A contribution to the theory of prime modules

By

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0.1 Abstract

This thesis is aimed at generalizing notions of rings to modules. In particular, notions of completely prime ideals, s-prime ideals, 2-primal rings and nilpotency of elements of rings are respectively generalized to completely prime submodules and classical completely prime submodules, s-prime submodules, 2-primal modules and nilpotency of elements of modules. Properties and radicals that arise from each of these notions are studied.

0.2 Acknowledgement

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> So he humbled you, allowed you to hunger and fed you with manna which you did not know nor did your fathers know, that He might make you know that man shall not live by bread alone but man lives by every word that proceeds from the mouth of the LORD - Deuteronomy 8:3.

St. James church of Port Elizabeth was there to feed my soul with the word and provided a home away from home.

Lastly, I am grateful to my wife Rose and my children Angel and Jerome for their love and for once again enduring my absence.

0.3 Declaration

I, David Ssevviiri with student number 209914513, hereby declare that the Thesis for Students qualification to be awarded is my own work and that it has not previously been submitted for assessment or completion of any postgraduate qualification to another University or for another qualification.

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David Ssevviiri

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

A prime number is a positive integer that has exactly two distinct divisors: one and itself. Equivalently, $p \in \mathbb{Z}^+$ is prime if

$$\text{for all } a, b \in \mathbb{Z}^+, ab = p \text{ implies } a = p \text{ or } b = p. \tag{1.1}$$

Primes are used to carry out a multitude of computations which are impossible with other numbers. This is evidenced in theorems such as: Fermat's little theorem, Wilson's theorem, fundamental theorem of arithmetic, etc. Primes are used in cryptography - this is due to the difficulty of factoring very large numbers into their prime factors. Many open problems about primes are standing. The popular ones are: obtaining a function generating all primes, Goldbach's conjecture - "every even integer greater than two can be expressed as a sum of two primes", the twin prime conjecture - "there are infinitely many twin primes", the Riemann hypothesis which is one of the seven Millennium problems, etc.

The importance attached to prime numbers among other reasons motivated their generalization to prime elements in commutative rings and prime ideals in rings. An element p of a commutative ring R is prime if it is a non-zero non-unit, and for any

$$a, b \in R, \quad p/ab \ \Rightarrow p/a \text{ or } p/b, \tag{1.2}$$

where p/b means p divides b. In terms of the ideal (p) generated by a prime p in the ring $\mathbb{Z}$ of integers, we restate the definition as:

$$\text{if } a, b \in \mathbb{Z} \text{ such that } ab \in (p), \text{ then } a \in (p) \text{ or } b \in (p). \tag{1.3}$$

Statement 1.3 suggests the following definition [60, p. 62]. An ideal $\mathcal{P}$ of a commutative ring R is a prime ideal if

$$\text{for any } a, b \in R \text{ such that } ab \in \mathcal{P}, \text{ then } a \in \mathcal{P} \text{ or } b \in \mathcal{P}. \tag{1.4}$$

From Statements 1.3 and 1.4, we can deduce that, in a ring of integers, if an ideal is generated by a prime number, then that ideal is also prime. For a general ring R (not necessarily commutative), an ideal $\mathcal{P}$ of R is said to be a prime ideal if

$$\text{for any ideals } \mathcal{A}, \mathcal{B} \text{ of } R \text{ such that } \mathcal{AB} \subseteq \mathcal{P}, \text{ then } \mathcal{A} \subseteq \mathcal{P} \text{ or } \mathcal{B} \subseteq \mathcal{P}. \tag{1.5}$$

If R is commutative, definitions (1.4) and (1.5) coincide, otherwise when (1.4) is used on a non-commutative ring, we say $\mathcal{P}$ is a completely prime ideal of R. Any completely prime ideal is prime but the converse does not hold in general. The ring $M_n(F)$ of all $n \times n$ matrices $(n > 1)$ over a field F is prime but for $E_{1n} \in M_n(F)$ where E_{1n} is the matrix with 1 in the $(1, n)$ coordinate but zeros elsewhere, we have: $E_{1n}^2 = 0$ and $E_{1n} \neq 0$. This shows $M_n(F)$ is not completely prime.

Other but less popular "primes" in rings include: uniformly strongly prime [66], strongly prime [42], unitary strongly prime [31], weakly prime [2], super-prime [79], [16, p.24], [15, p.346], strictly prime (also called strongly prime of bound one), [28], [43] and s-prime [78] among others. “Prime” ideals are used in rings to define and characterize radicals. For example, the prime radical of a ring (also called the Baer’s lower nil radical) is the intersection of all prime ideals of that ring and the (Köthe) upper nil radical of a ring is the intersection of all its s-prime ideals [78]. Of course, these radicals have other equivalent characterizations. The prime radical of a ring coincides with the set of all strongly nilpotent elements of that ring and if the ring is commutative, the strongly nilpotent elements are indistinguishable from the nilpotent elements. The upper nil radical of a ring is the sum of all nil ideals of that ring.

Prime ideals are used in the localization of commutative rings. The prime radical is used in algebraic geometry to study the celebrated Hilbert’s Nullstellensatz. Primary ideals which are a generalization of prime ideals are used in the so-called primary decomposition of rings, i.e., when every ideal of a given ring is a finite intersection of primary ideals. For instance, in a Noetherian ring, every ideal is an intersection of primary ideals.

Primeness exists for more algebraic structures. These include: semi-rings, near-rings (see [61] and [67]), modules over rings and modules over near-rings (see [47] and [48]) among others. In this thesis, emphasis is put on “primes” for modules over rings. From now on, a “module” will mean a “module over a ring”. Andrunakievich and Rjabuhin in [3] and Dauns in [28] were among the first to study prime modules. If P is a prime submodule of ${}_RM$, $m \in M \setminus P$ and R is a commutative ring, then $(P : m) := \{r \in R \ : \ rm \in P\}$ is a prime ideal of R called the associated prime ideal of M/P. In commutative algebra, associated prime ideals play a central role in the theory of primary decomposition, see [5], [23] and [57].

Other definitions of “prime” modules exist. Among them, we have: classical prime (or weakly prime), see [6], [7], [9], [10], [11], [13]; strongly prime defined differently in each of [28], [42], [62] and [63]; and weakly prime in [30] which is also different from that in [6], [7] and [9].

Despite the already many existing "primes" for modules in literature, there was still a gap. Some "primes" from rings, for instance, completely prime and s-prime had not been yet generalized to modules. This motivated part of our study. We have two generalizations of completely prime rings (domains) to modules, namely: completely prime modules and classical completely prime modules; they are respectively studied in chapters two and three. We generalize the notion of s-prime ideals in chapter four by defining s-prime submodules. Using s-prime submodules, we define the Köthe upper nil radical for modules.

By drawing motivation from how nilpotent and strongly nilpotent elements of rings are defined, in chapter five we define nilpotent and strongly nilpotent elements of modules. We show that, the set of all strongly nilpotent elements of a unital module defined over a commutative ring coincides with the classical prime radical of that module. We give examples of situations where this result still holds even when R is not commutative. This generalizes a result by Levitzki in rings, namely: the set of all strongly nilpotent elements of a ring coincides with the prime radical of that ring. In the same chapter, the structure of non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$ is characterized. When adjoined with the zero element, the non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$ form a topological ideal of the ring of p-adic integers.

We recall that, the prime radical of a non-commutative ring need not coincide with the set of all nilpotent elements of that ring. However, if the two coincide, then the ring is called 2-primal. Among other characterizations of 2-primal rings, we have: a ring R is 2-primal if and only if its prime radical is equal to its generalized nil radical (also called the completely prime radical). It is this that motivates our definition of 2-primal modules. Chapter six is devoted to studying 2-primal modules. We obtain several conditions that bridge the gap between completely prime radical and prime radical of a module just like what happens for the ring case. It however remains open to us whether our definition of 2-primal modules can be expressed in terms of nilpotent elements of modules.

If R is a ring, $R[x]$ the collection of all polynomials over R in the indeterminate x is a ring called the polynomial ring. If M is a module over a ring R, $M[x]$ the collection of all polynomials over M in the indeterminate x is a module over the ring $R[x]$. We call $M[x]$ the polynomial module. In chapter seven, we study properties of radicals of polynomial modules. More specifically, we show that radicals of the polynomial module induced by different "primes" satisfy both the polynomial equation as well as the Amitsur property and are polynomially extensible.

Just like some of the questions (e.g., about nilpotent elements of modules and 2-primal modules) studied in this thesis emerged from my masters treatise [73], a couple of questions have emerged while putting this thesis together; they have been posed in chapter eight which is the last chapter. Although we did not have immediate answers to them, we feel some should not be difficult to investigate. This however, suggests to us that more could still be done in trying to extend notions of "prime" and radicals from rings to modules. In the same chapter, we show that, when a module is defined over a commutative Artinian ring, all the nine module radicals studied in this thesis are indistinguishable.

Except with otherwise explicit mention, all rings are associative and not necessarily unital. All modules are left modules. Let R be a ring and M an

R-module. If N is a submodule of M, I is an ideal of R and I is an essential ideal of R, we respectively write $N \leq M$, $I \lhd R$ and $I \lhd \cdot R$. If $N, P \leq M$ such that $N \nsubseteq P$, we write $(P : N)$ to mean the ideal $\{r \in R \ : \ rN \subseteq P\}$. By $(P : m)$ where $P \leq M$ and $m \in M \setminus P$, we mean $\{r \in R \ : \ rm \in P\}$. $\langle m \rangle$ is the submodule of ${}_RM$ generated by $m \in M$, i.e., $\langle m \rangle = \mathbb{Z}m + Rm$. If R is unital then $\langle m \rangle = Rm$, otherwise $Rm \subseteq \langle m \rangle$ but $\langle m \rangle \not\subset Rm$ in general. If a is an element of a ring R, by (a) we denote the ideal of R generated by a. $\mathbb{Z}$ and $\mathbb{Z}^+$ are used to denote integers and positive integers respectively.

Lastly, we point out that, in the process of producing this thesis, the following papers were written: [36], [37], [38], [39], [40], [41] and [74]. In particular, [37], [40], [36], [39] and [74], [38] and [41] form the basis of chapters 2, 3, 4, 5, 6 and 7 respectively.

Chapter 2

Completely prime submodules

In this chapter, we generalize completely prime ideals in rings to submodules in modules. The notion of multiplicative systems of rings is generalized to modules. Let N be a submodule of an R-module M. Define $\text{co}.\sqrt{N} := \{m \in M \ : \ \text{every multiplicative system of } M \text{ containing } m \text{ meets } N\}$. It is shown that $\text{co}.\sqrt{N}$ is equal to the intersection $\beta_{co}(N)$ of all completely prime submodules of M containing N. We call $\beta_{co}(M) = \text{co}.\sqrt{0}$ the completely prime radical of M. If R is a commutative ring, $\beta_{co}(M) = \beta(M)$ where $\beta(M)$ denotes the prime radical of M. β_{co} is a complete Hoehnke radical which is neither hereditary nor idempotent and hence not a Kurosh-Amitsur radical. The torsion theory induced by β_{co} is discussed. The module radical $\beta_{co}({}_RR)$ and the ring radical $\beta_{co}(R)$ are compared. We show that the class of all completely prime modules, ${}_RM$ for which $RM \neq 0$ is special.

A proper submodule P of an R-module M with $RM \not\subseteq P$ is a prime submodule of M [3], [28] if for all ideals $\mathcal{A}$ of R and submodules N of M such that $\mathcal{A}N \subseteq P$, we have $N \subseteq P$ or $\mathcal{A}M \subseteq P$. If R is a commutative ring, this definition is equivalent to: for all $a \in R$ and every $m \in M$, if $am \in P$ then $m \in P$ or $aM \subseteq P$. We call this the definition of a completely prime submodule P of a module ${}_RM$. Several authors have discussed prime submodules in modules over commutative rings, e.g., [6], [7], [46] and [58] among others. In general (for example when R is not commutative), the two definitions above need not be equivalent - the later implies the former but not conversely. Simple modules (and maximal submodules) are always prime but need not be completely prime. This justifies our study of completely prime submodules in this chapter, drawing motivation from how completely prime ideals in rings are defined. Although some authors, e.g., [3, p.1792 No. 6], [29], [76, p.1840] mentioned about completely prime modules, to the best of our knowledge none has ever studied them in detail.

If I is a completely prime ideal of a ring R, the complement $R \setminus I$ is a multiplicative system, i.e., closed under multiplication. We generalize this notion to modules and show that if P is a completely prime submodule of M, the complement $M \setminus P$ of P is a multiplicative system of M.

2.1 Properties of completely prime submodules

Definition 2.1.1 A proper submodule P of an R-module M with $RM \not\subseteq P$ is a completely prime submodule if for each $a \in R$ and every $m \in M$ such that $am \in P$, we have $m \in P$ or $aM \subseteq P$.

An R-module M is completely prime if the zero submodule of M is a completely prime submodule of M. In general, an R-module M/P is a completely prime module if and only if P is a completely prime submodule of M.

Example 2.1.1 Every torsion-free module is a completely prime module.

Proof: Suppose $am = 0$ for $a \in R$ and $m \in M$. If $m = 0$, we are through. Suppose $m \neq 0$, by definition of torsion-free modules, $a = 0$ and $aM = 0$. ∎

An R-module M is reduced [52], [69] if for all $a \in R$ and $m \in M$, $am = 0$ implies $\langle m \rangle \cap aM = 0$.

Example 2.1.2 A simple module which is reduced is completely prime.

Proof: Suppose $am = 0$. If $m = 0$, we are through. Suppose $m \neq 0$, then $0 = aM \cap \langle m \rangle = aM \cap M = aM$. ∎

Proposition 2.1.1 *If $1 \in R$ and $P \lhd R$, then P is a completely prime ideal of R if and only if P is a completely prime submodule of ${}_RR$.*

Proof: Suppose $am \in P$ for $a \in R$ and $m \in M = R$. By definition of a completely prime ideal, $a \in P$ or $m \in P$ such that $aM \subseteq P$ or $m \in P$. Conversely, if for $a, b \in R$, $ab \in P$, by definition of completely prime submodules, $a \in aR \subseteq P$ or $b \in P$. ∎

Corollary 2.1.1 *R is a domain (a completely prime unital ring) if and only if ${}_RR$ is a completely prime module.*

In general, Proposition 2.1.1 holds for all rings R such that for all $a \in R$, $a \in aR$. The following proposition offers several other characterizations of completely prime modules.

Proposition 2.1.2 *Let M be an R-module. For a proper submodule P of M, the following statements are equivalent:*

1. *P is a completely prime submodule of M;*
2. *for all $a \in R$ and every $m \in M$, if $\langle am \rangle \subseteq P$, then either $\langle m \rangle \subseteq P$ or $\langle aM \rangle \subseteq P$;*

3. *$(P : M) = (P : m)$ for all $m \in M \setminus P$;*

4. *$\mathcal{P} = (P : M)$ is a completely prime ideal of R, and $(P : m) = (\bar{0} : \bar{m}) = \mathcal{P}$ for each $m \in M \setminus P$;*

5. *the set $\{(P : m) \ : \ m \in M \setminus P\}$ is a singleton;*

6. *$(P : F) = (P : M)$ for all subsets $F \subseteq M \setminus P$.*

Proof: $1 \Rightarrow 2$. If $\langle am \rangle \subseteq P$ for $a \in R$ and $m \in M$, then $am \in P$. By 1, $m \in P$ or $aM \subseteq P$. Hence, $\langle m \rangle \subseteq P$ or $\langle aM \rangle \subseteq P$.
$2 \Rightarrow 3$. Suppose $a \in (P : M)$ and $m \in M \setminus P$. Then, $aM \subseteq P$. So, $am \in P$ and $a \in (P : m)$. Now let $a \in (P : m)$ with $m \in M \setminus P$. Then $am \in P$ and $\langle am \rangle \subseteq P$. By 2, $aM \subseteq \langle aM \rangle \subseteq P$ since $\langle m \rangle \nsubseteq P$ as $m \notin P$. It follows that $a \in (P : M)$.
$3 \Rightarrow 1$. Let $am \in P$ for $a \in R$ and $m \in M$, then $a \in (P : m)$. If $m \in P$ we are through. Suppose $m \notin P$. By 3, $a \in (P : M)$ such that $aM \subseteq P$.
$1 \Rightarrow 5$. Let $m_1 \neq m_2$, $m_1, m_2 \in M \setminus P$. We show that $(P : m_1) = (P : m_2)$. Let $a \in (P : m_1)$, then $am_1 \in P$. By 1, $aM \subseteq P$. Thus, $am_2 \in P$ and hence $a \in (P : m_2)$. Similarly, if $a \in (P : m_2)$, we get $a \in (P : m_1)$.
$5 \Rightarrow 4$. $\mathcal{P} = (P : M) = \cap\{(P : m) \ : \ m \in M \setminus P\} = (P : m)$ for all $m \in M \setminus P$ by 5. But $(P : m) = \{r \in R \ : \ rm \in P\} = \{r \in R \ : \ r\bar{m} = \bar{0}\} = (\bar{0} : \bar{m})$, where $\bar{m} = m + P$. Let $ab \in (P : m)$ with $a, b \in R$ and $m \in M \setminus P$. Then $abm \in P$. If $b \in (P : m)$, we are through. Suppose $b \notin (P : m)$, i.e., $bm \notin P$, then by 5, $a \in (P : bm) = (P : m)$ and hence $\mathcal{P} = (P : m)$ for all $m \in M \setminus P$ is a completely prime ideal of R.
$4 \Rightarrow 1$. Similar to $3 \Rightarrow 1$.
$6 \Rightarrow 3$. Take $F = \{m\}$ where $m \in M \setminus P$.
$3 \Rightarrow 6$. $(P : F) = \cap_{m \in F}(P : m) = \cap(P : M) = (P : M)$.

■

Corollary 2.1.2 *If P is a completely prime submodule of ${}_RM$, then $(P : m)$ is a two sided ideal of R for all $m \in M \setminus P$.*

2.2 Multiplicative systems of modules

Definition 2.2.1 Let ${}_RM$ be a module. A nonempty set $S \subseteq M \setminus \{0\}$ is called a multiplicative system of ${}_RM$ if for each $a \in R$, $m \in M$ and for all $K \leq M$ such that $(K + \langle m\rangle) \cap S \neq \emptyset$ and $(K + \langle aM\rangle) \cap S \neq \emptyset$, then $(K + \langle am\rangle) \cap S \neq \emptyset$.

Corollary 2.2.1 *Let M be an R-module. A submodule P of M is completely prime if and only if $M \setminus P$ is a multiplicative system of M.*

Proof: $(\Rightarrow)$. Suppose $S = M \setminus P$. For $a \in R$, $K \leq M$ and $m \in M$ suppose $(K + \langle m\rangle) \cap S \neq \emptyset$ and $(K + \langle aM\rangle) \cap S \neq \emptyset$. If $(K + \langle am\rangle) \cap S = \emptyset$, then $\langle am\rangle \subseteq P$ and since P is completely prime $\langle m\rangle \subseteq P$ or $\langle aM\rangle \subseteq P$. Thus, $(K + \langle m\rangle) \cap S = \emptyset$ or $(K + \langle aM\rangle) \cap S = \emptyset$, a contradiction.
$(\Leftarrow)$. Let $a \in R$ and $m \in M$ such that $\langle am\rangle \subseteq P$ but $\langle m\rangle \nsubseteq P$ and $\langle aM\rangle \nsubseteq P$. Then, $\langle m\rangle \cap S \neq \emptyset$ and $\langle aM\rangle \cap S \neq \emptyset$. By definition of a multiplicative system, $\langle am\rangle \cap S \neq \emptyset$ such that $\langle am\rangle \nsubseteq P$, a contradiction. ■

Proposition 2.2.1 *For any proper submodule P of ${}_RM$, and $S := M \setminus P$, the following statements are equivalent:*

1. *P is a completely prime submodule of M;*
2. *S is a multiplicative system of M;*
3. *for all $a \in R$ and every $m \in M$, if $\langle m\rangle \cap S \neq \emptyset$ and $\langle aM\rangle \cap S \neq \emptyset$, then $\langle am\rangle \cap S \neq \emptyset$;*
4. *for all $a \in R$ and every $m \in M$, such that $m \in S$ and $\langle aM\rangle \cap S \neq \emptyset$ then $am \in S$.*

Lemma 2.2.1 *Let M be an R-module, $S \subseteq M$ a multiplicative system of M and P a submodule of M maximal with respect to the property that $P \cap S = \emptyset$, then P is a completely prime submodule of M.*

Proof: Suppose $a \in R$ and $m \in M$ such that $\langle am \rangle \subseteq P$. If $\langle m \rangle \nsubseteq P$ and $\langle aM \rangle \nsubseteq P$ then $(\langle m \rangle + P) \cap S \neq \emptyset$ and $(\langle aM \rangle + P) \cap S \neq \emptyset$. Since S is a multiplicative system of M, $(\langle am \rangle + P) \cap S \neq \emptyset$. Since $\langle am \rangle \subseteq P$, we have $P \cap S \neq \emptyset$, a contradiction. Hence, P must be a completely prime submodule. ■

Definition 2.2.2 Let R be a ring and M an R-module. For $N \leq M$, if there is a completely prime submodule containing N, we define

$$\text{co.}\sqrt{N} := \{m \in M \ : \text{ every multiplicative system of } M \text{ containing } m \text{ meets } N\}.$$

We write $\text{co.}\sqrt{N} = M$ when there are no completely prime submodules of M containing N.

Theorem 2.2.1 *Let M be an R-module and $N \leq M$. Then, either co.$\sqrt{N} = M$ or co.$\sqrt{N}$ equals the intersection of all completely prime submodules of M containing N, which is denoted by $\beta_{co}(N)$.*

Proof: Suppose co.$\sqrt{N} \neq M$. Then, $\beta_{co}(N) \neq \emptyset$. Both co.$\sqrt{N}$ and N are contained in the same completely prime submodules. By definition of co.$\sqrt{N}$ it is clear that $N \subseteq$ co.$\sqrt{N}$. Hence, any completely prime submodule of M which contains co.$\sqrt{N}$ must necessarily contain N. Suppose P is a completely prime submodule of M such that $N \subseteq P$, and let $t \in$ co.$\sqrt{N}$. If $t \notin P$, then the complement of P, $C(P)$ in M is a multiplicative system containing t and therefore we would have $C(P) \cap N \neq \emptyset$. However, since $N \subseteq P$, $C(P) \cap P = \emptyset$ and this contradiction shows that $t \in P$. Hence co.$\sqrt{N} \subseteq P$ as we wished to show. Thus, co.$\sqrt{N} \subseteq \beta_{co}(N)$. Conversely, assume $s \notin$ co.$\sqrt{N}$, then there exists a multiplicative system S such that $s \in S$ and $S \cap N = \emptyset$. From Zorn's lemma, there exists a submodule $P \supseteq N$ which is maximal with respect to $P \cap S = \emptyset$. From Lemma 2.2.1, P is a completely prime submodule of M and $s \notin P$. ∎

Proposition 2.2.2 *Let R be a ring and $\mathcal{P} \lhd R$, $\mathcal{P} \neq R$. The following statements are equivalent:*

1. *$\mathcal{P}$ is a completely prime ideal of R;*

2. *there exists a completely prime R-module M such that $\mathcal{P} = (0 : M)_R$.*

Proof: (1) $\Rightarrow$ (2). Let $\mathcal{P}$ be a completely prime ideal and $M = R/\mathcal{P}$. M is an R-module with the usual operation. If $p \in \mathcal{P}$ and $x \in R$ then, $p(x + \mathcal{P}) = px + \mathcal{P} = \mathcal{P}$. Hence, $\mathcal{P} \subseteq (0 : M)_R$. If $a \in (0 : M)_R$, $a(r + \mathcal{P}) = \mathcal{P}$ for all $r \in R$, hence $aR \subseteq \mathcal{P}$ and since $\mathcal{P}$ is a completely prime ideal we get $a \in \mathcal{P}$, hence $(0 : M)_R = \mathcal{P}$. M is completely prime, for if $a \in R$ and $m \in M = R/\mathcal{P}$ such that $am = \bar{0}$ then $m = m_1 + \mathcal{P}$ and $am_1 \in \mathcal{P}$. Since $\mathcal{P}$ is completely prime, we have $a \in \mathcal{P}$ or $m_1 \in \mathcal{P}$ and it follows that $aM = \bar{0}$ or $m = \bar{0}$.
(2) $\Rightarrow$ (1). Follows from Proposition 2.1.2. ∎

Corollary 2.2.2 *A ring R is a completely prime ring if and only if there exists a faithful completely prime R-module.*

Example 2.2.1 If R is a domain, ${}_RR$ is a faithful completely prime module.

Example 2.2.2 If I is a completely prime ideal of R, R/I is a completely prime R-module.

2.3 Preradicals and radicals

The terminology of radicals is that of [75]. Throughout this section rings have unity and all modules are unital left modules. A functor γ: R-mod $\to$ R-mod is called a preradical if $\gamma(M)$ is a submodule of M and $f(\gamma(M)) \subseteq \gamma(N)$ for each homomorphism $f : M \to N$ in R-mod. A radical γ is a preradical for which $\gamma(M/\gamma(M)) = 0$ for all $M \in R$-mod. A preradical is hereditary or left exact if $\gamma(N) = N \cap \gamma(M)$ whenever $N \leq M \in R$-mod (equivalently, if γ is a left exact functor). N is a characteristic submodule of M if $f(N) \subseteq N$ for every $f \in \mathrm{Hom}_R(M, M)$.

Proposition 2.3.1 [64, Proposition 1] *Let $\mathcal{M}$ be any nonempty class of modules closed under isomorphisms, i.e., if $A \in \mathcal{M}$ and $A \cong B$, then $B \in \mathcal{M}$. For any $M \in \mathcal{M}$ define*

$$\gamma(M) = \cap\{K \ : \ K \leq M, M/K \in \mathcal{M}\}.$$

It is assumed that $\gamma(M) = M$ if $M/K \notin \mathcal{M}$ for all $K \leq M$. Then

1. *$\gamma(M/\gamma(M)) = 0$ for all modules M;*

2. *if $\mathcal{M}$ is closed under taking non-zero submodules, γ is a radical;*

3. *if $\mathcal{M}$ is closed under taking essential extensions, then $\gamma(M) \cap N \subseteq \gamma(N)$ for all $N \leq M$.*

In particular, γ is a left exact radical if $\mathcal{M}$ is closed under non-zero submodules and essential extensions.

For any module M, we define the completely prime radical $\beta_{co}(M)$ as $\mathrm{co}.\sqrt{0}$. From Theorem 4.2.1, we have

$$\beta_{co}(M) = \cap\{K \ : \ K \leq M, M/K \text{ is completely prime}\}$$

which is a radical by Proposition 2.3.1 since completely prime modules are closed under taking non-zero submodules.

Let $\beta(M)$ be the prime radical of M (the intersection of all prime submodules of M).

Theorem 2.3.1 *If R is a commutative ring, then $\beta_{co}(M) = \beta(M)$.*

Proof: If R is a commutative ring, prime and completely prime submodules are indistinguishable. ■

Proposition 2.3.2 *For any R-module M,*

1. *$\beta_{co}(M)$ is a characteristic submodule of M;*

2. *If M is projective, then $\beta_{co}(R)M = \beta_{co}(M)$.*

Proof: Follows from [19, Proposition 1.1.3]. ■

Proposition 2.3.3 *For any $M \in R$-mod,*

1. *if $M = \bigoplus_{\Lambda} M_\lambda$ is a direct sum of submodules M_λ $(\lambda \in \Lambda)$, then*

$$\beta_{co}(M) = \bigoplus_{\Lambda} \beta_{co}(M_\lambda);$$

2. *if $M = \prod_{\Lambda} M_\lambda$ is a direct product of submodules M_λ $(\lambda \in \Lambda)$, then*

$$\beta_{co}(M) \subseteq \prod_{\Lambda} \beta_{co}(M_\lambda).$$

Proof: Follows from [19, Proposition 1.1.2]. ■

The following examples show that the preradical β_{co} is not hereditary.

Example 2.3.1 Consider the $\mathbb{Z}$-module $\mathbb{Z}_4$, $\beta_{co}(\mathbb{Z}_4) = 2\mathbb{Z}_4$ and $\beta_{co}(2\mathbb{Z}_4) = (0)$, hence $\beta_{co}(\mathbb{Z}_4) \cap 2\mathbb{Z}_4 = 2\mathbb{Z}_4 \not\subseteq \beta_{co}(2\mathbb{Z}_4) = (0)$.

