EXPLORING TEACHING PROFICIENCY IN GEOMETRY OF SELECTED EFFECTIVE MATHEMATICS TEACHERS IN NAMIBIA

A thesis submitted in fulfilment of the requirements for the degree of

DOCTOR OF PHILOSOPHY
(Mathematics Education)

of

RHODES UNIVERSITY, GRAHAMSTOWN, SOUTH AFRICA
(Faculty of Education)

By

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NOVEMBER, 2013

ABSTRACT

Quality mathematics education relies on effective pedagogy which offers students appropriate and rich opportunities to develop their mathematical proficiency (MP) and intellectual autonomy in learning mathematics. This qualitative case study aimed to explore and analyse selected effective mathematics teachers' proficiency in the area of geometry in five secondary schools in five different Namibia educational regions. The sample was purposefully selected and comprised five mathematics teachers, identified locally as being effective practitioners by their peers, Education Ministry officials and the staff of the University of Namibia (UNAM). The schools where the selected teachers taught were all high performing Namibian schools in terms of students' mathematics performance in the annual national examinations. The general picture of students' poor performance in mathematics in Namibia is no different to other sub-Saharan countries and it is the teachers who unfortunately bear the brunt of the criticism. There are, however, beacons of excellence in Namibia and these often go unnoticed and are seldom written about. It is the purpose of this study to focus on these high achievers and analyse the practices of these teachers so that the rest of Namibia can learn from their practices and experience what is possible in the Namibian context. The mathematical content and context focus of this study was geometry. This qualitative study adopted a multiple case study approach and was framed within an interpretive paradigm. The data were collected through individual questionnaires, classroom lesson observations and in-depth open-ended and semi-structured interviews with the participating teachers. These interviews took the form of post lesson reflective and stimulated recall analysis sessions.

An adapted framework based on the Kilpatrick, Swafford and Findell's (2001) five strands of teaching for MP was developed as a conceptual and analytical lens to analyse the selected teachers' practice. The developed coding and the descriptive narrative vignettes of their teaching enabled a qualitative analysis of what teachers said contributed to their effectiveness and how they developed MP in students. An enactivist theoretical lens was used to complement the Kilpatrick et al.'s (2001) analytical framework. This enabled a deeper analysis of teacher teaching practice in terms of their embodied mathematical knowledge, actions and interactions with students.

The analysis of data showed that three of five strands, conceptual understanding (CU), procedural fluency (PF) and productive disposition (PD), were addressed regularly by all five participating teachers. Evidence of addressing either the development of students' strategic competence (SC) or adaptive reasoning (AR) appeared rarely. Of particular interest in this study was that the strand of PD was the glue that held the other four strands of MP together. PD was manifested in many different ways in varying degrees. PD was characterised by a high level of content knowledge, rich personal experience, sustained commitment, effective and careful preparation for lessons, high expectations of themselves and learners, collegiality, passion for mathematics and an excellent work ethic. In addition, the teachers' geometry teaching practices were characterised by making use of real-world connections, manipulatives and representations, encouraging a collaborative approach and working together to show that geometry constituted a bridge between the concrete and abstract. The findings of the study have led me, the author, to suggest a ten (10) principles framework and seven (7) key interrelated factors for effective teaching, as a practical guide for teachers.

This study argues that the instructional practices enacted by the participating teachers, who were perceived to be effective, aligned well with practices informed by the five strands of the Kilpatrick et al.'s (2001) model and the four concepts of autopoesis, co-emergence, structural determinism and embodiment of the enactivist approach. The study concludes with recommendations for effective pedagogical practices in the teaching of geometry, and opportunities for further research.

Keywords:
Teaching proficiency, geometry, mathematical proficiency (MP) strands, effective teachers, mathematical knowledge for teaching for mathematical proficiency, effective teaching practices, Enactivism, embodiment and structural determinism.

ACKNOWLEDGEMENTS

The road to completing this doctoral thesis has certainly been a long and circuitous one, and along the way I learned from the support of many people at different times and in several universities and institutions. I wish to express my sincerest gratitude to the following people who contributed towards the completion of this thesis.

- First and foremost, I thank the higher being, Almighty God, for availing me the opportunity to come to the Republic of South Africa to pursue my studies at Rhodes University: Where Leaders Learn. Whilst at Rhodes, I learned a number of invaluable things which are not contained in this thesis such as love, virtues of values, different cultures and customs, including tolerance and resilience which kept me focused to this piece of work. During all this time, I was able to make life-long friends and network with scholars from other academic disciplines whose input and scholarly support have been a surmountable blessing and kept me in motion.
- I want to begin by thanking the five teachers who voluntarily and willingly participated in my study and shared their visions of effective teaching practices with me. I am indebted to this special group of Namibian mathematics teachers who invested their times in me and gave generously of their insights and expertise to contribute to the study. I thank the principals for giving me access to their schools and mathematics classrooms. They held high standards for themselves and the children they taught, and believed that hard work, commitment, discipline and the passion for mathematics could make a difference.
- My deepest gratitude goes to Prof Marc Schäfer, my supervisor, for his eagerness to engage with me, time to read everything I wrote, his outstanding supervision support and invaluable guidance throughout the PhD journey. Prof, you always trusted me, and was ready to accept extended and ad hoc sessions without excuses. Your positive influence has afforded me the opportunity to grow in my knowledge and understanding of mathematics education and embrace the world of research.
- I have been exceedingly fortunate to study at Rhodes University, and my work could not have proceeded without the forbearance of remarkable individuals. I would like to recognise each one and thank them individually and collectively for their vigorous support and engagement.
    - My deep thanks to Dr Caroline van der Mescht for your insightful reading of several drafts and valuable guidance, your ability to help me to understand my

work and see mathematical points of significance, and for your jumping into my work with both feet.

  - A big thank you to Rob Kraft, for your tremendous guardianship and for the helpful advice you provided along the way.
  - I also wish to thank Prof Mellony Graven for her input and valuable discussion when I was working on my proposal and data analysis.
  - I am also grateful to Prof Rob O'Donoqhue for the interest that he showed in my work and for his assistance in making Enactivist theory of cognition more intelligible to me.
  - Special thanks are also extended to Prof Hamsa Venkatakrishnan (University of Witwatersrand), Prof Stephen Lerman (South Bank University, London), Prof Elaine Simmt (University of Alberta), Prof Jérôme Proulx (Université du Québec à Montréal), Prof Merriam Sharan (University of Georgia), Prof David Stephens (University of Brighton) and Prof Mike Askew (Monash University, Melbourne), who gave me valuable advices. Their thoughtful comments and willingness to share their research expertise are greatly appreciated.

- My thinking was shaped by a generous group of scholars at Rhodes, including Dr Mike Mhlolo, Dr Charles Chikunda, Dr Clayton Zazu, Dr Duncan Samson, Debbie Stott, Peter Pausigere, Daniel Sabai, Helen Averbuch and Nyameka Kangela. It would be hard to imagine a more vibrant and insightful group of scholars to have studied with.
- I would also like to make special mention to the late Dr Humphrey Uyouyo Atebe, who was a source of great intellectual excitement. I could not ask for a better friend, a mentor and a partner in my research. May his soul rest in eternal peace!
- I am also thankful to Judy Cornwell for her editorial support, and willingness to proofread, edit and refine my thesis.
- Finally and perhaps most importantly, I would like to recognise and sincerely thank my family and friends, especially my son Abed Tashiya Tate-ati and my two adorable daughters Grace Popyeni and Well-Best Tulela for their committed and loving support, encouragement, patience and belief in my ability during this endeavour. Thank you, sweetheart, Scholastika Oletuwene. You have been very patient with me the whole time I was trying to complete my thesis, and the sacrifices made were as much mine as they were yours.

DEDICATION

This dissertation is dedicated to my late parents always in my heart

My beloved mother, **Beatrix-Maria Shekupe Aindongo** (1948-2012),
who passed on when I was busy shaping up my research.

and

My father, **Liberius Angalo Stephanus** (1942-1997),
who passed on, on the Christmas Eve a day I arrived home from school during my second year of undergraduate studies, of whom I am very, very proud and who I hope was proud of me.

When I was young, my father always told us that mathematics opens doors and careers. We could be anything we want to be as long as we did well in mathematics. I am grateful to my dad for helping me to get to grips with mathematics.

Thank you for taking those first steps with me!

DECLARATION OF ORIGINALITY

I, **Gervasius Hivengwa Stephanus**, (student number g10s2968) declare that this Doctoral dissertation *exploring teaching proficiency in geometry of selected effective mathematics teachers in Namibia* is my own work written in my own words. Where I have drawn on the words or ideas of others, these have been acknowledged in the manner required by the Rhodes University Department of Education Guide to referencing.

Gervasius Hivengwa Stephanus November 2013

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(Signature) (Date)

TABLE OF CONTENTS

LIST OF ABBREVIATIONS AND ACRONYMS

AR	Adaptive Reasoning
BETD	Basic Teacher Education Diploma
BSc	Bachelor of Science
CCK	Common Content Knowledge
CU	Conceptual Understanding
DNEA	Directorate of National Examination and Assessment
ELCIN	Evangelical Lutheran Church in Namibia
FoE	Faculty of Education
FRF	First Rand Foundation (Mathematics Chair at Rhodes University)
HED	Higher Education Diploma
HoD	Head of Department
ICT	Information Communication Technology
IGCSEO/H	International General Certificate of Secondary Education Ordinary/Higher
IRF	Initiation-Response-Feedback
JSC	Junior Secondary Certificate
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
LCE	Learner Centred Education
MEC	Ministry of Education and Culture
MEd	Masters of Education degree
MEO	Mathematics Education Officer
MKT	Mathematical Knowledge for Teaching
MoE	Ministry of Education
MTP	Mathematical Teaching Proficiency
MSc	Master of Science
MST	Mathematics, Science and Technology
MP	Mathematical Proficiency
MTEP	Mathematics Teacher Enrichment Programme
NAMCOL	Namibia College of Open Learning
NCTM	National Council of Teachers of Mathematics
NIED	National Institute for Educational Development
NSSC	Namibia Senior Secondary Certificate

NSSCH	Namibia Senior Secondary Certificate Higher level
NSSCO	Namibia Senior Secondary Certificate Ordinary Level
OECD	Organisation for Economic Co-operation and Development
OER	Omusati Education Region
OHP	Overhead Projector
PCK	Pedagogical Content Knowledge
PD	Productive Disposition
PF	Procedural Fluency
PS	Permanent Secretary
SADC	Southern African Development Community
SC	Strategic Competence
SCK	Specialised Content Knowledge
SMK	Subject Matter Knowledge
UK	United Kingdom
UNAM	University of Namibia

DEFINITION OF KEY CONCEPTS/TERMINOLOGIES USED IN THIS STUDY

In this particular section I define and present a summary of the fundamental concepts and terminologies underpinning this study. Full discussion of these terms is included in the literature review in Chapter Two.

- **Effective mathematics teachers**

For the purpose of this study, effective mathematics teachers are those whose learners have consistently performed well in the national mathematics examinations (JSC and NSSC) for the last three years and perceived effective by peers and Ministry of Education officials.

- **Effective teaching**

This study explores the idea of teacher quality in the context of effective pedagogy in mathematics teaching and mathematics teachers' classroom practices (interactions), and adopts a definition provided by Kilpatrick et al. (2001) and his associates to mean the following:

- Teaching strategies that enable students to build on their existing proficiencies, interest, understanding and experiences, and
- The ability to invite students to explain their solutions and solution methods to others, listen to and respect one another, accept and evaluate different viewpoints and engage in an exchange of mathematical thinking and perspectives in a variety of contexts (Kilpatrick et al., 2001).

- **Teacher effectiveness**

Effectiveness is defined, in this study, as how well teachers are able to promote authentic learning and to develop mathematical proficiency in their learners (Kilpatrick et al., 2001).

- **Mathematical proficiency (MP)**

In this study, the use of the term "mathematical proficiency" is consistent with Kilpatrick et al.'s (2001) definition. It refers to five interwoven strands: *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning* and *productive disposition*[1].

- **Teaching for mathematical proficiency (MTP)**

In this study, I use the term *mathematical teaching proficiency* (MTP) as defined by Kilpatrick et al. (2001), who defined MTP as "teacher's *effectiveness* in consistently helping students learn worthwhile mathematical content, and teacher's *versatility*: being able to work effectively with a wide variety of students in different environments and across a range of mathematical content" (p. 369).

- **Classroom practices**

For this study, this term is used interchangeably with *classroom interactions*. It thus refers to interaction practices and verbal and non-verbal classroom communications between the teacher and the learners, and among the learners themselves (Lewis, 2009).

- **Instructional practice**

I use Cohen and Ball's (2001) definition of instructional practice to mean "interactions that involve the teacher, the pupils and the learning content" (p. 75). That is, instructional practice is more than just teaching and what the teacher does. It thus includes how the teacher works in interaction with the learners and learning content, and these interactions proceed inside multiple contexts: the school setting and culture, the classroom environment and learners' backgrounds. Explicitly, when I refer to *instructional practice*, in this study, I am concerned with the instructional routines inside the mathematics classroom but am also cognizant that these contexts permeate the classroom in ways that are both observable (visible) and undetectable (invisible) to the researcher.

- **Teaching practice**

Teaching practice, as distinct from instructional practice, then, refers to a specific kind of teaching practice that unfolds in real time in the mathematics classroom in the presence of the students. It refers specifically to "the teacher, what the teacher does or works on, the teacher's set of habitual acts or actions and the teacher's professional mode of being" (Lewis, 2009, p. 4).

- **Co-emergence and structural determinism/coupling**

Co-emergence and structural determinism are enactivist concepts underpinning this study and are part of the theoretical framework. This research project looks at how teachers' mathematical classroom discourses co-emerge or co-evolve with students' ideas. Co-emergence refers to the mathematical ideas that appear, show up or arise in the processes of mathematics teaching and learning due to the interactions between the teacher and the learners or interactions among learners themselves (Miranda, 2004).

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[1] While conceptual understanding (CU) refers to "an integrated and functional grasp of mathematical ideas" (p. 118) that involves the organisation of knowledge into a coherent whole, procedural fluency (PF) denotes the flexible, quick and accurate performance of appropriate procedures. Strategic competence (SC) refers to the ability to formulate, represent and solve mathematical problems. Adaptive reasoning (AR) is the capacity of logical thought, reflection, explanation and justification of the relationships among concepts and situations. Hence, it is not enough for the teacher to only know the algorithms involved in solving a mathematics task. The teacher also needs to be able to justify and explain the logic behind his solution process. Productive disposition (PD) is the ability "habitual inclination to see mathematics as sensible, useful and worthwhile coupled with a belief in diligence and one's own efficacy" (p. 116).

LIST OF FIGURES

LIST OF TABLES

CHAPTER ONE

INTRODUCTION AND THE CONTEXT OF THE STUDY

"Mathematics is one of humanity's great achievements. By enhancing the capabilities of human mind, mathematics has facilitated the development of science, technology, engineering, business and government" (Kilpatrick, Swafford and Findell, 2001, p. 1).

1.1. Introduction to the study

Mathematics is at the heart of many effective careers (National Council of Teachers of Mathematics (NCTM), 1998). It supports learning in a diverse range of fields of study, whether in science or the humanities (Kho, 2007). At the individual level mathematics also "underpins many aspects of our everyday activities, from making sense of information in the newspaper to making decisions about personal finances" (p. 3). Today, an understanding of Mathematics, Science and Technology (MST) is very important in the workplace. Pillay (2006) comments that "as routine mechanical and clerical tasks become computerised, more and more jobs require high level skills that involve critical thinking, problem solving, communicating ideas and collaborating effectively" (p. 584). Many of these jobs, for example engineering, build on skills developed through high-quality MST education. Research evidence also reveals that achievements in MST are increasingly recognised as one of the most reliable indicators for measuring socio-economic and geo-political development among nations (Atebe, 2008). Globally, a strong grounding in mathematics is essential to provide a highly educated work force that supports innovations and technology-driven economics (Ang, 2008). Thus, there is a lot of emphasis placed on the provision of good quality MST education. It is against this background that Namibia, as a developing country, needs quality mathematics education in order to compete favourably in the ever growing global world of technology. The promotion of quality and effective mathematics education in Namibian schools is essential for attaining the technical, scientific and economic goals and transform the society into a knowledge-based economy (Nambira, Kapenda, Tjipueja and Sichombe, 2009). The "Namibian Vision 2030" (Office of the President, 2004, p. 2) encapsulates the aspirations of Namibia in this regard, and it seems valuable to explore these goals further.

The Vision 2030's overall goal is to improve the quality of life of the Namibian people to match the level of their counterparts in the developed world. The vision is that "[B]y the year 2030, Namibia should be a prosperous, industrialised and knowledgeable economic-based nation enjoying peace, harmony and political stability" (Office of the President, 2004, p. 2).

But without an educated workforce, this vision might well remain a mirage. A key component of the vision is an emphasis on skills development and high quality education that prepares learners to take advantage of a rapidly changing global environment. This includes development in MST education so Namibia needs mathematics teachers who are able to continually improve their teaching practices and nurture a productive passion for mathematics. In this regard, *quality teaching* is critical to the Namibian national vision. Therefore, information on the nature and quality of teachers' classroom instructional practices is vital in order to improve both teaching quality, and levels of mathematical proficiency and achievement among students.

This study thus sought to analyse the teaching practice of mathematics teachers who are perceived to be effective in the classroom. The working definition of ***effective*** or ***effective teaching*** that is used in this study is a broad one. Effective teachers, for the purpose of this study, are those teachers whose learners have been consistently performing well in the national mathematics examinations. Further, they are teachers who have a high standing and good reputation in the mathematics education community, including the Namibian Ministry of Education. Askew, Brown, Rhodes, William and Johnson (1997) defined effective numeracy teachers as highly performing mathematics teachers who have the knowledge and awareness of interrelations between areas of the mathematics curriculum that they teach, and "their classes of pupils had, during the year, achieved a high average gain in numeracy in comparison with other classes from the same year group" (p. 2). The latter part of this definition is consistent with the selection of effective teachers in this study.

The focus on "effective teachers" within the Namibian context is in part motivated by the intent to present credible role models to others within Namibia and beyond and to disseminate their repertoire of strategies and approaches so that others might learn from them. The selection was, in part, made on the basis of teachers whose learners have been performing well in the mathematics national examinations over the past three years (2008-2010). The literature shows that the teacher remains an important and indispensable resource in the provision of quality education. The quality of interaction between teachers and learners directly relates to the performance of learners. Dimarco (2010) argues that action in the mathematics classroom is a catalyst in the process of developing students' mathematical proficiency to progress into mathematics at university level. This action involves effective classroom learning communities that encourage mutual engagement in rigorous mathematics via quality mathematical interactions (ibid). Kilpatrick, Swafford and Findell (2001) expound that quality mathematical interactions potentially build students' "mathematical dispositions and intellectual autonomy" (p. 316). Building Namibian's human capital so that it has the proficiency, agency and confidence to choose and utilise mathematics in flexible ways hinges on effective teaching. Notions of teacher effectiveness and effective mathematics teaching will be unpacked further in Chapter two.

1.2. The context of the study: Namibia

Namibia is a post-colonial nation which reformed its education system after independence in 1990. One of the major aims of that education transformation was to redress inherent imbalances and improve the quality of education. Article 20 of the Namibian Constitution stresses all Namibian learners have a right to quality education. The educational reform post 1990 prioritised the teaching of mathematics in order to fulfil the major goals of Namibian education, viz. *access, quality, equity, democracy* and *lifelong learning* (Namibia. Ministry of Education and Culture [MEC], 1993). The educational reform in Namibia also proposed pedagogical changes in the classroom, namely the introduction of Learner Centred Education (LCE) shortly after independence (Kapenda, 2007; MEC, 1993). LCE implies a shift in the teacher's role from that of a classroom expert to being a facilitator of learning (*ibid*). This shift had a direct effect on the mathematics curriculum by emphasising mathematical thinking, reasoning, problem-solving skills and proficiency as well as learning mathematics with sufficient understanding.

Despite the education transformation, the teaching and learning situation in Namibian schools remains poor (Stephanus, 2008). Schools comprise two broad categories: a relatively small group of well performing schools in terms of learners' mathematical and academic achievement. These are often former white schools such as the Etosha secondary school in the Oshikoto region, Oshigambo high, a mission school in Ohangwena region, and some urban senior secondary schools. These high performing schools have established a culture of success that sustain them, and which is passed on from one generation of teachers and learners to another. One could suppose that the teachers in these schools use effective teaching and learning strategies that stimulate students' interests and understanding in mathematics. I thus wish, in this study, to explore and document effective teaching practices (with respect to teachers' teaching proficiency) that are effective in order to assist schools and teachers in Namibia to learn from these practices. Namibia's school leaving examinations occur at the end of Grade 12. "These examinations determine students' eligibility for further study and job opportunities" (Brodie, 2009, p. 3) as well as monitor school performance over time. A close look at the 2010 examination results in Table 1 shows that while 39% of secondary schools offer grade 12, only 561 students out of 28757 students who sat for the grade 12 Namibia Senior Secondary Certificate (NSSC) examinations, or 2% only of those who wrote mathematics, did so at the higher level. But what is notable is that the results of this small minority were very good, viz. 84% achieved over 50% in the final examination.

A second category of schools include those schools mostly characterised by low performance. These schools are located mostly in rural areas, and some are in urban areas. The results in Table 1 are an example of this. The bulk of learners (48%) of those who did mathematics did mathematics at the ordinary level. But only 37.7% of this cohort scored above 50% in the examination.

Table 1.1: 2010 Grade 12 Namibia Senior Secondary Certificate (NSSC) examination results

Namibia Senior Secondary Certificate Ordinary/ Higher (NSSCO/H) levels	**Number of schools who wrote 2010 NSSC examination**	**Total no. of candidates who sat for 2010 NSSC Grade 12 examination**	**Total no. candidate who sat for 2010 NSSC Grade 12 Mathematics examination**	**No. learners who achieved A - D grades (NSSCO)and 1-3 points (NSSCH) in Mathematics examination**	**No. of learners who achieved E-G grades (NSSCO) or no point in Mathematics examination (NSSCH)**	**No of candidates (learners) who were ungraded in Mathematics examination**	**Average percentage of learners who performed above 50% in Math**	**Average percentage of learners who performed below 50% in Math**
NSSCO	152 (61%)	19917 (69%)	13704 (48%)	5306 (37%)	7209 (51%)	1189 (8.3%)	37.7%	61.3%
NSSCH	96 (39%)	8840 (31%)	561 (2%)	428 (76%)	90 (16%)	43 (0.3%)	84%	16%
Totals	248 (100%)	28757 (100%)	14265 (50%)	5734 (100%)	7299 (51%)	1232 (8.6%)		

As a Mathematics Education Officer (MEO) in the Education Directorate of Omusati region, my official duties include, inter alia coordinating, facilitating and monitoring the implementation of the ministerial policies and curriculum, developing teaching and learning materials, and implementing training programmes for staff professional development at a regional, circuit and cluster level. In addition, in my role as a MEO I have many opportunities to visit mathematics classrooms and observe teaching practices. The question I continue to ask myself is: What makes a good or effective mathematics teacher in Namibia?

My classroom observation shows that, despite the widespread disappointing results of pupils and the bad reputation that mathematics teachers have in Namibia (Stephanus and Schäfer, 2011), there are scattered pockets of effective mathematics teaching. These isolated cases of success largely motivate and drive this research, and the study aims to take a closer look at them. So, central to this investigation is the question: What are the characteristics of the instructional practices of effective mathematics teachers in Namibia, and what are their views on their teaching of mathematics? In order to answer this overarching question, I would like to make four introductory points.

Firstly, the focus of this study is effective mathematics teachers. This study is an attempt to gain insight into the mathematical teaching proficiency (MTP) and its components possessed and demonstrated by selected teachers in their classroom instructional practices. Five effective mathematics teachers were identified and selected on the basis of consistently high learner performance in the Junior Secondary Certificate (JSC) and Namibia Senior Secondary Certificate (NSSC) examinations over a period of three years (2008-2010), and on recommendation of the officials from the Education Ministry and the University of Namibia (UNAM) staff at the faculty of education, their peers as well as education regional Mathematics Advisory teachers.

Secondly, this doctoral research seeks to discover what makes these selected teachers, who have been defined as effective and successful teachers of mathematics do when they use MTP to teach particular concepts in the topic of geometry. It is imperative, therefore, that we understand what effective mathematics teaching looks like, and how effective teachers teach for mathematical proficiency (MP). One of the assumptions of this study is the notion that the result of MTP is learners who are mathematically proficiency. The expectation, therefore, is that these teachers are promoting MP in their learners. Hence, I want to explore and understand ways in which these teachers promote MP in their classroom instructional practices. The notion of MTP has been used as a conceptual framework for this study.

Thirdly, the notion of proficiency in mathematics is equated with more than just the ability to solve mathematical tasks. There are several other proficiencies that are crucial aspects of teaching for MP, such as logical reasoning, conceptual understanding and problem solving skills (Kilpatrick et al., 2001). Other core mathematical processes integral to teaching for MP include communication of mathematical ideas and proof. This includes logical thinking skills and the ability to move from abstract concepts to symbols, and the ability to make connections within mathematical ideas and with contexts outside of mathematics (Kilpatrick et al., 2001).

Fourthly, as Pillay (2006) says “it is productive to think about and understand the kind of mathematical work that is enacted in the practice of teaching” (p. 9). Presently there appears to be little consensus amongst policy makers, curriculum designers and teachers on what constitutes good mathematics teaching practice (Douglas, 2009) or what exactly constitutes the MP necessary for effective teaching in Namibian schools. The route then is to investigate this mathematical work by studying teachers involved in authentic classroom practice.

1.3. The importance of the mathematics domain in which this study is located

In researching teachers’ classroom practices, I specifically focus on the concept of geometry. Spotlighting this specific mathematics content area is in part triggered by the view that concentrating on one aspect of mathematics in a classroom would prove to be more revealing than focusing on mathematics in general (Shulman, 1987). The assumption is that these selected teachers would have developed MTP needed to effectively teach geometric concepts. Therefore, it would be useful to study how teachers who are perceived as effective teach geometry in particular. The area of geometry is particularly important in the mathematics curriculum since it is one of the core content components within the Namibian Mathematics Curriculum in which practical real-life examples and contexts are emphasised. I will explicate further on geometry as a study topic in Chapter two.

1.4. The research goal and questions

The objective of this study is to analyse selected secondary school teachers’ geometry teaching in Namibia who are deemed to be effective teachers of mathematics. In order to achieve this goal, I seek to answer the following research question: What are the teaching proficiency characteristics of selected effective mathematics teachers?

In seeking answers to this key question the following interrelated sub-questions were investigated in relation to different areas of teaching for MP:

1. How do these proficiency characteristics inform teachers’ classroom practices?
2. What are the teaching proficiency characteristics that are similar and different across the teaching practices of the selected teachers?
3. What personal experiences and characteristics enable them to be effective?
4. What contextual factors shape these effective practices?

5. What mathematical proficiency and pedagogical content knowledge of geometry do mathematics teachers who are considered effective have, in solving scenario-based geometry tasks?
6. What factors outside of Kilpatrick et al.'s (2001) analytical framework characterise effective teaching practice?

1.5. The conceptual and theoretical frameworks for this study

The study makes use of an adapted model of Kilpatrick et al.'s (2001) mathematical teaching proficiency (MTP) as an analytical tool to help conceptualise the various dimensions of effective teacher practice of teaching mathematics. Specifically, I adapted Kilpatrick et al.'s (2001) five strands of teaching for mathematical proficiency to analyse teacher practice. The Kilpatrick et al. (2001) framework builds on Shulman's (1987) dimensions of general pedagogical models of teaching competence. I found this model useful to analyse effective teachers' teaching proficiency because it was based on the notion of mathematical proficiency - a theoretical concept that is easily operationalized. The Kilpatrick et al.'s (2001, p. 380) conceptual and analytical framework entails five interwoven and interdependent strands of MTP that guides both the data collection and data analysis. These are:

- *Conceptual understanding* (CU) of core knowledge that encourages comprehension of concepts, operations and relations as required in the practice of teaching;
- *Procedural fluency* (PF) in carrying out basic instructional routines;
- *Strategic competence* (SC) in planning effective instruction and solving problems that arise during instruction;
- *Adaptive reasoning* (AR) in justifying and explaining one's instructional practices and in reflecting on those practices so as to improve them, and
- *A productive disposition* (PD) towards mathematics, the teaching, the learning and the improvement of practice.

As Kilpatrick et al. (2001) argue learners need to think mathematically to learn successfully and become proficient in mathematics. In "using the term mathematical proficiency (MP), Kilpatrick et al. (2001) point to the complexity of the mathematics *per se* that learners need to acquire, which in turn has implications for what, mathematically, teachers need to know" (Pillay, 2006, p. 7). Kilpatrick et al.'s (2001) model of teaching for MP guides my thinking and research into what selected effective teachers do within their classroom instructional practices in the context of their effective actions with learners. Hence, my study is situated within this broad conceptual framework of mathematical knowledge of teaching for MP.

The study also draws on elements of an enactivist worldview as an additional theoretical perspective or vantage point. As a useful extension of constructivism, enactivism was used to complement the Kilpatrick et al.'s (2001) model to conduct an analysis of effective teaching practices. Thus, the theoretical framework underpinning my study is that mathematical proficiency for teaching is embedded in the practice of teaching. Hence, the Kilpatrick et al.'s (2001) model of teaching proficiency provides a useful structure for data analysis within a broader theory of enactivism. Elements of enactivism are infused into the framework of Kilpatrick et al.'s (2001) model to provide a rich and powerful analytical tool of analysis of the teaching practices of effective teachers. The key philosophical assumption of this study is the view that the process of teaching or learning is a collaborative practice of acting, interacting and reacting to the stimuli of one's context and environment (Stephanus and Schäfer, 2011; Proulx, 2009). Thus, teaching is seen as a complex phenomenon and is created in and through interactions as teachers talk to students and students talk to teachers.

1.6. The research methodology

This study is oriented within the interpretive research paradigm and employs a qualitative method of data collection and analysis. The design chosen for the investigation of the study is a multiple case study focusing on a total of five mathematics teachers (two males and three females) from five Namibian high schools. The five teachers were purposefully selected. Sampling was carried out in two stages. The first stage comprised purposeful sampling (Creswell, 2007) where I initially selected 10 mathematics teachers - who had consistently achieved the top results in the Grade 10 and 12 national examinations for the last three years (2008-2010). The second stage involved contacting these 10 teachers and inviting five volunteers to participate in the study. My choice of the volunteers was influenced by (1) their voluntary participation and willingness to share teaching practice and experiences and (2) their qualifications. A case study methodology was most appropriate for my work, and a variety of data collection techniques (or tools) were utilised during the process. These included a teachers' geometry scenario-based questionnaire, classroom lesson observations and field notes, and structured individual and focus groups interviews. These methods generated rich data from which I was able to develop descriptions and conclusions. The data analysis was embedded in the data collection. Hence, the process of data collection and analysis ran concurrently through six phases and four interrelated stages.

Phase I of the study involved planning and negotiating access to the research sites for data collection. In this particular phase I sought the Education Ministry Permanent Secretary's permission to meet with regional education directors, school principals and selected mathematics teachers. Phase II of the data generation process involved the administration of biographical questionnaires to obtain participating schools' and teachers' personal profiles. Phase III of the data gathering process took the form of a geometry questionnaire administered to teachers, based on scenario-based tasks as a base-line to determine their pedagogical content knowledge and mathematical proficiency. Phase IV entailed filming geometry lessons in the mathematics classroom to explore how the selected five teachers taught geometry and characterise their teaching proficiency according to Kilpatrick et al.'s (2001) teaching proficiency model. Each geometry lesson was video-recorded and field notes were taken. In Phase V, post observation reflective interviews were conducted with each observed teacher to explore further their mathematical teaching practices and interactions with their learners, and to capture additional dimensions of their classroom practice. All interviews were audio-recorded. Phase VI comprised the analysis phase and consisted of four interrelated stages. The four stages involved the analysis of the geometry scenario-based questionnaire, a review and analysis of lesson videos, stimulated recall analysis session with the participating teachers and finally an analysis of stimulated recall analysis session data. This included the analysis of a series of vignettes which served to illustrate the teachers' geometry teaching practices and proficiency based on the models of my conceptual and theoretical frameworks. A more detailed description of the research methodology is provided in Chapter three.

1.7. Significance of the study

As mathematics education research grows globally, theories of teaching and learning are more available to researchers. That is why in exploring teaching proficiency, I made use of elements of Maturana and Varela's (1992) theory of enactivism and Kilpatrick et al.'s (2001) analytical tools. The need for evidence on the nature and quality of effective secondary school mathematics instructional practices is particularly acute in Namibia given the current anxieties about teachers' lack of pedagogical and content knowledge. As Adler (2005) comments "At the most basic level, we have yet to comprehend how to make mathematics learnable and enjoyable by all children" (p. 5). It is thus important to analyse and describe the practices of effective mathematics teachers in order to learn from them. The use of Kilpatrick et al.'s (2001) model of teaching proficiency is novel in Namibia and it is anticipated that this work could contribute to developing other frameworks to analyse teaching proficiency. The use of enactivism as a vantage point to contextualise Kilpatrick et al.'s (2001) model is also new and it is hoped that this approach will inspire other researchers to pursue a similar design in their work.

Furthermore, the work reported in this study differs from existing research in many ways. Rather than studying each case study teacher as a separate entity, I employed cross analysis and as they all had different education backgrounds and levels of mathematical knowledge, this enabled me to gain a rich understanding of how their teaching proficiency is expressed and manifested in their classrooms. This research has contributed to developing instruments to analyse teachers' teaching practices.

1.8. The structure of the thesis

The thesis consists of eight chapters.

- In **Chapter One** I introduce the study and describe its context.
- In **Chapter Two**, I present the literature review in relation to effective teaching practices and learning theories, with specific reference to the Kilpatrick et al.'s (2001) model of teaching for mathematical proficiency and enactivism. Specifically, I explain how the pedagogical concepts of the Kilpatrick et al.'s (2001) model and the enactivist theory of teachers' practice and proficiency are manifested in the actual classroom teaching to indicate how I use them in my narrative account. I also describe the kind of pedagogical approaches that engage learners and lead to desirable outcomes.
- In the methodology **Chapter Three**, I explain the research methods and design including sampling, and data gathering and interpretations in relation to the six qualitative phases and four interrelated stages of empirical enquiry.
- In **Chapter Four** I profile the participants and their respective schools. I also present and analyse how the participants engaged with the geometry scenario-based questionnaire.

- **Chapter Five** presents an analysis of the teachers' understanding of their own effectiveness and perceptions of factors that contribute towards their effective teaching practice through the lens of Kilpatrick et al.'s (2001) model.
- **Chapter Six** presents the qualitatively analysed data from the teachers' teaching practice. It is organised into three main parts. Part I of this chapter begins by presenting each teacher's practice through the lens of Kilpatrick et al.'s (2001) model. Part II and III present the analysis of each teacher's practice and reflections on their teaching practice through the lens of enactivism.
- In **Chapter Seven** I discuss the findings that are published in Chapter four through Chapter six paying particular attention to teaching proficiency characteristics that are similar and different across the teaching practices of the participating teachers. I also elaborate on how these characteristics or findings relate to my conceptual and theoretical underpinning.
- **Chapter Eight** is the concluding chapter where I consolidate the study findings with reference to the original research questions and within the contexts of the conceptual and methodological frameworks. I further interrogate both the limitations and significance of the study. Final conclusions are drawn and recommendations made for improvement of practice and further research. I conclude the thesis with some personal reflections on the research process.

CHAPTER TWO

LITERATURE REVIEW AND THE THEORIES UNDERPINNING THE STUDY

2.1. INTRODUCTION

The purpose of this chapter is to examine the concepts, theories and literature relevant to my study. I discuss the conceptual and theoretical frames on which the study is based and present a review of the literature relating to the effective teaching of mathematics and the characteristics of teacher effectiveness. The key concepts of the Kilpatrick et al.'s (2001) model of teaching proficiency and enactivist theory that underpin this study's conceptual framework and theoretical vantage points are also discussed in detail.

2.2. STRUCTURE OF THE LITERATURE REVIEW

This literature review is structured into four main parts. The first part begins by describing effective teaching of mathematics in the context of how "effectiveness" is defined and reported. It considers a series of key questions. What is meant by *effectiveness*? What does an effective teacher look like? What are the key features of effective teaching? In what ways does teachers' effectiveness differ from other ways of thinking about mathematical proficiency and skills?

The second part outlines the *concept of geometry as a teaching domain and study topic* in order to familiarise readers with some of the special features of geometry and how it is taught and learned. The nature of geometry (including its relevance and importance), the reasons for it being included in the school mathematics curriculum, how it can best be taught and learned, and the complex challenges that face both the Namibian education system and the teaching of geometry are also examined.

The third part of the literature review articulates examples of recent research on focussed instructional approaches (learning/pedagogic processes) that I argue underpin *effective teaching of mathematics*. The aim is to present selected instructional strategies or approaches which have been profiled as effective in mathematics classroom practices and to discuss how such domains of knowledge relate to effective teaching practices as well as what this implies for the field of mathematical teaching proficiency. Specifically, this discussion addresses the nature of the mechanism of interactions that teachers use to develop strands of mathematical proficiency.

The fourth part is devoted to examples of *interrelated models on effective and proficient mathematics teaching*. The discussion starts with the contributions of Shulman (1987), Ma (1999), Williams (2011), Ball and Hill (2008) and Hill et al. (2008). All of these authors contribute elements to models of effective teaching and learning, some as precursors of, and contributors to, the later model of Kilpatrick et al. (2001). I then move on to a detailed discussion of two related models concerned with the formulation and application of learning theories, namely Kilpatrick et al. (2001) model of teaching proficiency and Maturana and Varela's (1992, 1987) enactivist theory of embodied cognition.

2.3. DIFFERENT PERSPECTIVES OF EFFECTIVE TEACHING

There are many features or characteristics of effective teaching which have been compiled from descriptions of exemplary teachers (Clarke and Clarke, 2004) or from reviews of school effectiveness research (Stephens, 2009; Sammons, Hillman and Mortimore, 1995).

2.3.1. DEFINING AN EFFECTIVE TEACHER: AN ELUSIVE CONCEPT

Before engaging with the literature on the teaching of mathematics, two important points need to be made by way of introduction. The main enquiry that this study endeavours to answer is *what makes an effective mathematics teacher,* as effective is often associated with performance in mathematics. But, in reviewing the literature, I discovered that defining an effective mathematics teacher is an *elusive concept* and has been subjected to a number of different interpretations. In addition to effectiveness, the concept of what a successful or effective teacher is-is ill-defined. Nevertheless the discourse on effective mathematics teaching has permeated the education landscape for many years. The notion of effectiveness is "controversial and ranges from a personal idea or construct to a statement that is universal and generic" (Schäfer, 2010, p. 510).

Effectiveness has to be based on some measure of students' performance in mathematics. Askew, Brown, Rhodes, William and Johnson (1997) defined effective numeracy teachers as highly performing teachers who have the knowledge and awareness of interrelations between the areas of the mathematics curriculum that they teach, and "their classes of pupils had, during the year, achieved a high average gain in numeracy in comparison with other classes from the same year group" (p. 2). They further concluded that effective teachers are those who have a rich network of connections between different mathematical concepts or ideas and are able to select and use teaching strategies that are both efficient and effective. Kilpatrick et al. (2001) provide a unique language to describe the subject of mathematics education, although they do not define “effectiveness” specifically. This team came up within their five characteristics of proficient teaching, which I understand to have the same meaning as effectiveness. The term “proficient” comes from their experience and knowledge of the field, and builds on all their work on the standards, and is more an ideological position than anything else (Personal communication with Lerman, July, 2012). Consistent with Kilpatrick et al.’s (2001) findings, “proficient mathematics teachers are those who use teaching strategies and approaches that connect different areas of mathematics and different ideas in the same area of mathematics using a variety of words, symbols and diagrams; use pupils’ descriptions of their methods and their reasoning to help establish and emphasise connections and address misconceptions; emphasise the importance of using mental, written and electronic methods of calculation that are the most efficient for the problem at hand” (Askew et al., 1997, p. 2), and encourage discussion in both whole classes and small groups or with individual pupils. For the purpose of this study, my interest was not only in the effective teachers mentioned above, but also in teachers whose pupils consistently achieve good results in the National Grade 10 and 12 mathematics examinations. Further, I sought teachers who had a reputation amongst the wider Namibian community of being effective teachers.

Some research studies (Ball and Bass, 2003) define teacher effectiveness in terms of student achievement, some research (Stephens, 2009) focuses on teachers’ high performance ratings from supervisors, while others (Weiss, Pasley, Smith, Banilower and Heck, 2003) focus on strategies around sub-elements in the mathematics classroom such as the use of questioning and/or manipulations, while others present holistic views of learning mathematics. This variety of views made me ask the question: Why MIGHT we have such a variety and range of different outcomes to these studies on what is purported to be the characteristics of effective teaching of mathematics? Definitions of an effective teacher are thus dependent upon a variety of factors such as outlined by Sammons, Hillman and Mortimore (1995):

- The focus of the studies might differ. For example, a particular study may focus on individual and different components of the mathematics curriculum, or it may focus on attitudes or dispositions of teachers or learners, each producing different outcomes;
- The educational institution under consideration may have its own definition of teacher effectiveness depending on the school’s goals and ethos, thus shaping the focus of the study;
- The teaching practice within the classroom setting may differ in terms of what is observed or studied and thus generates differently shaped outcomes;

- The research methods applied may very well shape the results of what is considered effective;
- The nature of the subject being studied may well derive the characteristics of what appears to be effective, and/or
- There may be certain learning theories that are preferred by some researchers in terms of what the researchers may regard as good or effective mathematics teaching.

As effective teaching is often associated with students' achievement in some way, Sullivan, Clarke and O'Shea's (2010) and Clarke and Clarke's (2004) criteria are, in my opinion, values-led. Thus for the context of this chapter, specifically the theoretical descriptions of effective mathematics teachers and teaching uses defining characteristics from Sullivan et al. (2010) which include five critical attributes:

- *Clarity*: the teachers are able to identify big ideas that underpin the concepts that [they] are seeking to teach and communicate to students that these are the goals of the teaching;
- *Building on experience*: they are able to build on what the students know both mathematically and experientially, including creating and connecting students with stories that contextualise and establish a rationale for the learning;
- *Variety and challenge*: for example, the teachers engage "their students by utilising a variety of rich and challenging tasks that allow students opportunities to make decisions, and which use a variety of forms of representation" (Sullivan, 2011, p. 32);
- *Interacting and adapting*: this means that they interact with students while they engage in the experience, encourage students to interact with each other through asking and answering questions, and specifically plan to support students who need it and challenge those who are ready, and
- *Grouping*: by adopting pedagogies that foster communication and mutual responsibilities, they encourage students to work in small groups and also use reporting to the class by students as a learning opportunity.

Consistent with the Kilpatrick et al.'s (2001) characteristics of proficient teaching, Clarke and Clarke's (2004) study found that effective teachers are those who appear to:

- "focus on important mathematical concepts and ideas in a manner clear to students (**conceptual understanding**);

- draw from and use a range of *materials, tools, representations* and *contexts* for the same concept (**conceptual understanding**);
- make *connections* between mathematical concepts and ideas as the situation arises in class (**conceptual understanding**);
- encourage students to *share their [mathematical] thinking through various techniques*, and *use students' individual thinking* to build conceptual understanding (**procedural fluency**);
- select mathematical tasks that encourage *different possibilities and strategies*, and that are engaging to students (**strategic competence**);
- use a range of *teaching approaches*, including small and whole group instruction, which require varying roles of the teacher (**strategic competence**);
- use a *multitude of assessment methods* to inform their thinking (**strategic competence**);
- have high but *reasonable expectations* for students (**strategic competence**);
- encourage *reflecting on learning* in students and practice reflective thinking themselves (**adaptive reasoning**), and
- personally believe that mathematics is valuable and are confident in their own mathematical knowledge (**productive disposition**)" (Clarke and Clarke, 2004, p. 75).

These identified characteristics of effective teachers align well with the key dimensions of Kilpatrick et al.'s (2001) model of teaching proficiency which will be delineated in more detail in 5.2. These strands of mathematical actions constitute the kind of knowledge necessary for proficiency in mathematics. Thus, effective mathematics teachers must have a complete understanding of the subject themselves. For example, teachers who possess procedural fluency, but not conceptual understanding, are not likely to be able to present alternative methods of understanding the concept of gradient or long division. These teachers can help students use the algorithm, but may not be able to address the underlying role of place value to assist students who struggle with the traditional algorithm. Similarly, teachers who do not possess strategic competence will be hard pressed to support alternative solution strategies in their classrooms. Thus, in expecting teachers to successfully intervene in students' development of mathematical proficiency, teachers are encouraged to correlate their classroom instructional practices with conventional mathematical ideas and advice.

2.4. THE CONCEPT OF GEOMETRY

An overriding decision made in terms of my research context is the question of which mathematical content to focus on. The focus of this study is teachers' geometry classroom instructional practices in Namibian schools. The aim is to observe and report specifically on geometry classroom instructional practices so as to provide some explanations for teacher's proficiency in teaching geometry. The selection of geometry as the topic of study is significant because it is one of the core content components within the Namibian mathematics Curriculum in which practical real-life examples and contexts are very important in terms of its applicability to all learners, especially from grade 8 to 12. This is because both teachers and learners experience geometry in everyday life and, therefore, they have everyday knowledge about geometry apart from general mathematical knowledge or other mathematical domains.

As this study is about teacher effectiveness using geometry as the backdrop to understanding teachers' teaching practice, the models of effectiveness, vis-à-vis Shulman, Kilpatrick et al. (2001) framework and enactivism feed into each other. As mentioned earlier, Kilpatrick et al. (2001) never used the word effectiveness to talk about good teaching. They used the word "proficient". Thus, both the Kilpatrick et al.'s (2001) model and the enactivist perspective provide a unique language to look at effectiveness. Kilpatrick and his colleagues argue that mathematical proficiency or knowing is acquired through the interactions between the teacher and the learners, and is socially constructed. Moreover, teacher classroom instructional practice is guided by philosophies of people, learning theories and teaching approaches as is geometry. Thus, the Kilpatrick et al. (2001) framework of teaching proficiency and an enactivist theory of embodied mathematical cognition are relevant to the discourse of teachers' teaching practice, and both go a step further informing geometry teaching in particular. An adapted model of Kilpatrick et al. (2001)'s teaching (mathematical) proficiency guided the data collection. Enactivism contextualises the Kilpatrick framework to provide a depth of analysis to the exploration of effective teachers' teaching practice and the interrelationship between embodied processes of mathematical understanding (knowing) and teacher effectiveness in the pedagogical context of the mathematics classroom.

In the following section I define what geometry is and discuss a number of pertinent issues relating specifically to effective geometry teaching and learning. According to Atebe (2008, p. 12), geometry is the "science which treats properties of the shape, size and position of figures based on definitions, axioms and postulates: these granted all the rest follows by pure reasoning". That is, Atebe conceptualises geometry as the study of the properties of spatial objects and the relations (of constructible plane figures) between those properties. Battista (2007, p. 843), claims that geometry as a core component of the mathematics curriculum component and is "a complex interconnected network of concepts, ways of reasoning and representation systems that is used to conceptualise and analyse physical and imagined spatial environments". Schäfer (2003) recognises this by suggesting that "there is a tendency in the current thinking to embrace a broader view of geometry" (p. 2). In Schäfer's view, geometry is fundamental to mathematical cognition or learning in general, to developing spatial thinking and also to mathematics *per se*. For example, both mathematics teachers and learners "describe and represent their experiences with shape, space, time and motion using all available senses" (p. 2) in the form of geometry. In my view, geometry is a wonderful tool to govern our adaptive reasoning and deepen our conceptual understanding of mathematical ideas in two, three and even four dimensions. It is these central tenets of effective teaching of mathematics that I would like to explore in order to see how sequencing of classroom instructional activities allows the development of sound mathematics practice and mathematical reasoning or proficiency. Hence, the goals of teaching geometry are to:

- "develop spatial awareness, geometrical intuition and the ability to visualise;
- provide a breadth of geometrical experiences in two- and three- dimensions;
- develop knowledge and understanding of and, the ability to use, geometrical properties and theorems;
- encourage the development and use of conjecture, deductive reasoning and proofs;
- develop skills of applying geometry through modelling and problem solving in real world contexts;
- engender a vigorous and positive attitude to mathematics as well as its teaching and learning;
- develop useful ICT skills in specifically geometrical contexts, and
- develop an awareness of the historical and cultural heritage of geometry in society, and of the contemporary applications of geometry" (Battista, 2007; Jones, 2002, p. 19).

All these aspects, including the definitions of geometry and a consideration of the aims of teaching geometry, tend to make geometry a demanding topic to teach effectively. Thus, I will briefly explicate further on the importance of geometry, its inclusion in the school mathematics curriculum (the landscape of geometry), key ideas in teaching and learning geometry effectively, and geometry education in Namibian schools. Barriers or challenges to geometry education will be also addressed.

2.4.1. The importance of geometry and its impact on the Namibian Vision 2030

In an increasingly interdependent world, the critical role of the discipline of mathematics education and, thus, geometry in the context of the scientific skills and knowledge that we need to cope with the speed of global change cannot be underestimated. The inclusion of high-quality Mathematics, Science and Technology (MST) education in the 21st century curriculum and millennium development goals (OECD, 2004) state that "the objectives of personal fulfilment, employment and full participation in societal development increasingly require that all persons be mathematically, scientifically and technologically literate" (OECD, 2004, p. 11). For this reason, I use the concept of geometry that is "concerned with the capacity to analyse, reason and communicate effectively, solve and interpret problems" (French, 2004, p. 6) in real world contexts involving quantitative, spatial, probabilistic and other concepts. Ideally, geometry enhances our capacity to identify and understand the role that mathematics plays in real world contexts demanding problem solving, flexibility and multilevel thinking. It helps to form well-founded judgements and engages with mathematics in ways that meet the needs of all individuals' lives as constructive and reflective citizens.

The potential power of geometric applications, reasoning and spatial relationships are to activate our mathematical knowledge, abilities and competencies to solve real life problems successfully (Battista, 2007). As geometry plays a significant role in the lives of individuals and society as a whole, the practical application of geometry and critical mathematical thinking in particular should be encouraged in the mathematics classroom. This makes it imperative that Namibian mathematics education should equip learners with the geometric skills necessary for achieving mathematical proficiency, career aspirations and for attaining personal fulfilment. Geometry compels the human brain to reason and formulate problems, conjectures and to seek methods to solutions (Kilpatrick et al., 2001), which is vital to the future life opportunities and achievements of Namibian children. Geometry is a key instrument in diverse fields, i.e. architecture, engineering, communication, biology, geography, physical science, life and health science, business and commerce, law, agriculture, medicine and music (The Namibian Newspaper, 2011). In light of the Namibian Vision 2030 policy framework, the intention of the curricular discipline of geometry is to "encourage an approach to teaching and learning mathematics that gives strong emphasis" (OECD, 2004, p. 16) to mathematical processes associated with creating multiple alternative development strategies for confronting problems in real world situations and using the relevant mathematical knowledge to solve these problems; making these strategies implementable and problems amenable to mathematical treatment and sustainable development; and evaluating the solutions in the original problem contexts (Office of the President, 2004). According to Bleeker and Goosen (2009, p. 29), geometry can be an "effective tool in guiding learners towards abstract reasoning which we need in the wake of globalisation, with its challenges and opportunities".

2.4.2. The landscape of geometry: reasons for its inclusion in the school mathematics curriculum

Essentially, the world is built of shapes and spaces and geometry is its mathematics (Jones, 2002). The context of geometry teaching in Namibian secondary schools is based on the paradigm of a learner-centred approach in which the teacher may apply either of the teaching models or perspectives discussed earlier. Kilpatrick et al. (2001), for example, suggested teachers should teach and learners learn geometry to develop geometric understanding and reasoning.

French (2004), for example, explains that geometry has a vital role in the wider mathematics curriculum as it harnesses learners' spatial awareness, develops learners' reasoning skills and stimulates/informs and challenges their concept/facility. As some kind of a dynamic construct, geometry knowledge needs to be articulated if the study is to explore and observe teachers' classroom geometry teaching practices. In the next section, I discuss briefly the teaching and learning of geometry in Namibian schools.

2.4.3. Key ideas in teaching and learning geometry effectively

There is a considerable amount of research that focuses on the teaching and learning of geometry (Atebe, 2008; Schäfer, 2003; Jones, 2002; Clements, 2001) and how best to teach it. It is beyond the scope of this literature presentation to attempt to summarise it all. Instead, only select issues will be addressed. In my opinion geometry is a vital component of the mathematics curriculum. Being composed of interesting problems and surprising theorems, it is open to different teaching approaches. Research has shown that "geometry has a long history, intimately connected with the development of mathematics" (Jones, 2002, p. 5).

Geometry appeals to our visual, aesthetic and intuitive senses (Schäfer, 2003). In my view, geometry can be the area that captures the learner's interest, in a way that is stimulating and engaging, and consequently leads to mathematical proficiency. Hence, teaching geometry effectively could lead to more students enjoying and being successful in mathematics.

Jones (2002) contends that "teaching geometry effectively involves knowing how to recognise interesting geometrical problems and theorems, appreciating the history and cultural context of geometry and understanding the many and varied uses to which geometry is put" (p. 122). This means appreciating what a full and rich geometry education can offer to students when the mathematics curriculum is often dominated by other considerations, i.e. the demand for algebra. The study of geometry contributes to helping learners develop mathematical proficiency such as skills of visualisation, critical thinking, intuition, perspective, problem solving, conjecturing, deductive reasoning, and logical argument and proof (Atebe, 2008; Schäfer, 2003; Clements, 2001). Spatial reasoning is also important in geometry and other curricular areas. Thus, "a strong sense of spatial relationships and competence in using the concepts and language of geometry also support students' understanding of key mathematical concepts such as measurement" (Ontario. Ministry of Education, 2005, p. 3). Schäfer (2003, p. 18) defines spatial conceptual skills as the ability to "identify, describe, analyse, explain and justify objects and relationships of both a concrete and abstract nature". On account of this definition, I contend that spatial cognition uses both the Kilpatrick et al.'s (2001) model and an enactivist approach to learning and it encourages interactions, exploration and learning by doing. As Proulx (2008) points out, the environment provides the individual with information about some opportunities for action.

"Piaget's theory about cognitive development posits that a person needs to achieve a certain level (i.e. formal operational stage) to be able to reason formally, understand and construct proofs" (Rowland, 2000, p. 5). The Van Hiele model also suggests those students' geometrical understanding progresses through various levels, which cannot be bypassed (Ibid). Research supporting Van Hiele levels found that most students enter high school geometry with a low Van Hiele level of understanding (Atebe, 2008). Thus, teaching geometry effectively is complex since teaching formal proofs requires at least a Van Hiele level 4.

Jones (2002) contends that "in order to teach geometry effectively and give some coherence to classroom tasks, it is important to ensure that the early years of secondary school encourage learners to develop a higher disposition for the subject by providing opportunities to investigate spatial ideas and solve real life problems. There is also a need to ensure that there is a good understanding of the basic concepts and language of geometry in order to provide foundations for future work and to enable students to consider geometrical problems and communicate their ideas" (p. 11). Teachers should engage students in tasks that use descriptions, demonstrations, investigations and justifications in order to develop the necessary reasoning skills and confidence.

2.4.4. Geometry education in Namibian schools

> …Truly I begin to understand that although logic is an excellent instrument to govern our reasoning, it does not compare with the sharpness of geometry in awakening the mind to discovery… (Hart and Picciotto, 2001, p. ix).

This quote from Hart and Piccioto accentuates the significance of geometry and foregrounds why geometry in one form or another continues to be a cherished component of school mathematics in the Namibian education system. The current mathematics curriculum in Namibian schools aims to promote increased understanding and proficiency in mathematics. It calls for learning experiences, particularly interactive classroom instructional practices that support learners' active participation in developing mathematical proficiency. However, teachers' classroom mathematics instructional practices differ in quality and opportunities for learners' active involvement and participation as teachers regularly employ a multitude of different teaching approaches to provide a successful environment for their students and/or when making choices when they select tasks for students to work on. Clearly, the Namibian National Mathematics curriculum is offered at four different school phases, each phase with a different syllabus (Mathematics syllabus, NIED, 2008). The geometry curriculum is for students from grade four through twelve and beyond. The teaching focuses on the properties of configurations of geometric objects, namely points, (straight) lines and the properties of plane shapes such as triangles, quadrilaterals and circles being the most basic of these. In order to provide learners with a coherent mathematical proficiency, the geometry curriculum is organised around key learning competencies with an emphasis on 'process standards of communication, reasoning, representation, connection and problem solving' (NCTM, 1989). In grades 8 to 10, the focus is on developing mathematical understanding and fluency by describing three dimensional shapes and analysing their properties, including surface areas and volumes (Mathematics syllabus, NIED, 2008). The objectives of geometry teaching in grade 11 and 12 is consistent with that of grade 8 to 10, but here the focus is more on the analysis of two- and three-dimensional spaces and shapes by using spaces, distance and angles.

The goal of geometry teaching as reflected above may seem plausible, especially in the wake of new trends in mathematics education. The shift in focus from teacher-centred instruction to a more learner-centred one develops increasingly complex ways of mathematical reasoning and problem solving. Both the Kilpatrick et al.'s (2001) model and enactivist teaching perspectives accept the learners as equal partners in the learning process. There is, though, little empirical evidence that Namibian learners are acquiring the prescribed outcomes of being able to produce conjectures and generalizations relating to geometrical shapes and their properties. The teaching proficiency that inform geometry instructional practices, how teachers and learners interact and work on tasks in class, how teachers' mathematical proficiency (knowledge) is implicated in instructional practices and how those practices relate to effective teaching and learning, are the focus of this doctoral study.

Provisions and demands of the new Namibian curriculum suggest a drastic change from traditional modes of teaching to reform (learner-centred) teaching, which focuses on the nurturing of mathematical proficiency and development of interactive and shareable classroom instructional practices. It is important to note that teachers' instructional activities should be interlinked to bring about learners' active participation. It is, thus, very important that "ways are found not only to suggest changes in teachers' practices, but also to provide necessary support and assistance for such desired changes to manifest in the [mathematics] classroom" (Sanni, 2007, p. 39). One way to achieve this is through the execution or presentation of geometry lessons that support the nurturing of all five strands of mathematical proficiency in learners (Kilpatrick et al., 2001) and the development of a community of mathematical practice which is the main focus of this thesis. However, the success of the geometry teaching programme also depends on effective subject management, and collective involvement of learners in the mathematics teaching and learning process (NIED, 2008).

In summary, seeing that Kilpatrick et al.'s (2001) model of mathematical proficiency stresses conceptual and procedural knowledge as well as mathematical reasoning in influencing advancement in learning and teaching geometry, it is imperative that 'teachers redeem' the time, so to say. It could be that mathematics teachers do not recognize the potential for these competencies in a geometry lesson.

2.4.5. Barriers/Challenges to teaching and learning secondary school geometry

Namibian mathematics education is not alone in being inundated with problems. For years after independence, learners' mathematics performance and examination results are not encouraging as per the public expectation (Stephanus, 2008). With the advent of the new system (LCE), the curriculum was also subjected to changes in order to respond to the needs of the Namibian nation. One of the key changes made in the secondary school mathematics education syllabus was the exclusion (omission) of the *formal geometric proof* or *axiomatic geometry* (Gaoseb, 2009), albeit the curriculum reform efforts' call on mathematics teachers to provide all students with rich opportunities and experiences in geometry (NCTM, 1998). In reality, the preamble of LCE and exclusion of formal proofs from the secondary school mathematics syllabus actually influenced teachers to put "more emphasis in their teaching practices on inductive reasoning than on deductive reasoning" (Gaoseb, 2009, p. 79). Within the Namibian grade 8-12 mathematics curriculum, *geometric proofs* have been removed from the current syllabus, yet as Gaoseb (2009) notes, no interventions have been put in place to help teachers cope with the change. Research internationally has shown that the essence of school mathematics lies in geometry proofs, and that, *it is an essential component of doing, communicating and recording mathematics* (Van de Walle, 2007). In addition, mathematics teachers might hide behind the notion of the learner centred approach (LCE) by just giving learners more mathematics activities to do without clear explanations. Geometry is a helpful component of mathematics education in developing students' deductive, analogical and spatial reasoning, yet, it is a relatively small section in the Namibian Grade 8 to 12 mathematics curriculum and often it is not taught effectively or given proper attention. This, indeed, has direct implications on the curriculum and geometry teaching and learning.

