for some function $h(t, x, y)$. Rewriting (55) leads to

$$\begin{aligned} 0 \;=\; & U(c) - \delta J + (y-c)J_x + \frac{\partial J}{\partial t} + xrJ_x + y\alpha J_y - \frac{1}{2}\lambda_S^2\frac{J_x^2}{J_{xx}} - y\beta\rho\lambda_S\frac{J_x J_{xy}}{J_{xx}} \\ & + \frac{1}{2}y^2\beta^2 J_{yy} + \frac{1}{2}h^2 J_{xx} - y\beta\rho J_{xy} \end{aligned} \tag{57}$$

Applying the separation (6) and simplifying the resulting equation, three types of terms occur: (i) terms involving $(x+yF)^{1-\gamma}$, (ii) terms involving $(x+yF)^{-\gamma}$, and (iii) terms that involve neither $(x+yF)^{1-\gamma}$ nor $(x+yF)^{-\gamma}$. The first, second, and third terms have to cancel out separately, otherwise the separation is wrong (F or g would depend on x or y, which violates the assumption that both functions are only time-dependent). To be more precise, we rewrite (57) as

$$0 = \mathcal{H}_1\, g + \mathcal{H}_2\, g + \mathcal{H}_3\, g + U(c) - cg^{\gamma}(x+yF)^{-\gamma},$$

[12]See, e.g., Duffie (2001, p. 343). Notice that the PDE differs from a Hamilton-Jacobi-Bellman equation only because the controls π_S and c are fixed.

where

$$\mathcal{H}_1\, g = \left(\frac{\delta}{\gamma-1} - \frac{\gamma}{\gamma-1} g_t/g + r + \frac{\lambda_S^2}{2\gamma}\right) g^\gamma (x+yF)^{1-\gamma},$$
$$\mathcal{H}_2\, g = (1 + F' - (r - \alpha + \beta\rho\lambda_S)F)\, y g^\gamma (x+yF)^{-\gamma},$$
$$\mathcal{H}_3\, g = -\frac{\gamma}{2}\left(h^2 - 2y\beta\rho h F + y^2\beta^2F^2\right) g^\gamma (x+yF)^{-\gamma-1}.$$

Depending on the choice of c, the terms $U(c)$ and $cg^\gamma(x+yF)^{-\gamma}$ can be included into $\mathcal{H}_1\, g$, $\mathcal{H}_2\, g$, or $\mathcal{H}_3\, g$ which then have to be zero separately. To show this, we distinguish between two cases. Let $\bar{c}$ be a deterministic function.

1st case: $c = (x+yF)\bar{c}/g$. Then $U(c)$ and $cg^\gamma(x+yF)^{-\gamma}$ is of the same type as the terms of $\mathcal{H}_1\, g$. But then $\mathcal{H}_3\, g \neq 0$ for $|\rho| \neq 1$ and thus the separation (6) is violated.

2nd case: $c \neq (x+yF)\bar{c}/g$. Then $U(c)$ and $cg^\gamma(x+yF)^{-\gamma}$ cannot be included into $\mathcal{H}_1\, g$, $\mathcal{H}_2\, g$, or $\mathcal{H}_3\, g$ at the same time. However, since for $\gamma > 1$ and $|\rho| \neq 1$, we have

$$\mathcal{H}_3\, g < 0 \quad \text{and} \quad U(c) - cg^\gamma(x+yF)^{-\gamma} < 0,$$

the separation (6) is again violated. □

Proof of Theorem 3. In the retirement phase, the market is complete and Theorem 1 applies. By dynamic programming, we can write the indirect utility function in the artificially completed market before retirement as

$$J(t,x,y) = \max_{(c,\pi_S,\pi_I)} \mathrm{E}_t\left[\int_t^{\tilde{T}} e^{-\delta(s-t)} \frac{1}{1-\gamma} c_s^{1-\gamma}\, ds + \frac{1}{1-\gamma} g^{\mathrm{com}}(\tilde{T})^\gamma X_{\tilde{T}}^{1-\gamma}\right].$$

The Hamilton-Jacobi-Bellman equation associated with this problem is

$$\delta J = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J + \mathcal{L}_4 J, \tag{58}$$

where $\mathcal{L}_1 J$, $\mathcal{L}_2 J$, and $\mathcal{L}_3 J$ are given by (48)–(50), and

$$\mathcal{L}_4 J = \max_{\pi_I}\left\{ J_x x \pi_I \lambda_I + \frac{1}{2} x^2 J_{xx} \pi_I^2 + J_{yx} y x \beta\sqrt{1-\rho^2}\pi_I \right\}.$$