Example 2.3.2 Take a $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ where p is a prime number. By [14, Remark 4.6], $\beta_{co}(M) = \mathbb{Z}_{p^\infty}$ and if N is a proper submodule of M, N has a (maximal) completely prime submodule, say P. Thus, $\beta_{co}(N) \subset P \subset N = \beta_{co}(M) \cap N$ and $\beta_{co}(M) \cap N \not\subseteq \beta_{co}(N)$.

However, for direct summands, we have the following proposition.

Proposition 2.3.4 *Let $M = X \oplus Y$*

1. *If P is a completely prime submodule of X, then $P \oplus Y$ is a completely prime submodule of M;*
2. *$\beta_{co}(M) \cap X \subseteq \beta_{co}(X)$.*

Proof:

1. Let $a \in R$ and $m = (m_1, m_2) \in M = X \oplus Y$. Suppose $am \in P \oplus Y$. Then $am_1 \in P$ and $am_2 \in Y$. Since P is a completely prime submodule of X, we have $aX \subseteq P$ or $m_1 \in P$. $aX \subseteq P$ implies $aM = aX \oplus aY \subseteq P \oplus Y$. If $m_1 \in P$, then $m = (m_1, m_2) \in P \oplus Y$.
2. If Q is any completely prime submodule of X, then Q is a completely prime submodule of $M = X \oplus Y$. So, $\beta_{co}(M) \subseteq Q \subseteq X$. Thus, $\beta_{co}(M) = \beta_{co}(M) \cap X \subseteq Q$. Hence, $\beta_{co}(M) \cap X \subseteq \beta_{co}(X)$.

■

Corollary 2.3.1 *For any direct summand N of M, $\beta_{co}(M) \cap N = \beta_{co}(N)$.*

Remark 2.3.1 It follows from Proposition 2.3.1 and Examples 2.3.1 and 2.3.2 that completely prime modules are not closed under taking essential extensions.

Definition 2.3.1 A functor γ: R-mod $\to$ R-mod is called a Hoehnke radical [29, p. 454] if $f(\gamma(M)) \subseteq \gamma(f(M))$ for every homomorphism $f : M \to f(M)$ and moreover, $\gamma(M/\gamma(M)) = 0$ for all $M \in R$-mod. γ is complete if $\gamma(K) = K \leq M \in R$-mod implies $K \subseteq \gamma(M)$. γ is idempotent if $\gamma(\gamma(M)) = \gamma(M)$ for all $M \in R$-mod. A Kurosh-Amitsur radical is a complete idempotent Hoehnke radical.

Theorem 2.3.2 *The completely prime radical β_{co} is a complete Hoehnke radical which is not Kurosh-Amitsur.*

Proof: β_{co} is a Hoehnke radical by the nature of its definition, cf., [29, (4), p. 455]. All preradicals are complete, cf., [29, (3) p.455]. The radical β_{co} is not idempotent since $2\mathbb{Z}_4 = \beta_{co}(\mathbb{Z}_4) \neq \beta_{co}(\beta_{co}(\mathbb{Z}_4)) = (0)$, and hence not Kurosh-Amitsur. ■

2.3.1 Torsion theory induced by the radical β_{co}

Definition 2.3.2 [75, p.139] A torsion theory in the category of modules R-mod is a pair $(\mathcal{T}, \mathcal{F})$ of classes of modules in R-mod such that,

1. $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$.

2. If $\mathrm{Hom}(C, F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$.

3. If $\mathrm{Hom}(T, C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

Define $\mathcal{T}_{\beta_{co}} = \{M : \beta_{co}(M) = M\}$ and $\mathcal{F}_{\beta_{co}} = \{M : \beta_{co}(M) = 0\}$. $\mathcal{T}_{\beta_{co}}$ is a torsion class, $\mathcal{F}_{\beta_{co}}$ is a torsion-free class and the pair $(\mathcal{T}_{\beta_{co}}, \mathcal{F}_{\beta_{co}})$ a torsion theory, see [75, p.140]. $\mathcal{T}_{\beta_{co}}$ coincides with the class of modules with no completely prime submodules.

Proposition 2.3.5 [75, Prop. 2.1] $\mathcal{T}$ *is a torsion class for some torsion theory if and only if it is closed under quotient objects, direct products and extensions.*

Corollary 2.3.2 *For any $M \in R$-mod and $N \leq M$, $\mathcal{T}_{\beta_{co}}$ is closed under quotients, direct products and extensions.*

By Example 2.3.2, $\mathcal{T}_{\beta_{co}}$ is not closed under taking submodules.

Corollary 2.3.3 *The following statements hold.*

1. $\bar{\beta_{co}}(M) = \sum\{N : N \leq M$ *and* $\beta_{co}(N) = N\}$ *is an idempotent preradical,* $\bar{\beta_{co}} \subseteq \beta_{co}$, $\mathcal{T}_{\beta_{co}} = \mathcal{T}_{\bar{\beta_{co}}}$ *and* $\bar{\beta_{co}}$ *is the largest idempotent preradical contained in* β_{co}.

2. $\hat{\beta_{co}}(M) = \cap\{N : N \leq M, M/N \in \mathcal{F}_{\beta_{co}}\}$ *is a radical.* $\beta_{co} \subseteq \hat{\beta_{co}}$, $\mathcal{F}_{\beta_{co}} = \mathcal{F}_{\hat{\beta_{co}}}$ *and* $\hat{\beta_{co}}$ *is the least radical containing* β_{co}.

Proof: Follows from [19, Proposition, 1.1.5]. ■

Proposition 2.3.6 *If $M \in \mathcal{T}_{\beta_{co}}$, then for each non-zero homomorphic image N of M there exists $K \leq N$ such that $0 \neq K \in \mathcal{T}_{\beta_{co}}$. This is the module analogue of (R1) in [33, Theorem 2.1.5].*

Proof: Since $\mathcal{T}_{\beta_{co}}$ is closed under quotients, the result follows from [75, Prop. 2.5]. ■

Example 2.3.3 Any completely prime module M is β_{co}-torsion-free.

Remark 2.3.2 All said about completely prime (sub)modules under section of preradicals also holds for prime (sub)modules. Thus, the prime radical $\beta(M)$ is a complete Hoenhke radical which is neither hereditary nor idempotent (hence not Kurosh-Amitsur). Furthermore, prime modules are not closed under taking essential extensions. However, if we define a faithful prime radical, $\beta_0(M)$ as,

$$\beta_0(M) = \cap\{P : P \leq M, M/P \text{ is faithful and prime}\}$$

β_0 is a Kurosh-Amitsur radical, cf., [64, Section 3]. Furthermore, the class of all faithful prime modules is closed under essential extensions. This leads us to the following.

Question 2.3.1 Is the class of all faithful completely prime modules closed under essential extensions?

2.3.2 Comparison of the radicals $\beta_{co}({}_RR)$ and $\beta_{co}(R)$

Lemma 2.3.1 *For any associative ring R, $\beta_{co}({}_RR) \subseteq \beta_{co}(R)$.*

Proof: Let $x \in \beta_{co}({}_RR)$ and I be a completely prime ideal of R. From Proposition 2.2.2, we have R/I is a completely prime R-module. Hence, $x \in I$ and we have $x \in \beta_{co}(R)$, i.e., $\beta_{co}({}_RR) \subseteq \beta_{co}(R)$. ■

Remark 2.3.3 In general the containment in Lemma 2.3.1 is strict.

Example 2.3.4 Let $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{Z}_2 \right\}$ and $M = {}_RR$. (0) is a completely prime submodule of ${}_RR$. Hence, $\beta_{co}({}_RR) = 0$. $(0 : R)_R$ is a completely prime ideal of R but $(0 : R)_R \neq (0)$. For if $b \neq 0$, $b \in \mathbb{Z}_2$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R = 0$. Hence, $\beta_{co}(R) \subseteq (0 : R)_R$. But since $(0 : R)_R(0 : R)_R = 0$ we have $(0 : R)_R \subseteq \beta_{co}(R)$. Hence, $\beta_{co}(R) = (0 : R)_R \neq 0$.

Lemma 2.3.2 *For any ring R and any R-module M we have*

$$\beta_{co}(R) \subseteq (\beta_{co}(M) : M)_R.$$

Proof: $(\beta_{co}(M) : M)_R = (\bigcap_{P \leq M} P : M) = \bigcap_{P \leq M} (P : M)$, where P is a completely prime submodule of M. Since $(P : M)_R$ is a completely prime ideal of R for each completely prime submodule P of M, we get $\beta_{co}(R) \subseteq (\beta_{co}(M) : M)_R$, i.e., $\beta_{co}(R)M \subseteq \beta_{co}(M)$. ■

The containment is strict: let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ for some prime number p. $\beta_{co}(M) = \mathbb{Z}_{p^\infty}$ and $\beta_{co}(R) = (0)$, i.e., $\beta_{co}(R)M = (0)$.

Proposition 2.3.7 *For any ring R, $\beta_{co}(R) = (\beta_{co}({}_RR) : R)_R$.*

Proof: From Lemma 2.3.2, $\beta_{co}(R) \subseteq (\beta_{co}({}_RR) : R)_R$. Since $\beta_{co}({}_RR) \subseteq \beta_{co}(R)$ we have $\beta_{co}(R) \subseteq (\beta_{co}({}_RR) : R) \subseteq (\beta_{co}(R) : R)$. Let $x \in (\beta_{co}(R) : R)$. Hence $xR \subseteq \beta_{co}(R) = \bigcap_{\mathcal{P} \text{ completely prime in } R} \mathcal{P} \subseteq \mathcal{P}$ for all completely prime ideals $\mathcal{P}$ of R. Since $xR \subseteq \mathcal{P}$ for $\mathcal{P}$ completely prime, we have $x \in \mathcal{P}$ and $x \in \beta_{co}(R)$. Hence, $(\beta_{co}(R) : R) \subseteq \beta_{co}(R)$ and we are done. ■

Proposition 2.3.8 *For all R-modules M,*

1. *$\beta_{co}(M) = \{x \in M \ : \ Rx \subseteq \beta_{co}(M)\}$.*

2. *If $\beta_{co}(R) = R$, then $\beta_{co}(M) = M$, i.e., if R has no completely prime ideals, then M has no completely prime submodules.*

Proof:

1. Since $\beta_{co}(M) \leq M$, we have $R\beta_{co}(M) \subseteq \beta_{co}(M)$. Conversely, let $x \in M$ with $Rx \subseteq \beta_{co}(M)$. Hence $Rx \subseteq P$ for all completely prime submodules P of M. Since P is also a prime submodule, we have $x \in P$ and hence $x \in \beta_{co}(M)$.

2. $R = \beta_{co}(R)$ gives $R \subseteq (\beta_{co}(M) : M)$ from Lemma 2.3.2. Hence $RM \subseteq \beta_{co}(M)$ and from (1), we have $M \subseteq \beta_{co}(M)$, i.e., $M = \beta_{co}(M)$.

■

Proposition 2.3.9 *Let R be any ring. Then, any of the following conditions implies $\beta_{co}(R) = \beta_{co}({}_RR)$.*

1. *R is commutative;*

2. *$x \in xR$ for all $x \in R$, e.g., if R has an identity or R is Von Neumann regular.*[1]

Proof:

1. Since R is commutative, it follows from Proposition 2.3.7 and Lemma 2.3.8 that $\beta_{co}(R) \subseteq \beta_{co}({}_RR) \subseteq \beta_{co}(R)$ and $\beta_{co}(R) = \beta_{co}({}_RR)$.

2. Let $x \in \beta_{co}(R)$, then from Proposition 2.3.7, $xR \subseteq \beta_{co}({}_RR)$ and since $x \in xR$, we get $x \in \beta_{co}({}_RR)$ such that $\beta_{co}({}_RR) = \beta_{co}(R)$.

■

2.4 A special class of completely prime modules

A class ρ of associative rings is called a special class if ρ is hereditary, consists of prime rings and is closed under essential extensions, cf., [33, p.80]. Andrunakievich and Rjabuhin in [3] extended this notion to modules and showed that prime modules, irreducible modules, simple modules, modules without zero divisors, etc form special classes of modules. De La Rosa and Veldsman in [29] defined a weakly special class of modules. We follow the definition in [29] of a weakly special class of modules to define a special class of modules.

[1] A ring R is Von Neumann regular if for every $r \in R$, there exists $x \in R$ such that $r = rxr$.

Definition 2.4.1 For a ring R, let $\mathcal{K}_R$ be a (possibly empty) class of R-modules. Let $\mathcal{K} = \cup\{\mathcal{K}_R \ : \ R \text{ a ring}\}$. $\mathcal{K}$ is a special class of modules if it satisfies:

S1 $M \in \mathcal{K}_R$ and $I \lhd R$ with $I \subseteq (0 : M)_R$ implies $M \in \mathcal{K}_{R/I}$.

S2 If $I \lhd R$ and $M \in \mathcal{K}_{R/I}$, then $M \in \mathcal{K}_R$.

S3 $M \in \mathcal{K}_R$ and $I \lhd R$ with $IM \neq 0$ implies $M \in \mathcal{K}_I$.

S4 $M \in \mathcal{K}_R$ implies $RM \neq 0$ and $R/(0 : M)_R$ is a prime ring.

S5 If $I \lhd R$ and $M \in \mathcal{K}_I$, then there exists $N \in \mathcal{K}_R$ such that $(0 : N)_I \subseteq (0 : M)_I$.

Following similar techniques of [80], we get:

Theorem 2.4.1 *Let $\mathcal{M} = \cup\mathcal{M}_R$ be a special class of modules. Then,*

$$\mathcal{J} = \{R \ : \ \textit{there exists } M \in \mathcal{M}_R \textit{ with } (0 : M)_R = 0\} \cup \{0\}$$

is a special class of rings. If $\mathcal{R}$ is the corresponding special radical then,

$$\mathcal{R}(R) := \cap\{(0 : M)_R \ : \ M \in \mathcal{M}\}.$$

Theorem 2.4.2 *Let $\mathcal{J}$ be a special class of rings and for every ring R, let*

$$\mathcal{M}_R = \{M \ : \ M \textit{ is an } R\textit{-module}, RM \neq 0 \textit{ and } R/(0 : M)_R \in \mathcal{J}\}.$$

If $\mathcal{M} = \cup\mathcal{M}_R$, then $\mathcal{M}$ is a special class of modules. If r is the corresponding special radical and M is any R-module, then

$$r(M) := \cap\{P \leq M \ : \ M/P \in \mathcal{M}_R\}.$$

For the completely prime modules, we have

Theorem 2.4.3 *Let R be any ring and*

$$\mathcal{M}_R := \{M \ : \ M \text{ is a completely prime } R\text{-module with } RM \neq 0\}.$$

If $\mathcal{M} = \cup \mathcal{M}_R$, then $\mathcal{M}$ is a special class of R-modules.

Proof:

S1 Let $M \in \mathcal{M}_R$ and $I \lhd R$ with $IM = 0$. M is an R/I-module via $(r+I)m = rm$. We show $M \in \mathcal{M}_{R/I}$. Let $a + I \in R/I$ and $m \in M$ such that $(a+I)m = 0$, then $am = 0$ such that $aM = 0$ or $m = 0$ since $M \in \mathcal{M}_R$. Thus, $M \in \mathcal{M}_{R/I}$.

S2 Let $I \lhd R$ and $M \in \mathcal{M}_{R/I}$. M is an R-module w.r.t. $rm = (r+I)m$ for $r \in R$, $m \in M$. Let $a \in R$, $m \in M$ such that $am = 0 \Leftrightarrow (a+I)m = 0$. $(a+I)M = 0$ or $m = 0$ since $M \in \mathcal{M}_{R/I}$. Thus, $aM = 0$ or $m = 0$ and $M \in \mathcal{M}_R$.

S3 Suppose $M \in \mathcal{M}_R$, $I \lhd R$ and $IM \neq 0$. Let $a \in I$, $m \in M$ such that $am = 0$. Since $M \in \mathcal{M}_R$, $m = 0$ or $aM = 0$. Therefore, $M \in \mathcal{M}_I$.

S4 Let $M \in \mathcal{M}_R$. Hence $RM \neq 0$. Since $(0 : M)_R$ is a completely prime ideal of R, it is a prime ideal and $R/(0 : M)_R$ is a prime ring.

S5 Let $I \lhd R$ and $M \in \mathcal{M}_I$. $(0 : M)_I$ is a completely prime ideal of I. We have $(0 : M)_I \lhd I \lhd R$ and since $I/(0 : M)_I$ is a completely prime ring we have $(0 : M)_I \lhd R$. Choose $K/(0 : M)_I \lhd R/(0 : M)_I$ maximal w.r.t. $I/(0 : M)_I \cap K/(0 : M)_I = 0$. Then $I/(0 : M)_I \cong (I + K)/K \lhd \cdot R/I$. Since the class of completely prime rings is essentially closed, R/I is completely prime. Let $N = R/K$. N is an R-module and $RN \neq 0$. From Proposition 2.2.2 we have $(0 : N)_R = K$. $(0 : N)_I \subseteq (0 : M)_I$, for let $x \in (0 : N)_I$, then $xR \subseteq I \cap K \subseteq (0 : M)_I$ and it follows that $x \in (0 : M)_I$ thus, $\mathcal{M}$ is a special class.

■

Corollary 2.4.1 *If $\mathcal{M}_{co}$ is the special class of completely prime modules, then the special radical induced by $\mathcal{M}_{co}$ on a ring R is given by*

$$\beta_{co}(R) = \cap\{(0 : M)_R \ : \ M \textit{ is a completely prime } R\textit{-module}\}$$

$$= \cap\{I \triangleleft R \ : \ I \textit{ is a completely prime ideal}\}$$

$$= \mathcal{N}_g(R) \textit{ the generalized nil radical.}$$

Chapter 3

Classical completely prime submodules

We define and characterize classical completely prime submodules which are a generalization of both completely prime ideals in rings and reduced modules (as defined by Lee and Zhou in [52]). A comparison of these submodules with other "prime" submodules in literature is done. If $\text{Rad}(M)$ is the Jacobson radical of M and $\beta^c_{cl}(M)$ the classical completely prime radical of M, we show that for modules over left Artinian rings R, $\text{Rad}(M) \subseteq \beta^c_{cl}(M)$ and $\text{Rad}(_RR) = \beta^c_{cl}(_RR)$.

A ring R is completely semiprime (reduced) if and only if for all $a \in R$, $a^2 = 0 \Rightarrow a = 0$. An R-module M is reduced if for all $a \in R$ and every $m \in M$, $am = 0$ implies $\langle m \rangle \cap \ aM = 0$. Our definition of a reduced module is a generalization of that in [52], where Rm is used in the place of $\langle m \rangle$. We state an equivalent but more handy definition for a reduced module.

Definition 3.0.2 An R-module M is reduced if for all $a \in R$ and every $m \in M$, $a^2m = 0$ implies $a\langle m \rangle = 0$.

To show that these two definitions are indeed equivalent, we need Lemma 3.0.1 first.

Lemma 3.0.1 *The R-module M is reduced if and only if:*

1. *for all $a \in R$ and $m \in M$ such that $am = 0$, then $a\langle m\rangle = 0$ and*

2. *$a^2m = 0$ implies $am = 0$.*

Proof: Suppose M is reduced, $a \in R$, $m \in M$ and $am = 0$. By definition of reduced modules, $a\langle m\rangle \subseteq \langle m\rangle \cap aM = 0$ giving (1). Now let $a \in R$ and $m \in M$ such that $a^2m = a(am) = 0$. Since M is reduced, we have $am \in \langle am\rangle \cap aM = 0$ which establishes (2). For the converse, let $a \in R$ and $m \in M$ such that $am = 0$. If $x \in \langle m\rangle \cap aM = 0$, then $x = m_1 = an$ for some $m_1 \in \langle m\rangle$ and $n \in M$. From (1), $am_1 \in a\langle m\rangle = 0$. But $am_1 = ax = a^2n = 0$. From (2), we have $a^2n = 0$ implies $x = an = 0$. Hence, $\langle m\rangle \cap aM = 0$. ∎

Proposition 3.0.1 *An R-module M is reduced if and only if for all $a \in R$ and every $m \in M$, if $a^2m = 0$, then $a\langle m\rangle = 0$.*

Proof: ($\Rightarrow$) is clear from Lemma 3.0.1. For the converse, assume that, if $a^2m = 0$ then, $a\langle m\rangle = 0$. Let $a \in R$ and $m \in M$ such that $am = 0$ and $x \in \langle m\rangle \cap aM$. Now, $x = m_1 = am_2$ for some $m_1 \in \langle m\rangle$ and $m_2 \in M$. Since $am = 0$, we also have $a^2m = 0$ and from our assumption, $a\langle m\rangle = 0$ and hence $am_1 = 0$ since $m_1 \in \langle m\rangle$. Now, $am_1 = 0 \Rightarrow a^2m_2 = 0$, so that $a\langle m_2\rangle = 0$ (again from our assumption). Consequently, $x = am_2 \in a\langle m_2\rangle = 0$ and hence $\langle m\rangle \cap aM = 0$ which completes the proof. ∎

Definition 3.0.1 of a reduced (completely semiprime) module motivates the following two definitions:

Definition 3.0.3 A proper submodule P of an R-module M with $RM \nsubseteq P$ is completely semiprime if for all $a \in R$ and every $m \in M$, $a^2m \in P$ implies $a\langle m\rangle \subseteq P$.

Definition 3.0.4 A proper submodule P of an R-module M with $RM \nsubseteq P$ is classical completely prime if for all $a, b \in R$ and every $m \in M$, $abm \in P$ implies $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$.

An R-module M/P is a classical completely prime (resp. completely semiprime) module if and only if P is a classical completely prime (resp. completely semiprime) submodule of M. Thus, an R-module M is classical completely prime (completely semiprime) if and only if the zero submodule is a classical completely prime (completely semiprime) submodule of M.

Example 3.0.1 A free module M over a domain R is classical completely prime.

Proof: Suppose $abm = 0$ for some $a, b \in R$ and $m \in M$. Then $abm = ab\sum_{i=1}^{n}(r_im_i) = \sum_{i=1}^{n}(abr_i)m_i = 0$ for some $r_i \in R$ and $m_i \in M$. Since M is free $abr_i = 0$ for all $i \in \{1, \cdots n\}$. For $m \neq 0$, there is at least one $j \in \{1, \cdots n\}$ such that $r_j \neq 0$. Now $abr_j = 0$ implies $a = 0$ or $b = 0$ (since R is a domain) such that $a\langle m\rangle = 0$ or $b\langle m\rangle = 0$. ■

Example 3.0.2 A torsion-free module M over a domain R is classical completely prime. It follows that flat modules over domains (and hence projective modules over domains) are classical completely prime modules.

Proof: Suppose for $a, b \in R$ and $m \in M$, $abm = 0$. If $m = 0$, $a\langle m\rangle = 0$ and $b\langle m\rangle = 0$. Let $m \neq 0$, then $ab = 0$ since M is torsion-free. Hence, $a = 0$ or $b = 0$ since R is a domain. Therefore, $a\langle m\rangle = 0$ or $b\langle m\rangle = 0$. The last part is due to the fact that flat modules are torsion-free, see [75, Example 1, p.15] and projective modules are flat modules. ■

Example 3.0.3 Every submodule P of a module M over a division ring R is a classical completely prime submodule.

Proof: Suppose $a, b \in R$ and $m \in M$ such that $abm \in P$. If $ab = 0$, $a = 0$ or $b = 0$ such that $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. Suppose $ab \neq 0$, then, $m \in (ab)^{-1}P \subseteq P$.[1] Thus, $a\langle m\rangle \subseteq P$ and $b\langle m\rangle \subseteq P$. ■

Example 3.0.4 Any prime (sub)module over a commutative ring is a classical completely prime (sub)module.

3.1 Investigation of properties

In this section, we investigate properties exhibited by classical completely prime (semiprime) submodules. First, we introduce notions of symmetric and IFP submodules that will prove useful later in the sequel.

Lambek in [51, p.364] called a module M symmetric if $abm = 0$ implies $bam = 0$ for $a, b \in R$ and $m \in M$. We call a submodule P of an R-module M symmetric if $abm \in P$ implies $bam \in P$ for $a, b \in R$ and $m \in M$. So, a module M is symmetric if its zero submodule is symmetric. From [17], a right (or left) ideal I of a ring R is said to have the insertion-of-factor-property (IFP) if whenever $ab \in I$ for $a, b \in R$, we have $aRb \subseteq I$. We call a submodule N of an R-module M an IFP submodule if whenever $am \in N$ for $a \in R$ and $m \in M$, we have $aRm \subseteq N$. A module is IFP if its zero submodule is IFP.

[1] $(ab)^{-1}$ is here used to mean the inverse of ab in R

Proposition 3.1.1 *For any submodule P of an R-module M,*

$$\textit{completely semiprime} \Rightarrow \textit{symmetric} \Rightarrow \textit{IFP}.$$

Proof: Let $abm \in P$. $(bab)^2 m \in P$ and P completely semiprime gives $bab\langle m\rangle \subseteq P$. Thus, $(ba)^2 m = bab(am) \in bab\langle m\rangle \subseteq P$ and again P completely semiprime gives $bam \in ba\langle m\rangle \subseteq P$. For the second implication, let $am \in P$ for $a \in R$ and $m \in M$. Then $Ram \subseteq P$ and P symmetric implies $aRm \subseteq P$. ■

Example 3.1.1 A module M over a left duo ring R (a ring whose all left ideals are two sided) is fully IFP (every submodule of M is IFP) but it need not be symmetric.

Proof: Let $P \leq M$, $a \in R$ and $m \in M$ such that $am \in P$, then $a \in (P : m)$. $(P : m)$ is a left ideal of R but since R is left duo, we have $(P : m) \triangleleft R$ and $aR \subseteq (P : m)$ such that $aRm \subseteq P$. Hence, P is IFP. $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is a left quasi duo ring (i.e., every maximal left ideal of $\mathbb{Z}_2$ is two sided). By [44, Prop. 2.1], any n-by-n upper triangular matrix ring R over $\mathbb{Z}_2$ is left quasi duo. Hence, every submodule of the module ${}_RR$ ($R = \mathbb{Z}_2$) is IFP. We show that the zero submodule of ${}_RR$ is not symmetric. Take $m = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$, $a = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix}$, and $b = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{pmatrix} \in R$; $abm = (0)$ but $bam \neq (0)$. ■

Example 3.1.2 A submodule P of a module M over a commutative ring R is symmetric but it need not be completely semiprime.

3.1.1 Properties of underlying ring

We give information classical completely prime (completely semiprime) submodules of ${}_RM$ reveal about the underlying ring R. Propositions 3.1.2 and

3.1.3 indicate that there is a one to one correspondence between completely semiprime (classical completely prime) submodules P of the module ${}_RM$ and completely semiprime (completely prime) ideals of R of the form $(P : m)$ for all $m \in M \setminus P$.