2.5. KEY FOCUSSED APPROACHES OF EFFECTIVE MATHEMATICS TEACHING

In this section, I draw on international research findings and summaries of recommendations about mathematics teaching actions (McGraner, VanDeyden and Holdeheide, 2011; Sullivan, 2011; Hill et al., 2008; Franke, Kazemi and Battey, 2007; NCTM, 2000) to present key principles or approaches that underpin the effective teaching of Mathematics. Specifically, I discuss eight interconnected instructional approaches of teaching that have been illustrated as exemplifying effective classroom instructional practices that support meaningful learning in mathematics. These pedagogic approaches entail problem-based learning approaches, reasoning and proofing, making connections, classroom communication and questioning (mathematical discourse), using representations, sufficient practice and worthwhile or high cognitive level mathematical tasks. Although these approaches operate separately, there is some overlap in their definitions. Effective mathematics teaching and tasks call particularly on one or more of these learning approaches. It is assumed, therefore, that "effective strategies that teachers employ will vary according to both the object of the learning and the needs of the students" (Ontario. Ministry of Education, 2005, p. 2). These instructional approaches provide a basis for discussing the ways in which different mathematical strands or competencies are invoked in response to different kinds and levels of cognitive demands imposed by different mathematical problems and tasks (OECD, 2004). In this way, they describe the mathematical processes teachers and students need to learn and apply as they work to achieve the expectations outlined within Kilpatrick et al.'s (2001) model, which will be discussed later in Section 2.6.2.1.

2.5.1. Problem solving approach

Mathematics per se is an activity of solving problems. Problem solving is central to effective mathematics programmes. For this reason, effective teaching is based on the idea that students learn mathematical concepts most effectively when they are given opportunities to investigate mathematical concepts and ideas through problem solving, and are guided carefully into an understanding of mathematical principles involved (Ontario. Ministry of Education, 2005). By learning to solve mathematics problems and by learning via problem solving, students are given opportunities to connect mathematical concepts/ideas, thereby developing conceptual understanding. Wagner (2005) argues that when planning mathematics programmes, effective teachers provide activities and assignments that encourage students "to learn basic concepts, skills and structures of mathematics through problem solving" (p. 31). Thus, "teachers need to use rich mathematics problems and present situations that provide a variety of opportunities for students to develop mathematical understanding through problem solving" (Ontario. Ministry of Education, 2005). Teachers who teach via relevant and meaningful problem solving experiences appear to help students to develop conceptual understanding and in turn they perform best on procedural knowledge. As a mainstay of effective teaching practice, problem solving enables teachers to "help students become more confident in their ability to do mathematics; to use the knowledge they bring to school and connect mathematics with situations outside the classroom" (p. 11); to reason, communicate ideas and apply concepts, strategies and procedures, and find enjoyment in mathematics. This suggests that "learning is meaningful when students engage in collaborative processes of knowledge construction" (Niessen, 2007, p. 31). The teacher's role is thus not to instruct students but to facilitate the process in order to enable students to attain the necessary base of knowledge.

Problem solving plays an important part in developing mathematical proficiency in learners. Research suggests that "real life problems are taken as its point of departure because these provide a powerful learning surrounding to students" (Niessen, 2007, p. 9). Thus, allowing mathematics teaching to be problem based means allowing children to wonder why things are, to inquire, to search for solutions and to resolve incongruities (Wagner, 2005). This implies that both mathematics instruction and learning should start with stimulating problems, challenging dilemmas and questions for students. One strategy that may be useful for encouraging students to engage with mathematical problems and to persist, even if challenged, is the use of multiple, highly cognitive level tasks. Indeed, mathematics teachers should "use a range of tasks that allow students abundant opportunities to solve problems, explain reasoning, build their [conceptual] understanding and develop the necessary fluency" (Sullivan, 2011, p. 39). Additionally, teachers should use challenging tasks and maintain the demand of the tasks to support students in persisting and posing mathematical problems either for themselves or for others in the class to solve. Learning through problem solving is significantly better for improving students' performance in conceptual understanding, adaptive reasoning and strategic competence (Samuelsson, 2010). In this sense, effective teachers should teach problem solving skills in the context in which they will be used. They should also give practice of similar problem solving strategies across multiple contexts to develop a solid understanding of mathematical concepts in students (Suurtamm, 2010). To summarise, according to these texts, teaching and learning through problem solving "increases opportunities for the use of critical thinking skills, [namely, discovering own solution strategies], estimating, evaluating, classifying, assuming, recognizing relationships, hypothesizing, offering opinions with reason, and making judgements" which are mathematically precise (Kilpatrick et al., 2001).

2.5.2. Reasoning and explaining approach

The primary aim of effective mathematics teaching practice in Brodie's (2010) views is to "teach students to think and reason mathematically" (p. 3). "Teaching to think [mathematically]" implies "developing the ability of the students to investigate, question and explain information imparted" (Wilson, Thomas and Stonson, 2005, p. 83). This perspective is consistent with the NCTM's mathematics Standards, which says "the vision of mathematics is to see teachers encouraging students to probe for ideas, thoughts and expressions" (NCTM, 1989, p. 10). The NCTM mathematics teaching principle calls for effective classroom instructional practices to be based on students' mathematical thinking, reasoning, imaginations and actions (NCTM, 2000). Thus, the NCTM's proviso of *effective mathematics teaching* specifically calls on mathematics teachers to have a sound knowledge of mathematical concepts they impart, and "to be able to draw on that knowledge with flexibility" (p. 5) in their classroom instructional practices or tasks (NCTM, 2006). Hence, all learning, particularly new learning, should be embedded in well-chosen contexts for learning. This means contexts that are broad enough to allow students to investigate their mathematical understandings, identify and develop relevant supporting skills, and gain experience with varied and interesting applications of the new mathematical concepts. Such rich contexts for learning open the door for students to see the key principles or big ideas of mathematics, i.e. pattern or relationship. This understanding of key principles may very well enable and encourage students to embrace and use mathematical reasoning throughout their lives. And, because mathematical reasoning must be the primary focus of students' communication, it is important for teachers to select instructional strategies that elicit this from their students (Ontario. Ministry of Education, 2005).

Additionally, instructions embracing the mathematical reasoning process support a deeper understanding of mathematics by enabling students to make sense of the mathematical concepts they are learning. Such instructional processes involve exploring phenomena, developing ideas, making mathematical conjectures and justifying results (Kilpatrick et al., 2001). This teaching model requires mathematics teachers to draw on students' natural ability to reason, so as to help them learn to reason mathematically, for instance, by encouraging students to consider other students' viewpoints and justifying their choices of problem solving approaches. Teachers may also encourage students "to reason from the evidence they find in their explorations and investigations or from what they already know to be true, and to recognize the characteristics of an acceptable argument in the mathematics classroom" (Ontario. Ministry of Education, 2005, p. 9). In addition, teachers may "help students to revisit conjectures that they have found to be true in one context to see if they are always true. For instance, when teaching students about decimals, teachers may guide students to revisit the conjecture that multiplication always makes things bigger" (Ontario. Ministry of Education, 2005, p. 9).

According to Givvin, Jacobs, Hollingsworth and Hiebert (2009), effective teaching practices also afford students “a large degree of responsibility and control over their own learning”, engage students in processes that involve mathematical reasoning and conceptual thinking, enable students to solve mathematical problems by exploring on their own, and “provide opportunities for discovery and individual problem solving” (p. 11). An aspect of my study will involve looking at the instructional practices of effective teachers in making sense of their mathematical proficiency for teaching in particular Namibian classrooms. Essentially, the key elements emerging from the literature examining effective mathematics teaching practices are a “clear focus on mathematical concepts and thinking, an emphasis on valuing children’s strategies to solution and encouraging them to share their strategies and solution” (Stephens, 2009, p. 29).

2.5.3. Connecting and applying concepts approach

Connecting learning or mathematical concepts to students’ real world experiences is another effective approach of introducing and teaching new mathematical concepts. Introducing concepts through linkages enables students to relate new ideas to a context of past learning. This means, teachers should build on what students know mathematically and experientially. This practice entails “creating and connecting students with stories that both contextualise and establish a rationale for the learning” (Sullivan, 2011, p. 37). As Speer (2011) argues, “a healthy combination of student action and linking new materials to previously learned mathematics concepts, procedures and practical experiences will set the stage to help students feel more comfortable in their knowledge and understanding of the new concept or procedures” (p. 316). Furthermore, seeing the relationships among mathematical concepts and procedures helps students to develop conceptual understanding and general mathematics principles.

Kilpatrick et al. (2001) advocate that effective instructional approaches and learning activities, drawing on students’ prior knowledge, are likely to capture their interest and encourage meaningful practice both inside and outside the mathematics classroom. This means, “students’ interest will be engaged when they are able to see the connections between the concepts they are learning and their application” in real-life situations (Ontario. Ministry of Education, 2005, p. 8). Understanding gained through linkages and concept connected to real life situations give learners power over mathematics (Ball and Hill, 2008). This power is likely to increase students’ disposition and comfort level in their ability to reason mathematically, set up problems, value the variety of approaches and techniques, make the conceptual leap from concrete to abstract reasoning and develop meaningful understanding of fundamental mathematical concepts in problems (Speer, 2011).

2.5.4. Questioning approach

According to Weiss et al. (2003), a primary means to encourage *sense-making* in mathematics lessons is *teacher questioning* (p. xii). That is, effective instructional practices or high quality lessons frequently include questioning used effectively to find out what students already know about the concept addressed (Hill et al., 2008; Weiss et al., 2003; Kilpatrick et al., 2001), and to provoke deeper thinking and monitor emerging, new mathematical ideas (Weiss et al., 2003). These questioning techniques often include *probing students for elaboration, explanation, justification* or *generation of new questions* or *conjectures* (Ball and Bass, 2009; Kilpatrick et al., 2001). "Expressing and defending their thinking, beliefs and opinions as well as questioning others' ideas help students recognise, clarify and repair inconsistencies in their own thinking" (Webb, Franke, Ing, Chan, De, Freund and Battey, 2008, p. 361). However, Weiss at al. (2003) caution that effective questioning is not the only means of helping students make sense of Mathematics concepts. But, relevant, practical and accessible mathematics examples of real life contexts that teachers give in lessons help students connect mathematical concepts to their experiences as a way to enhance their mathematical understanding. They further report that "purposeful and thought-provoking teacher's demonstrations or student activities coupled with discussion or writing about observations and ideas can also be used to promote sense-making" (p. 1) of mathematical or geometrical concepts. Thus, effective mathematics lessons should include sufficient opportunities for sense-making. Such a platform will help students to experience phenomena, conduct investigations, work out problems or exercises and connect new concepts to their existing mathematical knowledge. Brodie (2010), for example, states that "teachers' questioning style provides an important methodological lens for understanding these relationships among mathematical proficiency strands" (p. 1).

2.5.5. Sufficient practice approach

The value of guided and/or independent practice for learning mathematics effectively has long been recognised. Larson (2002) contends that an effective mathematics lesson should entail the *introduction, development of the concept/skill, guided practice, summary* and *independent practice* phases (p. 1). According to him, as the development phase continues, it is necessary for students to move from learning activities to initial practice under the guidance of the teacher and gradually move into the independent practice phase. During these phases, the teacher should give students certain mathematical problems to work on alone or in groups and then discuss these problems together. This practice enables the teacher to check on the learners' understanding of concepts and, depending on the discussion, might indicate a need for additional teaching and exploration of the mathematical concepts. Effective instructions that facilitate students' conceptual understanding, therefore, suggest students should be offered opportunities to practice skills or procedures (Sullivan, 2011), being what Kilpatrick et al. (2001) describe as procedural fluency and/or strategic competence. It is important to note that teachers carefully choose and effectively incorporate tasks that seek to develop conceptual understanding and fluency into their lessons. Practice is necessary for students to master any skill and mathematics is no exception. Kilpatrick et al. (2001) argue that "significant instructional time should be devoted to developing concepts and methods, and carefully directed practice with feedback should be used to support student learning" (p. 11).

2.5.6. Representation and manipulative approach

Mathematics manipulatives such as pattern blocks, counters, geometric solids etcetera are also a valuable aid to effective classroom practices (Franke et al., 2007). Manipulatives are interactive, visual and concrete representations of "a dynamic object that presents opportunities for constructing mathematics knowledge" (Moyer-Packenham, 2011, p. 236). Manipulatives develop representational fluency when teachers link symbolic, pictorial or concrete representations. For example, placing a "90°" besides a picture of a right angle. As the literature suggests, students' mathematical concepts evolve through direct interaction with the environment (Proulx, 2008), and manipulative materials provide a vehicle through which this can happen. The use of concrete materials enhances students' learning and conceptual development, while improving their disposition towards mathematics teaching (Thompson, 1992). Instructional practices that make use of tasks using manipulatives and models afford learners access to language and mathematical terminology (Franke et al., 2007). Thus, the proverb: *I hear and I forget, I see and I remember, I do and I understand* is a justification that the use of manipulative materials and is a key to learning about mathematical concepts.

Shulman (1987) and Ma (1999) support this by asserting that for the pedagogical content knowledge and profound knowledge of a particular concept to come into play, teachers should have a deep understanding of the mathematics they plan to teach, and should know how a particular manipulative may be used to support the development of a specific mathematical concept. This claim resonates robustly with Askew et al. (1997) who state that effective teachers use manipulatives to enhance the mathematical concept being developed in helping children construct knowledge and connect mathematical ideas. Kilpatrick et al. (2001) point out that "physical materials are not automatically meaningful to students and need to be connected to the situations being modelled" (p. 7). An effective teacher mediates students' understanding of the representations and serves as a bridge between the concrete and the abstract. One of the aims of this study is to gain a better understanding of how effective teachers use mathematics manipulatives.

2.5.7. Facilitation of a classroom mathematical discourse approach

The nature of mathematical discourse is a central feature of effective classroom practice. Among the best supported suggestions for effective teaching in the literature is that teachers must carefully plan for classroom discourse as a part of their mathematics instructional process (Chen, Benton, Cicatelli and Yee, 2004). Mathematics discourse is a vision of the classroom dialogue that actively involves students in tasks and uses students' explanations to develop mathematical ideas. This vision expects that effective teachers should strongly guide and engage students in mathematical arguments (Choppin, 2007). That is, teachers should articulate or enact their classroom practices in ways that "actively involve students and use their ideas as sources of mathematical ideas in the classroom community of learners" (p. 33). Ideally, teachers initiate students into the practice of mathematical argumentation and, through this practice, they intellectually engage students with other students' ideas by reflecting and building on the mathematical explanations of their peers (Choppin, 2007). Additionally, Franke et al. (2007) mentioned in McGraner et al. (2011) posit that developing mathematical conversations allows teachers to continually learn from their learners. That is, mathematical conversations that centre on learners' ideas can provide teachers a window into learners' thinking in ways that learners' individual work cannot do alone. They further advise that effective teachers should provide opportunities for learners to share their mathematical thinking and problem-solving processes, justify and formulate conjectures publically, and evaluate multiple solution strategies. To ensure high-quality mathematical discourses or discussion, teachers must structure conversations and communication around particular problems and solution methods, engage all students in classroom discussion and respond to a variety of student responses in ways that illuminate and amplify essential mathematical [understanding] (McGraner et al., 2011, p. 56).

Research further suggests that whole class discussion or argument can be effective when teachers encourage "openness to students' solution methods, including a need to share, explain and debate their work with others in the classroom" (Wagner, 2005). This means, effective learning occurs when students' thinking and reasoning are encouraged in class discussions. Clarke and Clarke (2004) illustrate the importance of social aspects of the mathematics classroom that focus on conceptual development. According to them, a "lesson that promotes mathematics discourse consists of teacher-led discussion of problems posed in a whole class setting, collaborative small group problems posed in a whole class setting, collaborative small group problem solving" (p. 109) and an ensuing discussion in which students justify the problem interpretation and solutions they developed during individual work. In the process, teachers recruit students into the discourse by eliciting their explanations and justifications, and skilfully use students' ideas as the basis of ensuing discussion (Chen et al., 2004).

Kilpatrick and his team argue that "fostering students' communication skills is an important part of the teacher's role in the classroom. Through skilfully led classroom discussions, students build mathematical understanding and consolidate their learning of key mathematical concepts. Small-group or whole-discussions provide students with the opportunity to ask questions, make conjectures, share and clarify ideas, suggest and compare strategies and explain their reasoning (Kilpatrick et al., 2001). As they discuss ideas with their peers, students learn to discriminate between effective and ineffective strategies for problem solving. In the process, the teacher can prompt them to explain their thinking and the mathematical reasoning behind a solution or the use of a particular strategy by asking: "how do you know, why do you do or say so, which method is right and why"? (Ontario, Ministry of Education, 2005)

According to Short, Singh, Yarrow and Millwater (2000), an effective mathematics teacher constructs the lesson through a *dialogue with the students*. As the dialogue develops and unfolds, the teacher and students interact with one another, step by step within an almost predictable pattern of moves, counter moves and activities as the lesson is constructed piece by piece (Short et al., 2000). In such a lesson, proficient teachers use clear language to explain to students in lucid and systematic ways the relationship in meaning among mathematics terms and concepts. Likewise, effective teaching practices appear to be "premised on teachers using information about students' mathematical thinking to guide classroom discussion. Among the teaching practices mathematics teachers can use to obtain information about their students' mathematical thinking is to ask them to explain their strategies for solving problems, to generate multiple problem-solving strategies and to compare their problem-solving strategies" (Webb et al., 2008, p. 2). My view strongly resonates with Schoenfeld (2002), who posits that learning in the classroom is a social activity. Hence, the role of the social interactions that occur in the mathematics classroom and the way it impacts on effective learning of mathematical concepts is another focus of this doctoral study.

2.5.8. High cognitive level tasks approach

The key decision that teachers make in developing mathematical proficiency or specialised mathematics is the choice of classroom tasks. Effective instructions use worthwhile mathematical tasks such as investigations and problem solving for students to learn with deep understanding. Worthwhile mathematical tasks, according to Koontz (2005), are those tasks that are based on learners' understandings, interests and experiences, and which provide a range of ways for diverse learners to learn sound and significant mathematics. For that reason, effective teachers use a range of types of tasks that allow students opportunities to solve problems, explain their solving strategies and reasoning, build their conceptual understanding and develop the necessary fluency. As Sullivan (2011) argues, "the mathematical tasks that are the focus of classroom work and problem solving determine not only the level of thinking by students, but also the nature of the relationship between the teacher and the students" (p. 31). In light of the required mathematical actions, described by Kilpatrick et al. (2001), it is impossible for teachers to foster conceptual understanding, procedural fluency, strategic competence and adaptive reasoning in learners without providing them with high-quality mathematical tasks that are designed to foster metacognitive development and have a social component. According to Wagner (2005), effective tasks are motivating, engaging and incorporate real world applications and contain important mathematical content that require learners to demonstrate their understanding. Such tasks encourage the use of challenging problem situations to introduce new concepts through conceptual learning instead of procedural learning (p. 5).

High cognitive level classroom tasks are those that focus on developing students' procedural fluency, those that use representations or models, and those that use authentic contexts or contextualised practical problems (Kilpatrick et al., 2001). They are more open-ended and progressively increase the complexity of the demand on students, emphasise methods over answers, facilitate connections between topics/concepts, support cooperative group work, build on what learners bring to the lesson and explore common misconceptions (Sullivan, 2011). It is important that good fluency tasks offer learners opportunities to practice skills or procedures, think and reason mathematically. In this study I would like to see how teachers use a range of mathematics classroom tasks that allow learners opportunities to build their understanding of concepts, demonstrate their problem solving abilities, explain and justify their mathematical thinking and reasoning, and develop the necessary fluency, as explained by Kilpatrick et al. (2001).

2.5.9. Reflections on key focussed approaches of effective mathematics teaching

I have hitherto presented a synthesis of eight specific elements or key principles of effective interaction in the mathematics classroom that have been recommended to be significant for development of students' mathematical proficiency. These principles support each other and should be viewed as a whole, not in isolation. As a set, these elements underpin much of the discussion about the conceptual and theoretical frameworks that follows.

2.6. CONCEPTUAL AND THEORETICAL FRAMEWORKS INFORMING THE STUDY

2.6.1. Strategic models of effective mathematics teaching

This section of the literature review presents a selection of "strategic models" for teaching mathematics effectively that focus on key theoretical concepts underlying this research project, and some elements which contribute to such models. Essentially, these models tend to encompass the strategic views of what effective teaching practices are and explain in detail characteristics/features of effective mathematics teaching approaches as elaborated in the preceding sections.

I will thus begin with the discussion around Shulman's (1987, 1986) and Ma's (1999) work who sought to contribute key elements towards strategic models of learning. This is followed by a discussion on two interdisciplinary models of effective mathematics learning and teaching. These constitute the work of Kilpatrick's (2001) model of teaching proficiency and

Maturana and Varela's (1992) theory of enactivism. In addition to being used as conceptual and theoretical frames, I also use them as my analytical lens for this study.

2.6.1.1. Shulman's and Ma's perspectives on teachers' competencies and knowledge of mathematics

Within this research, Shulman's and Ma's academic work on knowledge required for teaching mathematics successfully are precursors to Kilpatrick et al.'s (2001) model of teaching for mathematical proficiency, which inspired my thinking and research into the teaching practice of effective teachers of mathematics. Kilpatrick et al. (2001) adopted a composite and comprehensive view of successful learning which they termed *mathematical proficiency*, while Shulman's (1987) and Ma's (1999) concepts of *mathematical content/pedagogical knowledge* and *profound mathematical understanding for teaching* are important aspects of effective or successful mathematics teaching and learning as well as of improved student achievement. This knowledge is important for teachers' own mathematical practices and by implication provides the necessary tools for teacher teaching practices (Brodie, 2009; 2010). I will now explore Shulman's (1987) concept of "subject and pedagogical content knowledge" as well as Ma's (1999) notion of "profound understanding of fundamental mathematics", and how other related studies or models on mathematics education interpret the importance of such a knowledge base for teaching. Shulman (1987) discerned three different types of mathematical knowledge needed for effective teaching:

- *"Subject matter knowledge*: knowledge of mathematics being taught, the amount and organisation of the subject matter *per se* in the teacher's mind;
- *Specialised content knowledge*: knowledge of generic teaching strategies viz. planning questioning, grouping, assessing and general factors that might have an impact on learning. Other elements include whether the teacher has all the learners engaged, his/her use of proper classroom management techniques, and the quality of instructional materials;
- *Pedagogical content knowledge*: knowledge of ways of representing and formulating the subject that makes it comprehensible to others. This includes the degree to which teacher can appropriately integrate the use of the instructional techniques with the mathematical concepts being taught and its effectiveness on learners learning. This also includes the use of correct language to clearly convey mathematical ideas".

In her widely reported comparative study of the knowledge of basic mathematical concepts among American and Chinese teachers, Ma (1999) pointed to four aspects of knowledge for effective teaching. These are:

- Knowledge of basic mathematical ideas, which is the mathematical ideas that are pertinent to school mathematics;
- The ability to make connections between these mathematical ideas;
- A capacity to create and use multiple representations of these mathematical ideas in teaching, and
- A deep knowledge of the curriculum continuum.

Such a knowledge base for teaching enables mathematics teachers to be competent and skilfully perform powerful pedagogical acts, which enable learners to understand subject matter concepts (Schulman, 1987, 1986). Ma cited in Kilpatrick et al. (2001) said:

> "…One thing is to study whom you are teaching. The other thing is to study the knowledge you are teaching. If you can interweave these two things together nicely, you will succeed…believe me, it seems to be simple when I talk about it, but when you really do it, it is very complicated, subtle and takes a lot of time. It is easy to be a school mathematics teacher, but it is difficult to be a good mathematics teacher…" (p. 370).

These show that there is a difference in being a *mathematics teacher* and being an *effective mathematics teacher*. In addition, the complexity that Ma alludes to, and the notion of teachers' fundamental mathematics understanding or teaching proficiency resonates with Shulman's (1987) teacher mathematical knowledge which he defines as a *particular form of content knowledge that embodies the aspects of mathematics content most germane to its teachability* (p. 9). By talking about profound understanding of fundamental mathematics in a different context, Ma (1999) referred to collaborative partnerships between the teacher and learners whereby she does not "regard improvement of teachers' mathematical knowledge as necessarily preceding improvement of learner learning", but rather claims that "interacting with each other should support the improvement of the other" as the two are "interdependent in the teaching and learning processes" (p. 146). She calls for a context where the same idea of profound mathematical understanding will work in both mathematics learning of the teachers and in their classroom teaching practices. Shulman (1987) argues that effective mathematics teachers possess a unique blend of mathematical content knowledge and pedagogical content knowledge for teaching geometry topics (concepts) to students. Kilpatrick et al. (2001), on the other hand, claim teachers' mathematical proficiency or knowledge is an important aspect of effective teaching and learning as well as of improved learners' achievements. Teachers should be knowledgeable and act as facilitators of learning (Kamwi, 2001). Such knowledge plays a pivotal role in their classroom instructional practices, in developing conceptual understanding in students and in enhancing mathematical learning, understanding and problem solving.

In an explanation of Shulman's categorisation of teacher mathematical knowledge, many scholars (Williams, 2011; Ball and Bass, 2009; Hill, Blunk, Charalambos, Lewis, Phelps, Sleep and Ball, 2008; Ball and Hill, 2008) argued that the importance of teachers' content and pedagogical knowledge is influential in providing quality instruction and learning and affording learners opportunities to engage in high level conceptual thinking. The literature has shown that the higher the qualifications of teachers the better they teach the secondary mathematics curriculum (Hill et al., 2008). In addition, Ball, Bass and Hill (2004) urge highly qualified mathematics teachers to simplify, compress and clarify their mathematical understanding for students to internalise concepts easily.

Ball and her colleagues distinguish "(i) common content knowledge and specialised content knowledge within Shulman's subject matter knowledge, and (ii) knowledge of content and students and knowledge of content and teaching within Shulman's pedagogical content knowledge" (Williams, 2011; Hill et al., 2008; Ball et al.,2004). Like Shulman, they also recognise the importance of curriculum knowledge and horizon knowledge as domains within mathematical knowledge for teaching. They define horizon knowledge as what equips the teacher with the knowledge of what s/he currently teaches relates to and might be useful to what learners will learn in some subsequent grade levels. This knowledge also helps teachers to "set the foundation for what will come later" (Sanni, 2009, p. 51). The notion of mathematics *teachers' pedagogic knowledge*, advanced by Askew et al. (1997), adds another element of knowledge of how teachers teach or students learn mathematics or mathematical concepts. Such mathematical knowledge enhances quality effective teaching of mathematics which ultimately strengthens and deepens students' mathematical understanding. This mathematical proficiency is likely to be observed through teachers' mathematics classroom work, a focus of this study.

In this study my focus includes teachers' mathematical content knowledge and pedagogical content knowledge (PCK) for the purpose of understanding how the two categories of knowledge interact in shaping and informing effective mathematics teachers' classroom practices, and the way these teachers operationalize such knowledge in their mathematics classroom situations. My choice of PCK is also inspired by an understanding that teachers' classroom teaching actions are bound up with their content knowledge and pedagogy. Thus, such a study focusing on teachers' mathematical proficiency for teaching that focuses on both content knowledge and profound understanding of fundamental mathematics allows for an understanding of how effective mathematics teachers comprehend the subject matter, and how they are able to "elucidate subject matter in new ways, reorganise and partition it, clothe it in [their classroom mathematics] activities and emotions, in metaphors and exercises, and in examples and demonstrations, so that it can be grasped by students" (Sanni, 2009, p. 9; Shulman, 1987). This understanding of teachers' mathematical proficiency or knowledge in practice is central for the purpose of this doctoral study.

In refining "teachers' mathematical knowledge base for teaching", reputable mathematics educators have differently characterized this unique kind of teacher professional knowledge or teaching ability. This ability leads to a "holistic understanding of fundamental mathematical concepts" such as *mathematics knowledge for teaching* (Hill et al., 2008, p. 431). Carpenter, Fennema and their colleagues as mentioned in Stephens (2009) use the term *cognitively guided instruction* that aims to understand and build on students' mathematical ideas, thinking and reasoning.

Thus, I would like to focus my classroom observation and investigation on the characterisation of mathematical processes related to the mathematical thinking that effective secondary school mathematics teachers use in their everyday work of teaching. Mathematics teachers require mathematical thinking in analysing lesson concepts, planning effective lesson activities for a specified aim and anticipating student' responses (Chick, Pham and Baker, 2006b). The argument here is that teachers' mathematical thinking is not just required for lesson execution, but also for conducting a good mathematics lesson. Akin to mathematical reasoning (Brodie, 2009), "mathematical thinking makes a difference to every minute of the lesson" (Chick et al., 2006b, p. 140). Thus, the resources/blend of mathematical knowledge and skills that effective mathematics teachers bring to the mathematics classroom discussion and tasks is my way into analysing their mathematical thinking. I thus support the view expressed by Chick et al. (2006b) that "if students' ability to think mathematically is an important outcome of schooling" (p. 143), then it is clear that teachers' mathematical thinking should feature prominently in any mathematics lesson.

In this study, “mathematical knowledge for teaching” refers to both individual teacher’s expertise and mathematical content knowledge that support his classroom teaching practice (Ma, 1999), that is, knowledge of how and why a particular mathematical procedure works and how best to define the mathematical concept or term for a specific grade level. This knowledge also refers to knowing specific errors children are likely to make in a specific content area. Such knowledge may further imply ‘mathematical quality of instructions’, which I understand as richness of the mathematics of a lesson, including mathematical explanation, justification and related observable lesson artefacts. In addition, because a teacher as a person possesses diverse life experiences and knowledge that she brings to the mathematics class, these are factors that make up a total person and affect their interest and response to innovation and change and reform (Sproston, 2005, p. 31).

It is logical to assume that a mathematics teacher with a good grasp of mathematical concept knowledge imparts this positively to develop students’ mathematical understanding. Such a teacher can expose students to varied, cognitively challenging mathematical tasks and adventurous mathematical encounters (Taylor, 2008; Howie, 2004). Conversely, Taylor (2008) explains that a teacher with a weak mathematical concept knowledge is limited for s/he “cannot teach what s/he does not know and is, therefore, constrained by this knowledge lack in his/her classroom instructional practices” (p. 24). The literature locally and internationally confirms that while teacher’s mathematical content knowledge for teaching is crucial, it is not sufficient. There is also a need for “subject content knowledge for teaching” (Williams, 2011). This knowledge concerns the ability of teachers to enable students to understand the mathematical concepts they teach.

According to Williams (2011, p. 19), “experienced [mathematics] teachers possess a unique blend of [mathematical] content knowledge and pedagogical knowledge for teaching particular topics to certain groups of students that they develop over time”. From a programmatic perspective, the concept of PCK provides credence to the notion that the art of teaching is a lot more complex than simply the transmission of knowledge and skills from a teacher to students. It is rather the management of, and sensitivity to, many different variables which merge to create an environment conducive to learning. Teacher’s PCK empowers the teacher and encourages the democratic learner-centred methods and activities that engage students in high level mathematical conceptual thinking.

This study highlights claims that teachers' mathematical knowledge and scaffolding practice play a significant role in the teaching and learning of mathematics, and geometry specifically. Using the work of researchers like Williams (2011), Hill et al. (2008), Ball et al. (2004) and Shulman (1987) to illustrate the kinds of understanding entailed in expert teachers' knowledge and classroom practices, I expound that good teacher mathematical proficiency for teaching yields benefits both for classroom instructional practice and the development of mathematically proficient learners (Kilpatrick et al., 2001). Hill et al. (2008) examined "the dynamics of [mathematical] knowledge by studying the relationship between five teachers' mathematical knowledge for teaching and the mathematical quality of their instruction. *Mathematical knowledge for teaching,* according to them, means not only the mathematical knowledge common to individuals working in diverse professions but also the subject matter knowledge that supports that teaching, for instance, why and how specific mathematical procedures work, how best to define a mathematical term or concept for a particular grade level, and the types of errors students are likely to make with particular content". "M*athematical quality of instruction* refers to a composite of several dimensions that characterise the rigor and richness of the mathematics of the lesson, including the presence or absence of mathematical errors, mathematical explanation and justification, mathematical representation and related observables" (Hill et al., 2008, p. 431). The present case study is part of this mathematical knowledge for teaching, and so, teachers' subject matter knowledge (SMK) and intuitions are the main prerequisite for teaching proficiency that broadens conjecturing, explorations and investigation procedures in the mathematics class. Shulman's domains of mathematical knowledge for effective teaching that guided my study are exemplified in Figure 2.1.

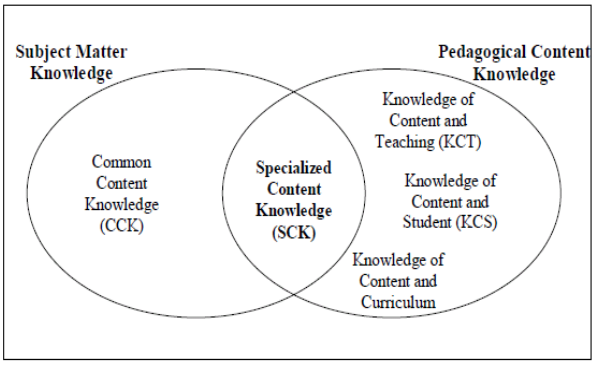


Figure 2.1: Mathematical knowledge required for effective teaching
Source: (Hill, Ball and Schilling, 2008, p. 377)

Moreover, Butterfield and Chinnappan (2010) and Hill et al. (2008) discuss four dimensions of Shulman's domains of mathematical knowledge for teaching (MKT). These domains include common content knowledge (CCK), specialised content knowledge (SCK), knowledge of content and students (KCS) and knowledge of content and teaching (KCT) as shown in Figure 2. They define CCK as mathematical knowledge and skill possessed by a well-educated adult. SCK refers to knowledge of how to use alternatives to solve a variety of mathematical problems, articulate mathematical explanations and demonstrate representations. KCS refers to knowing about mathematics and knowing about students. That is, knowledge of how to anticipate what the student is likely to think, and relate mathematical ideas to developmentally appropriate language used by students. KCT entails such knowledge that combines knowing about mathematics and knowing about teaching. This means knowledge of how to sequence content for instruction, determine instructional advantages of different representations, pause for clarification and when to ask questions, analyse errors, observe and listen to students' responses, prompt, pose questions and probe with questions and select appropriate tasks (Butterfield and Chinnappan, 2010, p. 110).

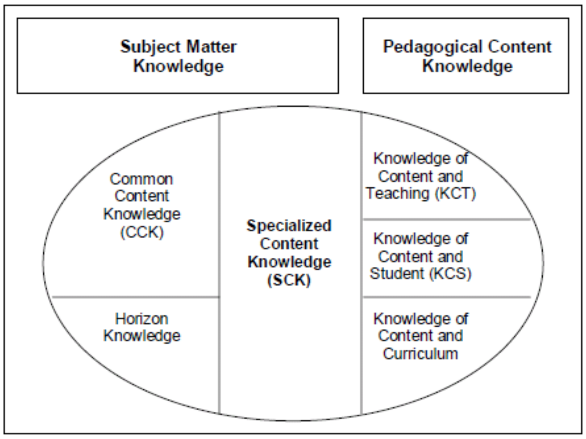


Figure 2.2: Schematic representation of domains of knowledge for teaching mathematics
Source: (Ball, Hill and Bass, 2005, p. 16)

Howie (2004) opinionated that "nobody can get into someone's head to examine the knowledge that she/he may have; whether you believe that knowledge resides in the head of the individual or within social interaction" (p. 27). Also, there are aspects of teachers' practices that only seem to find expression once teachers actually engage with their instructional business in the mathematics classroom. Hence, the critical links between the mathematical content knowledge and classroom instructional practices, for many teachers, is implicit and yet has a significant bearing on both teachers' professional knowledge and practice. As I have chosen to work with effective mathematics teachers, who bring along distinctive idiosyncrasies in this study, I am particularly concerned with the teachers' mathematical teaching proficiency (i.e. mathematical content knowledge and pedagogical knowledge) that translates into their classroom instructional practices and the way perceived effective mathematics teachers scaffold their mathematics classroom work. Teachers' mathematical proficiency, particularly SMK, is central to orchestrate classroom practice, but it seems it can be rendered ineffective if the related PCK is not in place. Howie's (2004) study on interpreting South African teachers' mathematical knowledge, in the context of learner-centred practice, indicates that "the mathematics teacher being able to draw effectively on his/her SMK facilitates aspects of classroom scaffolding" (p. 115). This principle relates to making the teaching and learning of mathematics a meaningful and motivating process, and it can be selected and used appropriately to achieve different lesson ends (Howie, 2004). For example, this principle can include a teacher's ability to "explore/make explicit what is known, challenge/extend students' mathematical thinking, demonstrate the use of a mathematical instrument, or to assist students arrive at a key generalisation" (Stephens, 2009, p. 30). In particular, drawing on SMK enables teachers to scaffold and make more informed decisions about how they meet the learning needs of all students in the most appropriate way possible. Hence, SMK enables a range of classroom interaction patterns or scaffolding practices similar to approaches discussed in 2.5.1 to 2.5.8, as characterised by the following key elements:

- **"Modelling**: the teacher explains, demonstrates and offers behaviour for imitation or shows students what to do and/or how to do it;
- **Collaborating**: the teacher responds to students' suggestions, invites comments and opinions to what they are doing, accepts critique, and works interactively with students in-the-moment on a task to jointly achieve a solution;
- **Excavating**: the teacher systematically explores students' mathematical ideas, understandings, explanations and questions to find out what students know to make the known more explicit;
- **Guiding**: the teacher observes, listens, monitors students as they work on tasks, asks questions designed to help students see connections and articulate conjectures;
- **Convince me**: the teacher actively seeks evidence, encourages students to be more specific. For instance, the teacher may act as if he does not understand what students are saying, encourages them to provide data, explain or justify their ideas;

- **Extending**: the teacher sets significant challenges, uses open-ended questions to explore the extent of students' understanding, facilitate generalisations, and provide a context for further probing or learning;
- **Orienting**: the teacher sets the scene, poses a problem, establishes a context, invokes relevant prior knowledge and experience and provides a rationale necessary at the beginning of a lesson or new task or idea;
- **Probing**: the teacher probes students' mathematical understanding using a specific question or task designed to elicit a range of problem solving strategies, presses for clarification and identifies possible areas of need;
- **Focusing**: the teacher focuses on a specific gap (i.e. concept, skill, strategy) that students need to progress. Similarly, the teacher maintains a joint collective focus and provides an opportunity for students to bridge the gap themselves;
- **Reflecting**: the teacher orchestrates a recount of what was said, a sharing of mathematical ideas and strategies. This typically occurs during whole lesson discussion share time of a lesson where learning or a concept is made explicit, key strategies are articulated, valued and recorded, and
- **Apprenticing**: the teacher provides opportunities for more learned students to operate in a teacher capacity, endorses student-to-student interactions" (Stephen, 2009, p. 31).

The tenets of effective mathematics teaching practice that I have outlined thus far and that I will draw on in my data analysis and interpretations is used in very specific contexts. I believe this knowledge impacts directly on the forms of mathematical and teaching proficiency that will become evident and the way in which it is used in this study. Furthermore, the qualities of effective teaching identify common elements of teaching practices that characterise teacher effectiveness in the mathematics classroom. This knowledge is useful in this study and will be used to describe the "mathematical actions of teaching", actions taken by teachers or students that require the application of teaching and mathematical proficiency…proficiency with content, proficiency in classroom mathematical activities, and proficiency with teaching (Kilpatrick et al., 2001). With respect to these proficiencies, I focus on teachers' teaching proficiency in relation to their mathematical knowledge for the purpose of understanding how these blends of knowledge interact in shaping and informing teachers' classroom instructional practices.

In the next section, I describe the conceptual and theoretical frameworks that underpin this study. I commence with an in-depth discussion on Kilpatrick et al.'s (2001) model of teaching proficiency, and the relationship between its teaching proficiency strands or competencies with respect to effective mathematics teachers' classroom practices. Each of these concepts, I suggest, informs and is informed by the others. I then address the theory of enactivism as a theoretical vantage lens.

2.6.2. Models of Kilpatrick et al. (2001) teaching proficiency and Enactivism

This section presents in some detail Kilpatrick et al.'s (2001) teaching proficiency model and enactivism as a conceptual framework and theoretical vantage point that underpin and inform this study. The enactivist perspective emphasises mathematical knowing as a process of doing, reasoning and actions. It also stresses the linking of mathematics to everyday life experience in accordance with the learner centred education (LCE). According to Proulx (2009), the enactivist view encourages learning and construction of knowledge by means of a collaborative process in a complex mathematics classroom. Therefore, any given learning situation needs to encompass the teacher, the learner, the learning content and the particular context that co-create it in order for some form of interactions to occur.

The enactivist perspective suggests a move or shift from individual constructivist assumptions such as learner centred education to a more practice-meaning-making approach modelled along complexity theory (Proulx, 2008). Thus the emergent enactivist theory is a post-social constructivist model (Reid, 1996). Enactivism relates to Kilpatrick et al.'s (2001) framework in that teaching and learning are interrelated, and that they are effective and meaningful when the teacher and students engage in collaborative processes of knowledge construction. In Canada, for example, some educators believe that enactivism has significant implications for mathematics teaching (Proulx, 2009; Davis, 1996), as it views the act of knowing (cognition) as "a complex co-evolving process of systems interacting and affecting each other in the teaching dynamic" (Li, 2010, p. 4). The inclusion of enactivism allows the broadening of Kilpatrick et al.'s (2001) model to move beyond mere accounts of teaching for mathematical proficiency and also focus on the ways of meaning-making or interpretation of mathematics as the teacher interacts with the students in the classroom. Enactivism, then is a useful extension of constructivism and is used in this study to complement the Kilpatrick et al.'s (2001) teaching model in order to analyse effective teaching practices, and how people make meanings as a process of embodied mathematical cognition as discussed in 5.3.2.2, since teaching is "what teachers do, say and think with learners concerning the learning content in a particular [environment], in time" (Douglas, 2009, p. 518; Thames, 2009).

2.6.2.1. Kilpatrick et al.'s (2001) teaching proficient model as a conceptual framework

2.6.2.1.1. Introduction

Kilpatrick et al.'s (2001) teaching proficiency framework has been used in numerous countries with substantial success in mathematics education research. Some of these countries are the United States (Kilpatrick et al., 2001), South Africa (Brodie, 2010) and Zimbabwe as per my personal communication with a member of the Rhodes PhD research team (Zazu, October 2010).

This research is a case study that focuses on the *teaching proficiency* of selected effective mathematics teachers. However, presently there appears to be little consensus amongst policy makers, curriculum designers and teachers on what constitutes good mathematics teaching practice (Douglas, 2009) in Namibian schools. The Kilpatrick et al.'s (2001) framework of teaching proficiency provides a way to think about mathematics teaching and learning in that it encompasses the key features of knowing and doing mathematics (Kilpatrick at al., 2001). Teaching proficiency implies expertise in handling mathematical ideas and creating varied interaction patterns with students in the mathematics classroom. Kilpatrick and his colleagues point out that those teachers who are mathematically proficient have a solid understanding of mathematical concepts, are fluent in performing operations, exercise a selection of strategic knowledge, reason clearly and maintain a positive outlook towards geometry. The five strands of teaching proficiency constitute the knowledge, skills, abilities and beliefs that effective mathematics teachers should be able to capitalise on in teaching mathematics. In addition, effective teachers should be able to present students with mathematical tasks that incorporate all mathematical actions in an integrated manner so that each action reinforces the others in their teaching practices.

Interestingly, conceptual understanding and procedural fluency are the primary focus in school mathematics, with evidence that teachers focus more on procedures than on concepts. According to Brodie (2010), both strategic competence and adaptive reasoning add reasoning, justifying, making connections, exploring, verifying, and clarification, to the first two strands. Mathematical reasoning, which is a core competence in geometry, includes formulating, testing and justifying conjectures. Proficient teachers should thus stimulate learners' structure (as stressed by enactivists) into thinking and justifying about mathematical ideas. That is, effective mathematics teachers should create opportunities for their students to discuss, evaluate and mutually agree on mathematical concepts or ideas being taught. Likewise, effective teaching practice implies identifying learners' expressions of mathematical ideas and responding to them in appropriate ways. So teachers will be unable to engage their learners in productive conversations about multiple ways to solve mathematical problems if they themselves can only solve them in a single way. Conversely, teachers with a weak conceptual knowledge of geometry tend to demonstrate procedures to learners and then give them opportunities to practice them. So, teachers' teaching proficiency and actions appear to affect what is taught and ultimately learned. Accordingly, Kilpatrick et al. (2001) contend that social contexts in which instructional practices occur are important, and therefore the pedagogical challenge for teachers is to manage classroom practices so as to develop mathematical proficiency. I, therefore, argue that teachers should teach mathematics in ways that enable learners to reason mathematically.

Kilpatrick et al.'s (2001) framework consists of five interwoven strands of teaching proficiency or desirable mathematical actions, which form the core conceptual frame and analytical tool for this study. These are:

2.6.2.1.2. Strand 1: Conceptual Understanding (CU)

This teaching proficiency, i.e. "knowing why", entails comprehension of mathematical concepts, symbols, operations and relations. This also involves the core mathematical knowledge required in the practice of teaching (Kilpatrick et al., 2001). Teachers' ability to justify claims, use symbol notations efficiently, defining terms precisely, and make generalisations are examples of effective practices. Conceptual understanding also refers to teachers' ability to explain problem-solving methods and connect them to real-life situations. CU is crucial for an integrated and functional grasp of mathematical ideas and concepts such as shapes, spaces, measurements, patterns and functions in geometry and trigonometry. Similarly, CU implies representing the mathematics content in different ways and connecting it to everyday life in meaningful ways. It helps us to know the distinction between new mathematical ideas and why these ideas are important in real world contexts. Hiebert and Lefevre (1986) define conceptual knowledge as "knowledge that is rich in relationship" (p. 3). Thus, teaching practices in this context should focus on the development of meaningful and intellectually challenging tasks that enhance students' conceptual understanding. In the context of this study, skilled teachers should be able to model a situation/concept to make it easier to understand and solve problems related to it. In addition, teachers' instructions and instructional activities should be geared towards enabling learners to "build new mathematical knowledge through problem-solving, solve problems that arise in mathematics and in other contexts, and monitor and reflect on the process of mathematical problem-solving" (NCTM, 2000, p. 31). Kilpatrick et al. (2001) advise that children need to have multiple opportunities to fully grasp mathematical concepts, and perform various mathematical tasks.

Moreover, the notion of conceptual understanding as discussed by Schäfer (2012) is a "critical component of mathematical proficiency that is necessary for anyone to [teach] or 'learn' mathematics successfully. This implies teachers' understanding of mathematical knowledge [proficiency] that not only revolves around isolated facts but includes an understanding of the different contexts that frame and inform these facts" (p. 1).

From an enactivist perspective, conceptual understanding implies knowing as enactive representation drawing on physical experiences based upon embodied thoughts and visual imagery (Holton, 2010). From this perspective, CU is understood as a social learning process (interaction) grounded within a person, a topic or a particular environment. That is, mathematical knowing or understanding occurs through shared actions. Hence, enactivists understand mathematical knowing and understanding or conceptual knowledge as manifested actions that occur through social relationships that bind the teacher and the learners together in the practice of teaching (Proulx, 2009). An example is an agent (a teacher or a team) helping an individual (learner) to understand concepts and construct meaning.

2.6.2.1.3. Strand 2: Procedural Fluency (PF)

Procedural or mathematical fluency refers to knowledge or skills of procedures, i.e. knowing when and how to use procedures flexibly, accurately, efficiently and appropriately in different ways and contexts (Kilpatrick et al., 2001). PF also refers to "having factual knowledge and concepts that come to mind readily" (Sullivan, 2011, p. 7). In the context of this study, this refers to symbols, rules, heuristics and formulas as well as knowledge used in solving mathematical problems or carrying out elementary instructional (teaching) routines. It also refers to being fluent in procedures from other disciplines of mathematics such as constructing shapes, measuring space, constructing algebraic equations/functions and/or describing data. For example, drawing a triangle or measuring an angle using a protractor.

Another example of the way PF works is in mathematical language and definitions. That is, a teacher or student should know or note what it means by terms such as "right angle, parallel line, chord, diameter, isosceles triangle, twice or at least". That is to say "if the learner can readily recall key definitions and facts, then these facts can facilitate problem solving and other actions" (Sullivan, 2011, p. 7). According to Hiebert and Lefevre (1986), sound procedural fluency involves conducting a series of actions in a certain order/manner and in a smooth/skilful way, using mathematics skills reliably to solve problems, and generate examples to test mathematical ideas and understandings. Enactivists view PF as action sequences in solving routine and non-routine mathematical problems. From this viewpoint, PF is viewed as doing mathematics as part of the embodiment of mathematical understanding, symbols and shapes (Davis, 1996). Thus, procedural fluency involves deployment of imagined reason or action to solve mathematics problems. In this study, I am interested in how teachers develop mathematical fluency in problem solving in their students and allow more capacity for other mathematical actions.

2.6.2.1.4. Strand 3: Strategic Competence (SC)

This teaching strand concerns the ability to plan effective instructions and instructional activities as well as solving problems that arise during instructions (Kilpatrick et al., 2001). This competency involves "a set of critical control process that guide an individual to effectively recognise, formulate and solve problems" (Sullivan, 2011, p. 7). This competence refers to the teacher's ability to formulate, represent and solve mathematical problems, both numerically and symbolically, as they arise in the context of any domain of mathematics (i.e. geometry, algebra, statistics, etc.). In the mathematics classroom context, this implies a teacher's ability to involve students engaging in devising problem solving strategies or in engaging them in high cognitive level mathematics classroom tasks. That is to say, SC implies teacher's competence to engage students in tasks for which solutions are not known in advance. In this study, I am interested in looking at the way effective teachers apply and adapt a variety of appropriate strategies to help learners solve mathematical problems.

In conjunction with the enactivist structural determinist framework, SC is an "enactment of routine instructional activities that support the generation of new knowledge" (Niessen, 2007, p. 6) that *co-emerges* with the learners' ways of knowing. In the classroom, both the teacher and the learners construct meaning and learn from each other. In a way, the teacher role is to help extend the students' repertoire of problem solving strategies. Using the notion of structural determinism will enable me to examine how instructions relate to student learning. Specifically, structural determinism appears to posit that teaching does not directly cause student learning to take place but rather, teaching influences students' cognitive thinking/process which, in turn, influence learning. Thus, SC calls for the teacher to be proficient and co-evolving and actively engaging with the learners individually or in a group. So SC involves active dialogue, asking questions that trigger learners' actions and critical thinking, and relating mathematical concepts to current contexts and learners' daily life experiences or prior mathematical knowledge.

2.6.2.1.5. Strand 4: Adaptive Reasoning (AR)

Kilpatrick et al. (2001) define adaptive reasoning as the "capacity for logical thought, reflection, explanation and justification" (p. 380). This competence involves justifying and explaining one's instructional practices and in reflecting on those practices in order to continuously improve them (Kilpatrick et al., 2001). This teaching strand or competence demands teachers, as critical reflective practitioners, to effectively plan and present particular mathematical problems to their students. Teachers should also discuss mathematical thinking/reasoning that students manifest in learning processes and teaching dynamics. From my perspective of teaching, AR implies an understanding of what has been learnt and being able to use it in current contexts. It is about the capacity to think logically about relationships among concepts and situations, and to reason adaptively. Similarly, it implies valid reasoning and justification, since teaching geometry requires teachers to be able to teach relationship, logic and proof. The notion of enactivist *co-emergence* and *co-evolution* in relation to AR inspires mathematics sense-making, mathematical thinking, image making and problem solving as a process of mathematical reasoning and actions. In classroom practice, the teacher and learners are both seen as a source of mathematical knowledge that bring their own intuitions and learn from each other. Thus, students' explanations and justifications of their mathematical ideas provide a clear basis for mathematical understanding and problem solving skills. This further implies that the way students explain their mathematical ideas provides a means to recognise other aspects that can be used in order to arrive at a specific answer to a particular problem. Consistent engagement in such thinking practice leads to deeper understanding of concepts as well as the ability to demonstrate complex problem solving and adaptive reasoning skills. Hence, it demands teachers pose challenging mathematical tasks, bring students' ideas into more purposeful engagement with one another and consider reasoning as mathematical understanding or knowing.

2.6.2.1.6. Strand 5: Productive Disposition (PD)

Productive disposition is about seeing and identifying with the usefulness of mathematics in people's general ways of being. Kilpatrick and his colleagues define this teaching skill or strand as:

> "The tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner [teacher] and doer of mathematics applies equally to all domains of mathematics" (p. 142).

This quotation shows the all-encompassing nature of this strand of teaching proficiency. PD thus stresses teacher's passion and enthusiasm towards mathematics as subject, teaching, learning and improvement of practice (Kilpatrick et al., 2001). It emphasises all habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (knowledge and learning) to maintain students' curiosity. In my view, a love for problem-solving is an integral part of all mathematics learning and productive mathematical disposition.

Consistent with the enactivist perspective, PD refers to the teachers' dynamism across instructional practice and how it shapes student learning. From this perspective, PD is about relationship-centred teaching that embraces both the teacher and learners as partners and recognises and honours the social nature of learning, and diversity, and the mutual trust and partnership the teacher brings to the mathematics classroom.

As is argued by Kilpatrick et al. (2001), all these strands of teaching competence are intertwined and contribute to a balanced curriculum. Kilpatrick et al. (2001) warn that mathematical proficiency in any domain of mathematics such as geometry is "not a one-dimensional trait, and it cannot be achieved by focusing on just one or two of these strands" (p. 116). Kilpatrick et al. (2001) argue that teachers should interweave the five critical strands of teaching proficiency and develop each strand simultaneously with the others. Thus, it is the aim of this study to analyse how effective teachers' classroom instructional practices relate to Kilpatrick et al.'s (2001) strands of proficient teaching. Figure 2.3 shows the features or concepts that were analysed and synthesised from the data.

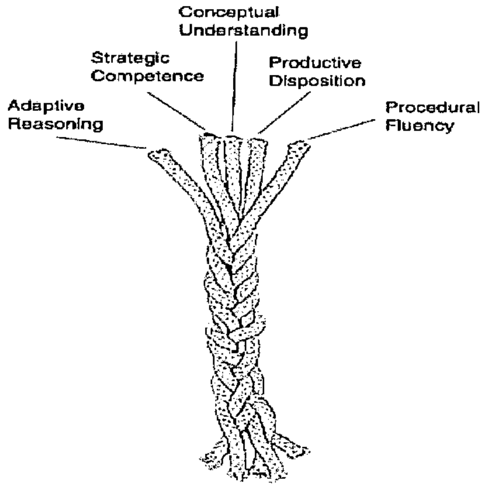


Figure 2.3: Kilpatrick, Swafford and Findell et al.'s (2001) model of teaching proficiency
Source: National Research Council (Kilpatrick, Swafford and Findell, 2001, p. 117)

This conceptual framework in Figure 2.3 guided the analysis of collected data as well as the discussion of the results in Chapters Four to Seven. In order to carry out this research, the conceptual framework was adapted because my study is based on mathematics education and it emphasises proficient mathematics teachers' classroom instructional practices (interaction). Teachers' strategic competence ameliorates the challenges learners face in the learning process and teaching dynamic, and what eventually becomes the enacted lesson. For this reason, mathematics teachers need to be able to:

- design instructional activities that can effectively build on pupils' existing knowledge, engage pupils' preconception or bridge commonly held conceptions and mathematical understanding;
- encourage and allow pupils to use their own informal problem solving strategies, at least initially, and then guide their mathematical thinking towards more effective strategies and advanced understanding;
- excite and encourage mathematical discourse so that pupils can clarify their problems solving strategies for themselves and others, and compare the benefits and limitations of alternate approaches to mathematics problems, and
- use a variety of structured tasks to guide pupils through generating and testing hypotheses, and asking them to explain clearly their hypotheses and conclusions to help deepen their mathematical understanding (Kilpatrick et al., 2001).

This study is therefore guided and driven by Kilpatrick et al.'s (2001) intertwined five components of the social learning theory with mathematical and teaching proficiencies placed at the centre. These theoretical approaches formulated my conceptual framework, and guided and illuminated this study within the broader theory of enactivism.

2.6.2.1.7. A critique of Kilpatrick et al.'s (2001) model of teaching proficiency

Every theory has its highlights and lowlights. Kilpatrick et al.'s (2001) model of teaching for mathematical proficiency was originally welcomed as being helpful in relation to the learning activities and experiences of teachers, and a useful framework to help conceptualize the various dimensions of the practice of teaching mathematics. However, given the complexity of the teaching and learning process, a body of criticism has subsequently built up. Some of the general critical observations that have been made about the model are:

(1) Although research evidence tends to support that Kilpatrick et al.'s (2001) model of teaching proficiency is useful in developing understanding of the knowledge base needed for teaching, and knowledge of students and how they learn, a doubt about its relevance to practice emerged in the context of classroom instructional practices. Groth and Bergner (2006), for example, used the Kilpatrick et al.'s (2001) model to assess teachers' motivation and observed that it lacked procedural directions for implementing its recommendations in a classroom setting.

(2) There appears to be a paradox in the literature regarding the impact of the model on teaching practice. Groth and Bergner (2006), for instance, observed that "constraints in the school setting due mainly to students and administrators would prevent one from successfully implementing the recommendations of the model" (p. 2). These negative critiques appear to be indicative of Kennedy's (1997) category pertaining to the nature of schools.

(3) Another criticism is that the Kilpatrick et al.'s (2001) teaching approach tends to look at the social and cultural aspects only as aids to individual learning. That is, the model does not attend to the socio-cultural contexts within the teacher-learner interactions that take place. Hence, it ignores that culture and society are important agents in the teaching and learning process. In contrast, enactivist theory emphasises "learning as participation" in particular classroom events with which the teacher and learners may be more or less familiar.

(4) Another weakness is that the Kilpatrick et al.'s (2001) model does not seem to emphasise the notion of embodied cognition, which foregrounds actions such as guessing and imagining,

and contends that knowledge construction cannot be separated from the act of being in the cognition process. Hence, the enactivist perspective incorporates a sense of doing and places the focus on mathematical actions being undertaken during the learning process.

(5) The model has also neglected key ideas such as the co-emergence and co-evolution of the teacher and student or mathematical ideas. These enactivist ideas stress both the teacher and the learner as being the source of mathematical knowledge and/or having something in common to learn from each other. Similarly, these ideas emphasise mathematical knowing or cognition as arising in the interactions of the individual with his environment, which the model in question has largely ignored.

Because of these limitations or gaps in Kilpatrick et al.'s (2001) model of teaching proficiency, I turned to enactivism because it provided me with a cohesive theoretical framework to examine teaching through an analysis of teachers' and learners' participation in a broader context. Thus, learning, from an enactivist perspective, is assumed as occurring through structural changes that an individual undergoes when participating in the classroom events. From an enactivist perspective, it is assumed that *mathematics knowing or cognition is a complex co-evolving process arising in the interactions between a person and his or her environment* (Li et al., 2010; Proulx, 2008). In that sense, the role of the teacher in the mathematics classroom is that of a "perturbator" in order to encourage, provoke and trigger students into thinking differently about mathematical concepts that may not form part of their personal construct/ideas. This has implications for both the learners and the teacher as they are significant parts of the collective classroom environment. Enactivism helped me see how effective teachers encourage the creation of a mathematical community of practice amongst their learners, and the way teachers and learners interact and affect each other as part of a larger collective system. I explored how these frameworks relate and contribute to my study, and how these perspectives help me to understand the nature of effective teachers' classroom mathematics practice in relation to their teaching proficiency.

2.6.2.2. ENACTIVISM AS A THEORETICAL VANTAGE POINT

2.6.2.2.1. Introduction

In this section, I first provide an overview of enactivism that underpins this research. I discuss the relationships between enactivism and situated cognition and complexity science.

2.6.2.2.2. Overview of Enactivism as a theoretical lens

As this research is partly situated within the enactivist perspective, its focus is not merely on the teachers' experiences, but also on the relations that bind both the teacher and students together within enacted webs of many interacting elements and persons. I explain enactivism as it is defined by researchers in the field of mathematics education (Stephanus and Schäfer, 2013, 2011; Proulx, 2009, 2008; Begg, 2000; Davis and Sumara, 1997; Maturana and Varela, 1986).