Since we again conjecture a solution of the form $J(t,x,y) = \frac{1}{1-\gamma} g(t)^\gamma (x + yF(t))^{1-\gamma}$, we obtain (51)–(54) and the optimal consumption and stock investment stated in (18)–(19). The first-order condition for π_I implies that

$$\pi_I = -\lambda_I \frac{J_x}{xJ_{xx}} - \beta\sqrt{1-\rho^2}\frac{yJ_{xy}}{xJ_{xx}}, \tag{59}$$

which with the conjectured J leads to (20) and

$$\mathcal{L}_4 J = g^\gamma(x+yF)^{1-\gamma}\frac{\lambda_I^2}{2\gamma} - g^\gamma(x+yF)^{-\gamma} yF\sqrt{1-\rho^2}\beta\lambda_I + \frac{\gamma}{2} g^\gamma (x+yF)^{-\gamma-1} y^2F^2\beta^2(1-\rho^2).$$

Substituting the expressions for $\mathcal{L}_i J$ back into the HJB-equation (58), we see that the terms involving $(x+yF)^{-\gamma-1}$ cancel out. The terms involving $(x+yF)^{\gamma}$ disappear due to the fact the function $F(t)$ defined in (17) satisfies the ordinary differential equation $F' - r_F^{\text{art}} F + 1 = 0$ and $F(\tilde{T}) = 0$. All the remaining terms involve $(x+yF)^{1-\gamma}$. For these terms to cancel out, we need $g(t)$ to satisfy the ordinary differential equation $g' - r_g^{\text{art}} g + 1 = 0$ and $g(\tilde{T}) = g^{\text{com}}(\tilde{T})$. This is satisfied by the function stated in (16). □

Proof of Theorem 4. After retirement, the market is complete and the solution from Theorem 1 applies. By dynamic programming, the indirect utility function $J(t,x,w)$ in the artificially completed market is therefore

$$J(t,x,w) = \max_{c,l,\pi_S,\pi_I} \mathrm{E}_t \left[\int_t^{\tilde{T}} e^{-\delta(s-t)} \frac{1}{1-\gamma} c_s^{\xi(1-\gamma)} [1-l_s]^{(1-\xi)(1-\gamma)}\, ds + \frac{1}{1-\gamma} g^{\text{com}}(\tilde{T})^{\gamma} X_{\tilde{T}}^{1-\gamma} \right].$$

Given the wage dynamics (31) and the wealth dynamics

$$dX_t = X_t \left[(r + \pi_{St}\sigma_S\lambda_S + \pi_{It}\lambda_I)\, dt + \pi_{St}\sigma_S\, dW_t + \pi_{It}\, d\tilde{W}_t \right] + (l_t w_t - c_t)\, dt, \quad t \le \tilde{T},$$

the Hamilton-Jacobi-Bellman equation associated with the dynamic maximization problem becomes

$$\delta J = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J + \mathcal{L}_4 J, \tag{60}$$

where

$$\begin{aligned}
\mathcal{L}_1 J &= J_x(rx+w) + \frac{\partial J}{\partial t} + J_w \alpha w + \frac{1}{2} J_{ww} w^2 \beta^2, \\
\mathcal{L}_2 J &= \max_{c,l} \left\{ \frac{1}{1-\gamma} c^{\xi(1-\gamma)} [1-l]^{(1-\xi)(1-\gamma)} - J_x(c + [1-l]w) \right\}, \\
\mathcal{L}_3 J &= \max_{\pi_S} \left\{ J_x x \pi_S \sigma_S \lambda_S + \frac{1}{2} x^2 J_{xx} \pi_S^2 \sigma_S^2 + J_{wx} w x \beta \rho \sigma_S \pi_S \right\}, \\
\mathcal{L}_4 J &= \max_{\pi_I} \left\{ J_x x \pi_I \lambda_I + \frac{1}{2} x^2 J_{xx} \pi_I^2 + J_{wx} w x \beta \sqrt{1-\rho^2} \pi_I \right\}.
\end{aligned}$$

We handle each of these terms separately and then combine them afterwards.