Proposition 3.1.2 *For $P \leq {}_RM$, the following statements are equivalent:*

1. *P is a completely semiprime submodule of M;*
2. *$(P : m) = (P : \langle m\rangle) = (\bar{0} : \bar{m})$ is a completely semiprime ideal of R for every $m \in M \setminus P$, where $\bar{m} = m + P$;*
3. *for all $m \in M \setminus P$, $(P : m) \lhd R$ and for all $a \in R$ if $a^2m \in P$, then $am \in P$;*
4. *for all $m \in M \setminus P$, $(P : m) \lhd R$ and for all $a \in R$ if $\langle a^2m\rangle \subseteq P$, then $\langle am\rangle \subseteq P$;*
5. *for all $a \in R$ and every $m \in M$, if $\langle a^2m\rangle \subseteq P$, then $\langle a\langle m\rangle\rangle \subseteq P$.*

Proof: (1) $\Rightarrow$ (2). Since $(P : m)$ is always a left ideal of R for all $m \in M \setminus P$, we show that if $a \in (P : m)$, then $aR \subseteq (P : m)$. Suppose $a \in (P : m)$, then $Ram \subseteq P$ and from Proposition 3.1.1, we have $aRm \subseteq P$ and therefore, $aR \subseteq (P : m)$ as required. Let $m \in M \setminus P$, $(P : m) = \{r \in R : rm \in P\} = \{r \in R : r\bar{m} = \bar{0}\} = (\bar{0} : \bar{m})$. The inclusion $(P : \langle m\rangle) \subseteq (P : m)$ is clear. Suppose $x \in (P : m)$, then $xR \subseteq (P : m)$. Hence, $x\langle m\rangle \subseteq P$ and we have $x \in (P : \langle m\rangle)$. Lastly, suppose $a^2 \in (P : m)$, i.e., $a^2m \in P$. Then, $am \in a\langle m\rangle \subseteq P$ since P is a completely semiprime submodule of M. Thus, $a \in (P : m)$.
(2) $\Rightarrow$ (1). Let for all $a \in R$ and $m \in M$, $a^2m \in P$. Then, $a^2 \in (P : m)$ which implies $a \in (P : m)$ by definition of a completely semiprime ideal of a ring R. Thus, $aR \subseteq (P : m)$ and $aRm \subseteq P$. Therefore, $a\langle m\rangle = \mathbb{Z}am + aRm \subseteq P$ and

P is a completely semiprime submodule of M.
(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) and (5) $\Leftrightarrow$ (1) are trivial. ∎

Corollary 3.1.1 *An R-module M is reduced if and only if for every $0 \neq m \in M$, $(0 : m)$ is a completely semiprime two sided-ideal of R.*

Proposition 3.1.3 *For a proper submodule P of an R-module M, the following statements are equivalent:*

1. *P is a classical completely prime submodule of M;*
2. *for every $m \in M \setminus P$, $(P : m) = (P : \langle m\rangle) = (\bar{0} : \bar{m})$ is a completely prime ideal of R;*
3. *for all $m \in M \setminus P$, $(P : m) \lhd R$ and if $a, b \in R$ such that $abm \in P$, then $am \in P$ or $bm \in P$;*
4. *for all $m \in M \setminus P$, $(P : m) \lhd R$ and if $a, b \in R$ such that $\langle abm\rangle \subseteq P$, then $\langle am\rangle \subseteq P$ or $\langle bm\rangle \subseteq P$;*
5. *for all $a, b \in R$ and every $m \in M$, if $\langle abm\rangle \subseteq P$, then $\langle a\langle m\rangle\rangle \subseteq P$ or $\langle b\langle m\rangle\rangle \subseteq P$.*

Proof: (1) $\Rightarrow$ (2). Every classical completely prime submodule of M is a completely semiprime submodule of M. We have seen in Proposition 3.1.2 that $(P : m)$ is an ideal of R and $(P : m) = (P : \langle m\rangle) = (\bar{0} : \bar{m})$. Let $a, b \in R$ and $m \in M \setminus P$ such that $ab \in (P : m)$, i.e., $abm \in P$. Now, P classical completely prime submodule gives $am \in a\langle m\rangle \subseteq P$ or $bm \in b\langle m\rangle \subseteq P$. Hence, $a \in (P : m)$ or $b \in (P : m)$.
(2) $\Rightarrow$ (1). Let for $a, b \in R$ and $0 \neq m \in M$, $abm \in P$, i.e., $ab \in (P : m)$. $(P : m)$ a completely prime ideal of R gives $a \in (P : m)$ or $b \in (P : m)$. Hence, $(am \in P$ and $aRm \subseteq P)$ or $(bm \in P$ and $bRm \subseteq P)$ such that $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) and (5) $\Leftrightarrow$ (1) are trivial. ∎

The zero divisor set of ${}_RM$ [7, p.316] is the set

$$Zd(M) := \{r \in R \ : \ \text{there exists } 0 \neq m \in M, \ \text{with } rm = 0\}.$$

The following proposition provides us with two other ways of constructing completely prime ideals of a ring R from a submodule P of an R-module M.

Proposition 3.1.4 *Let P be a classical completely prime submodule of an R-module M. Then,*

1. *for any $m, n \in M \setminus P$ either $(P : n) \subseteq (P : m)$ or $(P : m) \subseteq (P : n)$;*
2. *$Zd(M/P)$ is a completely prime ideal of R;*
3. *for all submodules K and L of M not contained in P, $(P : L) \subseteq (P : K)$ or $(P : K) \subseteq (P : L)$;*
4. *$(P : K)$ is a completely prime ideal of R for all submodules K of M such that $K \nsubseteq P$.*

Proof:

1. Assume $n, m \in M \setminus P$. Then, $(P : n)(P : m) \subseteq (P : n) \cap (P : m) \subseteq (P : n + m)$. We know that, $(P : n + m)$ is a completely prime ideal of R and hence a prime ideal of R. So, we have $(P : n) \subseteq (P : n + m)$ or $(P : m) \subseteq (P : n + m)$. If $(P : n) \subseteq (P : n + m)$, then $(P : n) = (P : n) \cap (P : n + m) \subseteq (P : m)$. Similarly, if $(P : m) \subseteq (P : n + m)$, we get $(P : m) \subseteq (P : n)$.
2. By definition, $Zd(M/P) = \bigcup\limits_{m \in M \setminus P} (P : m)$. But $\{(P : m)\}_{m \in M \setminus P}$ form a chain of completely prime ideals of R. We see that $Zd(M/P)$ is the largest of all the $(P : m)$'s and hence a completely prime ideal of R.

3. $(P:K)(P:L) \subseteq (P:K) \cap (P:L) \subseteq (P:K+L)$. Hence, $(P:K) \subseteq (P:K+L) \subseteq (P:L)$ or $(P:L) \subseteq (P:K+L) \subseteq (P:K)$.

4. To show that $(P:K)$ is a completely prime ideal of R, it is enough to show that it is both prime and completely semiprime as an ideal of R. If P is classical completely prime, by Theorem 3.4.1 it is classical prime (see definition 3.4.2) and hence $(P:K)$ is a prime ideal of R for all $K \leq M$ such that $K \not\subseteq P$. Suppose $a^2 \in (P:K)$ for $a \in R$ and $K \leq M$ with $K \not\subseteq P$, then $a^2k \in P$ for all $k \in K$. By hypothesis, $a\langle k\rangle \subseteq P$ for all $k \in K$. Thus, $aK \subseteq P$ such that $a \in (P:K)$.

■

Proposition 3.1.5 below and its corollaries provide more justification for our definition of classical completely prime submodules.

Proposition 3.1.5 *If $1 \in R$ and $P \lhd R$, then P is a completely prime ideal of R if and only if P is a classical completely prime submodule of ${}_RR$.*

Proof: Suppose P is a completely prime ideal of R and for any $a, b \in R$ and $m \in {}_RR$, $abm \in P$. By definition of a completely prime ideal, $a \in P$ or $b \in P$ or $m \in P$. Thus, $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. Conversely, suppose the ideal P of R is a classical completely prime submodule of ${}_RR$. Let for any $a, b \in R$, $ab \in P$. Since $1 \in R$ by definition of classical completely prime submodule, $ab.1 \in P$ implies $aR \subseteq P$ or $bR \subseteq P$ such that $a \in P$ or $b \in P$. ■

Remark 3.1.1 In general, Proposition 3.1.5 holds for all rings R such that $a \in aR$ for all $a \in R$. A comparison of Propositions 3.1.5 and 2.1.1 demonstrates yet another reason why both completely prime as well as classical completely prime submodules generalize completely prime ideals.

Corollary 3.1.2 *If $1 \in R$, then R is a domain if and only if ${}_RR$ is a classical completely prime module.*

Corollary 3.1.3 *If $1 \in R$ and $P \lhd R$, then P is a completely semiprime ideal of R if and only if it is a completely semiprime submodule of ${}_RR$.*

Corollary 3.1.4 *A unital ring R is reduced if and only if ${}_RR$ is a reduced module.*

3.1.2 Homomorphic images

Proposition 3.1.6 *Let M be an R-module, $N, P \leq M$ such that $N \subseteq P$. If $f : M \to M/N$ is the canonical epimorphism, then P is a classical completely prime submodule of M if and only if $f(P)$ is a classical completely prime submodule of M/N.*

Proof: Elementary. ∎

Remark 3.1.2 If $N \not\subseteq P$, P classical completely prime submodule of M does not in general imply $f(P)$ is a classical completely prime submodule of M/N (and hence classical completely prime is not in general closed under homomorphic images). See Example 3.1.3 below.

Example 3.1.3 The $\mathbb{Z}$-module $\mathbb{Z}$ is a classical completely prime module, cf., Corollary 3.1.2 and $N = 8\mathbb{Z}$ is a submodule of $M = {}_{\mathbb{Z}}\mathbb{Z}$. By [1, Example 2.5], M/N is not a reduced module (i.e., not a completely semiprime module) and hence not a classical completely prime module.

Proposition 3.1.7 *Let $f : R \to A$ be a ring epimorphism and M an A-module, then M is an R-module and ${}_AM$ is classical completely prime if and only if ${}_RM$ is classical completely prime.*

Proof: Define a function from ${}_RM$ to ${}_AM$ by $rm = f(r)m$. This function turns M into an R-module whenever M is an ${}_AM$ module. Suppose ${}_AM$ is classical completely prime and for all $r, s \in R$ and $m \in M$, $rsm = 0$. Then, $0 = rsm = f(r)f(s)m$. Since ${}_AM$ is classical completely prime, $f(r)\langle m\rangle_A = 0$ or $f(s)\langle m\rangle_A = 0$. Suppose $f(r)\langle m\rangle_A = 0$, f onto and definition of f imply $r\langle m\rangle_R = \mathbb{Z}rm + rRm = \mathbb{Z}f(r)m + f(rR)m = \mathbb{Z}f(r)m + f(r)f(R)m = f(r)[\mathbb{Z}m+Am] = f(r)\langle m\rangle_A = 0$. Similarly, if $f(s)\langle m\rangle_A = 0$, we get $s\langle m\rangle_R = 0$. Thus, ${}_RM$ is classical completely prime. Assume ${}_RM$ is classical completely prime and for all $a, b \in R$ and $m \in M$, $abm = 0$. f onto implies there exists $r, s \in A$ such that $a = f(r)$ and $b = f(s)$, i.e., $f(r)f(s)m = rsm = 0$. By assumption, $r\langle m\rangle_R = 0$ or $s\langle m\rangle_R = 0$. If $r\langle m\rangle_R = 0$ (resp. $s\langle m\rangle_R = 0$), the fact that f is onto leads to $a\langle m\rangle_A = 0$ (resp. $b\langle m\rangle_A = 0$). Hence, ${}_AM$ is classical completely prime. ■

3.1.3 Properties of submodules and direct summands

Proposition 3.1.8 *If M is a classical completely prime module, then any submodule N of M is also a classical completely prime module.*

Proof: Elementary. ■

Proposition 3.1.9 *For an R-module M, the following statements are equivalent:*

1. *M is a classical completely prime module;*
2. *Each direct summand of M including the zero summand is a classical completely prime submodule of M.*

Proof: (1) $\Rightarrow$ (2). Suppose $M = K \oplus P$ where K and P are submodules of M. Let $m \in M$, $a, b \in R$ such that $abm \in P$. If $b\langle m\rangle \subseteq P$, we are through. Suppose $b\langle m\rangle \nsubseteq P$. Then, $\langle m\rangle \nsubseteq P$ and $abm \in \langle m\rangle \subseteq K$. Thus, $abm \in P \cap K = 0$, i.e., $abm = 0$. By (1), $a\langle m\rangle = 0$ or $b\langle m\rangle = 0$. We assumed that $b\langle m\rangle \nsubseteq P$, so $b\langle m\rangle \neq 0$. Therefore, $a\langle m\rangle \subseteq P$ and P is a classical completely prime submodule of M.

(2) $\Rightarrow$ (1). $M \cong 0 \oplus M$. From (2), the zero summand is a classical completely prime submodule of M. Therefore, M is a classical completely prime module.

■

3.2 Classical multiplicative systems

Definition 3.2.1 Let R be a ring and M an R-module. A nonempty set $S \subseteq M \setminus \{0\}$ is called a classical multiplicative system of M if, for all $a, b \in R$, $m \in M$ and for all submodules K of M, if $(K + a\langle m\rangle) \cap S \neq \emptyset$ and $(K + b\langle m\rangle) \cap S \neq \emptyset$, then $(K + \{abm\}) \cap S \neq \emptyset$.

Proposition 3.2.1 *Let M be an R-module. A proper submodule P of M is classical completely prime if and only if its complement $M\backslash P$ is a classical multiplicative system.*

Proof: Suppose $S := M\backslash P$. Let $a, b \in R$, $m \in M$ and K be a submodule of M such that $(K+a\langle m\rangle)\cap S \neq \emptyset$ and $(K+b\langle m\rangle)\cap S \neq \emptyset$. If $(K+\{abm\})\cap S = \emptyset$, then $abm \in P$. Since P is classical completely prime, $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. It follows that $(K + a\langle m\rangle) \cap S = \emptyset$ or $(K + b\langle m\rangle) \cap S = \emptyset$, a contradiction. Therefore, S is a classical multiplicative system in M. For the converse, let $S := M\backslash P$ be a classical multiplicative system in M. Suppose for $a, b \in R$ and $m \in M$, $abm \in P$. If $a\langle m\rangle \nsubseteq P$ and $b\langle m\rangle \nsubseteq P$, then $a\langle m\rangle \cap\ S \neq \emptyset$ and $b\langle m\rangle \cap\ S \neq \emptyset$. Thus, $abm \in S$, a contradiction. Therefore, P is a classical completely prime submodule of M. ■

Proposition 3.2.2 *Let M be an R-module, P be a proper submodule of M, and $S := M\backslash P$. Then, the following statements are equivalent:*

1. *P is a classical completely prime submodule of M;*
2. *S is a classical multiplicative system of M;*
3. *for all $a, b \in R$ and $m \in M$, if $a\langle m\rangle \cap\ S \neq \emptyset$ and $b\langle m\rangle \cap\ S \neq \emptyset$, then $abm \in S$;*
4. *for all $a, b \in R$ and $m \in M$, if $\langle a\langle m\rangle\rangle > \cap\ S \neq \emptyset$ and $\langle b\langle m\rangle\rangle \cap\ S \neq \emptyset$, then $\langle ab\rangle \cap\ S \neq \emptyset$.*

Lemma 3.2.1 *Let M be an R-module, $S \subseteq M$ a classical multiplicative system of M and P a submodule of M maximal with respect to the property that $P \cap S = \emptyset$. Then, P is a classical completely prime submodule of M.*

Proof: Suppose $a \in R$ and $m \in M$ such that $\langle abm\rangle \subseteq P$. If $\langle a\langle m\rangle\rangle \nsubseteq P$ and $\langle b\langle m\rangle\rangle \nsubseteq P$ then $(\langle a\langle m\rangle\rangle + P) \cap S \neq \emptyset$ and $(\langle b\langle m\rangle\rangle + P) \cap S \neq \emptyset$. By definition of a classical multiplicative system S of M, $(\langle abm\rangle + P) \cap S \neq \emptyset$. Since $\langle abm\rangle \subseteq P$, we have $P \cap S \neq \emptyset$, a contradiction. Hence, P must be a classical completely prime submodule. ∎

Definition 3.2.2 Let R be a ring and M an R-module. For $N \leq M$, if there is a classical completely prime submodule of M containing N, we define clc.$\sqrt{N} := \{m \in M\ :$ every classical multiplicative system of M containing m meets $N\}$. We write clc.$\sqrt{N} = M$ when there are no classical completely prime submodules of M containing N.

Theorem 3.2.1 *Let M be an R-module and $N \leq M$. Then, either clc.$\sqrt{N} = M$ or clc.$\sqrt{N}$ equals the intersection of all classical completely prime submodules of M containing N, which is denoted by $\beta^{c}_{cl}(N)$.*

Proof: Suppose $\text{clc.}\sqrt{N} \neq M$. Then, $\beta^{c}_{cl}(N) \neq \emptyset$. Both $\text{clc.}\sqrt{N}$ and N are contained in the same classical completely prime submodules. By definition of $\text{clc.}\sqrt{N}$ it is clear that $N \subseteq \text{clc.}\sqrt{N}$. Hence, any classical completely prime submodule of M which contains $\text{clc.}\sqrt{N}$ must necessarily contain N. Suppose P is a classical completely prime submodule of M such that $N \subseteq P$, and let $t \in \text{clc.}\sqrt{N}$. If $t \notin P$, then the complement of P, $C(P)$ in M is a classical multiplicative system containing t and therefore we would have $C(P) \cap N \neq \emptyset$. However, since $N \subseteq P$, $C(P) \cap P = \emptyset$ and this contradiction shows that $t \in P$. Hence $\text{clc.}\sqrt{N} \subseteq P$ as we wished to show. Thus, $\text{clc.}\sqrt{N} \subseteq \beta_{co}(N)$. Conversely, assume $s \notin \text{clc.}\sqrt{N}$, then there exists a classical multiplicative system S such that $s \in S$ and $S \cap N = \emptyset$. From Zorn's lemma, there exists a submodule $P \supseteq N$ which is maximal with respect to $P \cap S = \emptyset$. From Lemma 3.2.1, P is a classical completely prime submodule of M and $s \notin P$. ∎

3.3 Complete systems

Definition 3.3.1 Let R be a ring and M an R-module. A nonempty set $T \subseteq M \setminus \{0\}$ is called a complete system if, for all $a \in R$, $m \in M$ and for all submodules K of M, if $(K + a\langle m\rangle) \cap T \neq \emptyset$, then $(K + \{a^2m\}) \cap T \neq \emptyset$.

Corollary 3.3.1 *Let M be an R-module. A proper submodule P of M is completely semiprime if and only if $M\backslash P$ is a complete system.*

Proposition 3.3.1 *Let M be an R-module, P be a proper submodule of M, and $T := M\backslash P$. Then, the following statements are equivalent:*

1. *P is completely semiprime;*
2. *T is a complete system;*
3. *for all $a \in R$ and $m \in M$, if $a\langle m\rangle \cap T \neq \emptyset$, then $a^2m \in T$;*

4. *for all* $a \in R$ *and* $m \in M$, *if* $\langle a\langle m\rangle\rangle \cap T \neq \emptyset$, *then* $\langle a^2 m\rangle \cap T \neq \emptyset$.

Remark 3.3.1 Every classical multiplicative system is a complete system but not conversely.

3.4 Comparison with “primes” in literature

In this section, we compare classical completely prime (resp. completely semiprime) submodules with prime (resp. semiprime) and classical prime (resp. classical semiprime) submodules.

Definition 3.4.1 [28] $P \leq {}_RM$ with $RM \not\subseteq P$ is a semiprime submodule of M if for $a \in R$ and $m \in M$ such that $aRam \subseteq P$, then $am \in P$.

Definition 3.4.2 [10, p.338] $P \leq {}_RM$ with $RM \not\subseteq P$ is classical prime if for any $N \leq {}_RM$ and any $\mathcal{A}, \mathcal{B} \lhd R$ such that $\mathcal{A}\mathcal{B}N \subseteq P$, then $\mathcal{A}N \subseteq P$ or $\mathcal{B}N \subseteq P$. $P \leq_R M$ is classical semiprime if for every $\mathcal{A} \lhd R$, and $N \leq M$ such that $\mathcal{A}^2N \subseteq P$, then $\mathcal{A}N \subseteq P$.

Propositions 3.4.1 and 3.4.2 are modifications of [10, Proposition 1.1] and [10, Proposition 1.2] to suit a not necessarily unital module.

Proposition 3.4.1 *Let* $P \leq {}_RM$, *the following statements are equivalent:*

1. P *is a classical prime submodule of* M;

2. *for all* $a, b \in R$ *and every* $m \in M$, *if* $(a)(b)m \subseteq P$, *then* $(a)m \subseteq P$ *or* $(b)m \subseteq P$;

3. *for all* $a, b \in R$ *and every* $m \in M$ *such that* $aRb\langle m\rangle \subseteq P$, *then* $a\langle m\rangle \subseteq P$ *or* $b\langle m\rangle \subseteq P$.

Proposition 3.4.2 *Let $P \leq {}_RM$, the following statements are equivalent:*

1. *P is a classical semiprime submodule of M;*
2. *for all $a \in R$ and every $m \in M$, if $(a)^2m \subseteq P$, then $(a)m \subseteq P$;*
3. *for all $a \in R$ and every $m \in M$, if $aRa\langle m\rangle \subseteq P$, then $a\langle m\rangle \subseteq P$.*

Remark 3.4.1 In literature, classical prime is used interchangeably with weakly prime, cf., [7], [9], [10], [11], [13]. We here use classical prime instead of weakly prime. In defense of our nomenclature, weakly prime modules exist in [30] when used in a totally different context - a context which generalizes the notion of weakly prime ideals for rings to modules. To the best of our knowledge, classical prime has never been used by other authors to mean something different. Our "classical semiprime" is what is called "semiprime" in [10], our nomenclature reflects that classical semiprime is derived from classical prime. Lastly, our "semiprime" is the semiprime in [28].

Theorem 3.4.1 *For any $P \leq {}_RM$, we have the following implications:*

(a) in general			*(b) when P is IFP*		
		prime	*prime*		
		$\Downarrow$	$\Downarrow$		
classical	$\Rightarrow$	*classical*	*classical*	$\Leftrightarrow$	*classical*
completely prime		*prime*	*completely prime*		*prime*

Proof: (a). Classical completely prime $\Rightarrow$ classical prime. Let for all $a, b \in R$ and every $m \in M$, $(a)(b)m \subseteq P$. Then, $abm \in P$ and P classical completely prime in M implies $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. Thus, ($am \in P$ and $aRm \subseteq P$) or ($bm \in P$ and $bRm \subseteq P$) so that $(a)m = (\mathbb{Z}a + Ra + aR + RaR)m \subseteq P$ or $(b)m = (\mathbb{Z}b + Rb + bR + RbR)m \subseteq P$. Hence, P is a classical prime submodule

of M. Prime $\Rightarrow$ classical prime see, [73, Prop. 4.1.11].
(b). Suppose P is IFP, we show that classical prime implies classical completely prime. Suppose P is a classical prime submodule of M. If $a, b \in R$ and $m \in M$ such that $abm \in P$, then $aRbm \subseteq P$ and $aRb(Rm) \subseteq P$ so that $aRb\langle m\rangle \subseteq P$. This implies, either $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$ by definition of classical prime submodule. So, P is classical completely prime. ■

Remark 3.4.2 From [13] a prime module is classical prime but not conversely even when a module is defined over a commutative ring. It is also true that over a commutative ring, completely prime is the same as prime and classical prime is the same as classical completely prime. Hence, a classical completely prime module need not be completely prime.

Theorem 3.4.2 *P is a completely prime submodule if and only if it is both a prime and a completely semiprime submodule.*

Proof: Any completely prime submodule is both prime and completely semiprime. Suppose P is both prime and completely semiprime. Let $am \in P$ for any $a \in R$ and $m \in M$. Because P is completely semiprime, it is IFP and hence $a\langle m\rangle \subseteq P$. P prime implies $m \in P$ or $aM \subseteq P$. Thus, P is completely prime. ■

Theorem 3.4.3 *P is a classical completely prime submodule of an R-module M if and only if P is both a classical prime and a completely semiprime submodule of M.*

Proof: Every classical completely prime submodule is completely semiprime. From Theorem 3.4.1, classical completely prime submodules are classical prime. For the converse, assume P is both a completely semiprime and a classical prime submodule of M. Now, let $a, b \in R$ and $m \in M$ such that

$abm \in P$. By Proposition 3.1.1, P is IFP. Hence, $aRb\langle m\rangle \subseteq P$. P classical prime implies $a\langle m\rangle \subseteq P$ or $b\langle m\rangle \subseteq P$. ∎

Example 3.4.1 Every maximal submodule P of an R-module M is a classical prime submodule but there exist modules with maximal submodules which are not classical completely prime. Let $\mathcal{A} \lhd R$ and $N \leq M$ such that $\mathcal{A}N \subseteq P$, where P is a maximal submodule of M. If $N \subseteq P$, we are through. Suppose $N \nsubseteq P$. Then, $M = N + P$ so that $\mathcal{A}M = \mathcal{A}N + \mathcal{A}P \subseteq P$. This shows P is a prime submodule and hence a classical prime submodule. We construct a maximal submodule which is not classical completely prime. Let $R = (M_2(\mathbb{Z}), +, .)$ be a ring of all 2-by-2 matrices with integral entries and $(M_2(\mathbb{Z}_2), +)$ be a group of all 2-by-2 matrices with entries from the ring $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. Then, $M_2(\mathbb{Z}_2)$ is an $M_2(\mathbb{Z})$-module and

$$P = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$$

is a maximal submodule of $M_2(\mathbb{Z}_2)$. Now, let $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $m = \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}$. $abm = 0 \in P$ but $a\langle m\rangle \nsubseteq P$ and $b\langle m\rangle \nsubseteq P$ since $am \notin P$ and $bm \notin P$. Therefore, P is a maximal submodule of $M_2(\mathbb{Z}_2)$ but not a classical completely prime submodule of the $M_2(\mathbb{Z})$-module $M_2(\mathbb{Z}_2)$.

In regard to Example 3.4.1, we point out that, although it is not true in general, we can find non-commutative rings for which every maximal submodule is classical completely prime. To illustrate this, we use left (quasi) duo rings. A ring R is called left (quasi) duo if every left (maximal left) ideal of R is two sided.

Proposition 3.4.3 [69, Proposition 3.6] *R is a left quasi-duo ring if and only if each simple R-module M is reduced.*

Proposition 3.4.4 *If R is a left quasi-duo ring, then each maximal submodule P of M is a classical completely prime submodule of M.*

Proof: Let P be a maximal submodule of M and R a left quasi-duo ring. M/P is simple and from Proposition 3.4.3, it is reduced. Hence, P is a completely semiprime submodule of M. Since every maximal submodule of M is classical prime, it follows from Theorem 3.4.3 that P is a classical completely prime submodule of M. ■

Remark 3.4.3 It is not possible to get an example like Example 3.4.1 for a ring R which is a collection of all upper triangular matrices over $\mathbb{Z}$. This is because, upper triangular matrix rings are left quasi-duo and from Proposition 3.4.4, maximal submodules are always classical completely prime.

It is clear from Example 3.4.1 that simple modules are not always classical completely prime. We give another example to show that simple modules are not always classical completely prime. It makes use of Lemma 3.4.1.

Lemma 3.4.1 *For a simple and reduced module ${}_RM$, $am = 0$ implies $aM = 0$ for all $a \in R$ and $0 \neq m \in M$.*

Proof: Suppose $am = 0$. Since M is simple and reduced, we have $0 = aM \cap \langle m \rangle = aM \cap M = aM$. ■

Example 3.4.2 Let $M = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ where entries of matrices in M are from $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and $R = M_2(\mathbb{Z})$. ${}_RM$ is a simple module which is not classical completely prime.

Proof: Let $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$,

$$rM = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} a & a \\ c & c \end{pmatrix}, \begin{pmatrix} b & b \\ d & d \end{pmatrix}, \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} \right\} \subseteq M$$

for any $a, b, c, d \in \mathbb{Z}$. The would be nontrivial proper submodules, namely; $N_1 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \right\}$, $N_2 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ and $N_3 = \left\{ \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}, \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \right\}$ are not closed under multiplication by R since, for a and c odd, $rN_1 \nsubseteq N_1$, for b and d odd, $rN_2 \nsubseteq N_2$ and for a odd but b, c, d even, $rN_3 \nsubseteq N_3$. Take $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \in R$ and $m = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \in M$, $am = 0$ but $aM \neq 0$ since $a = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix} \neq 0$. By Lemma 3.4.1, M is not reduced and hence not classical completely prime. ■

Since over commutative rings classical completely prime submodules and classical submodules are indistinguishable, we have:

Example 3.4.3 [13, Example 1] Assume that R is a unital commutative domain and $\mathcal{P}$ is a non-zero prime ideal in R. $\mathcal{P} \oplus 0$ and $0 \oplus \mathcal{P}$ are classical completely prime submodules in the free module $M = R \oplus R$, but they are not prime submodules.

3.5 Comparison of "semiprimes"

Theorem 3.5.1 *For any submodule P of an R-module M,*

$$\textit{completely semiprime} \;\Rightarrow\; \textit{semiprime} \;\Rightarrow\; \textit{classical semiprime.}$$

Proof: Suppose for $a \in R$ and $m \in M$, $aRam \subseteq P$, then $(a^2)^2m \in P$ and P completely semiprime implies $a^2m \in a^2\langle m\rangle \subseteq P$. Hence, $am \in a\langle m\rangle \subseteq P$ and P is semiprime. Now, suppose $aRa\langle m\rangle \subseteq P$ but $a\langle m\rangle \nsubseteq P$. Then, there exists $m_1 \in \langle m\rangle$ such that $am_1 \notin P$. By definition of semiprime submodules, $aRam_1 \nsubseteq P$ and so $aRa\langle m\rangle \nsubseteq P$ which is a contradiction. Therefore, whenever $aRa\langle m\rangle \subseteq P$, we have $a\langle m\rangle \subseteq P$ and semiprime $\Rightarrow$ classical semiprime. ∎

The reverse implications in Theorem 3.5.1 are not true in general. The simple module M constructed in Example 3.4.2 is semiprime (because all simple modules are prime) but it is not completely semiprime. For the second implication, a counter example was constructed by Hongan in [43, p.119].