The term "enact" means "to work in or upon or "to act or perform" (Begg, 2000; Proulx, 2008). "Enactivism refers to the idea of knowing in action" (Niessen, 2007). People come to know about the world by interacting with it *bodily, experientially* and *cognitively* (p. 45). This means that an enactivist stance conflates identity, actions and knowledge so that there is a combination of being, doing and knowing (Niessen, 2007; Begg, 2000). As constant interaction is such an important feature in the enactivist worldview, I agree with Begg (2000) that enactivism holds a "relational ontology, meaning that all social realities and knowledge of self, others and things are viewed as interdependent or co-dependent constructions existing and known only in relation to each other" (Begg, 2000, p. 145). Enactivism is a view of cognition in which it is assumed that [mathematical] knowing, understanding or knowing occurs through the actions a person performs, both mentally and physically (Proulx, 2008). Its focus is on the process by which an organism and its environment are interacting. Enactivism as a theory of knowing (Proulx, 2009) is an emerging philosophical worldview and is currently gaining momentum in the mathematics education research arena. It builds on the traditional constructivist paradigm that underpins the Namibian education system. Proulx (2008), for instance, suggests that "enactivism might be viewed as an extension of constructivism, although there are significant differences" as well (p. 3). Hence, I contend that the theory of embodied cognition called enactivism might be considered one aspect of embodied constructivism, with additional social and radical aspects, but does not supersede constructivism at all.

Enactivism as an ecological paradigm was initially utilized in biology and pure science, but more recently, has been applied in social-science domains such as mathematics education, economics and politics to study multifaceted systems with the view to understanding complex perspectives, contexts and circumstances. As an ecological paradigm, an enactivist perspective views the body, mind, actions, souls, emotions, imaginations, experiences and the world *per se* as inseparable and indissolubly interconnected (Proulx, 2009; Begg, 2000; Reid, 1996; Maturana and Varela, 1992). Varela et al. (1991) use the term “enactive” to designate this view of mathematical knowledge and evoke the view that what is known is brought forth through bodily actions such as acting, doing and/or reasoning. The enactivist *theory of cognition* within mathematics emerged and developed from the work of Merleu-Ponty (1962), Maturana and Varela (1986) and Varela, Thompson and Rosch (1991), and then was recently, extended by numerous scientists and researchers such as Li, Clark and Winchester (2010); Holton (2010); Proulx (2009); Begg (2000) and Reid (1996). While this is a recent perspective, it draws from diverse domains including philosophy, phenomenology, complexity theory and evolutionary biology (Begg, 2000).

Conceptually, enactivism suggests that teaching and learning are complex and interrelated. “Mathematics knowing” is understood as manifested actions as teachers interact with students and vice versa. Hence, enactivists argue that “learning is through the learner’s [and teacher’s] acts and is acted upon by the world, and understanding is embedded in doing” (Li, et al., 2010, p. 16). “All doing is knowing and all [mathematical] knowing is doing” (Maturana and Varela, 1992, p. 26). From this perspective, learning is seen as the co-evolution of the knower and known that transform both as they interact with the natural world outside the formal learning environment. This implies the teachers’ interactions with the students during, say, the mathematics lesson set the stage for co-evolution. In a learning sense, co-evolution describes how children’s ideas and actions, as well as the interrelated aspects of learning tasks change each other over time. For example, conceptual and procedural knowledge co-evolve in that understanding as one improves or enhances the other (Kilpatrick et al., 2001). Specifically, the generation of problem-solving procedures is based on conceptual understanding. In the context of this study, this implies that a student who has greater conceptual understanding of the properties of triangles, for instance, may figure out a procedure to find another side length of a right-angled triangle given the other two sides.

Should a learner be unable to progress on any action the teacher and the learner could co-evolve along the spectrum of authentic responsive participation, collaboration, partnership and a shareable learning process and any new ideas that is generated could help them solve the problem (Mennin, 2010).

In contrast to the Kilpatrick et al. (2001) teaching approach, the theoretical stance of the enactivist approach emphasises the way that an organism and the human mind organise themselves by interacting with the environment (Davis and Sumara, 1997). The enactivist emergent worldview is premised on the notion that the human mind is inseparable from the world around it (Lozano, 2005). That is, *mind is seen more as a process that results from the ongoing flow of information between the environment and the many aspects of the body that interacts with it* (Davis, 2004). Varela et al. (1991, p. 122) state "as mindfulness grows appreciation for the components of experience grows".

Similar to Kilpatrick et al.'s (2001) model of proficient teaching, the enactivist *theory of cognition* is critical of the notion that "cognition is [fundamentally] not seen as a representation of the world [outside the observer (learner)], but rather, as an ongoing bringing forth of a world through the process of living itself" (Lozano, 2005, p. 25). Varela et al. (1991) as mentioned in Samson (2011) use "cognition" to describe this new perspective-enactivism-as an "embodied and co-emergent interactive process where the emphasis is on *mathematical knowing* as opposed to [mathematical] *knowledge*" (p. 39). Hence, cognition is viewed as the enactment of a mind on the basis of the variety of actions that a living being performs in the learning process. It is within this context that I situate my research project.

In enactivism, *knowing* is a learning process that requires actions, exploration, justification and language, through interactions with the unfolding circumstances such as mathematical tasks. This implies that teaching in a dynamic mathematics classroom is neither seen as an act of transmitting knowledge, nor is the learner seen as a passive recipient of knowledge. Teaching is seen as integrally involved with the doing of mathematics and fostering the attainment of mathematical proficiency, which is similar to what is described in the Kilpatrick teaching approach.

Proulx (2009) states that it is the *structure of the learner* that allows him/her to make sense of learning and mathematical concepts. Daniel (2005) argues that the "changes that result from the interaction between the living being and its environment are brought about by the disturbing agent but determined by the structure of the disturbed system" (p. 3). This implies that "learning should be understood not in terms of a sequence of [action], but rather in terms of an ongoing structural dance: a complex choreography of events which cannot be fully disentangled and understood, let alone reproduced" (Begg, 2000, p. 13).

The enactivist view has its roots in phenomenology, and Merleau-Ponty (1967) as cited in Hudgens-Haney (2010), suggests that the enactivist's notion of embodiment has this double sense:

> "It encompasses both the body as a lived, experiential structure and the body as the context or the milieu of cognitive mechanisms. Embodiment in this double sense has been virtually absent from cognitive science, both in philosophical discussion and in hands-on research. He (Merleau-Ponty) thus claims that we cannot investigate the circulation between cognitive science and human experience without making this double sense of embodiment the focus of our attention. This claim is not primarily philosophical. On the contrary, the point is that both the development of research in cognitive science and the relevance of this research to lived human concerns require the explicit thematization of this double sense of embodiment" (p. 42).

I therefore argue that the enactivist view of cognition is fundamental to the way of making sense of teachers' teaching practice and how the teacher, together with students, continually evolve within the classroom environment as they engage with geometric concepts. For the purpose of this current study, approaching teacher classroom teaching practice and mindreading from an enactivist perspective provides the right sort of emphasis to better understand these teaching practices and their embodied (enacted) mathematical understanding. It can help to determine what teachers' embodied views of cognition reveal about their teaching proficiency of mathematics and effectiveness.

The enactivist view has implications for classroom practice, for example, assumptions about *learning geometry means doing geometry* (Begg, 2000), of which I believe solving everyday life or contextual problems are an essential part. In addition, enactivism implies teaching and learning approaches should focus on creating many opportunities for students to learn geometrical concepts, and that the teaching-learning process should be highly interactive (Maturana and Varela, 1992). Given these dimensions and qualities, an enactivist framework is believed "to offer a dynamic new vision for examining learning and performance, and enable us to see the field of mathematics education in a new light" (Li et al., 2010, p. 16). Enactivism offers the following potential contributions to teaching practice (Holton, 2010):

- "It provides a theoretical grounding as well as a more solid, concrete empirical foundation for some of the concepts to come out of constructivism. For example, the notion of the embodied nature of teaching and learning and its role in learning and mathematical understanding;
- It offers a deeper and richer discourse about teacher practice and the relationship between the teachers and the students;

- It supports the learning environment and teaching that involve not only the words and sentences a teacher utters/writes on the board during the lesson, but also all the hands and arms gestures, body movements and facial expressions the teacher and learners perform in the mathematics classroom, and
- Its notion of embodied cognition perhaps humanises students and the learning even more than the constructivist paradigm" (p. 3).

In tandem with the Kilpatrick et al.'s (2001) proficient teaching model for mathematical proficiency, *effectiveness depends on enactment, mutual and interdependent interactions of the three elements of instructional triangle*: *mathematical content, the teacher, and the learners* as instruction unfolds (Kilpatrick et al., 2001, p. 9). From the enactivist perspective, Maturana and Varela (1992) have identified several theoretical concepts that are useful in analysing the teaching and learning of mathematics. For the purpose of this study, I will use and discuss the following four underlying enactivist concepts of teacher practice and proficiency:

(1) autopoesis
(2) embodied cognition
(3) co-emergence, and
(4) structural determinism and coupling.

These four fundamental enactivist theoretical concepts constitute the theoretical vantage point of this research. I use them firstly to enhance Kilpatrick et al.'s (2001) model in finding out how teachers' embodied perceptions of their mathematical proficiency and effectiveness support their own classroom teaching for mathematical proficiency during geometry lessons. Secondly, they enable me to ascertain how teachers as perturbators encourage learning and construction of knowledge and make sense of new experiences and challenges, while attending to emerging conversations and interpretations through shared activities. I discuss each of the concepts in greater depth, firstly by providing a brief definition and description and, secondly, by indicating the role that each has in the teaching and learning of mathematics.

2.6.2.2.2.1. The concept of autopoesis

Fundamental to the enactivist approach is the concept of autopoesis, which is premised on the notion of sense-making, of how living systems produce and consume meaning. Thus, autopoesis refers to the way living systems [teacher and learners] address and engage with the domain in which they operate and defines life as the ability to self-produce as opposed to reproduce (Lucas, 2005):

> "The stability of the internal medium is a primary condition for the freedom and independence of certain living bodies in relation to the environment [milieu] surrounding them". (p. 1)

Closely related to the radical variety of constructivism (Reid, 1996), the idea of enactivist autopoesis entails that properties of complex living systems are "self-organising, self-producing, dynamic and interactive". Self-organisation is a process where new structure (patterns) and properties appear (emerge) in a complex system under particular conditions without central authority or external elements imposing it through planning (Mennin, 2010). This coherent pattern arises from the local interaction of the elements that make up the system. The interaction is achieved in a way that all elements act at the same time, without any of the elements being a coordinator/facilitator. For this reason, teaching and learning in complex classrooms is seen as self-organised adaptation to changing circumstances. I thus wish to explore the notion of mathematical proficiency in a practical context where teachers together with the learners interact and experience the processes of negotiation towards meaning making in a way that is consistent with the autopoetic notion of sense-making.

As Reid (1996) maintains, complex systems are systems that possess capabilities of producing and maintaining themselves through their own interactions. He further argues that the "components of autopoetic systems must be dynamically related in a network of ongoing interactions" (p. 2). This implies that autopoetic components should respond actively to unfolding circumstances, and interact in ways which are continually changing so as to allow the system to continue existing. Mennin (2010) argues that "shared assumptions about the group tasks and expectations about learning provide [precise] boundaries for the interactions of participants" (p. 163). In my view, group work or discussion based-learning in the mathematics classroom are good examples of a complex system or autopoetic entity. McMillan (2004) describes components of autopoetic systems as learning what the world is able to offer them, in order to anticipate their future existence. This, in an enactivist view, means complex dynamic systems learn to adapt to changes in circumstances, and consequently reorganize themselves as they gain experiences (McMillan, 2004). From this perspective, the teacher and learners, as human/living beings are autopoetic physically and intellectually. This means during the mathematics lesson, the teacher together with the students work as a group on certain mathematical concepts, tasks or problems. Similarly, teachers do assign students to groups to share ideas, intuitions and experiences on given mathematical tasks. The essence of autopoesis appears to contribute to the teaching practice, and shows how the teacher assists learners (who operates adequately to their needs) to undergo continuous structural change so that they keep acting adequately in the changing teaching and learning situation. In terms of the enactivist approach and teacher classroom instructional practice, "learning is precisely this continual change" (Reid, 1996, p. 2) which allows both the teacher and learners to continue interacting even though the teaching-learning process is constantly changing. Thus, it could be argued that an autopoetic system is about how we organise our schools, plan our lessons or how we learn or teach mathematics.

In the present study, the teacher and learners in the mathematics classroom have become an autopoetic system as they continuously interact to bring meaning and authentic learning forth. Hence, understanding a mathematics classroom as well as the groups that arise within it as a complex, self-organising and self-updating autopoetic unity or system (Davis and Simmt, 2003, Davis, 2006) might give me a chance to see how teachers and learners bring together their *being*, *identities*, *actions*, *knowledge*, *doing* and *knowing* (Davis et al., 1996).

2.6.2.2.2.2. The concept of embodied cognition

Enactivism emphasises the notion of embodied cognition. It foregrounds bodily action such as gesturing, nodding, imagining and guessing, explaining and expressions in the cognition process (Reid, 1996). The concept of embodied cognition assumes that it is impossible to separate the construction of knowledge from the act of being. We actively construct knowledge through our perceptual-motor capacities and our dynamic interactions with the environment, i.e. cognition is constructive. Varela et al. (1991) argue at length that for example "colour provides a paradigm of a cognitive domain that is neither pre-given nor represented but rather experiential and enacted". In particular, they maintain that our ability to see colours results from the active interplay of various sensor motor modalities. In other words, we construct knowledge through our unique embodiments, the capacities of which are, in turn, shaped through these interactions. Begg (2000) argues that cognition is constructive since it involves projecting bodily schemas and combining these schemas to create a metaphorical understanding of the world or mathematical concepts.

Mathematical knowledge thus has to be seen in terms of our evolving sensory-motor representations as opposed to restricting these representations to just a fixed set of perceptual capacities. “Embodied cognition theorists contend that thought results from an organism’s ability to act in its environment. More precisely, what this means is that as an organism learns to control its own movements and perform certain actions, it develops an understanding of its own basic perceptual and motor-based abilities, which serve as an essential first step toward acquiring more complex cognitive processes, such as language” (Internet Encyclopaedia of Philosophy, 2012). The work of developmental psychologists Thelen and Smith, who have used the principles of embodied cognition to understand how infants learn from their body movements, have been valuable to the field of education (Begg, 2000). Their empirical findings support the claim that “thought grows from action and that activity is the engine of change” (p. 31). In light of the above discussion, the important role of embodiment can therefore not be neglected from the child’s active learning process. Mathematically, this will allow me to consider not only the formal ideas that emerge from actions, but also to scrutinise those actions and how students are enabled to construct or discover mathematical ideas for themselves.

The notion of embodied cognition implies that people come to know about the world by interacting with it *bodily*, *experientially* and *cognitively* (Niessen, 2007). This means that human beings individually are “simultaneously biological and social beings that experientially embody both cognitive and physical dimensions within their actions” (p. 14). As continuous interactions are such an important feature in enactivism, one could claim that it holds an ontological embodiment, which implies that “the world which is given in perception is the concrete inter-subjectively constituted life-world of immediate experience” (Li et al., 2010, p. 6). Enactivists view *embodiment* as the “developing process of our interaction with the real world” (Li et al., 2010, p. 7). Thus, embodiment is not simply our beliefs, but also our ways of living and experiencing the world that involves our sensory and motor processes, perceptions and actions. Mennin (2010, p. 163) states that each person embodies a different history or experience that structurally enables him/her to sense and identify gaps in their understanding, or what could be described as “the need to know”. He further states that teachers and learners strive to reach the frontier of their understanding through individual and collective exploration, exchange and collaboration. In classroom practice, this means that the teacher and learners promote a “community of learners of mathematics” where the teacher learns from the learners and the learners learn from each other as well as from the teacher through unformulated exploration, unstructured interactions and undirected movement (Davis, 1996).

The concept of enactivist embodied cognition is viewed as a reflection in which the body and mind are brought together. It also refers to the process of active living systems or information processor as opposed to meaning producer (Li et al., 2010). For this reason, an enactivist perspective understands mathematical knowing (or cognition) as an embodied interactive process co-emergent with the environment in which the person acts. This means, mathematics understanding is observed as a doubly embodied on-going action in an environment (Kieren, 1996). Capra (1996), for example, highlights two fundamental double senses of embodied cognition: physiological and phenomenological. In terms of the former, cognition is seen as dependent upon the kinds of experience that come from having a body as physical with various sensory-motor capacities. This means embodiment regards the body not only as lived structure to experience, but also the setting for conscious actions or decision. In the latter, individual sensory-motor capacities are considered to be embedded in a more encompassing biological and cultural context. The "philosophy of phenomenology argues that phenomena are inseparable from the context in which they exist and from the person [that observes] them" (Niessen, 2007, p. 13). The notion of embodiment, in this study, refers to the person's structure that determines the actions which a person takes. As Kieren (1996, p. 1) explains "person-in embodiment" implies the occasion and space for actions as provided by the environment. Moreover, the notion of embodiment implies that our bodies are embedded in a wider context. That is to say, for the learner to continue existing as such in a given environment, he/she needs to act or reason mathematically in certain ways. This means, as a learner of mathematics, the learner has to demonstrate to the teacher that they embody mathematical understanding, and that he/she is able to do certain things and carry out certain actions before encouraged to learn new mathematical concepts.

My enactivist lens will allow me to see the teaching and learning of geometry as a complex process, which is the product of histories of interactions (Proulx, 2009) through which teachers' and learners' embodied structure, that is, their actions, thoughts and emotions are manifested and shaped. This will also help explain teachers' and learners' embodied understanding of mathematics, and existing instructional practices and techniques that have been effective or successful. Namely, teachers' embodied experiences and actions such as listening will enable understanding of their communication, dialogue and collective action and learners' interpretations or participations in the teaching and learning process.

Further, the enactivist embodiment approach appears to emphasise the notion of active learning and considers the learner as an active learner. The primary conditions for this active learning process are allowing for the activity or mathematical understanding to be grounded in the learner's embodiment as opposed to merely mental processes and allowing the learner to create something in the process of learning as opposed to mere interaction (Li et al., 2010; Proulx, 2009; Begg, 2000; Davis and Sumara, 1997; Maturana and Varela, 1992). Moreover, the embodiment principle should have personal meaning to the learner and invoke reflexivity of its own role in the knowledge construction process. The objects and components of the teachers' classroom instructional practice/activity should have a high ceiling to sustain the learner's enthusiasm in the learning process and the learner's sensory-motor representations should evolve as a result of his visual sense of the world.

2.6.2.2.2.3. The concept of co-emergence

A central element of enactivism and this study is the idea of co-emergence. Co-emergence is when something new such as mathematical ideas or thought co-emerges or comes into being as a result of the interaction of two or more people. That is, when people interact within, or on, a particular social structure, a new social practice may co-emerge, i.e., irreducible to the sum of its parts and having its own properties and powers (Begg, 2000). Thus, the co-emergence framework contends that the process of learning (or change) not only depends on the interactions between the environment or the teacher and the learners, but that the two are inseparable (Li et al., 2010). Co-emergence in complex adaptive systems such as learning is premised on the ideas that the systems represented by the learner and the context are inseparable (Fenwick, 2004). According to Proulx (2009), humans interconnect with the systems in which they act through a series of *structural couplings*. That is to say, when two systems [in this case the teacher and learners] coincide, the *perturbation* of one system stimulates or excites responses in the structural dynamics of the other (Proulx, 2009; Davis and Sumara, 1997; Maturana and Varela, 1992). This further implies that the "resultant coupling creates a new transcendent unity of action and identities that could not have been achieved independently by either participant" (Fenwick, 2004). Teaching in the mathematics classroom, for instance, can be described as "a collective activity in which interaction both enfolds and renders visible the teacher and students, the objects mediating their [mathematical] actions and dialogue, the problem space that they define together and the merging plan or solution they devise" (p. 247). Furthermore, the contribution that each participant makes changes his/her actions, the interactions and the emerging object of the focus, the other participant, and subject position within the collective activity. Varela et al. (1991) call this “mutual specification”, a fundamental dynamic of the system as it constantly engages in joint action and interaction. From this perspective, teaching or learning is understood in terms of co-emergence, as the learner and environment [teacher] emerge together in the process of cognition. This means, teachers and learners have embodied experiences which they bring to the learning and both learn from each other. Consequently, they both generate mathematical understanding and interpretations through their participation in the shared lesson activities as opposed to predetermined learning actions.

An example of co-emergence of the learner and the environment in the mathematics classroom could be the coincidence of two stimuli that excite the conveyance of a response from one learner to the other. This denotes actions that occasion modification of the learner’s structure. Hence, co-emergence disputes the notion of “cause and effect” since what a learner does in the mathematics classroom is “caused” both by his own dynamic structure and environment’s constraint (Reid, 1996). The co-emergence of the learner and teacher could imply the joint action and interaction they both engage in. This means what the leaner will know at the end of the lesson develops through his interaction with the teacher and everyone else in the classroom. Thus, the theoretical dimension of co-emergence seems to encourage the view beyond an individual learning to understand mathematics knowing or actions as constantly enacted via co-emergence.

2.6.2.2.2.4. The concept of structural determinism and coupling

Structural determinism/coupling is another key idea of an enactivist worldview, and is closely related to Maturana and Varela’s (1992) concept of ‘natural drift’ and Charles Darwin’s evolutionary perspective. Fundamental concepts of Maturana and Varela’s biological perspective of cognition, which have informed my ideas of learning in this study, are those related to structure and self-organisation. According to Lozano (2005), structure refers to components and relations that actually constitute a particular unit and makes its organisation real. Self-organising denotes existing relations among the components of a system belonging to a particular group or class (p. 25). Learning, in structural determinist views, is a self-organised adaptation to changing classroom or teaching circumstances. It is a process through which the organism structure changes while it continues existing in a given environment (Maturana, 1987). In the context of this study, learning or mathematical knowing occurs as students change their structure in complex processes of interaction with the teacher and the environment. Lozano (2005) notes that the teacher and students need to preserve their organisation while they continue existing as a complex adaptive system in the mathematic classroom. Thus, teachers should note that to teach or learn mathematics implies an ability to act in ways that can be considered mathematical. Similarly, they should understand teaching and "learning as dynamical [and self-organising processes] requiring continuous feedback from multiple diverse interactions". "Reflection and feedback promote [effective teaching] and learning of mathematics" (Mennin, 2010, p. 160).

In a structural determinist perspective, "learning is not seen as a causal event determined by the external stimuli, but rather arises from the learner's own structure as she/he interacts with the unfolding circumstances and the mathematics teacher within the classroom environment" (Proulx, 2008, p. 9). This, for me, seems to indicate that anything offered to the learner in the mathematics classroom is, at most, a trigger. In a mathematics classroom, these could be explanations, questions, clarifications or mathematical ideas/tasks that trigger learners' mathematical understanding/knowing as manifested through their actions and interactions with the teacher. In this study, whatever is done or taking place in the mathematics classroom is seen as part of the teachers' and learners' structures. Proulx (2009) makes a pertinent comment that the effect of what a teacher says or poses to the learner is determined by the learner's own knowledge (structure) rather than predefined by the teacher. Hence, whatever the teacher says or does verbally or nonverbally in the classroom is believed to influence the learning process, invoke learner's response and therefore play a pivotal role in the teaching and learning of geometry.

The study will investigate how teachers' geometry classroom instructional practices influence students' mathematical problem-solving procedures, and how these practices about problem-solving procedures influence students' conceptual understanding of geometry concepts.

Structural determinism and coupling as key ideas of an enactivist worldview assume learning as arising from the learner's own structure as she/he interacts with unfolding circumstances within the complex classroom environment (Proulx, 2008). Here, I introduce the notion of the structural determinism of a living learner to resonate with Capra's (2002) sentiment that "it is not the environment or entities in the environment that determines which perturbations an organism [learner] responds to, but the organism itself specifies those responses" (p. 2). The argument made here in light of the enactivist structural coupling or determinist perspective is that the teacher should interact with the students and allow them to work with peers or groups they have naturally formed through a sense of structural coupling. Drawing from such a notion of the readily occurring processes of structural coupling or determinism, enactivists appear to advocate the view of deliberately assigning students to pairs or groups in the classroom for discussions or conversations. Miranda (2004) notes: “allowing students to work in coupled groups, however might be necessary to, sometimes, allow ‘divorce’ between the already existing couples and to create room for many new and deliberate ‘couples’ to form” (p. 21). This may also result in more complex possibilities for mathematical ideas/discourses and hence provide me with a better perspective of the dynamics of how effective mathematics teachers take part and involve their students in an interactive teaching and learning process in a dynamically complex mathematics classroom. Specifically, the enactivist framework principle will enable me to better understand how teachers’ and students’ language and actions as visible manifestations of their embodied (enacted) mathematical understanding influence teaching and learning in the geometry classroom.

The enactivist theory has similarities with the Kilpatrick et al.’s (2001) framework, which highlights aspects of learning and teaching as related and complex, and considers teacher and students co-evolving as part of a complex holistic classroom environment. In such complex situations, the teacher influences what is learnt by interacting with students, while developing a history together of a mutual relationship through co-evolving and co-adaptation to each other (Proulx, 2009). Both Kilpatrick et al.’s (2001) framework and the enactivist perspective consider mathematical understanding to be embedded in a co-emerging relationship binding teacher and students together; and call for the teacher to be proficient and active in the learning process and teaching dynamic, where his/her actions trigger students’ thinking, actions and responses to mathematical concepts/tasks. The knowledge, beliefs, interactions and actions of both the teacher and learners affect what is taught and ultimately learned in the mathematics classroom. That is, the teacher’s attention and responses to the learners further shape the course of instruction.

2.6.2.3. Enactivism and complexity science in education

One of the important goals of this doctoral study is to look at the complexity of the interface between teachers’ mathematical teaching proficiency and classroom practice. Therefore I need to examine how enactivism and complexity theory could assist me in achieving this goal. The enactivist philosophical worldview can be said to co-emerge with the complexity perspective, in that they are both involve reconstructing long-established theories on teaching and learning, with a particular interest in mathematics education. The similarities of the enactivist approach and complexity science theory suggest that they add to an understanding of each other. Such theories, argues Johnson (2001), have largely contributed to the split that emerged in perspectives, and assist in moving academic discourses far away from the dichotomous challenge initiated by the views of Descartes and Newton such as *mind versus body, self versus other, knower versus knowledge, human versus nature, teacher versus students, classroom versus real world* and *school versus community of practice* (Begg, 2000). Though enactivism and complexity science (as contemporary theories of learning) are almost akin, they "differ in terms of their objects of study and the scopes of their assertions" (Davis and Simmt, 2003, p. 142). Hence, complexity theory has become more relevant to such deliberate social projects as Mathematics education or schooling. In particular, complexity theory has highlighted that, by attending to a particular matter, idea or action, a teacher can greatly increase the likelihood of complex transcendent possibilities within the mathematics classroom (Davis and Simmt, 2003). As a mathematics education researcher, I become interested in how enactivism drawing on complexity science theory informs the teaching practice and social learning in the mathematics classrooms.

Despite the broad range of phenomena that are captured under the umbrella of complexity, it appears that they all have several features in common with enactivism: in particular, *certain necessary but insufficient conditions* must be met in order for a system to arise and maintain "its fitness" within a dynamic context, that is, to learn (Davis and Simmt, 2003; Megan, 2008). As one of the main tenets of mathematics cognition, the notion of "survival of the fittest" based on Charles Darwin's idea of evolution assumes that "superior ideas will supersede inferior ones as organising agents seek to understand the world" (Sumara and Davis, 1997, p. 408). Therefore, in articulating the mathematics classroom as a complex learning environment, Davis and Simmt (2003) describe the five essential and interdependent, but inadequate, complexity conditions that must be met in order for [such] a system to arise and maintain its fitness within dynamic mathematics contexts, that is, to learn. The five conditions are: "(1) *internal diversity*, (2) *redundancy*, (3) *decentralised control*, (4) *organised randomness* and (5) *neighbour interactions*" (p. 147). Davis and Simmt (2003) describe "internal diversity" as unique qualities along with characteristics and experiences that define a person as an individual, including his/her communication style. They use "redundancy" to characterise an important difference between various approaches to the teaching of mathematics. "Decentralised control" refers to when the system itself decides what is and is not acceptable, and that appropriate actions can only be conditioned by external authorities rather than being imposed. “Organised randomness”, on the one hand, is a structural condition that helps to determine the balance between redundancy and diversity among agents. This implies that the structures that define complex systems [such as mathematics classrooms] maintain a delicate balance between sufficient organisation to orient agents’ actions and sufficient randomness to allow for flexible and varied responses (Davis and Simmt, 2003). Such situations neither imply that everyone does the same thing nor does their own thing, but a matter of everyone participating in a joint or collective coordinated task, project or action. According to Davis and Simmt (2003, p. 156), “neighbour interactions” are not the interactions among physical bodies or social groupings, but rather, the neighbours that must bump against one another are mathematical ideas, intuitions, hunches, feelings, queries and other manners of action presentation. From the complexity perspective, both the teacher and learners create together the conditions that will allow mathematical actions to be adequate. Learning outcomes cannot be predetermined or predicted, but the criteria for the adequateness of actions are, at least in part, specified by teachers and students. With this in mind, I am interested in exploring the way that effective mathematics teachers create contexts or possibilities in which certain mathematical actions related to the learning of different geometrical concepts can be fostered.

To a certain degree, “power and authority” from both the enactivist and complexity viewpoints should be distributed and shared by the teacher and learners as participants across the mathematics classroom. As Davis and Simmt (2003) advise, “[power] authority in the mathematics classroom does not only reside within the actions of deciding what and who is right or wrong but also with the notion of who creates the [mathematical] idea or knowledge that become part of the collective” (Miranda, 2004: 15). With this shift toward a pragmatic of transformation, I believe that both enactivism and complexity science have become not just a valuable means to interpret, but a source of practical advice to mathematics teachers, educators and mathematicians.

2.6.2.4. Reflection on the power and potential use of some enactivist theoretical concepts in my data analysis

In earlier sections, I outlined and defined my conceptual and theoretical frameworks and indicated that they are structurally similar, both in enhancing teacher competence in the dynamic processes of mathematics teaching and learning. These are both the subject of this study and a synthesising tool that can be used for construing Kilpatrick et al.'s (2001) model and enactivist theory as similar layers of a unified structure. I view the five strands of Kilpatrick et al.'s (2001) teaching proficiency model and the four enactivist theoretical dimensions as similar. The interconnectedness of key concepts of the Kilpatrick et al.'s (2001) model and enactivist theory provides the context within which to explore, analyse and elicit participating teachers' perspectives of effective teaching, and how these concepts shape their instructional practice. The accounts and suggestions that I give at this juncture are clear illustrations of my voice and enactivist (mathematical) thinking and are, essentially, the truth in terms of the power of how or what I am looking for in the classroom teaching practice of selected effective mathematics teachers in Namibia. What follows are three examples of how I will apply the enactivist theoretical concepts of structural determinist, autopoesis and co-emergence perspectives to look at my collected data.

Within the concept of structural determinism and structural coupling, learning or the learner's mathematical actions is not seen as caused or determined by the teacher. Rather, learning or actions arise from the learner himself as he interacts with his own thoughts, other peers or the mathematics teacher over given mathematical tasks, or offers his ideas or explanation as a result of his mathematical conceptual sense-making. Thus, from an enactivist viewpoint, learning or understanding is dependent on, but not determined by, the teacher, peers or teaching. In terms of co-emergence and adaptive reasoning, for example, I want to look into how mathematics classroom discourses emerge and evolve; paying attention to any interactive features that might arise among the students or between the students and the teacher, and how emerged mathematical ideas or actions lead to learning possibilities for both the teacher and learners. I will also see how mathematics teachers occasion or trigger learners to think mathematically and explain their mathematical thinking or ideas while interacting and acting with the peers and the teachers as they, at the same time, construct their own mathematical understanding (Miranda, 2009). How the teacher follows up with exercises for whole-class or pair work discussion after making some explanatory notes or examples is also of interest. The notion of co-emergence in this instance implies that the teacher's (learner's) actions lead to learners' (teacher's) reactions and new actions and all these together open up the possibility for further shared mathematical actions.

To bring in the notion of autopoesis, I would like to see how mathematics teachers assign or allocate students to group work, create opportunities for, listen to and extend learners, do or solve mathematical exercises or problems together with students, and how they stimulate students to offer mathematical ideas or take part in lesson activities on their own. Similarly, I will observe how teachers try to involve all students to brainstorm their embodied mathematical understanding and pick up on students' mathematical actions and ideas that emerge or co-emerge during the discussion, conversation or argument. These given examples are reflections of the meaning of autopoesis, co-emergence and structural determinism. These concepts have the potential to be used in my analysis of recorded lesson videos.

2.7. CONCLUDING COMMENTS

The purpose of this chapter was to establish the conceptual and theoretical framework that underpins the study. Over the course of the chapter, therefore, I discussed common perspectives and characteristics of effective teaching of mathematics, the concept of geometry as study topic and two related theoretical frameworks of the Kilpatrick et al.'s (2001) model and enactivist theory of embodied cognition that inform this study. I have also shown the potential relationships of these elements to the findings of the study. I now turn to the methodology I employed to gather and analyse my data.

CHAPTER THREE

RESEARCH DESIGN AND METHODOLOGY

"There are in fact a number of reasons for selecting a methodological approach, but one's decision often expresses values about what the world is like, how one ought to understand it and what the most important threats to that understanding are" (Firestone, as cited in Niessen, 2007, p. 22).

3.1. INTRODUCTION

Educational research is a process of gaining a better perspective of the complexities of human experience (Merriam, 2009). My goal in undertaking this case study was to analyse selected effective secondary school teachers' geometry teaching in Namibia, in order to obtain an in-depth perspective on their classroom instructional practices or activities and understand their teaching proficiency characteristics, as defined by Kilpatrick et al. (2001). My particular focus was on teachers' mathematical teaching proficiency (MTP), classroom pedagogy and teacher mathematical interactions with students. Yin (2009) and Creswell (2007) regard educational research to be essential as it provides legitimate information, knowledge and principles in an area that is little understood to guide the decision-making, thinking and discussion process in education. The study was sparked by both local and international discourses about effective mathematics teaching. The preceding chapter concluded that, to understand effective mathematics teachers' classroom practices in Namibian schools and their teaching proficiency characteristics, an understanding of the relationship between the two is needed. It is therefore important to clarify this relationship as well as to increase the understanding of how selected effective teachers construct and unfold mathematical proficiency (MP) in their teaching practices. The study examines this relationship by exploring teacher MTP mainly through the use of videotapes to assess the depth of their classroom teaching practices.

The chapter is split into three parts. In the first part, the research orientation including the epistemology, ontological perspectives and methodology used in my research are examined. The second part of the chapter gives an account of the research methods and design as well as the analytical devices developed for the case studies based on the Kilpatrick et al. (2001) conceptual framework. It also describes the research approach followed in qualitative case study research selected for this investigation. The final part of the chapter articulates the sample and sampling techniques along with participant selection criteria, after which instrumentation, data collection and data analysis process and pertinent strategies pertaining to ethics, validity and trustworthiness, used to enhance the quality of the research, are described.

3.2. ORIENTATION TO THE RESEARCH

The philosophical assumptions underlying this research come from the interpretive tradition, and the study employs qualitative methods of data collection and analysis. According to Cohen, Manion and Morrison (2007), the interpretive research approach is characterised by a concern for the individual, in this case, effective mathematics teachers, to understand the subjective world of their lived experiences as the basis of their mathematical actions, interactions and activities (Reid, 1996). Interpretive approaches focus on action (Thom, 2000). According to the interpretive advocates, human action is inseparable from meaning, and experiences are classified and ordered through interpretive frames (Cohen et al., 2007). The chosen paradigm affords me the opportunity to interpret and comprehend participants' experiences within a set context…a context within which "researcher and research environment are seen to co-emerge" (Reid, 2002, p. 3). From the enactivist viewpoint, the teachers and learners are an integral part of the context as opposed to being placed in it, and so the context inherently changes as learning occurs, teachers teach and learners learn and vice versa (Lozano, 2005). Advocates of interpretive approaches note that an "interpretive research approach is characterised by a particular *ontology*, *epistemology* and *methodology*" (Lincoln and Guba, 2003; Yin, 2003; Jenkins, 2007; Andrade, 2009). As Jenkins (2007) argues:

> …A researcher working in this tradition assumes that [participants'] subjective experiences are real and should be taken seriously (ontology)…this means, a researcher can understand participants' experiences by interacting with them and listening to what participants are telling him (epistemology)…and that qualitative research techniques are best suited to this task (methodology)… (p. 36)

This assertion implies a subjective epistemology and the ontological belief that reality is socially constructed (Creswell, 2007). Begg (2002) captures the essence of interpretive qualitative research in descriptions involving "reality in interaction, the researcher and the researched as two equally important elements of the same situation, and studying reality from the inside not the outside" (p. 51). In my study, the observation of the interactions between the teachers and students in the mathematics classrooms and between these participating teachers is paramount. It is through these interactions that the reality of participants' lived experiences and the processes of their teaching practice or teaching proficiency are discovered. Important too are the interactions between the participating teachers and myself to reflect on their teaching practices. The challenge in my research was thus to capture the complexity of the participants' interactions, their lived experiences and teaching practices, reflecting upon these practices and suggesting new ideas that would expand educational boundaries. I thus assume that the epistemological relationship between my participants as co-researchers and I was an interactive, reciprocal process (Thom, 2000). I interacted and worked together with the participating teachers to create a mutual dialogue and discourse on effective teaching practices throughout the research process.

I chose enactivism as my theoretical epistemology. Enactivist perspective, by its very nature, values learning as occurring in all domains of human existence (Begg, 2002). Specially, it encourages researchers to "reflect on the teaching practices to understand the purpose of all actions and develop appropriate approaches to learning that value the full human development of proficient mathematical students holistically, as learning is holistic" (p. 51). Enactivism was applied to this study because teachers have reasons and principles for their actions, which are not necessarily evident in observed lessons, and may not align with my own. From this viewpoint, teachers come with the view of mathematics teaching and how to teach it effectively, which they put into practice. In the following sections, I present several aspects of case studies as a research method. These include the study design and implementation and how their robustness was achieved.

3.3. RESEARCH METHODOLOGY

The need to investigate and analyse geometric classroom instructional practices "within its real-life context" (Yin, 1984, p. 23) leads me to adopt a qualitative case study methodology in order to gain "intensive, holistic description and analysis" (*Ibid*) of teachers' teaching practice, and how they go about teaching for MP in their mathematics classrooms. Case study research methodology is defined as "an empirical inquiry that investigates a contemporary phenomenon within its real-life context, when the boundaries between phenomenon and context are not clearly evident, and in which multiple sources of evidence are used" (Yin, 1984, p. 23). The research strategy adopted was to conduct multiple case studies in secondary schools. I focused on a total of five effective mathematics teachers from five Namibian high schools in five different education regions. Multiple-case design with real-life events shows numerous sources of evidence through replication rather than sampling logic (Yin, 2009). In fact, as Yin argue, "multiple case studies are the preferred research strategy when *how* or *why* questions are being posed, when the investigator has little control over the events, and when the focus is on a contemporary phenomenon within its real-life context" (p. 13). One important characteristic of (multiple) case studies is its power to allow the researcher to use or choose multiple sources of evidence and different data collection techniques and tools (Yin, 2009). In this study, multiple case studies typically enabled me to focus on a "single unit within which there may be several examples, events or situations that may be exemplified by numerous case studies" (Merriam, 1988, p. 46). Stake (1994) argues that "case studies are not a methodological choice, but a choice of object to be studies" (p. 236). An important point to make here is that the main thrust or idea of this research project was not to judge or evaluate teacher compliance with the curriculum but to learn from the participating teachers how they perceive and enact effective teaching practices. These are teachers from effective or high performing secondary schools.

I adopted a qualitative research approach for this particular study because it was appropriate for exploring the teaching practice of effective mathematics teachers. Such an approach enabled me to collect rich, holistic information from participants' perspectives in a naturalistic classroom environment using multiple and appropriate methods that maximised information adequacy, efficacy and ethical considerations (Merriam, 1998; Thom, 2000). A qualitative research strategy was appropriate here as it offered a means of investigating complex collective interactions consisting of multiple events of potential importance (Merriam, 1998) in understanding how selected effective teachers teach geometry. A key feature of qualitative and interpretive case study methods is that the researcher is directly involved in the process of data collection and analysis (Cohen et al., 2007). Being the primary human instrument mediating the collection of data and data analysis, I adopted a qualitative method strategy in order to understand teachers' instructional practices in relation to their teaching proficiency using the Kilpatrick et al.'s (2001) model of teaching for mathematical proficiency. I also found qualitative research methodology particularly valuable from an enactivist perspective as it allowed me to understand complex interactions, tacit practices and often hidden beliefs and values. Within the context of this qualitative study, teaching and learning in a traditional mathematics classroom environment are interdependent, and researchers strive to explore effective teaching practices in all its complexity in a variety of settings (Davis and Sumara, 1997). This understanding is particularly germane for the process of this investigation in order to gain a deeper insight into the embodied knowing, processes and skills of effective teachers, and the manner in which these teachers' lived experiences are manifested in a wide variety of classroom environments. This also enabled me to understand the aspects of classroom practices that proved effective for the teachers when generating or motivating discussion, explaining and/or asking questions around certain mathematical (geometric) concepts and ideas.

Specifically, the study adopted a number of characteristics typical of qualitative research as expanded on below (Yin, 2009; Creswell, 2007; Cohen et al., 2007; Pillay, 2006).

- *Participant perspectives*: the focus on the participants' perspectives of effective teaching was emphasised. Hence, this study not only presents the several meanings individual mathematics teachers attach to their experiences but also investigates the classroom learning processes that formed these meanings.
- *Naturalistic settings*: by studying people, events or classroom practice in the complex mathematics classroom settings, the study aimed to explore the particular teaching practice context and its impact on learning as understood by the participants in their own environment.
- *Multiple data sources*: in order to enhance the validity of my research, I collected data from different sources for triangulation purpose. Triangulating data means comparing and cross-checking the consistency of information derived at different times by different means with qualitative methods to examine the topic of interest in depth.
- *Researcher as key instrument*: as the main research data collection tool, I personally gathered and transcribed primarily lesson video recordings and interview data. This process enabled me to become immersed in the collected data and develop deeper insights for analysis.
- *Richness of data and description*: The findings of the study were delivered through rich descriptions of the context, actors and events. This account was intended to unpack and to understand the complexity of actual teaching practice in the classroom and, thus, to assist the reader in experiencing the classroom events vicariously.
- *Interpretive*: All the features mentioned thus far are inextricably bound up with the interpretive feature of this type of inquiry. That is, not only the participants and the researcher but also the readers make interpretations of interpretations based on their own understanding of the issues explored in this study.

These aspects of qualitative research matched the requirements of my methodology. It should be noted that my personal assumptions about educational research, experience, knowing and identity inevitably have played a role in selecting a qualitative methodology for this research.

Enactivism is the epistemology I have chosen as I wish to explore and discover meaning (Merriam, 2009) as it is created by the teachers in this case study.

3.3.1. The unit of analysis

The issue being investigated in these educational, interpretive, and qualitative multiple case studies was how effective teachers' teaching proficiency, mathematical knowing and actions shaped their classroom instructional practices. Multiple case studies typically use cases that are similar in some ways. The cases become members of a group or examples of a phenomenon (Stake, 2006, p. 6). The *unit of analysis* being examined was selected effective mathematics teacher teaching practices and their experiences. Thus a case study approach was particularly suited to this study because it allowed me to "explore one specific case from multiple sources of data provided information about the phenomena of interest and other related factors in the context, thereby providing detailed quality data for analysis" (Yin, 2009, p. 13). As I compared the teaching practice across the case study teachers (Stake, 2000), I sought both what was common and what was different or significant about my case studies. This provided an opportunity for triangulation relying on the comparative multiple case logic of replication and extension of theoretical perspective (Merriam, 2009).

3.3.2. Sample and sampling techniques

> "The quality of a piece of research not only stands or falls by the appropriateness of methodology and instrumentation but also by the suitability of the sampling strategy that has been adopted" (Pillay, 2006, p. 50).

This extract suggests that one of the key elements of an effective case study research method is the adoption of a suitable sampling strategy that ensures the selection of appropriate participants to generate rich data cases. For that reason, care was taken when doing sampling for this study. As Cohen et al. (2007) point out, sampling in qualitative research is often deliberate, usually bearing specific criteria in mind and choosing specific cases which are best suited to the needs of the researcher. Merriam (1988, p. 61) emphasises that "purposeful sampling is based on the assumption that the researcher wants to discover, understand and gain insights into and therefore must select a sample from which the most can be learned". Hence, in this study, I employed a *purposive or purposeful sampling* method/technique (Creswell, 2007) to identify appropriate teachers from different schools in different education regions. This enabled me to generate meaningful and relevant data to support my findings.

The study was conducted with five purposively selected teachers from five different schools. The National Institute of Educational Department (NIED) staff in Okahandja, Ministry of Education (MoE) officials at the Head office in Windhoek and staff in the Faculty of Education (FoE) at the University of Namibia (UNAM) as well as Regional Education Mathematics Advisory teachers were approached for assistance in identifying possible "effective mathematics teachers" to take part in this study. Although I only needed five teachers, I requested the MoE and UNAM officials to identify 10 teachers, from whom I selected five.

Sampling was carried out in two stages:

- In the **first stage**, I approached and engaged the Ministry of Education personnel to select 10 teachers in Namibia who have consistently achieved the top results in the Grade 10 and 12 national examinations over the last three years. Archived statistical information (examination results) on Junior Secondary Certificate (JSC/grade 10) and Namibian Senior Secondary Certificate (NSSC/grade 12) students' performance in mathematics over a period of three years was used in selecting 10 teachers/schools.
- The **second stage** involved contacting these 10 identified teachers and inviting five volunteers to participate in my study. The study objective was clearly articulated and I emphasised that the ultimate objective of this study was not to evaluate their teaching and compliance with the curriculum but to analyse factors that contribute to their effective teaching practice. After giving out copies of the information sheet to the teachers (see Appendix C), they were asked to decide whether or not they would like to be a part of the study.

Certain criteria had to be met in selecting the final participating teachers from the list of contacted teachers. The prime selection criteria used were:

- voluntary participation and willingness to share teaching practices and experiences;
- mathematically qualified secondary school (Grade 8 to 12) mathematics teachers, namely, those who have a minimum Basic Teacher Education Diploma (BETD) teaching qualification;
- high performing teachers in terms of learners' mathematics performance who have a high standing and 'good reputation' in the mathematics education community of practice as determined by their significant peers, and

- the participating teacher and school had to be committed to the project for the entire duration of the project.

The teachers come from high schools in five different educational regions of Namibia. There are 13 educational regions in Namibia. The participants showed a high level of interest and enthusiasm.

3.3.2.1. Participants

My participants were two males and three females. Even though gender was not a central factor in this research, it is important to point out that the dominance of female mathematics teachers in my sample of participating teachers was neither purposive nor deliberate, and is not representative of mathematics teachers in Namibia. Table 3.1 below summarises information about the participating teachers. For the purpose of this study, the participants and schools have been categorised or coded as follows. Teacher 1 is referred to as Demis of school A, Teacher 2 as Jisa of school B, Teacher 3 as Ndara of school C, Teacher 4 as Emmis of school D and Teacher 5 as Sann of school E. I used pseudonyms in order to protect the participants in terms of privacy, confidentiality and anonymity. However, as Lankshear and Knobel (2004) suggest "assuring confidentiality and anonymity is actually quite difficult to put into practice as some schools are readily identifiable because they are easily recognised in the region" as they performed well (p. 110).

Table 3.1: School type and teachers' demographic information in my sample

School	School type	Teacher and Name	Sex	Age	Levels of study	Teaching experience in years
A	Public	T1: Demis	Female	> 40	Gr.12, BSc, HED	20
B	Private	T2: Jisa	Female	30-40	Gr.12,B.Sc, MSc, BEd	10
C	Public	T3: Ndara	Male	30-40	Gr. 12, B. Science	6
D	Private/Day	T4: Emmis	Male	> 40	Gr. 12, BEd, MEd	15
E	Private/Day	T5: Sann	Female	30-40	Gr. 12, HED	8

As the age distribution in the table above shows, two of the five teachers were forty years old or older, and three of them were over thirty years old. The table further shows that four (80%) of the teachers in the sample had more than six years teaching experience. My sampled schools are spread across five regions in Namibia, namely, two (B and C) in the northern regions, two (A and D) in central regions and one (E) in the coastal regions.

3.4. QUALITATIVE DATA COLLECTION PROCESS

Mills (2003) argue that "qualitative research uses narrative and descriptive approaches for data collection to understand the way things are and what they mean from the perspective of the research respondents" (p. 4). She further points out that researchers require data collection instruments that are sensitive to underlying meaning when gathering and interpreting them. What follows is a description of tools that I developed for generating data and analysing effective teaching in the Namibian context.

3.4.1. Developing analytical tools for analysing effective teaching practices

Initially I focused on designing six phases of data generation and analysis. For each of the phases, instruments were carefully designed, face-validated by experienced researchers, and some piloted with Rhodes University students. Spurred by new ideas about how to analyse or measure the mathematical quality of teachers' classroom practice or knowledge evident in practice (Ball et al., 2004), these instruments were designed to measure both the MTP features of instructions and the ways in which mathematical concepts are conveyed to students. The instruments further measure whether the geometric concepts or content presented to students is mathematically rigorous with respect to elements of the Kilpatrick et al.'s (2001) model of MTP operationalised during instructions.

I then devoted some time to the piloting and validation of the developed instruments. Guided by Kilpatrick et al.'s (2001) teaching proficiency model and the enactivist foundations of the study, I developed an analytical rubric comprising descriptions of MP and the teaching techniques that I considered might occur or manifest by participating teachers. These teaching proficiency descriptions, in the form of a classroom observation checklist, provided the lens for gathering and analysing the data from teacher-student interactions occurring in the lesson videos and interviews with teachers in the fifth and sixth phases of the study. Collected data, from each teacher that participated, focused on the emerged interactions between him/her and the students and the classroom environment/situation at large. However, I did not have the chance to interview any of the teachers during the piloting process, so my first experience of interviewing arose whilst in the process of collecting the actual data for this study. I felt that I was sufficiently prepared theoretically and heeded the numerous cautions about the difficulties of interviewing (as initially discussed with the project supervisor). However, I pretested and piloted the geometry scenario-based questionnaire with several South African secondary school mathematics teachers and some Rhodes University students studying towards Masters of Education degrees (MEd) before finalising the one used in this study. A copy of the final questionnaire can be found in Appendix H.

3.4.2. Phases of data generation and analysis strategies

The data analysis was not a separate phase from data collection which can be marked out at some singular time during the inquiry (Cohen et al., 2007). That is, the data analysis was embedded in (commenced during) data collection. The field work and data collection period spanned a period of eight months, commencing in April 2011 and ending in November that year. A wide variety of data collection techniques (or tools) were utilised in order to make sure this multiple case study research was sound (see Table 3.5 on pages 101 to 102 for more detail). I collected five interconnected types of data from each participant. These included biographical information, the paper-and-pen geometry scenario-based questionnaire, video-recordings of classroom practices and interviews in the form of a post-lesson reflective and stimulated recall analysis session. The data collecting and analysis process was spread over six phases. Below is a detailed description of each phase:

3.4.3. Phase I: Planning and negotiating access for actual data collection

In the first phase, I negotiated access to the research site with the MoE head office. I first sought the permission of the MoE Permanent Secretary (PS), who allowed me to meet with the Regional Directors of Education, who authorized me to carry out research in their regions. After securing official approval (Appendix B), I personally approached school principals and the selected mathematics teachers and, arranged meetings with them where my research purpose was clearly articulated, verbally and in writing. Permission was also obtained in collaboration with other necessary persons in authority, including Circuit Inspectors of Education.

Although I had PS approval and was committed to a democratic and collaborative research process, I was anxious that telling potential participants I wanted to find out more about their mathematical content knowledge and teaching proficiency, and how these influence their classroom practices, might be threatening to the participants. Thus, during my familiarisation visits to schools, I tried not to be naive or patronising, and carefully explained my interest and intention. I tried to inspire teachers to participate in this study and constantly assured them that the objective of the study was not to evaluate them, and that I wished to learn from them and from their practices about what made them *effective* or *successful* in teaching mathematics. Furthermore, I indicated that the study would be beneficial to them in terms of their reflections, building a community of practice, learning from each other and feeling affirmed. Moreover, I stressed that they would have access not only to their own filmed lessons but also to other teachers' lessons as well as any papers written for conferences and the final thesis. Suitable dates and time schedules for actual classroom observations and interviews were negotiated with each participating teacher. After explaining all of this and giving them copies of the information sheet for participation and a detailed timeline summary of activities, the five teachers completed the consent form and agreed to participate fully in the study. Overall, the purpose of phase one was to identify and ascertain research settings, create a rapport and good working relationship with teachers involved, and negotiate access to their classrooms.

3.4.4. Phase II: Administration of biographical questionnaire: School/teacher profile

After gaining consent from all the teachers and schools to proceed with the research, I conducted a contextual analysis of their schools and their classroom resources. The purpose of the contextual profile was to understand my five participants' working environment and their relationships to this environment.

During this phase, I used an **initial profile questionnaire** to gain insight into each individual teacher's personal details pertaining to their mathematical background, qualifications and teaching experience. Each teacher was asked to complete the profile questionnaire which consisted of four separate sections.

- The first section of the questionnaire dealt with the teachers' personal biographical information, including age, teaching qualifications and years of teaching experience within the secondary school phase, including specific grades.
- The second section of the questionnaire asked teachers about the total number of learners at the school as well as in their classrooms.
- The third section questioned teachers about how they would describe themselves in terms of their teaching reputation, their mathematical content knowledge and their passion for mathematics that they bring to the teaching and learning process in the classroom.
- The fourth and last section addressed teachers' personal perceptions about their school's motto or mission, resource contexts and support, learner admission criteria as well as typical features that make each of their schools a beacon of excellence or

significantly different from other schools both regionally and nationally. (See Appendix E for the questionnaire).

3.4.5. Phase III: Administration of the geometry scenario-based questionnaire

This phase aimed to, specifically, measure the participating teachers' personal mathematical knowledge and their knowledge of how to teach for MP. This was done to explore and establish the first four elements of MP for teaching, namely, the CU, PF, SC and AR in geometry of the participating teachers. These proficiency strands are understanding, fluency, and problem solving and reasoning, that is, knowledge of the mathematical content. They describe how mathematical content is explored and developed and provide the language to the developmental aspects of the learning of mathematics. A further intention was to assess teachers' teaching self-efficacy towards geometry. Research has shown that teacher CU, PF (Kilpatrick et al., 2001) and self-efficacy (Matoti, 2011) are important dynamic constructs which influence a teacher's ability to teach effectively.

This instrument consisted of items sourced from the Mathematics Teachers Enrichment Programme (MTEP) under the First Rand Foundation (FRF) Mathematics Chair at Rhodes University, and some previous studies on mathematical reasoning (Proulx, 2008) and teachers' task in relation to teachers' knowledge (Sanni, 2009). Some questions were taken from mathematics textbooks. These items were modified slightly, where necessary, to suit the context of the teachers in the study. The questionnaire contained twelve scenario-based questions in geometry. These items included two on trigonometry (bearings and trigonometry ratios), two on geometrical constructions, one on angle properties (locus), two on graphs and Cartesian coordinates and five on perimeters, areas and volumes (geometrical terms and relationships). In line with the grade 8 to 12 mathematics curriculum syllabus content, the scenarios which were used to explore teachers' content knowledge in the context of their classroom teaching practice, interweave conceptual understanding and procedural fluency in geometry.

The scenario-based questions (see Appendix H) were aimed at drawing out specific themes related to teachers' geometric proficiency, yet allowing for a variety of solving skills. Teachers' responses to the scenarios enabled me to analyse and discern the teachers' mathematical proficiency of geometric concepts in relation to their CU and PF, but also revealed their different solving methods, thinking, reasoning and mathematical skills. Cohen et al. (2007) advises that "test items that have multiple methods to answer allow for exploration of patterns and trends [which help to describe what is happening in the teaching and learning context] and provide a measure of respondents' perceptions, feelings and responses about issues of particular concern to the evaluator" (p. 146).

The questionnaire was administered at the teachers' respective schools. The task which formed the focus of this questionnaire required participating teachers to solve mathematical problems by analysing scenarios involving key geometric concepts frequently encountered in the teaching and learning of school geometry. The questionnaire items were made up of structured questions (Stake, 1995) specifically aimed at establishing a sense of teachers' geometrical expertise in their teaching of mathematics. Some of the questions had multiple parts. I used a group of written open-ended questions in order to obtain qualitative responses from my teachers. I hoped that, in completing this task, teachers would reveal their own preferences for approaches to geometry teaching and learning, and some ideas relating to their mathematical proficiency or daily classroom teaching practices. As the questionnaire was dominated by open-ended questions, it was made clear that the emphasis was not on right or wrong answers, but on the solving procedures and processes or approach to untangling each of the scenarios. I also stressed to them that approaches to issues such as those presented in the scenarios are relative to individual teachers and their classroom peculiarities.

3.4.6. Phase IV: Classroom observation (video recording)

The primary data source for this study was the classroom observation of teaching (video recording). Thus, in this phase, I report on the classroom observation as well as the analytical tool I developed to identify and analyse mathematical teaching proficiency features of the classroom work of effective teachers. Observation of lessons provides direct access to social interaction and phenomena (Cohen et al., 2007). It offers a permanent account of a transient situation, and it greatly enriched my research data by supplementing the data gathered from the questionnaires and interviews. Classroom observations focused on the teachers' actual instructional practices during lessons involving geometry. Cohen et al. (2007) note that collecting observational data:

> Enables us (researchers) to understand the context of programmes, to be open-ended inductive, to see things that might otherwise be unconsciously missed, to discover things that participants might not freely talk about in interview situations, to move beyond perception-based data (e.g. opinions, etc.), and to access personal knowledge. (p. 369)

In this study, the emphasis was to seek evidence on how geometry is taught in these Namibian high schools. The analysis of lesson videos is discussed under phase VI stage II.

Each selected teacher was observed teaching three geometry sessions or lessons in each school in order to ensure that they did not just prepare in response to my call (see Table 3.3). This enabled me to investigate similarities and differences in classroom interactions among those teachers. With these data, I was also able to compare mathematics teaching to see if there were any pedagogical differences or similarities between these teachers when teaching geometry. Rather than only focusing on the teacher throughout the classrooms, my focus was also on the students' responses and ideas that emerged during the lessons. I spent on average two weeks in each classroom of the five teachers. These two weeks occurred over a period of eight months. On average, each lesson lasted 40 minutes. Most lessons (8 out of 15) were observed in the morning since most schools teach mathematics lessons before midday. In total, fifteen (15) instructional sessions/lessons were thus video recorded and analysed in order to see how effective teachers are teaching vis-a-vis the Kilpatrick et al.'s (2001) analytical framework. Table 3.2 below shows the number of lessons observed by school/teacher and grade level.

Table 3.2: Number of lessons observed by school (teacher) and grade level

School (Teacher)	**Grade**				
	8	**9**	**10**	**11**	**12**
A (Teacher 1)			XX	X	
B (Teacher 2)			XXX		
C (Teacher 3)				XXX	
D (Teacher 4)			XXX		
E (Teacher 5)			XXX		

Specifically, this phase involved non-participant observation. A non-participant observer, according to Somekh and Lewis (2005), assumes a neutral, unassuming position while observing subjects as they engage in their natural everyday activities. To ensure that my role of non-participant observer was not compromised, I remained detached from the participants. Detailed field notes were taken, as this is another important means of recording observation.

3.4.6.1. Videotaping - Filming geometry lessons in the classrooms

During a lesson the thread of instructional practice is hard to follow, because it differs for each participating teacher, and without a video recorded experience leaves no trace (Lewis, 2007). Observation does not always reveal the most crucial acts in the process of teaching as they are quite complex to discern. As I was interested in the totality of effective teachers' classroom practice, I collected *records of practice* by means of digital recordings. The video camera focused on the interaction between the teachers and students and how the teachers communicated with students to promote mathematical proficiency. It also focused on the kinds of tasks that were given to students or teachers and students worked on together.