Substitution of all relevant derivatives of the conjectured J in (33) into the expression for $\mathcal{L}_1 J$, we obtain

$$\begin{aligned}
\mathcal{L}_1 J = g^{\gamma}(x+wF)^{1-\gamma} &\left\{ \frac{\gamma}{1-\gamma}\frac{g_t}{g} + \frac{\gamma}{1-\gamma}\alpha\frac{w g_w}{g} - \frac{\gamma}{2}\beta^2 \left(\frac{w g_w}{g}\right)^2 + r + \frac{\gamma}{2(1-\gamma)}\beta^2 \frac{w^2 g_{ww}}{g} \right\} \\
&+ g^{\gamma}(x+wF)^{-\gamma} w \left\{ 1 + F'(t) + (\alpha - r)F + \gamma\beta^2 \frac{w g_w}{g} F \right\} - \frac{\gamma}{2}\beta^2 w^2 F^2 g^{\gamma} (x+wF)^{-\gamma-1}.
\end{aligned}$$

The first-order conditions for the maximization over c and l in $\mathcal{L}_2 J$ imply that

$$c = \xi^{1-\xi(\gamma-1)/\gamma}(1-\xi)^{-k} w^k J_x^{-1/\gamma}, \qquad 1 - l = \frac{1-\xi}{\xi} w^{-1} c.$$

With the conjecture for J, we have $J_x = g^\gamma (x + wF)^{-\gamma}$. Substituting this into the above expressions for c and l, we obtain (35) and (36), and find

$$\mathcal{L}_2 J = -\frac{\gamma}{\gamma - 1} \xi^{-\xi(\gamma-1)/\gamma}(1-\xi)^{-k} w^k g^{\gamma-1}(x + wF)^{1-\gamma}.$$

The first-order condition for π_S in $\mathcal{L}_3 J$ implies (53). With the conjectured J, we have $J_{xx} = -\gamma g^\gamma (x - wF)^{-\gamma-1}$ and $J_{wx} = \gamma g^{\gamma-1}(x + wF)^{-\gamma-1}(g_w(x + wF) - gF)$, so that we get the optimal stock investment stated in (37). Tedious, but straightforward, computations lead to

$$\begin{aligned}\mathcal{L}_3 J = {} & g^\gamma (x + wF)^{1-\gamma} \left\{ \frac{\lambda_S^2}{2\gamma} + \rho\beta\lambda_S \frac{w g_w}{g} + \frac{1}{2}\gamma\rho^2\beta^2 \left(\frac{w g_w}{g}\right)^2 \right\} \\ & - g^\gamma (x + wF)^{-\gamma} wF \left\{ \rho\beta\lambda_S + \gamma\rho^2\beta^2 \frac{w g_w}{g} \right\} + \frac{1}{2} g^\gamma (x + wF)^{-\gamma-1} w^2 F^2 \gamma\rho^2\beta^2.\end{aligned}$$

The first-order condition for π_I in $\mathcal{L}_4 J$ implies (59). Substituting in the derivatives of the conjectured J, we easily get (38), and after further straightforward computations, we find

$$\begin{aligned}\mathcal{L}_4 J = {} & g^\gamma (x + wF)^{1-\gamma} \left\{ \frac{\lambda_I^2}{2\gamma} + \sqrt{1-\rho^2}\beta\lambda_I \frac{w g_w}{g} + \frac{1}{2}\gamma(1-\rho^2)\beta^2 \left(\frac{w g_w}{g}\right)^2 \right\} \\ & - g^\gamma (x + wF)^{-\gamma} wF \left\{ \sqrt{1-\rho^2}\beta\lambda_S + \gamma(1-\rho^2)\beta^2 \frac{w g_w}{g} \right\} \\ & + \frac{1}{2} g^\gamma (x + wF)^{-\gamma-1} w^2 F^2 \gamma(1-\rho^2)\beta^2.\end{aligned}$$