Corollary 3.5.1 *If P is an IFP submodule of M, then*

$$\textit{completely semiprime} \;\Leftrightarrow\; \textit{semiprime} \;\Leftrightarrow\; \textit{classical semiprime.}$$

Proof: It is enough to show that classical semiprime $\Rightarrow$ completely semiprime, the rest follows from Theorem 3.5.1. Let $a^2m \in P$, where $a \in R$ and $m \in M$. For P IFP, $aRa\langle m\rangle \subseteq P$. By definition of classical semiprime, $a\langle m\rangle \subseteq P$ and P is completely semiprime. ∎

A ring R is left (right) permutable [20, p.258], if for all $a, b, c \in R$, $abc = bac$ ($abc = acb$). R is permutable if it is both left and right permutable. Commutative rings and nilpotent rings of index ≤ 3 are left (right) permutable. A ring R is medial [20], if for all $a, b, c, d \in R$, $abcd = acbd$. A left (right) permutable ring is medial but not conversely. A unital medial ring is indistinguishable from a commutative ring. A ring R is left self distributive, denoted by LSD (resp. right self distributive, denoted by RSD) if for all $a, b, c, d \in R$, the identity: $abc = abac$ (resp. $abc = acbc$) holds. LSD rings are left permutable, see [34, Corollary 2.2]. Left (right) permutable rings and medial rings exist in abundance; according to Birkenmeier and Heatherly in [20, p.258], they are a special type of PI-rings and also exist as special subrings of every ring. Furthermore, if R is a non-commutative medial (left permutable, right permutable or permutable) ring, then the ring of polynomials (resp. formal power series or formal Laurent series) over R is a medial (left permutable, right permutable or permutable) ring which is not commutative, see [20, p.262-263].

Theorem 3.5.2 *If P is a classical semiprime submodule of ${}_RM$ and R is a medial (left permutable, right permutable or LSD) ring then each of the following statements implies P is a completely semiprime submodule of ${}_RM$:*

1. *M is finitely generated,*
2. *M is free,*
3. *M is cyclic.*

Proof: We prove only the case for M cyclic, the proofs for other cases are similar. Suppose $a^2m \in P$ for $a \in R$ and $m \in M$, $R^2a^2m \subseteq P$. R medial implies $RaRam \subseteq P$. Since M is cyclic, $m = rm_0$ for some $r \in R$ and $m_0 \in M$. $RaRarm_0 \subseteq P$ and $R^2aRarm_0 \subseteq P$. Again, R medial leads to $RaRaRm \subseteq P$. It follows that $RaRa\langle m\rangle \subseteq P$. Since P is classical semiprime, $Ra\langle m\rangle \subseteq P$, i.e., $Ra \subseteq (P : \langle m\rangle)$. P classical semiprime implies $(P : \langle m\rangle)$ is a semiprime ideal of R and hence $a \in (P : \langle m\rangle)$, i.e., $a\langle m\rangle \subseteq P$. ■

Corollary 3.5.2 *If P is a prime (resp. classical prime) submodule of ${}_RM$ with R medial (left permutable, right permutable or LSD), then each of the* *following statements implies P is completely prime (resp. classical completely prime).*

1. *M is finitely generated,*
2. *M is free,*
3. *M is cyclic.*

Proof: Suppose P is prime (resp. classical prime) then P is classical semiprime. By Theorem 3.5.2, P is completely semiprime. It follows from Theorem 3.4.2 (resp. Theorem 3.4.3) that P is completely prime (resp. classical completely prime). ∎

3.6 Intersections of "completely prime" submodules

Proposition 3.6.1 *For any module ${}_RM$, $\beta_{co}(M)$ and $\beta^c_{cl}(M)$ are completely semiprime submodules.*

Proof: Let $a^2m \in \beta_{co}(M)$ for $a \in R$ and $m \in M$, then $a^2m \in P$ for all completely prime submodules P of M. Since any completely prime submodule is completely semiprime, we have $a\langle m\rangle \subseteq P$ for all completely prime submodules P of M. Hence, $a\langle m\rangle \subseteq \beta_{co}(M)$ and $\beta_{co}(M)$ is completely semiprime. A similar proof shows that $\beta^c_{cl}(M)$ is also a completely semiprime submodule. ∎

Remark 3.6.1 If P is a completely semiprime submodule of ${}_RM$, then P need not be an intersection of completely prime submodules. We know that over a commutative ring, prime is the same as completely prime and semiprime is the same as completely semiprime, see Corollary 3.5.1. Now, let $R = \mathbb{Z}[x]$ and

$F = R \oplus R$. If $f := (2, x) \in F$ and $P = 2R + Rx$ which is a maximal ideal of R, then $N = Pf$ is a semiprime submodule of F which is not an intersection of prime submodules, see [46, p.3600].

Question 3.6.1 Is every completely semiprime submodule of a module an intersection of classical completely prime submodules?

3.7 The radicals $\beta^c_{cl}(M)$ and $\beta_{co}(R)$

Let $\mathcal{M}^c$ be the class of all completely prime rings, i.e., rings which have no non-zero divisors. Then $\mathcal{M}^c_R$ is the class of all classical completely prime R-modules. We have $\mathcal{R}_c = \mathcal{N}_g$, the generalized nil radical which we shall call the completely prime radical of R (denoted by $\beta_{co}(R)$) with

$$\beta_{co}(R) := \cap\{I \triangleleft R \ : \ I \text{ is a completely prime ideal}\}.$$

The corresponding classical completely prime radical for the R-module M will be denoted by

$$\beta^c_{cl}(M) := \cap\{N \leq M : M/N \in \mathcal{M}^c_R\}.$$

Since each classical completely prime submodule of an R-module M is also classical prime submodule, we have $\beta_{cl}(M) \subseteq \beta^c_{cl}(M)$ where $\beta_{cl}(M)$ is the classical prime radical (the intersection of all classical prime submodules of M). If M is an R-module over a commutative ring, then the two radicals coincide.

Proposition 3.7.1 *For any ring R, $\beta^c_{cl}({}_RR) \subseteq \beta^c_{co}({}_RR) \subseteq \beta_{co}(R)$.*

Proof: Follows from Lemma 2.3.1 and the fact that any completely prime module is classical completely prime. ∎

Lemma 3.7.1 *For any R-module M, we have*

$$\beta_{co}(R) \subseteq (\beta^c_{cl}(M) : M)_R \subseteq (\beta_{co}(M) : M)_R.$$

Proof: We have $(\beta^c_{cl}(M) : M) = (\bigcap_{S \leq M} S : M)_R = \bigcap_{S \leq M} (S : M)_R$ where M/S is a classical completely prime module. Since $(S : M)_R$ is a completely prime ideal, we get $\beta_{co}(R) \subseteq (\beta^c_{cl}(M) : M)$. The next inclusion holds because $\beta^c_{cl}(M) \subseteq \beta_{co}(M)$. ■

Remark 3.7.1 The containment in Lemma 3.7.1 is in general strict. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ for some prime number p. Now $\beta^c_{cl}(M) = \mathbb{Z}_{p^\infty}$ and $\beta_{co}(R) = 0$, i.e., $\beta_{co}(R)M = (0)$.

Lemma 3.7.2 *For any ring R, we have*

$$\beta_{co}(R) = (\beta^c_{cl}({}_RR) : R)_R = (\beta_{co}({}_RR) : R)_R.$$

Proof: Follows from [29, Proposition 4.6]. ■

Recall that for an R-module M, we have the Jacobson radical $\text{Rad}(M)$ of the module M defined as:

$$\text{Rad}(M) = \cap\{K \leq M \ : \ K \text{ is a maximal submodule of } M\}.$$

Theorem 3.7.1 *Let M be a module over a left Artinian ring R. Then*

$$Rad(M) \subseteq \beta^c_{cl}(M) \text{ and } Rad({}_RR) = \beta^c_{cl}({}_RR) = \beta_{co}({}_RR).$$

Proof: From [18, Cor. 4.3.17, p.178], $\text{Rad}(M) = \text{Jac}(R)M = \beta_{co}(R)M$ and from the fact that $\beta_{co}(R) \subseteq (\beta^c_{cl}(M) : M)_R$ we get $\text{Rad}(M) \subseteq \beta^c_{cl}(M)$. Again from [18, Cor. 4.3.17, p.178], and Lemma 3.7.2, $\text{Rad}({}_RR) = \text{Jac}(R)R = \beta_{co}(R)R = \beta^c_{cl}({}_RR) = \beta_{co}({}_RR)$. ■

Chapter 4

s-prime submodules

In this chapter, a generalization of the notion of an s-system of rings to modules is given. Let N be a submodule of M. Define $\mathcal{S}(N) := \{m \in M$: every s-system containing m meets $N\}$. It is shown that $\mathcal{S}(N)$ is equal to the intersection of all s-prime submodules of M containing N. We define $\mathcal{U}(M) = \mathcal{S}(0)$. This is called (Köthe's) upper nil radical of M. We show that if R is a commutative ring, then $\mathcal{U}(M) = \beta(M)$ where $\beta(M)$ denotes the prime radical of M. We also show that if R is a left Artinian ring, then $\beta(M) = \mathcal{U}(M) = \text{Rad}(M) = \text{Jac}(R)M$ where $\text{Rad}(M)$ denotes the Jacobson radical of M and $\text{Jac}(R)$ the Jacobson radical of the ring R. Furthermore, we show that the class of all s-prime modules ${}_RM$ for which $RM \neq 0$ forms a special class of modules.

Behboodi in [10] and [11] generalized a notion of (Baer's) lower nil radical to modules. We here generalize a notion of (Köthe's) upper nil radical to modules. s-prime ideals in rings were introduced in [78], where it was pointed out that the intersection of all s-prime ideals in a ring R is the upper nil radical of R. In this chapter, we generalize the notion of s-prime ideals in rings to submodules in modules. An ideal $\mathcal{P}$ of a ring R is an s-prime ideal if for any

$\mathcal{A}, \mathcal{B} \triangleleft R$ and any $x \in \mathcal{AB}$ such that $x^m \in \mathcal{P}$ for some $m \in \mathbb{N}$, then $\mathcal{A} \subseteq \mathcal{P}$ or $\mathcal{B} \subseteq \mathcal{P}$. A set S of elements of a ring R is called an s-system [78] if $S = \emptyset$ or S contains a multiplicative system S^* such that for every element $s \in S$, we have $(s) \cap S^* \neq \emptyset$, where $\emptyset$ is the void set and (s) the ideal of R generated by $s \in R$. An ideal $\mathcal{P}$ of a ring R is an s-prime ideal if its complement $C_R(\mathcal{P})$ is an s-system. An ideal $\mathcal{P}$ of a ring R is semi s-prime if and only if for any $\mathcal{A} \triangleleft R$ and $x \in \mathcal{A}$ such that $x^m \in \mathcal{P}$ for some $m \in \mathbb{N}$, then $\mathcal{A} \subseteq \mathcal{P}$. $\mathcal{P}$ is an s-prime ideal of R if and only if $\mathcal{P}$ is both prime and semi s-prime ideal of R. s-prime for rings has been referred to as strongly prime by some authors, e.g., see [70, Definition 2.6.5, p.170].

4.1 s-prime modules

Definition 4.1.1 A proper submodule P of M with $RM \nsubseteq P$ is called an s-prime submodule if the following is satisfied: If $\mathcal{A} \triangleleft R$ and $N \leq M$ and for every $x \in \mathcal{A}$ there exists $n \in \mathbb{N}$ such that $x^n N \subseteq P$, then $N \subseteq P$ or $\mathcal{A}M \subseteq P$. An R-module M is s-prime if the zero submodule of M is an s-prime submodule of M. In general, an R-module M/P is an s-prime module if and only if P is an s-prime submodule of M.

Example 4.1.1 Every simple module is an s-prime module.

Proof: Suppose $N \neq 0$ and $\mathcal{A}M \neq 0$ for some $\mathcal{A} \triangleleft R$ and $N \leq M$. Since M is simple, we can find an element $m \in M$ such that $\mathcal{A}m$ is a non-zero submodule and as such $\mathcal{A}m = M$. Hence, we can get an element $a \in \mathcal{A}$ such that $am = m$. This gives $a^k m = m$ not zero for any positive integer k. Hence, since $N \neq 0$ we have $N = M$ and consequently $a^k m$ is an element of $a^k N$, i.e., $a^k N \neq 0$ for every $k \in \mathbb{N}$. ∎

Remark 4.1.1 Every maximal submodule is an s-prime submodule.

Proposition 4.1.1 *The following statements are equivalent:*

1. *A proper submodule P of M is an s-prime submodule of M.*

2. (a) *P is a prime submodule of M and;*

   (b) *for every $\mathcal{A} \lhd R$ such that $\mathcal{A} \not\subseteq (P : M)$ there exists $a \in \mathcal{A} \setminus (P : M)$ such that $a^n M \not\subseteq P$ for all $n \in \mathbb{N}$.*

Proof: ($\Rightarrow$) (a). Let $\mathcal{A} \lhd R$, $N \leq M$ such that $\mathcal{A}N \subseteq P$. For all $x \in \mathcal{A}$, $xN \subseteq P$. Since P is s-prime, $N \subseteq P$ or $\mathcal{A}M \subseteq P$ and P is a prime submodule of M.
(b). Let $\mathcal{B} \lhd R$ such that $\mathcal{B} \not\subseteq (P : M)$, i.e., $\mathcal{B}M \not\subseteq P$. Let $m \in M$ such that $\mathcal{B}m \not\subseteq P$. Now, since P is an s-prime submodule, there exists $b \in \mathcal{B}$ such that $b^n\mathcal{B}m \not\subseteq P$ for all $n \in \mathbb{N}$. Hence $b^n M \not\subseteq P$ for every $n \in \mathbb{N}$.
($\Leftarrow$). Let $\mathcal{A} \lhd R$ and $N \leq M$ such that $N \not\subseteq P$ and $\mathcal{A}M \not\subseteq P$. Let $n \in N$ such that $n \notin P$. Now, since P is prime, we have $(P : M) = (P : \langle n \rangle)$ is a prime ideal. From (b), there exists $a \in \mathcal{A}$ such that $a^k M \not\subseteq P$ for every $k \in \mathbb{N}$. Hence, $a^k \notin (P : M) = (P : \langle n \rangle)$ for every $k \in \mathbb{N}$. Thus, $a^k\langle n \rangle \not\subseteq P$ and $a^k N \not\subseteq P$ for every $k \in \mathbb{N}$, so P is an s-prime submodule of M. ■

Corollary 4.1.1 *For any ring R, P is an s-prime submodule of M if and only if P is a prime submodule of M and $\mathcal{U}(R/(P : M)_R) = (0)$, where $\mathcal{U}(R)$ is the upper nil radical of the ring R.* [1] *In other words, $R/(P : M)$ has no non-zero nil ideals.*

Corollary 4.1.1 gives a perfect generalization of the notion of s-prime for rings (see [70, Definition 2.6.5 p.170]) to modules.

[1] Upper nil radical of a ring is the ideal generated by all nil ideals of that ring. It is the largest nil ideal of the ring.

Proposition 4.1.2 *If R is a unital ring, then R is s-prime if and only if ${}_RR$ is an s-prime module.*

Proof: We know that R is a prime ring if and only if ${}_RR$ is a prime module. R prime implies $(0:R)=0$. Hence, whenever R is prime, $\mathcal{U}(R)=0$ if and only if $\mathcal{U}(R/(0:R))=0$. It follows that: R is prime and $\mathcal{U}(R)=0$ if and only if ${}_RR$ is prime and $\mathcal{U}(R/(0:R))=0$, i.e., R is s-prime if and only if ${}_RR$ is s-prime. ■

Definition 4.1.2 A submodule P of an R-module M is a semi s-prime submodule if for all $\mathcal{A} \triangleleft R$ and every $N \leq M$ with $a \in \mathcal{A}^2$ and $N \not\subseteq P$ such that $a^nN \subseteq P$ for some $n \in \mathbb{N}$, then $\mathcal{A}N \subseteq P$.

Proposition 4.1.3 *The following statements are equivalent:*

1. *P is a semi s-prime submodule of M;*
2. *for all $\mathcal{A} \triangleleft R$ and every $N \leq M$ if $\mathcal{A}N \not\subseteq P$, then there exists $a \in \mathcal{A}^2$ such that $a^nN \not\subseteq P$ for all $n \in \mathbb{N}$;*
3. *$\mathcal{U}(R/(P:N))=0$ for all $N \leq M$ with $N \not\subseteq P$.*

Theorem 4.1.1 *A submodule P of ${}_RM$ is s-prime if and only if P is prime and semi s-prime.*

Proof: Similar to that of Proposition 4.1.1. ■

Proposition 4.1.4 *For any module ${}_RM$,*

1. *$\beta^s(M)=\cap\{P : P \leq M,\ P$ s-prime submodule of $M\}$ is a semi s-prime submodule;*
2. *if P is semi s-prime, then $(P:N)$ is a semi s-prime ideal of R for any $N \leq M$ with $N \not\subseteq P$.*

Proof:

1. Let $\mathcal{A} \lhd R$, $N \leq M$ such that $\mathcal{A}N \not\subseteq \beta^s(M)$, then there exists an s-prime submodule P such that $\mathcal{A}N \not\subseteq P$. By definition of an s-prime submodule, there exists $a \in \mathcal{A}$ such that $a^n N \not\subseteq P$ for all $n \in \mathbb{N}$. Hence, for $b = a^2 \in \mathcal{A}$, we have $b^m N \not\subseteq \beta^s(M)$ for all $m \in \mathbb{N}$.

2. Let $\mathcal{A}^2 \lhd R$ and $a \in \mathcal{A}$ such that $a^n \in (P : N)$ for some $n \in \mathbb{N}$ and $N \leq M$ with $N \not\subseteq P$. Then, $a^n N \subseteq P$ and since P is semi s-prime, $\mathcal{A}N \subseteq P$, i.e., $\mathcal{A} \subseteq (P : N)$. Hence, $(P : N)$ is a semi s-prime ideal of R.

■

Proposition 4.1.5 *Any semi s-prime submodule is classical semiprime.*

Proof: Let P be a semi s-prime submodule of ${}_RM$. Suppose for all $\mathcal{A} \lhd R$ and every $N \leq M$, $\mathcal{A}^2 N \subseteq P$. Then $a^2 N \subseteq P$ for all $a \in \mathcal{A}$. P semi s-prime implies $\mathcal{A}N \subseteq P$ and hence P is classical semiprime.

■

Proposition 4.1.6 *For modules over a commutative ring,*

$$\textit{semi } s\textit{-prime} \Leftrightarrow \textit{semiprime} \Leftrightarrow \textit{classical semiprime.}$$

Proof: Suppose R is commutative and P is classical semiprime. Let $\mathcal{A} \lhd R$ and $\mathcal{A}N \not\subseteq P$ for some $N \leq M$. Then $\mathcal{A}^2 N \not\subseteq P$ and there exists $a \in \mathcal{A}^2$ such that $aN \not\subseteq P$. P semiprime implies $(P : N)$ is semiprime and because R is commutative $(P : N)$ is completely semiprime, hence $a^n \notin (P : N)$ for all $n \in \mathbb{N}$, i.e., $a^n N \not\subseteq P$ for all $n \in \mathbb{N}$ which shows that P is semi s-prime. The rest follows from Proposition 4.1.5 and Corollary 3.5.1. ■

From Proposition 4.1.1, every s-prime module is prime. The following is an example of a prime module which is not s-prime.

Example 4.1.2 We use the construction and computation in [45, Example 1.2 and Proposition 1.3]. Let S be a domain, n be a positive integer and R_n be the 2^n by 2^n upper triangular matrix ring over S. Define a map $\delta : R_n \to R_{n+1}$ by $A \to \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Then, R_n is considered as a subring of R_{n+1} via δ. Let $D_n = \{R_n, \delta_{nm}\}$ (with $\delta_{nm} = \delta^{m-n}$ whenever $n \leq m$) be the direct system over $I = \{1, 2, 3, \cdots\}$ and let $R = \varinjlim R_n$. From [45, Proposition 1.3], R is a prime ring with $\mathcal{U}(R) \neq 0$. Hence, a prime ring which is not an s-prime ring. If we let $M = {}_RR$, then M is a prime module which is not an s-prime module.

The following Lemma is a generalization of [11, Proposition 1.1 (2)]. In [11], it was given for unital modules. We here modify it to suit a not necessarily unital module. In the place of Rm, we use $\langle m \rangle$ and the two coincide when M is unital.

Lemma 4.1.1 *Let M be an R-module. For a proper submodule P of M, the following statements are equivalent:*

1. *P is a prime submodule of M;*
2. *for all $a \in R$ and every $m \in M$, if $(a)m \subseteq P$, then either $m \in P$ or $(a)M \subseteq P$;*
3. *for all $a \in R$ and every $m \in M$, if $a\langle m \rangle \subseteq P$, then either $m \in P$ or $aM \subseteq P$;*
4. *for each left ideal $\mathcal{A}_l$ of R and every $m \in M$, if $\mathcal{A}_l\langle m \rangle \subseteq P$, then either $m \in P$ or $\mathcal{A}_lM \subseteq P$;*

5. *for each right ideal $\mathcal{A}_r$ of R and every $m \in M$, if $\mathcal{A}_r m \subseteq P$, then either $m \in P$ or $\mathcal{A}_r M \subseteq P$;*

6. *$\mathcal{P} = (P : M)$ is a prime ideal of R, and $(P : \langle m \rangle) = (\bar{0} : \langle \bar{m} \rangle) = \mathcal{P}$ for each $m \in M \setminus P$;*

7. *the set $\{(P : \langle m \rangle) \ : \ m \in M \setminus P\}$ is a singleton.*

Remark 4.1.2 If R is a commutative ring and M an R-module, each prime submodule P of M is an s-prime submodule of M.

Proof: Let P be a prime submodule of M and $a^n \langle m \rangle \subseteq P$ for $n \in \mathbb{N}$, $a \in R$ and $m \in M$. If $m \in P$, we are through. Suppose $m \notin P$. Since P is prime, we have $a^n M \subseteq P$ by Lemma 4.1.1 (3). Hence, $a^n \in (P : M)$. But $(P : M)$ is a prime ideal of a commutative ring, thus $a \in (P : M)$ and $aM \subseteq P$. ■

Remark 4.1.3 Where as we have seen in Proposition 4.1.4 that any intersection of s-prime submodules is a semi s-prime submodule, the converse does not hold in general. For over a commutative ring, prime is the same as s-prime (see Remark 4.1.2) and semiprime is the same as semi s-prime, (see Proposition 4.1.6). In [46, p.3600] an example of a module over a commutative ring was constructed (also given in Remark 3.6.1) to show that semiprime submodules need not be intersections of prime submodules.

Proposition 4.1.7 offers several other characterizations of s-prime modules.

Proposition 4.1.7 *Let M be an R-module. For a proper submodule P of M, the following statements are equivalent:*

1. *P is an s-prime submodule of M;*

2. *for every $a \in R$ and for every $m \in M$, if $a^n \langle m \rangle \subseteq P$ for some $n \in \mathbb{N}$, then $m \in P$ or $aM \subseteq P$;*

3. *P is a prime submodule of M and for every $\mathcal{A} \triangleleft R$ with $\mathcal{A} \not\subseteq (P:M)$ there exists $a \in R$ such that $a^n M \not\subseteq P$ for every $n \in \mathbb{N}$;*

4. *$\mathcal{P} = (P:M)$ is an s-prime ideal of R and $(\bar{0} : \langle \bar{m} \rangle) = (P : \langle m \rangle) = \mathcal{P}$ for every $m \in M \setminus P$.*

Proof: (1) $\Rightarrow$ (2). Suppose P is an s-prime submodule of M and for all $a \in R$ and every $m \in M$ $a^n \langle m \rangle \subseteq P$ for some $n \in \mathbb{N}$. Now, $a \in (a) \triangleleft R$ and $\langle m \rangle \leq M$. By (1), $m \in \langle m \rangle \subseteq P$ or $aM \subseteq (a)M \subseteq P$.
(2) $\Rightarrow$ (1). Suppose there exists $\mathcal{A} \triangleleft R$, $N \leq M$ such that $N \not\subseteq P$ and $\mathcal{A}M \not\subseteq P$. Then there exists $m \in N \setminus P$ and $aM \not\subseteq P$ for some $a \in \mathcal{A}$. By (2), $a^n \langle m \rangle \not\subseteq P$ for all $n \in \mathbb{N}$. Thus, $a^n N \not\subseteq P$ for all $n \in \mathbb{N}$ and P is an s-prime submodule of M.
(1) $\Rightarrow$ (3). Follows from Proposition 4.1.1.
(3) $\Rightarrow$ (4). Suppose $\mathcal{A}, \mathcal{B}$ are ideals of R such that $\mathcal{A} \not\subseteq (P:M)$ and $\mathcal{B} \not\subseteq (P:M)$. By (3), P is a prime submodule of M. Thus, $(P:M)$ is a prime ideal and $\mathcal{AB} \not\subseteq (P:M)$. By (3), there exists $a \in \mathcal{AB} \subseteq R$ such that $a^n \notin (P:M)$ for all $n \in \mathbb{N}$ and therefore $(P:M)$ is an s-prime ideal of R. By Lemma 4.1.1 (6), $\mathcal{P} = (P : \langle m \rangle) = (\bar{0} : \langle \bar{m} \rangle)$ for every $m \in M \setminus P$.
(4) $\Rightarrow$ (1). Suppose there exists $\mathcal{A} \triangleleft R$ and $N \leq M$ such that $N \not\subseteq P$ and $\mathcal{A}M \not\subseteq P$. Then there exists $m \in N \setminus P$ and $\mathcal{A} \not\subseteq (P:M)$. By (4), $\mathcal{A} \not\subseteq (P : \langle m \rangle)$. Since $(P : \langle m \rangle)$ is a semi s-prime ideal of R, we have $x^n \notin (P : \langle m \rangle)$ for all $n \in \mathbb{N}$, i.e., $x^n \langle m \rangle \not\subseteq P$ and $x^n N \not\subseteq P$ for all $n \in \mathbb{N}$. ∎

Proposition 4.1.8 *For any module ${}_R M$, completely prime $\Rightarrow$ s-prime.*

Proof: Suppose $a^n \langle m \rangle \subseteq P$ for some $n \in \mathbb{N}$. Then, $a^n m \in P$. Because P is completely prime, it is classical completely prime such that $am \in a\langle m \rangle \subseteq P$. By definition of completely prime we have $aM \subseteq P$ or $m \in P$. ∎

Example 4.1.3 A simple module is s-prime but it need not be completely prime.

4.2 s-systems and s-prime radical

In [10], Behboodi introduced the notion of an m-system for modules and used it to characterize the prime radical for a module, cf., [10, Definition 1.3]. We now introduce a notion of an s-system as a generalization of an m-system and then use it to characterize the upper nil radical for modules.

Definition 4.2.1 Let R be a ring and M an R-module. A nonempty set $S \subseteq M \setminus \{0\}$ is called an s-system if, for each $\mathcal{A} \lhd R$ and for all $K, L \leq M$, if $(K + L) \cap S \neq \emptyset$ and $(K + \mathcal{A}M) \cap S \neq \emptyset$, then there exists $x \in \mathcal{A}$ such that $(K + x^n L) \cap S \neq \emptyset$ for every $n \in \mathbb{N}$.

It is easy to see that every s-system is an m-system.