"Video recordings, as a visual source, capture actions, physical posturing and gestures" (Opie, 2004, p. 15) from which effective practice can be identified and discussed. Hence, in this study, video recording was a particularly rich data source for the following reasons:

- video recording captures the lesson content and almost all events that take place in the classroom, which includes visual and verbal content, and chances of distorted information are limited (Opie, 2004).
- video data is more versatile than any other form of data because it opens an opportunity for multiple views to make discoveries, and it can also be kept as a valuable resource for teaching of mathematics, provided ethical compliance processes are followed (Opie, 2004).
- it provides relatively unprocessed data as opposed to field notes, for example, that first pass through one observer's interpretive frame. It is contextually rich and can capture social complexity in a way that other sources may not (Pillay, 2006).
- it also provides a rich source of data about what is going on in the classroom action during analysis at a later time, and conveys relational qualities that are harder to see in written artefacts (Lewis, 2007).

The video recordings were useful in providing case materials for each participating teacher during the stimulated recall analysis sessions.

3.4.6.2. Kilpatrick classroom observation coding instrument

Table 3.3 on page 86 shows the coding instrument used to analyse the lesson video transcriptions. I generated a set of observable indicators (codes) that represent a realistic reconstruction of the five strands of MTP. Each strand is recognised by phrases indicating observable indicators that describe or show how each strand was exemplified. Merriam (1998) advises that "codes or categories should reflect the research purpose … in effect; codes are the answers to the research questions" (p. 183). Hence, the process of searching for patterns in this study lead to an in-depth analysis of lesson video transcriptions to understand the complex link between participating teachers' teaching practice and teaching proficiency in teaching geometry.

Table 3.3: Summary of the analytical framework for analysing lessons

The five mathematical teaching proficiency strands of adapted Kilpatrick et al.'s (2001) model	Code	As defined by Kilpatrick et al. (2001, p. 380)	Observable indicators of the FIVE strands of mathematical teaching proficiency The Teacher:
1.Conceptual Understanding	**CU**	Conceptual understanding can be thought of as part of the "Knowing why" of mathematical knowledge. Understanding of core knowledge or teaching that encourages comprehension of mathematical concepts, operations and relations as required in the practice of teaching. Competency in this strand is defined in terms of "being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes".	**CU1**: uses mathematically appropriate and comprehensible definitions and language **CU2**: provides accurate explanations of concepts that are useful to learners (e.g. teacher explains ideas or procedure to deepen the learners' conceptual understanding or elaborate on geometric concepts under considerations or related concepts i.e. understanding of locus, angles, theorem, quadrilateral properties through problem solving) **CU3**: emphasises the links or connections between different geometric concepts, ideas, such as the interrelationships of properties of shapes **CU4**: makes reference to interesting contexts, giving/using examples from learners' contexts that they can relate to. **CU5**: makes links to learner's prior knowledge. **CU6**: makes conceptual links to other areas of mathematics such as algebra **CU7**: engages learners in tasks that are conceptually rich **Non examples**: Giving explanation of a concept in a form of telling and does not make conceptual links (showing a lower or insufficient conceptual understanding with some errors), and asking questions that require learners to guess what teachers is asking.
2.Procedural Fluency	**PF**	Procedural fluency can be thought of as part of the "Knowing when and how" of mathematical knowledge, i.e. quick recall and accurate execution of procedures and "fluency in carrying out basic instructional routines". Teaching that encourage knowledge or skills of rules, algorithms, or procedures used to solve mathematical problems or tasks flexibly, accurately, efficiently and appropriately in different ways and contexts (knowledge of step-by-step instructions that prescribe how to complete a task). Knowledge of when and how to use procedures appropriately and skills in performing them efficiently and accurately.	**PF1**: asks learners questions that solicit the procedures for solving a problem or the next step in the process of solving mathematical tasks or problems under discussion. **PF2**: asks learners to explain and justify their answers or methods for solving problems. **PF3**: explains procedures and provides algorithms to solve mathematical problems at stake (i.e. how a procedure should be used) **PF4**: elaborates on solving procedures suggested by learners. **PF5**: encourages learners to use mathematical formulae, procedures and techniques accurately **PF6**: encourages learners to use mathematical formulae, procedures and techniques correctly **PF7**: encourages learners to use mathematical formulae, procedures and techniques appropriately

			PF8: encourages multiple procedures in solving problems to develop learners' procedural fluency and understanding of geometric concepts **Non examples**: Performing strict procedures or algorithmic type of problem that have no connections to the underlying geometric concepts or ideas and lacking justification or explanations (call on learners to recall procedures, fact or definition).
3.Strategic Competence	**SC**	Strategic competence can be thought of as part of the "Knowing 'what' to teach, 'when' and 'how' to teach it". Teacher's ability in planning effective instructions and instructional activities as well as formulating, representing and solving problems that arise during instructions. Competency in this strand is linked to what commonly called problem solving and problem formulation. A theme characteristic in the above conception is the heuristics embedded in the problem solving process.	**SC1**: plans the lesson carefully **SC2**: provides tasks/activities that allow multiple solving strategies and evaluation of different solution method strategies. **SC3**: emphasise and encourages learners' engagement with the solution of non-routine tasks i.e. formulating own theorems in circle geometry **SC4**: encourages learners to discuss and solve problems collaboratively **SC5**: asks probing questions for learners to reflect critically and provide critical reasoning and argumentation (to move the lesson on). **SC6**: represents ideas carefully using multiple representations/notations such as mapping graphical, symbolic representations, algebraic notations and pictures. **SC7**: engages with expected and unexpected learners' mathematical ideas and suggestions **Non examples**: Using (or forcing students to use) a particular problem solving strategy to solve routine problem rather than developing strategies with students by allowing them to determine a way to solve non-routine problems that make sense to them
4.Adaptive Reasoning	**AR**	Reasoning in justifying, explaining one's instructional practices and reflecting on those practices so as to improve them. Teaching that encourages or emphasises capacity for logical thought, for reflection on, explanation of and justification of mathematical arguments. That is, teacher's capacity to think logically about the relationship among concepts and situations, and ability to consider and select from alternatives.	**AR1**: provides situations and activities that require logical reasoning. **AR2**: asks questions that solicit learners to explain or justify their solution strategies **Examples**: why are two angles equal, how you did that, why, why are you saying so, what would be the reason, how did you come to that answer to encourage the learners' development and articulation of justification and argumentation. **AR3**: engages with learners that encourages reflection. **AR4**: encourages learners to think deductively. **AR5**: provides learners with geometric activities that require and emphasise deductive reasoning **AR6**: invites constructive criticism and feedback from learners. **Non examples**: Explanations of concepts or procedures are not immediately followed with informal proof, justification and/or deductive reasoning. Learners are not encouraged to consistently explain or justify their answers or claims.

5.Productive Disposition	**PD**	Disposition towards mathematics, teaching, learning and the improvement of practice. Teaching that encourage or instil habitual inclination to see senses in mathematics, to perceive it as both useful and worthwhile subject to be learned, coupled with a belief in the value of diligent work and in one's own efficacy, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics.	**PD1**: provides learners with homework tasks to encourage learners to do Mathematics outside of the classroom. **PD2**: encourages and affirms learners **PD3**: adopts an attitude with learners that making mistakes in the mathematics classroom is OK **PD4**: teaches with passion and enthusiasm **PD5**: expects learners to be enthusiastic **PD6**: expects learners to work hard **PD7**: motivates learners by making the lessons interesting with hands-on activities. **PD8**: has high expectations of learners **PD9**: enforces the notion that everybody has a contribution to make in the math classroom. **Non examples**: Have a negative disposition a lot of the time, e.g., displays anger, aloofness, and sarcasm. Does not praise effort and performance or encourage learners to persist or persevere, and constantly seeking the correct answers from certain learners.

Note: All fifteen lessons observed were subjected to this analytical tool (Adapted from Kilpatrick et al., 2001).

3.4.7. Phase V: Individual interview schedule: Post lesson reflective interviews

I was concerned that only using classroom observation instruments might cause me to miss other aspects of classroom practice not included on the observation protocol but which could distinguish the practices of effective teachers. Hence, in this Phase I conducted post observation reflective interviews with each teacher to further ascertain their mathematical practices and interactions with learners, and to capture additional dimensions of classroom practices. The individual interview schedule was semi-structured in nature (Stake, 1995; Cohen et al., 2007). Teachers were asked to walk me through their geometry lessons than me talking it through and, consequently, I explored with teachers:

- critical events, actions or incidents that emerged during their classroom practice,
- influencing factors that contribute to their mathematical stories,
- teaching proficiency and their mathematical actions and interactions with students,
- artefacts of instructional practice such as lesson presentation and development including mathematical tasks, mediation of knowledge and use of resources for teaching and learning, as well as what they think made their lessons effective.

The interview questions were posed in such a way as to align them with the notion of teaching for MP, as advocated by Kilpatrick et al. (2001). The schedule of questions for this semi-structured interview with each teacher also included questions related to notions of *autopoesis*, *embodiment*, *co-emergence* and *structural determinism* as described in Chapter Two.

These interviews provided an ideal space for teachers to talk about their teaching proficiency and how they become effective, and for me to ask probing questions to gain insights I could use as the basis for further inquiry (Yin, 2008). I regarded this interview as a conversational partnership in which I assisted a process of reflection. Interview sessions lasted from forty minutes to sixty minutes, depending entirely on how much an interviewee had to say or share. To "understand the complexity of the relationships between the pedagogy and the teachers' lived experiences, one must first be able to *listen* to the teachers' voices" (Miranda: 2009, p. 53). The interview with teachers was an attempt to listen to the Namibian mathematics teachers' voices as they articulated understanding of their own effectiveness. Interview transcripts were given back to the respondents as part of the process of member-checking during the joint participative analysis interview session at NIED. Refer to Appendix G for an interview schedule. The interview transcripts are appended in a CD.

Speaking informally to the teachers and asking them various questions about why their schools were effective in teaching or what made them successful enabled me to collect primary or first-hand information. The advantage of doing this was that teachers had comprehensive views on their own effectiveness as well as views on their schools' highlights and lowlights. The disadvantage of this is that some teachers, particularly principals, might have a biased opinion about their practice and, thus, may not have been willing to share their effective practices or tell the truth when it comes to a particular question.

3.4.8. Phase VI: Data Analysis

A challenge of this research was to manage the large amount of data collected in the five lengthy phases of data generation. The data analysis was inseparable from the data collection. Data was analysed using descriptive narratives (Cohen et al., 2007) at each stage of analysis. This was done in part to help inform the next stage of data collection and analysis with some insights gained from prior data collection. It also encouraged early and continuous engagement with the data. Graven (2002) notes that, after an initial reading of the data, the researcher should make an assertion and choose an excerpt from the data and write a narrative [descriptive] vignette that supports that assertion. According to her, the narrative vignette provides the base for effective reporting of fieldwork and provides a means to identify and verify assertions, concepts and ideas that emerge through the study. Following these recommended methods, I generated these narrative vignettes as I went along. For example, I examined samples of instructional practices including both lesson artefacts and teachers' commentaries on the lessons that were intended to display how each of the strands of MTP could be identified in the lesson. I then wrote a series of narrative or descriptive vignettes as I proceeded and at each stage of data analysis. Whilst data from all the teachers informed the research process, the more in-depth analysis was done over a period of eighteen months.

Patton (2002) defines data analysis as a process through which we make sense of raw data and communicate their essence of what it reveals. This process involves organising, accounting for, and explaining the data in terms of the participants' definitions of the situation, noting patterns, themes, categories and regularities (Cohen et al., 2007). Creswell (2007) argues that data analysis in qualitative research should occur concurrently with data collection as this enable the researcher to test any emerging conclusions. In my analysis, for example, the data from the geometry questionnaire needed to be analysed first, before I observed the teachers in the mathematics classroom and embarked on the second level of analysis of classroom observation data. Similarly, data from lesson observation recordings needed to be analysed first to set the stage for the post observation reflective interviews with each teacher. As a result, this phase ran concurrently with phase V and focused on the analysis and interpretation of the filmed lessons.

According to Creswell (2007), qualitative research allows participants to explain, discuss and reflect on their own experiences and teaching practices. Bearing in mind that my sample consisted of effective teachers, they also acted as co-researchers and I was able to have fruitful discussions with them concerning their views and perceptions of the observed lessons. From an enactivist perspective, "there are interpretations and interpretations of interpretations" (Reid, 1996, p. 206). That is, the participating teachers and I were involved in a process of co-evolution of ideas and co-emergence. In this way participants share their collective understandings and inter-subjectivity, which arise and co-emerge through their dialogue (Davis, 1996). This means that the data were interpreted by several researchers. This also means that the data were interpreted many times by me as a researcher. In this sense, data from the teachers' geometry questionnaire, lesson videos and interviews co-emerged from the researchers via a blending of collective and reflective conversation.

The analysis of data firstly dealt with the description of each case based on the data collected via the different instruments. That is, data from all data sources were thoroughly reviewed and transcribed verbatim and, thereafter, were coded using an adapted version of Kilpatrick et al.'s (2001) model of teaching proficiency, and enactivist concepts (Maturana and Varela, 1992). Each of the data sources was colour coded to look for similarities and differences as well as odd occurrences. Codes based on the Kilpatrick et al. (2001) pedagogical concepts were assigned to lesson video and interview transcripts, to explore if and how the teachers addressed the five strands of teaching for mathematical proficiency. The enactivist concepts enabled me to explore how teachers' mathematical knowing and actions shaped their instructional practices in the mathematics classroom.

After codes had been assigned, a clustering procedure (Creswell, 2007) was used to gather data into categories. Emerging themes were then summarised to investigate which mathematical strands were entailed in the teachers' teaching practice. Excerpts that showed some evidence of teachers' perspectives and mathematical teaching proficiency were used to elucidate teachers' understanding of their own effectiveness. This vertical analysis enabled me to write a particular descriptive account for each teacher so that I could do a horizontal analysis and look for commonalities and differences across the five teachers. The findings chapter represents the negotiated consensus of critical friends with regard to how the data were coded and categorised. The process of data analysis and interpretation ran concurrently through four interrelated stages, as discussed below:

3.4.8.1. Stage I: Analysis of teacher geometry scenario-based questionnaire

In this first stage of data analysis, I carefully read and examined data generated through the scenario-based questionnaire in order to explore and unpack teachers' content geometry knowledge and mathematical thinking, reasoning and proficiency. Specifically, in this stage, I deconstructed teachers' CU, PF, SC and AR with regard to their responses to the questionnaire. Analysis of teachers' responses on the questionnaire items involved scrutinizing the solving methods applied to identify characteristics of teachers' mathematical content knowledge and preferred teaching approaches. Specifically, my interest was to reveal how they confronted the scenarios, what strategies they used which made sense to them, and to see whether their selected solution strategies were similar to or different from each other. Responses to items were presented for each teacher in a tabulated format from which written interpretations were made.

3.4.8.1.1. Coding scheme for analysing teacher responses to geometry scenario-based questionnaire

The geometry questionnaire was not just about getting the right answers, but it was about the process of how the participating teachers answered the problems. For this reason, I coded teachers' responses to the scenario-based items according to the coding given in the table below. Hence, teachers' MP was measured in relation to their responses to the questions and the kinds of mathematical competencies that they demonstrated. In order to help me recognise the first four strands (CU, PF, SC and AR), I expanded the definitions and summarised, in detail, the descriptions of what Kilpatrick et al. (2001) meant by these strands. I excluded the productive disposition (PD) strand from my coding, because this involves a long term habitual attitude rather than an action, and this was difficult to infer from scenario-based questionnaire. However, PD was evident in my analysis of the lesson video recordings. Noting that these strands are interwoven, the summary descriptions of the coding, as given in Table 3.4 below, represent a synthesis of the MP descriptions for each question item in the questionnaire.

Table 3.4: Coding instrument (Adapted from Kilpatrick et al., 2001)

The four mathematical proficiency strands of adapted Kilpatrick et al.'s (2001) model and their coding	Kilpatrick et al. (2001, p. 116) descriptions of strands	Observable indicators of the FOUR strands of mathematical proficiency (MP) Evidence of what teachers can typically do:
Conceptual Understanding (CU)	Integrated and functional grasp of mathematical ideas, concepts, operations, and relations which enable one [student or teacher] to learn new ideas by connecting those ideas to what he/she already knows.	**CU1**:demonstrates comprehension of mathematical concepts, operations, relations and notations (efficient and integrated mathematical understanding of the problem) **CU2**: uses models and multiple representations flexibly **CU3**: identifies relationships among concepts **CU4**: explains concepts clearly based on her investigations, and makes connections to meanings underlying those concepts
Procedural Fluency (PF)	The skill in carrying out procedures flexibly, accurately, efficiently and appropriately.	**PF1**: shows appropriate, efficient and accurate execution of mathematical procedures and algorithms, including those that require sequential decisions, to determine the answer (accurate computation) **PF2**: carries out routine procedures in explicit ways **PF3**: selects and applies problem-solving strategies accurately, efficiently, flexibly and appropriately. **PF4**: employs basic algorithms, formulae, procedures or conventions accurately, efficiently, flexibly, appropriately.
Strategic Competence (SC)	The ability to formulate, represent and solve mathematical problems.	**SC1**: formulates and carries out a plan of problem solving using drawings, sketches and other representations **SC2**: solves problems using appropriate solving strategies **SC3**: evaluates solving strategies for accuracy and efficacy **SC4**: organises information appropriately when solving problems and precisely communicates his/her actions and reflections regarding his/her findings, interpretations to the scenarios or situations.
Adaptive Reasoning (AR)	The capacity for logical thought, reflection, explanation and justification.	**AR1**: communicates proper mathematical thinking and reasoning through a sequence of calculation steps or explanations and arguments based on his/her interpretations **AR2**: justifies and/or explains responses, answers or steps **AR3**: reflects on and explains concepts or solving procedures using accurate mathematical language and notations **AR4**: tests conjectures and makes generalisation from results

I coded teachers' responses to scenarios as **CU (Conceptual Understanding)**, for teachers who demonstrated clearly and efficiently an understanding of the mathematical concepts, relations, notations and procedures (**CU1**). A pedagogical content knowledge related response or action was coded as **CU2** when the teacher represented mathematical situations in different ways or used multiple representations, models or real objects. Similarly teachers' responses that used sketches to talk about or explain the concept to learners (**CU4**), or illustrated an efficient strategy to solve the problem or explain the relationships that exists between concepts (**CU3**) were coded as **CU**.

Teachers' responses that were characterised by knowledge of procedures, and when and how to use them appropriately, and skills in performing them accurately and efficiently were coded as **PF (Procedural Fluency)**. For example, direct, explicit execution of procedures (**PF1**), giving the formulae required for engaging with the tasks (**PF2**), carrying out routine procedures to provide solutions to tasks or employing basic algorithms accurately, flexibly and efficiently (**PF4**). In the same way, teachers' responses to tasks that showed evidence of the use of multiple procedures with connections, or without necessarily making connections to the meaning that underlies both the procedures and concepts involved, were also classified as **PF**. Thus, the responses by teachers who mentioned that all they needed to do was to measure the length EK, BD or perpendicular height *h*, etc. of a shape (to get a certain length), were coded as **PF**.

Teachers' responses that show application of mathematical concepts and skills, selection and use of key concepts or skills to determine the operation needed (**SC4**) were recorded as **SC (Strategic Competence)**. For example, when the teacher said, "he/she will draw the learners' attention to different shapes, ask learners to build shapes from card, investigate closed figures or other numbers, and discuss it together", this was interpreted as teacher involvement in problem formulation and solving together with the learners as one solution would emerge.

Responses that showed reasoning, justification and explanation (**AR1**) about relationships among concepts and situations or applications of principles and strategies were considered as **AR (Adaptive Reasoning)**. Responses that reflected sufficient reasons or explanations of particular procedures used or why students' responses were in/correct, and a justification of solutions were coded as **AR**. Attempts to strategically reflect on or justify responses, answers or steps, reason through the problem at hand (**AR3**) and validate measurement, test conjectures by proving and estimating dimensions (**AR4**) were also noted as **AR**.

As these strands are interlinked, I also anticipated teachers would explore given scenarios, use some drawings, representations and models, and explain their solving strategies for arriving at the answers (see Appendix I for teachers' responses).

3.4.8.2. Stage II: Review and analysis of lesson videos

In stage II, I watched and reviewed the qualitative classroom observation data (videotaped lessons) thoroughly to capture teachers' actual teaching, and to formulate some possible questions to help me engage with the teachers about their teaching practice during the post lesson reflective interviews. I watched the lesson videos over and over to refresh my memory as I needed to provide the detail I wanted to include in the lesson descriptions. I transcribed and detailed the data generated through lesson observations so I could document unique patterns of their teaching practice.

I then reviewed the lesson transcripts a number of times alongside the developed classroom lesson observation checklist/schedule, based on the Kilpatrick et al.'s (2001) analytical framework. I used this checklist to reveal teachers' geometry classroom practice, and emerging patterns of interactions which surfaced in the lessons from the lesson videos. The focus here was on their teaching proficiency in relation to their geometry knowledge and teaching strategies. The interconnectedness of the enactvist concepts of mathematical actions provided the context within which I analysed teachers' perspectives of effective teaching and how it shaped their classroom practices. In my analysis, the concepts of enactivism were also used to analyse not only the teachers' utterances, but also the nature of mathematical actions that shaped teachers' classroom interactions with students. This analysis afforded me the opportunity to discover how the sampled teachers position or see themselves in relation to mathematical content knowledge and teaching proficiency, mathematical representation of ideas and disposition towards teaching mathematics.

I analysed and coded each lesson transcript using different coloured highlighter pens to look for similarities and differences and identify themes so that the data could remain in context and provide a visual indication of emerging categories. These categories were used to create a visual representation of the competencies and traits required of effective mathematics teachers. To establish the validity of the codes, I coded all lesson transcripts using the same rubric or codes, and each lesson transcript was coded separately.

3.4.8.3. Stage III: Structured and open-ended interviews – analysis with the teachers

During my visit to Hokkaido University, Japan in 2009, one important lesson I learned was the "Japanese practice of lesson study" which shaped my methodology design. The aim of core lesson study activities is for teachers to collaboratively interact and reflect on a small number of study lessons to examine their practice in a real context. Lesson study is typically characterised as classroom situated, context based and learner-focused (Suh, 2007). The goal of a lesson study is to improve the effectiveness of the experiences that teachers provide to their students.

This joint analysis with participating teachers forms both the data-gathering and data-analysis stage. In this stage, I finalised and confirmed the data collection process through "*stimulated recall participative analysis sessions"* (Gass and Mackey, 2000). The stimulated recall method uses visual and audio cues to help the individual accurately remember and verbalise the thoughts and decision-making process he/she experienced during the taped episodes (Thom, 2000). "It is a retrospective self-reporting technique that provides access to the participants' thinking, revealing the influences (aims, intentions, plans, expectations, interpretations, perceptions) that direct their behaviours" (Thom, 2000, p. 51). Stimulated recall interviews are commonly used to gather data about thought processes in the classroom (Gass and Mackey, 2000). Advantages of this method are that participants are encouraged to reflect and share their experiences and expertise spontaneously, using their own language. The video clips also allow the researcher to observe interactions between people and, when combined with the focus group interview process, another level of information gathering possibly missing from the individual interviews may be obtained (Thom, 2000).

For this joint lesson analysis session, I designed a one-day workshop with all participating teachers. All five teachers were required to attend this workshop or meeting offsite at NIED, where collaborative viewing of their lesson video footages was conducted. My decision to involve the teachers in a joint lesson analysis was to establish a good team spirit and collegiality as co-researchers and to introduce these teachers to the Kilpatrick et al.'s (2001) teaching framework for MP as a tool for analysing their practice within the Namibian Mathematics Curriculum. I employed a "*collaborative approach*" for the teachers to analyse their own teaching practices. A private room was used for the viewing of the video footages with teachers. The analysis started at 8h00 in the morning and adjourned at around 17h30. Collaborative discussions related to the viewing of the video footage were audio taped.

As an enactivist researcher, I attempted to make use of this interaction in order to identify emerging patterns of teaching practices and interesting moments in the mathematics classrooms and to clarify my understanding of the teaching and learning processes of geometry (Reid, 1996). This interview session provided a platform for participating teachers to share practices of effective teaching, to create unlikely connections and align their teaching efforts and experiences with discourses of effective mathematics teaching. The joint participative analysis session focused on what the teachers highlighted during the initial interviews, and I asked teachers to elaborate on those points.

During this session, I found that the participants appeared to enjoy this process, readily making comments about what they were doing and thinking at the time of the lesson observation taping. The schedule of the interview questions for this joint analysis stage with teachers was divided into five successive tasks (see Appendix J for the complete interview schedule), as discussed next.

First Task: The first task in this analysis was to **member-check (analyse) the interview transcripts** with each teacher. Participants were provided with the transcripts of the first interview session and were asked to add any omission to the transcripts, or to delete any inaccuracies. After perusing the transcripts, teachers were asked to extract common issues and differences that emerged from the discussion I had with each of them. I also asked teachers to pay attention to their responses, highlighting in red the points that I could not capture clearly on the voice recorder.

Second Task: The second task was lengthier and took nearly the whole day. During this task, **teachers viewed their own and the other teachers' geometry lesson videos** to analyse in depth, the nature of the quality of mathematics instructions across the five lessons or classrooms, and then to reflect on those lessons. The purpose was to provide snapshots of geometry teaching as they exist in mathematics classrooms in a variety of contexts in Namibia. Essentially, each teacher was asked to select one geometry lesson out of the three lessons I observed and video recorded, which he/she though best reflected his/her daily classroom teaching practice, and that the group might discuss. These were lessons that all of us, as a group of co-researchers, viewed and reflected on. Permission was then given to the teachers to view each selected lesson video for the entire lesson time. Teachers watched and analysed lesson videos in class. During the viewing of the lesson footages, I paused, rewound or forwarded the videos as requested by teachers. This was done so that teachers could clarify their intentions, actions or discuss emerging critical incidents. For the most part, this analysis cycle involved the following steps, which were repeated for each of the selected lesson videos observed:

- Collaborative viewing of video recordings by the sampled teachers and the researcher for the purpose of observing and gathering data, and
- Collaborative discussion of what happened in those selected lessons and why those lessons in particular stood out, mainly for the purpose of data analysis and interpretation.

Third Task: The third task looked at the nature of the quality of mathematics instructions. For this task, I asked the five teachers to reflect on their **own competencies and the competencies of other teachers** by watching the lesson videos and, then to have a focus group discussion. I gave the teachers four questions to guide them in deconstructing the most important competencies needed in teaching mathematics effectively, as evidenced from their lesson videos. These questions were: (1) what are the most important competencies needed in teaching mathematics effectively or successfully, (2) how do you construct effectiveness in your mathematics teaching, (3) how do those constructions of effectiveness in mathematics teaching inform your classroom practice, and (4) what are the teaching proficiency characteristics that are similar and different across your teaching practices or lesson videos?

Fourth Task: In the fourth activity of the joint analysis, I used the **lesson observation schedule** on teaching for MP as a conceptual framework to collate the observations and experiences that the teachers shared. I provided teachers with a short ten minute outline of the Kilpatrick et al.'s (2001) analytical tool. After a brief discussion, each participant was given a classroom lesson observation checklist, including important events or mathematical actions in the lesson. Specifically, teachers were asked to indicate, on a four-point Likert scale, whether they strongly disagreed (1), disagreed (2), agreed (3) or strongly agreed (4) with the provided mathematical actions to measure the extent to which their classroom instructions or mathematical interactions with learners align with the Kilpatrick et al.'s (2001) analytical tool, or the way mathematical concepts were presented in their teaching practices. I regarded this observation schedule as a powerful tool for gaining insight into their understanding of teaching for MP. Nevertheless, one of the teachers handed back one unfilled checklist to me, providing various reasons for doing so (saying it was infeasible to evaluate his own practice).

Fifth task: The final part of the joint analysis tasks and data generation was **reflection on the research process**. Each teacher was afforded the opportunity to share their own experience of the workshop. It was rewarding for the five teachers to reflect on the research process and to share with me what they had learned from this process, how useful they had found the observation schedule or analytical tool, and to record what they felt were the weaknesses of the Kilpatrick et al. (2001) analytical tool or classroom observation protocol.

The joint analysis interview session, for the most part, allowed teachers to share their teaching practices as lived experience. The lesson analysis interviews also provided me with an opportunity to delve into detailed analysis and to discover valuable and unique insights from the teachers' perspectives of their teaching practice. Interviews also enabled me to dissect the teachers' mathematical reasoning and personal visions for mathematics teaching. According to Sanni (2009, p. 78), interviews enable the researcher "to explore complex issues in detail and prompt for clarification". Hence, the stimulated recall sessions focused on questions such as:

- how do you deal or address learners' misconceptions of mathematical ideas in your classroom?
- how do you ensure that students do understand solving methods, procedures or formulas employed or agree with other methods used?
- why did you do that, what approach do you use and why or what else could you do?
- can you give examples of mathematical proficiency that are reflected in your practice?
- suppose a student poses a question, how do you answer him/her or how do you direct questions from one student to another?

3.4.8.4. Stage IV: Analysis of stimulated recall analysis session data

In the last stage, I finalised the data analysis process. That is, I analysed the data generated through the stimulated recall participative analysis session with teachers in relation to all data. The lesson video and interview transcripts were used to analyse and describe teachers' mathematical actions, in particular, actions that were used to help children develop MP. The audiotapes of my interviews with teachers were a pleasure to transcribe as I found them all most articulate. They spoke fluently and eloquently, and generally expressed their views with ease. I needed to do very little to explain my questions or prompt their responses. However, the interview with Teacher 2 was a little disappointing in that she did not elaborate on her ideas as much as I had expected. I also wondered about the extent to which her use of English as a second language hindered her in providing more detailed responses and whether this, together with her marked accent, would have an impact in her classroom interaction with her learners.

The enactivist theoretical lens was also used to uncover additional ways in which teachers' mathematical teaching proficiency manifests in their classroom instructional practice, and how their mathematical teaching proficiency intersects with their other characteristics to produce effective interactions with the learners. The aim was to provide a deeper perspective of the elements of teachers' daily geometry classroom practices that lead to successful teaching for mathematical proficiency and improvement in student learning.

Measures were also taken to ensure the trustworthiness of the study. For example, I discussed the findings with critical friends to confirm and verify emerging themes from the transcripts. The transcripts of the lesson videos and interviews with the five case study teachers were coded, analysed and discussed thoroughly.

3.5. PROBLEMS WITH THE CLASSROOM OBSERVATION SCHEDULE

The classroom observation schedule was not without its problems or shortcomings. The findings from a joint analysis with the five teachers raised some questions about the efficiency of the tool. Additionally, criticisms were raised by several teachers about the efficiency of such a lesson observation protocol. For example, the teacher from school E indicated that the tool did not reflect teachers' stress. Also, a school A teacher noted, it did not address teachers' dress code, but most agreed this was not a relevant issue. Regarding the analysis, there was a question included in the observation schedule which addressed whether the 'teacher is friendly or teaches with passion and enthusiasm'. Some teachers, particularly the school D teacher, felt that the rating of either agreeing or disagreeing was too limited, and suggested a grading of 0 to 10 so that one was not coerced to disagree which would not yield sufficiently specific data. Taking the teachers' comments into account, the classroom observation checklist was improved.

Table 3.5 below provides a summary of the phases of data collection and analysis stages.

Table 3.5: Summary of procedures and techniques used for data collection and analysis in the study

Phase	Instrument	Reason/Purpose	Emerging Data
I.	Planning for actual Data Collection	Negotiate access to schools, get to know participants and creates rapport with them, hold meetings with school principals and teachers involved and explain the purpose of the research, obtain teachers' specific details, and decide upon date for actual classroom observations.	
II.	Biographical Questionnaire	Questionnaire to teachers in all five respective schools (*contextual profiling*). Collect biographical information about participating teachers and general information about their school and teaching, and gain insight into each individual teacher's personal details pertaining to his/her qualifications and teaching experiences as well as their participation willingness.	Biographical data
III.	Teachers' geometry content knowledge questionnaire.	Explore and establish teachers' geometry conceptual understanding and procedural fluency based on Kilpatrick et al.'s (2001) model (draw out specific themes related to teachers' geometry proficiency and self/teaching efficacy).	Quantitative and Qualitative data

IV.	Classroom Observations (filming lessons)	Investigate selected teachers' actual teaching in geometry classrooms in order to document teachers' geometric instructional practices and what they value as effective teaching practice. In total, 15 lessons were videotaped to look for evidence of how teachers' teaching proficiency played out in their classroom practices, in order to construct narrative of key moment or teaching proficiency that shape those lessons.	Video-tape recordings and lesson transcripts (Qualitative)
V.	Interviews with each observed teacher based on preliminary analysis of lesson videos	To ascertain from each of the five teachers their mathematical stories as learners and teachers of mathematics with a particular focus on actual events of their classroom practices. This platform provided me with an opportunity to member-check and test out some of my tentative or preliminary ideas of the emerged themes of what happen during the lessons.	Interview transcripts (textual) (Qualitative)
VI.	Data analysis and Interpretations	**Stage I - Analysis of geometry content knowledge questionnaire**: examine data generated through teachers' geometry content knowledge questionnaire to explore teachers' content/geometry proficiency, mathematical skills and reasoning.	Teachers' mathematica l proficiency classified according to 4 strands of Kilpatrick analytical tool (Qualitative)
		Stage II - Review of lesson videos: Review and transcribe qualitative classroom observation data in order to capture teachers' actual teaching and document patterns of practices or evidence of how teachers' teaching proficiency played out in their practices.	Teachers' teaching proficiency classified according to the 5 strands of Kilpatrick analytical tool (Qualitative)
		Stage III - Interviews-analysis with the teachers: Co-analysis of video footages with 5 teachers and member-checking of interview transcripts. Run a one day workshop where classroom observation protocol was spelt out to teachers; use a "Collaborative approach": stimulated recall participative analysis session to jointly analyse videos and negotiate teachers' instructional practices and actions and reasons for their actions or interactions with students. The aim is to ascertain from each of the 5 teachers their mathematical stories, get their assistance to confirm or correct my interpretations of certain particular teaching proficiency strands, probe more on any emerging ideas and actions, and get evidence of teachers' productive disposition with a focus on actual events of their classroom practices.	Qualitative (textual), audiotape recordings and transcripts
		Stage IV – analysis of stimulated-recall interview data: Analyse data generated through stage III (stimulated recall participative analysis session) and triangulate my data to get deeper insight into teachers' teaching practices.	Interview transcripts (textual) (Qualitative)

3.6. CONSIDERATIONS CONCERNING RIGOUR: VALIDITY AND RELIABILITY

In this study, rigour is concerned more specifically with the *validity* and *reliability* of the research process, that is, the *appropriateness*, *meaningfulness* and *usefulness* of the claims that I, a researcher, make (Sanni, 2009). Given this need for integrity, I worked rigorously with the data collection and data analysis as well as with the existing theory and literature in order to make careful links between them and evolve a qualitative work worthy of the trust and confidence of the readers.

In the context of my study, "validity is described as the degree to which a method, a test or a research tool actually measures what it is supposed to measure" (Merriam, 2009, p. 354). Reliability, on the other hand, is an important consideration in that it may be useful as an indicator of *trustworthiness* or *quality* of the research. Thus, Creswell (2007) further describes reliability as "the extent to which a test, method or instrument gives unswerving results across a range of settings used by a range of researchers" (p. 274). Reliability simply means dependability, stability, consistency and accuracy (Lincoln and Guba, 2003). It is the "extent to which a measuring instrument yields a similar result on repeated applications with different subjects in different contexts" (Atebe, 2008, p. 125).

3.6.1. Ensuring validity of my study

As Schäfer (2003, p. 69) notes "validity is not seen as part of a final product control process or verification, but rather a continuous process of credibility, growth and understanding". From this perspective, data validation is a complex and rather multifaceted, and thus requires careful consideration. Viewed as the key issue and requirement for both qualitative and quantitative research methods, validation of data in this study was deemed more necessary in the context of my work to "accurately describe the phenomena being researched" (Cohen et al., 2007, p. 113). Validity issues that I needed to be aware of, or deal with, are discussed below.

3.6.1.1. Researcher bias

I needed to be conscious of and make explicit any assumptions, values, beliefs and theories that I bring to the research process in order to avoid leading the interviewees through my questions and actions. It is important that I acted with integrity at all times (Maxwell, 2008).

3.6.1.2. Rich data

Rich data were collected through the geometry questionnaire, non-participant classroom lesson observations of teaching and intensive interviews with the research participants. Also, as a researcher, I excluded any of my own experiences within the community of practice and focused on the teachers' viewpoints.

3.6.1.3. A team of expert researchers and critical friends

The help of critical friends was beneficial in shaping up my "research ideas and enhancing reflection on the research process" (Muir, 2010, p. 43). As part of a PhD research discussion group where we share our research ideas and work, a team of peers (PhD fellows) and expert researchers (research-minded people) have been involved in the research and the data validation process in one way or another. This is a space where colleagues can question the research process, the data and emerging findings. Such a team of critical friends was fundamental and explicitly facilitated the validation process of the research findings.

Most importantly, my supervisor for this study played an integral role in the validation of the research process, by scrutinising my arguments with regard to the data analysis and presentation, classroom observation schedule/tool and interview questions. All the way through the process, he offered different critical perspectives to issues but also enriched both the research process and my engagement with the project. As Schäfer (2003) concedes, "validation does not imply that the goal is to achieve uniformity in the criticism, but rather that the views are valid, meaningful and useful" (p. 72). Throughout the research process, the findings, suggestions and alternative explanations that emerged from the supportive team of critical friends were taken into account and acted upon as a means of establishing clarity in the research.

3.6.1.4. Progressive subjectivity

As another validation technique, my subjectivity, interpretation and ideas cannot be "given privilege over that of anyone else" (Schäfer, 2003, p. 72). For Schäfer, the matter is not whether the constructs are subjective or not, but rather whether they have been sufficiently opened to the light of criticism.

3.6.1.5. Collegial team of co-researchers (member checking)

This validation modus operandi sometimes referred to as participant validation, involved a team of five participating teachers as co-researchers and several expert researchers to tease out and verify my analysis and interpretation of themes. As Schäfer (2003) notes, member checking is "[a] crucial technique for establishing credibility" (p. 73). Member checking was achieved via feedback from the participants once the interviews had been transcribed to ensure that the data accurately reflected the participants' experiences. This technique was applied particularly during phase VI; stage III, of this research project. Further, at the level of analysis, findings were shared with several critical colleagues. Such a collegial team of co-researchers comprised:

- five participating teachers as co-researchers who were involved in the research as effective teachers, and who were given copies of the interview transcripts to help secure the interpretive validity of the research.
- a mathematics educator and a post-doctoral research fellow in teacher professional development. His research work focused on mathematics argumentation and was firmly embedded in an interpretive research paradigm. His data analysis was on the interpretation of lesson video recordings of teachers in action.

3.6.1.6. Triangulation

I used multiple sources of data. In this research teachers' biographical and geometry questionnaires, classroom observations and interviews were used as data for analysis. In addition, in an attempt to ensure validity, my proposed data collection tools draw from recently established models on teaching proficiency or robust mathematics instructions and developed classroom observation instruments relevant to the practice of teaching. After constructing a geometry questionnaire, experts in geometry were requested to cross-check them. The test items were selected and adapted from standardized examinations that have passed through various validity-testing processes. A comprehensive lesson observation tool or checklist was developed based on Kilpatrick et al.'s (2001) model and the enactivist framework. This was crafted and then face-validated with the assistance of specialists at the Rhodes University Department of Education. In addition, I implemented a procedure proposed by Atebe (2008). In this framework, a consultative panel of independent observers was involved to view the videotaped lessons and make comments on the manner in which information was captured and the quality of data interpretation. In this process, I concentrated on achieving conformability, credibility and transferability (Cohen et al., 2007).

3.6.2. Ensuring reliability of my study

What follows are some of the measures that I took to ensure the reliability of my research instruments. In terms of the geometry content knowledge questionnaire, I sourced items that have been used before and did some modification, where necessary, to suit my study needs, before conducting a reliability test (piloting) for the individual items. Further, I piloted both the lesson observation schedule and interview questions. This was achieved by viewing an existing recorded mathematics lesson video with Rhodes University Masters of Education students. With regard to the analysis strategies and interpretation of audio transcripts of both lessons and interviews with teachers, I involved all five participating teachers plus other researchers in the analysis and checking of collected qualitative data. I therefore verified my interpretations of what I observed in their lessons, (a) the emerging patterns of their individual lessons and (b) interactions across the five teachers' lessons or classrooms.

3.7. ETHICAL CONSIDERATIONS

The subject of ethics is important in the educational research process for a number of reasons. From a broad, collective point of view, ethics has to do with the application of moral principles to prevent harming or wronging others (Opie, 2004). Therefore, I needed to be familiar with the pertinent issues involved to avoid any problem or criticism of my research work, and to proceed ethically without compromising the validity of the research (Cohen et al., 2007). As ethics involves consideration of right and wrong in the practice of research (Remenyi, 1998), this study was conducted in a manner which promotes ethics and integrity. The work of Cohen et al. (2007) raised some useful checklists or tips relating to ethical issues for me to think about before conducting my research, for example, obtaining and/or ensuring informed consent, access and acceptance, privacy, anonymity and confidentiality. I will briefly reflect on each of these issues in relation to how they were addressed in this study:

- *formal permission:* Receiving the consent of the research participants protect their rights to self-determination (Cohen et al., 2007). In preparation for the fieldwork component of this study, permission and approval were obtained from all relevant authorities. Also, being an education officer, my position of authority might have intimidated the teachers. In order to overcome this, a thorough outline of the research and its purpose were clearly spelled out to encourage participants to take part.
- *voluntary informed consent*: Informed consent and formal institutional approval were sought from all potential and identified research participants. In addition, the consent form, describing the research objectives and procedures and the study benefits to the participants, was signed by each of the participating teachers before the first phase of data collection began. Participants' right to voluntary participation was meticulously explained, but the teachers were told that they could choose not to participate in the research, and that they could withdraw from the research process at any stage without there being any recriminations.

- *guarantees of anonymity*: At the start of the research I explained to teachers that I would not use their names for the sake of anonymity. However, I did wish to acknowledge their individual contributions so all the five participants gave their permission to have their real names associated with the evidence and appear in the final report. The privacy of my participants was assured, while their schools' identities as well as video and interview transcriptions were kept anonymous by using pseudonyms or fictitious names as discussed in the description of my sample.
- *Openness with the informants*: As I needed to be sincere, open and avoid moralistic judgements, I spent time developing rapport with the participants and making them feel that they could trust me. This means, in this study, participants were made aware of exactly why the research evidence was required and exactly what will be done with it once the research has been completed. I also explained (and opened for question) the nature of the research process involved, the benefit to myself (as researcher) and the possible benefit for teachers (contribution to research on teacher learning).
- *trust and confidentiality*: All aspects of the study, including the results, were strictly confidential, and only the researchers had access to information on participants.
- *respect for truth and accuracy*: As a researcher who wanted teachers to talk about their own experiences as effective mathematics teachers, I made sure that my research participants felt respected. Accountability, in case study research, is related to the perceived ability of the researcher to provide a deep description of context specific events and the perspectives that comprise the cases. While all case study researchers have their own subjective accounts of the experiences in the field, they need to be conscious of their obligation to represent the interpretations and perspectives of their subjects. Stake (1995) notes:

> …the interpretations of the researcher are likely to be emphasised more than the interpretations of those people studied, but the qualitative case researcher tries to preserve the multiple realities, the different views, and even the contradictory view of what is happening… (p. 12)

Further, individual interviews with each of the five teachers were held after school hours at a convenient time and venue and the data was authenticated by means of member checking.

3.8. Limitations of the study

This research is a study of a unique case and has clear limitations. The case study's purpose does not claim to represent the world, rather, it attempts to provide a rich description of the complexity of teacher practice [and teaching proficiency] as it exists at a particular time, for a specific group of people within a particular context (Peedo, 2003). The participants were not chosen to be representative of teachers in Namibia thus, the findings from this study may not generalise to teachers outside of the sample.

A final limitation of this study is the fact that enactivism, like effective practices, is still in its infancy. The literature on an enactivist worldview is scarce and disseminated over many disciplines, which made it difficult to locate. As I progressed through the study, I felt constrained by time. Many facets of enactivist theory still need to be forged from empirical observations. It is thus my conviction that the limitations of this study, in general, and both the methodological and procedural limitations, in particular, will spur other mathematics educators and prospective researchers to explore and investigate further.

The use of each of the data gathering tools and techniques in relation to my research questions is shown in Table 3.6.

Table 3.6: Match of data collection instruments and techniques and research questions

Research questions (study purpose)	Data collection instruments and techniques				
	Biographical Questionnaire	**Geometry scenario based questionnaire**	**Lesson Observations**	**Post lesson reflective interviews**	**Stimulated recall analysis interviews**
1. What are the teaching proficiency characteristics of selected effective mathematics teachers?		X	X	X	X
2. How do these proficiency characteristics inform teachers' classroom practice?			X	X	X
3. What are the teaching proficiency characteristics that are similar and different across the teaching practices of selected teachers?			X	X	X
4. What are teachers' personal experiences and characteristics enabling them to be effective?	X			X	X
5. What contextual factors shape these effective practices?				X	X
6. What mathematical proficiency and pedagogical content knowledge of geometry do mathematics teachers who are considered effective have, and need to demonstrate, in solving scenario-based geometry tasks?		X			
7. What factors outside of Kilpatrick et al.'s (2001) analytical framework characterise effective teaching practice?				X	X

3.9. CONCLUSION TO THE CHAPTER

In this chapter, I discussed the methods and rationale that were used to address the main research question: *what are the teaching proficiency characteristics of selected effective mathematics teachers*? The study is oriented within a qualitative research conceptual framework, and is situated within an interpretive paradigm. A case study methodology was adopted for its flexibility, suitability to naturalistic contexts, and its focus on gathering rich data from multiple cases. Details of the choice of the methodology, the chosen overall paradigm and data collection instruments used have been presented. The unit of analysis connected to the objective of this thesis has been specified. In addition, the selection criteria of the participants were described. Because of the chosen strategy of an intimate interplay between the data collection and analysis in this study, this chapter has offered a more than usual amount of detail on the methods and how they were implemented. The chapter ended with some consideration of questions of validity, ethics and limitations of this study. In the next chapter, I present the analysis of the collected data in accordance with the analytical approaches and theoretical framework that have been used for approaching effective teaching practices of this thesis.

CHAPTER FOUR

DATA PRESENTATION AND ANALYSIS A

TEACHER PROFILES AND TEACHER KNOWLEDGE

Effective teaching and learning are about knowing and knowledge, about being able to estimate the veracity of knowledge … It is also about the questions how we acquire knowledge and students should come to know (Niessen, 2007, p. 12).

4.1. INTRODUCTION

The overriding goal of this study was to explore and analyse selected effective mathematics teachers' teaching proficiency in the area of geometry. I observed and interviewed five mathematics teachers and had six interrelated research questions. The main research question was: What are the teaching proficiency characteristics of selected effective mathematics teachers? I had five data collection tools: (1) a biographical information questionnaire, (2) geometry content knowledge questionnaire in the form of scenario-based items, (3) classroom lesson observations and video recordings, (4) post lesson reflective interviews and (5) stimulated recall analysis sessions. In order to conduct my in-depth narrative analysis and maintain coherence in interpreting the results, an organised and systemic fashion of data analysis was undertaken. Separate chapters are devoted to the simultaneous presentation and analysis of each set of data that contributed to my understanding of effective teaching practice of selected mathematics teachers. Accordingly, the presentation of these findings is structured and discussed in **three main chapters** (Chapters four, five and six) which correspond to the phases of data generation and analysis as well as the research questions of this study.

This particular chapter presents an analysis of the teacher profiles and their contexts as well as teachers' mathematical content knowledge. It is structured and discussed in **two main sections**. The first section is split into two main parts. In the first part of this section, I analyse the contexts of the participating schools from which I collected the data. The second part of the section provides the profiles of each of the five participants. The last section is an analysis of their pedagogical content knowledge (PCK) through the lens of Kilpatrick et al.'s (2001) model. Each section closes with a summary discussion or conclusion synthesising the findings.

4.2. PROFILES OF THE PARTICIPANTS AND THEIR CONTEXT

In this first section I am going to introduce the participating teachers and their school contexts in Namibia. This section coincided with Phase II results and addresses the following two research questions: *What are teachers' personal experiences and characteristics enabling them to be effective* and *what are contextual factors shaping these effective practices*?

4.2.1. Part I: The profiles of the participating schools in which I worked

By using the initial school and teacher background questionnaire, the schools will be described in terms of their:

- location
- contextual factors (school type, historical contexts, institutional factors and cultural contexts)
- demographics of teachers and learners

The geographical position of the school in which the research was conducted is important. Contextual factors such as historical context may give the reader, a glimpse of the conditions in Namibia under which teachers' classroom practices were studied and analysed. Institutional factors and culture are also important in showing school practices that have been effective and the differences in resources and socio-economic profiles across the five schools in which the participants work. This is important as it clarifies the nature of the schools as well as the contexts in which geometry instruction takes place.

4.2.1.1. School A (Teacher 1: Demis)

School location

School A is situated in a town in central-north Namibia near the Waterberg National Park. This particular area was settled by German colonists in 1884. The German influence is evident in the Germanic architecture of the school buildings.

Contextual factors

School type

School A is a co-educational state secondary school (grade 8 to grade 12) which formerly served white children. It is partly-funded by the state and partly by donors from the private sector (local and international trusts, businesses, foundations and organisations). Thus the school is relatively well resourced and maintained. Teachers have their own classrooms, and students move from classroom to classroom for different disciplines. The school has two separate functional computer laboratories (one for the teachers and the other for the learners) and separate science laboratories for Biology and Physics/Chemistry. The school fees are N$ 400 per trimester per learner.

Historical factors

After independence, the school changed its medium of instruction from Afrikaans to English, with Afrikaans as a compulsory subject. The school has sustained an impressive record of academic excellence in Mathematics and other subjects. Many of the students who graduated from this school study at local and South African Universities. The principal says the reason for its success and reputation is due to the teachers' commitment and students' hard work.

Institutional factors

School A has a hierarchical structure of administration and management with a School Board and Parent Association, a Principal and Heads of Department (HoDs), with Subject Heads and teachers. As with other public schools, the school's end of year examinations for grade 10 and 12 are regulated by the Directorate of National Examinations and Assessment (DNEA) under the Ministry of Education. According to the principal, the school has been rated as one of Namibian's best schools. For example, in the 2009 grade 12 national examinations, the school was ranked in the top five out of over 1700 full-time schools and training institutions in Namibia.

Culture

At the time of data collection, the school was under the leadership of a vibrant and dynamic female principal who had been schooled there and taught there for more than 17 years. "Learning from one another by being with one another" was her core leadership philosophy to "championing the education system and governing the human resources" while also believing that "greater autonomy strengthens accountability" (2nd June 2011, Personal communication with the principal). The focus of the school is on rigorous academic training.

Demographics of teachers and learners

School A is made up of the principal, 28 teachers including one private German language teacher and 740 learners (of which 205 were in the hostel) in grades 8 to 12. Mathematics is compulsory for all students. The school had four qualified mathematics teachers including the HoD, who was involved in this study. Although the school currently attracts more and more non-resident students, the majority of learners are overwhelmingly white English and

Afrikaans speaking from family units with good education and a high economic background but the principal has plans to make it more inclusive. At the time of this observation period, the class size averaged 25 learners.

4.2.1.2. School B (Teacher 2: Jisa)

School location

School B is located in a northern town of Namibia, which has undergone rapid growth and development in recent years, and is much smaller than school A.

Contextual factors

School type

Founded 70 years ago, school B is a co-educational private secondary school for grade 8 to 12 students from lower and middle working-class families. It is state aided and run by the Roman Catholic Church. The church provides for the educational infrastructure and pays some salaries for additional teaching staff (e.g. nuns from India) including institutional workers. The Education Ministry provides essential facilities such as textbooks, stationery and pays salaries for permanently employed teachers. In addition, the school receives some financial and material support from non-governmental organisations and individual benefactors. Teachers have their own classrooms and students rotate from one class to another for different subjects. The school has two computer laboratories which are sufficiently equipped with functional computers and internet facilities. The school library is however moderately equipped and is being run by well-trained learners assisted by a librarian. The school fees per learner are N$ 300 per trimester and N$ 900 per year.

Historical factors

Being owned by the church, school B is characterised by a strong family atmosphere which was encouraged by the nuns since its inception. Due to the close link with the church, religious activities and basic education formed important aspects of the learners' everyday lives.

Institutional factors

School B also has a hierarchical management structure with a School Board, a Principal who oversees department heads who in turn supervise the subject heads and their teachers. As with government secondary schools, school B follows the Namibian official school curriculum regulated by DNEA.

Culture

The focus of the school is on whole learner development but also on rigorous academic training. This includes double periods and extra afternoon classes in Mathematics, Science and English as well as Saturday school and holiday programmes. Mathematics is a compulsory subject for all grades. Since 2005, the school performance has been outstanding, and the school was awarded trophies both nationally and regionally. To encourage learners to work harder, the school adopts a "mastery test programme" as a monitoring tool which is displayed in all offices and classrooms and is strictly monitored. In order to further learning the school involves learners in several national and regional extra mural activities such as Mathematics Olympiad, English debating and essay competitions.

Demographics of teachers and learners

The school employs 16 teachers. There are two qualified Mathematics teachers of whom one was a HoD. At the time of this study, the total enrolment from grade 8 through 12 was 365 learners. The largest class consisted of 36 learners.

4.2.1.3. School C (Teacher 3: Ndara)

School location

School C is located in a small town in northern Namibia, which is often flooded during the rainy season.

Contextual factors

School type

School C is an Evangelical Lutheran Church in Namibia (ELCIN) co-educational, state supported 'rural' boarding school for grades 8 to 12. The students are from lower-working class and working class families from four of the Namibian northern regions. The school is partly funded by the government but also relies heavily on a strong local support base, private sector, self-organised fundraising events and annual school fees to cover the salaries of additional staff and resources. Each teacher has his/her own classroom and students rotate from classroom to classroom for different lessons, it has a well-equipped library and computer laboratory. The current school fees stand at N$ 7 590 per year which is divided in three trimesters of N$ 2 530 per learner.

Historical factors

School C was founded in 1960 under the leadership of a Finnish national who was its first principal. Significantly, the school is surrounded by some special features which make it a tourist attraction. The school buildings adjoin a river which flows from Angola to the Kunene River. On its shores there is a famous fig tree "omukwiyugwemanya" (whose name is derived from the fact that it grows on a rock).

Institutional factors

The school has a hierarchical structure for school administration and management with a School Board, Principal, HoDs and teachers. School C also functions as a Cluster Centre. In addition, the school follows the national curriculum which is endorsed by the DNEA.

Culture

Teachers at school C are very caring, encouraging learners to persist, praising efforts and performance. There is individual attention and students are taught to work on the three "Ds": *Discipline, Dedication* and *Determination*. Like other schools in my sample, school C is renowned for producing top results for students in the country. Afternoons are used for two hour long supervised homework sessions and Mathematics extra lessons.

Demographics of teachers and learners

The school has an international teaching staff of 18 teachers including some from the Southern African Development Community (SADC) region such as Zambia, Zimbabwe and Nigeria. The school receives voluntary teachers from the United States every two years, who primarily cater for the mathematics and science disciplines. The school has 356 learners who all do mathematics. The school average class size is 20 learners, which is well below the recommended Namibian average of 35 per class.

4.2.1.4. School D (Teacher 4: Emmis)

School location

School D is situated in a former white suburb in the heart of the Namibian capital city Windhoek. The area where the school is located is populated by relatively wealthy black, white and Indian families.

Contextual factors

School type

School D is a co-educational private school owned by the Catholic Church. It largely caters for 'upper-middle class' families. The school has no boarding establishment; hence a 'day school' and parents drive their children to and from the school every day. The school caters for both primary (grades 1 to 7) and secondary phases (grade 8 to 12) located on adjoining property. The school is not subsided by the government but it is well-resourced and supported by a financially viable parent body as well as international organisations. Thus the school has world class facilities of all kinds, including an excellent library, laboratories and mini-labs in every classroom, and extensive sports facilities. At the time of this study, the school fees were N$ 28 140 for grade one, N$ 36 240 for grade ten and N$ 41 370 for grade twelve per year. The fees for expatriates were notably higher (i.e. N$ 49 250 for grade one per year).

Historical factors

Founded in 1962, school D is internationally recognised for its academic excellence. Traditionally, the school was established as a Catholic all boys' school with the ideal of nurturing the development of pupil's full potential. After independence the school became co-educational.

Institutional factors

The primary school is functionally and administratively run by a HoD who reports to the overall school principal. The secondary school follows the curriculum of the NSSC which is endorsed by the DNEA and University of Cambridge, and examinations are written at both the Higher and the Ordinary levels. The academic programme emphasises Mathematics, Science and English as first language.

Culture

School D is committed to providing an all-round, international, relevant education for both boys and girls in a sociable and caring environment. It is well known for its academic rigour and is regarded as one of the top academic schools in the country - a reputation that the school seems to guard and maintain. The majority of grade 12 graduates are selected for admission to top Southern African and other international Universities. Students seem especially committed to hard work because of the reputation of the school and parental expectations. Students are encouraged to participate in the Southern African English, Mathematics, Science and Afrikaans Olympiads, and are typically placed in the top 20.

Demographics of teachers and learners

School D services and caters for the upper middle class. The school has an international teaching staff consisting of 52 full-time and part-time teachers from the SADC region (i.e. Zimbabwe, South Africa) and outside the region, such as, Australia. In total the school had eight (8) qualified Mathematics teachers for both the primary and secondary sections. The school accommodates 587 pupils of which 362 were in the secondary phase. Class size averaged 22 learners, which is below the recommended average of 35.

4.2.1.5. School E (Teacher 5: Sann)

School location

School E is located in one of the holiday destination coastal towns of Namibia.

Contextual factors

School type

School E is a co-educational private day-school with a population from the upper and middle-class backgrounds. The school has two sections: the primary (grade 1 to 7) and secondary (grade 8 to 12) phases. As with other sampled schools, mathematics is compulsory for all grades. The school is fairly well resourced, with two-storey classroom blocks in fairly good condition. The school has two laboratories for Science and Biology which are well-resourced, and two excellent air-conditioned computer centres each with computers with internet access. The school fees for the year are N$ 1050 per month, which is N$ 12 600 per learner per year.

Historical factors

School E was founded in 1961 by the Roman Catholic mission and became a state school in 1995. Initially the school drew its pupils from English, Afrikaans and Germany speaking families. After independence the school opened its doors to all Namibians and rapidly became popular with black and coloured pupils from upper-class families. Former pupils have distinguished themselves in the academic, business, educational, religious and political fields in Namibia.

Institutional factors

With the home and the church being the core education partners, the school operates as a "professional learning centre". It engages the entire spectrum of professionals in coming together for learning within a supportive, self-centred community. The school prides itself on its commitment to a discipline “code of conduct”, as well as a hard work ethic and excellence.

Culture

Teachers at school E praised the team spirit and “togetherness” at the school saying they feel at home with fellow teachers and pupils. Teachers have high expectations for all students. Since its foundation the school has enjoyed outstanding grade 10 and 12 results, often obtaining a 100% pass rate. The institution prides itself in its rich and diverse variety of extra-mural activities, i.e. athletics, rugby and many others. A variety of enrichment activities are also offered for each grade level, viz. academic and sport tours and educational excursions. While school E is well known for its academic and sporting achievement it does emphasise personal growth and leadership development.

Demographics of teachers and learners

The school population is racially mixed. At the time of this study, the staff had grown to a teaching team of 40, with six qualified Mathematics teachers. The school had an interesting mix of white and black children (630 learners), with white pupils still predominating.

4.2.1.6. Summary and conclusion of the general profiles of the five schools

The five schools comprise two private and three state schools. The five co-educational schools are however more similar than they are different. They are renowned in their respective regions as effective schools and offer impressive learning environments. They are the best high schools and among the top 10 schools in the country according to the DNEA ranking. They all regularly achieved pass rates of 80% and above in the grade 10 and 12 national examinations, both regionally and nationally. They are all functional in that they are focused on quality teaching and time on task, in contrast to some other schools in Namibia. This means that each school starts on time; teachers do not dodge lessons and conduct lessons on time; teachers commit much of their time to conceptual teaching (as will be shown later in Chapter Six); teachers and learners adopt an attitude of hard work and teamwork; there are regular and flexible teacher meetings to reflect on learners’ performance, needs and progress, and there are incentives for teachers such as tours and additional financial bonuses. In addition, management closely monitors the quality of teaching. According to teachers, principals are supportive of efforts to improve teaching and learning in their schools, provide advice and explain work ethics.

As can be expected in the Namibian context, the socio-economic context of the participating schools is fairly high, though varied to some extent as indicated in the school profiles. School fees vary in my sampled schools from N$1 200 per year to N$12 000 per year in public schools and from N$28 000 to N$72 000 in independent schools.