When we substitute the above expressions back into the HJB-equation (60), we first note that the terms involving $(x + wF)^{-\gamma-1}$ cancel out. Collecting terms involving $(x + wF)^{-\gamma}$, we see that they also cancel, because of the fact that $F(t) = F^{\mathrm{art}}(t)$ satisfies the ordinary differential equation $F'(t) - r_F^{\mathrm{art}} F(t) + 1 = 0$. All the remaining terms involve $g^\gamma (x + wF)^{1-\gamma}$. For our conjecture to be verified, we therefore need these terms to cancel as well, which implies that the function $g(t, w)$ has to satisfy the partial differential equation

$$\frac{1}{2}\beta^2 w^2 g_{ww} + \left(\alpha - \frac{\gamma-1}{\gamma}\beta[\rho\lambda_S + \sqrt{1-\rho^2}\lambda_I]\right) w g_w - r_g^{\mathrm{art}} g + g_t + \xi^{-\xi(\gamma-1)/\gamma}(1-\xi)^{-k} w^k = 0.$$

In order to ensure that $J(\tilde{T}, x, w) = \frac{1}{1-\gamma} g^{\mathrm{com}}(\tilde{T})^\gamma x^{1-\gamma}$, we need $g(\tilde{T}, w) = g^{\mathrm{com}}(\tilde{T})$. It is easily verified that the solution is given by (34). □

Proof of Theorem 5. In the retirement phase, the market is complete and it is well-known[13] that the solution is

$$J^{\text{com}}(t,x,r) = \frac{1}{1-\gamma} g^{\text{com}}(t,r)^{\gamma} x^{1-\gamma}, \quad t \geq \tilde{T},$$

where

$$g^{\text{com}}(t,r) = \int_t^T e^{-\tilde{D}(s-t) - \frac{\gamma-1}{\gamma}\mathcal{B}(s-t)r}\, ds + \varepsilon^{\frac{1}{\gamma}} e^{-\tilde{D}(T-t) - \frac{\gamma-1}{\gamma}\mathcal{B}(T-t)r}$$

with

$$\begin{aligned}\tilde{D}(\tau) = &\left(\frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2}\tilde{\Lambda}^2\right)\tau + \frac{\vartheta + \frac{\gamma-1}{\gamma}\lambda_B\sigma_r}{\kappa}\frac{\gamma-1}{\gamma}\left(\tau - \mathcal{B}(\tau)\right) \\ &- \frac{\sigma_r^2}{2\kappa^2}\left(\frac{\gamma-1}{\gamma}\right)^2\left(\tau - \mathcal{B}(\tau) - \frac{\kappa}{2}\mathcal{B}^2(\tau)\right),\end{aligned} \tag{61}$$

where $\tilde{\Lambda}^2 = \lambda_B^2 + \left(\frac{\lambda_S}{1-\rho_{SB}^2}\right)^2 - 2\rho_{SB}\lambda_B\frac{\lambda_S}{1-\rho_{SB}^2}$. Fix λ_I and write the indirect utility function before retirement as

$$J(t,x,y,r) = \max_{(c,\pi_S,\pi_B,\pi_I)} \mathrm{E}_t\left[\int_t^{\tilde{T}} e^{-\delta(s-t)}\frac{1}{1-\gamma}c_s^{1-\gamma}\, ds + \frac{1}{1-\gamma} g^{\text{com}}(\tilde{T},r)^{\gamma} X_{\tilde{T}}^{1-\gamma}\right].$$

The Hamilton-Jacobi-Bellman equation associated with this problem is

$$\delta J = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J + \mathcal{L}_4 J, \tag{62}$$

where

$$\begin{aligned}\mathcal{L}_1 J = &\frac{\partial J}{\partial t} + J_x r([x+yF] - yF) + J_x y + J_y y\alpha + \frac{1}{2}J_{yy}y^2\beta^2 \\ &+ J_r(\vartheta - \kappa r) + \frac{1}{2}J_{rr}\sigma_r^2 - J_{ry}y\beta\rho_{YB}\sigma_r,\end{aligned} \tag{63}$$

$$\mathcal{L}_2 J = \max_c \left\{\frac{1}{1-\gamma}c^{1-\gamma} - cJ_x\right\}, \tag{64}$$

$$\begin{aligned}\mathcal{L}_3 J = \max_{\pi_S,\pi_B} \Big\{ &J_x x(\pi_B\sigma_B\lambda_B + \pi_S\sigma_S\lambda_S) + \frac{1}{2}x^2 J_{xx}(\pi_B^2\sigma_B^2 + \pi_S^2\sigma_S^2 + 2\pi_B\pi_S\rho_{SB}\sigma_B\sigma_S) \\ &+ J_{xy}xy\beta(\rho_{YB}\pi_B\sigma_B + \hat{\rho}_{YS}\pi_S\sigma_S) - J_{xr}x\sigma_r(\pi_B\sigma_B + \pi_S\rho_{SB}\sigma_S)\Big\},\end{aligned} \tag{65}$$

$$\mathcal{L}_4 J = \max_{\pi_I}\left\{J_x x\pi_I\lambda_I + \frac{1}{2}x^2 J_{xx}\pi_I^2 + J_{xy}xy\beta\hat{\rho}_Y\pi_I\right\}. \tag{66}$$