Corollary 4.2.1 *Let M be an R-module. A submodule P of M is s-prime if and only if $M \setminus P$ is an s-system of M.*

Proof: $(\Rightarrow)$. Suppose $S := M \setminus P$. Let $\mathcal{A} \lhd R$ and $K, L \leq M$ such that $(K + L) \cap S \neq \emptyset$ and $(K + \mathcal{A}M) \cap S \neq \emptyset$. If there exists $n \in \mathbb{N}$ such that $(K + x^n L) \cap S = \emptyset$ for every $x \in \mathcal{A}$, then $x^n L \subseteq P$ for every $x \in \mathcal{A}$. Since P is s-prime, we have $L \subseteq P$ or $\mathcal{A}M \subseteq P$. It follows that $(L + K) \cap S = \emptyset$ or $(\mathcal{A}M + K) \cap S = \emptyset$, a contradiction. Therefore, S is an s-system of M.
$(\Leftarrow)$. Suppose that for every $\mathcal{A} \lhd R$, $N \leq M$ and $x \in \mathcal{A}$ there exists $n \in \mathbb{N}$ such that $x^n L \subseteq P$. If $L \nsubseteq P$ and $\mathcal{A}M \nsubseteq P$, then $L \cap S \neq \emptyset$ and $\mathcal{A}M \cap S \neq \emptyset$. Hence, there exists $a \in \mathcal{A}$ such that $a^n L \cap S \neq \emptyset$ for every $n \in \mathbb{N}$. Hence, there exists $a \in \mathcal{A}$ such that $a^n L \nsubseteq P$ for every $n \in \mathbb{N}$, a contradiction. Therefore, P is an s-prime submodule of M. ■

Proposition 4.2.1 *Let M be an R-module, P a proper submodule of M and $S := M \setminus P$. Then, the following statements are equivalent:*

1. *P is an s-prime submodule of M;*

2. *S is an s-system of M;*

3. *for every $\mathcal{A} \lhd R$ and for all $N \leq M$, if $N \cap S \neq \emptyset$ and $\mathcal{A}M \cap S \neq \emptyset$, then there exists $a \in \mathcal{A}$ such that $a^n N \cap S \neq \emptyset$ for every $n \in \mathbb{N}$;*

4. *for every $a \in R$ and every $m \in M$, if $\langle m \rangle \cap S \neq \emptyset$ and $aM \cap S \neq \emptyset$, then $a^n \langle m \rangle \cap S \neq \emptyset$ for all $n \in \mathbb{N}$.*

Proof: (1) $\Leftrightarrow$ (2) follows from Corollary 4.2.1.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clear.
(4) $\Rightarrow$ (1). Suppose $a \in R$ and $m \in M$ such that $a^n \langle m \rangle \subseteq P$ for some $n \in \mathbb{N}$. Suppose $m \notin P$ and $aM \nsubseteq P$. Then, $\langle m \rangle \cap S \neq \emptyset$ and $aM \cap S \neq \emptyset$ and from (4), $a^t \langle m \rangle \cap S \neq \emptyset$ for all $t \in \mathbb{N}$. Hence also $a^n \langle m \rangle \cap S \neq \emptyset$, i.e., $a^n \langle m \rangle \nsubseteq P$ a contradiction. From Proposition 4.1.7 (2), P is an s-prime submodule of M. ■

Lemma 4.2.1 *Let M be an R-module, $S \subseteq M$ an s-system and P a submodule of M maximal with respect to the property that $P \cap S = \emptyset$. Then, P is an s-prime submodule of M.*

Proof: Suppose $\mathcal{A} \lhd R$ and $L \leq M$ such that for every $a \in \mathcal{A}$, $a^n L \subseteq P$ for some $n \in \mathbb{N}$. If $L \nsubseteq P$ and $\mathcal{A}M \nsubseteq P$ then $(L+P) \cap S \neq \emptyset$ and $(\mathcal{A}M+P) \cap S \neq \emptyset$. Since S is an s-system, there exists $b \in \mathcal{A}$ such that $(b^k L + P) \cap S \neq \emptyset$ for every $k \in \mathbb{N}$. Since $b^t L \subseteq P$ for some $t \in \mathbb{N}$, we have $P \cap S \neq \emptyset$, a contradiction. Hence, P must be an s-prime submodule. ■

Definition 4.2.2 Let R be a ring and M an R-module. For $N \leq M$, if there is an s-prime submodule containing N, then we define

$$\mathcal{S}(N) := \{m \in M \ : \ \text{every } s\text{-system of } M \text{ containing } m \text{ meets } N\}.$$

We write $\mathcal{S}(N) = M$ whenever there is no s-prime submodules of M containing N.

Theorem 4.2.1 *Let M be an R-module and $N \leq M$. Then, either $\mathcal{S}(N) = M$ or $\mathcal{S}(N)$ equals the intersection of all s-prime submodules of M containing N.*

Proof: Suppose $\mathcal{S}(N) \neq M$. This means

$$\beta^s(N) := \cap\{P \ : \ P \text{ is an } s\text{-prime submodule of } M \text{ and } N \subseteq P\} \neq \emptyset.$$

Both $\mathcal{S}(N)$ and N are contained in the same s-prime submodules. By definition of $\mathcal{S}(N)$ it is clear that $N \subseteq \mathcal{S}(N)$. Hence, any s-prime submodule of M which contains $\mathcal{S}(N)$ must necessarily contain N. Suppose P is an s-prime submodule of M such that $N \subseteq P$, and let $t \in \mathcal{S}(N)$. If $t \notin P$, then the complement of P, $C(P)$ in M is an s-system containing t and therefore we would have $C(P) \cap N \neq \emptyset$. However, since $N \subseteq P$, $C(P) \cap P = \emptyset$ and this contradiction shows that $t \in P$. Hence $\mathcal{S}(N) \subseteq P$ as we wished to show. From this we have $\mathcal{S}(N) \subseteq \beta^s(N)$. Conversely, assume $s \notin \mathcal{S}(N)$, then there exists an s-system S such that $s \in S$ and $S \cap N = \emptyset$. From Zorn's lemma, there exists a submodule $P \supseteq N$ which is maximal with respect to $P \cap S = \emptyset$. From Lemma 4.2.1, P is an s-prime submodule of M and $s \notin P$, as desired. ∎

Proposition 4.2.2 *Let R be a ring and $\mathcal{P} \lhd R$, $\mathcal{P} \neq R$. The following statements are equivalent:*

1. *$\mathcal{P}$ is an s-prime ideal of R;*

2. *there exists an s-prime R-module M such that $\mathcal{P} = (0 : M)_R$.*

Proof: (1) $\Rightarrow$ (2). Let $\mathcal{P}$ be an s-prime ideal and let $M = R/\mathcal{P}$. M is an R-module with the usual operation. Let $p \in \mathcal{P}$ and $x \in R$. Then, $p(x + \mathcal{P}) = px + \mathcal{P} = \mathcal{P}$. Hence, $\mathcal{P} \subseteq (0 : M)_R$. Now suppose $a \in (0 : M)_R$.

Then $a(r+\mathcal{P}) = \mathcal{P}$ for all $r \in R$, hence $aR \subseteq \mathcal{P}$ and since $\mathcal{P}$ is an s-prime ideal we get $a \in \mathcal{P}$, hence $(0 : M)_R = \mathcal{P}$ and M is an s-prime module.
(2) $\Rightarrow$ (1). Follows from Proposition 4.1.7. ∎

Corollary 4.2.2 *A ring R is an s-prime ring if and only if there exists a faithful s-prime R-module.*

Example 4.2.1 If R is a domain, then ${}_RR$ is a faithful s-prime module since every domain is an s-prime ring.

Throughout the remaining part of this section rings have unity and all modules are unital left modules.

For any module M, we define the upper nil radical $\mathcal{U}(M)$ as $\mathcal{S}(0)$, i.e.,

$$\mathcal{S}(0) := \{m \in M, \text{ every } s\text{-system in } M \text{ which contains } m \text{ also contains } 0\}.$$

From Theorem 4.2.1, we have

$$\mathcal{U}(M) = \cap\{K \ : \ K \leq M, M/K \text{ is } s\text{-prime}\}$$

which is a radical by Proposition 2.3.1 since s-prime modules are closed under taking non-zero submodules.

Proposition 4.2.3 *For any R-module M,*

1. *$\mathcal{U}(\mathcal{U}(M)) = \mathcal{U}(M)$, i.e., $\mathcal{U}$ is idempotent;*
2. *$\mathcal{U}(M)$ is a characteristic submodule of M;*
3. *If M is projective then $\mathcal{U}(R)M = \mathcal{U}(M)$.*

Proof: Follows from [19, Proposition 1.1.3]. ∎

Proposition 4.2.4 *For any $M \in R$-mod,*

1. *if $M = \bigoplus_{\Lambda} M_\lambda$ is a direct sum of submodules $M_\lambda (\lambda \in \Lambda)$, then*

$$\mathcal{U}(M) = \bigoplus_{\Lambda} \mathcal{U}(M_\lambda);$$

2. *if $M = \prod_{\Lambda} M_\lambda$ is a direct product of submodules $M_\lambda (\lambda \in \Lambda)$, then*

$$\mathcal{U}(M) \subseteq \prod_{\Lambda} \mathcal{U}(M_\lambda).$$

Proof: Follows from [19, Proposition 1.1.2]. ■

Theorem 4.2.2 *If R is a commutative ring and M an R-module, then*

$$\beta(M) = \mathcal{U}(M).$$

Proof: If R is commutative, every prime submodule is s-prime, see Remark 4.1.2. ■

Remark 4.2.1 In general, $\mathcal{U}(R)M \subset \mathcal{U}(M)$ even over a commutative ring. For let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$. Since R is commutative, $\mathcal{U}(M) = \beta(M)$. From [11, Example 3.4], we have $\beta(M) = \mathbb{Z}_{p^\infty}$. But $\beta(R) = \mathcal{U}(R) = 0$. Hence, $0 = \mathcal{U}(R)M \subset \mathcal{U}(M) = \mathbb{Z}_{p^\infty}$.

Theorem 4.2.3 *If M is a module over a left Artinian ring R, then*

$$\mathcal{U}(M) = \mathit{Rad}(M) = \mathit{Jac}(R)M.$$

Proof: Since every maximal submodule is s-prime, we have $\mathcal{U}(M) \subseteq \mathrm{Rad}(M) = \mathrm{Jac}(R)M$. Since R is left Artinian $\mathcal{U}(R) = \mathrm{Jac}(R)$. Hence, $\mathcal{U}(M) \subseteq \mathrm{Rad}(M) = \mathrm{Jac}(R)M = \mathcal{U}(R)M \subseteq \mathcal{U}(M)$. ■

4.3 Upper nil radical of the module $_RR$ and the ring R

Lemma 4.3.1 *For any associative ring R, $\mathcal{U}(_RR) \subseteq \mathcal{U}(R)$.*

Proof: Let $x \in \mathcal{U}(_RR)$ and I be an s-prime ideal of R. From Proposition 4.2.2, we have R/I is an s-prime R-module. Hence, $x \in I$ and we have $x \in \mathcal{U}(R)$, i.e., $\mathcal{U}(_RR) \subseteq \mathcal{U}(R)$. ∎

Remark 4.3.1 In general the containment in Lemma 4.3.1 is strict.

Example 4.3.1 Let $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{Z}_2 \right\}$ and $M = {_RR}$. It is easy to check that (0) is an s-prime submodule of $_RR$. Hence, $\mathcal{U}(_RR) = 0$. Now, we have $(0 : R)_R$ is an s-prime ideal of R, $(0 : R)_R \neq (0)$. For if $b \neq 0$, $b \in \mathbb{Z}_2$, then $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R = 0$. Hence, $\mathcal{U}(R) \subseteq (0 : R)_R$. But since $(0 : R)_R(0 : R)_R = 0$ we have $(0 : R)_R \subseteq \mathcal{U}(R)$. Hence, $\mathcal{U}(R) = (0 : R)_R \neq 0$.

Lemma 4.3.2 *For any ring and any R-module M we have*

$$\mathcal{U}(R) \subseteq (\mathcal{U}(M) : M)_R.$$

Proof: We have $(\mathcal{U}(M) : M)_R = (\bigcap_{S \leq M} S : M) = \bigcap_{S \leq M} (S : M)$, where S is an s-prime submodule of M. Since $(S : M)_R$ is an s-prime ideal of R for each s-prime submodule S of M, we get $\mathcal{U}(R) \subseteq (\mathcal{U}(M) : M)_R$, i.e., $\mathcal{U}(R)M \subseteq \mathcal{U}(M)$.
∎

The containment is strict: let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ for some prime number p. $\mathcal{U}(M) = \mathbb{Z}_{p^\infty}$ and $\mathcal{U}(R) = (0)$, i.e., $\mathcal{U}(R)M = (0)$.

Proposition 4.3.1 *For any ring R, $\mathcal{U}(R) = (\mathcal{U}({}_RR) : R)_R$.*

Proof: From Lemma 4.3.2, $\mathcal{U}(R) \subseteq (\mathcal{U}({}_RR) : R)_R$. Since $\mathcal{U}({}_RR) \subseteq \mathcal{U}(R)$ we have $\mathcal{U}(R) \subseteq (\mathcal{U}({}_RR) : R) \subseteq (\mathcal{U}(R) : R)$. Let $x \in (\mathcal{U}(R) : R)$. Hence $xR \subseteq \mathcal{U}(R) = \bigcap_{\mathcal{P}\ s\text{-prime in } R} \mathcal{P} \subseteq \mathcal{P}$ for all s-prime ideals $\mathcal{P}$ of R. Since $xR \subseteq \mathcal{P}$ for $\mathcal{P}$ s-prime, we have $x \in \mathcal{P}$ and $x \in \mathcal{U}(R)$. Hence, $(\mathcal{U}(R) : R) \subseteq \mathcal{U}(R)$ and we are done. ■

Proposition 4.3.2 *For all R-modules M,*

1. *$\mathcal{U}(M) = \{x \in M \ : \ Rx \subseteq \mathcal{U}(M)\}$.*
2. *If $\mathcal{U}(R) = R$ then $\mathcal{U}(M) = M$.*

Proof:

1. Since $\mathcal{U}(M) \leq M$, we have $R\mathcal{U}(M) \subseteq \mathcal{U}(M)$. Conversely, let $x \in M$ with $Rx \subseteq \mathcal{U}(M)$. Hence $Rx \subseteq P$ for all s-prime submodules P of M. Since P is also a prime submodule, we have $x \in P$ and hence $x \in \mathcal{U}(M)$.
2. $R = \mathcal{U}(R)$ gives $R \subseteq (\mathcal{U}(M) : M)$ from Lemma 4.3.2. Hence $RM \subseteq \mathcal{U}(M)$ and from (1), we have $M \subseteq \mathcal{U}(M)$, i.e., $M = \mathcal{U}(M)$.

■

Proposition 4.3.3 *Let R be any ring. Then, any of the following conditions implies $\mathcal{U}(R) = \mathcal{U}({}_RR)$.*

1. *R is commutative;*

2. $x \in xR$ for all $x \in R$, e.g., if R has an identity or R is Von Neumann regular.

Proof:

1. Since R is commutative, it follows from Proposition 4.3.1 and Lemma 4.3.2 that $\mathcal{U}(R) \subseteq \mathcal{U}({}_R R) \subseteq \mathcal{U}(R)$ and $\mathcal{U}(R) = \mathcal{U}({}_R R)$.
2. Let $x \in \mathcal{U}(R)$, then from Proposition 4.3.1, $xR \subseteq \mathcal{U}({}_R R)$ and since $x \in xR$, we get $x \in \mathcal{U}({}_R R)$ such that $\mathcal{U}({}_R R) = \mathcal{U}(R)$.

■

4.4 A special class of s-prime modules

Theorem 4.4.1 *Let R be any ring and*

$$\mathcal{M}_R := \{M \ : \ M \text{ is an } s\text{-prime } R\text{-module with } RM \neq 0\}.$$

If $\mathcal{M} = \cup \mathcal{M}_R$, then $\mathcal{M}$ is a special class of R-modules.

Proof:

S1 Let $M \in \mathcal{M}_R$ and $I \lhd R$ with $IM = 0$. Now M is an R/I-module via $(r+I)m = rm$. We show $M \in \mathcal{M}_{R/I}$. Let $A/I \lhd R/I$, $N \leq M$. If $(x+I) \in A/I$ and $(x+I)^n N = 0$ for some $n \in \mathbb{N}$, then $(x^n+I)N = 0$ such that $x^n N = 0$. Because, M is R s-prime, we have $N = 0$ or $AM = 0$. Thus $N = 0$ or $(A/I)M = 0$. Therefore, M is an R/I s-prime module.

S2 Let $I \lhd R$ and $M \in \mathcal{M}_{R/I}$. M is an R-module w.r.t. $rm = (r+I)m$ for $r \in R$ and $m \in M$. Let $A \lhd R$, $N \leq M$ and $x \in A$ such that $x^n N = 0$ for some $n \in \mathbb{N}$. Then, $(x^n + I)N = 0$ such that $(x+I)^n N = 0$. Since $M \in \mathcal{M}_{R/I}$, we have $N = 0$ or $(a+I)M = 0$ for all $a \in A$., i.e., $N = 0$ or $AM = 0$ and $M \in \mathcal{M}_R$.

S3 Suppose $M \in \mathcal{M}_R$ and $I \lhd R$ with $IM \neq 0$. Let $J \lhd I$ and $N \leq M$, if $x \in J$ and $x^n N = 0$ for some $n \in \mathbb{N}$, then $x \in I$ and because $M \in \mathcal{M}_R$, we have $N = 0$ or $IM = 0$ such that $N = 0$ or $JM = 0$. Therefore, $M \in \mathcal{M}_I$.

S4 Let $M \in \mathcal{M}_R$. Hence $RM \neq 0$. Since $(0 : M)_R$ is an s-prime ideal of R, $R/(0 : M)_R$ is an s-prime ring.

S5 Let $I \lhd R$ and $M \in \mathcal{M}_I$. Since M is an s-prime I-module, $(0 : M)_I$ is an s-prime ideal of I. Now, $(0 : M)_I \lhd I \lhd R$ and $I/(0 : M)_R$ an s-prime ring implies $(0 : M)_I \lhd R$. Choose $K/(0 : M)_I \lhd R/(0 : M)_I$ maximal with respect to $I/(0 : M)_I \cap K/(0 : M)_I = 0$. Then, $I/(0 : M)_I \cong (I + K)/K \lhd \cdot R/K$ by [33, Lemma 3.2.5]. Since $I/(0 : M)_I \lhd \cdot R/K$ and $I/(0 : M)_I$ an s-prime ring R/K is s-prime. Let $N = R/K$. N is an R-module. Clearly, $RN \neq 0$. From Proposition 4.2.2, we have $(0 : N)_R = K$. We show $(0 : N)_I \subseteq (0 : M)_I$. Let $x \in (0 : N)_I$. Then $xR/K = 0$, i.e., $xR \subseteq K$. Now, $xR \subseteq I \cap K$ and from definition of $K/(0 : M)_I$, we have $xR \subseteq I \cap K \subseteq (0 : M)_I$. Hence $xRM = 0$ and since $xIM \subseteq xRM$ we have $xI \subseteq (0 : M)_I$ and $(0 : M)_I$ is a prime ideal of I implies $x \in (0 : M)_I$. Hence, $(0 : N)_I \subseteq (0 : M)_I$.

■

Proposition 4.4.1 *If $\mathcal{M}_s$ is the special class of s-prime modules, then the special radical induced by $\mathcal{M}_s$ on a ring R is $\mathcal{U}$.*

Proof: Let R be a ring. From Proposition 4.2.2, we have

$$\mathcal{U}(R) = \cap\{(0 : M)_R \ : \ M \text{ is an } s\text{-prime } R\text{-module}\}$$

$$= \cap\{I \lhd R \ : \ I \text{ is an } s\text{-prime ideal}\}.$$

■

Chapter 5

Nilpotency of module elements

We define nilpotent and strongly nilpotent elements of a module M and show that the set $\mathcal{N}_s(M)$ of all strongly nilpotent elements of M over a commutative unital ring R coincides with the classical prime radical $\beta_{cl}(M)$ the intersection of all classical prime submodules of M. We also characterize non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$. If B_k is a set of non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$, $B_k^0 = B_k \cup \{0\}$ is a non-unital ring. When considered as a $\mathbb{Z}$-module, B_k^0 is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and $N_p = \lim\limits_{\leftarrow} B_k^0$ is a compact topological ideal of the ring $\mathbb{Z}_p$ of p-adic integers.

A non-zero element a of a ring R is nilpotent if there exists $k \in \mathbb{Z}^+$ such that $a^k = 0$. Let $\mathcal{N}(R)$ denote the set of all nilpotent elements of R. If R is commutative, the set $\mathcal{N}(R)$ is an ideal of R. A ring R is completely semiprime (reduced) if and only if for all $a \in R$, $a^2 = 0 \Rightarrow a = 0$, i.e., if and only if R has no non-zero nilpotent elements. Consider $\neg(a^2 = 0 \Rightarrow a = 0)$ for some $a \in R$ which is logically equivalent to $(a^2 = 0$ and $a \neq 0)$. This translates as R not completely semiprime (R not reduced) is equivalent to there exists a non-zero element a in R which is nilpotent. We use the same approach to define nilpotent elements in modules. A non-zero module M over a commutative ring is semiprime [12] if $a^2m = 0$ implies $am = 0$ for all $a \in R$ and every $m \in M$. If R is commutative, this definition is equivalent to the following statements:

1. for all $a \in R$ and every $m \in M$, if $a^2m = 0$, then $a\langle m\rangle = 0$;

2. for all $a \in R$ and every $N \leq M$, if $a^2N = 0$, then $aN = 0$;

3. for all $a \in R$ and every $m \in M$, if $aRam = 0$, then $am = 0$;

4. for all $\mathcal{A} \lhd R$ and every $N \leq M$, if $\mathcal{A}^2N = 0$, then $\mathcal{A}N = 0$.

However, if R is not commutative all the statements turn out to be different and we have:

Definition 5.0.1 A module M over a ring R is said to be

1. completely semiprime if for all $a \in R$ and every $m \in M$, whenever $a^2m = 0$, $a\langle m\rangle = 0$;

2. weakly completely semiprime if for all $a \in R$ and every $m \in M$, if $a^2m = 0$, then $am = 0$;

3. classical completely semiprime if for all $a \in R$ and every $N \leq M$, whenever $a^2N = 0$, $aN = 0$;

4. semiprime if for all $a \in R$ and every $m \in M$, whenever $aRam = 0$, $am = 0$;

5. classical semiprime for all $\mathcal{A} \lhd R$ and every $N \leq M$, whenever $\mathcal{A}^2N = 0$, $\mathcal{A}N = 0$.

Proposition 5.0.2 shows us where weakly completely semiprime and classical completely semiprime lie in the chart of implications for module "semiprimes".

Proposition 5.0.2 *For any module M over a ring R, we have the following implications:*

$$\begin{array}{ccccc}
\textit{completely} & \Rightarrow & \textit{weakly completely} & \Rightarrow & \textit{classical completely} \\
\textit{semiprime} & & \textit{semiprime} & & \textit{semiprime} \\
\Downarrow & & & & \Downarrow \\
\textit{semiprime} & & \Rightarrow & & \textit{classical semiprime}
\end{array}$$

Proof: Completely semiprime $\Rightarrow$ weakly completely semiprime, see Lemma 3.0.1. Weakly completely semiprime $\Rightarrow$ classical completely semiprime. Let $a^2N = 0$ for $a \in R$ and $N \leq M$. Then $a^2n = 0$ for all $n \in N$. By definition of weakly completely semiprime submodules, $an = 0$ for all $n \in N$ such that $aN = 0$ and M is classical completely semiprime. Classical completely semiprime $\Rightarrow$ classical semiprime. Suppose $\mathcal{A}^2N = 0$ for $\mathcal{A} \lhd R$ and $N \leq M$, then $a^2N = 0$ for all $a \in \mathcal{A}$. By definition of classical completely semiprime submodules, $aN = 0$ for all $a \in \mathcal{A}$ such that $\mathcal{A}N = 0$. For the remaining implications (together with counter examples to show that the converses are not true in general), see chapter three. ∎

Now that we have the definition of weakly completely semiprime modules, we can paraphrase Lemma 3.0.1 as: a unital module M is completely semiprime if and only if M is weakly completely semiprime and IFP.

Example 5.0.1 Any module over a division ring is weakly completely semiprime, for if $a^2m = 0$ and $a \neq 0$, then $am = a^{-1}(a^2m) = 0$. If $a = 0$, the result follows trivially.

Example 5.0.2 A weakly completely semiprime module over a ring which is not right duo need not be completely semiprime. For if $a^2m = 0$ where $a \in R$ and $m \in M$, then $am = 0$ by definition of weakly completely semiprime modules. It follows that $a \in (0 : m)$. But since R is not right duo, aR need not be contained in $(0 : m)$. If $aR \nsubseteq (0 : m)$, $aRm \neq 0$ and $a\langle m\rangle \neq 0$.

Example 5.0.3 A fully faithful module (a module whose all submodules are faithful) over a reduced ring R is classical completely semiprime but it need not be weakly completely semiprime.

Example 5.0.4 Let M be a module over a fully semiprime ring (a ring whose all ideals are semiprime), then M is classical semiprime but it need not be classical completely semiprime. For if $\mathcal{A}^2N = 0$ for $A \lhd R$ and $N \leq M$, $\mathcal{A}^2 \subseteq (0 : N)$. By hypothesis, $\mathcal{A} \subseteq (0 : N)$ and $\mathcal{A}N = 0$. On the other hand, if $a^2N = 0$ with $a \in R$ and $N \leq M$, $a^2 \in (0 : N)$. A semiprime ideal need not be completely semiprime, hence it is possible that $a \notin (0 : N)$ such that $aN \neq 0$ in which case M fails to be classical completely semiprime.

From among the different definitions of "completely semiprime" modules, we get one (Definition 5.0.1(2)) which we think is the most suitable to use when defining a nilpotent element of a module. Our choice is motivated by the fact that Definition 5.0.1(2) involves only module elements (unlike the other definitions) just like its analogue for rings. In a manner similar to that for rings, we say an R-module M is not weakly completely semiprime if there exists a non-zero element m in M which is nilpotent.

5.1 Nilpotent elements of modules

Definition 5.1.1 A non-zero element m of an R-module M is nilpotent of degree k if there exists $a \in R$ and $k \in \mathbb{N} \setminus \{1\}$ such that $a^k m = 0$ and $am \neq 0$.

We take the zero element of a module to be nilpotent.

A left ideal I of a ring R is completely semiprime if $a^2 \in I$ implies $a \in I$ for all $a \in R$. An element m of a module ${}_RM$ is nilpotent if and only if $(0 : m)$ (the annihilator of m) is not a completely semiprime left ideal of R.[1] Thus, every left ideal $(0 : m)$ of R for all $0 \neq m \in M$ is completely semiprime if and only if the module ${}_RM$ is weakly completely semiprime.

Example 5.1.1 $\bar{1}$ is a nilpotent element of every $\mathbb{Z}$-module $\mathbb{Z}/n^k\mathbb{Z}$ for all positive integers k, n greater than 1 since $n^k.\bar{1} = \bar{0}$ but $n = n.\bar{1} \neq \bar{0}$.

Example 5.1.2 If $a \in R$ is nilpotent (with degree $n \geq 3$) in the ring R, then a is nilpotent (with degree $n - 1 \geq 2$) in the module ${}_RR$.

Proof: Suppose a is nilpotent in R. Then, $a^{n-1} \neq 0$ and $a^n = 0$ for some positive integer n. Now, $a^{n-1}a = 0$ and $0 \neq a^2$ since by hypothesis the least degree of nilpotency of a is 3. ∎

Remark 5.1.1 If m is a non-zero nilpotent element of the module ${}_RR$ and a in definition 5.1.1 coincides with m, then m is a nilpotent element of the ring R since $0 = a^k m = a^k a = a^{k+1}$.

Example 5.1.3 The $\mathbb{Z}$-modules $A_k = \mathbb{Z}/p^k\mathbb{Z} = \{0, 1, 2, ..., p^{k-1}\}$ where p is a prime integer and $1 \neq k \in \mathbb{Z}^+$ consist of nilpotent elements.

Proof: If $1 < k$, there exists $m \in \mathbb{Z}/p^k\mathbb{Z}$ such that pm is not a multiple of p^k. Thus, $pm \neq 0$ but $p^k m = 0$. Hence, m is nilpotent in $\mathbb{Z}/p^k\mathbb{Z}$. ∎

Proposition 5.1.1 *The non-nilpotent elements of the $\mathbb{Z}$-module $A_k = \mathbb{Z}/p^k\mathbb{Z}$ where $1 \neq k \in \mathbb{Z}^+$ are $\{p^{k-1}, 2p^{k-1}, 3p^{k-1}, ..., (p-1)p^{k-1}\}$, i.e., they are $p - 1$ in number and are all multiples of p^{k-1}.*

---

[1] A left ideal I of a ring R is completely semiprime if for all $a \in R$, $a^2 \in I$ implies $a \in R$.

Proof: Let $m = np^{k-1}$ where $n \in \mathbb{Z}^+$. Then, $a^k m = 0$ in A_k only when either a or n is a multiple of p. If $a = qp$ for some $q \in \mathbb{Z}^+$, $am = (qp)(np^{k-1}) = nqp^k = 0$. If $n = sp$ for some $s \in \mathbb{Z}^+$, then $am = aspp^{k-1} = asp^k = 0$. Hence, all multiples of p^{k-1} are non-nilpotent in $\mathbb{Z}/p^k\mathbb{Z}$. ∎

Corollary 5.1.1 *The number of nilpotent elements in the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$, $1 < k \in \mathbb{Z}^+$ is $p^k - p + 1$. Moreover, this number is always odd.*

Call a weakly completely semiprime module weakly reduced, then all reduced modules are weakly reduced. Over a commutative ring reduced and weakly reduced modules are indistinguishable. Furthermore, a module M is weakly reduced if it has no non-zero nilpotent elements.