All participating schools have a dedicated international teaching staff and relatively small class sizes. They all offer mathematics at higher level and mathematics is compulsory for all grades. This is the schools' decision to offer mathematics for all grades. They are reasonably well resourced in terms of instructional materials and have effective school-based assessment policies significantly different from the Ministry of Education official assessment norms. All these factors explain "professionalism", which is about being effective, creative and passionate about the mathematics teaching and learning, no more, no less! Table 4.1 below shows the summary description of the teachers' schools.

Table 4.1: Descriptions of the participating teachers' schools

Schools	**Teachers**	**School type**	**Residence type**	**Number of teachers**	**Number of Mathematics teachers**	**Learner: teacher ratio**	
						Population	**Ratio**
A	T1: Demis	Public	Boarding/Day	29	4	740/29	26:1
B	T2: Jisa	Private	Boarding/Day	16	2	365/16	23:1
C	T3: Ndara	Public	Boarding	18	4	356/18	20:1
D	T4: Emmis	Private	Day	52	8	587/52	11:1
E	T5: Sann	Private	Day	40	6	630/40	16:1

In the previous section I provided a description of each of the five schools from which I collected classroom data. I will now spend some time describing the five teachers in their individual context.

4.2.2. Part II: The individual teacher profile

Qualitative researchers aim to gather in-depth understandings of human behaviour and the reasons that govern such behaviour (Cohen et al., 2007). That is, qualitative research methods investigate the *why* and *how* of decision making as opposed to just *what, where* or *when*. Hence, a small but more focused sample is often preferred to a large sample. With this in mind, before I start with the analysis of their PCK, I introduce the research participants who kindly donated their time and expertise, and their vision of effective teaching. I will profile each teacher in terms of their:

- educational background
- professional experience
- interpretations of their own reputation
- role in the school, and
- mathematics classroom context

Details of the teachers were obtained from the initial background questionnaire (Appendix E: Section A) and interviews (Appendix G).

4.2.2.1. Teacher 1: Demis (School A)

Educational background

My first impression of Demis, a Namibian descendant, was that she was exuberant, energetic, vibrant and passionate about mathematics. Her professional qualifications include a three year bachelor degree in science (BSc) with mathematics as one of her majors, and a Higher Education Diploma (HED) in mathematics education. She spent time in the United States on an exchange program looking at mathematics teaching. She has earned subject teaching credential awards in numerous secondary schools and at many official subject occasions.

Professional experience

Demis started teaching in 1990. Her diverse teaching career has spanned two decades. She has taught mathematics in various schools around Windhoek and held various leadership positions such as subject head and HoD. Demis has been on the National Mathematics Curriculum Panel for over 10 years. She is also an author of grade 8 to 10 mathematics textbooks prescribed by the Namibian Education Ministry.

Interpretations of her reputation

In the interviews, Demis claimed to be well grounded in both mathematical concepts and their pedagogy. Specifically, she underscored, “engaging students in investigations and problem solving”, “connecting mathematical tasks to real life and practical contexts for students to see mathematics as worthwhile” and “having students talk and reason about taught concepts”. Demis also described herself as being committed to the mathematics teaching and bringing passion to the teaching and learning of mathematics. She is of the opinion that “her own teaching experience” is the most important factor in developing competence to teach effectively.

Role in the school

Demis described her role as being that of a facilitator. In addition to being both a HoD of the Mathematics and Science department and mathematics subject head, she also teaches mathematics grade 8, 10, 11 and 12. Demis serves the school on a variety of committees related to her teaching subject. These include the mathematics subject examination and regional mathematics and science fair committees.

Classroom context

Demis' mathematics classroom environment was very invigorating. Her classroom was neat and well organised with many mathematically based posters and charts pasted all over the wall. There are two chalkboards that Demis uses simultaneously when in class. Demis had many mathematics textbooks including her own published materials from which she selects tasks for students

4.2.2.2. Teacher 2: Jisa (School B)

Educational background

Jisa, an Indian nun, is a committed and passionate secondary school mathematics teacher. She has the following qualifications: a Bachelor degree of Science (BSc) and a Master's degree of Science (MSc) coupled with a Bachelor degree in Mathematics Education.

Professional experience

Jisa was in her second year of teaching at this school. Jisa' teaching experience includes teaching mathematics at grade 10 to 12 at her new school. She also taught mathematics in different school settings in her motherland in her ten year teaching career.

Interpretations of her reputation

Jisa said that she is an expert and experienced mathematics teacher with sufficient content knowledge. She feels that she is teaching mathematics with an emphasis on students' divergent thinking.

Role in the school

Jisa' core responsibilities are that of teaching mathematics to grades 10 to 12, and she also offers extra mathematics lessons in the afternoon.

Classroom context

Jisa' Mathematics classroom is neat, orderly and bright, with mathematics posters, charts, models and other instructional aides. Each learner has his own mathematics textbook, mathematical instruments and a calculator. The classroom was well-lit.

4.2.2.3. Teacher 3: Ndara (School C)

Educational background

Ndara, a Zimbabwean descendant, holds a grade 12 certificate 'A' level and a Bachelor degree of Science (BSc) in Mathematics and Statistics. At the time of my study he was studying towards a Post Graduate diploma in Mathematics with UNAM.

Professional experience

Ndara is a vibrant, committed and versatile mathematics teacher from Zimbabwe. He has a strong rapport with his learners, but his demeanour is decidedly business-like. His teaching subjects are mathematics and physics. Ndara has six years' experience teaching mathematics and physical science in secondary school. At the time of data collection, he was teaching grade 10 Additional Mathematics as well as Mathematics Ordinary and Higher levels in grade 11 to 12.

Interpretations of his reputation

Ndara referred to himself as being very good at both mathematics content and teaching it. He also valued sharing practices and experiences with other teachers. Ndara sees himself as successful in teaching acknowledging that "effective teachers are like 'pastors'". That is, his standing or role is to legitimate the profession and make mathematics education a symbol of expertise. Indeed, Kilpatrick et al. (2001) espouse that effective teaching depends on a knowledgeable, flexible and reflective practitioner (teacher).

Role in the school

Ndara is a mathematics subject head. His additional responsibilities include providing expert advice on effective ways of teaching mathematics, and helping teachers with strategies that shape effective teaching.

Classroom context

Ndara' mathematics classroom is typical of many Namibian public school classrooms, viz. limited to chairs and desks and chalkboard but also limited in other mathematics materials. Each learner had his own mathematics textbooks, calculators and mathematical instruments.

In addition his classroom houses the physical science laboratory. There is enough furniture, various mathematics textbooks and science materials.

4.2.2.4. Teacher 4: Emmis (School D)

Educational background

Emmis, a Cuban graduate, originally from Zimbabwe is a well-qualified and experienced mathematics teacher. He holds an A-Level certificate and a Masters of Education degree (MEd) in Mathematics and Computer Science.

Professional experience

Over the past 15 years Emmis has taught mathematics 'A'-level and 'O'-level as well as Mathematics International General Certificate of Secondary Education, Ordinary and Higher (IGSCEO/H) in many different school settings both in his homeland and metropolitan areas such as Windhoek. At the time of this study Emmis was teaching Mathematics (Additional, Ordinary and Higher levels) to grades 10, 11 and 12.

Interpretations of his reputation

Emmis takes the stance that mathematics should be enjoyable. He has a charming sense of humour and is dynamic, energetic, vibrant and enthusiastic about mathematics. Emmis described himself as a good motivator with a sound knowledge of mathematics subject matter and a strong notion of teaching geometry. He said he "plans carefully for lessons, makes the purpose and content explicit, uses systematic assessment and feedback, makes connections among concepts, and encourages learners to think about concepts while modelling what he wants learners to do/achieve".

Role in the school

Emmis was HoD of the Mathematics and Science department. He also plays a role in the overall school management, for example including advising the principal in the appointment of new teaching staff.

Classroom context

Emmis' classroom environment was particularly exciting in terms of teacher-learner ratio (22 learners), and the available space for the teacher to circulate and provide individual attention to each learner. His classroom was well resourced with varnished desks, chairs and cupboard, sliding chalkboard and many mathematics textbooks and materials, including his own teaching manual which was also given to each learner as a guidebook. Each learner had at least two mathematics textbooks, mathematical instruments and modern scientific calculator.

4.2.2.5. Teacher 5: Sann (School E)

Educational background

Sann, a South African descendant, is a qualified mathematics teacher and holds a Higher Education Diploma (HED) Senior Primary.

Professional experience

Over the last 15 years Sann had taught in a variety of school settings including the United Kingdom (UK). She taught mathematics in several schools, most notably high schools, where she taught Mathematics Ordinary and Higher levels and more recently she taught Afrikaans. During the observation period of this study Sann was teaching grade 10 Additional Mathematics, grade 11 Mathematics Ordinary level and grade 12 Mathematics Ordinary and Higher levels.

Interpretations of her reputation

Sann is a lively and vibrant teacher who has a partner-like attitude towards her learners. She believes that creating enthusiasm for mathematics enables pupils to see mathematics as benefiting them in life and for the labour force. Sann described herself as having a wealth of knowledge and expertise in the teaching and learning of mathematics. For example she said "I know the subject content very well. I use different resources to broaden my mathematical knowledge. My reputation at school is good as I always work to ensure a positive reputation. I like to build humour into my lesson while trying to influence the learners in the same way". The HoD described her as a good teacher who does an excellent job.

Role in the school

Sann is a Mathematics subject head. Her administrative duties entail helping other mathematics teachers with subject administration; ensuring that their teaching is of a high standard, moderating mathematics question papers, and offering workshops to share her teaching expertise with other teachers in the school.

Classroom context

Sann' classroom environment is well maintained, bright, colourful and orderly and has two chalkboards (one at the front and another at the rear). Chairs and desks were all properly varnished, facing both the rear and the front of the room to enable learners to see both chalkboards. There was an overhead projector which Sann uses as her main teaching resource. The wall is abundantly covered with a rich collection of carefully displayed mathematics posters i.e. graphs, geometric shapes and charts showing key mathematical principles. The teacher's desk had several mathematics textbooks on top of it.

4.2.2.6. Summary of the general profiles of the five teachers

To summarise, the participants are mostly female (3 of the 5), over the age of thirty years and reasonably well qualified. Four of the five have teaching qualifications in mathematics from either colleges of education or university, and have taught mathematics for at least five years. One of them did not have a teaching qualification but has a Master's degree in Pure Mathematics and Statistics. Four of them are expatriates. All have a good standing in the community and are committed teachers, and their learners are generally high performers in the national examinations.

They all claimed they possess a wide range of mathematical content knowledge. They expressed confidence in their teaching practices. They claimed to have a positive relationship with their learners. They said they have the abilities, attitudes, skills and dispositions which they seek to make relevant to students. They also draw from their personal experiences and from the teaching methods of their former teachers to enhance their learners' mathematical understanding. According to them, they work well with their principals and other teachers in their schools. While acknowledging that their effectiveness is partly shaped by effective management and leadership, they also claimed to develop clear expectations for academic performance, rules, procedures and guidelines as part of their responsibilities to create a successful learning environment and support mastery of the learning content. All five teachers in this study see themselves as part of a large community of practice of mathematics educators and learners. They share common goals, effective practices and visions of change which provide a common identity with which they can relate to as effective teachers of mathematics.

The study now turns to the analysis of the teachers' mathematical proficiency (MP) and pedagogical content knowledge (PCK) according to the Kilpatrick et al.'s (2001) model.

4.2.3. Analysis of teachers' mathematical proficiency (MP) and PCK

In this second part of section A, I analyse the teachers' responses to the items in the second questionnaire which I called a geometry scenario-based questionnaire. This was designed to assess and explore their geometry content knowledge. These findings correspond to Phase III of data gathering. Data from this source were collected, analysed and interpreted in order to explore the following research question.

- *What mathematical proficiency and pedagogical content knowledge of geometry do mathematics teachers who are considered effective have, and need to demonstrate, in solving scenario-based geometry tasks?*

The analysis of teachers' responses showed how each participating teacher interpreted and solved the scenario items, and describes their reactions to students' suggestions and disputes. The data analysis that follows represents the findings of this questionnaire in relation to the first four strands of MP, and is based primarily on qualitative description. Deconstructing teachers' content knowledge in a rich and qualitative manner involves lengthy descriptions and analysis. For the purpose of this thesis I will only present teachers' responses very briefly owing to the subsequent lengthy descriptions and analysis. I only analyse and report in detail on two scenario tasks (2 and 12), and show how teachers responded to those tasks and provide evidence of the four Kilpatrick's strands. The teachers' geometry content knowledge questionnaire is presented in Appendix H with their responses to this questionnaire presented in Appendix I.

4.2.3.1. Scenario items

Scenario 2 and 12 which are discussed in detail were presented as follows:

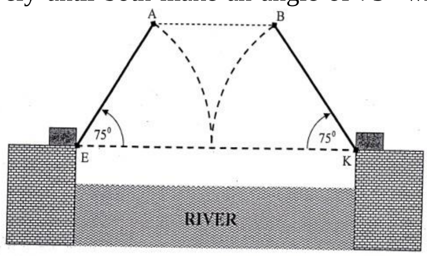
Scenario 2

A group of students in your class regularly pose problems for one another to solve in class. One of them had brought a problem, which the others could not solve. The problem was to find/determine the length of AB (rounded to 2 decimal places) as shown in the diagram below. The diagram illustrates a bridge positioned over a river. The bridge is made up of two arms AE and BK equal in length, such that AE = BK = 25 metres. The arms AE and BK rotate upward about E and K respectively until both make an angle of 75° with the horizontal EK in order to allow ships to pass through.

"It is not possible to find the answer", said one of the students. "Let's go to the teacher". They all agreed to come to you.

(a) Without providing the solution to the problem, how would you help the students to understand this problem by suggesting a number of "hints"?

(b) What is your solution to this problem?



This task measures teachers' CU, PF, AR and SC. The task in the scenario is practical in that using a draw bridge involving a trapezium is a familiar context for them. The scenario is intended to assess the manner in which teachers maintain precision in the process of solving a problem while attending to the details in the diagram. The task also addresses important applications of the trigonometry ratios which are recurrently tested in the Namibia Grade 10 and 12 national examinations. Central to understanding and completing this task is making connections to elements of mathematical practices as stated in the Kilpatrick et al. (2001) analytical framework. There is a diversity of mathematical strategies that can be used to solve this task, including algebraic geometric series. The pedagogic point is that each of these strategies allows teachers an opportunity to reveal their preferred problem solving approach and the way they emphasise mathematical ideas arising from their learners' investigations in their real classroom teaching. Since the procedure to solve this item was not suggested in the task, a number of strategies were identified by the teachers that might be of use to the students in understanding such a problem.

The student in scenario 2 said it was not possible to find the length of AB. To answer the first part of the question which requires teachers to help the students understand the problem, the hints to the students could be: to construct a right-angled triangle and a rectangle inside the trapezium such that opposite sides of a rectangle are equal. Teachers can then apply trigonometry ratios to the right-angled triangle to get $AB = 37.06m\ (2dp)$. In general, it was observed that all participating teachers enjoyed this task as they got the same answer and were procedurally correct in their steps.

In answering the first part of the question, Demis, Ndara, Emmis and Sann gave good hints, which is to divide the diagram to create a rectangle in between and two right-angled triangles within the trapezium (CU2, CU1) and to ask students to use trigonometry ratios to get to the answer (PF3, PF4). Conceptually, they suggested that they would first ask learners for the length of EK, and then how far apart A and B would be when horizontal in the new diagram they sketched. Coming to the second part, procedurally, these teachers arrived at the answer through the same hints they gave to the students. That is, they drew constructions lines to create a rectangle and two right-angled triangles within a trapezium and then applied the trigonometric ratios (Sine and Cosine rule) accurately, correctly and appropriately (PF1; PF2) to find the length of AB. It was noted that Jisa immediately recognised the relationship between EK, AE and BK (CU3) without drawing new diagrams as the other four teachers did. In terms of the hints to the student, Jisa gave the whole formula to get to the answer (PF1), instead of leading the students to an understanding of how to achieve the solution. In her solving approach, she did indicate x, which I believe should have been indicated on the diagram which she did not sketch as the others did.

Scenario 12
A cylindrical tin full of engine oil has a diameter of 12 cm and a height 14 cm. The oil is poured into a rectangular tin 16 cm long and 11 cm wide. What is the depth of the oil in the rectangular tin?

Prepare a marking scheme to assist other Mathematics teachers in the region to assess their students' responses to this question in the end-year examinations. The marking scheme needs to reflect how you give credit for conceptual AND procedural understanding.

This last scenario is a procedural task focusing on teachers' fluency in doing step by step procedures in working out the task. It also focuses on teachers' strategic competence in problem solving to formulate and facilitate procedures as it requires teachers to prepare a marking scheme to assist assessment of students' responses to this task. As the scenario calls for the use of procedures, it also requires teachers to investigate the conceptual meaning and explanation of volume of a cylinder and rectangular prism. Since this scenario requires two or more procedures, it calls for connections to the underlying meaning of concepts. Hence this scenario measures teachers' CU, PF and SC as it places additional algebraic demand on the teachers, as they have to relate the volume of a cylinder to the volume of the rectangular prism. It also requires teachers to experience that volume, as a geometric concept, is much more than just memorising and applying its formulae. For example, by sketching the cylindrical and rectangular tins, the volume of oil in a cylinder tin can be computed as $\pi r^2 h = (3.142)(6^2)(14) \approx 1583.57cm^3$. The volume of oil in a rectangular tin is then calculated to give $16 \times 11 \times h = 176hcm^3$. The two volumes can now be equated to determine the depth (h) of the oil in a rectangular tin as follow $176h = 3.142 \times 36 \times 14 \Rightarrow h = \frac{3.142\times36\times14}{176} \approx 8.99cm$. For the purpose of assessment, a mark could be given for the formula of finding the volume of oil in a cylindrical tin and another mark for correct substitution into the formula. An accuracy mark could be given for finding the value of $3.142 \times 6^2 \times 14$ to at least three decimal places. Again one mark could be given for the formula $16 \times 11 \times h$, another mark for the equation $16 \times 11 \times h = 3.142 \times 36 \times 14$ and the final mark for the correct value of h. In total this would be six (6) marks for this task.

Responding to this scenario, all teachers recognised that this problem arises in real life and the classroom. For example, Demis, Ndara and Emmis sketched the diagrams of a cylinder and rectangular prism respectively (CU2). They then executed mathematical procedures and algorithms correctly, accurately and appropriately (PF3) to determine the depth of oil in a rectangular tin. Without sketching the diagrams of a cylinder and prism, Jisa and Sann applied mathematical understanding along with mastery of operations (CU1) by employing mathematical procedures and formulae flexibly, accurately and correctly (PF4) to determine the depth of oil in a rectangular tin.

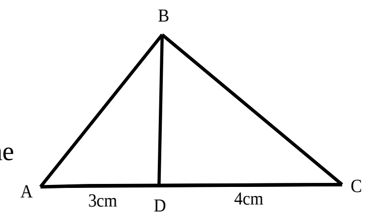
Only four teachers provided a conceptual explanation of how they could assess learners' responses to this task. For example, Demis allocated six (6) marks for finding the volume of oil in a cylindrical tin, i.e. one mark each for the formula, correct substitution and correct value. She also gave a mark for the formula of the volume of oil in a rectangular tin and another mark for correct substitution plus a mark for the value of h. Jisa assigned five (5) marks in total for this task. Two marks for the correct volume formula of oil in a cylinder and correct answer given to two decimal places, one mark for the correct volume formula of oil in a rectangular tin, one for correct substitution into the formula and another mark for the correct value of h. In total Ndara gave three (3) marks to this task by clearly specifying the credits for the steps. For example, he allotted one mark for the correct volume formula, substitution and answer of the oil in a cylindrical tin, another mark for the volume formula, substitution and expression of the oil in a rectangular tin plus a mark for the correct value of h. Sann indicated how she could assess the students' responses to this scenario task. She allocated six (6) marks in total to this task. A mark for seeing the value of the radius and two marks for the correct volume formula of oil in a cylinder, substitution and answer given to two decimal places. She also gave one mark for showing or writing in word that the volume of oil in a rectangular tin must be equal to 1583.567 (the oil volume in a cylindrical tin). In addition she awarded two marks for a follow through if, for example, the learner calculated the volume of oil in a cylindrical tin wrongly but have employed method and formulae efficiently and flexibly to get to the ensuing answer.

The following ten scenario tasks contained in the questionnaire are discussed very briefly:

Scenario 1

Imagine that three learners in your class come to you with a mathematical argument. The argument is about how to find the length of BD (in simplest surd form) in the diagram below with the following information:

- Δ ABC is a right-angled triangle (right angle at B).
- The perpendicular line BD intersects AC at D.
- AD = 3cm and DC = 4cm.



One of the learners says the length of BD is the same as the length of CD.
Another one says BD is the same as AB.
The third student says BD is two times the area of ΔABC divided by seven.

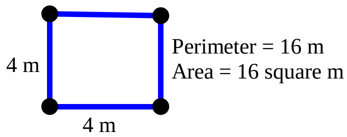
(a) How would you respond to each of the learner's responses?

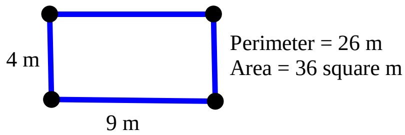
(b) Clearly explain and illustrate one correct strategy to solve the problem (recognising that there may be more than one way to solve it).

This question focuses on teachers' PF. It also focuses on teachers' pedagogical content knowledge which Kilpatrick et al. (2001) refers to SC. The question in the scenario focuses on the application of the Pythagoras theorem which participants encounter in their daily teaching of geometry, and so the context and the task would be familiar to them. According to the first student, $BD = CD$. This implies that $BD = 4cm$ and that ΔBDC is an isosceles triangle, hence $BD = 3cm$ and therefore $3cm = 4cm$, which is a grave contradiction. According to the second student, $BD = AB$. This means ΔABD is an isosceles triangle, hence $\angle ADB = \angle BAD$ which is inconsistent. This scenario item was answered by all teachers in respect of each learner's response. To answer the three students, teachers used information based on their investigations of the problem situation (CU4), and worked strategically (SC1) using accurate mathematical procedures and skills (PF1, PF4), thinking or reasoning skills (AR1), and appropriately concluded that the third student is correct.

Scenario 3

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told them. She explained that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you these pictures to prove what she is doing, and then asks you: *Is this always the case*?





(a) How would you respond to this student?

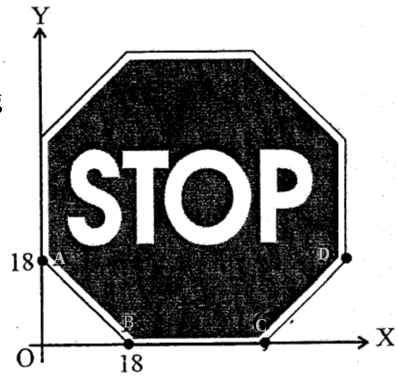
(b) Explain in detail how you would plan and implement a set of lessons that deals with "this theory".

The task in this scenario is primarily abstract or theoretical and can be approached by using a combination of inductive and deductive approaches through testing and justifying conjectures. It thus requires teachers to support or refute a conjecture or proposition made by the learner. Appropriate conclusions may be reached by considering various possibilities of increasing or decreasing the side lengths of various closed shapes. It thus assesses teachers' mathematical practice, which Kilpatrick and his colleagues termed SC, and teachers' content knowledge or CU and AR. By responding to the student, teachers clearly and precisely constructed viable arguments to support their own reasoning and critique the reasoning or finding of the student in question. Teachers also constructed quality counter-examples to explore and evaluate the truth of the student's proposition, thereby helping students to gain conviction or certainty about the effect of increasing the perimeter of a shape on its area. Similarly, teachers analysed the scenario by making conjectures and building logical progression of statements. Specifically, they used multiple representations to encourage the students to experiment themselves while reflecting on and explaining concepts through a sequence step or accurate mathematical language and notation (AR3). Procedurally, teachers attempted to achieve conviction by using a more generalised case (i.e. investigation and discovery) and were conclusive in their investigations.

Scenario 4

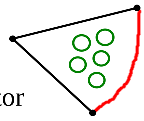
Half of your class find it difficult to internalise the concept of a linear equation. To help them understand this in teaching geometry, you choose a "regular octagon" stop sign, which is ubiquitous on the Namibian roads, and place it in a Cartesian plane as shown in the figure below.

Show your class at least two (2) ways of finding the linear equation of the line CD. Clearly illustrate your steps.



This item was included in the questionnaire to explore or measure teachers' understanding of mathematical concepts, procedural knowledge and their pedagogic approaches to teaching the concept of linear equation. The item reflects a real-world mathematical task, and allows for multiple approaches and demonstrations of conceptual knowledge and skills such as analysing information presented in the scenario. All five teachers answered this question successfully, though none of them illustrated at least two steps. Procedurally, Demis used the formula of the straight line equation and then found the values of the m (gradient) and c (y-intercept) of line CD, which finally led her to the answer. Jisa and Emmis recognised that by extending lines AB and CD, the y-intercept for CD would be -36, and the gradient (m) of AB and CD would be -1 and $+1$ respectively (CU2). Ndara used construction or sketch without a procedure at a different depth of understanding (CU4) to determine an interior angle of a regular octagon, the angle between line CD and the x-axis and a $45°$ right-angled triangle (PF2). Sann immediately recognised that the distance length between two vertices of a regular octagon is constant (CU3). She then used the Pythagoras rule and the linear equation formula (PF4) to find the lengths of line AB and line BC as well as the coordinates of C and D respectively.

The fifth item for consideration was as follows:

Scenario 5

Imagine a sector of a circle with five small identical circular holes of diameter 1 cm, as shown in the figure on the right corner. The radius of the circle is 8.4 cm and the angle subtended at the centre is 60°. Your grade 10 students were to calculate the area of the sector (and justify their answers), and three students came with the initial thoughts below:

Student A: The area of the sector is the same, because it will cover exactly the same portion without the circular holes.

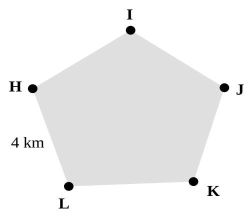
Student B: It is the area of the sector minus the area of the five circular holes, because if you cut out the sector from a paper/cardboard, the circular holes will not be part of it.

Student C: The area is the area of the sector plus the area of the circular holes, because they are all plane surface and the area is the total of all the areas.

How would you help each of these students so that they understand the correctness or incorrectness of their responses? Also provide the correct answer with a full explanation.

This question focuses on teachers' conceptual and procedural knowledge. The item also draws on knowledge and skills that are articulated in pedagogic knowledge or strategic competence as it requires teachers to look for both general methods and shortcuts to get to the answer. As it stands, the question has some generalised mathematical thinking. It thus measures teachers' mathematical reasoning as well. To do this, the teachers could judge three response options and then provide the correct answers with a full explanation or justification. In this case only student B is correct. It was important to note that all five teachers responded to this scenario task in similar ways, and gave carefully formulated explanations. Demis, for example, explicitly argued that student A may be correct depending on whether the question states without the holes, which can be subtracted, though she did not say the student is incorrect. She also argued that student B is correct and procedurally illustrated her methods to attest the student response (CU4). She stated that student C may be correct if the question says with holes, which are then cut out and not added without giving sufficient reasons. For Jisa, both students A and C are wrong and student B is right, and she gave reasons to explain and justify students' responses (AR2). By responding to all three students, Ndara suggested a number of strategies to reach the conclusion with sufficient reasons provided (AR2). Emmis said student B is correct and suggested using student A and B situations to explain why student C is wrong (AR2). His conclusion was that student A may be right or wrong depending on the situation, which is with holes or without holes, though, in this case, there is one situation presented. He also concluded that student C is incorrect giving precise reasons, which is subtracting and not adding the area. Sann concluded that student B is correct and her reasoning is sensible (AR2). Similarly, she concluded that student C is wrong which is correct.

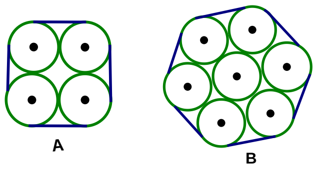
Scenario 6
Imagine five village headmen in northern Namibia who want to install a water pump so that it is *equidistant* from all five villages. The five villages (H, I, J, K and L) form a regular pentagon, and the distance between adjacent villages is **4 km** as shown in the diagram below.
A learner from the house of one of the headmen suggests the distance from any village to the water pump is 3.4 km.
How would you react to this suggestion? Is this learner correct?



This scenario item is situated in a daily life context of the participating teachers, and therefore is classified as measuring teachers' CU and PF as well as their SC. Again, the item is not intended to provide a setting for the teachers to demonstrate a particular data analysis, but to draw conclusions in a realistic setting using a range of problem solving techniques. The competencies needed for the task in this scenario are certainly relevant and part of the geometrical constructions that teachers encounter in the teaching of geometry. The scenario specifically requires teachers to do constructions of locus, isosceles or right-angled triangles within a regular pentagon to conclude appropriately.

By engaging with this scenario task, all teachers appropriately specified mathematical practices associated with the demand of this item. For example, all five teachers responded to this scenario through constructions using a ruler and protractor (CU2), and argued that the learner is correct. In this way teachers immediately investigated by drawing triangles or simply calculating the interior angle of a pentagon, and then applied the trigonometry principles (PF4) to prove the learners' answer. Specifically, they drew an isosceles triangle to find the centre angles of a pentagon by dividing the angles at the centre by 5 to get 72°. In some cases teachers constructed the bisector of all interior angles of a pentagon using straight edges and compass to find the point, where all angle bisectors meet, which is equidistant from all villages.

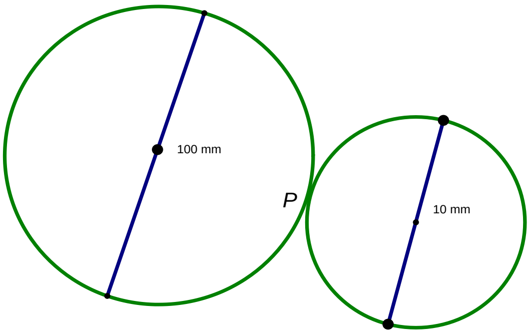
Scenario 7
The diameter of all the pencils in both A and B is **7 mm**. How would you help your learners to find the lengths of the elastic bands that hold the pencils together? For each diagram illustrates your strategies.



This procedural task assesses teachers' SC central to problem solving, facilitating procedures and to make connections to the meaning underlying the concept of measuration of plane shapes. The scenario is incorporated to assess how teachers could work with similar mathematical tasks and manage to maintain or increase the capacity for logical thoughts, reflection and explanation in their teaching practice. The scenario task also measured the extent to which teachers may recognise that there is a meaning behind the usual procedures or algorithms prominently used in mathematics, and that it is important to know "how" but also "why" those procedures work in real life contexts. As in class, to solve this task, teachers are required to employ multiple procedures in their solution which are not readily explicit in the task statement. They also need a deep understanding of the concepts of a circle and the relationship among its principal parts (i.e. radius, diameter, sector arc length, angles subtended at the centre by each sector, etc.) which are also not explicit in the statement of the task.

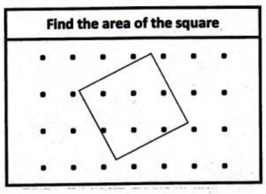
By answering this question, all five teachers recognised that the required procedure is not single, but multiple within a single task or problem. They hence analysed the tasks in both diagrams A and B, constructed and used mathematical models to interpret and solve the problem. Such as in diagram A, Demis, and Emmis applied mathematical understanding along with mastery of mathematical operations (CU1) and relationships (CU3), and carried out procedures in explicit ways (PF2) to work out the arc length. They then used the isosceles triangle base angles and the angle of the sector subtended by the elastic band at the centre of the circle to find the total length of the elastic band. In both diagrams, Jisa placed particular importance on the relationships (CU3) between the sector of a circle at each corner or circumference of a quarter circles, radius/diameter in the whole diagram and straight edges. For diagram A Ndara formulated and carried out a plan of problem solving using drawings and other sketches (SC1) as well as appropriate solving strategies (SC2) to arrive at the answer. He competently drew straight lines connecting seven pencil midpoints of the shape in diagram B which he then divided into a hexagon and several isosceles triangles. After that he used properties of the interior angles of a hexagon and the principle of isosceles triangle to calculate the size of angle at a point in B which is vertically opposite to the sector formed by the corner of circles. Sann recognised the sector or arc length at each corner of diagram A, which would make a full circle when put together and showed that the distance between the centres of two adjacent pencils is $7mm$ (CU3). She also showed that there were $6\ lengths$ of $7mm$ each from the centre of each pencil to the next in diagram B.

Scenario 8
Two friction wheels are of diameter 100 mm and 10 mm respectively. They touch at P and rotate without slipping [as shown below]. You asked your students to calculate the number of turns made by the small wheel when the large wheel rotates through 70°. But, none of your students was able to solve this problem. How would you explain the problem to your students (describe in detail your recommended strategies with a full explanation of each step)?



This simple shape item is classified mainly as a "procedural task" since it measures teachers' knowledge of procedures and concepts involved. The scenario further allows teachers to demonstrate their SC or pedagogical content knowledge. On the other hand, this question also calls on higher-level understanding of the concept of a circle and the meaning underlying the concepts of the diameter and the length of arc or circumference of a circle. Although the solution strategy is not explicit in the statement of the task, the task suggests the solution pathway requiring procedures, but with connection to the meaning that underlie mathematical concepts involved. Thus, by engaging with this task, all five teachers clearly analysed the relationships (CU3, CU4) between the two wheels in similar ways, applied mathematical procedures (PF1) and successfully concluded that the small wheel turns $1.94\ times$ which is only one complete turn.

Scenario 9
The problem on finding the area of the square in the diagram below is highlighted as an exercise task in the NSSC Mathematics textbooks. A group of learners from another Mathematics teacher class in the school come to you with this Mathematics dispute: "this problem has no right answer, unless you measure the spaces between the dots and then rotate the square".
(a) How would you find the area of the square?
(b) How would you react to the learners' dispute?



This scenario about finding the area of a square specifically focuses on teachers' CU and AR. The scenario is about working on the square area conceptually. Thus, the question requires teachers to generate possible strategies to solve the task, and then evaluate how effective they themselves or those strategies are to get to the answer. The aim here is to have teachers realise that there is more to area than formula, and have them make sense of what it would mean to work within the structure and construct chains of mathematical reasoning without specific guidance being provided in the scenario statement. By doing this, teachers are required to make a reasonable estimate of the area of a square as the problem in the scenario involves a more sophisticated idea, that of comparing the perimeter and area of a square in relation to the grid dotted rectangle. This further provides access to teachers' understanding of the concept of area and the relationship between the dimensions of the square and the grid dots that form a rectangle.

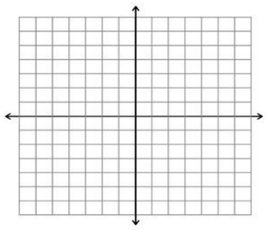
The learners' suggestion was that if the square is rotated in such a way that the vertices might coincide or overlap with the grid dots, then the area remains the same. The learners are making some sense as they seem to realise that by rotating the square, the area remain the same and hence the basic algorithms or formulae could be directly applied. Though some teachers attempted to investigate with formula, they got stuck and irritated due to possibly the absence of dimensions on the square. What is also surprising about the teachers' responses is that even though many recognised that the sketch is not drawn to scale, they supposed it was to scale.

Scenario 10

A quadrilateral has the following coordinates: M(-2; -1), A(0; 5), T(6; 3) and H(x; y), and is to be drawn on the Cartesian plane below. Point S is the midpoint of MT. The diagonals MT and HA of the shape MATH are perpendicular and bisect each other.

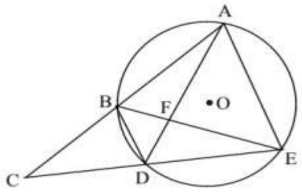
Student A suggests that the gradient of HA equals to - ½.
Student B suggests that MATH is a square.
Student C suggests that MATH is a kite.
Student D suggests that the coordinate of S is (2; 1).
Student E suggests that the equation of line MH equals to 3y – x = 5.

(a) Which of these suggestions are correct?
(b) Put yourself into the shoes of each student and suggest how each student arrived at his or her suggestion.



This item asks teachers to respond to a mathematical task situated in a teaching context, hence is classified as measuring teachers' SC, CU and PF. The scenario item is procedural in that a teacher has to look for a strategy to employ in solving this problem or responding to students' suggestions. The teacher has also to assess how effective she/he or the procedure employed is. Additionally, the item requires more analytical thinking and reasoning as the task in this scenario explores the rule for manipulation of formulae and linear equations logically. All teachers answered this question, and to a certain extent gave varied reasons about how each student arrived at his suggestion. Teachers sketched a quadrilateral with the diagonals bisecting each other (CU2, CU4). They then formulated and solved linear equations using algebraic methods (SC1). This allowed them to judge students' approaches in solving this problem or how students arrived at their answers and to identify correspondences between approaches suggested by students.

Scenario 11
In the given diagram below, O is the centre of the circle, BD: DE: DA: AB = 2: 3: 4: 3. The line segments CBA, CDE, AD and BE are straight. Find angle DFE.
Write down as many solutions to this problem as you can and provide a marking memo for each solution.



This scenario is mainly a procedural task and allows teachers to draw on their CU in order to execute the mathematical procedures involved. The scenario specifically requires teachers to solve this task by drawing on competencies of congruency of triangles in a cyclic quadrilateral in relation to the principle of "angle-angle-side and side-angle-side" theorems and as well as the relationship between angles subtended in the same segment by the same chord or arc. In this way, the task provides access to teachers' understanding of the theorem of secant from an external point of a circle that makes two chords. As evident in its statement, the scenario seeks an understanding of how ΔABD relates to ΔEBD and so ΔADE and ΔABE in relation to chords AE, BD, BE and AD. Therefore it requires the comparison of the lengths of the given chords that form a quadrilateral ABDE and interrelationship of angles in congruent triangles. However, only two teachers (Demis and Ndara) attempted to answer this question with varied answers. Demis recognised that $\Delta ABD \equiv \Delta EBD$ and thus $AD{=}BE{=}4$ (CU3). She then applied the Cosine rule and found that $\angle FAE{=}46.6°$, $BF = 2$ and assumed that ΔBDF is an equilateral triangle. Ndara recognised that angles subtended by the same chord in the same segment are equal (CU3). In addition, he applied the Sine rule and used triangle properties to prove that $\angle FAE = \angle FEA = 46.6°$, and hence in ΔFAE, $\angle FAE + \angle FEA = 93.2°$ which is equals to ∠DFE (exterior angle).

4.2.3.2. Summary of teacher knowledge of mathematical proficiency in geometry

Thus far, I have analysed MP and PCK characteristics of teachers in my sample. This analysis was conducted for the purpose of understanding the content knowledge of the participating teachers of mathematics in terms of the first four strands of MP. Based on this analysis, I provided evidence of teachers' mathematical proficiencies and their pedagogical content knowledge.

I employed the coding system as discussed in Chapter Three (see section 3.4.8.1.1.). However, the coding of responses with respect to the strands of MP was made difficult by the nature of the teachers' responses. In the context of the teachers' responses to the scenario-based items, it was difficult to assess CU and PF with a sufficient degree of validity and independently of each other. It was not explicitly clear how I could find out whether a teacher's response was rooted in conceptual or procedural knowledge or, to different degrees, in both. I thus argue that for my purpose the strands of Kilpatrick et al. (2001) analytical framework are not fine-tuned enough to discern and adequately describe (characterise) effective teachers' geometry content knowledge in relation to their responses to the scenario items. In particular, I argue that the four strands of MP are so interwoven that they cannot be measured independently of one another.

Despite the above dilemma, the item analysis of the questionnaire showed that the teachers' responses varied slightly over question items. When I looked at the tabulated teachers' responses (see Appendix I), all the teachers got the same answers but used different solving approaches. Teachers also interpreted and responded differently to the same mathematical scenarios and completed the same scenario items in different ways. However, there is one interesting finding for the five teachers. With respect to how well these teachers answered the scenario questions, their responses showed that all five teachers have a high level of MP or knowledge that would make them effective mathematics teachers. The data further indicated that these teachers clearly understand the content and demonstrated the first four strands of MP, namely, CU, PF, SC and AR. In terms of their PCK, these teachers are highly qualified, proficient in geometry and are able to see mathematics in a pedagogical context. By engaging with the same or different scenarios, some teachers showed predominantly CU, some predominantly showed PF while in other cases teachers demonstrated SC as well as AR. It could be argued that teachers' productive disposition (PD) towards mathematics was also evident through their different approaches to solving the tasks in the scenarios. This is because teachers' responses revealed their habitual or rather preferred problem solving strategies when teaching geometry in their mathematics classrooms.

Specifically the analysis indicated that all five teachers were proficient in terms of CU. The data revealed that the case study teachers were able to apply this knowledge to scenario questions related to investigations or questions that are more contextualised in real life situations. This implies teachers' internalisation of CU allows them to venture into investigative tasks or teaching. Teachers did also use trial and error methods to solve some questions.

The teachers were very proficient in PF. In this regard teachers were able to explain and justify conceptually and procedurally their mathematical solutions. This means, teachers could relate their calculations to the underlying mathematical meanings, concepts and ideas involved in the tasks. However, when some teachers were confronted with tasks slightly out of the ordinary and unfamiliar to them, such as tasks 3, 9 and 11, they found the tasks fairly difficult. It is important however to remember that CU and PF are not mutually exclusive. This is supported by recent studies that have shown that one supports the other (Rittle-Johnson et al., 2001).

The teachers were very proficient with SC in formulating, representing and solving most of the mathematical problems. They were procedurally fluent with solving mathematical problems, even though some teachers' solutions showed procedural errors in some cases or steps. When these teachers engaged with the scenario tasks that required SC, their responses showed evidence of CU in PF or vice-versa. Additionally, their responses to various scenario items seemed to indicate that the teachers in my sample teach geometric concepts, in a variety of formats and forms of representations.

With respect to AR, teachers displayed understanding of the meanings of mathematical concepts in scenarios they engaged with and reasoned mathematically with those concepts. They further noticed the links with other areas of mathematics, reasoned through sequences of calculation steps, and justified and explained their results or responses. They also made interpretations of their results/solutions as well as making generalisations or conjectures with counter-examples.

4.2.4. CONCLUSION TO THE CHAPTER

In this chapter, I presented the profiles of the participating teachers and a description of their contexts in which the study took place. I also provided an analysis of their mathematical proficiency and pedagogical content knowledge in geometry content. The analysis of the teachers' mathematical content knowledge and pedagogic approaches vis-à-vis the first four strands of Kilpatrick et al.'s (2001) model as assessed by the scenario-based geometry tasks indicated that the teachers have a solid and good understanding of MP and mathematics content. The analysis further showed that the five teachers have a rich pedagogical knowledge. The next chapter presents and discusses the findings and analyses of these mathematics teachers' own understanding of their effectiveness and perceptions of the multitude of factors that contribute towards their effective teaching practice using the lens of Kilpatrick framework.

CHAPTER FIVE

DATA PRESENTATION AND ANALYSIS B

TEACHERS' PERCEPTIONS OF THEIR EFFECTIVENESS

5.1. INTRODUCTION

In this chapter I present the participating teachers' personal perceptions of factors that contribute towards their effective practice. I mainly make use of Kilpatrick et al.'s (2001) model to answer the following research question: *What are teachers' personal experiences and characteristics that enable them to be effective and what contextual factors shape these effective practices*? I also considered factors outside Kilpatrick et al.'s (2001) model in my analysis. For this question, much of my evidence related to Phase V of the interview findings. For the scope of this chapter I only present the interviews with Teacher 1 (Demis), Teacher 4 (Emmis) and Teacher 5 (Sann). This is because these teachers provided particularly rich data. The five strands of Kilpatrick framework provided a useful structure to extract and analyse teachers' perception of their effectiveness and explore the nature of their teaching proficiency in more depth.

5.2. ANALYSIS OF TEACHERS' PERCEPTION OF THEIR EFFECTIVENESS

5.2.1. The teachers' view of mathematical proficiency (MP)

One of the questions in the post reflective interviews allowed teachers to articulate their understanding of the term *mathematical proficiency*. Demis reminded me that MP is a very broad concept of effectiveness to define and encompasses many teaching related issues. For example, she said:

> *Proficiency is a broad concept and it looks so much like it is a professional job. It includes the competency that you think an effective teacher should have* (Demis).

Another teacher added:

> *I think MP is like having sufficient knowledge in mathematical concepts, being able to tackle mathematical problems without a lot of conceptual understanding problems and being able to interrelate concepts. Basically it means, in my own opinion, having a sound knowledge of the subject content* (Emmis).

In agreement, another teacher pointed out that:

In general, I think it is…to take the mathematical concepts and carry them over to learners so that they understand what is actually meant (Sann).

The remaining teachers understood MP in terms of a sound understanding of mathematical concepts without violating mathematical principles. Other phrases that cropped up repeatedly in the teachers' articulation of MP included strong subject content knowledge and understanding of mathematical ideas. Whatever the most acceptable definition of the term in question might be, it was explicitly clear in these teachers' responses that they had strong feelings about their mathematical proficiencies.

5.2.2. The teachers' understanding of their own effectiveness

Looking at their responses using the strands of Kilpatrick et al.'s (2001) model, all five teachers shared similar views and understanding of their effectiveness as useful teachers of mathematics. For example, Demis articulated that:

> *Effectiveness is a combination of many competencies. I do not think it is just about practice or reasoning. But it is about doing it…you know…being involved the whole time doing it and working with the weaker one* (Demis).

Teachers described their effectiveness in terms of their strong pedagogical content knowledge (PCK). They said their flexible PCK helps their students become mathematically proficient and comfortable in the world of mathematics. They appeared to focus their PCK on what students should be able to do in terms of the mathematical content, skills and procedures and to reason mathematically. For example, teachers explained their effectiveness in terms of the key strands of conceptual understanding and fluency in carrying out basic instructional routines and the strategic competence to solve problems that come up during their lessons. When I asked them which one of these (conceptual and procedural knowledge) is important in their teaching of mathematics or geometry in particular, Emmis and Sann reiterated that:

> *It is very difficult to divorce any of the two…because if the concept is understood then the procedure has to be followed. And if the procedure is not understood or cannot be followed then it means the concept cannot be well understood* (Emmis).
>
> *If you do not have or understand the concept, you would not know which procedure to follow. So I put much emphasis on the concepts before actually do the procedure because if the learner does not know or understand a concept that angles around a certain point add up to* $360°$ *then she would not know that she has to subtract the value given to her to get one or two angles required* (Sann).

The teachers also appeared to spend an extensive amount of time planning their lessons. So their effective planning was shaped by their views on how to teach mathematics, interact with students and support them to make sense of mathematical ideas. Their adaptive reasoning to justify, explain and reflect on their classroom practice, and their willingness to try new methods was also evident in their responses. Their productive disposition and passion for mathematics appeared as an imperative focus of their teaching. What the interview data highlighted is that teachers' effectiveness and proficiency for teaching mathematics was also linked to many factors as well as the broader and professional learning community in which they worked.

The notion of effective mathematics teaching is central to this thesis and is multifaceted. There are key mutually dependent enabling factors/influences of teacher effectiveness as identified by the participants themselves. These *external* and *internal enabling factors* or characteristics should not be regarded as independent of each other, and I draw attention to various links between these factors which may help to provide a better understanding of possible mechanisms of school effectiveness. These factors are discussed with regard to how the interview data were coded and categorised together with the strands of the Kilpatrick framework.

5.2.2.1. External factors

5.2.2.1.1. Influence of effective school management, leadership and efficient organisation

Almost every teacher indicated effective school management or leadership to be a key factor in enabling them to be more effective. Teachers are of the opinion that effective management is more committed to instructional leadership, effective teaching and the development of the school. Similarly, they revealed that good performance and efficient organisation goes hand in hand with efficient school management. They indicated that good instructional measures implemented by school management ensured that quality assurance measures are supported in their schools, and that the performance standards are set and maintained. Effective management or leadership in this context means "being clear about what needs to be done in terms of school aims and policies, taking steps to ensure that the work is done well, taking steps to evaluate whether things are working as well as using the evaluation to make the necessary changes" (Namibia. MoE, 2005a, p. 2). Teachers implicitly underscored the role that principals play and identified their management style, their efficient organisation, and their relationship to the vision, values and goals of the school and their approach to change as significantly imperative for effective teaching, as manifested in the excerpt below.

> *So, management is number one. Like here at [this school], I think the management is very good…serious people…people who are dedicated to their work. It is like their lives depend on this job. And that makes the whole system now very effective…very, very effective (Teacher 4: School D). Yah, personally if I tell you the truth, I do enjoy my teaching because there is enough support from the management* (Emmis).

5.2.2.1.2. Influence of teacher collegiality, collaboration and teamwork

The participating teachers further emphasised the importance of networking with other professionals. Teachers indicated that collaboration with other teachers within and outside the school directly supports and assists their instructional strategies and selection of teaching materials. This teacher collaboration platform typically involved two or more teachers working together to determine what learners need to know and what or how they should be taught. The participating teachers explained that they routinely work in partnership with other teachers of the same grade level, in order to provide a consistent program for their learners. A number of the participating mathematics teachers characterised such collaborative meetings as formal planning sessions whereas others portrayed these meetings as informal idea exchanges. In both cases, teacher collegiality, collaboration and teamwork enabled a strong interactive culture of sharing of teaching strategies and materials for use in lessons. The quotations below illustrate the above.

> *The other thing we do here is that we have to share ideas with other colleagues within the school* (Jisa).
>
> *We had a group of teachers from Omaruru. They come (sic) and visited us for a week. They were in our classrooms. We were giving them worksheets, questions (sic) papers and everything that we were giving to the kids. They could observe how we are teaching and I think that this is a good idea to go around and actually observe other teachers of how (sic) to they are teaching and what they are doing…because if I do not know how to teach certain concepts, I won't find it in the textbook. In the textbook I find guidelines, but I have to see how it is being done before I will be able to do it properly* (Sann).

5.2.2.1.3. Effectiveness of the links with the broader community

The data showed that effective schools are underpinned by a broader principle of working with the community with the purposes of ensuring that the relevance of teaching and learning and the safety of students in the schools are not compromised. In this context, a shared and committed school leadership requires commitment from all education partners or stakeholders. Hence, the links with the broader community helps to lay a foundation of excellence. Essentially, the links with parents and other well-performing schools sustain effective teaching. These links also enable school managers to conduct school self-evaluation, secure other school- or community-based support, and involve parents in school activities and co-ordinate effective teaching practices. It is also vital for collaborating with all education stakeholders involved and teaching and learning benefit. The argument is that networking helps to create a community of practice and improve classroom practice through professional development activities. This provides the tools for effective ideas/practices and a chance to reflect on their practice with colleagues with the objective of obtaining 100% pass rates in national examinations. The excerpt below illuminates the above.

"Well…you know good results are always from a lot of factors…I think we start with the schools, go to the teachers themselves, and go to the management of the school including parents…But in general especially parents are more involved in the school [governance] and they should not say there is too much callings if they are called to schools. In fact the school calendar shows that we have a stipulation on consultative meetings with parents. Parents should visit the school, they should see their children work and they should see the progress of their children, and they should together with the school map out the way to help the learners with their work, which is being monitored by both the schools and the parents" (Ndara).

5.2.2.2. Internal factors

5.2.2.2.1. Conceptual understanding of core knowledge

In the interview, teachers indicated that their educational backgrounds and experience were useful in developing understanding of the knowledge base needed for effective teaching. In most cases, teachers reported that their pre-service and in-service training may have contributed to their knowledge of, and confidence with, various pedagogical practices. Teachers also emphasised teaching important core concepts so that their learners develop authentic understanding of those concepts. Teachers said, for example:

> *As far as the subject content is concerned I think I am well equipped. The training that I went through has prepared me enough for me to handle the concepts at this grade level* (Ndara).
>
> *I was trained and I think I was trained effectively. I was trained how to plan … I was trained how to develop concepts…how to introduce concepts and how to create a contradiction in the [mind of] learners … create the thinking mind in the learner* (Emmis).
>
> *Well, I think the fact that I had proper mathematics teachers also made a big difference* (Sann).

5.2.2.2.2. Conceptual approach to teaching

Analysis of interview data suggested that all participating teachers have the necessary knowledge for conceptual teaching as determined by the Education Ministry officials. Three conceptual approaches to teaching strategies that participating teachers employed to promote learners' conceptual understanding were identified. The first approach entailed techniques that make mathematical concepts more alive and engage learners to see connections between mathematics and the real world. The second teaching strategy suggested by teachers builds on learners' existing ideas and extends them to engage with new concepts. All teachers involved learners through practical work. Teachers gave their learners a lot of practice to encourage them to become aware of the underlying meanings of the concepts in their thinking. Teachers implied that *Practice makes perfect* and is necessary to master any skill. Teaching for conceptual understanding is typical of effective teaching (Kilpatrick et al., 2001).

A third approach to effective teaching that teachers mentioned referred to creating a democratic classroom environment within which students are seen as active co-participants and members of collaborative groups, and where they feel confident and able to express and discuss their views openly. Such an environment can only be created through the teacher being sensitive to learners' needs, feelings and ideas as well as an effective manager of a class group. Teacher also pointed to the need to provide learners with comprehensible explanations aimed at deepening their conceptual understanding and making connections between different concepts. The teachers said:

> *We also give learners extra practice and extra time of mathematics lessons and this makes our learners to understand concepts easily* (Jisa).

5.2.2.2.3. Strategic competence

The second broad area participants identified as central to their effectiveness was their strategic competence in planning and presenting particular mathematical problems to their students. They said this was helped by their own experience, their mathematical backgrounds and the type of quality teachers they had, either during their secondary school or their tertiary education, contributed to their knowledge of significant pedagogical strategies that shaped their current practices. These pedagogical strategies include: rich questioning to deepen learners' conceptual understanding, interlinking concepts, use of hands-on or multiple teaching approaches (investigations, practical, discoveries) to engage learners with the mathematical concepts, and interacting with their learners while involving them in doing their own practical learning. Some of the evidence for this is found in the following excerpts:

> *I like practical lessons...my yesterday lesson (sic) was on circle theorem…I involve them in practical (sic) to derive the theory, to discover the theorems so that they can use them in future and in any other development* (Emmis).

> *Explain the general rules. So the whole time when you teach, you have to take the students back to the general rule...go to more basics* (Demis).

> *I use (sic) to relate it to the real-life situations of the learners...contextualisation in real-life situation is the best* (Ndara).

5.2.2.2.3.1. School policy on homework

All participating teachers emphasised the importance of homework. At all the participating schools, homework is compulsory. From the perspectives of the teachers collectively, it was found that homework is a daily activity and given for a number of reasons. Firstly, homework is given to cultivate a sense of learner responsibility towards learning. Secondly, teachers assign homework for learners to do more practice. For all the teachers, it was important that homework enabled them to see what learners can do or not do and to enable their learners to hone their skills and comprehend the mathematical concepts they are taught. Teachers also viewed the homework as an extension of the executed lesson during which learners are engaged in individual practice or seatwork. For example:

> *But yah…homework in our school is a daily thing* (Emmis).

5.2.2.2.3.2. Structured and well-planned lessons

The interview data highlighted the importance of structured and well-planned lessons. The participating teachers took it upon themselves to make their lessons interesting and to motivate learners to feel excited about learning and to make connections to the mathematical concepts under discussion. The data showed that the participating teachers planned well and presented the lesson content or materials in a well-organised manner. Most teachers felt that well-planned lessons influence learners to take responsibility for their own learning, and this is certainly a legitimate position. For instance, teachers reported using carefully planned introductory questions, correcting homework and then posing some old examples to check learners' facility with prerequisite skills. They also presented some new examples, asked students to complete some illustrative tasks and posed further questions in sets of similar complexity. Some teachers provided clear definitions of concepts, demonstrated the link between concepts and assessment strategies, posed a problem to start a discussion, occasionally asked learners to justify and explain their mathematical ideas, thinking or methods to solution and/or situated mathematics within a realistic context to engage students with concepts. In addition, teachers indicated posing further examples to the class to check both the students' accuracy and their capacity to explain the process they used, before they set further examples for homework. One of the teachers said:

> *Part of it is the "Structure"…the way that we structure or sequence our work or concepts in a clear and proper sequential or hierarchical order. I am not just going to do this and that, but the ORDER of concepts is most important. For me the structure helps a lot towards effectiveness…there must be a coherent structure among concepts to be taught* (Demis).

5.2.2.2.4. Adaptive reasoning

Adaptive reasoning was also evident from the teachers' responses. Teachers asked open-ended questions to build learners' understanding of concepts while focusing learners' attention on key elements of the learning content. For example, teachers reported encouraging mathematical reasoning in their teaching by asking "why" questions that solicits answers that learners had to explain and justify. "Why" questions about facts or procedures that students engage with are rather frequent, being asked in order to give mathematical meaning to ideas or procedures, meaning of steps or solution methods and, explain or clarify responses of the learners. According to Kilpatrick et al. and his colleagues, such teaching foregrounds adaptive reasoning, which is the capacity for logical thought, reflection, explanation and justification, and in reflecting on those practices in order to improve on them (Kilpatrick et al., 2001). Similarly, teachers indicated bringing in investigations to enhance learners' mathematical reasoning. For example:

> *Another similarity is that we all encourage reasoning by asking questions like "why", "tell me why", "how did you get that", "why a procedure works", "why a solution method makes sense" or "why the answer is correct" to see if they really understand or grasped taught concepts* (Demis).

5.2.2.2.4.1. Influence of being self-reflective practitioners and role models

The participating teachers further acknowledged their own strengths, challenges, values and beliefs with respect to being effective teachers of mathematics. The interview data revealed that the teachers are reflective about their own teaching practice and constantly evaluate the effect of their own teaching on learners. All but one teacher indicated that they have technical skills and reflective skills that allow them to examine and evaluate their own teaching. They also acknowledged that they are role models for learners. All teachers at the five schools are of the opinion that all learners can learn, teaching can be, should be and often is facilitative of learning. They also believe that effective teachers have a wide spectrum of mathematical knowledge, attitudes, abilities and skills as well as integral teaching experiences. For example, Ndara and Jisa said:

> *"Right, in terms of say teaching mathematics I think it is important to be a role model…it means that a teacher should demonstrate a high level of understanding of the subject content. The teacher should [be exemplary] himself. When he comes to the class he should not just move with worked examples. I should say a teacher should go to the lesson without worked examples, you should know the subject. Because if you go there and start working from the paper, how would you motivate your learners that do not go to board with the book to copy your answer. How would you do that? It should start with you. You should be the role model. Yes!* (Ndara).

> *"Then…the first one is good subject knowledge, then good communication skills, then very good and friendly approaches, the same day…strictness and also good behaviours, then good presentation and then good friendship as well as very trustworthy. Well trust…according to my age I must be very trustful*" (Jisa).

5.2.2.2.5. *Productive disposition*

5.2.2.2.5.1. Influence of commitment, hard work, discipline and passion for mathematics on teacher effectiveness

It is interesting to note that commitment and hard work amongst the teachers and the learners, and the passion for mathematics have a far-reaching influence on teachers' effectiveness and learners' performance in the mathematics classroom. On many occasions, teachers indicated that they are committed, hardworking and consistently enforce fair, clear and well-understood rules to maintain discipline among their learners. They also indicated having a strong passion for mathematics teaching. This trait was evident from the way teachers described themselves during the interviews. The following extracts are typical of teachers' commitment, hard work and disposition.

> *I am sure this goes with a commitment of the teacher. Actually when a person is appointed to the job, the first thing you are commanded is to work hard and deliver on the mandate entrusted with you...you should motivate your learners, you should make your learners do the work, and you should also make sure that they [learners] in as much as they can understand the concepts. I think these are the basics of which a person may [well] achieve the target* (Ndara).

> *I think the first thing is just the general part of personality that I have a passion for mathematics. So, for me it is very important that you must love mathematics and that you must transfer that love for mathematics over to your students* (Demis).

5.2.2.2.5.2. Influence of supporting learners and creating lifelong learners

The participant teachers acknowledged their responsibility to support all learners at the five schools, which is indicative of creating lifelong learners. The data showed that effective teachers support personal growth, self-awareness and positive self-concept in learners. Teachers indicated that they purposefully set high standards and then supported learners in their achievement of them. Further, all five mathematics teachers recognised their responsibility to create lifelong passionate learners of mathematics. Through their classroom instructional practices, they engaged students in investigation, exploration, critical and divergent thinking, productive and challenging mathematics conversations and meaningful problem solving scenarios. Teachers indicated that they used a variety of teaching strategies and assessment appropriate to mathematics learning goals, while promoting multiple solving strategies in the mathematical tasks they gave to their learners.