We conjecture a solution of the form $J(t,x,y,r) = \frac{1}{1-\gamma}g(t,r)^{\gamma}(x+yF(t,r))^{1-\gamma}$. Substituting the

[13] See Sørensen (1999) for the case of terminal wealth only and see Wachter (2002) and Liu (2007) for how to extend such solutions to intermediate consumption.

relevant derivatives into $\mathcal{L}_1 J$, we obtain

$$\begin{aligned}\mathcal{L}_1 J =& g^\gamma (x+yF)^{1-\gamma}\left(r + \frac{\gamma}{1-\gamma}\frac{g_t}{g} + \frac{\gamma}{1-\gamma}\frac{g_r}{g}(\vartheta-\kappa r) - \frac{\gamma}{2}\sigma_r^2\left(\frac{g_r}{g}\right)^2 + \frac{\gamma}{2(1-\gamma)}\sigma_r^2\frac{g_{rr}}{g}\right)\\ &+ g^\gamma (x+yF)^{-\gamma} y\left(F_t + (\alpha - r)F + F_r(\vartheta - \kappa r - \beta\rho_{YB}\sigma_r) + \gamma\sigma_r[\sigma_r - \beta\rho_{YB}]\frac{g_r}{g}F_r + \frac{1}{2}\sigma_r^2 F_{rr} + 1\right)\\ &+ g^\gamma (x+yF)^{-\gamma-1} y^2\gamma\left(-\frac{1}{2}\beta^2F^2 - \frac{1}{2}\sigma_r^2F_r^2 + \beta\rho_{YB}\sigma_r F_r F\right).\end{aligned}$$

As in the preceding proofs, the first-order condition for c implies that $c = J_x^{-1/\gamma}$, which leads to (43) and

$$\mathcal{L}_2 J = \frac{\gamma}{1-\gamma} g^{\gamma-1}(x+yF)^{1-\gamma}.$$

Concerning $\mathcal{L}_3 J$, the first-order conditions for π_S, π_B form a system of two equations. Solving those we find

$$\begin{aligned}\pi_S &= -\frac{J_x}{xJ_{xx}}\frac{\lambda_S}{\sigma_S(1-\rho_{SB}^2)} - \frac{yJ_{xy}}{xJ_{xx}}\frac{\beta(\hat{\rho}_{YS} - \rho_{SB}\rho_{YB})}{\sigma_S(1-\rho_{SB}^2)},\\ \pi_B &= -\frac{J_x}{xJ_{xx}}\frac{\lambda_B(1-\rho_{SB}^2) - \rho_{SB}\lambda_S}{\sigma_B(1-\rho_{SB}^2)} + \frac{J_{xr}}{xJ_{xx}}\frac{\sigma_r}{\sigma_B} - \frac{yJ_{xy}}{xJ_{xx}}\frac{\beta(\rho_{YB} - \rho_{SB}\rho_{YS})}{\sigma_B(1-\rho_{SB}^2)},\end{aligned}$$

and inserting the derivatives of the conjectured indirect utility function we obtain (44) and (45). Very tedious, but straightforward, computations lead to

$$\begin{aligned}\mathcal{L}_3 J =& g^\gamma (x+yF)^{1-\gamma}\left(\frac{\gamma}{2}\sigma_r^2\left(\frac{g_r}{g}\right)^2 - \sigma_r\lambda_B\frac{g_r}{g} + \frac{1}{2\gamma}\left[\lambda_B^2 + \left(\frac{\lambda_S}{1-\rho_{SB}^2}\right)^2 - 2\rho_{SB}\lambda_B\frac{\lambda_S}{1-\rho_{SB}^2}\right]\right)\\ &+ g^\gamma (x+yF)^{-\gamma} y\left(\sigma_r\lambda_B F_r - \beta(\lambda_Y - \hat{\rho}_Y\lambda_I)F - \gamma\sigma_r^2\frac{g_r}{g}F_r + \gamma\rho_{YB}\sigma_r\beta\frac{g_r}{g}F\right)\\ &+ g^\gamma (x+yF)^{-\gamma-1} y^2\gamma\left(\frac{1}{2}\sigma_r^2F_r^2 + \frac{1}{2}\beta^2(1-\hat{\rho}_Y^2)F^2 - \rho_{YB}\sigma_r\beta F F_r\right).\end{aligned}$$