Example 5.1.4 A torsion-free module M over a reduced ring R is reduced (and hence weakly reduced). For if $m \neq 0$ and $a^k m = 0$ for some $a \in R$ and $k \in \mathbb{Z}^+$, then $a^k = 0$ because M is torsion-free and $a = 0$ since R is reduced. Thus, $a\langle m \rangle = 0$. So, modules such as ${}_{\mathbb{R}}\mathbb{R}$, ${}_{\mathbb{Z}}\mathbb{Z}[i]$, ${}_{\mathbb{Z}}\mathbb{C}$, ${}_{\mathbb{Z}}\mathbb{R}$, and ${}_{\mathbb{Z}}\mathbb{Z}$ are reduced.

Example 5.1.5 The $\mathbb{Z}$-modules $\mathbb{Z}/p\mathbb{Z}$ where p is a prime number are reduced. For if $a^2 m = \bar{0} \in \mathbb{Z}/p\mathbb{Z}$, either a or m is a multiple of p such that $am = \bar{0}$. In general, the $\mathbb{Z}$-modules $\mathbb{Z}/(\prod_{i=1}^{n} p_i^{k_i})\mathbb{Z}$ where p_i's are prime numbers, $k_i \in \mathbb{Z}^+$ and $k_i > 1$ for at least one i consist of nilpotent elements and the $\mathbb{Z}$-modules $\mathbb{Z}/(\prod_{i=1}^{n} p_i)\mathbb{Z}$ where the p_i's are distinct prime numbers are reduced.

A finite sum of nilpotent elements of a module M is not necessarily nilpotent in M, even when M is defined over a commutative ring R. $\bar{1}$ and $\bar{3}$ are nilpotent in the $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z}$ since $2^3.\bar{1} = \bar{0}$ but $2.\bar{1} \neq \bar{0}$ and $2^3.\bar{3} = \bar{0}$ but $2.\bar{3} \neq \bar{0}$. Their sum $\bar{4}$ is not nilpotent in the same $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z}$. If m is a nilpotent element of an R-module M (even when R is commutative), rm for $r \in R$ is not in general a nilpotent element of M. $\bar{2}$ is nilpotent in the $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z}$, but the product $2.\bar{2} = \bar{4}$ is not nilpotent in $\mathbb{Z}/8\mathbb{Z}$. Thus, the set $\mathcal{N}(M)$ of all nilpotent elements of an R-module M is in general not a submodule of M even when R is commutative.

One is then led to ask whether there are classes of modules for which the set $\mathcal{N}(M)$ is a submodule of M. The answer is yes. All reduced modules have $\mathcal{N}(M) = 0$ and nil modules (i.e., when $\mathcal{N}(M) = M$) exist, see Example 5.1.6.

Let $M^* = \mathrm{Hom}_R(M, R)$ and $T = \sum\limits_{f \in M^*} \mathrm{Im} f$ be the trace of M in R. M is nondegenerate [49] if $Tm \neq 0$ for every $0 \neq m \in M$.

Example 5.1.6 A nondegenerate module over a nil ring is nil.

Proof: By definition, $Tm \neq 0$ for all $0 \neq m \in M$, i.e., for all $0 \neq m \in M$ there exists $t \in T$ such that $tm \neq 0$. R nil implies $t^k m = 0$ for some $k(t) \in \mathbb{Z}^+$ and hence M is nil. ∎

However, we do not know of a situation where $\mathcal{N}(M)$ is a non-zero proper submodule of M.

Proposition 5.1.2 *Nilpotent elements of modules are preserved by module monomorphisms.*

Proof: Suppose m is nilpotent in ${}_RM$ and $f : M \to N$ is a module monomorphism. There exists $k \in \mathbb{Z}^+ \setminus \{1\}$ and $a \in R$ such that $a^k m = 0$ and $am \neq 0$. Now, $0 = f(0) = f(a^k m) = a^k f(m)$. Because f is injective, $0 \neq f(am) = af(m)$. ∎

We recall that an element a of a ring R is called strongly nilpotent if every sequence $a_1, a_2, a_3, \cdots$ such that $a_1 = a$ and $a_{n+1} \in a_n R a_n$ (for all n) is eventually zero. We generalize this notion to modules.

Definition 5.1.2 An element m of an R-module M is strongly nilpotent if $m = 0$ or for every sequence $a_1, a_2, a_3, \cdots$ where $a_1 = a$ and $a_{n+1} \in a_n R a_n$ for all n, we have $a_k m = 0$ for some $k \in \mathbb{Z}^+ \setminus \{1\}$ and $am \neq 0$.

Proposition 5.1.3 *Every strongly nilpotent element of the module ${}_RM$ with $1 \in R$ is nilpotent but not conversely.*

Proof: If $m = 0$, the result follows trivially. Suppose $m \neq 0$ is strongly nilpotent in ${}_RM$, i.e., for every sequence $a_1, a_2, \cdots$ where $a_1 = a$ and $a_{n+1} \in a_n R a_n$ for all n, we have $a_k m = 0$ for some $k \in \mathbb{Z}^+ \setminus \{1\}$ and $am \neq 0$. Take the sequence $a_1, a_2, a_3, \cdots$ such that $a_1 = a$, $a_2 = a^2$, $a_3 = a^4$, $\cdots a_k = a^{2^{k-1}}$. Then, $a^{2^{k-1}} m = 0$ but $am \neq 0$ such that m is nilpotent. $\bar{1}$ is nilpotent in a $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$, (cf., Example 5.1.1) but it is not strongly nilpotent. To see this, take the sequence $a_1, a_2, a_3, a_4, \cdots = \{4, 4^2, 4^4, 4^8, \cdots\}$. Note that $a_{n+1} \in a_n R a_n$ for all $n \in \mathbb{N}$ and $a_k.\bar{1} = 0$ for all $k \in \mathbb{N}$. Since $am = 4 \times \bar{1} = \bar{0}$, $\bar{1}$ is not strongly nilpotent in $\mathbb{Z}/4\mathbb{Z}$. ∎

Corollary 5.1.2 *If $1 \in R$ and $\mathcal{N}_s(M)$ is the set of all strongly nilpotent elements of M, then $\mathcal{N}_s(M) \subseteq \mathcal{N}(M)$ and this containment is in general strict even when R is commutative.*

Example 5.1.7 If a is a strongly nilpotent element of a ring R such that $am \neq 0$ then m is also a strongly nilpotent element of the module ${}_RM$.

Proposition 5.1.4 *Strongly nilpotent elements of modules are preserved by module monomorphisms.*

Proof: Suppose $0 \neq m \in {}_RM$ and for every sequence $a_1, a_2, a_3, \cdots$ where $a_1 = a$ and $a_{n+1} \in a_n R a_n$ for all n, we have $a_k m = 0$ for some $k \in \mathbb{Z}^+ \setminus \{1\}$ and $am \neq 0$. Let f be a module monomorphism, then $a_k f(m) = f(a_k m) = f(0) = 0$. Since $am \neq 0$ and f is injective, $0 \neq f(am) = af(m)$ and $f(m)$ is strongly nilpotent. ∎

The envelope of the zero submodule of a module ${}_RM$ is the set $E_M(0) := \{m \in M \ : \ m = rn$ and $r^k n = 0$ for some $r \in R, k \in \mathbb{Z}^+, n \in M\}$. Let $\langle E_M(0)\rangle$ and $\langle \mathcal{N}(M)\rangle$ denote the submodules of M generated by $E_M(0)$ and $\mathcal{N}(M)$ respectively. The zero submodule of ${}_RM$ is said to satisfy the radical formula (s.t.r.f) if $\langle E_M(0)\rangle = \beta(M)$.

Theorem 5.1.1 *For a module M over a ring R, we have:*

1. $\langle E_M(0)\rangle \subseteq \langle \mathcal{N}(M)\rangle$;
2. *if R is nil, then* $\langle E_M(0)\rangle = \langle \mathcal{N}(M)\rangle$;
3. *if R is nil, then the zero submodule of ${}_RM$ s.t.r.f if and only if*

$$\beta(M) = \langle \mathcal{N}(M)\rangle.$$

Proof: 1) If $m \in \langle E_M(0)\rangle$, $m = r_1m_1 + \cdots + r_km_k$ for $r_i \in R$ and $m_i \in E_M(0)$ with $k \in \mathbb{Z}^+$ and $1 \leq i \leq k$. Since each $m_i \in E_M(0)$ is of the form $m = s_in_i$ with $s_i \in R$ and $n_i \in \mathcal{N}(M)$. It follows that $m \in \langle \mathcal{N}(M)\rangle$.
2) It is enough to prove the reverse inclusion. Suppose $x \in \langle \mathcal{N}(M)\rangle$, $x = r_1m_1 + \cdots + r_km_k$ for $r \in R$ and $m_i \in \mathcal{N}(M)$ for all $i \in \{1, \cdots, k\}$. R nil implies there exists a positive integer t_i for each $i \in \{1, \cdots, k\}$ such that $r_i^{t_i} m_i = 0$. It follows that $r_im_i \in E_M(0)$ for each i and hence $x \in \langle E_M(0)\rangle$.
3) Follows from 2) above and the definition of submodules that s.t.r.f.

■

5.2 Generalization of Levitzki result of rings to modules

Lemma 5.2.1 *If P is a classical prime submodule of a module M over a commutative ring R and $rm \notin P$ for $r \in R$ and $m \in M$, then $rRrm \not\subseteq P$.*

Proof: Suppose $rRrm \subseteq P$. Then, $rRr\langle m\rangle \subseteq P$. By definition of classical prime submodules, $rm \in r\langle m\rangle \subseteq P$. ■

In the following theorem, we give a module analogue of the Levitzki result for rings, i.e., the set of all strongly nilpotent elements of R coincides with the prime radical of R, cf., [33, Theorem 2.6].

Theorem 5.2.1 *The set $\mathcal{N}_s(M)$ of all strongly nilpotent elements of a module M over a commutative unital ring R coincides with the classical prime radical $\beta_{cl}(M)$.*

Proof: Suppose for $1 = r_1 \in R$ and $m \in M$, $m = r_1m \notin \beta_{cl}(M)$. Then, $m = r_1m \notin P$ for some classical prime submodule P in M. By Lemma 5.2.1, $Rm = r_1Rr_1m \nsubseteq P$. Thus, we can find $r_2 \in r_1Rr_1 = R$ for which $r_2m \notin P$. Again, by Lemma 5.2.1, $r_2Rr_2m \nsubseteq P$. So, there exists $r_3 \in r_2Rr_2$ such that $r_3m \notin P$. By repeating this procedure, we obtain a sequence $r_1, r_2, r_3, \cdots$ such that $r_{n+1} \in r_nRr_n$ but there is no $k \in \mathbb{Z}^+$ for which $r_km = 0$. Therefore, m is not strongly nilpotent. This proves that $\mathcal{N}_s(M) \subseteq \beta_{cl}(M)$. Conversely, suppose m is not strongly nilpotent and an infinite sequence S of non-zero elements r_nm (defined above) exists. Using Zorn's lemma, we may choose a submodule P of M maximal with respect to having $P \cap S = \emptyset$. Suppose $\mathcal{A}, \mathcal{B}$ are ideals of R and N is a submodule of M such that $P \subset \mathcal{A}N$ and $P \subset \mathcal{B}N$. Then $\mathcal{A}N \cap S \neq \emptyset$ and $\mathcal{B}N \cap S \neq \emptyset$. Hence, $r_im \in \mathcal{A}N$ and $r_jm \in \mathcal{B}N$ for some $i, j \in \mathbb{Z}^+$. If $k = \max\{i, j\}$, $r_km \in \mathcal{B}N$. Hence, $\mathcal{A}r_km \subseteq \mathcal{AB}N$. So, $r_{k+1}m \in r_k\mathcal{A}r_km \subseteq \mathcal{AB}N$. This implies $\mathcal{AB}N \cap S \neq \emptyset$. Thus, P is a classical prime submodule of M which does not contain $r_{k+1}m$. Therefore, $m \notin \beta_{cl}(M)$ which proves the second inclusion $\beta_{cl}(M) \subseteq \mathcal{N}_s(M)$. ■

A zero submodule of M weakly satisfies the radical formula (w.s.t.r.f) [65] if $\beta_{cl}(M) = \langle E_M(0)\rangle$. It is easy to see that any module that s.t.r.f also w.s.t.r.f.

Corollary 5.2.1 *If M is a module over a commutative unital ring R, then*

1. *$\mathcal{N}_s(M)$ is a submodule of M;*

2. *$\mathcal{N}_s(M) \subseteq \beta(M)$;*

3. *the zero submodule of ${}_RM$ w.s.t.r.f if and only if $\mathcal{N}_s(M) = \langle E_M(0) \rangle$.*

Proof: $\beta_{cl}(M)$ the intersection of all classical prime submodules of M is a submodule of M. Since by Theorem 5.2.1 $\mathcal{N}_s(M) = \beta_{cl}(M)$, $\mathcal{N}_s(M)$ is a submodule of M. Any prime submodule is classical prime. It follows that $\beta_{cl}(M) \subseteq \beta(M)$. By Theorem 5.2.1, $\mathcal{N}_s(M) \subseteq \beta(M)$. 3 follows directly from the definition of modules that w.s.t.r.f and the fact that $\mathcal{N}_s(M) = \beta_{cl}(M)$. ■

A module is compatible [13] if its prime submodules coincide with its classical prime submodules. Multiplication modules, semisimple modules, and a field of fractions of a domain are multiplication modules.

Corollary 5.2.2 *If a compatible module over a commutative unital ring s.t.r.f (or w.s.t.r.f) then $\langle E_M(0) \rangle = \mathcal{N}_s(M) = \beta_{cl}(M) = \beta(M)$.*

Proof: For compatible modules, there is no distinction between prime submodules and classical prime submodules. Hence, compatible modules s.t.r.f if and only if they w.s.t.r.f (i.e., $\langle E_M(0) \rangle = \beta(M) \Leftrightarrow \langle E_M(0) \rangle = \beta_{cl}(M)$). If in addition to being compatible the module is unital and defined over a commutative ring, then from Theorem 5.2.1, we have $\langle E_M(0) \rangle = \mathcal{N}_s(M) = \beta_{cl}(M) = \beta(M)$. ■

Remark 5.2.1 Lemma 5.2.1 demands for a classical prime submodule which is semiprime. Hence, Theorem 5.2.1 Corollaries 5.2.1 and 5.2.2 still hold when instead of R being commutative, we have:

1. M fully IFP (e.g., when R is left duo),
2. M fully symmetric,
3. M is finitely generated (cyclic or free) and R is medial (LSD, left permutable or right permutable).

Proof: Follows from Corollary 3.5.1 and Theorem 3.5.2. ∎

5.3 Non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$

In this section, we characterize the structure of non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$. Table 5.3 below gives examples of non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$.

Prime number	The $\mathbb{Z}$-module	Number of non-nilpotent elements in the module	The non-nilpotent elements in the module
2	$\mathbb{Z}/2^2\mathbb{Z}$	1	$\bar{2}$
	$\mathbb{Z}/2^3\mathbb{Z}$	1	$\bar{4}$
	$\mathbb{Z}/2^4\mathbb{Z}$	1	$\bar{8}$
	$\mathbb{Z}/2^5\mathbb{Z}$	1	$\overline{16}$
3	$\mathbb{Z}/3^2\mathbb{Z}$	2	$\bar{3}$, $\bar{6}$
	$\mathbb{Z}/3^3\mathbb{Z}$	2	$\bar{9}$, $\overline{18}$
	$\mathbb{Z}/3^4\mathbb{Z}$	2	$\overline{27}$, $\overline{54}$
5	$\mathbb{Z}/5^2\mathbb{Z}$	4	$\bar{5}$, $\overline{10}$, $\overline{15}$, $\overline{20}$
	$\mathbb{Z}/5^3\mathbb{Z}$	4	$\overline{25}$, $\overline{50}$, $\overline{75}$, $\overline{100}$
	$\mathbb{Z}/5^4\mathbb{Z}$	4	$\overline{125}$, $\overline{250}$, $\overline{375}$, $\overline{500}$
p	$\mathbb{Z}/p^2\mathbb{Z}$	$p-1$	$\bar{p}, 2\bar{p}, 3\bar{p}, ..., (p-1)\bar{p}$
	$\mathbb{Z}/p^3\mathbb{Z}$	$p-1$	$\bar{p}^2, 2\bar{p}^2, 3\bar{p}^2, ..., (p-1)\bar{p}^2$
	$\vdots$	$\vdots$	$\vdots$
	$\mathbb{Z}/p^k\mathbb{Z}$	$p-1$	$(\bar{p})^{k-1}, 2(\bar{p})^{k-1}, ..., (p-1)(\bar{p})^{k-1}$

Table 5.1: Non-nilpotent elements in the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$

Proposition 5.3.1 *Let* $B_k = \{np^{k-1}\}_{n=1}^{p-1}$ *for* $k \in \{2, 3, 4, \cdots\}$,[2] *then*

1. *for a given prime* p, $|B_k| = |B_{k+1}|$ *for all* $k \in \{2, 3, 4, \cdots\}$;

2. $\sum\limits_{n=1}^{p-1} np^{k-1} = \begin{cases} 2^{k-1} & \textit{if} \quad p = 2; \\ 0 \ (\textit{mod}\ p^k) & \textit{if} \quad p \neq 2. \end{cases}$

Proof:

1. 1 is evident from how B_k is defined, i.e., each B_k for a given prime p consists of $p - 1$ elements.

2. If $p = 2$, then B_k has only one element 2^{k-1}. Suppose $p \neq 2$, $\sum\limits_{n=1}^{p-1} np^{k-1} = p^{k-1}p(\frac{p-1}{2}) = p^k(\frac{p-1}{2}) = 0 \pmod{p^k}$.

■

Proposition 5.3.2 *Define* B_k^0 *as* $B_k^0 = B_k \cup \{0\}$. *Then,* B_k^0 *is a ring (without unity) under addition modulo* p *and multiplication modulo* p.

Proof: If $a, b \in B_k^0$, then $a = np^{k-1}$ and $b = mp^{k-1}$ for some $m, n \in \mathbb{Z}^+$. $a+b = np^{k-1}+mp^{k-1} = (n+m)p^{k-1}$. If $n+m \leq p$, $(n+m)p^{k-1} \in B_k^0$ otherwise by division algorithm $n + m = rp + s$ for some $r, s \in \mathbb{Z}^+$ and $0 < s < p$. So, in this case, $(n + m)p^{k-1} = (rp + s)p^{k-1} \equiv sp^{k-1} \pmod{p}$. Therefore, in both cases $a + b \in B_k^0$. The identity element is 0, the additive inverse of np^{k-1} is $(p - n)p^{k-1}$ for $n \in \{1, 2, 3, .., p - 1\}$. Associativity is inherited from $\mathbb{Z}$. If $a, b \in B_k^0$, then $ab = (np^{k-1})(mp^{k-1})$ for some $n, m \in \{1, 2, 3, .., p - 1\}$. This implies $ab = nmp^{2(k-1)} \equiv 0 \pmod{p}$ since $2(k - 1) > 1$ for all $k \geq 2$. ■

[2] Note that B_k is the set of all non-nilpotent elements of the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$.

Although the rings $\mathbb{Z}/p\mathbb{Z}$ and B_k^0 have the same number of elements and elements of B_k^0 are got by multiplying those of $\mathbb{Z}/p\mathbb{Z}$ by p^{k-1}, the two rings are not isomorphic. The former is unital and not nil but the latter is non-unital and nil. However, the two rings coincide if $k = 1$.

Proposition 5.3.3 *Define $\psi_k : B_{k+1}^0 \longrightarrow B_k^0$ by $\psi_k(np^k) = np^{k-1}$. ψ_k is a ring isomorphism from B_{k+1}^0 to B_k^0.*

Proof: ψ_k is well defined, for if $np^k = mp^k$, then $n \equiv m \pmod p$. This implies $np^{k-1} \equiv mp^{k-1} \pmod p$ and so $\psi_k(np^k) = \psi_k(mp^k)$. $\psi_k(np^k + mp^k) = \psi_k([n+m]p^k) = (n+m)p^{k-1} = np^{k-1} + mp^{k-1} = \psi_k(np^k) + \psi_k(mp^k)$. $\psi_k([np^k][mp^k]) = \psi_k([nmp^k]p^k) = \psi_k(0p^k) = 0p^{k-1} = 0 = nmp^{2(k-1)} = (np^{k-1})(mp^{k-1}) = \psi_k(np^k)\psi_k(mp^k)$. ψ_k has kernel $pB_k^0 \equiv 0 \pmod p$, hence ψ_k is injective. Lastly, for all $np^{k-1} \in B_k^0$ there is $np^k \in B_{k+1}^0$ such that $\psi_k(np^k) = np^{k-1}$. Thus, ψ_k is surjective. ∎

For an indexed set I, the collection $\{R_i : i \in I\}$ of rings together with ring homomorphisms, $\psi_i : R_i \to R_{i-1}$ is called a projective system (inverse system) if whenever $i < j$, we have a homomorphism f_{ji} from R_j to R_i and if $i \leq j \leq k$, then $f_{kj} \circ f_{ji} = f_{ki}$. A sequence (x_i) in the direct product $\prod R_i$ is said to be coherent if it respects the maps ψ_i in the sense that for every i we have $\psi_{i+1}(x_{i+1}) = x_i$. The collection of all coherent sequences is called the inverse limit of the inverse system. The inverse limit is denoted by $\varprojlim(R_i, \psi_i)$ or just $\varprojlim R_i$ if no confusion is likely to arise.

$\cdots \xrightarrow{\psi_k} B_k^0 \xrightarrow{\psi_{k-1}} B_{k-1}^0 \to \cdots \xrightarrow{\psi_3} B_3^0 \xrightarrow{\psi_2} B_2^0$ is a projective system indexed by integers greater than 1. As an example, consider B_k^0 with $p = 5$:
$\cdots \xrightarrow{\psi_4} B_4^0 = \{5^3, 2\times 5^3, 3\times 5^3, 4\times 5^3, 0\} \xrightarrow{\psi_3} B_3^0 = \{5^2, 2\times 5^2, 3\times 5^2, 4\times 5^2, 0\} \xrightarrow{\psi_2} B_2^0 = \{5, 2\times 5, 3\times 5, 4\times 5, 0\}$. $\psi_2 : B_3^0 \to B_2^0$ is defined by $\psi_2(0) = 0, \psi_2(5^2) =$

$5, \psi_2(2\times 5^2) = 2\times 5, \psi_2(3\times 5^2) = 3\times 5$ and $\psi_2(4\times 5^2) = 4\times 5$. Clearly, ψ_2 is injective and surjective. For B_k when $p=5$ we get the following sequences:

$$\begin{array}{ccccccccccc}
\cdots & \to & 5^m & \to & \cdots & \to & 5^3 & \to & 5^2 & \to & 5 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \to & 2\times 5^m & \to & \cdots & \to & 2\times 5^3 & \to & 2\times 5^2 & \to & 2\times 5 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \to & 3\times 5^m & \to & \cdots & \to & 3\times 5^3 & \to & 3\times 5^2 & \to & 3\times 5 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \to & 4\times 5^m & \to & \cdots & \to & 4\times 5^3 & \to & 4\times 5^2 & \to & 4\times 5
\end{array}$$

In general, we have sequences defined by $\psi(np^m) = np^{m-1}$ across B_m's with an infinite number of elements but convergent to np. We also have sequences defined by $f(np^m) = (n-1)p^m$ within B_m with a finite number of elements (equal to $p-1$) and convergent to p^m.

Lemma 5.3.1 *If $a_1, a_2, \cdots, a_m$ is a complete system of residues modulo m, and if r is a positive integer with $(r, m) = 1$, (i.e., r is relatively prime to m) then $ra_1 + s, ra_2 + s, \cdots, ra_m + s$ is also a complete system of residues modulo m for any $s \in \mathbb{Z}$.*

Theorem 5.3.1 *Let $N_p = \lim\limits_{\leftarrow}(B_k^0, \psi_k)$, then N_p is a compact topological ideal of the ring $\mathbb{Z}_p$ of p-adic integers. Furthermore, N_p consists of sequences of the form $(\cdots, np^k, \cdots, np^3, np^2, np, 0)$, where $n \in \mathbb{Z}/p\mathbb{Z}$.*

Proof: Let $A_k = \{0, 1, 2, \cdots, p^k - 1\}$ and $B_k^0 = \{0, p^{k-1}, 2p^{k-1}, ..., (p-1)p^{k-1}\}$. Since $p^k - 1 \geq (p-1)p^{k-1} = p^k - p^{k-1}$ for all $k \in \mathbb{Z}^+$ and every $a \in B_k^0$ is a positive integer less than $p^k - 1$, and hence $a \in A_k$, we have $B_k^0 \subseteq A_k$ for all k. Therefore, $N_p = \lim\limits_{\leftarrow}(B_k^0, \psi_k) \subseteq \lim\limits_{\leftarrow}(A_k, \phi_k) = \mathbb{Z}_p$ where $A_k = \mathbb{Z}/p^k\mathbb{Z}$ and ϕ_k is a homomorphism from A_k to A_{k-1}. To show that $N_p \lhd \mathbb{Z}_p$, it is enough to show that $B_k^0 \lhd A_k$ for each k. Since $\{0, 1, 2, \cdots, (p-1)\}$ is a complete system of residues modulo p and for any $r \in A_k$, $(r, p) = 1$, by Lemma 5.3.1, $\{0, r, 2r, \cdots, (p-1)r\}$ is also a complete system of residues modulo p. So, $B_k^0 r = rB_k^0 \equiv B_k^0 \pmod p$ for all $r \in A_k$. Since B_k^0 are rings, their inverse limit N_p is also a ring. If we give $\prod\limits_{k \geq 2} B_k$ the product topology and B_k the discrete topology, the ring N_p inherits a topology which turns it into a compact space since it is closed in a product of compact spaces. ■

Corollary 5.3.1 *The ideal N_p (of the ring $\mathbb{Z}_p$) has no invertible elements.*

Proof: Follows from [71, Chap II, Proposition 2(a)] and the fact that every element of N_p is of the form $(\cdots, np^k, \cdots, np^3, np^2, np, 0)$, $n \in \mathbb{Z}/p\mathbb{Z}$. ■

Corollary 5.3.2 *N_p is an integral domain and a complete metric space.*

Proof: Since $\mathbb{Z}_p$ is an integral domain, cf., [71, p.12], its ideal N_p is also an integral domain. For the rest we follow the proof in [71, p.12, Proposition 3]. Every element x of N_p is of the form $x = p^n y$ where $y \in A_k$ and n is the p-adic valuation of x denoted by $v_p(x)$. The ideals $p^n N_p$ form a basis of neighborhoods of 0; since $x \in p^n N_p$ implies $v_p(x) \geq n$, the topology on N_p is defined by the distance $d(x, y) = e^{-v_p(x-y)}$. Since N_p is compact [cf. Theorem 5.3.1], it is complete. ■

Proposition 5.3.4 *B_k^0 and $\mathbb{Z}/p\mathbb{Z}$ are isomorphic $\mathbb{Z}$-modules.*

Proof: Since B_k^0 and A_k are abelian groups, they are $\mathbb{Z}$-modules and $\varphi_k(np^{k-1}) = n \pmod p$ is a module isomorphism from B_k^0 to A_k. φ is well defined, for if $np^{k-1}, mp^{k-1} \in B_k^0$ and $np^{k-1} = mp^{k-1}$, then $n \equiv m \pmod p$ and hence $n \pmod p = m$ (mod) p which implies $\varphi_k(np^{k-1}) = \varphi_k(mp^{k-1})$. $\varphi_k(np^{k-1} + mp^{k-1}) = \varphi_k([n+m]p^{k-1}) = (n+m) \pmod p = n \pmod p +$

$m \pmod p) = \varphi_k(np^{k-1}) + \varphi_k(mp^{k-1})$. For all $a \in \mathbb{Z}$, $\varphi_k(a[np^{k-1}]) = \varphi_k([an]p^{k-1}) = an \pmod p = a\varphi_k(np^{k-1})$. $\varphi_k(np^{k-1}) = 0 \Leftrightarrow n \pmod p = 0 \Leftrightarrow n = \bar{0}$. Thus, Ker $\varphi_k = \bar{0}$ and φ_k is injective. Since the $\mathbb{Z}$-modules B_k^0 and A_k are of the same size and φ_k is injective, by the pigeon hole principal φ_k is surjective. ■

Chapter 6

2-primal modules

A notion of 2-primal rings is generalized to modules by defining 2-primal modules. We show that the implications between rings which are reduced, IFP, reversible, semi-symmetric and 2-primal are preserved when the notions are extended to modules. Like for rings, 2-primal modules bridge the gap between modules over commutative rings and modules over non-commutative rings; for instance, for 2-primal modules, prime submodules coincide with completely prime submodules. Completely prime submodules and reduced modules are both characterized. A generalization of 2-primal modules is done where 2-primal and NI modules are a special case.