5.3. SUMMARY OF INSIGHTS GLEANED FROM INDIVIDUAL INTERVIEWS WITH TEACHERS

Both the external and internal factors presented here illustrate what teachers perceived as contributing towards their effective practice, and have implications for effective teaching. All five teachers shared a similar understanding of their effectiveness in teaching mathematics in Namibia. The pedagogical conclusion with regard to this data set is that effective teaching practices in complex mathematics classroom settings have a strong influence on students' learning and understanding of mathematical concepts. Evidence for the significance of these factors is derived from interviews with the five teachers who participated in this study and some informal discussions with other teachers in surveyed schools. An attempt has been made to support this evidence with quotations from the interview transcript where one can see the strong degree of overlap in their opinions.

5.4. CONCLUSION TO THE CHAPTER

The analysis in this chapter provided evidence of teachers' understanding of their own mathematical knowledge and effectiveness of teaching for mathematical proficiency in terms of the five strands of Kilpatrick et al.'s (2001) model. In the next chapter, I present, analyse and discuss teachers' teaching practice through the lens of Kilpatrick et al.'s (2001) model and enactivism.

CHAPTER SIX

DATA PRESENTATION AND ANALYSIS C

TEACHER PRACTICE

6.1. INTRODUCTION

This chapter presents and analyses the data pertaining to the teaching practices of the participating teachers by using Kilpatrick et al.'s (2001) model of teaching for mathematical proficiency and enactivist theory. The data is drawn from fifteen lesson video recordings of the lessons of the participating teachers. This presentation is done in a chronological order of visited schools and is consistent with the conceptual and theoretical frameworks outlined in Chapter Two. As indicated in Chapter One, the presentation of these findings is organised into three main parts. The first part reports on analyses and interpretations of classroom practices through the lens of Kilpatrick et al.'s (2001) framework. The second and final parts use the lens of enactivism to analyse what transpired in the fifteen lessons and the interview findings.

6.2. ANALYSIS OF TEACHER PRACTICE

6.2.1. Part I: Analysis of teacher practice through the lens of the Kilpatrick et al.'s (2001) model

In this section I analyse the teacher's geometry lesson videos using the observable indicators of the five strands of Kilpatrick et al.'s (2001) analytical tool and allow the data to tell its own story by using direct quotes from the teachers. As these teachers are deemed to be effective, I am not judging them against the five strands of MTP, I simply wish to provide a narrative of their lessons in terms of Kilpatrick et al.'s (2001) five strands of proficiency. Hence, I will report on their teaching stories that transpired during the lessons. In order to indicate whether these indicators or codes were apparent, I first look at each strand separately. I then take different vignettes from the lesson videos which demonstrate relevant aspects of teaching proficiency in relation to that particular strand. These lesson video descriptions and selected vignettes include both the instruction that took place as well as the teachers' interactions and utterances with the learners. Using these classroom lesson video recordings in my descriptive narrative analysis, I note what stood out in the fifteen lesson videos with a particular focus on my first two research questions, namely:

- *What are the teaching proficiency characteristics of selected effective mathematics teachers, and what are the teaching proficiency characteristics that are similar and different across the teaching practices of the selected teachers*?

The analytical tool I employed allowed me to go beyond a simple reconstruction and description of each lesson to consider the stories concerning what was being taught and to describe how the selected teachers incorporated MP in their teaching. The Kilpatrick et al. (2001) framework proved inadequate for all the actions and interactions between the teacher and the learners. In order to capture those, I engaged an enactivist framework.

6.2.1.1. Kilpatrick et al.'s (2001) analytical framework as a means to analyse lessons

The five strands of teaching for MP are *Conceptual Understanding* (CU), *Procedural Fluency* (PF), *Strategic Competence* (SC), *Adaptive Reasoning* (AR) and *Productive Disposition* (PD) as tabulated in Table 3.4. A number of observable indicators were crafted to operationalize the five strands. The indicators helped me to recognise and discern the different strands, for example, in the conceptual understanding (CU) strand, the indicators looked for evidence of the teacher's ability to make links to learners' prior knowledge, and provided accurate explanations of concepts, ideas and procedures or steps.

6.2.1.2. The analysis: lesson vignettes

The lesson video transcripts for this study were structured into vignettes. In this way, I was able to code whether the teachers' mathematical interactions with learners were conceptual, procedural, strategic, adaptive or productive. The codes used were CU, PF, SC, AR or PD[2] respectively. Kilpatrick et al. (2001) argue that all strands are of equal importance and are interdependent and interwoven. Accordingly, my expectation was that effective teachers structure classroom activities so that most of the five constituent strands of MTP are emphasised and manifested in one way or another.

At this juncture it must be noted that the above systematisation provides an initial indication of how to recognise the strands in the observed lessons. I acknowledge that it was not possible to pigeonhole the lesson videos into one specific strand because the strands overlap within and across lessons. For example the distinction between CU and PF is not always clear cut. At times, they appear to operate independently, yet at other times they complement each other. Kilpatrick et al. (2001) emphasised the need for the acquisition and application of both

[2] The acronyms introduced in this chapter will be used substantially in the remainder of this doctoral research.

CU and PF, but it is important to achieve the right balance between conceptual knowledge and procedural knowledge. Procedural knowledge has meaning only when it is linked to conceptual understanding (Hiebert and Lefvre, 1986). At the secondary school phase of teaching and learning, developing learners' procedural knowledge has positive effects on their conceptual understanding, and conceptual understanding is a prerequisite for the learners' ability to generate and select appropriate procedures (Ally, 2011; Wong and Evans, 2007).

The structure of the lesson video descriptions is arranged as follows.

- Firstly I provide a brief and general description of each lesson concentrating on the flow of activities and the types of events that were apparent during the lesson.
- Secondly, I organise each lesson event into a series of vignettes or episodes in relation to the indicators of each strand which were dominant. These provide evidence of teachers' mathematical interactions that promote the development of a particular strand of MP in learners.

The sequence of my analysis is not arbitrary but is framed by Kilpatrick et al.'s (2001) five strands of teaching for MP, where the sum or whole is greater than its individual parts. Hence a vignette is defined only as a segment of the lesson within which MP comes into play or was exemplified as these strands enable a broad understanding of best practice.

That said, I faced four dilemmas in my analysis of the video data that are worth mentioning:

- ❖ how to select certain parts of the lesson video data to analyse and interpret. To overcome this dilemma, I viewed all the videos several times to get a *feel* for how the different lessons played out. After this, lesson transcripts were created to capture all the teacher interactions with the learners. The transcripts were then scrutinized several times for evidence of the five strands of MP. As incidents were observed the lesson transcripts were annotated to explicate the evidence warranting the identification of the particular strand observed. With each viewing the annotations were refined and decisions clarified.
- ❖ how to frame my analysis in accordance with the Kilpatrick et al.'s (2001) analytical tool. I needed to design and construct a suitable rubric (see Table 3.4) that enabled me to analyse the individual lessons. However, I decided to analyse the video data as thoroughly as possible to reveal vignettes that showed evidence of the five strands.

- ❖ how to structure the dissertation to make it accessible to the reader. I structured my narrative based on the striking issues I uncovered after several re-reads of the lesson data.
- ❖ how much of the lesson video data I should present for the reader to make sense of the entire lesson. This required making decisions about what piece of the data to include and exclude, and to find the episodes that were most characteristic in terms of the five strands of Kilpatrick et al.'s (2001) model.

Given the lengthy analysis and complex overlapping of the five strands, I only present the in-depth analysis of three teachers namely Demis, Emmis and Sann in this dissertation. I selected the lesson videos of Demis, Emmis and Sann because they showed rich evidence of the Kilpatrick's five strands and shared commonalities of interesting individual lesson cases.

6.2.1.3. Demis (Teacher 1)

6.2.1.3.1. Lesson Video 1: Grade 10 Measurement of plane shapes

General description of the lesson

Demis' first geometry lesson focused on the perimeter and area of two dimensional combined shapes and took about 43 minutes. The lesson started with a review of homework given in the previous lesson. This introduction connected students to the concepts of the topic at hand. Demis divided students into groups of four, with two students each being given a specific role to carry out in that group. Each group was presented with two prepared square grid papers (graph worksheets) which contained a rectangle and a circle each, along with other materials such as ropes/strings, scissors, rulers and marker pens. She then assigned a series of four activities that used physical manipulatives to enable students to explore perimeter and area of combined shapes in a meaningful hands-on approach, and to gradually develop students' conceptual understanding of these two basic concepts.

The **first activity** focused on cutting out given shapes, and constructing the two combined shapes composed of a circle and a rectangle of chosen dimensions. The activity aimed to give students the opportunity to use physical manipulatives in order to build connections between mathematical concepts and representations, fostering more precise and richer understanding. The **second activity** required students to determine the perimeter of rectangles drawn on the square grid papers. This activity aimed to support students' understanding that the term 'perimeter' simply refers to the number of unit squares of chosen size along the boundary of a particular shape or region. Students were required to determine the perimeter of a rectangle that they cut out from the grid by identifying the number of unit squares on the edge of the rectangle. The **third activity** moved the focus from perimeter to area, but was conceptually linked to the previous activities. This activity aimed to visually enhance students' understanding that the term 'area' simply refers to the number of unit squares of chosen size that cover a particular shape which are enclosed within a specified boundary. The **fourth activity** moved the focus from the perimeter and area of single shapes (rectangle and circle) to that of constructed combined shapes composed of a rectangle with two half circles inside or outside. This activity involved much practice by students and took up almost two-thirds of the lesson time. The emphasis of this process was for students to share their different strategies for determining the number of squares covered within the constructed shapes, with a particular emphasis on finding the most economical strategy of working out both the perimeter and area. However, this task was completed in the next lesson.

Vignette 1: The development of students' Conceptual Understanding and Productive Disposition

To begin with the first activity and to ensure that learners were using the correct mathematical terminology, Demis opened the lesson by asking if students had an idea of what the terms "perimeter" and "area" are. The following transpired.

1 Demis: Tell me quickly, what is the perimeter? (**CU5**)
2 Student 1: The perimeter is a length around the figure.
3 Demis: Excellent (**PD2**), the perimeter is a distance around the figure (**CU1; CU2**). And what is the area?
4 Student 2: Area is the space inside.
5 Demis: That is good (**PD2**), area is the inside space of the figure or boundary. (**CU1; CU2**)

This exchange showed evidence of CU, with some elements of PD at play. This was conceptual in nature because Demis engaged the learners with a clear introduction of two key concepts, namely perimeter and area. Demis asked questions that solicited previously learned definitions of perimeter and area in order to obtain clarity about what students knew (lines 1 and 3). Also, in lines 3 and 5, Demis provided accurate explanations of ideas and terminologies that were useful to learners (CU2). While Demis challenged her learners to articulate their mathematical ideas (line 1 and 3), she also used mathematically appropriate and comprehensible definitions and language (CU1) to explain the relationship between the perimeter and the area. In this way, Demis demonstrated fluency in mathematical language as she provided precise and accurate explanations of the difference between perimeter and area in that a perimeter is the distance around the figure (line 3) while the area is the inside space of the figure (line 5).

With regard to elements of PD within this vignette, Demis affirmed the learners' responses by saying "excellent" (line 3) and "that is good" (line 5), which is PD2; to create a positive productive disposition towards mathematics or interest in the mathematical ideas they were engaged with. This finding resonates strongly with Kilpatrick et al. (2001) who claim that students are more likely to hold productive dispositions (autonomy, belief that mathematical competence is malleable rather than fixed) in a mathematics classroom in which the teacher transfers responsibility to students, solicits multiple solution strategies, provides process scaffolding and presses for conceptual understanding (CU).

Next Demis engaged learners in a practical hands-on task that was conceptually rich. For this task, she asked her students to cut out the rectangle and a circle on the two square grid papers (graph sheets) given to them. Learners examined the given shapes in pairs and cut them out to continue with the activity. The following ensued:

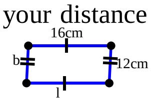
6 Demis: So, look at that rectangle you have cut out from the graph sheet, how are you going to work out the distance around the figure...what are you going to do? (**CU4**)
7 Student 3: We are going to count the square blocks.
8 Demis: Okay, you are going to count the blocks (**PD2**). Can we hear what other people are going to do, how would you determine the perimeter of this rectangle? **(CU4**)
9 Learner 4: We measure the distance end to end of one side of a rectangle, and then do the same to the other sides.
10 Learner 5: We use the ruler to measure the distance around the four sides of a rectangle.
11 Learner 6: We multiply the length and breadth distances by two and add them up to determine the perimeter of a rectangle.

The interaction that unfolded in this second episode showed that Demis addressed the conceptual basis of measurement of plane shapes in a variety of ways. For example, the development of students' CU of perimeter was typically manifested through Demis' exposition, various forms of questioning and whole-class reflections on the conceptually-focused rich task (line 6). As evident in lines 6 and 8, Demis made reference to interesting contexts, giving or using concrete materials from learners' contexts that they could relate to (CU4). She did this through asking learners to explain their methods for measuring the perimeter of the rectangle drawn on the square grid paper without using the formula (line 6 through 11) so students could figure out how to determine the perimeter of a rectangle and operationalize its definition.

Vignette 2: The development of students' Procedural Fluency

A considerable amount of lesson time was spent on addressing students' procedural skills (PF). During the whole-class reflection, Demis focused learners' attention on the concept of perimeter that demanded procedural knowledge, to move the lesson on and get learners ready for the next phase of the learning process. For example, she asked learners to determine the procedure for finding the perimeters of the rectangle that they cut out from the graph sheet. The first stage of the procedure involved counting the square blocks on the square grid paper where the rectangle was drawn. This conversation went as follows.

12 Demis: [*While moving around the class*] I can see some people have numbered the blocks which is a good thing. [*She then drew a rectangle on the board*] okay, I just want to know, what is your distance here, how many square blocks or centimetres are here [*pointing to the length side*]? (**PF1**)



13 Students: 16 cm (*length*).

14 Demis: 16 cm so basically the perimeter is $16cm$ plus…what are the blocks here [*width*]? (**PF3**)
15 Students: $12cm$
16 Demis: [*Writing on the board*] so we can say $P = 2l + 2b$ (**PF3**). Now your perimeter will be two times $16cm$ plus two times $12cm$, (**CU2**) that is equal to $2(16cm) + 2(12cm)$, so that will be $32cm + 24cm$ (**PF4).** People, can you think about algebra, is this like or unlike terms (**PF6; CU6**)?
17 Students: Like terms.
18 Demis: Like terms (**PD2**). What did we do in algebra when we have like terms, who can explain to me? (**CU6**)
19 Students: We add them together and keep the variables the same.
20 Demis: We add the numbers and keep the variables the same. So what will this answer be?
21 Students: $56cm$
22 Demis: [*Pointing to the coefficients 32 and 24*] do you see now the variables of cm in the solution, so $32cm + 24cm = 56cm$ and I stay at cm (**PF6**). Are you with me?

In the above interaction, the strand of PF dominated this vignette, but CU was also evident. In line 12, Demis asked questions that solicited the procedure for determining the perimeter of a rectangle, which is PF1. She also explained and elaborated on the procedure suggested by the learners (PF4) and then focused their attention on the number of square blocks to determine the dimensions of a rectangle (line 16). As she did this, Demis offered accurate mathematical explanations to give meaning to the ideas and procedures under discussion. For example, she explained how the formula or procedure should be used and why the solution method makes sense (line 16). Here, Demis encouraged her learners to use mathematical procedures and formulae accurately and appropriately (PF6) as she explained and showed them how the procedures should be used by making conceptual links or reference to algebra, which is CU6 (lines 16 and 22). This further indicated Demis' attention to the development of CU as she was able to provide accurate mathematical explanations, which is CU2, to give meaning to formula, steps and procedures (line 16). Such explicit explanations enabled students to exhibit skills of procedural fluency when they determined both the dimensions and perimeter of a rectangle as they were working on the given task.

As the lesson progressed, Demis launched the second activity. She allowed learners to continue working on the square grid paper (graph sheet) and asked them to share and justify their strategies for determining the area of the same rectangle under discussion.

23 Demis: Okay, if you are going to work out the area, what are you going to do? **(PF1)**
24 Students: Area is length times breadth.
25 Demis: Length times breadth, which means you are still going to count that blocks again. That is excellent. But before you use the formula, want you to understand this concept (**CU2**). [*Starts dividing the rectangle drawn on the board into rows and columns to show and explain how to find the space covered*] look at your rectangle; these are the rows and columns, these are the rows and there are 12 rows. These are the columns, and how many columns are there? (**PF3**)
26 Students: 16 $columns$
27 Demis: So, if you want to see all these small blocks [*inside the rectangle*], then you must take the length times the breadth. So, that is why its area is $16cm$ multiplied by $12cm$ and this gives us $16cm \times 12cm = 192$ ... Okay, people think of algebra again (**CU6**). In algebra if we have $a \times a$, what would be the answer? (**PF3; CU2**)
28 Students: a squared $[a^2]$
29 Demis: So a^2, then this will be $192cm^2 \Rightarrow 16cm \times 12cm = 192cm^2$. That is why this answer is squared centimetres (cm^2) and the answer for the perimeter is in centimetres (cm). Why is this one in cm^2? To get cm^2, we add them up like $16cm^1 \times 12cm^1$ to gives us $92cm^2$. (**PF3; CU2**)

I classified this vignette as showing PF because the task was largely procedural in nature. My analysis of this vignette showed that Demis provided accurate mathematical explanation of procedures and solutions, which is PF3 (lines 25, 27 and 29). For example, she explained the procedures for working out the area of a rectangle, by attending to the meaning of the steps involved in the procedures rather than simply listing these steps (lines 27 and 29). She also explained how the answer for the area compared to that of the perimeter (line 29).

I also identified two utterances illustrative of Demis' attention to learners' prior conceptual knowledge and mathematical proficiency when she made conceptual links to algebra (CU6) when explaining procedures to deepen learners' CU (lines 25 and 29). For example, in line 27, Demis reminded the learners to think of what they have done in algebra in terms of like- and unlike-terms. This is an indication of her attention to promoting conceptual knowledge through procedural knowledge (line 27 and 29).

Vignette 3: The development of students' Productive Disposition

Another important strand of MP that was evident in this lesson was that of PD. While modelling the solution for the area of a rectangle (line 29), Demis intentionally wrote meter (m) unit instead of centimetres (cm) as appeared in the sum below (line 30).

30 Demis: Okay, $Area\ of\ rectangle = length \times width = 16m \times 12cm = 192cm^2$ (**PD3**).

Some learners immediately detected this error in the calculation and Demis confirmed the error and rectified it. This is evidence of promoting PD, as Demis showed the learners that making mistakes in her mathematics classroom is permissible, which is, PD3 (line 30). In the interviews, when I asked her to reflect on her lesson, Demis said she made that error deliberately to help learners keep the same variables and build connections between the two concepts under consideration. This was evident when she reminded the learners of algebra concepts they covered in previous grades, asking them what a multiplied by a $[a \times a]$ would be when working out $16cm \times 12cm$ (line 27). Towards the end of the lesson, Demis provided positive feedback for learners to see the worth of the lesson (line 31). She also assigned a homework task to encourage learners to do mathematics outside of the classroom, which is PD1 (line 31). The homework task was purposely selected to allow learners to understand the concepts and the way they appear in national examinations.

> 31 Demis: okay, people we are going to continue with this task tomorrow. But before we go, check here, we said the perimeter is the distance around the figure. So, the perimeter of figure 1 will be:
> $P = halfcircle + length + halfcircle + length$
> The perimeter of figure 2 will be:
> $P = length + halfcircle + length + halfcircle$
> That is exactly the same. It supposes to give you the same answer (distance). And think about why your perimeter distances were not the same for tomorrow discussion. Please check your homework (**PD1**) on page 220. It is just the revision of this work on perimeter and area. Okay, good bye class.

6.2.1.3.2. Lesson Video 2: Grade 10 Measurement of plane shapes

General description of the lesson

This lesson built on the first lesson and focused mainly on the last activity. Demis engaged students in exploring the relations of the dimensions of the two constructed shapes, and the notions of perimeter and area. This mathematical activity required students to determine the perimeter and area formulae through practical (or problem) situations in order to explore and discover mathematical relations. The task was conceptually challenging to students because it included real world context for students to engage in a rich mathematical conversation. Students discussed and worked on the task collaboratively, thereby forging a conceptual link with the conventional formula for determining the perimeter and area of combined shapes: $Perimeter/Area = rectangle \pm circle$. Specifically, the task invited students to figure out (or derive) the strategy by moving beyond the procedural knowledge to consider some mathematical principles that would help them understand mathematical relations and concept meanings. During small group discussion, students helped each other identify the dimensions of constructed shapes making connections to the procedures discussed earlier.

As students were working through this activity, Demis moved the room monitoring, assisting and directing students, and sometimes stopping to answer questions from students. Consequently, each group constructed two rectangles, one with two half-circles inside and the other with two half-circles outside. The nature of this activity also illustrated the helpful direction from the teacher and typical learner participation in this lesson.

Vignette 1: The development of students' Conceptual Understanding

Demis allowed her students to recall what they did during the previous lesson. The connection was clear and the students approached the new activity as a continuation of the earlier one. She then asked two students in each group of four to:

a) cut the circle into halves that they initially cut out from the square grid paper,
b) align and paste the half-circles inside/outside the original rectangle they cut out, and
c) measure the perimeter and discuss their findings with the two students in a group.

As Demis was moving from desk to desk, the following conversation transpired.

1 Demis: Remember each group has a circle and the rectangle that you cut out, one for two people. You have to figure out the plan. That is why I have brought you ropes and scissors. The first thing we are going to do is to work out the distance around the figure but not by using the formula (**CU7**). I want us to do estimation of that distance first. So, let me ask you, what is meant by estimation? (**CU4**; **CU5**)
2 Student: it means "not exact"
3 Demis: okay, it means not exact, any other idea? (**CU4**; **CU5**)
4 Student: it means "more or less"
5 Student: it means "guessing"
6 Demis: Do you agree with them?
7 Student: yes, it means "not accurate"
8 Demis: yah, you tell me nice (**PD2**). It is more or less…it is not 100% correct (**CU1**).

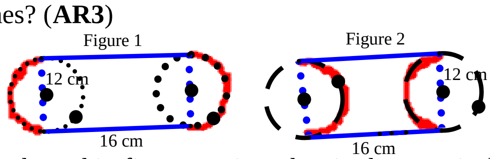
In this vignette, which I coded mainly as CU, Demis engaged learners in a task that is conceptually full (CU7) and mathematically meaningful (line 1), for them to make and extend their conjectures based upon connections between different representations. For example, in lines 1 and 3, Demis made links to learners' knowledge of previously discussed concepts, which is CU5, to provide accurate explanations of concepts that are useful to learners and enable them to see the interrelatedness of the mathematical concepts she was teaching. Since Demis enabled learners to connect their mathematical understanding of the concept "estimate" with *not exact* (line 2), *more or less* (line 4), *guessing* (line 5) or *not accurate* (line 7), this evidence suggests that she was able to use mathematically appropriate and comprehensible definitions and language (CU1) to promote CU. This further suggests that Demis made conceptual links or reference to interesting contexts, using examples from learners' contexts that they can relate to, which is CU4 (line 3). Demis also reiterated the learners' responses in the process by emphasising a key mathematical element, the use of the definition of "estimate", to support learners' CU (line 1). Another important event of CU in this vignette is that, by integrating the learners' own ideas in the lesson, Demis seemed to communicate to them that, in her classroom, their own ideas and methods are valued (lines 2, 4, 5 and 7). In line 8, Demis encouraged and affirmed learners (PD2) when she said "you tell me nice". This is a crucial feature of PD, which Kilpatrick et al. (2001) identify as an important feature of a mathematics classroom that is set up as a community of practice.

Interaction in this episode showed a strong CU as Demis was able to explain the concept meanings, represent mathematical concepts in different situations and knew how different representations could be used in different ways (lines 1 through 8). Connections are useful when the teacher links related concepts and methods in appropriate ways (Kilpatrick et al., 2001).

Vignette 2: The development of students' Adaptive Reasoning

Demis then proceeded with a whole-class discussion requiring her students to share their ideas about the broken lines appearing in the shaded part of the constructed two-dimensional shapes which she also drew on the board. The conversation went as follows.

9 Demis: We were busy yesterday with measuration. Do you have your shapes you constructed yesterday, may I please see them? So, we are going to focus on the perimeter and then focus on the area. Let me redraw these figures quickly. Usually in the examination when they do it like this, they will give you this part shaded with the broken lines like this [*pointing to the second figure*]. Whenever they want you to work out the perimeter of the shaded part, the shaded part will be given like this. Why do you think they make it like broken lines? (**AR3**)



10 Student 1: They show you that it is a rectangle.
11 Demis: That is the good one reason. But if you look at this figure again, what is the practical reason? (**AR3**; **AR4**)
12 Student 2: They show you that, that part is a half circle.
13 Demis: That is also very good to show that it is a half circle. But so to say they show that there is nothing (**CU2**). Do you see anything there? Do you see that side anymore? (**AR3**)
14 Students: No.
15 Demis: It is not there. That is the good reason why they make it a broken line. Now, I want to know from you, very, very important…what do you think when the other groups were measuring their figures and you were measuring yours, was the distance around those figures the same? (**AR2**)
16 Students: No.
17 Demis: It was more less the same. But it was actually supposed to be the same, because in the first figure, I go the length here and a length there, and then I go a half circle here and there. In the second figure I go a half circle here and a half circle there, and then I go length and I go length there as well. But, why the difference in your answers was so little likes this? (**AR2**)
18 Students: It was due to the way we stretch the rope.
19 Demis: Very good, very good, good one idea. You see some of you when you look at these similar shapes [*she then overlaps similar shapes to show the differences resulting from the way learners cut out the figures*] you can see that this corner is longer than this one. Do you see the difference in corners?

The interaction that unfolded in this episode showed evidence of AR. That is, Demis provided situations and activities that required logical reasoning. For instance, in lines 9 and 11, she engaged with learners through questioning that encourage reflection (AR3), and the ability to justify mathematical thinking, reasoning or solutions (AR2). This was evident verbally by posing questions such as "why do you think they make it like a broken line" (line

9), "what is the practical reason" (line 11), or "why there is a difference in your answers" (line 13). Demis also sought to offer clarification of what she meant by a dotted line or what she meant by her labelling and why. I thus classified this incident in the *adaptive reasoning category* because Demis encouraged her learners to think deductively, which is AR4 (line 11). Demis also justified her conclusion using deductive reasoning (line 19). In this regard she worked from the definition of the perimeter and presented an explanatory argument as to why the difference in the learners' answers was a little uneven.

Vignette 3: The development of students' Strategic Competence

Demis then focused students' attention on mathematical procedures, thereby forging a conceptual link to the conventional formula for determining the perimeter and area of constructed shapes. The following episode shows how Demis engaged her class in a procedure-focused discussion in which all five strands of MP were evident.

20 Demis: It is always good people especially to structure your working plan. When you are doing geometry, it is very, very important that you present your work nicely. The perimeter of this combined figure 1 will be $P = l + \frac{1}{2}circle + l + \frac{1}{2}circle$ (**SC6**). Someone also said yesterday, Ms, can't we take these two half circles as a one full circle. That is excellent, because two half circles make one full circle (**CU3**). So one length plus another length will be equal to what? (**PF1**)
21 Students: two lengths plus a circle.
22 Demis: $P = 2l + 1\ circle$ (**PF3**). Can we do the same with the perimeter of figure 2?
23 Students: Yes.
24 Demis: Yes this equals to $P = \frac{1}{2}circle + l + \frac{1}{2}circle + l$, which is the same as $P = 2l + 1\ circle$, okay (**SC6**). People most of you as you have seen today I was most satisfied. After you show the plan that you are going to do, then just do it like this. So, what is the formula for the circle perimeter? (**PF1**)
25 Students: The perimeter of the circle is $2\pi r$
26 Demis: Okay, people remember like what I always say let us put the brackets around the variables to make our working easy. $2\pi r = 2(\pi)(r)$ Can you still remember this?
$\Rightarrow P = 2l + 1\ circle$ (**PF3**)
$\Rightarrow P = 2(l) + 2(\pi)(r)$ (**PF5**)
$\Rightarrow P = 2(\,) + 2(\,)(\,)$. (**PF6**; **PF7**)
Just to get your work nicely done, you do it like this. You still write the brackets but you clear the "inside". There is nothing inside the brackets. Then in the place of l you put $16cm$, and then in the place of π you put 3.14. When you go to grade 11, we will use π as 3.142 or $\frac{22}{7}$.

This third vignette, typical of longer sequence of interactions, suggests that the focus had shifted to address students' SC, allowing multiple solving strategies and evaluation of different solution strategies. In line 20, for instance, Demis engaged learners in a discussion to find ways to devise a working plan, represent ideas carefully using multiple representations and notations (i.e. symbolic representations, algebraic notations, formulae) (SC6) and to solve the problem. In her attempt to explain the procedures (PF3) and find the formula that allowed them to find the perimeter of combined figures, Demis showed a flexible approach to problem solving in explicit ways (line 22). For instance, she represented the perimeter formula carefully using multiple representations such as symbolic representations and algebraic notations, which show SC6 (line 24). Within this vignette, Demis showed high facility with SC in formulating, representing and solving mathematical problems (line 26). This is because her main focus was to explain and structure the solving working plan, formulate the perimeter formula expression based on the rectangle and circle, and substitute into the formula to work out the perimeter of the geometric shapes.

As the lesson progressed, Demis shifted attention from the perimeter to the areas of two shapes under consideration. At this point she focused learners' attention on the difference in the area formulae of the two shapes. This event was captured as follows.

27 Demis: We come now to the areas of these two figures. As you see figure 1 is bigger and figure 2 is smaller. Who can give me a plan of finding the area of this figure 1? What can we do with the rectangle and circle, should we "*add*" them or "*subtract*" them? (**CU6**)
28 Students: "Add them".
29 Demis: Okay, excellent, that mean area of figure 1 is: $A = rectangle + 1\ circle$ (**SC6**; **PF3**). This helps you to keep your mind getting into a certain direction to the solution. We have this circle here because this half circle and this half circle make one full circle (**CU2; CU3**). So, what is the formula for the area of a rectangle? (**PF1**)
30 Students: $length \times breadth$
31 Demis: $A = rectangle + 1\ circle$ so the area for a rectangle is:
$A\ (rectangle) = l \times b + 1\ circle$ (**SC6**). And what is the formula for the area of a circle? (**PF1**)
32 Students: πr^2
33 Demis: Okay, $A = rectangle + 1\ circle$ which is $A = l \times b + \pi r^2$. People again we just want to concentrate on this substitution. This is actually algebra work we are dealing with now. So, $A = l \times b + \pi r^2$ if we create brackets as we did before this will be $A = [(\,) \times (\,) + (\,)(\,)^2]$ (**PF5; PF6; PF7**). Let's just fill in the inside (**PF3**). $A = [(16) \times (12) + (3.14)(6)^2]$. Remember it is not 6 times 2, it is 6 times 6, it is 6 multiplied by itself. So, $A = [(16) \times (12) + (3.14)(36)]\ cm^2$. Instead of putting cm, cm, everywhere inside the big brackets, then you can just write it outside like this to indicate that your answer will be in squared centimetre [cm^2]. Okay, think of what we said yesterday because it is cm times cm. [*she then turns learners' attention to figure 2*] and this is one figure again. As you saw, the answers cannot be the same. So, to find the area of the shaded part is a rectangle minus the circle because I have been cutting the half circles out. Thus, *cutting out* means "*minus*", or getting out.
$A(shaded) = Recatngle - 1\ circle$ (**PF3**)
$\Rightarrow A(shaded) = l \times b - \pi r^2$ (**PF5; PF6; PF7**).
$\Rightarrow A(shaded) = (16 \times 12 - (3.14)(6)^2 cm^2$. The only difference in this formula is the 'minus sign', which means cutting out.

In this excerpt, procedural knowledge (PF), once again, occupied centre stage. For example, in line 27 through 33, the teacher used concrete, pictorial and symbolic representations to explain the procedures and provide algorithms to solve the mathematical problems at stake (PF3). Representations in this episode, however, were used mainly to show students the steps in procedures (lines 31 and 33). Demis used different representations using problem solving to facilitate correct (PF6), accurate (PF5) and appropriate (PF7) mathematical procedures for learners to notice the links with other areas of mathematics such as algebra (lines 33). For example, in lines 27 and 33, Demis provided accurate explanation of how to find the areas of two shapes using terms such as “cut out” or “add it” to emphasise the interrelationships of properties of geometric shapes (CU3). Demis demonstrated familiarity with mathematical procedures as these terminologies are aligned to PF by incorporating step by step procedures in the solving process (lines 29 to 33). From the discussion of MP (Kilpatrick et al., 2001), PF is not enough, and needs CU and the other strands. Here, I classified this third vignette as showing SC, since Demis identified useful strategies to enable the learners to think strategically, use mathematical formulae correctly and to remember the appropriate procedures to solve problems in the mathematics task. The episode also illustrated that the teacher’s decision to break down solving methods into bite-sized procedures pieces served to make connections between concepts and underlying meanings, thus offering students the opportunity to make sense of the mathematical ideas that are useful to them.

Demis clearly developed students’ SC as she let learners to explain the method or choose effective procedure for determining the rectangle perimeter without telling them what to do or doing the work for them. The mathematical discourse that Demis used was clearly that of *representing* (Kilpatrick et al., 2001) since the use of representations (i.e. constructed shapes, diagrams drawn on the board, formulae and dimensions of a rectangle) is another way in which the interrelationships could be represented (lines 27, 29, 31 and 33).

Vignette 4: The development of students’ Productive Disposition

Over the course of this lesson, Demis motivated learners by making the lesson interesting with hands-on activities, which is PD7. In that way, Demis nurtured learners’ natural curiosity by inquiring about their understanding of concepts before she shared her own understanding of those concepts. Interestingly, Demis used different representations to formulate mathematical formulae which were then discussed with learners. As the bell rang, she reminded students of the homework task to encourage them to practice mathematics outside of the classroom.

I found it useful to examine occasions where students were invited to solve contextualised mathematical tasks. That is, tasks typically located in real-world contexts from which students were expected to extract relevant information, formulate that information accurately before solving the tasks and relating the solution back to the original context. Central to Demis’ teaching was the development of mathematical power for all students, including the ability to explore, conjecture and reason logically, to solve non-routine problems and to connect ideas within mathematics (Kilpatrick et al., 2001). Hence, the notion of conceptual teaching that dominated the tasks and discourse in this class, varied substantially from the other teachers in my sample.

6.2.1.3.3. Lesson Video 3: Grade 11 trigonometry

General description of the lesson

On my last observation day, Demis taught a grade 11 mathematics higher level class. She commenced the lesson with a whole class discussion focusing on both the review and application of "trigonometric" functions and the use of the calculator to solve trigonometric functions (ratios). As the lesson unfolded, Demis informed her class that "trigonometry is one of the most important topics in mathematics higher level, adding that she was going to do a revision of last year's work focusing the lesson on the "90°" and "not 90° triangles". Demis carefully sequenced students' ideas, situating the introduction around explaining the meaning and definitions of key concepts in trigonometry and right-angled triangles. The learning activities focused mainly on supporting the students' construction of mathematical meaning and solving mathematical problems. Towards the end of the lesson, Demis linked the mathematics of trigonometry with everyday contexts in which learners could apply the skills they learned in the lesson. She then assigned a homework task.

My analysis of the participation of the class over the entire period showed that students were generally engaged in the mathematical concepts at hand. Based on the transcript of this lesson, I identify the teacher's mathematical actions that show evidence of the specific strands of teaching for MP.

Vignette 1: The development of students' Conceptual Understanding

Demis wrote the term "trigonometry" on the board, and began her third lesson by getting learners to define or say what they knew about trigonometry. This led to the following:

1 Demis: Tell me, what is "trigonometry" all about? (**CU5**)
2 Student 1: It is about Sine, Cosine rule and Tangent
3 Demis: OK, we learned about SOHCAHTOA (**CU4**), but trigonometry is about what again (**AR3**)?
4 Student 2: It is about triangles.
5 Demis: It is about triangles…Why do you say it is about triangles? (**AR2**)
6 Students: because trigonometry involves triangles
7 Demis: Okay, it is about triangles…why…that is because the word "tri…" means "three" sides or angles (**CU2**)

The introductory segment exemplified evidence of CU. As in line 1, Demis made links with learners' prior knowledge, which is CU5, as she solicited previously learned definitions of trigonometry. Using effective questioning and linking learners' prior knowledge is central to conceptual teaching and understanding and Demis afforded learners the opportunity to demonstrate and explain their mathematical ideas about what trigonometry (line 1) and the Pythagoras theorem (line 10). Demis used mathematically appropriate language when she mentioned “tri…” to remind the learners of the relationships between right-angled triangles and properties of the trigonometric ratios (line 7). She also demonstrated CU when she provided accurate explanations of the concept ‘trigonometry’ (CU2), linking it to the notion of SOHCAHTOA that learners could relate to, which is CU4 (line 3). My analysis further showed that elements of AR were evident within this first episode. For example, in lines 3 and 5, Demis encouraged reflections on SOHCAHTOA, and asked questions that encouraged learners to explain and justify their mathematical ideas, which are AR2 and AR3 respectively.

Demis clarified what SOHCAHTOA meant to the sides of a triangle, making reference to the diagram she drew on the board (see line 8 below). She focused learners’ attention on the words the “90°” triangles and “Not 90°” triangles to clarify SOHCAHTOA by relating its meaning to the positions and distinction between “opposite sides/angles”, “adjacent sides” and “hypotenuse sides” for explicit and greater understanding.

8 Demis: Okay, now if you look at the diagram, you will see that trigonometry divides into the “$90°\ triangles$” and the “$not\ 90°\ triangles$”. And the one that we start with tomorrow is the “$not\ 90°\ triangles$”. You should remember that there are some of the things that they ask you. If they give you a “$90°\ triangles$”, then they can ask you to find a “side” or an “angle”. These are two things that they can ask you to do (**CU2; CU3**). So, who can still remember, depending on the information given, that if they give you two sides and then ask you to find the other side, what did we say that you can do (**CU5**)?
9 Student 3: Use Pythagoras theorem.
10 Demis: It was Pythagoras’s theorem. [*Pointing to the poster on the wall*] you can see this theorem on this poster on the wall. What is Pythagoras all about, why is this Pythagoras theorem? Let’s follow this statement in this triangle ABC: $AB^2 = BC^2 + AC^2$, why do we do that? (**AR2**)
11 Student 4: It actually says if you take the area of those three squares and square each it give you that relationship.

The interaction that unfolded in the episode above showed that Demis provided accurate explanations and explained the meaning of concepts useful to learners, which is CU2, using diagrams and definitions of trigonometric concepts (line 8). In addition, Demis emphasised the links or connections between different geometric concepts and ideas such as the interrelationships of trigonometric ratios, which is CU3 (line 8).

Demis continued by pointing to a poster on the classroom wall displaying different trigonometry diagrams and formulae.

12 Demis: Excellent! Look at this poster. As you see, Pythagoras says if you take this side and make a square and take this side and make a square with it, then you take this side and make another square, then the area of this square plus the area of this square will be equal to the area of a big square. The area of a square is side times side. And that's why it says these two areas if we square them, they will give us this area. The Pythagoras theorem is referring to area. (**CU2**)

In the next phase of the lesson, Demis focused on the right-angled triangle that she drew on the board to provide accurate explanations of concepts under discussion. She drew a right-angled triangle ABC with $\angle ACB = 30°$ and AC hypotenuse side of length $5cm$. She then focused learners' attention on the "opposite side", "adjacent side", "hypotenuse side", "SOHCAHTOA" and then the "Pythagoras rule" for greater conceptual understanding. The following dialogue occurred.

13 Demis: If this angle ACB is 30°, AC is $5cm$, and then I ask you to work out this side [*AB*], can you still remember the ratios of a right-angled triangle sides? You need just to remember those words. That is why we are going to use SOHCAHTOA. There are three sides on this triangle, and we have some ways of naming these sides. There is one side which is fixed. How do we call that side? (**CU5**) 14 Students: The hypotenuse. 15 Demis: That is a hypotenuse. That is fixed. But the opposite and adjacent sides are not fixed. We name them from where we name it. Now this is a little bit confusing but we always say we have to stand at an angle given (**CU2**). But how do we work out opposite and adjacent sides, when do you know which is one?

The interaction in this episode revealed that Demis continued to make links with learners' prior knowledge, which is CU5 (line 13), and provided accurate explanations of useful concepts, CU2, (line 15) such as emphasising the mathematical importance of understanding the relationships between opposite and adjacent sides in trigonometry. This in turn led to conceptual links or references to interesting contexts, giving and using examples from learners' contexts that they could relate to, which is CU4 (lines 16 and 20).

16 Demis: Okay, if I am standing here, you see there, my class is <u>opposite</u> Ms J class. Ms K class is just <u>next to mine</u>, so what would I say, her class is <u>adjacent to mine</u> (**CU4**). So when Ms Demis comes and stands here [*pointing to ∠ACB in triangle ABC*], this side [*AB*] will be what…? 17 Students: Opposite side. 18 Demis: Opposite side. And the one [*BC*] <u>next to me when standing there</u>, what will that be? 19 Students: Adjacent side.

In this episode, Demis demonstrated CU of how to construct, interpret and apply the knowledge of trigonometry in familiar contexts by using real life example of the classroom blocks to clarify the position of a right angle triangle sides through using words like "opposite", "next to mine/me" and "adjacent to mine" (lines 16 and 18). These examples involved multiple representations such as her mathematics classroom and other fellow teachers' classrooms in the school to model the interrelationships between opposite, adjacent and hypotenuse sides of right-angled triangles. Specifically, she used these practical contexts to provide or elicit mathematical representations, explanation, justifications and explicitness around trigonometric ratios.

Demis used strategic examples from learners' contexts they could relate to, as well as mathematically appropriate and comprehensive language to enhance students' conceptual understanding, which is CU4 (lines 16 and 18). She used practical contexts and stories to engage learners with the lesson content, which is CU4 (lines 20 through 24).

20 Demis: Okay, let me give you a story to just refresh your memory. If there are three people want to play Chess game, how many people can play at a time? (**CU4**)
21 Students: Two.
22 Demis: Two people can play the Chess, the one given and the one asked. This is the one given and this is the one asked [*referring to triangle ABC*]. So in this triangle, this side [*hypotenuse*] is the one given and this side [*adjacent*] is the one asked. Okay, who stays out of the game?
23 Students: Opposite side.
24 Demis: Opposite side. Then the team players in this case, as you see, are adjacent and hypotenuse. So then you go to your SOH<u>C</u>**AH**TOA, this means **adjacent** and **hypotenuse**. So, this one will be Cos [i.e. $Cos\ 30° = \frac{adj}{hyp}$] (**CU2**).

With this particular example, Demis used the story of three people wanting to play 'chess' to model the trigonometric ratios, Cosine rule in particular, and explained that only two people can play at a time to make clear the relationship between opposite and adjacent sides. She did this to help learners to understand the relationships between the three sides of a right-angled triangle, i.e. how to name the *opposite*, *adjacent* and *hypotenuse* sides accurately and how to properly apply the Pythagoras theorem. This is evidence of CU in real practice as Demis was able to draw on such terminologies "trigonometry" and "SOHCAHTOA" for learners to gain conceptual clarity and procedural fluency on the meaning and difference between terms such as "opposite side", "adjacent side" "hypotenuse side" and "Pythagoras rule". The explanation of concepts that Demis offered were mathematically accurate and useful for the learners (CU2) to understand the concepts and ideas involved (line 24). In this way, Demis supplied her students with helpful clues by which to remember and perform trigonometry ratios accurately (line 22).

Vignette 2: The development of students' Procedural Fluency

Demis next drew a right-angled triangle on the board and continued to provide accurate explanations of the distinction between trigonometry ratios (Cosine, Sine and Tangent). She allowed students to indicate when they were certain to identify the sides of a right-angled triangle given enough information (e.g. one angle and one or two side lengths).

> 25 Demis: In this triangle ABC, AB is $3cm$, BC is $4cm$ and ∠BAC is given as x. If I ask you to work out that angle named x, step 1, we name our sides. So, if I name these sides, will I name them from C or will I name them from A? (**CU5**; **PF1**)
> 26 Students: You name them from A
> 27 Demis: So, I name these sides from A because now I name these sides from the angle asked, which is x. So, if I name these sides from A, what did I say about the sides over the $90°$. This will be the hypotenuse. You see in this example that this side BC is the opposite and this side AB is the adjacent. So, what will our team players be? In other words, who is playing the game? And who is out of the game? Hypotenuse is out of the game. So, what will our team name be? (**PF1**)
> 28 Students: Opposite and adjacent sides.
> 29 Demis: Our team players are going to be "opposite" and "adjacent". It is opposite over adjacent and should not be adjacent over opposite. If so, then mathematically we call it Cot. Okay, can you see this relationship? Make sure that you opt for the correct one on top. So, we have opposite over adjacent ($\frac{4}{3}$). So, when you press 2nd shift, you tell the calculator to go to the programs underneath. Please press for me shift, Tan, open bracket $\frac{4}{3}$, and then close bracket (**PF3**), what do you get?
> 30 Students: $53.1°$ (1 dp).

In lines 25 and 27, Demis tapped into the learners' prior knowledge using CU5 when she asked for ideas of the procedures for solving a problem and asked the learners what the next step should be in the process of solving the mathematical task (PF3). In line 29, Demis displayed knowledge of mathematical procedures when she offered an explanation of the procedures and provided algorithms to solve the mathematical problems in question (PF3).

Vignette 3: The development of students' Adaptive Reasoning

Later in the lesson, Demis provided learners with several tasks that required and emphasised deductive reasoning. The first task was done on the board and it involved a right-angled triangle discussed in the previous vignette (line 25). The second task involved a right-angled triangle with two side lengths given and required learners to work out the other side of a triangle using Pythagoras's theorem. The third problem was similar to the second task but required learners to find an angle. I will exemplify only three short examples from these tasks that elicited or showed adaptive mathematical reasoning (AR).

> 31 Demis: [*Task 1*] you get $53.1°$. If someone writes this answer $tanA = \frac{4}{3}\left(\frac{adj}{hyp}\right)$ like this $tanA = \frac{4}{3} = 53.1°$, is this answer mathematically correct? (**AR3**)
> 32 Student: Not correct.
> 33 Demis: Why not? This answer $53.1°$ is correct. But why is writing the answer $tanA = \frac{4}{3} = 53.1°$ not correct? (**AR2; AR3**)
> 34 Student: Because of the equal sign [=].
> 35 Demis: Very good. Listen people. If you just write equal sign it means that you say $tanA = 53.1°$.

> And $tanA$ is indicating that ratio which is equal to ($\frac{4}{3}$). So, if you just still keep on writing equal sign (=), you still say it is $tanA = 53.1°$, which is incorrect because $tanA$ is not equal to $53.1°$. It is angle A which is equal to $53.1°$ [$A = 53.1°$]. Okay, do not just write the equal sign; rewrite the angle to show that it is equal to that answer. (**CU2**)

Demis continued.

A	B
$x^2 = 62 + 82$ [$1st\ step$]	$x^2 = 6^2 + 8^2$
$= 36 + 64$ [$2nd\ step$]	$x^2 = 36 + 64$
$= \sqrt{100}$ [$3rd\ step$]	$x^2 = 100$
$= 10$ [$4th\ step$]	$x = \sqrt{100}$
A: $learner\ method$	B: $Demis\ procedure$

36 Demis: [Task 2] why did this person leave out that x^2 there [*pointing to a blank space in the last three steps in A*]? (**AR3**)

37 Student: it would be wrong to write x^2 there after putting a square root [$\sqrt{\ }$] on the other side of the equal sign.

38 Demis: that is good. Remember x^2 equals to 100, but not to the square root of 100 *[pointing to the* $\sqrt{100}$ *on the board]*. Though in the final examination, you have to write down x in your final answer [*writing down x^2 in the second step and x in the third and fourth step as shown in B*] (**CU2**).

Demis then engaged learners in justifying the reason for the difference in the answers:

$$FG^2 = FH^2 - GH^2$$
$$FG^2 = 2.6^2 - 1.2^2$$

39 Demis: Hans, tell me, why is this a "subtraction" [*pointing to the minus (-) sign between FH^2 and GH^2 on the board*]? (**AR2**)

40 Student (Hans): Because you have…one is bigger and the other one is smaller…yah, yah…the hypotenuse…because you have to plus two sides to get the hypotenuse side and you have to minus the hypotenuse from adjacent side to get to the answer.

41 Students: That is not how to get to the answer [*said this in chorus immediately after Hans finishes explaining to the teacher*].

42 Demis: This is almost very good! Yah, that means you add these two short lengths of the triangle to the long one, and if you are given a hypotenuse side, you have to reverse and then the reverse of addition is subtraction…you said it is the short side [*adjacent]* minus the long side [*hypotenuse*]. This is automatically wrong, but remembers it is always the long side minus the short side. (**AR7; CU2**)

The interaction showed evidence of AR. That is, Demis was able to provide students with worthwhile mathematical tasks that elicited mathematical reasoning. She also engaged learners through high level questioning that encouraged reflections and required learners to explain and justify their solution strategies which are AR3 (line 31). In line 33, interaction further displayed a high level of CU and PF. For example, the teacher confirmed learners' answers by providing accurate explanations of concepts, which is CU2 (lines 35, 38 and 42). This discussion helped to clarify the distinction between opposite and adjacent sides. It also encouraged learners to use their mathematical reasoning to justify their answers, and use procedures efficiently and appropriately.

Vignette 4: The development of students' Productive Disposition

Opportunities to create an environment conducive to fostering confidence and encouraging learners to persevere was observable, for example, organising the learning content so that it was meaningful to learners. At the end of the lesson, an assignment from the textbook was given for the homework task, which is PD1.

I now move on to Emmis' three lesson videos.

6.2.1.4. Emmis (Teacher 4)

6.2.1.4.1. Lesson Video 1: Grade 10 Circle geometry

General description of the lesson

Emmis' first videotaped lesson on circle geometry revolved around generating the rules or angle theorems in circle geometry. To ensure that learners were using the correct mathematical terminology, Emmis introduced the lesson by drawing a circle on the board, and then asked learners to identify/name and define key characteristics of the circle. Before proceeding, he first ensured that learners were familiar with the terms "diameter", "radius", "chord", "arc", "segment", and "sector" by eliciting learners' prior mathematical knowledge. Emmis was carefully attentive to differences between learners' contributions that were valid and those that were not relevant. He allowed more time for learners to identify the key parts of a circle before focusing on the rules or theorems in circle geometry. I assumed the lesson was carefully designed to engage learners actively in the conceptual and procedural development of the topic. Learners spent much of the lesson doing the real practice of mathematicians. This lesson video is discussed briefly below.

Vignette 1: The development of students' Conceptual Understanding

Emmis, in his introduction to circle geometry, drew a circle with an inclined chord on the board. He then defined the properties of a circle as follows.

1 Emmis: Good morning class. If you have a circle like this one and there is a line inside the circle like this one, what do you call this line? (**CU5**) 2 Students: It is called a chord. 3 Emmis: It is a chord. And so, what is the name of the longest chord inside the circle? (**CU5**, **CU3**) 4 Students: A diameter. 5 Emmis: [*He then drew a line passes through the centre of a circle*] so this is a chord as well. But it has a special name called "diameter" (**CU1; CU2**). Alright, let us concentrate or zoom on the chord again. What do we call this part of the circle? 6 Students: Arc. 7 Emmis: Arc. If we are dealing with this chord, this will be the minor arc [*shading the minor arc*]. And then obviously the bigger part, how will it be called…? (**CU5**) 8 Students: Major arc. 9 Emmis:Major arc. Now as you see we have the arc there which is the minor arc. And this is the major arc (**CU2**). But now you will see that this part also which I am shading with these crossing lines [*minor segment*], we can also name it or it got a special name so that we can have easy reference to these key parts of circle (**CU2**). This is called what…? (**CU5**) 10 Students: A sector

11 Emmis: A sector? Normally we refer to a sector when it is like this [*drawing another circle with two radii forming a sector*], when it has an angle which is formed at the centre like this (**CU2**). We are now talking about this one formed by the chord and an arc. It is called a segment. This one is the minor segment [*shading it*]. And of course, the bigger segment will be called the major segment (**CU2**). Okay, it is important that we remember these terminologies because we are going to be kind of deriving the theory [*theorems*]. We are going to be using also these terminologies here. Without these terminologies, it will be surely very difficult for us to be able to really formulate these theorems.

This first vignette, which I classified primarily as showing CU, suggests that Emmis used *questioning* to make explicit links with previously learned concepts and connect new ideas to students' prior or existing knowledge, which is CU5 (lines 1 to 9). In lines 3, Emmis emphasised the links or connections between different geometric concepts to offer an explicit definition and accurate explanation of useful concepts such as the interrelationships between the chord and diameter (CU3). He explicitly provided accurate definitions of terms such as "chord", "diameter", "sector" and "segment" with drawings on the chalkboard (lines 5, 9 and 11). In this instance, he used mathematically appropriate and comprehensible definitions and language (CU1) when he gave accurate definition of what a chord or diameter is (line 5). Again in line 11, Emmis correctly used mathematical terminology to provide accurate explanations of concepts (CU2) that are part of the theorems in circle geometry.

Vignette 2: The development of students' Procedural Fluency and Strategic Competence

Emmis continued by instructing his students to draw different sizes of circles with radii of their choice in order to formulate theorems in circle geometry. During this interaction, Emmis drew a circle and wrote "**Angle in a semi-circle**" on the board and said.

12 Emmis: Okay, we are going to formulate the theory now (**SC3**). This is the first theory that we are going to look at. Please take out your compasses and straight edged rulers plus your protractors. [*Gave instructions for learners to do the activity*] please, draw a circle, a quite fairly nice circle so that you can see and measure what you will draw inside it (**PF5; PF6; PF7**). If it is too small, it will be very difficult for you to actually make the conclusion that we want to have in the end. If all of you have drawn a circle, then draw a diameter, anyhow whether it is vertical, horizontal or inclined. So you should know that a diameter is a line that passes through the centre to the circumference of the circle (**CU2**).

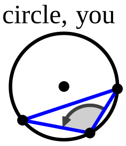
13 Student 7: So, I think you must first let us see how you do it.

14 Emmis: Alright, that is very correct. Okay, I will draw mine as well. Please pay particular attention to this. You will draw an angle using the diameter as one of the sides of the triangle inside the circle (**CU7**). The vertices of the triangle should be on the circumference of the circle, whichever way you draw it, this side or this side [*pointing to the semi-circles*]. What I mean is, this side of the triangle you have drawn will be on the circumference of the circle. But also make sure that this line must pass through the centre of the circle. And draw an angle [*in the semi-circle*] using one side of a triangle which will be on the circumference of the circle. (**CU2**)

Here, Emmis emphasised and encouraged learners' engagement with the solution of non-routine tasks, i.e. formulating their own theorems for a semi-circle, which is SC3 (line 12). This is evidence of strategic competence, which Kilpatrick et al. (2001) term as knowing what to teach, when and how to teach it.

As the lesson proceeded, Emmis provided clear steps for the mathematical procedures co-produced with students in order to develop their PF, deepen their understanding of geometric concepts (CU2) and encourage them to use techniques accurately, flexibly and appropriately (PF5, PF6, PF7) CU in order to establish that an angle subtended by the diameter is a right angle (lines 12 and 14), which is formulating theorems in semicircle (SC3). For example, he said:

15 Emmis: Use protractor to measure angle and write down what you get. Shona, what did you get?
16 Student (Shona): 90°
17 Emmis: Okay, that is what you got. Salomon, what did you get…100°?
18 Student (Salomon): I got 90°
19 Emmis:Waa, it is like all of you are getting the same angle. Okay, remember we all got different sizes of circle, different lengths of diameter. So, you can be rest assured that whenever you have a diameter like this and an angle drawn to the circumference like this, it will always give you 90°. Okay, let's formulate the theorem now. We are going to say (and write this down in your notebooks) "***An angle subtended by the diameter is* 90°**" (**SC3**). We can also rephrase this theorem differently. Some textbooks actually say "**An angle in the semi-circle is 90°"** (**SC3**). It will always be like that. But, you cannot say any angle which is subtended by any, any chord is 90°. If you see a chord like this and there is an angle drawn here, you cannot say this angle is 90° [*pointing to the chord in a circle below*]. When this line is a diameter, it should pass through the centre of the circle. And in the examination, we must tell you that this is a centre of the circle. If the examiner does not say this is a centre of the circle, you cannot assume that this line is a diameter or it looks as if it passes through the centre (**CU2**).



I also saw elements of PF and SC at play within this second vignette, and I coded them accordingly. In line 19, he provided accurate explanations and definitions of angles subtended by the diameter, which is CU2. He also supported his students in understanding how and why mathematical procedures work, making connections between those procedures and meanings underlying those concepts while integrating various strands of MP. Between lines 12 through 19, Emmis encouraged his learners to use mathematical procedures and techniques appropriately using rulers, compasses and protractors to draw a triangle inside the circle and measure an angle in a semi-circle, which is PF7. Mathematically, Emmis involved students in a rich mathematical task and real engagement with mathematical meaning through a meticulous step-by-step construction of a right-angled triangle in a semi-circle in order to allow them to gain an understanding and consolidation of key concepts and interconnected procedures displaying both PF and CU2.

Emmis continued with the whole-class discussion to engage learners with the solution of non-routine mathematical tasks to formulate more rules and theorems for finding angles in circle geometry. This time he engaged learners to establish that angles subtended by the same chord are equal. He wrote "**Angles subtended by the same chord**" on the board and then said.

21 Emmis: Draw another circle and indicate the centre (**CU7**). Draw a chord anywhere you want, and make sure the chord does not pass through the centre of the circle. It can look like this, but it does not need to be inclined like mine. Yours can be horizontal or vertical (**PF3**). Take your ruler and draw a triangle using this side [*chord*] as your base. Make sure this angle is at the circumference [*pointing to the angle in a circle he drew on the board*] (**PF5**). Then draw another angle using the same base. When you finish, please measure the sizes of those two angles you have drawn and write down their values. Make sure you measure those angles accurately (**PF5**). Alright, let's look at the results, what did you get?
22 Student 14: 30° and 30°
23 Student 15: 50° and 50°
24 Student 16: 40° and 40°
25 Student 17: 60° and 60°
26 Emmis:[*He then wrote down students findings on the board*] Okay you can tell me now, what conclusion can you formulate about this? (**SC3**)
27 Student 19: Angles subtended by the same chord are the same.
28 Emmis: Angles subtended by the same chord are equal. Okay, all of us are correct. Please conclude "**Angles subtended by the same chord are equal**". Some books say "**Angles subtended in the same segment are equal**" (**SC3**). And they will always be equal when are in the same segment. As you can see, this is the minor segment and this is the major segment (**CU2**).

This extract illustrated how Emmis addressed students' comprehension of mathematical concepts and procedures, which Kilpatrick et al. (2001) termed CU and PF. In lines 21 to 28, Emmis used a procedural knowledge approach to encourage the students to use mathematical procedures and techniques accurately (PF5) to appropriately formulate the theory or notion of angles subtended by the same chord, which is SC3.

After this task the lesson proceeded at a steady pace to afford time for developing ideas and more theorems in circle geometry. Subsequently, Emmis engaged his students in a similar task to investigate the relationship between "angles subtended at the centre and on the circumference of the same circle". Emmis wrote on the board **"Angles subtended at the centre and circumference of the circle**", and said.

29 Emmis: Okay, let's go to the next theorem (**CU7**). That is what we want to check now, those two angles, one at the centre and the other one at the circumference. And please make sure we want to conclude based on your results (**SC3**). Draw another circle. Put a dot on the centre (**CU2**). When you do this, draw an angle subtended at the centre like this. Draw another angle, this time, starting from the same, same points (here and there) exactly subtended at the circumference like this. Measure these two angles and write down what you have found (**PF3**). Okay, can you give me your results?
30 Student 22: 21° and 42°
31 Student 23: 32.5° and 65°
32 Student 24: 33° and 66°

The above extract clearly showed that as Emmis emphasised development of students' SC, he also addressed their CU and PF (line 29). For instance, he addressed SC by involving students in doing the task themselves, as they did previously, to measure angles in the same segment and establish that the angle at the centre is twice the angle subtended on the same chord (line 29). The fact that Emmis engaged learners in a task that is conceptually rich to formulate angle theorems in circle geometry that might be used in subsequent lesson suggests his ability to "devise novel solution strategies when needed" (Kilpatrick et al., 2001, p. 126).