The first-order condition for π_I implies that

$$\pi_I = -\lambda_I\frac{J_x}{xJ_{xx}} - \beta\hat{\rho}_Y\frac{yJ_{xy}}{xJ_{xx}},$$

which with the conjectured J leads to (46) and

$$\mathcal{L}_4 J = g^\gamma (x+yF)^{1-\gamma}\frac{\lambda_I^2}{2\gamma} - g^\gamma(x+yF)^{-\gamma} yF\hat{\rho}_Y\beta\lambda_I + g^\gamma(x+yF)^{-\gamma-1}\frac{\gamma}{2}y^2F^2\beta^2\hat{\rho}_Y^2.$$

Substituting the expressions for $\mathcal{L}_i J$ back into the HJB-equation (62), we see that the terms involving $(x+yF)^{-\gamma-1}$ cancel out. The terms involving $(x+yF)^{-\gamma}$ cancel out exactly when F satisfies the PDE

$$\frac{1}{2}\sigma_r^2 F_{rr} + \big(\vartheta - \kappa r + \sigma_r[\lambda_B - \beta\rho_{YB}]\big)F_r + F_t - (r - \alpha + \beta\lambda_Y)F + 1 = 0, \tag{67}$$

where

$$\lambda_Y = \lambda_B \frac{\rho_{YB} - \rho_{SB}\rho_{SY}}{1-\rho_{SB}^2} + \lambda_S \frac{\rho_{SY} - \rho_{SB}\rho_{YB}}{1-\rho_{SB}^2} + \lambda_I \hat{\rho}_Y.$$

The remaining terms all involve $(x+yF)^{1-\gamma}$. They will cancel out exactly when g satisfies the PDE

$$\frac{1}{2}\sigma_r^2 g_{rr} + \big(\vartheta - \kappa r + \frac{\gamma-1}{\gamma}\sigma_r \lambda_B\big) g_r + g_t - \left(\frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma} r + \frac{\gamma-1}{2\gamma^2}\Lambda^2\right) g + 1 = 0, \tag{68}$$

where $\Lambda^2 = \lambda_B^2 + \left(\frac{\lambda_S}{1-\rho_{SB}^2}\right)^2 - 2\rho_{SB}\lambda_B \frac{\lambda_S}{1-\rho_{SB}^2} + \lambda_I^2$. The appropriate values at time $\tilde{T}$ are $F(\tilde{T}, r) = 0$ and $g(\tilde{T}, r) = g^{\text{com}}(\tilde{T}, r)$. Note that these PDEs are very similar to the bond pricing PDE in the Vasicek model, which makes it natural to guess on exponential-affine solutions (integrated due to the constant term 1 appearing in both the above PDEs). It is straightforward to check that the solutions are indeed given by the expressions (41) and (42) with

$$A(\tau) = \frac{\vartheta + \sigma_r[\lambda_B - \beta\rho_{YB}]}{\kappa}(\tau - \mathcal{B}(\tau)) - \frac{\sigma_r^2}{2\kappa^2}\Big(\tau - \mathcal{B}(\tau) - \frac{\kappa}{2}\mathcal{B}^2(\tau)\Big) + (\lambda_Y \beta - \alpha)\tau, \tag{69}$$

$$\begin{aligned} D(\tau) = {} & \left(\frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2}\Lambda^2\right)\tau + \frac{\vartheta + \frac{\gamma-1}{\gamma}\lambda_B\sigma_r}{\kappa}\frac{\gamma-1}{\gamma}(\tau - \mathcal{B}(\tau)) \\ & - \frac{\sigma_r^2}{2\kappa^2}\left(\frac{\gamma-1}{\gamma}\right)^2 \Big(\tau - \mathcal{B}(\tau) - \frac{\kappa}{2}\mathcal{B}^2(\tau)\Big). \end{aligned} \tag{70}$$

This completes the proof. □

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