From chapter three we know that the following chart of implications is true:

$$
\begin{array}{ccccc}
\text{completely prime} & \Rightarrow & \text{prime} & \Rightarrow & \text{semiprime} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\begin{array}{c}\text{classical} \\ \text{completely prime}\end{array} & \Rightarrow & \text{classical prime} & \Rightarrow & \text{classical semiprime} \\
\Downarrow & & & & \Uparrow \\
\text{completely semiprime} & & \Rightarrow & & \text{semiprime.}
\end{array}
$$

Reddy and Murty in [68] defined a semi-symmetric ideal I of a near-ring R (and hence of a ring R) as one such that if $a^n \in I$ for some $a \in R$ and $n \in \mathbb{N}$, then $(a)^n \subseteq I$. We extend this notion to modules and call a submodule N of an R-module M semi-symmetric if for $a \in R$ and $m \in M$ such that if $a^2m \in N$, then $(a)^2m \subseteq N$.

If R is a ring, let $\beta(R)$, $\beta_{co}(R)$ and $\mathcal{N}(R)$ be the prime radical, generalized nil radical (also called the completely prime radical) and the set of all nilpotent elements of R respectively. An ideal I of a ring R is called 2-primal [21, Definition 1.1] if $\beta(R/I) = \mathcal{N}(R/I)$. Thus, a ring R is 2-primal if and only if the zero ideal is 2-primal, i.e., $\beta(R) = \mathcal{N}(R)$. 2-primal rings were first studied in [72] (although not so called at that time). Birkenmeier *et al.* in [21, Prop. 2.1] showed that a ring R is 2-primal if and only if $\beta(R) = \beta_{co}(R)$. The notion of 2-primal exists in literature for other algebraic structures such as near-rings, for example, see [41]. In this chapter we extend it to modules by defining 2-primal (sub)modules. We show that in modules, completely semiprime, symmetric, semi-symmetric, IFP and 2-primal bridge the gap between prime submodules and completely prime submodules; and that the implications between these notions for rings are preserved when they are extended to modules.

6.1 Notions that lead to 2-primal modules

Recall that for a module M over a ring R, $\beta_{co}(M)$ and $\beta(M)$ respectively denote the completely prime radical of M and prime radical of M.

Definition 6.1.1 A submodule P of an R-module M is 2-primal if

$$\beta_{co}(M/P) = \beta(M/P).$$

Thus, an R-module M is 2-primal if $\beta_{co}(M) = \beta(M)$.

Example 6.1.1 A module over a commutative ring is 2-primal.

According to Behboodi, an element m of an R-module M is nilpotent [10, Definition 2.2] if $m = \sum_{i=1}^{r} a_i m_i$ for some $a_i \in R$, $m_i \in M$ and $r \in \mathbb{N}$, such that $a_i^k m_i = 0$ $(1 \leq i \leq r)$ for some $k \in \mathbb{N}$; and m is strongly nilpotent if $m = \sum_{i=1}^{r} a_i m_i$ for some $a_i \in R$, $m_i \in M$ and $r \in \mathbb{N}$, such that for every i $(1 \leq i \leq r)$ and every sequence $a_{i1}, a_{i2}, a_{i3}, \cdots$ where $a_{i1} = a_i$ and $a_{in+1} \in a_{in} R a_{in}$ (for all n), we have $a_{ik} R m_i = 0$ for some $k \in \mathbb{N}$. The set of all Behboodi's strongly nilpotent elements of a module is a submodule and is denoted by $\text{Nil}_*(M)$, cf., [10, Definition 2.4]. Note that, these definitions are not equivalent to our definitions of nilpotent element and strongly nilpotent element introduced in chapter five.

Theorem 6.1.1 *Suppose R is a 2-primal ring and M is a projective R-module, then*

$$\beta_{co}(M) = \beta(M) = Nil_*(M).$$

Proof: Since β and β_{co} are both preradicals, for any projective R-module M, $\beta(R)M = \beta(M)$ and $\beta_{co}(R)M = \beta_{co}(M)$, cf., [19, Proposition 1.1.3]. If $\beta(R) = \beta_{co}(R)$, we get $\beta(M) = \beta_{co}(M)$. $\beta(M) = \text{Nil}_*(M)$ follows from [11, Theorem 3.8]. ■

Corollary 6.1.1 *A projective module over a 2-primal ring is 2-primal.*

Proposition 6.1.1 *If $_RM$ is both projective and faithful, then $_RM$ 2-primal implies R is 2-primal.*

Proof: Suppose $\beta_{co}(M) = \beta(M)$. M projective implies $\beta_{co}(R)M = \beta(R)M$. So, $\beta_{co}(R)M - \beta(R)M = 0$. Thus, $(\beta_{co}(R) - \beta(R))M = 0$. M faithful implies $\beta_{co}(R) - \beta(R) = 0$. Hence, $\beta_{co}(R) = \beta(R)$. ■

Proposition 6.1.2 *Let R be a ring such that $r \in rR$ for all $r \in R$ (e.g., if R is Von Neumann regular or if R has a right identity), then R is 2-primal if and only if the module ${}_RR$ is 2-primal.*

Proof: From Proposition 2.3.9, $\beta_{co}(R) = \beta_{co}({}_RR)$. A similar proof of the prime case yields $\beta(R) = \beta({}_RR)$. The rest is trivial. ■

We have not found any examples of 2-primal rings R for which the R-module is not 2-primal, we therefore have:

Conjecture 6.1.1 *If R is a 2-primal ring, the R-module M is 2-primal.*

The implications between symmetric, IFP, reduced, semi-symmetric and 2-primal for rings were given in [21, p.105], [55, p.314] and [56, p.494]. We show in Theorems 6.1.2 and 6.1.3 that they are (the implications) preserved by their module analogues.

Theorem 6.1.2 *For any submodule P of an R-module M,*

$$\begin{array}{ccccc} & & R \text{ commutative} & & \text{semi-symmetric} \\ & & \Downarrow & & \Uparrow \\ \text{completely} & \Rightarrow & \text{symmetric} & \Rightarrow & \text{IFP} \Leftrightarrow (P:S) \lhd R \\ \text{semiprime} & & & & \text{for any subset } S \text{ of } M. \end{array}$$

Proof: Completely semiprime $\Rightarrow$ symmetric $\Rightarrow$ IFP, see Proposition 3.1.1. IFP $\Leftrightarrow (P : S) \lhd R$ for any subset S of M. $(P : S)$ is always a left ideal. Let $a \in (P : S)$ for any subset S of M and $a \in R$. Then $aS \subseteq P$. P IFP implies $aRS \subseteq P$ and $aR \subseteq (P : S)$. Conversely, suppose $(P : S) \lhd R$ for any subset S of M and $am \in P$ for $a \in R$ and $m \in M$. Then $a \in (P : \{m\})$. By hypothesis, $(P : \{m\}) \lhd R$ and hence $aR \subseteq (P : \{m\})$ such that $aRm \subseteq P$. IFP $\Rightarrow$ semi-symmetric. Suppose $a^2m \in P$, then $a \in (P : am)$. Since $(P : \{am\}) \lhd R$, we have $(a) \subseteq (P : \{am\})$ such that $(a)am \subseteq P$, $(a)aRm \subseteq P$, $(a)Ram \subseteq P$ and $(a)RaRm \subseteq P$. Hence, $(a)^2m = (a)[\mathbb{Z}a + aR + Ra + RaR]m \subseteq P$. R commutative $\Rightarrow$ symmetric is trivial. ■

Corollary 6.1.2 *For any proper submodule P of an R-module M, the following statements are equivalent:*

1. *P is completely semiprime,*
2. *P is semi-symmetric and classical semiprime,*
3. *P is semi-symmetric and semiprime,*
4. *P is IFP and classical semiprime,*
5. *P is IFP and semiprime,*
6. *P is symmetric and classical semiprime,*
7. *P is symmetric and semiprime.*

Proof: It suffices to prove (2) $\Rightarrow$ (1), the rest follows from Theorem 6.1.2 and the fact that completely semiprime $\Rightarrow$ semiprime $\Rightarrow$ classical semiprime. Suppose for $a \in R$ and $m \in M$, $a^2m \in P$. Then $(a)^2m \subseteq P$ since P is semi-symmetric. P classical semiprime implies $(a)m \subseteq P$. Thus, $a\langle m\rangle \subseteq P$ and P is completely semiprime. ■

Remark 6.1.1 When $P = 0$ in Corollary 6.1.2, we get different characterizations of reduced modules.

Theorem 6.1.3 *For any submodule P of an R-module M,*

$$P \text{ semi-symmetric submodule } \Rightarrow P \text{ is a 2-primal submodule.}$$

Proof: It is enough to show that semi-symmetric + prime = completely prime. One direction is trivial. Suppose P is both prime and semi-symmetric. Let $a \in R$ and $m \in M$ such that $am \in P$. It follows from the definition of semi-symmetric submodules that $(a)^2 m \subseteq P$. By definition of prime submodules, $(a)M \subseteq P$ or $(a)m \subseteq P$. If $(a)M \subseteq P$, $aM \subseteq P$ and we are through. Suppose $(a)m \subseteq P$. If $m \in P$, we are through. Suppose $m \notin P$. Then $a \in (a) \subseteq (P : m)$. Since P is prime, $(P : m) = (P : M)$ and $a \in (P : M)$ such that $aM \subseteq P$. ■

We construct examples to illustrate that the implications in Theorems 6.1.2 and 6.1.3 are not reversible in general.

Example 6.1.2 Let R be the ring of all 4-by-4 upper triangular matrices over a field F. Let $a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $a^2 = 0$, but $(a)^2 \neq 0$. If $m = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in {}_RR$, $a^2 m = 0$ but $(a)^2 m \neq 0$. Thus, the zero submodule of ${}_RR$ is 2-primal by Proposition 6.1.2 and [21, Example 1.4] but not semi-symmetric.

IFP submodules need not be symmetric and symmetric submodules need not be completely semiprime, see Example 3.1.1 and Example 3.1.2 respectively.

Example 6.1.3 A ring R is reversible [25], [50], if $ab = 0 \Leftrightarrow ba = 0$ for all $a, b \in R$. A torsion-free module over a reversible ring is symmetric. Note that a reversible ring need not be commutative. Hence, symmetric modules need not be defined over commutative rings.

Corollary 6.1.3 *Each of the following statements implies that each submodule of a module M is 2-primal:*

1. *every semiprime submodule of M is completely semiprime,*
2. *every prime submodule of M contains a completely prime submodule of M,*
3. *every classical prime submodule of M is completely prime,*
4. *every prime submodule of M is completely prime.*

Proof: Elementary. ■

Example 6.1.4 If M is a finitely generated (cyclic or free) module over a medial (left permutable, right permutable or left self distributive) ring, then M is 2-primal.

Proof: Follows from Corollary 3.5.2 and Corollary 6.1.3(4). ■

Corollary 6.1.4 *For a proper submodule P of an R-module M, the following statements are equivalent:*

1. *P is a completely prime submodule of M,*
2. *P is a prime and a completely semiprime submodule of M,*
3. *P is a prime submodule of M and the module M/P is symmetric,*
4. *P is a prime submodule of M and the module M/P is IFP,*
5. *P is a prime submodule of M and $(P : S) \lhd R$ for all subsets S of M,*

6. *P is a prime submodule of M and the module M/P is semi-symmetric,*

7. *P is a prime and 2-primal submodule of M.*

Proof: 1⇒ 2 ⇒ 3 ⇒ 4 ⇒ 5 ⇒ and 6 ⇒ 7 follow from Theorems 6.1.2 and 6.1.3 respectively.
7⇒ 1. Suppose P is a prime and 2-primal submodule. Then M/P is a prime module such that $\beta(M/P) = (0)$. P 2-primal implies $\beta(M/P) = \beta_{co}(M/P) = (0)$. Since $\beta_{co}(M/P)$ is always a completely semiprime submodule of M/P, M/P is a reduced (equivalently completely semiprime) module. Thus, P is a completely semiprime submodule and hence completely prime. To see this, let $am \in P$ for $a \in R$ and $m \in M$. P completely semiprime implies P is IFP, hence $a\langle m\rangle \subseteq P$. P prime implies $m \in P$ or $aM \subseteq P$. Thus P is completely prime. ∎

Example 6.1.5 Reduced modules are 2-primal.

Example 6.1.6 If R is a left duo ring, the module $_RM$ is strongly 2-primal, (i.e., every proper submodule of M is 2-primal).

Proof: Follows from Example 3.1.1 and Corollary 6.1.4. ∎

Corollary 6.1.5 *For a classical semiprime submodule, the following statements are equivalent:*

1. *completely semiprime,*

2. *symmetric,*

3. *IFP,*

4. *semi-symmetric.*

Proof: It is enough to show that semi-symmetric implies completely semiprime. The rest follows from Theorem 6.1.2. Suppose $a^2m \in P$, by definition of semi-symmetric submodules, $(a)^2m \subseteq P$. By definition of classical semiprime submodules $(a)m \subseteq P$ such that $a\langle m\rangle \subseteq P$ and P is completely semiprime. ■

Modules over fully idempotent rings (rings whose all ideals are idempotent, [26], [27]) and modules over fully prime rings (rings whose all ideals are prime [22, p.5390]) are classical semiprime and hence satisfy Corollary 6.1.5.

Proposition 6.1.3 *For any module M, the following statements are equivalent:*

1. *M is 2-primal,*

2. *$\beta_{co}(M) \subseteq \beta(M)$.*

Proof: Elementary. ■

Proposition 6.1.4 *2-primal modules are closed under direct sums.*

Proof: Suppose $\{M_\lambda, \lambda \in \Lambda\}$ are 2-primal modules and $M = \oplus M_\lambda$. By Proposition 2.3.3 and Remark 2.3.2, $\beta(M) = \oplus\beta(M_\lambda) = \oplus\beta_{co}(M_\lambda) = \beta_{co}(M)$. ■

6.2 Generalization of 2-primal modules

By following a generalization of 2-primal near-rings in [41, Section 2] we have:

Definition 6.2.1 Let ρ be a Hoenhke radical of a module M for which $\beta \subseteq \rho \subseteq \beta_{co}$. A submodule N of an R-module M is ρ-primal if $\rho(M/N) = \beta_{co}(M/N)$. Thus, M is ρ-primal if $\rho(M) = \beta_{co}(M)$.

Example 6.2.1 If $\rho = \beta$ the prime radical of M, then M is 2-primal.

Example 6.2.2 Primeless modules (for instance the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ where p is a prime number) are ρ-primal for all Hoenhke radicals ρ such that $\beta \subseteq \rho \subseteq \beta_{co}$.

Proof: If M is primeless, it is completely primeless. Hence, $\beta(M) = \beta_{co}(M) = M$. Thus, $\rho(M) = \beta_{co}(M) = M$ for all $\beta \subseteq \rho \subseteq \beta_{co}$. ■

Example 6.2.3 Projective modules over ρ-primal rings are ρ-primal for all $\beta \subseteq \rho \subseteq \beta_{co}$.

Proof: Suppose M is projective and R is ρ-primal, i.e., $\rho(R) = \beta_{co}(R)$, then $\rho(M) = \rho(R)M = \beta_{co}(R)M = \beta_{co}(M)$. ■

A ring R is called NI (due to [54] and also studied in [45]) if its upper nil radical coincides with the set of all its nilpotent elements, i.e., $\mathcal{U}(R) = \mathcal{N}(R)$. The upper nil radical, $\mathcal{U}(M)$ of an R-module M is the intersection of all its s-prime submodules, where $P \leq M$ is s-prime if P is prime and the ring $R/(P : M)$ has no non-zero nil ideals, i.e., P is prime and $\mathcal{U}(R/(P : M)) = 0$. We say an R-module M is NI if $\mathcal{U}(M) = \beta_{co}(M)$.

Example 6.2.4 If $\rho = \mathcal{U}$ the upper nil radical of M then M is NI.

Definition 6.2.2 A submodule P of an R-module M is

1. l-prime if it is a prime submodule and the ring $R/(P : M)$ contains no locally nilpotent ideals, i.e., $\mathcal{L}(R/(P : M)) = 0$, where $\mathcal{L}$ is the Levitzki radical map.

2. strictly prime [28] if $RM \not\subseteq P$ and for each $m \in M \setminus P$ there exists $b \in R$ such that if $a \in R$ and $abm \in P$, then $aM \subseteq P$.

3. strongly prime [8] if $RM \not\subseteq P$ and for each $m \in M \setminus P$ there exists a finite set $F \subseteq R$ such that if $a \in R$ and $aFm \subseteq P$, then $aM \subseteq P$.

Proposition 6.2.1 *For any submodule P of an R-module M we have:*

$$\textit{completely prime} \Rightarrow \textit{strictly prime} \Rightarrow \textit{strongly prime} \Rightarrow \textit{prime}.$$

Proof: We prove the first implication, for the others see, [43, p.131]. Since $RM \not\subseteq P$, there exists $t \in R$ such that $tM \not\subseteq P$. If $atm \in P$ for $a \in R$ and $m \in M$, then by definition of completely prime submodules of M, $m \in P$ or $atM \subseteq P$. If $m \in P$, we are through. Suppose $m \in M \setminus P$, then $atM \subseteq P$. Since $tM \not\subseteq P$, there exists $m_1 \in M$ such that $tm_1 \notin P$, but since $atm_1 \in P$, we have $aM \subseteq P$ by definition of completely prime submodules. ■

Proposition 6.2.2 *Any strictly prime module is s-prime.*

Proof: We already know that if M is strictly prime, it is prime. Let $A \triangleleft R$ such that $AM \neq 0$. There exists $a \in A$ and $m \in M$ such that $am \neq 0$. Since M is strictly prime, we can find $b \in R$ such that if $r \in R$ and $rbam = 0$ then $rM = 0$. Now, $aM \neq 0$ (since $am \neq 0$) implies $abam \neq 0$ by definition of strictly prime modules. It follows that $0 \neq abam \in abaM$. $abaM \neq 0$ and $am \neq 0$ imply $ababam \neq 0$. By induction, we get $(ab)^n m \neq 0$ for all $n \in \mathbb{N}$ and hence $(ab)^n M \neq 0$ for all $n \in \mathbb{N}$. This shows that M is s-prime.

Example 6.2.5 Let $R = FS$ where F is a field and S a permutation group on the integers. R is a primitive group ring hence an s-prime ring and prime ring which is not strongly prime and hence not strictly prime, see [42, p.215]. If $M = {}_RR$, then M is an s-prime module and hence prime module which is not strongly prime and hence not strictly prime. Thus, a prime module need not be strongly prime and an s-prime module need not be strictly prime.

Example 6.2.6 Let F be a field and $R = \begin{pmatrix} F & F \\ F & F \end{pmatrix}$, then the module ${}_RR$ is strongly prime but not strictly prime, see [43, p.119].

Example 6.2.7 Let D be a simple domain with identity which is neither left nor right ore. The matrix ring $R = M_n(D)$, $1 < n < \aleph_0$ is strictly prime (also called strongly prime of bound one) but not completely prime, see [81, p.315, Sec. 4.12]. In [24, p.153] there is another example of a strictly prime ring R which is not completely prime. If we take $M = {}_RR$, then M is strictly prime but not completely prime.

Proposition 6.2.3 *If $\mathcal{L}(M)$, $\beta_s(M)$ and $\beta_{sc}(M)$ denote the intersection of all l-prime submodules of M, all strongly prime submodules of M and all strictly prime submodules of M respectively, then $\beta \subseteq \mathcal{L} \subseteq \mathcal{U} \subseteq \beta_{sc} \subseteq \beta_{co}$ and $\beta \subseteq \beta_s \subseteq \beta_{sc} \subseteq \beta_{co}$.*

Example 6.2.8 If $\rho = \beta_s, \beta_{sc}, \mathcal{L}$ in Definition 6.2.1 for a given module, then such a module is respectively $\beta_s, \beta_{sc}, \mathcal{L}$-primal.

Remark 6.2.1 A 2-primal module is ρ-primal for all $\beta \subseteq \rho \subseteq \beta_{co}$ but not conversely. For instance, NI modules need not be 2-primal. To see this, suppose R is unital. Then by a proof similar to that of Proposition 6.1.2, R is NI if and only if the module ${}_RR$ is NI. Direct limits of direct systems of NI modules of the form ${}_RR$ are NI, cf., [45, Prop. 1.1]. But direct limits of direct systems of 2-primal modules of the form ${}_RR$ need not be 2-primal, cf., [45, Example 1.2].

Theorem 6.2.1 *Let $\mathcal{M}_{cp}$ and $\mathcal{M}_p$ denote a class of completely prime modules and prime modules respectively. For any class $\mathcal{M}_\rho$ of some type of "prime" modules such that $\mathcal{M}_{cp} \subseteq \mathcal{M}_\rho \subseteq \mathcal{M}_p$, a submodule P of an R-module M is a completely prime submodule if and only if $M/P \in \mathcal{M}_\rho$ and $\rho(M/P) = \beta_{co}(M/P)$.*

Proof: If P is a completely prime submodule of M, $M/P \in \mathcal{M}_\rho$ and $\rho(M/P) = \beta_{co}(M/P)$. For the converse, the proof is similar to that of 7⇒1 in Corollary 6.1.4. ■

Corollary 6.2.1 *The following statements are equivalent for a proper submodule P of an R-module M:*

1. *completely prime submodule,*
2. *strongly prime submodule which is β_s-primal,*
3. *strictly prime submodule which is β_{sc}-primal,*
4. *s-prime submodule which is NI (i.e., $\mathcal{U}$-primal),*
5. *l-prime submodule which is $\mathcal{L}$-primal,*
6. *prime submodule which is 2-primal.*

Remark 6.2.2 For $P = 0$, we get more characterizations of reduced modules from Corollary 6.2.1.

Let $\bar{\rho}(N) = \cap\{P \leq M \ : \ N \subseteq P \text{ and } M/P \in \mathcal{M}\}$ for all $\beta \subseteq \rho \subseteq \beta_{co}$. For any $N \leq M$, $\rho(M/N) = \bar{\rho}(N)/N$. In particular, if $N = 0$, then $\rho(M) = \bar{\rho}(0)$.

Proposition 6.2.4 *Let M be an R-module, $\mathcal{M}$ a class of R-modules such that $\mathcal{M}_{cp} \subseteq \mathcal{M} \subseteq \mathcal{M}_p$ and ρ a Hoenhke radical associated with the class $\mathcal{M}$, then*

1. *$\rho(M/N) = \beta_{co}(M/N)$ implies $\bar{\rho}(N)$ is a completely semiprime submodule of M,*
2. *$\rho(M) = \beta_{co}(M)$ implies $\rho(M)$ is a completely semiprime submodule of M,*

Proof: 1) Let $a \in R$ and $m \in M$ such that $a^2m \in \bar{\rho}(N)$, $N \leq M$. Then $a^2(m+N) \subseteq \bar{\rho}(N)/N = \rho(M/N) = \beta_{co}(M/N)$. Since $\beta_{co}(M/N)$ is a completely semiprime submodule of M/N, $a(\langle m\rangle + N) = (a\langle m\rangle + N) \subseteq \bar{\rho}(N)/N$. Hence $a\langle m\rangle \subseteq \bar{\rho}(N)$ and $\bar{\rho}(N)$ is a completely semiprime submodule of M.
2) Follows from 1) if we let $N = 0$. ■

Corollary 6.2.2 *If M is a ρ-primal module, then $\rho(M)$ is a completely semiprime submodule of M for all $\beta \subseteq \rho \subseteq \beta_{co}$.*

Proposition 6.2.5 *Let N be a submodule of M such that $N \subseteq \rho(M)$. Then $\rho(M) = \beta_{co}(M)$ implies $\rho(M/N) = \beta_{co}(M/N)$.*

Proof: Since $N \subseteq \rho(M)$, $\bar{\rho}(N) = \rho(M)$. If $\rho(M) = \beta_{co}(M)$, then $\bar{\rho}(N) = \beta_{co}(M)$. Since $\beta_{co}(M)$ is a completely semiprime submodule of M containing N, we have $\bar{\rho}(N) = \beta_{co}(M) = \bar{\beta}_{co}(N)$. Thus, $\rho(M/N) = \bar{\rho}(N)/N = \bar{\beta}_{co}(N)/N = \beta_{co}(M)/N = \beta_{co}(M/N)$. ■

Chapter 7

The Amitsur property in modules

For a series of different "primes" in modules, if γ is a radical corresponding to one such "prime" and M is a module over a ring R, we show that $\gamma(M[x]) = \gamma(M)[x]$, $(\gamma(M[x]) \cap M)[x] = \gamma(M[x])$ and if $\gamma(M) = M$, then $\gamma(M[x]) = M[x]$, i.e., γ satisfies the polynomial equation, has the Amitsur property and is polynomially extensible. We also show that the module ${}_RM$ is 2-primal or ρ-primal (where ρ-primal is a generalization of 2-primal) if and only if so is the module ${}_{R[x]}M[x]$.

In this chapter unlike other chapters, all modules are left unital modules and the rings are both associative and unital. Let R be a ring and $R[x]$ the polynomial ring over R. If γ is a radical map, the properties: $(\gamma(R[x]) \cap R)[x] = \gamma(R[x])$; if $\gamma(R) = R$, then $\gamma(R[x]) = R[x]$; and $\gamma(R[x]) = \gamma(R)[x]$ are termed as: γ has the Amitsur property; γ is polynomially extensible and γ satisfies the polynomial equation respectively. Ring radicals with such properties were studied in [16], [32], [33], [35], [53] and [77]. The prime radical, Levitzki radical, upper nil radical, Jacobson radical, generalized nil radical, Brown-McCoy radical, antisimple radical and Behrens radical satisfy the Amitsur property. The prime radical and the Levitzki radical are polynomially extensible. The prime radical, uniformly strongly prime radical and Levitzki radical satisfy the polynomial equation. For definitions of the aforementioned ring radicals see [33]. In this chapter, we aim at obtaining module radicals that exhibit the aforementioned properties. The following classes of submodules will be useful for this purpose: prime, s-prime, completely prime, strictly prime, strongly prime, classical semiprime and completely semiprime.

7.1 The "going up" and "going down" theorems

The terminology used here is that of [59]. Let σ be a property of submodules defined for submodules in M and in $M[x]$, where M is an R-module and $M[x]$ the corresponding $R[x]$-module. A submodule with property σ may be called a σ-submodule. If σ is such that P a σ-submodule of ${}_RM$ implies $P[x]$ is a σ-submodule of ${}_{R[x]}M[x]$, then we say that the "going up" theorem holds for σ-submodules. If Q a σ-submodule in $M[x]$ implies that $Q \cap M$ is a σ-submodule in M, then we shall say that the "going down" theorem holds for a σ-submodule.

Let σ be a property for submodules defined in the R-module M and also in the $R[x]$-module $M[x]$. If N is a submodule in M, the closure of N relative to σ which we denote by N_σ is the intersection of all σ-submodules which contain N. Hence, closed submodules (relative to σ) are those submodules that are intersections of σ-submodules.