Vignette 3: The development of students' Adaptive Reasoning

Emmis then took the discussion to the next stage of **adaptive reasoning** to encourage learners to think deductively and conclude appropriately.

> 33 Emmis: Look at your results. What is the relationship between the angles subtended at the centre and angles subtended at the circumference? (**AR2**)
> 34 Student: Angles subtended at the centre are more or higher than angles subtended at the circumference.
> 35 Emmis: Okay, angles subtended at the centre are always going to be bigger than the angles subtended at the circumference if they are always subtended by the same chord. But now, how big or what is the really relationship between these angles (**AR3**; **AR4**)?
> 36 Student: Angles at the centre exactly double angles at the circumference.
> 37 Emmis: Okay, let's conclude now by saying (please write down this in your notebooks): "**Angle subtended at the centre is 'twice' angle subtended at the circumference by the same chord**" (**SC3**). This is true for all, all angles that we have measured here as you can see. For example you get 21° and 42°; 33° and 66° and so on. You can now believe in this theory. So, if you see that there is a big difference which is not of one angle twice the other, then it means there is a problem (**CU2**).

I coded this vignette in the adaptive reasoning category because Emmis engaged with his learners through higher level questioning that encouraged reflection (AR3) to justify their conclusion using deductive thinking or reasoning, which is AR4 (lines 33 and 35). Kilpatrick et al. (2001, p. 129) describe adaptive reasoning as "the capacity to think logically about the relationships among the concepts and situations". Emmis asked questions that required students to reflect critically and explain the relationships between the sizes of two angles under consideration (line 35). Emmis further provided accurate explanations as to why the angles subtended at the centre were twice the angles subtended at the circumference of the circle, which is CU2 (line 38). Hill et al. (2008), for example, argue that "teachers must hold in mind the central ideas around a particular topic, choose lessons that help build these ideas and then enact the lessons such that they relate to those ideas in a meaningful way" (p. 478).

Vignette 4: The development of students' Productive Disposition

In the course of this lesson video, Emmis affirmed learners' responses (PD2). He encouraged and motivated his learners by making the lesson interesting with hands-on activities (PD7). Before dismissing the class, Emmis assigned a similar set of problems for the homework task for students to experiment with various strategies and theorems they had developed and discussed in class. The homework task was also given to encourage learners to do mathematics outside of the classroom (PD1). As Kilpatrick et al. (2001, p. 131) argue developing a productive disposition requires "frequent opportunities to make sense of mathematics”. I thus argue that the hands-on mathematical activities that Emmis employed enabled learners to make sense of fundamental mathematical principles or theorems in circle geometry.

The next video description illustrates Emmis’ second observed lesson.

6.2.1.4.2. Lesson Video 2: Grade 10 Theorems in circle geometry

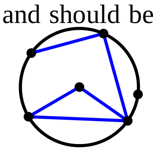
General description of the lesson

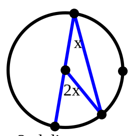
Emmis’ second lesson focused on the consolidation of rules and theorems in circle geometry. The entire lesson was focused on the homework tasks given in the last lesson to explain mathematical meanings and ideas. In particular Emmis focused learners’ attention on the concepts of angles in semi-circle, angles subtended by the same chord in the same segment and angles subtended at the centre and on the circumference of the circle. While working on the tasks, learners were asked to copy the tasks discussed in their notebooks so that they could correct wherever they had gone wrong. In the course of the lesson the teacher interacted with students through a questioning approach. This approach was used quite effectively to provide a quick check on students understanding of concepts. In the learning process the teacher challenged students’ thinking whilst encouraging them to explain different ways to find answers and justify or clarify their mathematical thinking.

Vignette 1: The development of students’ Conceptual Understanding

Emmis’ second observed lesson on circle geometry involved.

1 Emmis: What did we learn yesterday, which one can you remember? Lutenda! (**CU5**)
2 Student 1 (Lutenda): An angle subtended at the centre is twice as much as the angle subtended on the circumference of the circle.
3 Emmis: Subtended by the same chord. It must start from the same chord and should be in the same segment (**CU1**; **CU2**). [*He then drew two diagrams as shown below*].
Is this angle at the centre twice that angle at the circumference?[1st *diagram*]
4 Students: No.
5 Emmis: Why not? Linda! (**AR2**)
6 Student 3 (Linda): The two angles do not have the same base.


1st diagram



2nd diagram

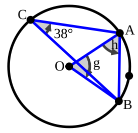
7 Emmis: Remember that they have to be subtended by the same chord. If this angle is $2x$, this one is going to be equal to x as long as they come from the same chord, one subtended at the centre, one at the circumference (**CU6**; **CU2**). And then the one subtended at the centre is twice the one subtended at the circumference.

In this first vignette, which I coded primarily as CU, Emmis conceptually focused on an increasingly categorisation of theorems in circle geometry. Through a questioning approach, the links to learners’ prior knowledge, which is CU5, was of necessity selective (line 1). On the basis of learners’ responses Emmis provided accurate explanations of useful concepts for students who were, as yet, unsure of the theorems they were expected to learn. For example, in lines 3 and 7, Emmis elaborated on angles subtended at the centre and on the circumference (CU2) using mathematically appropriate language and comprehensible definitions and representations, which is CU1. In that way, he made conceptual links to other area of mathematics such as algebra (CU6) by using $2x$ and x with two diagrams to enhance students' CU of the angles subtended at the centre and on the circumference (line 7). In line 5, he exemplified the strand of AR when he asked his learners to explain or justify their answers, and I coded this as AR2.

Vignette 2: The development of students' Adaptive Reasoning and Procedural Fluency

During the homework answer exchange, Emmis dealt with fifteen mathematical problems intended to address learners' PF. I use two examples of the homework tasks on angles that show Emmis' attention to developing learners' PF and AR. Emmis went to the board and invited learners to share their solving methods for finding angle sizes in various shapes involving circles. (It is useful to mention here that this extract is discussed further in relation to the aspect of enactivism).

8 Emmis: Okay, in this task let's look for the values of g and h. This angle is 38°. So what can we do here, what is the value of g? (**PF1**)



Taks 12

9 Students: g is equal to 76°.
10 Emmis: Okay, you are saying g is equal to 76°, why? (**PF2; AR2**)
11 Student 12: Because g is subtended at the centre and is twice the angle at circumference.
12 Emmis: Okay, because g is subtended at the centre and doubles the one subtended at the circumference. That is $g = 38° \times 2 = 76°$ (**PF4**). So, what is the value of h now (**PF1**)? If you tell me the value of h, I will respect you for a very long time (**PD6; PD8**). What is h? This is your time now. John!
13 Student (John): 30°
14 Emmis: Why? (**PF2; AR2**)
15 Students: Because it is an equilateral triangle.
16 Emmis: We said in an equilateral triangle, all angles are equal to what?
17 Student 13: 60° (degrees)
18 Emmis: But, you said 30°. So, you are saying something else (**AR3**). What is the value of h? (**PF1**)
19 Student14: h is 38°
20 Emmis: Why? (**PF2; AR2**)
21 Student 14: I divided by 2.
22 Emmis: Why did you divide by 2 (**AR2**)? [*He then drew triangle OAB separately on the board*] you have this triangle here, so you can see it clearly. We have 76° here. What are you saying now? (**AR3**)
23 Student 15: $h = 26°$
24 Emmis: So, you got 26°, but the question is "why"? (**PF2; AR2**)
25 Student 16: Then that angle is half of the other angle.
26 Emmis: Is this angle half, who said that? Look here, in mathematics we never assume that this angle is half of this angle [h] unless we tell you that this angle is equal to this (**CU2**). These diagrams are not the same. That reason is out. Please try, try you must reason alongside your listening. Like I said you have to step up your level best (**PD6**)
27 Student 17: Sir, sir, I think that angle [h] is 45°.
28 Emmis: Why? (**AR2**)
29 Student 18: Because the angle CAB is subtended by the diameter, so angle at the top is 90°.
30 Emmis:Which angle is 90°, this one [*pointing to CAB*]? There is no diameter here. The diameter has to

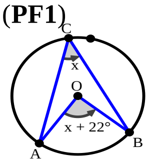
pass through the centre. Now, what can we do?
31 Student 19:I think h is equal to $32°$.
32 Emmis: Why? (**AR2**)
33 Student 19: Because the triangle should add up to 180°, and h and that angle opposite [*OBA*] are equal.
34 Emmis:Angle h and this angle [*pointing to OBA*] are equal. It is true your answer is correct and his answer is correct too. But why, why did you step up and say they are equal. You can stand in the court and say I know they are equal and there is no other way, they have to be equal. Why, why (**AR2**)? Chile!
35 Student (Chile): Because the distance from the centre to the circumference is always the same.
36 Emmis:[*Pointing to lines OB and OA in an isosceles triangle AOB*] because the distance from the centre to the circumference is always same. These two lines are equal (**CU2**). So, they form what? (**AR3**).
37 Students: Isosceles triangle
38 Emmis:Yeah, I spent more time on this question to make sure that you will get this. More questions of this nature will come, and never ever be tricked by that. The examiners will never ever tell or indicate to you that these two lines are equal. But you will have to know that it is isosceles triangle and therefore angle h is equal to this angle OBA (**CU2**). You divide by two, so h is equal to 180° minus $76°$ and then divided by 2, which is $h = \frac{180°-76°}{2} \Rightarrow h = 52°$. So, let's put the reason there. Because OBA is an isosceles triangle and $OA = OB = radius$. (**PF3**)

In the interactions above, I encountered the interconnectedness of the strands of MP and coded it accordingly. My analysis of this episode revealed that Emmis provided situations and higher-level tasks that required AR (logical reasoning and justification) as well as CU and PF. Emmis asked questions so learners needed to explain and justify their answers or their solution strategies (AR2) as well as the methods they used for solving problems (PF2) (see lines 10 through 24). For instance, he asked questions like "why is $g = 76°$" (line 10) and why h equals to $26°$ or $32°$ (lines 24 and 32). As evident in line 36, he engaged with learners and encouraged reflection (AR3). In line 12, he showed a high expectation of learners (PD8) to work hard (PD6). This was evident when he said "*if you tell me the value of* h*, I will respect you for a very long time*" (line 12).

Emmis also demonstrated good CU and showed facility with PF. His questions, comments and explanations were intended to direct students to particular investigations and explanations of the underlying meaning of mathematical concepts (CU2), and to find ways to understand and solve the problem at hand (PF1) (lines 8, 10, 12 and 14). Specifically, he posed a variety of questions, drawing on learners' prior knowledge, that solicited the procedures for solving a problem or the next step in the process of solving the mathematical tasks under discussion, which is PF1 (lines 12 and 18). Emmis further explained and applied the correct formula as he demonstrated the procedures for calculating the value of h (PF3) in order to clarify and enhance learners' CU of isosceles triangles formed by radii (line 38).

Emmis moved on to another task on angle subtended at the centre and on the circumference, extracting the procedure for solving the task and pressing learners to justify their answers and thinking or reasoning.

39 Emmis:Alright let's look at task 15. Can you tell me, what do they want us to do here? (**PF1**)
40 Students: They want us to find the value of x.
41 Emmis: So, what can we do to find x? (**PF1**)



Task 15

42 Student 1: x is $22°$ because $x + 22°$ must be double of the value of x, so we add them up when it is double.
43 Emmis: [*Smiling*] so, you are saying this x at the centre is $22°$. So that means the other x is also $22°$. Let's hear whether that is true. (**AR1**, **AR6**)
44 Student 2:We have to multiply the angle subtended at the centre by 2 because it is twice that x.
45 Emmis: So, you are saying this $2(x + 22°)$ (**CU6**). Ok I agree that this angle is "twice" the other one. But now you make it four times bigger. It is already twice bigger than this one. Let's say for example if this $(x + 22°)$ is $100°$ then this (x) must be $50°$. Now you multiply $100°$ by 2 again (**CU2**; **PF4**). But you now need to think the other way round (**AR3**). What can you do? (**SC5**)
46 Student:Sir, we could say $x + 22°$ is equal to $x + 22°$ divided by 2.
47 Emmis:You are saying $x + 22° = \frac{x+22°}{2}$. Look at this mathematics, how can you say $100°$ is equal to $100°$ divided by 2. This is just what you say $100° = \frac{100°}{2}$, this is fifty (**PF4; AR3**). Make sure when you put equal sign you are correct (**AR4, CU6**).
48 Student: Sir x is equal to $x + 22°$ over 2.
49 Emmis:You say x equals to $x + 22°$ over 2 [$x = \frac{x+22°}{2}$]. I agree with you because the value [*angle*] at the circumference is half the one at the centre of the circle. That is correct. Or that value x multiplied by 2 it becomes what…?
50 Students: $x + 22°$
51 Emmis:That is $2x = x + 22°$ and then you make x the subject, which becomes $x = 22°$.

Over the course of this task, Emmis constantly pressed students to develop conjectures thereby explaining and justifying their reasoning or answers (AR1). He also asked probing questions so students needed to reflect critically (AR3) in order to derive complete responses from them with accurate mathematical explanations and justification (lines 39 and 41). This discussion also strikes me as SC because Emmis engaged with students to provide critical reasoning and argumentation (SC5) (line 45). In line 43, Emmis invited constructive criticism and feedback from learners (AR6). As he elaborated on solving procedures suggested by the students (PF4), Emmis also displayed a sound skill in PF, which is crucial for developing students' SC (lines 39 and 41). At the same time, he made conceptual links to algebra, which is CU6, to encourage learners to formulate and solve algebraic equations, justify their work, and explain ideas in order to make their reasoning clear (lines 45 to 47).

Emmis' conversation with students in this vignette was good for mathematical thinking, reasoning and solving problems. It could be argued that the interaction in the above extract enabled students to reach a higher level of CU and abstraction, which in turn developed their capacity to argue, talk about ideas, interpret mathematical concepts, and solve mathematical problems collaboratively.

6.2.1.4.3. Lesson Video 3: Grade 10 Cyclic quadrilaterals

General description of the lesson

Emmis' third lesson was on angles in cyclic quadrilaterals where he investigated and formulated theorems. This included pairs of opposite angles in a cyclic quadrilateral are supplementary, and that an exterior angle at a vertex of a cyclic quadrilateral equals the interior opposite angle. As usual, the teacher began with a review of homework task given in the previous lesson, discussing and presenting solutions to six homework tasks. The homework tasks included problems in circle geometry taken from the mathematics textbook. The teacher modelled the solutions for the first two problems and invited students to share their solutions for the remaining problems on the board but was hindered by the available time. The idea was to get students to critically identify and explain how the problems in the homework task related to each other and to the theorems in circle geometry discussed in the previous two lessons. The teacher used a variety of whole-class, teacher-directed instructional strategies before asking students to work in pairs on the final task. There were a number of activities occurring so eventually the teacher moved the focus to formulating angle theorems in cyclic quadrilateral.

Vignette 1: The development of students' Conceptual understanding, Procedural Fluency and Strategic Competence

Emmis engaged his students in a conceptually rich task in order to address their CU, PF and SC. He wrote on the board "**Angles in a cyclic quadrilateral"** and the following interactions with the students occurred.

1 Emmis: Okay, take out your mathematical instruments to develop more theories, more theorems (**SC3**). We do not use "circle" this time around. I do not know why or what is the reason for using cyclic. I even think there is not a spelling mistake. That is how they call cyclic quadrilateral. What is a quadrilateral? (**CU5**)

2 Student 7: A four sided shape.

3 Emmis: A four sided shape. Do we all agree? (**AR6**)

4 Students: Yes.

5 Emmis:Okay, as usual, you draw a circle first (**SC3**). [*He then drew a circle on the board*] and then you draw a four sided shape with all its vertices on the circumference. [*He then drew a quadrilateral with vertices off the circumference*] you cannot draw a four sided shape inside the circle and it looks like this. It must have all its vertices on the circumference of the circle (**PF3**). So draw one inside the circle, and please measure all the angles inside the quadrilateral ABCD accurately (**PF5**). We are going to draw or make some very interesting conclusions. [*He then started moving from desk to desk to see what students were doing*] okay, let me start filling up your results on the board. What did you get for these angles? (**PD7**)

<u>Angles of a quadrilateral</u>:	**A**	**B**	**C**	**D**
<u>Student findings</u>:Student 1:	90°	77°	72°	81°
Student 2:	90°	90°	90°	90°
Student 3:	72°	85°	82°	95°
Student 4:	89°	97°	91°	83°

Now look at your own results and add them up to see what you get. I want you to look at the opposite

angles of your quadrilaterals-you can take angle A and C or B and D. When you add the opposite angles, what did you get? (**CU7**)
6 Students: 180°; 181°; 184°; 180°; 180°; 180°; 180°; 183°…
7 Emmis: So what is your conclusion then?
8 Student 16: Two opposite angles add up to 180°
9 Emmis: What is your conclusion again? (**SC3**)
10 Student 17: Opposite angles in a cyclic quadrilateral are equal to 180°.
11 Emmis: **Opposite angles in a cyclic quadrilateral are equal to 180°** (**CU1; SC3**).

An interesting interaction that happened in this vignette was the interconnectedness of all five mathematical strands that underpinned Emmis' classroom discussion with his students. In line 1, for example, Emmis exemplified CU through making links and connections to previously learned concepts and to learners' prior knowledge (CU5) in order to facilitate mathematical learning. He also showed SC in encouraging learners' engagement with the solution of non-routine tasks (SC3) (line 5), which emphasised theory formulation and deductive reasoning. In this sense, he provided an opportunity for his learners to engage in practical construction and practices of mathematicians such as hypothesising, making conjectures and justifying their ideas and findings (SC3) (lines 5 through 18). Of course, development of concepts or hands-on experience alone is not sufficient so Emmis explored the opportunity to address students' PF. Hereby Emmis encouraged learners to use mathematical procedures and techniques accurately, which is PF5 (line 5). He also used mathematically appropriate and comprehensible definitions and language (CU1) when he provided accurate explanations of opposite angles in a cyclic quadrilateral (line 18), in order to justify learners' answers (line 11). Thus, Emmis also supported the development of students' PD by making the lesson interesting with hands-on experience or activities (PD7) through a theoretically rich mathematical activity (CU7) (line 5).

Emmis then assigned similar problems or tasks involving angles in cyclic quadrilateral to help students see the links and connections between different geometric concepts and the interrelationships of properties of shapes in cyclic quadrilateral. During the course of these tasks, he engaged students in devising problem solving strategies through questioning. He also engaged with students to reflect critically on solving procedures suggested by others to encourage them to use mathematical formulae and procedures flexibly and appropriately.

In the following section, I focus on Sann' instruction and briefly illustrate the extent to which it fulfilled the five strands of teaching for mathematical proficiency (MTP).

6.2.1.5. Sann (Teacher 5)

6.2.1.5.1. Lesson Video 1: Grade 11 Geometrical terms and angle properties

General description of the lesson

After entering the class and greeting the learners, Sann commenced by saying *at the back we have Mr Stephen. He is visiting us and he would like to see how we are coping with mathematics in our school. We will go through geometry. Geometry composes of building blocks. That is in geometry, we have a point. If we have a set of points next to one another, they make a line*. Sann' first geometry lesson revolved around basic terminologies and properties of angles. The lesson took approximately 45 minutes and started with a review of basic terms and line properties of angles in geometry. These terms included "point", "parallel and perpendicular lines" and "angle types including right, vertically opposite, complimentary and supplementary angles" as well as "congruency". Angle properties existing between different plane figures such as triangles were also discussed. The teacher distributed two worksheets to each student. The first worksheet focused on geometric terms and angle properties and was split into three columns indicating the *geometrical terms, angle explanation* and *the drawings of angles*. The second worksheet included multiple problems requiring vocabulary recognition, rules and procedures as well as application of angles in various two-dimensional diagrams. I observed that the worksheets were designed to help students develop an understanding of fundamental angles and their application to solving problems in different drawings. The teacher projected all the worksheets onto the board by using an Overhead Projector (OHP).

Vignette 1: The development of students' Conceptual Understanding

Sann took the class through the first worksheet which was displayed on the board. After a brief explanation of the concept of geometry and definitions of different angles, she continued the discussion with a series of questions about parallel lines and angles in familiar contexts. For example:

1 Sann: We have straight lines in the class. If you have two straight lines next to one another, they form parallel lines. And if you look around the classroom there are all different parallel lines around us. Is there anybody who can tell me what in the class which are parallel lines? (**CU5**) 2 Student: Window frames 3 Sann: [*Pointing to the window*] window frames. Most of the straight lines in kit form parallel lines. What else? (**CU4)** 4 Students: Table, Door frames, Door edges, Chairs… 5 Sann: Chairs, everywhere around us, there are parallel lines. Parallel lines mean there are two lines next to one another, but never clash into one another and never go straight further away from each other, and the distance between them stays the same (**CU2**). [*Writing on the worksheet*] but whenever you are given this, there are two small arrows to indicate that they are parallel lines. You know that when you have more than one single parallel line, there are two single arrows to indicate that they are parallel.

The introductory exchange in this episode showed evidence of CU as Sann incorporated learners' prior knowledge about geometrical terms (CU5) right from the onset of the lesson (lines 1 and 3). The links to prior knowledge encouraged learners to think about the concepts they were engaged with. It also challenged learners to articulate and relate their mathematical ideas to things they knew (lines 3, 4).

As the lesson progressed, Sann explained the meanings of angles formed by two lines intersecting at right angle.

6 Sann: What are perpendicular lines? (**CU5**)
7 Student 6: They cross each other
8 Sann: They cross each other. But there is something specific about the cross?
9 Student 7: They form 90°
10 Sann: There is 90°, okay. Where are the perpendicular lines in this classroom? (**CU4**)
11 Student 8: Window frames.
12 Sann: The edges of the window frames are perpendicular lines. Where else? (**CU4**)
13 Student 9: The door edges.
14 Sann: The door edges up there, they are perpendicular lines. Your table edges, and the way I am standing now. The fact that I am not totally falling over, I am also perpendicular to the ground or class floor. I make the 90° angles. And we know that by the little squares that indicating this (**CU2**). That is why these lines make 90° [*pointing to the perpendicular lines on the worksheet on the OHP*].

This interaction showed evidence of CU that was conceptual in nature as Sann engaged her learners with a lively and contextual introduction to the concept of perpendicular lines in the everyday world. The learners engaged with the concept meaning (CU5) by eliciting practical examples and making connections to real world contexts (lines 1 and 4). As is evident in lines 3 and 12, Sann used practical examples from learners' contexts that they could relate to (CU4). Indeed, Sann used strategic observations and evidence around her mathematics classroom to develop and explain fundamental concepts clearly using real life contexts or examples to enhance students' CU (lines 6 to 13). Accurate definitions and examples of both parallel and perpendicular lines were fully explained and discussed with students rather than simply working from the information provided on the given worksheet (lines 5 and 14).

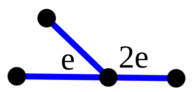
One particular example of CU that struck me was when Sann offered a *metaphor* to define and explain the meaning of perpendicular lines (line 14). In this particular instance, she represented the concept of perpendicular angles by relating it to the way she was standing on the classroom floor. For example, in line 14, she explained and demonstrated that as she was standing up straight and not falling over, she was perpendicular to the floor because her body posture formed angle 90° with the classroom base/floor (CU4).

18 Sann: Then "*Complimentary Angles*". If I give you a complimentary ticket to a show, it means it is for free, you do not need to pay for it. Now I go to the concert, and when I get there I have to take a chair and lie down [*while sitting down on the table*] I have to sit and watch. So, you sit up straight. So, complimentary angles add up to 90°. [*While sitting on the table*] when I sit like this, my body here make 90° angle with the table (**CU4**). Then "*supplementary angles*", if I supply you with something, say a pen. Am I only giving to you half of a pen?
19 Students: No
20 Sann: No, I give the whole pen to you, and it is in a 'straight line form'. Supplementary angles add up to 180° because you give the whole thing as it is. Whatever you give somehow, the line that I am giving it to you with is straight line. So the straight line that you are giving it with is your supplementary angle, which means 180 degrees as I lie down on the top of the table (**CU4**).

Again interaction was lively and developed students' geometrical understanding of concepts of complimentary and supplementary angles. Defining supplementary angles, by giving a pen to a student and then explaining that the line of a pen was a straight line of 180° is CU4 (line 23). In a similar incident, the teacher demonstrated a reflex angle (CU4) and showed how a reflex angle was formed when a learner who was pushed in the face bent backwards but never moved or fell down. She also used strategic examples and definitions to beef up students' CU.

Vignette 2: The development of students' Procedural Fluency

Sann led a whole-class discussion based on the second worksheet given to students at the start of the class. The activity focused on the properties of lines and angles to demonstrate how these mathematical terminologies are used to solve problems in different angle drawings. Most of these problems on the worksheet required algebraic knowledge, and Sann worked out some questions on the OHP transparency. She then asked students to continue with the task while moving around the class, guiding and helping learners individually. Sann did not do the work for them and students suggested different ways of solving the problems they were working on. Again learners were very competent in carrying out the task, and correctly worked out most of the problems.



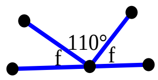
21 Sann: [*Continued with the exercise*] number 5, what is the value of e? (**PF1**)
22 Students: 60°,
23 Sann: let see how you get to that 60°. These two angles add up to what?
24 Students: 180°
25 Sann: So, $180° = e + 2e, \Rightarrow 180° = 3e$. Then 180° divided by 3 gives us 60° (**PF3**).

Lines 21, 23 and 25 captured evidence of Sann addressing the development of PF as she was able to quickly move to the use of different angle diagrams so that the students could see angles in diagrams to grasp the meaning of underlying concepts and practice the procedures connected to those concepts. For example, in lines 21, Sann promoted the use of rules as she asked learners questions that solicited the procedures for determining the values of angle e, which is PF1. She also provided further explanations on the procedures and problem solving methods, which are PF3 (line 25). As Kilpatrick et al. (2001) argue procedural fluency is not limited to the ability to use procedures, it also includes an understanding of when and how to use them which Sann was able to do to promote the development of PF of her students.

Vignette 3: The development of students' Adaptive Reasoning

Adaptive reasoning was also displayed during the second task during the work on the question in the diagram below from the activity worksheet. During this task the teacher asked for the meaning of 'f and f' angles and why they are equal.

26 Sann: Okay, let's go to number 6. [*Moves from desk to desk*] let's see where we can start? What is the meaning of f and f angles there? (**AR2**)



27 Students: They are equal.
28 Sann: Why are they equal? (**AR2**)
29 Students: They have the same value.
30 Sann: Okay, they have the same value. And so $180° = 110° + f + f$. This $110°$ was on this side of equation, now we bring it to that side, what can we do?
31 Students: Subtract
32 Sann: Subtract, and then $180° - 110°$ is…?
33 Students: $70°$
34 Sann: $70° = f + f$. And $f + f$ is…?
35 Students: $2f$
36 Sann: $70° = 2f$, what must I do with my 2?
37 Students: Divide both sides.
38 Sann: $\frac{70°}{2} = f$, what is $70°$ divided by 2?
39 Students: $35°$
40 Sann: $35° = f$, that is the size of each angle there.

Here Sann encouraged the students' development and articulation of their justification of mathematical ideas for example, in lines 26 and 28, thereby showing AR2. It was interesting to note that, throughout the lesson, Sann rarely gave direct answers. She often asked questions that encouraged the students to think mathematically and more deeply about their work. For example, in the box above, Sann seemed to facilitate forms of mathematical thinking that involved more than memorisation and recall of facts and procedures (see lines 30 to 40).

Vignette 4: The development of students' Productive Disposition

In my observation, Sann manifested a high positive disposition as she shared real life and practical examples explaining to students how mathematics relates to architecture and design in order to create mathematical confidence among her learners. In this way she exuded competence in her ability to work with mathematical ideas and real life examples. She joked from time to time and the students were happy with her kinaesthetic explanations of concepts.

As evident in the box below, Sann also provided learners with a homework task to encourage them to do and practice mathematics outside of the classroom, which is PD1 (line 41).

> 41 Okay, those tasks in numbers 1 to 3, exercise 24.1 on pages 187 to 188 will be your homework (**PD1**). Please do that, we will look at them tomorrow.

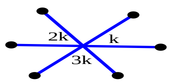
6.2.1.5.2. Lesson Video 2: Grade 11 Investigating angle properties

General description of the lesson

In this lesson Sann made an explicit links to concepts covered in the previous lesson on geometric terms and angle properties. The discussion built up progression from concept meanings to formulation of algebraic equations, thereby revitalising students' understanding of concepts, mathematical fluency and their problem solving skills. Sann passed out two worksheets to each learner to do the tasks individually and in pairs. The tasks on the first worksheet included problems based on familiar contexts involving angles at a point and on straight lines in addition to vertically opposite angles, co-interior angles, alternate angles and corresponding angles. The second worksheet included triangles and quadrilaterals made by parallel lines. These tasks were assigned to allow students to explain how they got their solutions to those tasks in order to build their understanding and develop the necessary fluency. Most of these tasks required multiple solution strategies. However, for the analysis of this lesson video, I chose only two tasks that show evidence and interconnectedness of CU, PF, SC and AR at play.

Vignette 1: The development of students' Adaptive Reasoning

Good morning class [Sann said]. I want us to continue with the worksheet that we were busy with yesterday. She then put the first worksheet on the OHP to discuss and demonstrate procedures for solving mathematical tasks.

> 1 Sann: Now let's look at task 10. I want to know something here. If this angle is k, and I want the value of this one [*pointing to the vertically opposite angle*], what will that value be? (**AR3**)
> 2 Students: k
> 3 Sann: Why is it k as well? (**AR2**)
> 4 Students: Opposite angles.
> 5 Sann: It is not just opposite angles, but vertically opposite angles (**CU1**)
>
> 
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The interaction with students that emerged in this episode revealed evidence of AR with some elements of CU at play. Line 1 clearly established that Sann was building on students' fluency in mathematical language and angle definitions to facilitate problem solving. For example, she encouraged reflection (AR3) to think more deeply about angles made by vertical lines. Also, in line 5, Sann demonstrated CU when she used mathematically appropriate definitions and language to explain that k and the other angle are 'vertically opposite angles', which is CU1.

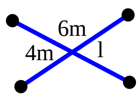
The importance of students' own ideas and methods hence prompted Sann to continue to ask students to think of the method and work out the value of k individually, as appears in the next episode below.

6 Sann: [*She then started moving around the class*] What are you going to do here? (**PF1**)
7 Students: Divide $360°$ by 12.
8 Sann: Why did you divide by 12? (**AR2**)
9 Students: Because there are $12k$.
10 Sann: Ok this means if you add up all the k, you get $12k$ and then you say $360°$ divide by 12 to get the value of k. So, $360° = 12k \Rightarrow 30° = k$ (**PF4**)

Although the strand of AR seemed to dominate the extract above, PF was also evident. For instance, in line 6, the teacher asked learners questions to extract the procedure for determining the value of k (PF1). She also explained and elaborated on the procedures suggested by the learners (PF4) to encourage them to use mathematical procedures correctly (line 10). At this stage, however, some of the students suggested different solving methods to get the value of k. For example, some students indicated that they used three angles (k, $2k$ and $3k$) on a straight line/angle and equate them to $180°$ to get the value of k. Others used the revolution angle property by adding the six angles and equating it to $360°$ to find the value of k. Such an interaction with the students was evidence of both SC and mathematical fluency (PF) in that the teacher provided tasks that allowed for multiple solving strategies and evaluation of different solution method strategies (SC2). She also encouraged multiple procedures in solving problems (PF8) as learners had showed concept understanding and carried out procedures flexibly to get the answer.

The emphasis on learners' own ideas and methods was also stressed in the next task. This problem had different variables and required learners to work out the values of m and l. Sann asked the class to think of the problem solving strategies to approach the task so communicated to her learners that what was valued in her lesson was not just the answers they give, but also various ways they could determine the values of the two variables (m and l) in relation to the two intersecting straight lines. The dialogue below is an evidence of this.

11 Sann: This one is similar to something that you had in your homework. Over here, before you start we have two variables here. [*Pointing to the angle vertically opposite to* $6m$] Okay, if this is $6m$, this angle would be…?
12 Students: $6m$
13 Sann: Why is it the same? (**AR2**)
14: Students: Because they are vertically opposite angles.



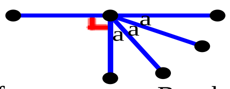
15 Sann: [*Writing it down on the diagram*] $6m$. This angle is $4m$ and this one is l, what can you tell me about $4m$ and l? (**AR4**)
16 Students: They are the same.
17 Sann: They are exactly the same. So instead of using l, you can use $4m$ [*altering* l *into 4m on the diagram*]. So, I want the value of m. If we add up these m, we get…? (**AR2**)
18 Students: $20m$
19 Sann: So, $360°$ divided by 20, it gives us?
20 Students: $18°$
21 Sann: $m = \frac{360°}{20} = 18°$.

Sann guided her students to recognise the meaning of the symbol *vertically opposite angles* (line 14), and this enabled them to devise a strategy to work out the value of m. Sann probed students for further explanation of their mathematical ideas, procedures and solutions by asking them to share how they had solved the problem (AR2), hence promoting their reasoning skills (lines 13 and 17). Similarly, she encouraged them to think deductively about the two angles $4m$ and l, which is AR4 (line 15). As she engaged them in the investigation, students were free to choose whatever method that made sense to them as they experimented with their ideas. For example, students used a straight line or revolution ideas to work out the task. Sann also gave the students a chance to determine whether or not their ideas were correct based on logic and not because she said so.

Vignette 2: The development of students' Strategic Competence

In the next segment, Sann focused on another problem on the worksheet. This time she asked students to practice individually. Sann went around checking and marking the students as they were solving the problem to determine the value of a in the task below. The following excerpt bears testimony to the fact that Sann explained all the information needed to get to the answer.

22 Sann: Okay, you are saying a equals to $30°$. I agree with you. But there are three possible ways to do this (**PF3**).



(i) You can start with a revolution if you want to. But then you have to realise that this is a straight line so on the top of the straight line we subtract $180°$ from $360°$ and then subtract again $90°$ from $360°$. Then what is left is divided by 3 to get the value a which is $30°$ [$360° - 180° = a + a + a + 90°$]. (**SC2**)

(ii) Or you say this is a straight line, which is $180°$ minus this $90°$ to get $90°$ divided by 3 to get $30°$ [$180° - 90° = 3a \Rightarrow a = \frac{90°}{3}$]. (**SC2**)

(iii) Or you can say this is $90°$ on the left side. This is a straight line, and then the right side must be equal to $90°$ divided by 3 to get $30°$ [$a + a + a = 90° \Rightarrow a = \frac{90°}{3}$]. (**SC2**)

I classified this episode as showing SC because the task that Sann engaged learners with allowed multiple solving strategies and evaluation of different solution method strategies

(SC2). For example, in line 22, Sann explained a variety of solving strategies (PF3) in order to clarify and discuss students' mathematical thinking, reasoning and problem solving methods and justify their responses. During this task, students suggested different values of angle a. For example, some students suggested $12°$ while others had given $30°$ as the value of angle a. As she did at the start of the lesson, Sann carefully justified students' responses by providing algorithms to solve the task at hand. Additionally, by providing explanations on procedures and problem solving methods, and having learners share their mathematical ideas, Sann provided an opportunity for learners to consider productive solving strategies and ideas that they could pursue. Listening to and clarifying students' solving strategies is, indeed, a key aspect of teachers' specialised content knowledge and of teaching mathematics using such mathematical tasks (Hill et al., 2008).

Vignette 3: The development of students' Productive Disposition

Throughout this lesson, Sann showed respect for students' ideas and contributions and allowed them to learn by doing written mathematics in their exercise/work books. She also gave students worthwhile tasks to do that make them see the sense in mathematics and perceive themselves as effective learners and doers of mathematics (Kilpatrick et al., 2001).

6.2.1.5.3. Lesson Video 3: Grade 11 Angle properties of triangles

General description of the lesson

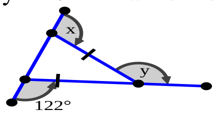
Sann' last lesson had a similar structure to the previous lessons. The introduction, consisting of reading through learners' homework assignments; the tasks in a form of an investigation, during which learners were given mathematics problems to do individually to recognise the relationships between angles in several triangles made by parallel lines, conjecturing and constructing new meanings and the whole class led discussion which enabled learners to share and integrate their ideas to reach a shared meaning of concepts taught. This lesson was typical of Sann' teaching style, and the dialogue it generated was noteworthy. During the entire lesson, the teacher was interacting with the students while moving around the class. For the analysis of this videotaped lesson, I chose only one task that showed evidence of CU, PF, SC and AR at play.

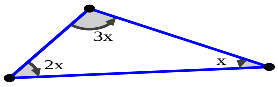
Vignette 1: The development of students' Procedural Fluency and Adaptive Reasoning

During this stage of the lesson, Sann continued to give high level mathematical tasks to the students. For example, in the tasks below there are no parallel lines but there are sides of either equal or uneven lengths that students should consider to find the values of x and y.

Hence, the episode below showed how Sann worked with the same tasks but increased the cognitive demand level of the tasks.

1 Sann: How did you worked out the values of x and y in the first diagram there? Can you explain and share your answers with others? (**PF2**)
2 Student 8: $180°$ minus $122°$ is $58°$…divided by 2 is $29°$ which is x. And $29°$ divided by 2 is $14.5°$ which is y.
3 Sann: Is she right? (**AR6**)
4 Students [*in chorus*]: No, no…she is wrong!
5 Sann: I do not agree with what she said as well.





The interaction that unfolded in this episode showed that Sann continued to ask questions to clarify students' understanding of the concepts and procedures. For example, in line 1, she asked questions where students needed to explain and justify answers or methods for solving problems under discussion (PF2). Interestingly, in line 3, Sann did not accept students' answers right away, but invited constructive criticism and feedback from the class (AR6). This helped students to learn to reason mathematically, build their confidence and rely on their understanding, thereby enhancing their PD and competence in AR.

My analysis of this lesson video indicated that Sann prepared tasks and activities that encouraged students to search for multiple solution strategies and increased the cognitive demand level by using assessment tasks that required no parallel lines. She helped students to progress in solving problems of different demands and ended the lesson with a shared mathematical meaning.

Vignette 2: The development of students' Productive Disposition

6 Circle 20 also. All the questions that you circle are for your homework. That is why we did not do them in class (**PD1**).

Development of students' PD was also evident as appears in the extract above. The lessons generally concluded with a brief summary of the main points and a homework task, which is PD1 (line 6).

6.2.1.6. Summary of insights gleaned from the teacher lesson videos

This part of Chapter Six provided evidence of the ways in which three teachers' classroom practice and their teaching proficiency supported the development of students' MP. The analysis of the lesson videos indicated that all five teachers in this case study were strong in conceptual and procedural teaching in mathematics. Kilpatrick et al. (2001) argue that PF and CU are interwoven. "Understanding makes learning skills easier, thus a certain level of skill is required to teach mathematical concepts for understanding"(p. 122). The strand of PD emerged as a "character trait in the service of these teachers' mathematics teaching" (p. 380), and was addressed in virtually every lesson. What emerged in the lesson videos was that PD was the strand that holds the other four strands of MP together. I observed fifteen lesson videos and noted that neither SC nor AR was prominent. There was little evidence of the teachers presenting or emphasising students' engagement with the solution of non-routine or authentic mathematical tasks. Hence, the analysis of lesson videos allowed me to conclude tentatively that despite the fact that these teachers showed evidence of strands of teaching proficiency in the geometry scenario-based questionnaire viz. CU, PF, SC and AR, in terms of their classroom teaching approach, they clearly taught at conceptual and procedural levels. In some cases, there were opportunities for development of SC or AR which some teachers did not engage with within the context of this case study. From my perspective, some teachers did not move their students into the space that enabled development of SC or AR.

There were some similarities regarding the nature of explanations of fundamental mathematical concepts in Jisa' and Ndara' classes. Both Jisa and Ndara attempted to develop mathematical concepts by using various representations and modelling, making conceptual links or references to interesting contexts and giving practical and real life examples from students' contexts that they could relate to. It was also worth noting that their demonstrations had an explicit quality that connected students to mathematical concepts. This included the use of precise and appropriate comprehensible language to provide accurate definitions and explanations of mathematical ideas, explicit use of symbols and notations, making real life connections for students to grasp meanings, and giving students the autonomy to solve problems and share their mathematical ideas. Effective instruction in practicing procedures and algorithms was equally evident in Jisa' and Ndara' three lessons, alongside mathematical reasoning and strategic competence through complex non-routine mathematical tasks. For example, they demonstrated the requisite procedural knowledge and skills in teaching geometry and therefore their teaching for PF promoted the acquisition of mathematical techniques as well as successful learning.

The study now turns to Part II. Once again, for the purpose of this dissertation I will only report on the analysis of three teachers, namely Demis, Emmis and Sann.

6.2.2. Part II: An in-depth analysis of teacher practice through the lens of enactivism

6.2.2.1. Introduction

From my analysis of lesson videos and post lesson reflective interviews, I was able to see the five strands of Kilpatrick et al.'s (2001) framework of teaching for mathematical proficiency (see sections 6.2.1.1 to 6.2.1.5). However, what was not evident was how the teachers' bodily actions interacted with the learners' learning. Some of what was unseen was the "teachers' or learners' judgement and reasoning in actions, but some were physically visible but overlooked nonetheless" (Lewis, 2009, p. 77).

The notion of enactivism enabled me to analyse these actions. The enactivist paradigm, described in the literature review and methodology chapters of this thesis, was a helpful lens to complement Kilpatrick et al.'s (2001) model to facilitate a deeper analysis of teacher practices. I now continue with the data analysis and interpretations based on the enactivist theoretical concepts of **autopoesis**, **embodiment** or **structural determinism** and **co-emergence** that shaped the teaching practices of the participating teachers. In this study the notion of autopoesis allowed me to describe essential properties of complex living systems and to understand how an agent (in this case the teacher and students) produce meaning. It also enabled me to understand how effective learning occurs through an agent's interaction with the environment. The notion of structural determinism offered me a way to understand teachers' embodied cognition, and to see how effective teachers engaged with problem posing and problem solving as well as mathematical explanations carried out in classrooms. The concept of co-emergence informed my data analysis by looking at the interactions or possibility for unpredictable shared mathematical actions or strategies in solving mathematical problems that arise as a result of the problem posing or mathematical tasks that co-emerge with the teachers' and students' ways of mathematical knowing[3]. This was done to provide an in-depth analysis and vignettes of teachers' classroom instructional practices and describe the complex process of the teaching and learning.

The data for this section is drawn from three sources and is organised as follows. Firstly, I will present vignettes of the lesson videos with analysis and interpretations consistent with the four concepts of enactivism in Demis' first, Emmis' second and Sann' first videotaped

[3] Mathematical knowing and knowledge, in this study, are used interchangeably. However, a clear distinction is worth noting here. Knowledge is something that can be possessed whilst knowing implies using knowledge in action. Hence, mathematical knowing here refers to knowledge becoming part of the teacher practice (Kazemi and Hubbard, 2008).

lessons. This analysis and the selected vignettes or episodes show how teachers addressed development of students' mathematical thinking and sense-making in the teaching and learning of geometry. I include vignettes to illustrate how spaces for engagement and mathematical interactions were co-constructed in classroom moments. A full description and analysis of all the lesson videos analysed according to the Kilpatrick et al.'s (2001) model is detailed in sections 6.2.1.1 to 6.2.1.5. So, in this analysis, I only present exchanges that best illustrate the relevant concepts of an enactivist approach.

Secondly, I present the analysis of data taken from the stimulated recall analysis session (focus group interview) with teachers. In this interview, teachers talked about their own instructional routines and individual idiosyncrasies. They also shared ideas of *becoming mathematicians*, for instance, who they really are, what they really do and what really makes them effective teachers of mathematics.

Thirdly, I present the findings from the individual post-lesson reflective and focus group interviews conducted with the selected mathematics teachers to assess their responses and perceptions of factors outside of Kilpatrick et al.'s (2001) analytical framework that are represent in effective teaching practices.

Sections 6.2.2.2 and 6.2.2.4 attempt to answer the following research question. *What factors outside of Kilpatrick et al.'s (2001) analytical framework characterise effective teaching practice? In other words, what factors might represent or affect effective teaching practice, and how might these factors be interpreted within the enactivist framework used in this study*?

6.2.2.2. Demis (Teacher 1)

6.2.2.2.1. Lesson Video 1: Grade 10 Measurement of plane shapes

In this grade 10 lesson on measurement of plane shapes, Demis used a useful strategy to develop students' conceptual understanding and to ground mathematical reasoning in a perceptual and situated learning process. Demis used an investigation approach/instruction requiring abstraction and analytical reasoning. In the vignettes below I describe significant ways in which the teacher encouraged the students, using an investigative approach, to enhance the use of the concrete materials. At the same time, the investigative approach provided a useful context for innovative instructional strategies.

Vignette 1: Teacher's enactment of structural determinism and embodied cognition principles

Demis, after establishing her Grade 10 students' prior knowledge, involved students in different stages of the learning process, from brainstorming to construction of combined two-dimensional geometric shapes and formulating the procedure for determining the perimeter and area of constructed shapes. She reconceptualised the pedagogy which was reflected in three different phases of the lesson. Firstly, because of the nature of the mathematical task, Demis invited her students to share ideas and co-construct two combined geometric shapes with her. Secondly, Demis adopted a more holistic approach to the learning process by integrating students' ideas, skills and prior knowledge. Thirdly, through the emergent collaborative interaction as well as mathematical discourses, Demis transformed the teaching from passive content consumption to active knowledge generation. As I observed, students were engaged and offered conjectures about the formula or procedures for determining both the perimeter and area of the constructed geometric shapes.

1 Demis: So, look at that rectangle you have cut out from the graph sheet, how are you going to work out the distance around the figure...what are you going to do?
2 Students: We are going to count the square blocks.

The mathematical discourse and learning experience that emerged in this episode extended from the teacher's knowledge structure to the knowledge structure of her students thereby co-ordinating her own knowledge structure with that of the students.

Line 1 showed the teacher engaging her students in an investigation situation/task inherent in mathematical problem solving. Special qualities of this lesson that illustrated embodiment were visual representations used to support learners' conceptual knowledge and embody mathematical procedures. The aim was to develop learners' conceptual understanding of the perimeter and area of combined shapes using concrete materials. The visual aids for this lesson were a rectangle and circle drawn on the square grid papers or graph sheets. These materials served as a support and as the basis for mathematical abstraction. The fact that the teacher's knowledge structure has properties that go beyond those of constituent elements (Reid, 1996), showed Demis' ability to construct the two combined shapes before they actually abstracted the perimeter and area formulae or procedures. Demis used the rectangle and circle that students cut out from the graph sheets to embody the mathematical procedure that served as models for abstracting, interpreting and conceptualise the perimeter and area formula. Demis involved her students in a discovery or investigation task to establish the relationship, from one representation system to another representation that built into two-dimensional shapes or diagrams (lines 3 to 7). Over the course of this task, Demis required her students to construct relevant system of relationships that as well helped them to notice the meanings of dotted lines in two dimensional combined shapes (lines 3 and 7).

3 Demis: We are going to focus on the perimeter and then focus on the area. Whenever they want you to work out the perimeter of the shaded part, the shaded part will be given like this. Why do you think they make it like broken lines?
4 Students: They show you that, that part is a half circle.
5 Demis: That is also very good to show that it is a half circle. But so to say they show that there is nothing. Do you see anything there? Do you see that side anymore?
6 Students: No.
7 Demis: It is not there. That is the good reason why they make it a broken line.
8 Demis: Okay, you are going to count the blocks. [*She then drew a rectangle on the board*] okay, I just want to know, what is your distance here, how many square blocks or centimetres are here [*pointing to the length side*]?
9 Students: 16 cm (*length*).
10 Demis: $16\ cm$ so basically the perimeter is $16cm$ plus…what are the blocks here [*width*]?
11 Students: $12cm$
12 Demis: [*Writing on the board*] so we can say $P = 2l + 2b$. Now your perimeter will be two times $16cm$ plus two times $12cm$, that is equal to $2(16cm) + 2(12cm)$, so that will be $32cm + 24cm$. People, can you think about algebra, is this like or unlike terms?
13 Students: Like terms.
14 Demis: Like terms. What did we do in algebra when we have like terms, who can explain to me?
15 Students: We add them together and keep the variables the same.
16 Demis: We add the numbers and keep the variables the same. So what will this answer be?
17 Students: $56cm$

The mathematical interaction that emerged in this episode showed Demis using visual representations as valuable examples of embodiment. As Varela et al. (1991) note teachers and learners have embodied experiences which they bring to the learning and both learn from each other. After brainstorming the required procedure, students realised that they could solve the problem by using what they had previously learned and their embodied knowledge. The first thing they needed to do, as they discovered, was to count the square blocks in reaction to the teacher's several questions: "what are we going to do" (line 1), "what are the blocks here" (line 8) or "what did we do in algebra" (line 14). All of these questions were discussed and the structure of the working plan was done until the solutions of two constructed shapes were reached.

The teacher then led the discussion to the next phase of the lesson, which was to compare the results of the two constructed shapes. After several moments of sharing ideas, some students discovered that the perimeters of both shapes were the same and told the teacher that the perimeters were somewhat different because of the way they cut out and combined the two shapes. This discovery allowed the teacher to engage her students in a discussion to determine the procedural rule or the structure of finding the perimeter and area formula (lines 18 to 20) and to solve mathematical tasks jointly as co-participants, problem solvers and co-constructors of knowledge. I argue that students' enhanced understanding of perimeter and area concepts and problems solving skills co-emerged with the teacher's embodied knowledge about effective teaching practice and situated learning in particular. The emerging mathematical discourse in the episode below (lines 18 to 20) contributed to the students' development of a wide range of strategies and actions that were extremely important for problem solving, deductive reasoning, co-operative work and creative thinking. The teacher was also inspired by the collaborative interaction with students and continued the discussion the following day/lesson.

18 Demis: okay, people we are going to continue with this task tomorrow. But before we go, check here, we said the perimeter is the distance around the figure. So, the perimeter of figure 1 will be:
$P = halfcircle + length + halfcircle + length$
The perimeter of figure 2 will be:
$P = length + halfcircle + length + halfcircle$
The perimeter of this combined figure 1 will be $P = l + \frac{1}{2}circle + l + \frac{1}{2}circle$. So one length plus another length will be equal to what?
19 Students: two lengths plus a circle.
20 Demis: $P = 2l + 1\ circle$. We can also do the same with the perimeter of figure 2, which equals to $P = \frac{1}{2}circle + l + \frac{1}{2}circle + l$, which is the same as $P = 2l + 1\ circle$. That is exactly the same. It supposes to give you the same answer (distance). And think about why your perimeter distances were not the same for tomorrow discussion.

Lines 18 and 20 showed evidence of situated learning, which according to Lave and Wenger (1991), emphasises the ideas that much of what is learned is specific to the situation in which it is learned. Situated cognition thus suggests that mathematical action is grounded in the concrete situations in which it occurs. Enactivist embodied cognition views abstraction of mathematical concepts via investigative instruction as effective if what is taught in the mathematics classroom involves concrete materials and representations that trigger the students' structure to determine their own actions and what is meaningful for them.

Vignette 2: Teacher enactment of co-emergence principle

Enactivism affirms that language expressed in mathematical discourses cannot be separated from cognition (Jaworski, 1994). Language is an inherent part of human interactions that create structural coupling and co-emergence. In vignette 1, language or mathematical discourse co-emerged and coincided with the classroom activity. In order to know how the teacher and students think mathematically, I needed to look at the way they spoke when they jointly solve mathematical problems. Mathematical thinking is inseparable from intrapersonal communication between the teacher and students. In this case, my analysis of the mathematical interactions also enabled me to look at the process of mathematical problem solving as well as the way the teacher and students talked about mathematical object (mathematizing). Mathematizing was manifested in the words, signs and drawings used, and all these involved problem solving. I thus classified vignette 1 as also showing co-emergence. The co-emergence of the teacher-student interactions (or the teacher and environment) also pointed to a strategy that potentially enabled the teacher to offer her students an appropriate mathematical task for investigation, abstraction and mathematical reasoning. Such investigation task embodies the teacher's embodied knowledge of geometry and pedagogy.

In vignette 1, students' mathematical understanding seemed to co-emerge with the world, which is the teacher's explanations, through their actions of building mental models, evaluating the results and revisiting mathematical formulae learned previously. Mathematical knowing through the teacher's enactment of co-emergence was evident in this vignette where the teacher fostered students' mathematical thinking through problem solving.

My analysis suggested that using the co-emergence principle afforded the teacher a space to constantly and actively engage her students in joint action and interaction (situated learning process) by doing the activity themselves. Such a learning process confirmed the enactivist view that learning occurs in a co-adaptive fashion where agents (teacher) and the world (her students) co-emerge through their ongoing interaction (Proulx, 2009). An enactivist approach to learning views autopoesis as having the potential to do what is readily perceived by an agent. In such complex dynamic autopoetic classroom environments, learning is determined by the *fit* between the agent's capacities and skills and the action relation properties of the environment itself (Li et al., 2010). In the episode in vignette 1 above, the teacher seemed to see the fit between her students' capabilities (to construct combined shapes and investigate further) and their environment (teaching them how to abstract the procedure for determining perimeter formula). I argue that such a fit enabled the teacher's embodied knowledge to connect with students' prior understanding, creating co-emergence of mathematical ideas in action.

From the third lesson observed, it was also evident that Demis demonstrated her knowledge of the content of school geometry by providing pertinent and connected contexts for her students, and by employing the ideas of storytelling. She used practical contexts to provide mathematical representations, explanations, justifications and explicitness around trigonometric ratios/functions. Consistent with the enactivist principle of structural determinism and embodied cognition this story structure seemed to provide a series of mathematical actions which emphasised clues to the mathematical meanings of trigonometric functions. This further indicated that Demis works from “having mathematics to being having known mathematics” (Personal communication with Elaine Simmt, August 2012).

The following section describes some important examples of autopoesis, co-emergence, structural determinism and embodiment principles in Emmis (Teacher 4)’s mathematics classroom.

6.2.2.3. Emmis (Teacher 4)

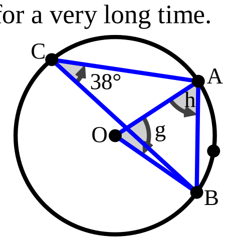
6.2.2.3.1. Lesson Video 2: Grade 10 Theorems in circle geometry

My analysis described in section 6.2.1.3.2, vignette 2, showed that there was some evidence of autopoesis, co-emergence, embodiment and structural determinism/coupling in the mathematical interactions within Emmis’ classroom routine/culture. The following vignettes illustrate my claims in terms of these enactivist concepts. For a full lesson description, refer to section 6.2.1.3.2.

Vignette 1: Understanding of a mathematics classroom as a complex dynamic autopoetic system of mathematical interactions

This vignette illustrated an example of **autopoesis** in Emmis’ classroom. In this lesson, Emmis was videotaped teaching circle geometry with a grade 10 class. The discussion was prompted by the teacher’s question: *What is h? If you tell me the value of h, I will respect you for a very long time* (line 1). During the course of the interaction, the students were led to the correct answer.

1. Emmis: So, what is h now? If you tell me the value of h, I will respect you for a very long time.
2. Student 1: $30°$
3. Emmis: Why?
4. Student 1: Because it is an equilateral triangle.
5. Emmis: It is an equilateral triangle because we have talked about an equilateral triangle now. We said in an equilateral triangle, all angles are equal to what…?
6. Student 2: $60°$
7. Emmis: But, you said $30°$. So, you are saying something else.
8. Student 3: h is $38°$
9. Emmis: Why?
10. Student 3: I divided by 2.
11. Emmis: So, if you do not know then it means you divide by 2. You have this triangle here. [*Drawing the small triangle OAB separately on the board*] let me write here so that it can be seen clearly. So, we have $g = 76°$ here. So you are saying what now?
12. Student 4: $h = 26°$.



Taks 12

13. Emmis: So, you got $26°$, but the question is "why"?
14. Student 4: Then that angle is half of the other angle.
15. Emmis: Is this angle half, who said that? [*Pointing to angle* h] look here in mathematics we never assume that this angle is half of this angle unless we tell you that this angle is equal to this. These diagrams are not the same. That reason is out. Please try, try you must reason alongside your listening. Like I said you have to step up your level best.
16. Student 5: Sir, sir, I think that angle [h] is $45°$.
17. Emmis: Why?
18. Student 5: Because the angle CAB is subtended by the diameter, so angle at the top is $90°$.
19. Emmis: Which angle is $90°$, this one [*pointing to angle CAB*]? There is no diameter here my dear. The diameter has to pass through the centre. Are you trying to tell me that I am the only one who knows?
20. Students: [*Very proudly and loudly*] no, no!
21. Emmis: Now…
22. Student 6: I think h is equal to $52°$.
23. Emmis: Why?
24. Student 6: Because the triangle [OAB] should add up to $180°$, and h and that angle opposite are equal.
25. Emmis: h and that angle [*pointing to OBA*] is equal, which is true. But why?
26. Student 7: That does not work.
27. Emmis: That is what he said. It does not work. And honestly let me tell you now. They are equal. It is true your answer is correct and his answer is correct too. But why, why did you step up and say they are equal. You can stand in the court and say I know they are equal and there is no other way, they have to be equal. Why, why? Chile!
28. Student 8 (Chile): Subtended by the same chord.
29. Emmis: Which chord now? [*Pointing to line AB*] this one is a chord. [*And then pointing to lines OB and OA*] and this one is not a chord and this one is not a chord too. [*Listening to one student explaining and justifying her ideas*] Okay, here you got the right reason…what did you say?
30. Student 9: Because the distance from the centre to the circumference is always the same.
31. Emmis: Because the distance from the centre to the circumference is always same. [*Pointing to the two sides OB and OA of an isosceles triangle AOB while indicating that the two sides are equal with short lines*]. So, these two lines are…?
32. Students: equal.
33. Emmis: So, they form what…?
34. Students: [*after realising that two sides are radii and are equal, some students then started saying aaahhh, yeah, yeah*] Isosceles triangle.
35. Emmis: Yeah, you have to know that it is isosceles triangle and therefore angle h is equal to this angle [*OBA*]. You divide by two, so h is equal to $180°$ minus $76°$ and then divided by 2, which is $h = \frac{180°-76°}{2} \Rightarrow h = 52°$. So, let's put the reason there. Because OBA is an isosceles triangle and $OA = OB = radius$.

Using Maturana and Varela's (1992) concept of autopoesis, I suggest that these mathematical interactions were autopoetic in nature because Emmis' interactions with the students was light-hearted (*joie de vivre*) which provided a space for meaning making showing that Emmis' classroom is a complex dynamic autopoetic system. In their interactions, the teacher created a space for the learners to construct meaning sufficient to proceed with solving the problem and to anticipate a solution that works. McMillan (2004) describes components of autopoetic systems as learning what the world is able to offer them in order to anticipate their future existence. I suggest this vignette showed "autopoesis" or "autopoetic behaviours of the mathematical communication between the teacher and students that are being "self-organising or self-propelling" (lines 3 to 35).

Consistent with the enactivist autopoetic notion, the dialogue that emerged in this episode seemed to have co-evolved and co-emerged out of the teacher's and students' personal histories (Reid, 1996), structures or experiences and their working together on the mathematical task under discussion (lines 5, 11 and 15). In this context, the teacher's actions can be interpreted as "triggering and/or catalysing the student's mind to retrieve what was in his mind". By catalysing the students' actions, the teacher actively stimulated them to effectively respond to him and communicate mathematically (lines 1 to 18). Evidence for this is that the teacher did not act as a source of knowledge as he commented: "are you trying to tell me that I am the only one who knows" (line 19). The teacher pressed for mathematical reasoning (lines 3, 9 and 23) creating the possibility for engaging students in mathematical dialogue/discussion to explore their mathematical ideas. For example, between lines 1 and 16, dynamic patterns of the mathematical discourses that arose as the teacher and students acted and interacted with one another showed the autopoetic behaviour of a complex dynamic system (Reid, 1996; Lucas, 2005; Mennin, 2010). That is, consistent with the enactivist autopoetic view, students did not only contribute to the mathematical classroom discourses that emerged (lines 4, 10 and 14), but they also initiated these mathematical discourses or interactions without the explicit questions/instructions of the teacher (lines 8 and 16). The ideas that emerged in this episode from the teacher and the students transformed into different discourses as they merged with each other (Davis and Simmt, 2003; Miranda, 2004). For example, in lines 2, 4, 6, 8 and 10, the students were not concerned about getting something wrong in front of their peers. In response, the teacher accommodated (acting adequately) the changing teaching and learning situation provided by the students' reaction as he worked on the mathematical problem with them. In an enactivist view, complex dynamic systems learn to adapt to changes in circumstances, and consequently reorganise themselves as they gain experiences (McMillan, 2004). Therefore, the teacher and students engaged in mathematics learning as a collective, social activity (Davis and Sumara, 1997), enhancing and learning from each other to make sense of their interpretations, and thus reorganised themselves into a learning community of practice. This understanding of dynamic co-participation from an enactivist viewpoint, is supported by Lave and Wenger (1991) who view effective learning as a 'social co-participation' in a community of practice that is directed towards expertise and determined by the cultural and social context in which it is enmeshed. In this connection, autopoesis stresses learning as co-created and provides for social processes such as communication and negotiation in which *physical cognitive agents,* as a means of enabling communication and encouraging learning, interact through environmental perturbations for which change is being sought (Khan and Gray, 2012).