Theorem 7.1.1 *Let σ be a property of submodules defined in ${}_RM$ and in*

${}_{R[x]}M[x]$ *such that the following are true:*

(a) M is a σ-submodule;

(b) if N is a submodule and B a σ-submodule in $M[x]$ such that $R[x]N \subseteq B$, then $N \subseteq B$;

(c) the “going up” and “going down” theorems hold for σ-submodules. Then for each submodule N in M

$$(N[x])_\sigma = (N_\sigma)[x].$$

Proof: Clearly (a) and (c) imply $M[x]$ is a σ-submodule in $M[x]$. Moreover, it follows easily from (c) that the “going up” and “going down” theorems hold also for closed submodules. Since $N \subseteq N_\sigma$, we have $N[x] \subseteq N_\sigma[x]$. But by the “going up” theorem for closed submodules, $N_\sigma[x]$ is a closed submodule in $M[x]$ and hence $(N[x])_\sigma \subseteq N_\sigma[x]$. To obtain the inclusion in the other direction, suppose $N[x] \subseteq C$ where C is a closed submodule in $M[x]$. Now, $C \cap M$ is a closed submodule in M by the “going down” theorem, and since $N \subseteq C \cap M$, we have $N_\sigma \subseteq C \cap M$. Now $R[x]N_\sigma[x] \subseteq C$ and by (b) $N_\sigma[x] \subseteq C$. Applying this result to the case in which $C = (N[x])_\sigma$, we have $N_\sigma[x] \subseteq (N[x])_\sigma$ and the proof is completed. ■

Theorem 7.1.2 *If M is a “prime” R-module in one of the following senses, then $M[x]$ is a “prime” $R[x]$-module in the same sense, i.e., the listed “primes” satisfy the “going up” theorem.*

1. *prime,*
2. *s-prime,*
3. *strictly prime,*

4. *strongly prime,*

5. *classical semiprime,*

Proof:

Prime: Let $f(x) \in R[x]$ and $m(x) \in M[x]$ such that $f(x)R[x]m(x) = 0$. Suppose $m(x) = \sum_{i=0}^{k} m_i x^i \neq 0$. Hence, we may assume $m_k \neq 0$. If $f(x) = \sum_{j=0}^{l} a_j x^j$, then from $f(x)R[x]m(x) = 0$, we have $\sum_h (\sum_{i+j=h} a_j r m_i) x^h = 0$ for all $r \in R$. Now, for all $r \in R$

$$a_0 r m_0 = 0 \tag{7.1}$$

$$a_0 r m_1 + a_1 r m_0 = 0 \tag{7.2}$$

$$\vdots$$

$$a_l r m_{k-1} + a_{l-1} r m_k = 0 \tag{7.3}$$

$$a_l r m_k = 0. \tag{7.4}$$

Since $a_l R m_k = 0$, $m_k \neq 0$ and M is prime, we have $a_l M = 0$ hence Equation (7.3) reduces to $a_{l-1} r m_k = 0$ for all $r \in R$. Hence, $a_{l-1} R m_k = 0$ and again M prime and $m_k \neq 0$ implies $a_{l-1} M = 0$. Continuing with this process, leads to $a_j M = 0$ for $j = 0, \cdots, l$ and we have $f(x)M[x] = 0$, i.e., $M[x]$ is a prime module.

s-**prime:** For $m(x) \in M[x]$ where $m(x) = m_0 + m_1 + \cdots + m_t x^t$, we define the least coefficient of $m(x)$ as $m_k \neq 0$, the coefficient of the term $m_k x^k$ such that $m_{k-1} = m_{k-2} = \cdots = m_1 = m_0 = 0$. Let $f(x) \in R[x]$ and $m(x) \in M[x]$ such that $(f(x))^n R[x]m(x) = 0$ for some $n \in \mathbb{Z}^+$. Suppose $m(x) = \sum_{i=0}^{k} m_i x^i \neq 0$. Let m_d be the least coefficient of $m(x)$. If $f(x) = \sum_{j=0}^{l} a_j x^j$, then from $(f(x))^n R[x]m(x) = 0$, we have $a_0^n r m_d = 0$ for all $r \in R$ such that $a_0 M = 0$ since ${}_R M$ is s-prime. $a_0 M = 0$ implies all the terms in the expansion of $(f(x))^n r m_d$ involving a_0 vanish. This leads to $a_1^n r m_d = 0$ for all $r \in R$. Again ${}_R M$ s-prime implies $a_1 M = 0$. By a similar argument all terms in expansion of $(f(x))^n r m_d$ that involve a_1 vanish. This leads to $a_2^n r m_d = 0$ for all $r \in R$ such that $a_2 M = 0$. By induction principle, $a_i M = 0$ for all $i \in \{0, \cdots, l\}$ such that $f(x)M[x] = 0$ and hence ${}_{R[x]}M[x]$ is s-prime.

Strictly prime: Let $0 \neq m(x) \in M[x]$ with $m(x) = \sum_{i=0}^{k} m_i x^i$ such that m_t is the least coefficient of $m(x)$. $m_t \neq 0$ and M strictly prime imply there exits $b \in R$ such that if $abm_t = 0$ for some $a \in R$, then $aM = 0$. Suppose $a(x)bm(x) = 0$ for some $a(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then

$$a_0 b m_t = 0 \tag{7.5}$$

$$a_0 b m_{t+1} + a_1 b m_t = 0 \tag{7.6}$$

$$a_0 b m_{t+2} + a_1 b m_{t+1} + a_2 b m_t = 0 \tag{7.7}$$

$$\vdots$$

$$a_n b m_k = 0. \tag{7.8}$$

$a_0 b m_t = 0$ implies $a_0 M = 0$ since M is strictly prime. Equation 7.6 reduces to $a_1 b m_t = 0$ such that $a_1 M = 0$ since M is strictly prime. Equation 7.7 becomes $a_2 b m_t = 0$ and we get $a_2 M = 0$. It is clear that we get $a_i M = 0$ for $0 \leq i \leq n$. Hence $a(x)M[x] = 0$ and ${}_{R[x]}M[x]$ is a strictly prime module.

Strongly prime: Let $0 \neq m(x) \in M[x]$ with $m(x) = \sum_{i=0}^{k} m_i x^i$ such that m_t is the least coefficient $m(x)$. Since $m_t \neq 0$ there exists $F \subseteq R \subseteq R[x]$ such that for $a \in R$, $aFm_t = 0$ implies $aM = 0$. Let $a(x)Fm(x) = 0$ for some $a(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then, we get

$$a_0 F m_t = 0 \tag{7.9}$$

$$a_0 F m_{t+1} + a_1 F m_t = 0 \tag{7.10}$$

$$a_0 F m_{t+2} + a_1 F m_{t+1} + a_2 F m_t = 0 \tag{7.11}$$

$$\vdots$$

$$a_n F m_k = 0. \tag{7.12}$$

From Equation 7.9 it follows that $a_0 M = 0$. Equation 7.10 becomes $a_1 F m_t = 0$ and consequently $a_1 M = 0$. Equation 7.11 becomes $a_2 F m_t = 0$ and hence $a_2 M = 0$. Continuing with this process leads to $a_i M = 0$ for $0 \leq i \leq n$. Hence $a(x)M[x] = 0$ and we are through.

Classical semiprime: Let $a(x)R[x]a(x)R[x]m(x) = 0$ with $a(x) = \sum_{i=0}^{l} a_i x^i$ and $m(x) = \sum_{j=0}^{k} m_j x^j$. Then $(\sum_{i=0}^{l} a_i x^i) r (\sum_{i=0}^{l} a_i x^i) s (\sum_{j=0}^{k} m_j x^j) = 0$ for all $r, s \in R$ such that for all $r, s \in R$

$$a_o r a_o s m_0 = 0 \tag{7.13}$$

$$a_0 r a_0 s m_1 + a_0 r a_1 s m_0 + a_1 r a_0 s m_0 = 0 \tag{7.14}$$

$$a_0 r a_0 s m_2 + a_0 r a_2 s m_0 + a_2 r a_0 s m_0 + a_1 r a_1 s m_0 + a_1 r a_0 s m_1 + a_0 r a_1 s m_1 = 0 \tag{7.15}$$

$$\vdots$$

$$a_{l-1} r a_l s m_k + a_l r a_{l-1} s m_k + a_l r a_l s m_{k-1} = 0 \tag{7.16}$$

$$a_l r a_l s m_k = 0. \tag{7.17}$$

From Equation 7.13, $a_0Ra_0Rm_0 = 0$, M classical semiprime implies $a_0Rm_0 = 0$. Equation 7.14 reduces to: $a_0Ra_0Rm_1 = 0$. M classical semiprime implies $a_0Rm_1 = 0$. $a_0Rm_0 = 0$ and $a_0Rm_1 = 0$ reduce Equation 7.15 to:

$$a_0ra_0sm_2 + a_1ra_1sm_0 = 0 \text{ for all } r, s \in R. \tag{7.18}$$

Multiply Equation 7.18 by a_0R, the second term vanishes since $a_0Rm_0 = 0$. So, we get $a_0Ra_0Ra_0Rm_2 = 0$. M classical semiprime implies $a_0Ra_0Rm_2 = 0$. Again, M classical semiprime implies $a_0Rm_2 = 0$. This reduces Equation 7.18 to $a_1Ra_1Rm_0 = 0$ such that $a_1Rm_0 = 0$. Repeating this process leads to $a_iRm_j = 0$ for all i, j. Hence $a(x)R[x]m(x) = 0$ and ${}_{R[x]}M[x]$ is classical semiprime.

■

Theorem 7.1.3 *Let M be an R-module and $M[x]$ the corresponding $R[x]$-module. If P is a prime (resp. completely prime, strongly prime, s-prime, classical semiprime and completely semiprime) submodule in $M[x]$, then so is $P \cap M$ in M, i.e., the "primes" listed in this theorem satisfy the "going down" theorem.*

Proof:

Prime: Let $a \in R$, $m \in M$ such that $aRm \subseteq P \cap M$. $aR[x]m \subseteq P$ since each element of $aR[x]m$ is a sum of terms belonging to P and since P is prime, we have $m \in P$ or $aM[x] \subseteq P$. Hence $m \in P \cap M$ or $aM \subseteq P \cap M$.

Completely prime: Let $a \in R$ and $m \in M$ such that $am \in P \cap M$. Then $am \in P$. Since $a \in R \subseteq R[x]$, $m \in M \subseteq M[x]$ and P is completely prime in $M[x]$, $m \in P$ or $aM \subseteq aM[x] \subseteq P$ such that $m \in P \cap M$ or $aM \subseteq P \cap M$.

Hence, $P \cap M$ is completely prime in M.

Strongly prime: Let P be a strongly prime submodule in $M[x]$ and suppose that there exists $m \in M \setminus (P \cap M)$ such that for every finite subset $F \subseteq R$, there exists $r \in R$ with $rFm \subseteq P \cap M$ but $rM \not\subseteq P \cap M$ (and hence $rM \not\subseteq P$). Let J be any finite subset of $R[x]$, say $J = \{z_1, \cdots, z_n\}$ with $z_i = \sum_{j=0}^{n_i} a_{ij} x_i^j$ where $a_{ij} \in R$ and $i = 1, \cdots, n$. Furthermore, let $F_i = \{a_{ij},\ j = 0, 1, \cdots, n_i\}$ and $F = \cup_{i=1}^{n} F_i$. Then, F is a finite subset of R and from our assumption there exists $r \in R$ such that $rFm \subseteq P \cap M \subseteq P$ and $rM \not\subseteq P$. Let $z_i \in J$ be arbitrary. Then $rz_i m = r(a_{i0} + a_{i1}x_i + a_{i2}x_i^2 + \cdots + a_{in}x_i^n)m = ra_{i0}m + ra_{i1}mx_i + \cdots + +ra_{in}mx_i^n$. From $rFm \subseteq P$, $ra_{ij}m \in P$ for $j = 0, 1, \cdots, n$. It follows that $rz_i m \in P$. But z_i was arbitrary, hence $rJm \subseteq P$. This contradicts the fact that P is a strongly prime submodule since $rM \not\subseteq P$ and consequently $P \cap M$ is a strongly prime submodule of M. ■

The s-prime, classical semiprime and completely semiprime cases can be proved in a way similar to that of the prime and completely prime cases.

Corollary 7.1.1 *If $Q[x]$ is a prime (resp. s-prime, strongly prime, classical semiprime) submodule of ${}_{R[x]}M[x]$, then Q is a prime (resp. s-prime prime, strongly prime, classical semiprime) submodule of ${}_RM$.*

Proof: In Theorem 7.1.3, if we use $Q[x]$ in the place of P, then we get $P \cap M$ as $Q[x] \cap M = Q$. ■

By using similar arguments as for the polynomial ring $R[x]$ and $I[x]$ where $I \lhd R$ and where $(R/I)[x] \cong R[x]/I[x]$ we get:

Proposition 7.1.1 *If M is an R-module and N is a submodule of M, then*

$$(M/N)[x] \cong M[x]/N[x]$$

where $N[x] = \{\sum_{i=0}^{k} n_i x^i \ : \ n_i \in N\}$ *is an* $R[x]$*-submodule of* $M[x]$*.*

Proposition 7.1.2 *If* M *is an* R*-module and* $M[x]$ *is the corresponding* $R[x]$*-module, then* P *is a prime (resp. s-prime strongly prime, classical semiprime) submodule of* M *if and only if* $P[x]$ *is a prime (resp. s-prime, strongly prime, classical semiprime) submodule of* $M[x]$*.*

Proof: This is clear from Theorem 7.1.2, Proposition 7.1.1 and Corollary 7.1.1. ∎

Proposition 7.1.3 *Let* M *be an* R*-module and* $M[x]$ *the corresponding* $R[x]$*-module.* P *is a completely prime submodule of* M *if and only if* $P[x]$ *is a completely prime submodule of* $M[x]$*.*

Proof: From [52, Theorem 1.6], ${}_RM$ is completely semiprime (reduced) if and only if ${}_RM[x]$ is completely semiprime. Hence from Proposition 7.1.1, we have P is a completely semiprime submodule if and only if $P[x]$ is a completely semiprime submodule of $M[x]$. But P is a completely prime submodule if and only if P is both prime and completely semiprime, see Theorem 3.4.2. From this and Proposition 7.1.2, the result follows. ∎

7.2 Module radicals with Amitsur property

If $\mathcal{M}$ is any of the following classes of modules: prime, s-prime, completely prime, strictly prime and strongly prime, then $\mathcal{M}$ is closed under taking non-zero submodules and by Proposition 2.3.1 γ is a radical.

As for rings, see [33, p. 276], [77, Definition 2.13] we say a radical map γ

1. has Amitsur property if $(\gamma(M[x]) \cap M)[x] = \gamma(M[x])$;

2. is polynomially extensible if $\gamma(M) = M$, then $\gamma(M[x]) = M[x]$;

3. satisfies the polynomial equation if $\gamma(M[x]) = \gamma(M)[x]$.

Theorem 7.2.1 *If $\beta(M)$, $\mathcal{N}(M)$, $\beta_{co}(M)$, $\beta_s(M)$ and $\beta_{sc}(M)$ respectively denote the prime radical, s-prime radical, completely prime radical, strongly prime radical and strictly prime radical of M, then*

1. $\beta(M[x]) = \beta(M)[x]$,

2. $\mathcal{N}(M[x]) = \mathcal{N}(M)[x]$,

3. $\beta_{co}(M[x]) = \beta_{co}(M)[x]$,

4. $\beta_s(M[x]) = \beta_s(M)[x]$ *and*

5. $\beta_{sc}(M[x]) \subseteq \beta_{sc}(M)[x]$.

Proof: This is a simple application of Theorem 7.1.1 and the fact that the prime (resp. s-prime, completely prime, strongly prime and strictly prime) radical of a module is the intersection of all the prime (resp. s-prime, completely prime, strongly prime and strictly prime) submodules. ■

Proposition 7.2.1 *Let γ be a module radical map such that $\gamma(M[x]) = \gamma(M)[x]$, then*

1. $(\gamma(M[x]) \cap M)[x] = \gamma(M[x])$, *i.e., γ satisfies the Amitsur property;*

2. *if $\gamma(M) = M$, then $\gamma(M[x]) = M[x]$, i.e., γ is polynomially extensible.*

Proof: $(\gamma(M[x]) \cap M)[x] = (\gamma(M)[x] \cap M)[x] = \gamma(M)[x] = \gamma(M[x])$. Suppose $\gamma(M) = M$, then by hypothesis, $\gamma(M[x]) = \gamma(M)[x] = M[x]$. ■

Theorem 7.2.2 *Let β, $\mathcal{N}$, β_{co} and β_s denote respectively the prime, s-prime, completely prime and strongly prime module radicals. Then, β (resp. $\mathcal{N}$, β_{co} and β_s) satisfies the Amitsur property and is polynomially extensible.*

Proof: Follows from Proposition 7.2.1 and Theorem 7.2.1. ∎

Theorem 7.2.3 *Let R be a ring such that $x \in xR$ for every $x \in R$, (i.e., R has a right identity). Let γ be any of the following radicals: prime radical, upper nil radical, generalized nil radical and strongly prime radical; then*

$$\gamma(R[x]) = \gamma(R)[x].$$

Proof: From [29, Prop. 4.8] we have that $\gamma(R) = \gamma({}_R R)$ for γ any of the above mentioned radicals. From Theorem 7.2.1, we have $\gamma(R[x]) = \gamma({}_{R[x]}R[x]) = \gamma({}_R R)[x] = \gamma(R)[x]$. ∎

Corollary 7.2.1 *Let R be any ring such that for every $x \in R$, $x \in xR$ and γ any of the following ring radicals: prime radical, upper nil radical, strongly prime radical and generalized nil radical; then*

1. *$(\gamma(R[x]) \cap R)[x] = \gamma(R[x])$, i.e., γ satisfies the Amitsur property;*

2. *if $\gamma(R) = R$, then $\gamma(R[x]) = R[x]$, i.e., γ is polynomially extensible.*

Theorem 7.2.4 *The module ${}_R M$ is 2-primal (resp. β_s-primal) if and only if so is ${}_{R[x]}M[x]$.*

Proof: We prove the 2-primal case, the other case can proved in a similar way. If $\beta(M) = \beta_{co}(M)$, from Theorem 7.2.1, $\beta(M[x]) = \beta(M)[x] = \beta_{co}(M)[x] = \beta_{co}(M[x])$. For the converse, suppose $\beta(M[x]) = \beta_{co}(M[x])$. Then $\beta_{co}(M) = M \cap \beta_{co}(M)[x] = M \cap \beta(M)[x] = \beta(M)$. ∎

Chapter 8

Conclusion

In this chapter, we bring together all the implications between different "primes" discussed in the preceding chapters; radicals corresponding to the "primes" are ordered. The nine module radicals studied in this thesis coincide whenever the module is defined over a commutative Artinian ring, see Theorem 8.1.1. Questions that emerge from the thesis are posed.

■

8.1 A summary of implications

From chapters two, three, four and six, we have the following implications:

(a) when a module is defined over an arbitrary ring, see Figure 8.1;

$$\begin{tikzcd}
\text{simple} \arrow[dd] & & \text{completely prime} \arrow[d] \arrow[ddr] & \\
& & \text{strictly prime} \arrow[dll] \arrow[d] & \\
s\text{-prime} \arrow[dr] & & \text{strongly prime} \arrow[d] & \text{classical completely prime} \arrow[ddl] \\
& l\text{-prime} \arrow[r] & \text{prime} \arrow[d] & \\
& & \text{classical prime} &
\end{tikzcd}$$

Figure 8.1: Implications between different “primes” in modules

(b) when a module is defined over a commutative ring, we have:

completely prime $\Leftrightarrow$ strictly prime $\Leftrightarrow$ strongly prime $\Leftrightarrow$ s-prime $\Leftrightarrow$

l-prime $\Leftrightarrow$ prime $\Rightarrow$ classical prime $\Leftrightarrow$ classical completely prime.

(c) P a “prime” submodule of ${}_RM$ implies the ideal $(P : N)$ of R is “prime” for any $N \leq M$ with $N \nsubseteq P$, see Table 8.1.

Submodule P of a module ${}_RM$		Ideal $(P : N)$ of R for any $N \leq {}_RM$ with $N \not\subseteq P$
completely prime submodule	$\Rightarrow$	completely prime ideal
classical completely prime submodule	$\Rightarrow$	completely prime ideal
s-prime submodule	$\Rightarrow$	s-prime ideal
strictly prime submodule	$\Rightarrow$	strictly prime ideal (strongly prime ideal of bound one)
strongly prime submodule	$\Rightarrow$	strongly prime ideal
prime submodule	$\Rightarrow$	prime ideal
classical prime submodule	$\Rightarrow$	prime ideal

Table 8.1: P "prime" implies $(P : N)$ "prime"

Remark 8.1.1 In this section, we only define strictly prime ideals and strongly prime ideals; and prove the implication: P strongly prime (resp. strictly prime) submodule of ${}_RM$ implies $(P : N)$ is a strongly prime (resp. strictly) ideal of R for any submodule N of M not contained in P. This is because all the other "prime" ideals and their corresponding implications were respectively defined and proved in the previous chapters. For the prime and classical prime submodules, see [10] and [11] respectively.

A ring R is left strongly (resp. strictly) prime if for each $0 \neq r \in R$ there exists a finite set $F \subseteq R$ (resp. $s \in R$) such that for $t \in R$, $tFr = 0$ (resp. $tsr = 0$) implies $t = 0$. An ideal I is a left strongly (resp. strictly) prime ideal of R if and only if R/I is a left strongly (resp. strictly) prime ring. We here omit the word "left" and call left strongly (resp. left strictly) prime rings/ideals strongly (resp. strictly) prime rings/ideals.

Proposition 8.1.1 *If P is a strongly (resp. strictly) prime submodule of ${}_RM$, then $(P : N)$ is a strongly (resp. strictly) prime ideal of R for any submodule*

N of M not contained in P.

Proof: Let $a \in R \setminus (P : N)$, then $aN \not\subseteq P$. So, $am \notin P$ for some $m \in N$. Now $0 \neq am \in M$ and P strongly prime implies there exists a finite subset $F \subseteq R$ such that if $b \in R$ and $bF(am) \subseteq P$, then $bM \subseteq P$. Now suppose $bFa \subseteq (P : N)$, i.e., $bFaN \subseteq P$, then $bF(am) \subseteq P$. It follows that $bM \subseteq P$ which implies $bN \subseteq P$, i.e., $b \in (P : N)$. The proof for the strictly prime case is similar. ■

From Figure 8.1, we get Figure 8.2 which shows the strength of module radicals. This Figure is what one would expect - compare it with strength of ring radicals in [81] and in [33, p.293].

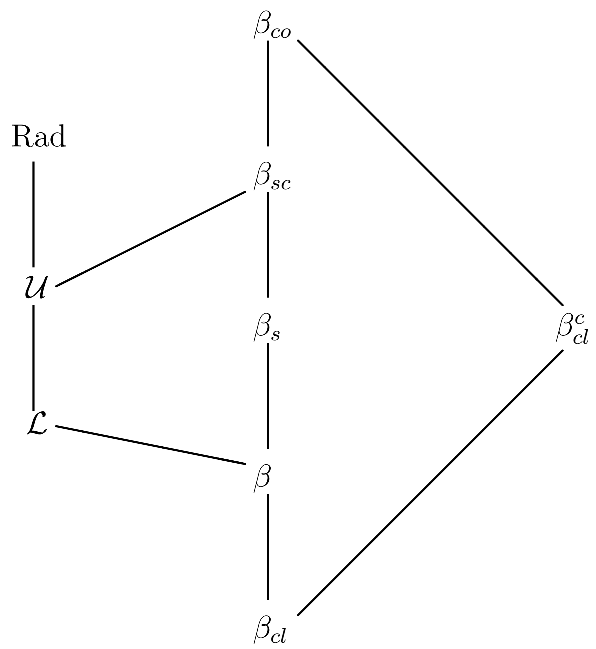

Figure 8.2: Strength of module radicals

A comparison of Figure 8.2 of module radicals and that of ring radicals in [81] indicates lots of gaps in the figure for module radicals. This compels us to suggest that, more should be done in terms of investigating module analogues for existing ring radicals - where this is not possible, an explicit declaration of its nonexistence should be made.

Theorem 8.1.1 *If ${}_RM$ is a module, then*

1. $\beta_{cl} = \beta = \mathcal{L} = \mathcal{U} = Rad$ *whenever R is left Artinian;*
2. $\beta_{co} = \beta_{sc} = \beta_s = \beta = \mathcal{L} = \mathcal{U}$ *and* $\beta_{cl} = \beta^c_{cl}$ *whenever R is commutative;*
3. $\beta^c_{cl} = \beta_{cl} = \beta = \mathcal{L} = \mathcal{U} = Rad = \beta_{co} = \beta_{sc} = \beta_s$ *whenever R is both Artinian and commutative.*

Proof:

1. From Figure 8.2, $\beta_{cl} \subseteq \beta \subseteq \mathcal{L} \subseteq \mathcal{U} \subseteq$ Rad. If R is left Artinian, by [9, Theorem 2.15] Rad $\subseteq \beta_{cl}$ from which the desired result follows.
2. For commutative rings, there is no distinction between:
   (a) completely prime, strictly prime, strongly prime, prime, l-prime and s-prime submodules;
   (b) classical prime and classical completely prime submodules.
3. Follows from 1) and 2) above.

■

Remark 8.1.2 In the place of commutative ring in Theorem 8.1.1 we can have:

1. M IFP or M symmetric,
2. ${}_RM$ cyclic (finitely generated, or free) with R medial (LSD, left permutable or right permutable).

8.2 Emanating questions

Work from this thesis raises the following questions:

Question 8.2.1 Can we have a 2-primal module characterized in terms of nilpotent elements (as defined in chapter five) of that module?

Question 8.2.2 Is the sum of nil submodules nil? If no, can we use this fact to prove Köthe's conjecture in negative?

Question 8.2.3 What is the relationship between the upper nil radical of a module M (which is defined as the intersection of all s-prime submodules of M) and the sum of all nil submodules of M? By a nil submodule, we mean a submodule whose all elements are nilpotent. For rings, these two notions coincide.

Question 8.2.4 Does the Jacobson radical (resp. Brown-McCoy radical, classical prime radical, strictly prime radical) of a module satisfy the Amitsur property and is polynomially extensible?

Question 8.2.5 The theory of "primes" in chapters 2, 3, 4 and in papers [9] and [10] is developed in a rather similar way. Isn't there a universal way of studying them?

Question 8.2.6 If P is a "prime" submodule of ${}_RM$, what can be said of the ring $\mathrm{End}_R(M/P)$? Can we obtain a table similar to Table 8.1?

Question 8.2.7 Under which conditions can one have reverse implications in Table 8.1?

Appendix I: Conferences, workshops and seminars attended

Results in this thesis were presented by the author in the following conferences, workshops and seminars:

1. South African Mathematical Society congress held at Stellenbosch University in October-November 2012.

2. Eastern Cape Post Graduate seminar held at Nelson Mandela Metropolitan University in October 2012.

3. Pure Mathematics workshop at Strathmore University, Nairobi Kenya in July 2012.

4. International Conference on the Theory of Radicals, Rings and Modules held at Sultan Qaboos University in the Sultanate of Oman in January 2012.

5. Pure Mathematics workshop at Strathmore University, Nairobi Kenya in July 2011.

6. First Kenyatta University International Mathematics Conference that was held in June 2011 at Kenyatta University in Nairobi Kenya.

Appendix II: Tables of symbols

Category	Symbol	Description
General	$=$	equal to
	$\cong$	isomorphism
	$\equiv$	congruence
	$\subset$	strict inclusion of sets
	$\subseteq$	contained or equal to (for sets)
	$\in$	member of (for elements)
	$\bigoplus$	direct sum
	$\prod$	direct product
	$\cap$	intersection of sets
	$\cup$	union of sets
	$\emptyset$	empty set
	$\lim\limits_{\leftarrow}$	inverse limit
	$\neq$	not equal to
	$\not\subset$	not a strict inclusion of sets
	$\not\subseteq$	neither contained nor equal to (for sets)
	$\notin$	not member of (for elements)
Sets of numbers	$\mathbb{R}$	reals
	$\mathbb{Z}$	integers
	$\mathbb{Z}^+$	positive integers
	$\mathbb{N}$	natural numbers
	$\mathbb{Z}_p$	p-adic integers
Rings and ideals	$\lhd$	ideal
	$\lhd\cdot$	essential ideal
	$\beta(R)$	prime radical of a ring R
	$\beta_{co}(R)$	completely prime radical of a ring R
	$\mathcal{U}(R)$	upper nil radical of a ring R
	$\mathrm{Jac}(R)$	Jacobson radical of a ring R
	$\mathcal{L}(R)$	Levitzki radical of a ring R
	$\mathcal{N}(R)$	set of all nilpotent elements of a ring R
	$\mathcal{N}_g(R)$	generalized nil radical of R

Table 8.2: Symbols defined

Category	Symbol	Description
Modules	$N \leq M$	N is a submodule of M
	$(N : M)$	annihilator of the factor module M/N
	$(0 : M)$	annihilator of the module M
	$\beta(M)$	prime radical of a module M
	$\mathrm{Rad}(M)$	Jacobson radical of a module M
	$\beta_{co}(M)$	completely prime radical of a module M
	$\beta_{cl}(M)$	classical prime radical of a module M
	$\beta_{cl}^{c}(M)$	classical completely prime radical of a module M
	$\beta_{sc}(M)$	strictly prime radical of a module M
	$\beta_{s}(M)$	strongly prime radical of a module M
	$\mathcal{U}(M)$	upper nil radical of a module M
	$\mathcal{L}(M)$	Levitzki radical of a module M
	$\mathcal{N}(M)$	nilpotent elements of a module M
	$\mathcal{N}_s(M)$	strongly nilpotent elements of a module M
	$E_M(0)$	envelope of the zero submodule of M
	$\langle m \rangle$	submodule generated by m

Table 8.3: More symbols defined

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