Another valuable example of autopoesis appeared between lines 22 and 35 where the teacher and his students were structurally coupled (complicit) in ways which enhanced each other. That is, students effectively learned what the teacher asked them to and so the teacher learned how the students explored and explained their mathematical ideas and how they confirmed the correct answers. Interestingly, the teacher was able to maintain the interactions, discussion and questions through the whole exchange (lines 23 and 33). Similarly, students responded actively to the unfolding circumstances, which were the teacher's questions and ways of pressing for mathematical reasoning. The analysis above revealed the teacher and students as living components of a complex autopoetic system, which are "dynamically related in a network of ongoing interaction" (Reid, 1996, p. 2).

Later in the episode, the autopoetic character of the mathematical interaction disappeared when the teacher confirmed that student 8 was right (line 29) and that the two answers were correct (line 27). This means the interactions between the teacher and students over this task ceased when the solution was found. Though the autopoesis of the discussion ended as students could not continue to explore their mathematical ideas (line 35), it was interesting to observe that the teacher was encouraging (*triggering*) the students' exploration of mathematical ideas, reasoning and ways of mathematical sense making. For example, when he said: "why" (lines 3 and 9), "you are saying something else" (line 7), "so you are saying what now" (line 11), "but then the question is why" (line 13) and "so they from what…" (line 33). Through this interaction, both the teacher and students revealed themselves as components of a complex dynamic autopoetic system as they were able to learn what each of them was able to offer to each. They also conveyed a sense of "mathematical being" when the teacher asked and students answered questions such as "why"? As Miranda (2004) argues no matter what we choose to call the type of mathematics [task] the students engage in, the students' experience with that mathematics [task] is not pre-given, but it emerges during classroom interactions.

The principle of **co-emergence** was also evident in the conversation in this vignette. For example, in line 5, the mathematical discourses that evolved enabled the co-emergence of new ideas that were taken up for further discussions. In this particular instance, the teacher asked the student to describe an equilateral triangle in relation to the size of its interior angles. I interpreted these mathematical interactions as integral to the students' and teacher's mathematical understanding, but not as entities separate from them (Proulx, 2009). The students had great mathematical ideas and the teacher motivated the discussion around those ideas.

Vignette 2: Teacher's exemplification of structural coupling or determinism in the mathematical discourses

The vignette below was taken from the grade 10 lesson on theorems in circle geometry. It showed Emmis and a student working on a mathematical task and is an example of **structural determinism** and **coupling**. In the episode a student explored a different solving strategy to reconstruct/find the value of x. The debate showed evidence of how an autopoetic entity determines what happens to it and within its structure (Maturana and Varela, 1992). Maturana as mentioned in Proulx (2008, p. 20) argues that "when two or more organisms interact recursively…each becoming a medium for the other, the result is mutual ontogenic structural coupling".

36. Student 10: But sir, I am sure my first method gives the same answer. My first idea I said $180°$ minus $90°$ plus $76°$ $[180° - (90° + 76°)]$.
37. Emmis: Ok, but you know it is wrong. And accidentally I do not even know how you involve in that $90°$, how did you get that $90°$?
38. Student 10: You see that the way that line [*CB*] lies looks like cutting the circle into two equal segment or passing through the centre.
39. Emmis: Like I was also saying it yesterday that you never ever assume that, that line passes through the centre unless we indicate that that line passes through the centre. You cannot just look at the diagram and assume that that line passes through the centre.

The student's explanation of his solution method (line 36) showed that students in this class were able to explore their interpretations independently of the teacher and this was shown again in line 38. Some enactivist theoreticians (Capra, 1996; Davis and Sumara, 2006; Proulx, 2009) observe that mathematical knowing arises from the individual's social interactions between the learning individual and its environment. This environment includes three fundamental mutual components, what Kilpatrick and his associates (2001) call an "instructional triangle" (p. 314). In vignette 2, for instance, these components are the teacher, the students and the content knowledge. Most importantly, this includes the language that arises from these complex interactions (Capra, 1996). Language plays a profound role in social interactions because it enables "structural coupling" between the knowledge systems of individuals as they recursively interact.

The mathematical task in vignette 1 continued with a discussion between the teacher and student 10 (lines 36 to 39). The teacher explained that the class must know that the triangle was an isosceles triangle, and showed them the procedure involved to find the value of h (line 35), this stimulated responses in the knowledge structure of the other students (Proulx, 2009). Thus vignette 2 showed evidence of how learning arising from the learner's own knowledge structure as he/she interacted with the unfolding circumstances and/or with the teacher in the complex classroom environment. In accordance with the notion of structural coupling, the learner's explanation of his solving method strategy (line 38) provided clear evidence of '*complicity*' between the teacher and the learner which conforms with procedural fluency. This means that the learner's mathematical understanding was always present in the teacher's actions (line 39). I thus argue that the mathematical meaning co-created in this extract was not inseparable from the teacher, the learner and anyone else in the classroom.

Vignette 3: Teacher's exemplification of co-emergence in the mathematical discourses

The vignette below was drawn from the third lesson as an indicator of my conceptualisation of an enactivist **co-emergence** framework. It showed how Emmis (the teacher) responded to unexpected questions or deviant ideas and explanations from students (lines 1 and 3).

1. Student 11: Sir, what does *subtended* [*by the diameter*] mean?
2. Emmis: Okay, I am surprised that I did not ask if you can explain what the word "subtended" mean.
3. Student 12: We can say "subtended by the diameter means…angle in a half circle" opposite the diameter.
4 Emmis: Alright, the meaning thus come out of what I was explaining. This is a word which you can only find in this talking of theorems because…of course I never come across it anywhere else apart from circle theorems. They say subtended even the way that I am standing sometimes…it is a bit confusing. But …there is…in fact that whenever an angle is formed in the circumference and is related to any chord, we can say "it is angle subtended *by* the diameter or some can say subtended *at* the circumference *by* the diameter".

This interaction showed evidence of the teacher's and students' mathematical understandings that "co-emerged or coincided" as a result of the perturbation (question and explanation) of one system (in this case, the student) that stimulated the other system's (the teacher's) response (Begg, 2000; Proulx, 2009). The notion that a teacher can become a *disturbing agent* in the mathematics classroom, and that the changes (learning) will be determined by the structure of the learners was made explicit in vignette 3. During my observations I saw that Emmis had a little influence on what actually was raised by the students and so his control over the learning outcome was limited. Hence, this made Emmis engage in *perturbation*. I classified this incident as an example of co-emergence, which emphasises teacher's knowledge co-emerging with students' ways of knowing. There were no compelling distractions from Emmis' plan for this lesson, although the student's question about "what the term subtended mean" might be a case in point (line 1) of a disturbing agent. Of interest here was that another student (line 3) who articulated her own mathematical ideas in response to the other student's and teacher's questions. The teacher made further reference to the students' explanation (line 4), although the teacher's explanation of the term "subtended" was surely worth a moment's pause during the lesson.

The teacher's response to the student's question (line 4) was to explain the term "subtended" as it was not made clear in the introduction. This opened up a space for students to share their mathematical ideas (line 2), and the student's response indicated that he interpreted the term in the same way as the teacher (lines 3 and 4). In this way, the student's explanation and/or mathematical knowing co-emerged with the teacher's explanation and/or mathematical understanding. This means the student's mathematical ideas come into being as a result of the interaction that unfolded between him and the teacher and anyone else in the classroom. Thus, in this class, the teacher's response to the student's explanation seemed to position the student as equally proficient with respect to both the mathematical concept and the type of participation in the classroom. Thus in this vignette, the teacher and the students are both sources of mathematical knowledge as both brought their own intuitions and learnt from each other.

Emmis also used actions, gestures and expressions in the teaching and learning process. For example, he demonstrated more concrete examples intended to make students figure out the right procedures, step/method to solutions or make them think mathematically. This provided clues to solve mathematical problems, communicate the solution to students, validate or even generalise the solution to the problem. Specifically, this was evidence of bodily actions accompanying speech utterances produced during the solution of mathematical problems. In this way when a student worked out a mathematical problem or shared his solution strategy on the board, the teacher bent backward to signal that the solution was correct or incorrect. This teacher's movement and bodily action in class were a routine to make students use procedures appropriately and perhaps to see the worth of mathematics. Actions, from an enactivist perspective, are not simply a display of understanding, but they are themselves understandings. Thus, what is fantasized, what is imagined, what is guessed at, and what is intuited are understood as being extremely important to meaning making and contributing to knowledge and what is learned (Sumara and Davis, 1997a).

I now move on to describe Sann' mathematical interaction with students in terms of the principles of embodiment, structural determinism and co-emergence.

6.2.2.4. Sann (Teacher 5)

6.2.2.4.1. Lesson Video 1: Grade 11 Geometrical terms and angle properties

I noted in section 6.2.1.4, the analogies, illustrations, practical and real life examples, explanations and demonstrations in Sann' first lesson. All these representations helped students to connect geometry to real-life situations and meaningful contexts. In this lesson, Sann made deliberate choices in her representation of geometrical terms and angle properties. The following vignette shows Sann using her body and other integral representations as valuable examples of structural determinism and embodiment (refer to section 6.2.1.4.1 for full lesson description).

Vignette 1: A closer look at Sann' enactment of the structural determinism principle

This vignette was taken from the first lesson and illustrates the use of structural determinism to deepen students' conceptual understanding of geometric terms and angle properties. The episode described below occurred in the first lesson where Sann explained the relationships between geometrical terms and angle properties such as acute, right angle, obtuse angle and so forth. Over the course of the lesson, Sann demonstrated how biological structure and human actions interact in the construction of the lesson. I view structure as various elements which an individual must contend with such as knowledge, and experiences in the classroom environment. Actions refer to the thinking and decisions that individuals take within the context or through the structures (Reid, 1996). For example, Sann' use and explanations of mathematical terminologies showed knowledge in action as revealed in demonstrations and choices in teaching.

1 Sann: Is there anybody who can tell me what in the classroom which is parallel lines?
2 Student 1: Window frames
3 Sann: [*Pointing to the window*] window frames. Most of the straight lines in it form parallel lines. What else?
4 Student 2: Table
5 Student 3: Door frames
6 Student 4: Door edges
7 Student 5: Chairs
8 Sann: Chairs, everywhere around us, there are parallel lines. Parallel lines mean there are two lines next to one another, but never clash into one another and never go straight further away from each other, and the distance between them are stays the same. Where are the perpendicular lines in this classroom?
9 Student 6: Window frames.
10 Sann: The window frames, the edges of the window frames are perpendicular lines. Where else?
11 Student 7: The door edges.

12 Sann: The door edges up there, they are perpendicular lines. Your table edges, and the way I am standing now. The fact that I am not totally falling over, I am also perpendicular to the ground or class floor. I make the 90° angles.

Sann' own mathematical meaning, knowing and descriptions were presented in a way designed to enable students to learn abstract mathematical concepts. She used analogies, illustrations, explanations and demonstrations (lines 8 and 12). Sann chose practical and contextual examples from her mathematics classroom for the optimal acquisition of mathematical concepts, procedures and essential vocabulary or terminology (lines 1 to 10). This embodied mathematical knowledge and the teacher's ability to explain what parallel lines or perpendicular lines look like in real life situations showed that Sann addressed mathematical terminologies both conceptually and concretely. In lines 8 and 12, Sann explained parallel and perpendicular lines using real life contexts and her body. This suggests that mathematical knowledge is internally represented information and that understanding occurs when representations of ideas are connected to situations in which they are introduced (Hiebert and Carpenter, 1992).

Vignette 2: A closer look at Sann' enactment of the embodiment principle

During the first lesson, Sann further used kinaesthetic (or tactile) means of her own body to represent mathematical concepts and enhance their meaningful internalisation by the students. For example, in the following episode, the teacher considered the relationship between the perceived world of embodiment and the conceptual world of mathematical symbolism with particular reference to the concepts of acute angle, right angle, straight angle, and reflex angle as well as complementary and supplementary angles (lines 13 to 15).

13 Sann: "**Acute angle**" is between 0° and 90° and it is the smallest angle that we make [*form*]. And you all know when you are living with a small puppy or kid you often like to say that is "cute". An acute angle is the smallest angle.
Then we have a **"straight angle"**. A straight angle and a straight line are exactly the same. That means angles on a straight line must add up to 180°. [*Lying down on the table*] like the way I am lying on the table.
The "**reflex angle**" is between 180° and 360°. Reflex is, if you stand like this and you bend backward [*showing with her body*]. Why do you they call it a reflex angle? [*She then asks one student to stand up and push him on the face as if she is hitting him very hard*] if I push him very hard, he is going to bend backward. So he is going to reflex to what I am doing to him. He is definitely not going to come to the front when I try to hit him [*showing with her body*]. So, that is reflex angle, it is more than 180° but smaller than 360°.
So, **complimentary angles** add up to 90°. [*While sitting on the table*] when I sit like this, my body here make 90° angle with the table and so the way you sit on the chair. Thus complementary angles add up to 90°.
Then "**supplementary angles**", if I supply you with something, say a pen. Am I only giving to you half of a pen?
14 Students: No
15 Sann: No, I give the whole pen to you, and it is in a straight line form. Supplementary angles add

up to 180° because you give the whole thing as it is. Whatever you give somehow, the line that I am giving it to you with is straight line. So the straight line that you are giving it with is your supplementary angle.

The interactions in this vignette showed that the teacher used embodied mathematical knowledge, relating the learning of angle properties to the physical phenomena and process of actions as mathematical concepts. She used kinaesthetic for developing imagery pertaining to targeted geometrical terms (line 13). She used her body to play a pedagogical role analogous to concrete artefacts. She refocused the development of mathematical concepts from actions to processes in a way that was more meaningful to students. Specifically, Sann did not just present her explanations and illustrations into the world of the physical, but she literally used her own body for her students to realise angle properties. This illustrates teaching and learning situated in physical contexts as the teacher delivered the learning materials within the students' direct experience. Hamilton (2005) shows that the embodied world has a variety of different meanings depending on the context in which the concept occurs and considers the way in which these are linked to mathematical symbolism. An enactivist view of embodiment posits that cognition is embodied (Proulx, 2008). That is, the way we act is a function of our body, its physical and temporal location and our interactions with the world and/or surrounding environment where we are. Accurate explanations and mathematically comprehensible definitions of geometric terms contained in this extract showed that effective bodily actions and well-designed embodied experiences help students to understand mathematical concepts because the teacher was more readily able to align her real world experiences with abstract representations and conceptual models (Li et al., 2010). This suggests that the image of learners' mental representation was built up in a way that used images from the immediate surrounding and/or materials that were important to learners at the time. As Khan and Gray (2012) argue representations or metaphors play a constitutive role in creating new knowing.

This vignette therefore demonstrated an embodied conception of angles, in particular the use and explanation of right angle and reflex angles given to students. This gave rise to the embodied interpretation of angles in geometry by using physical phenomena or representations. On the other hand angles and their properties situated in the context of body kinaesthetic leads more naturally to the embodied world of perceptions and actions which give physical interpretations of mathematical concepts. This practice helps children to conceptualise the concept of angles in geometry, as Hamilton (2005) stresses that body kinaesthetic as intelligence develops desired capabilities and thus is a valuable tool for approaching a concept or subject in a variety ways, and as Davis and Sumara (1997) suggest this is the desired result of the implementation of an enactivist approach to learning.

6.2.2.4.2. Summary of insights gleaned from the lesson video analysis based on enactivism

Part II of this chapter provided evidence of the manner in which the classroom practice of three teachers assisted in the development of students' MP in terms of their depiction of the enactivist approach. It also provided vignettes of teachers' geometry lessons to exemplify the ways in which teachers' embodied mathematical knowledge and interactions play a role in the enactment of the four components of enactivism. Given that our reality cannot be separated from our mind (Li et al., 2010), learning and teaching practice should not be viewed as isolated processes or events occurring only in the classroom. The learning world created by the mathematical interactions between the teachers and their students in these classroom vignettes showed the complexities of the everyday world in which teachers and students exhibit their embodied mathematical understanding. Whether by accident or design, the teachers' mathematical discourses, demonstrations and explanations in these vignettes were framed by enactivist concepts of **autopoesis**, **co-emergence** and **embodiment** or **structural determinism**.

6.2.3. Part III: Teacher reflection on their practice through the lens of enactivism

6.2.3.1. Introduction

This final part of Chapter six stands back and looks at the holistic nature of the evidence that is presented concentrating on the focus group (stimulated recall analysis session) discussion with all five mathematics teachers where they viewed their own filmed lessons and worked together to guide me to make connections in their teaching practices. In order to do this, I draw from the Phase VI, Stage IV focus group discussion, Phase IV classroom lesson observations and Phase V post lesson reflective interviews with each teacher. My focus here is primarily on the third task to pull all the threads together.

6.2.3.2. Third task: Nature of the quality of mathematics instructions

I gave the teachers four guiding questions in order to reveal the *proficiency* needed for effective teaching. We first watched the lesson videos and then the teachers reflected on their own competencies and the competencies of other teachers followed by a focus group discussion. The second main issue in this discussion concerned the general effectiveness in mathematics teaching. The third issue was about identifying particular instances of effectiveness in mathematics teaching that informed the teachers' classroom practice. We specifically looked for similarities and differences across teaching practices or lesson videos. I structure my analysis by starting with a discussion of the factors outside of Kilpatrick et al.'s (2001) model that influence the "competencies" required to teach mathematics effectively in conjunction with the enactivist theory.

Enactivism enabled me to understand teachers' teaching practice, embodied mathematical understanding and experiences more fully. Also, the application of these categories of enactivism during the focus group interviews allowed me to interpret what I observed in the mathematics classrooms. Specifically I used the notions of "*autopoesis*", "*structural determinism*" or "*embodiment*" and "*co-emergence*" to look at teacher practice and the factors that represent effective teaching in relation to teachers' responses and comments during the stimulated recall analysis session.

In the focus group interviews, all five teachers expressed varied and broad perspectives of their teaching practices that they believe connect, empower and engage the whole child as a co-participant. Although they did not articulate their ideas directly in enactivist terms, the accumulated understanding of their responses reflected an enactivist practice, in that "multiple domains of human existence or operation" were referenced, and as "effective members of successful schools, teachers' responses and understanding of factors that represent effective teaching practice built upon each other's understanding and co-emerged in the process of co-evolution of multiple ideas" (Hamilton, 2005, p. 155).

6.2.3.3. Factors that contribute to teachers' success and effectiveness from the perspective of enactivism

In this section, I present the factors outside the Kilpatrick et al.'s (2001) model as subheadings together with the teachers' responses. A general discussion follows. Eleven themes emerged from the interviews namely: (1) teachers' embodiment, (2) self-organisation and structuring, (3) teacher-student partnership, (4) use of resource textbooks, (5) teacher mathematical interactions with students, (6) learning community or practice, (7) ethical principles and values about effective teaching, (8) enabling learning opportunities, (9) understanding of geometry as doing geometry, (10) understanding of mathematics as a dynamic body of knowledge and (11) essential characteristics of effective teaching.

6.2.3.3.1. Teachers' embodied mathematical knowledge and structural determinism

My interest here was to see how the teachers' mathematical actions supported mathematical understanding that co-emerged with the students' ways of knowing and sense making. The responses to the question about *mathematical competences* for teaching mathematics effectively fell into two main categories: Firstly, those that related to teachers' personal attributes and secondly those that related to teaching and learning practices. These two aspects are inextricably entwined and so provide a description of both elements of effective teacher characteristics and embodied mathematical knowledge in terms of teaching strategies. For example, Demis said:

> *Basically to encourage [mathematical] reasoning, for me this is the most. We must try to get effective ways…more activities that encourage that reasoning…but again if you give activities to children, we must not provide too quickly answers…And all we must do is taking them through mathematical thinking, and again we must not do all the thinking for them* (Demis).

Emmis emphasised a teacher's solid mathematical knowledge by saying:

> *Having sufficient knowledge of mathematical concepts, being able to tackle mathematics without a lot of problems and being able to interlink concepts, are the most competencies* (Emmis).

As the interview progressed, teachers' responses put more emphasis on mathematics and real life or concrete examples, abstractness and the connected nature of mathematics as a tool for solving real life problems. For example, teachers pointed out:

> *I still agree that more mathematical concepts are a bit abstract and interconnected. But…going to the practical or application part of it enhances understanding of concepts. For some mathematics concepts, you cannot find practical examples. But where you can find practical examples, please do throw them in, while continue exploiting the environment as well in terms of appropriate teaching materials* (Ndara)
>
> *But…I think it is more effective if we are trying to give more practical examples in real-life. And if there are any practical examples in life, please do make use of them in your teaching* (Emmis)

There were a number of responses that reflected a combination of these attributes. For example, teachers highlighted a teacher's abilities to facilitate student participation and have a sense of humour. Emmis argued:

> *The moment I go to the class, and when I am really there I am a very encouraging teacher and very free with them. I go in there and we start the lesson by joking and we start chatting. But through this interaction I am challenging them a lot. So when I am challenging them like: "do you think you are good now", "do you think you know theorems". Then they reply: "yes" Sir, please bring them on. Most of them now they take it as a challenge they like it…because I am challenging them*! (Emmis)

Teachers also believed in giving a lot of practice. They talked about how they incorporated the previous years' question papers in the learning process. For example, Jisa and Emmis described ways of giving practice tasks and claimed that:

> *Questions taken from [previous] examination question papers are very good for showing them a number of items being asked in the final examination* (Jisa).
>
> *The more practice you give, the more homework you give and the more challenging work you give to them, the more you make them understand the concepts.* (Emmis)

Responses to the question about "how teachers' teaching proficiency characteristics informs their classroom practices" were specifically related to teachers' mathematical knowledge and pedagogical skills. Most teachers expressed that an effective mathematics teacher should be confident, enthusiastic and knowledgeable as this has an impacts on the students' interest and motivation. For instance, three teachers said:

> *If a teacher is confident, then s/he is mostly likely to build the same level of confidence and positive attitude in learners towards solving mathematics problem in the classroom and to see why mathematics is so wonderful* (Ndara).
>
> *This success depends on the atmosphere you create in the classroom. This is very important. It is about teaching with your heart, may be that sounds more like psychology, you know. But, it is like building self-confidence, bringing in may be instructor* (Demis).
>
> *If you know your learners and you know what they are interested in, you can focus some specific things [concepts or mathematical ideas] on what is happening in their daily life circumstance* (Sann).

In contrast, two teachers (Ndara and Sann) argued that a good teacher should be mathematically capable, trustworthy and a good listener.

> *I think firstly you need to have a really good knowledge of what you are teaching and you must be very comfortable with it. And you must try and think [listen to] how the learners will respond to mathematical ideas and questions* (Sann).
>
> *And then learners should not doubt you…because once they doubt you, they will lose confidence in you as a teacher. Learners should know that you are capable…if you do not have always answer for them, just tell them that you do not have an answer and you will find it for them* (Ndara).

Overall, the teachers' responses were consistent with aspects of quality mathematical pedagogical practices (Kazemi and Franke, 2004) and supported the notion of embodied cognition and structural determinism. Effective teachers "establish classroom spaces that are truly conducive to sharing, developing interrelationships that develop students' mathematical identities and create hospitable environment that make it possible to reason, communicate, reflect on and critique ideas" (Anthony and Walshaw, 2009, p. 539).

6.2.3.3.2. Teacher self-organisation and structuring as complex adaptive autopoetic system

In the interview, all the teachers shared their mathematical culture and deep connections with the essence of their autopoetic beings. In this regard, they acknowledged themselves as integral and connected to the teaching and learning process in their classrooms. The five teachers asserted that they were well organised and allowed their students some control in their learning in conjunction with an understanding of the learners' personalities and attributes. All five participating teachers believed that "they are really what they do" in their mathematics classrooms which is in keeping with the principle of autopoesis. They agreed that the key factors in developing effective mathematical actions are "coherent structure", "self-organisation" and "the ability to manage themselves and their students to reach specific goals". Demis and Emmis argued that:

> *Part of it is the "Structure"…the way that we structure or sequence our work or concepts in a clear and proper sequential or hierarchical order. I am not just going to do this and that…but the ORDER of concepts is most important…For me the structure helps a lot towards effectiveness…and then you can go wild. But there must be a 'coherent structure' among concepts to be taught* (Demis).
>
> *There are mathematics modules for grade 12 that I wrote for my learners which they use…because these concepts are mine; it is like I am part of the mathematics as well…So we need to keep the balance so that we ensure that before our learners leave matrix they are equipped with necessary skills* (Emmis).

Consistent with the autopoesis notion of sense-making, teachers' responses revealed that they used stories and metaphors to build mental representations; they explained mathematical concepts to students, directing their attention by using appropriate stories and metaphors or analogies. Sann explained:

> *I am always trying to, especially when dealing with new concepts, think of something realistic there in their everyday life situation. For example, when I am teaching "**negative indices**" in fractions, I am using a "**double storey-house**" thereby I ask them to identify where everything belongs or where things have to go or be. And then I let them to draw a little house where everything that they want is at the top or at the bottom. And then when you are at the bottom, you get upstairs…you have to move to the top. And when you are at the top, you have to move to the bottom. This is exactly the same as we deal with or move the negative indices. This practical example means that when you are a negative index at the bottom, then you have to move to the top. And when you are a negative index at the top, you have to come down…if they do it that way then they master the procedure and are likely to do it accurately. I have little*

> *stories like that one, and then they like it and that also help them to remember concepts* (Sann).

This extract reflects that the teachers' instructional practices are consistent with the fundamental premise of autopoesis and embodied cognition which encourages teachers to continuously interact in ways that promote meaning making and authentic learning. Lakoff and Johnson (1999) contend that through metaphors teachers build concepts that enable students to function and interact in the world. Sann' extract showed an awareness of the enactivist principle of embodiment. She was explicit in making concepts physical through meaningful demonstrations and examples.

6.2.3.3.3. The role of teacher-student partnership

The notion of "*partnership*" was evident among the participating teachers as a group. All five teachers reported a strong sense of partnership between their learners and themselves in their teaching. Teachers certainly understood that, as Demis explained, "They trust each other and are committed to a mutually shared academic goal". Emmis and Sann shared the same response:

> *There is a partnership between me and my learners and we are all kind of wanting to achieve the same objectives. My learners will never argue a lot about whatever I give them. They always think I am right for whatever I do…these are kids who really want me to mark their papers, who force me to mark their work* (Emmis).

> *In my class they know they are being rewarded when they are doing their homework. Also…as some learners might need more attention of teacher and others less, with all of them I am in a different partnership, but it is a 50/50 partnership* (Sann).

Similarly, the teachers pointed out that they extend their understanding of themselves as a learning community of practice in that their responsibilities to themselves and their students are hardly distinguishable. In particular, Sann recognized that she has to consciously share her personal accounts with students to keep their interest and connect with students.

> *A lot of time in class I actually tell things from my own life of what happens to me. This makes them feel comfortable with me and the subject* (Sann).

Hamilton (2005) notes that an environment where teachers and students can truly communicate about their issues, obstacles and achievements brings a rich understanding and satisfaction to the entire teaching and learning experience.

There is a strong parent-teacher relationship in the five schools participated in this study, and the teachers said they had a good relationship with parents and the wider community. I suggested that a good rapport between teachers and parents has a positive effect on teacher effectiveness and their school success, and Sann agreed saying:

> *As a 'professional learning community'…the teachers, the administrative staffs, the parents and the wider community…that is everybody should be involved in getting this child to be a positive, competent and contributing law abiding citizen one day* (Sann).

As members of effective schools, the teachers can be seen as an instance of "a connected, empowered learning community of practice" (Hamilton, 2005, p. 163).

6.2.3.3.4. Use of resource textbooks and modification of curriculum or restructuring tasks

All the teachers interviewed stressed that they used different resources that contained a variety of explanations of concepts, examples and exercises in addition to the prescribed textbooks. They also indicated that they often go beyond the syllabus in preparation for lessons in order to look for materials or mathematical tasks that are suitable for their specific needs. In terms of restructuring tasks, teachers acknowledged the need to assess the quality of instructional materials and modify them if necessary. From my observation, it seemed that participating teachers rarely used prescribed textbooks, and adapted and improvised teaching materials of their own.

> *Resource wise, I mean like our learners have about four modules or other mathematics textbooks which is more than enough…we have the data bank…and then access to past questions papers* (Ndara).
>
> *We go beyond the syllabus in preparation for lessons, but not teach beyond the syllabus. Also I never used the [prescribed] textbook…I use the NAMCOL mathematics materials. However textbooks that we need to use are those that contain more appropriate exercises and examples* (Emmis).

The teachers also expressed dissatisfaction with most aspects of the prescribed mathematics textbooks at this grade level. Specifically, Emmis reported his frustration with prescribed textbooks when he said:

> *The most problem with some of these textbooks is that there are not enough explanations and clear examples given for various concepts* (Emmis).

Three teachers said they used online materials and printed texts on mathematics education and research. The remaining two teachers used other textbooks to select appropriate mathematical tasks, new approaches to solving problems and innovative teaching practice.

6.2.3.3.5. Teachers' mathematical interactions with students: a significant factor for teachers

The participating teachers emphasized that it is important to interact with students, to attract their attentions and to listen to their answers and mathematical ideas. They also emphasised the importance of students interacting with each other. Three teachers (Demis, Sann and Ndara) shared similar views:

> *[Bodily] actions are important for as I said that is one of the elements of effective teaching in my class in that my whole body is teaching* (Demis).

> *I think it is really important to interact with the learners because you cannot just teach and hope that they understand. They have to tell you that they understand…so by interacting with them you can find out whether they know what they are doing, and then you can help them there* (Sann).

> *It is very important that they are interacting to explain their mathematical ideas, because at time learners have better ways of explaining to each other than the teacher…that is…learners have shortcuts perhaps that are very important that can help other learners to understand concepts better and move forward* (Ndara).

All five teachers also commented on the progress they had made in encouraging a *collaborative approach to learning* among their students and how much this is appreciated by them. The excerpt below encapsulates the mutual relationship with students in their teaching.

> *As you have seen I would explain the questions [problems] halfway, so that they [students] get an idea of how to get to the really answer. And then I leave them because I want them to try it themselves…explain to each other and to think how to carry on from there* (Emmis).

My analysis of teachers' responses to their interactions with students indicated that the participating teachers were primarily concerned with the educational coherence of the learning process, using "non-specifically knowledge related teaching strategies" (section 4.4.1.2.4) to foster the mathematical sense-making of their students. In order to understand teachers' actions while they carried out mathematics lessons or instructional routines, I focused on interactional patterns noted several times in a given mathematics situation. When watching the lesson video recordings, the teachers described their *questioning and explanation techniques* as allowing students to explain and justify why they solved problems in certain ways. They made an effort to understand how their learners think and reason about mathematical concepts and procedures. They used effective questioning that challenged students to move from relatively simple to more complex forms of mathematical thinking. For example, Emmis and Sann explained that:

> *In my class I am always saying to them…whoever will give me a proper reason why it is correct what he is saying…I will respect him forever* (Emmis).

> *…Because if they explain to me how they got an answer I know they understand it [concept or procedure] properly…* (Sann).

In order to ensure effective teaching and learning, and thus effective actions, Ndara reported employing general teaching strategies that can be replicated from one situation to another, which bring coherence to the learning experiences of his students.

> *Ya…that are my way of teaching…I normally give examples that are relevant. And normally I pull out past [mathematics] examination papers, so that the learners have hands-on practice. They should not be taken by surprise in the end when examination comes. They should be equipped in line of what the examination demands. I normally work like that* (Ndara).

In particular, when I asked him to reflect on his third lesson, he explained how he develops concepts as he said:

> *Ya, what I normally do is I teach a concept. And then I try to develop it gradually over time because the first thing before we do anything is that [sic]…the learner must master the basics of that topic [concept]. Then from the basics of that topic we gradually move to complex mathematical problems. That way I see it better because…if I teach mathematics I demonstrate how a mathematical problem is solved. And then I will leave the rest to my learners* (Ndara).

I can infer from these teachers' comments that their questioning and explanation styles were important in their daily teaching practices. Teachers' questioning and explanations allowed students to find coherence in the learning process and make sense of mathematical concepts.

6.2.3.3.6. Teacher's understanding of themselves as a learning community of practice

Two more themes appeared in the narrative, one related to teachers' own personal view of their mathematical abilities that make them effective teachers of mathematics. The other pertains to their willingness to share ideas with colleagues. The memories of schooling were similar for all participating teachers. All five teachers indicated that the knowledge they gained during their pre- and in-service trainings plays a fundamental role in their success. They also acknowledged that the way they were taught mathematics as secondary school teachers shapes much of how they teach mathematics nowadays. Teachers similarly mentioned learning a great deal in their mathematics classrooms while interacting with students as new challenging ideas pop up. Further, all, but one, indicated that they get together at school and department level to discuss effective teaching approaches and worthwhile tasks. All five participants collaborate with inexperienced teachers to help them out to consolidate their knowledge. This collaborative culture suggests that these teachers are leading lights of a learning community of practice which helps to lay the foundation for professional growth.

6.2.3.3.7. Teacher's ethical principles and values about effective teaching

In their focus group interviews, the teachers spoke about their own ethical principles and values that they think contribute to learners' mathematical understanding. In this regard, they shared the following principles:

- ❖ Preparing thoroughly for lessons and organising themselves, as a dynamically complex autopoetic system, in order to constantly interact with students and respond to their questions consciously. Through interactions with students, teachers take responsibility for the mathematical actions that occur in response to students' mathematical ideas, comments and questions;
- ❖ Working hard at developing trusting classroom communities for students. Equally important, teachers have high standards and expectations of themselves and of their students; they expect excellence, monitor performance and provide feedback; pay close attention to instruction, provide guidance to improve teaching and learning; align instructional practices with required standards, and adopt a collaborative work culture;
- ❖ Promoting conducive classroom environment and relationships that allow students to have a strong sense of belonging, think for themselves, ask questions and take intellectual risks or challenges in the learning process. Teachers believe such a learning environment presents students with the opportunity to develop and examine their resiliency when learning become challenging;
- ❖ Allowing students to develop a positive attitude or "productive disposition" (Kilpatrick et al., 2001) towards the learning of mathematics. Teachers believe this raises comfort levels and gives students greater confidence in their own capability to learn and make sense of mathematical concepts;
- ❖ Providing opportunities for students to solve challenging mathematics tasks, while soliciting or encouraging them to explain and justify their solution methods to others,
- ❖ Presenting more relevant content and learning activities that are practical and personalised to engage learners' interest and commitment. Teachers said they guide this process of developing strategies with students by allowing students to determine a

way to solve the problem that make sense to them rather than forcing them to use a particular strategy;

- ❖ Providing students with opportunities to work collaboratively with their peers in small groups to make sense of mathematical ideas, problem solving methods, and see themselves as mathematical learners/"doers of mathematics" (Kilpatrick et al., 2001);
- ❖ Introducing mathematical concepts in ways that enable students to build on their existing mathematical proficiency, interest and experiences;
- ❖ Designing mathematical tasks that enable them to interact with students while encouraging students to interact with each other and posing questions that challenge and extend students' mathematical thinking and reasoning;
- ❖ Providing opportunities for their students to explore mathematical concepts and ideas, including linking students' interests to real life examples and meaningful stories that contextualise the learning while employing a variety of forms of representation, and
- ❖ Giving more practice for students to perfect the mathematical procedures they are learning, and to improve their mathematical fluency, problem solving skills and conceptual understanding.

Another perceived aspect of their effective teaching was attributed to their use of appropriate teaching strategies with their learners. They all engaged learners in hands-on activities or investigations[4], critical and divergent thinking and problem solving. They also individualised instructional procedures in keeping with the aptitude of their learners. All the teachers agreed that effective teaching involves using a variety of strategies and techniques, and purposeful assessment is the primary link between the curriculum and instruction. They believe content and pedagogy are equally important, and it is their ability and skill that allows them to examine and evaluate their own teaching practice and the learners' learning which lead to them become more effective or expert teachers of mathematics.

These instructional strategies readily translate into the key ideas of an enactivist model of learning. Learning then becomes *situated, sharable, participatory and collaborative* and *action based* as the teachers situate learning or pose mathematical problems in the embodied knowledge or daily life experiences of their students (Davis and Sumara, 1997). Learning mathematical concepts with an enactivist approach becomes authentic as it is connected to

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[4] Approaching mathematics content through investigations helps students develop flexibility and confidence in approaching problems, fluency in using mathematical skills and tools (e.g. manipulatives), and proficiency in evaluating their solutions (Kilpatrick et al., 2001, p. 11).

something meaningful. Teachers' interactions and relationships with students are improved and the ownership of learning that is required in this approach means that students are led towards the critical consciousness required in order to take personal actions. The strategies employed by the teachers allowed their interactions with students to "lead the participants and to unfold within the reciprocal, co-determined actions of the person involved" (Davis and Sumara, 1997, p. 110).

6.2.3.3.8. Enabling learning opportunities that connect, empower and keep students on tasks

All the five teachers interviewed in this study acknowledged that they provided different learning opportunities to their students while enabling quality learning experiences that connected and empowered all diverse groups. Teachers indicated that they had "devoted approaches" such as using real world examples and guiding students to solve mathematical problems by acting out the solution to give them a deeper meaning of concepts. Teachers employed these approaches to connect, empower and keep their students focussed on tasks. I observed that all the teachers entered their classrooms with clearly articulated learning goals, well selected mathematical tasks and provided instant and appropriate feedback to their students. I argue that these teachers are truly helpful. They interacted with learners and gave them sufficient time to do class work. Specifically, teachers revealed different learning opportunities for their learners, for example:

> *Ok, as I said I do "investigation". I do a little bit of practical projects which is also nice* (Ndara).

> *…teaching during weekends…and I give my grade 10 learners some mathematical tasks from grade 11 and 12 textbooks…* (Jisa).

6.2.3.3.9. Teachers' understanding of teaching or learning geometry as doing geometry

When asked what "teaching or learning geometry means doing geometry" implied, a number of teachers articulated the hands-on activities or mathematical tasks that they gave to their students as particularly important in the teaching and understanding of geometric concepts. For example, Ndara remarked that:

> *Geometry is very practical. I think geometry is the topic that actually…if you say mathematics is all around us, then geometry is one of the best topics to do so…say for example, the concept of polygon and the formula for interior angles* $[(n-2)180°]$. *I will take polygon and divide it into triangles from one point. And then it shows them it is always having two-less…so I bring a lot of concept things into geometry to help them* (Ndara).

Emmis highlighted what Teacher 3 mentioned i.e. teaching geometry involved learners in doing geometry. He stated:

> *I think it means it is a practical topic that involves doing most of the time to learners. Since geometry involves a lot of constructions, then that is practical and normally it involves the learners not the teacher. So, the teacher should not centralise the learning or the lesson, but should rather dispatch that lesson to the learners so that they do themselves that lesson. Ok!* (Emmis)

Here again the teachers stressed the use of relevant examples and contexts and acknowledged the value of relating geometry to the practical real-life situation of the learners. This helped students to gain a richer or deeper understanding of geometric concepts that they experience in their everyday life and that they are already familiar with.

6.2.3.3.10. Teacher's understanding of geometry (mathematics) as a dynamic body of knowledge

All, but one, of the teachers reported that they engaged their students with concepts so that they came away with the understanding that mathematics is a dynamic body of knowledge generated and enriched by "investigation" as opposed to mathematical facts and procedures to be memorised. This means that participating teachers provided learning experiences for students that clearly depict mathematics or geometry in particular as investigative in nature.

> *When we talk of geometry, the important thing is that we have to link it up with everything happening in our life. So, actually relating things practically to what they can see in order to better their understanding. In most cases I focus on what they can see in terms of buildings, roofs on the structure, traditional homesteads…even show them what is happening on TV as it brings up things that are three-dimensional. So that is one of the dynamism that we are talking about* (Ndara).

The next section (6.2.3.3.11) of the chapter discusses not only the participants' views of essential effective teaching practices, but also the collective teaching proficiency features that emerged as the participants noticed the similarities and differences in their teaching practices between what they knew and what they came to know while engaged with the research process.

6.2.3.3.11. Essential teaching proficiency characteristics of effective mathematical interactions

At the end of data collection phase, during the stimulated recall analysis session, each teacher was asked to comment on their teaching proficiency characteristics that are **similar** or **different** as evidenced from their individual interview transcriptions and lesson videos. All the teachers shared images of being effective and *becoming mathematicians with their* *students*, particularly stressing asking questions "why" for mathematical justification and explanations. They also emphasised students' mathematical understanding by connecting and applying mathematical concepts to real life situations/examples and students' daily experiences while interacting with their students. The following comments from the interview transcripts of the discussion describe teaching proficiency characteristics that are similar across the five teachers' classrooms.

Demis appreciated the ways they encourage mathematical reasoning by asking "why" questions. For example, she remarked that:

> *Such questioning allows students to understand and grasp taught concepts, and to explain and justify their solving methods and answers* (Demis).

Jisa explained that:

> *Most of us employ learner centred teaching. We are also more interacting with kids which is very important. Another useful practice is that we teach from simple or basics to complex concepts. Another important similarity is reinforcement of concepts understanding and motivation* (Jisa).

Ndara emphasised, in a very direct way, two fundamental aspects of mathematical practices. He said:

> *We all use different assessment strategies to assess students' understanding of concepts and also their weakness in different areas of geometry. A lot times at the start of the lessons, we have students brainstorming what they know about concepts for themselves. We are not providing them with things; they [learners] are providing each other with it* (Ndara).

Similarly, Emmis commented that:

> *Something that is similar: we use as many resource textbooks plus our own materials to expose learners to many examples. This is good for the learners to think about these concepts from a different perspective* (Emmis).

Sann emphasised the importance of checking homework and spending more time on concepts. She concluded that:

> *Also we all check the homework every day…spend more time or navigate through the things (concepts) that we think that they [learners] cannot do it. We just present it and they are all having courage to browse through it and do the rest. We never actually allow them to think that they are unable to do it* (Sann).

Demis distinguished effective teaching as a teacher-led lesson with a coherent structure. She expressed pride in being amusing and considered using actions as one of the greatest elements of effective teaching in her class as she explained:

> *My whole body is teaching…my eyes are teaching, my hands are teaching. I am bodily involved in this process…but this is more just to bring expressions, gestures to things* (Demis).

In contrast, Jisa believed that effective teaching occurred when she used divergent thinking questions to enable learners to conjecture, visualise or give precise description of the steps in a process.

> *I use [divergent] thinking or reasonable mathematical questions for the learners to think mathematically* (Jisa).

Ndara recognised contextual teaching as important in relating learners to what is happening in their environment to enrich their understanding of concepts. He thus argued that:

> *Some contextualisation in real life situations is the best…although for some of us, educators, is a challenge* (Ndara).

For Emmis, effective teaching practice happens when he provides interesting mathematical tasks, listens carefully to learners, and shares ideas with students and pays attention to individual learning needs. He is also able to transfer knowledge by linking ideas to enable learners to understand mathematical concepts.

> *As a teacher I think about what I am going to teach. A lot of teachers read what they are going to teach. They do not think about what they are going to teach themselves* (Emmis).

Sann tended to see effective teaching as focusing on student engagement by carefully listening to their mathematical ideas, providing real life and practical examples and frequently allowing them to work on mathematical tasks in small groups.

> *I listen carefully when explaining their ideas…I then say: "do you agree with what you say, are you sure", "what are you guys saying, is she right or wrong and why". I also like my learners to brainstorming, discussing questions that are sometimes very challenging* (Sann).

The role of an effective teacher, in secondary schools, is not only to foster the mathematical thinking and sense-making of their students. It is also to "educate children…to give them *cognitive values* embedded, for instance, in self-questioning or the inquiry process" (Cobb et al., 1991, p. 21).

I now move on to present a summary of other factors that the participating teachers indicated affected effective teaching practice.

6.2.3.3.12. Factors affecting mathematics teachers' teaching practices and effectiveness

In this last section of the chapter, I analyse and present data collected from interviews with teachers to address the following sub question: *What are other teachers' factors outside of Kilpatrick et al.'s (2001) analytical framework that might affect effective teaching practices*? In order to get a deeper understanding of related factors that affect teaching practices, the five mathematics teachers were asked to explain 'why' they thought Grade 10 and 12 results in Namibian schools are relatively poor. Another question was to explain why there were not more effective mathematics teachers (see Appendix G). I wanted to see whether there was agreement amongst the teachers on this issue. What follows is a succinct summary of these findings together with extracts.

6.2.3.3.12.1. Teacher's personal attributes

A strong finding was that all five teachers felt that the most important school related factor affecting effective teaching practices in the mathematics classroom is the teacher him/herself. They believed that high level achievement of mathematical proficiency depends on the personal attributes of the teacher. On a similar note, all but two teachers believed that their passion for and positive attitude towards mathematics foster a deeper understanding of geometry plus an awareness that mathematics learning can and should be enjoyable. Participating teachers claimed that the majority of teachers are not as comfortable and confident in their own mathematical knowledge and ability; nor do they show pride in individual students' success. For example:

> *Well it could be that the teachers are not as comfortable with the subject as most of the teachers in our school are* (Sann).
>
> *I think the main reason is these teachers are not efficient and effective enough in all mathematics topics* (Jisa).

Other personal traits were expressed as "teacher idleness and laziness".

> *Of course when it comes to teachers, there are teachers who are lazy…I do not want to rule out that* (Ndara).

In contrast, Ndara took a different view by saying effective teachers exist, but was critical of the students:

> *Yaa, well I would like to think that effective teachers are there, but let's also look at it from the learners' side. I think in comparison to where I come from I have seen something stranger in terms of the learners' behaviours. They drink until late in the night and then you cannot expect such a learner to go to school the next day* (Ndara).

6.2.3.3.12.2. Teachers' pedagogical content knowledge (PCK)

All five teachers identified their pedagogical content knowledge (PCK) as the crucial factor in the effective teaching of mathematics. Shulman (1987) describes this kind of knowledge as knowing the ways of representing and formulating the subject matter and making it comprehensive to students (Ball et al., 2008). They highlighted subject content knowledge (SCK) as a crucial attribute of effective mathematics teachers. For example, Demis argued that:

> *I think and I must say my own opinion…the biggest problem is that our people [teachers] lack content knowledge. They sometimes have to teach Grade 10 or Grade 12 even they still struggle with the basics of that grade level* (Demis).

6.2.3.3.12.3. Teacher training

All, but one, of the teachers interviewed in this study pointed to teacher training in both the subject content area and pedagogy as the strongest factor in effective teaching practices with positive educational outcomes. The participating teachers felt that the majority of teachers in Namibian schools were not well trained in mathematics content and that they are uncomfortable when teaching the subject.

> *I think the level of understanding of the teacher of the subject is very important…so actually when I talked to someone [teachers] they were telling me there has been some Colleges of Education that have been emphasising on classroom management rather than on subject content* (Ndara).

> *I do not think everybody was really [properly] trained in mathematics in a way…I do not think teachers feel that comfortable in the subject* (Sann).

The five teachers felt that most of the in-service mathematics teachers struggled to pass high school mathematics and their marks particularly in grade 12 examinations were not that good. Consequently, they continued to upgrade their examination grades/symbols for years through NAMCOL, and then went into teaching training as the last resort. They felt these teachers do not have a passion for the subject, and their low subject content knowledge affect their teaching competency throughout their careers. From the viewpoint of enactivist embodied cognition, Reid (1996) regards the substance of cognitive structure of an individual as it determines the individual's own actions on himself and his world and what he can. Ndara strongly resonated with this claim:

> *So if you take this teacher who was struggling with the subject in high school, and you give that teacher a subject to teach…I think that person [teacher] will also struggle like the learner because [he] is going to learn the subject [concepts] and at the same time the learners try to learn from him* (Ndara).

Participating teachers strongly indicated that mathematics teachers in many schools do not give enough work in mathematics such as additional exercises and homework assignments. It was also interesting to learn that three of the respondents argued that some teachers spend too much teaching time on concepts or topics that they are comfortable with, neglecting the rest. This means some mathematics teachers skim over some concepts that seem challenging to them. Consequently, learners may approach the end of year examinations, thinking that they are well prepared only to discover that there are some concepts which they did not cover or were not exposed to. For instance, Emmis said:

> *There are some teachers whom I talked to and confessed to me when I was teaching in the northern Namibia that they do not teach this and that topic because they do not know it very well. But they are teaching these kids who are expected to sit the examinations. Now how do you expect these kids to know more that topics?* (Emmis)

This finding is consistent with Carnoy et al. (2011, p. 119) who noted that "teachers with more mathematical content knowledge may feel more comfortable with the subject matter in the curriculum, hence may be more likely to spend more time teaching the curriculum materials. They may also cover a greater number of topics because they may be more familiar with a greater range of topics…they could also choose to devote more time to topics more challenging for students in their classes".

6.2.3.3.12.4. Teaching effectively is indeed very difficult

Another compelling factor that mathematics teachers mentioned as significantly affecting teaching practices relates to the notion of "teaching effectively or teacher effectiveness". Participating teachers claimed that teaching effectively is challenging, for example, they struggled to facilitate learner participation and elicit their contributions; to invite learners to listen to one another, to respect one another and themselves; to accept different viewpoints and to engage learners in an exchange of mathematical thinking and understanding. Establishing a good relationship with learners by interacting with them before, during and after the lessons so that they feel welcome and encouraged and motivated to be involved was another challenge the teachers face.

Other factors raised by the teachers were:

- ❖ All five teachers indicated overcrowded classroom as a factor that impedes teachers in assisting all learners during mathematics lessons.

- Both Demis and Emmis reported lack of commitment among the teachers and learners as key education stakeholders, lack of resources and purposeful assessment strategies in other schools as circumstances that retard progress.
- Demis, Ndara, Emmis and Sann were worried about the income they received compared to other professions and that the low teacher salary was clearly the most obvious reason that caused effective, well trained and highly qualified mathematics and science teachers to leave the teaching profession.
- Ndara felt that the labour law and teacher recruitment policy in Namibia generally restricted or locked out well qualified, trained and motivated teachers, particularly the non-locals to join the teaching profession.

6.2.3.3.13. Summary of insights gleaned from stimulated recall analysis session

I have provided a wide range of evidence to showcase the way in which teachers reflected on their own practices and effectiveness. This in-depth data analysis enabled me to explain factors outside of Kilpatrick et al.'s (2001) analytical framework that participating teachers regarded as representing and affecting effective teaching practices. The insight and knowledge gained were interesting. Teachers had high mathematical content knowledge and strong personal knowledge of themselves. They viewed mathematics as a dynamic body of knowledge. This research has shown me that there is a strong relationship and connection between **competencies**, **effectiveness informing practice**, **constructions of effectiveness** as well as **similarities and differences**. These were very evident in the classroom practice of all the teachers. Some teachers like Jisa and Ndara did talk about their embodied knowledge and effectiveness in a particular way whereas Demis, Emmis and Sann made connections between self-efficacy, theoretical ideas of competencies (effectiveness) and practice. Collectively, all the teachers marshalled their embodied mathematical understanding and knowledge in terms of their personal and daily classroom experiences in a community of practice.

6.3. CONCLUSION TO THE CHAPTER

The main goal of this chapter was to use the collected data to provide an in-depth narrative analysis of the participating teachers' geometry teaching practices. Teaching for MP was a focal point of their classroom instructional practices. A detailed analysis of their teaching videotapes, separately and in relation to each other, indicated that there were clear similarities and differences among their teaching practice and mathematical interactions with students.

As the analysis unfolded, the facets of these teachers' teaching proficiency in relation to the four enactivist concepts of teacher practice were clear to see. These components revealed the complexities of the teachers' everyday world and embodied understanding of mathematics. Further, the analysis of their responses from the interviews identified broad themes of key factors or interesting characteristics of effective teaching. These themes were further categorised into a mix of external and internal factors that are important to consider in relation to teacher effectiveness. In the next chapter, I pull these threads together to assess and present the teaching proficiency characteristics that were similar and different across the teaching practices of the five teachers. I also present an adapted model of instructional strategies for effective teaching that has emerged from this study.

CHAPTER SEVEN

REFLECTIONS ON THE FINDINGS

> *Instruction that develops mathematical proficiency is not simple, common, nor well understood. It comes in many forms and can follow a variety of paths* (Kilpatrick et al., 2001, p. 359).

7.1. INTRODUCTION

The focus of the reflection in this chapter is on the three critical research questions that framed the analysis:

- What are the teaching proficiency characteristics of selected effective mathematics teachers?
- How do these proficiency characteristics inform teachers' classroom practice?
- What are the teaching proficiency characteristics that are similar and different across the teaching practices of the selected teachers?

Chapters Four, Five and Six provided a detailed account and analysis of the qualitative data collected from questionnaires, classroom lesson videos and several interviews with the participating teachers. During the course of the data analysis a number of broad insights, which Merriam (2009) refers to as preponderances, emerged. These are explored in this chapter in terms of Kilpatrick et al.'s (2001) model of teaching for mathematical proficiency (MP), and the enactivist framework of embodied cognition. Kilpatrick et al.'s (2001) five strands of teaching for MP and enactivist concepts therefore comprise the *subheadings* in this particular chapter. The chapter also presents an 'expanded version' of Kilpatrick et al.'s (2001) model of MP, as a practical guide for teachers. This expanded version entails a detailed discussion of characteristics of effective pedagogy in mathematics, and of how the participating Namibian teachers developed their MTP and effectiveness.

7.2. REFLECTIONS ON THE GEOMETRY SCENARIO-BASED QUESTIONNAIRE FINDINGS

The questionnaire was used to determine the content capability and PCK of the selected teachers and to show how competent they were in teaching geometry. The teachers revealed a relatively strong mathematical content knowledge, and showed insights into particular teaching and problem solving methods. A strong mathematical identity, personal knowledge and skills, and a high disposition towards mathematics were revealed. The data from the lesson video footages and interviews confirmed that the integrity of the observed geometry lessons aligned with effective teaching practices as supported by teachers' responses in the geometry content knowledge questionnaire.

7.3. OBSERVING TEACHERS' PRACTICE

The mathematics lessons observed in the classroom represented the daily teaching practices of the five selected teachers, and captured the mathematical discourse patterns that dominated their instruction.

7.3.1. Reflections on the classroom lesson videos

My observation showed that the teachers' practice was indeed a complex phenomenon. The analysis of the lesson videos data indicated that aspects of the five strands were evident. There were similarities and differences in how the teachers facilitated students' learning in relation to the Kilpatrick et al.'s (2001) model or framework.

Firstly, the lesson video analyses indicated that three of the five strands, viz. conceptual understanding (CU), procedural fluency (PF) and productive disposition (PD) were addressed regularly by all five teachers. The strands of strategic competence (SC) and adaptive reasoning (AR) appeared rarely but were nevertheless present.

Secondly, evidence showed overwhelmingly that it was possible for the teachers to teach in a classroom environment where most of the five strands of MP were manifested and reinforced each other. However, little evidence was seen of these teachers presenting non-routine or authentic mathematical tasks or problems other than a vehicle for managing whole class discussions of mathematical concepts and procedures.

Thirdly, as the analyses unfolded it became apparent that the students' learning of mathematics was informed by more factors than just the ways in which their teachers addressed the five strands of Kilpatrick et al.'s (2001) model. There was ample evidence in the teachers' practice of the four enactivist theoretical concepts of *autopoesis, embodiment, structural determinism* and *co-emergence*. This was not surprising as recent studies have shown that "when broad and inclusive categories of analysis have been exploited, mathematics teachers, irrespective of their locations and particularities, appear to act similarly" (Andrews, 2011, p. 12).

Fourthly, it was apparent that while the teachers used real-world connections and contexts, multiple representations, manipulatives and employed expositions, why and how questioning, the **mechanism** that contributed to effective instruction and distinguished their classroom instructional practices were the “mathematical tasks” and their “interactions” with students. Kilpatrick et al. (2001) view “effective teaching and learning of mathematics as the product of interactions among the teacher, the students and the mathematics in an instructional triangle”. (p. 314)

7.3.1.1. The development of students’ conceptual understanding (CU)

All five teachers demonstrated a good understanding of conceptual knowledge of geometry. The data showed that development of students’ CU was exemplified in episodes of explanation (exposition), various forms of oral questioning and whole-class reflections on conceptually-focused rich mathematical tasks. From these observations, CU was one of the five strands of MP that gave meaning to concepts and understanding of relationships (Kilpatrick et al., 2001). Effective questioning was central to conceptual teaching and understanding. During classroom practice, teachers focused on the development of deep conceptual knowledge and connections between mathematical concepts to aid procedural skills. Researchers note that conceptually oriented teaching encourages students to make links within mathematics itself and between mathematics and the real world (Andrews, 2011). The teachers also showed *explicitness* about mathematical language, terms and terminologies in a number of ways. All five teachers used mathematically appropriate language and made conceptual connections between different geometric concepts as well as links to other areas of mathematics such as algebra. On many occasions, the teachers pointed to multiple representations of mathematical ideas to encourage a deeper understanding of mathematical concepts. The teachers also asked students to explain solutions and strategies, and to make their thinking and ideas explicit. It is however important to note that the development of CU overlapped iteratively with other strands of MP.

The next section attempts to identify “**uniqueness**” in the ways in which the five mathematics teachers developed this strand of MP in teaching geometry and comments on both positive and negative qualities of the lessons.

7.3.1.1.1. Demis (Teacher 1)

Demis prioritised more conceptual knowledge, focusing primarily on linking mathematically relevant representations to concepts in an explicit way. Through a conceptual teaching approach, Demis involved students in several hands-on activities with concrete manipulatives while pressing them to figure out mathematical ideas for themselves (refer to lesson video 1, vignette 4). These manipulatives included pictures/diagrams, manipulative models, real world situations, written symbols and oral language. Demis made use of these physical manipulatives or resources in order to establish concrete mathematical ideas, confirming Kilpatrick et al.'s (2001) notion that representations are interwoven. Resources provided opportunities for greater understanding when doing and working in the concrete as suggested by Clements (1999, p. 50):

> "[Effective] mathematics teachers use physical manipulatives to build what is referred to as 'integrated concrete ideas' that help children to engage in the construction of meaningful mathematical ideas".

7.3.1.1.2. Jisa (Teacher 2)

Jisa addressed students' development of CU differently. She presented concepts that were grounded in CU. For example, she used two moving rays attached to the point where the diameter and circumference intersect to illustrate and model the construction of angles in a semi-circle (refer to lesson video 1, vignette 1). She also used examples drawn from everyday situations in order to reinforce students' CU, such as her classroom window frames to explain mathematical connections and the relationship between parallel lines and quadrilaterals (refer to lesson video 3, vignette 1). By using realistic examples from everyday life contexts, Jisa displayed CU of teaching geometry, supporting the contention that effective teachers support students in creating connections between mathematics and everyday experiences (Anthony and Walshaw, 2009).

7.3.1.1.3. Ndara (Teacher 3)

Ndara' conceptual understanding was linked to conceptually-oriented teaching strategies. Across three lessons, Ndara explicitly talked about the meaning of mathematical language used by the students and defined loci terms such as "sets of points from a point", "sets of points from a line or two lines" and "sets of points that are equidistant from two points or lines", in order to make the vocabulary comprehensible. He demonstrated how to use these terms in different contexts including their applications (refer to lesson video 1). As new concepts and skills were learned, Ndara built on these in a steady progression of skills (National Mathematics Advisory Panel, 2008).

7.3.1.1.4. Emmis (Teacher 4)

Emmis, like Jisa, was observed teaching grade 10 circle geometry. In teaching angle rules or theorems in circle geometry, Emmis emphasised mathematical concepts and process (refer to lesson video 1, vignette 1, and lines 1 to 32). His conceptual understanding in teaching theorems in circle geometry showed that "next to the 'how' of the procedures also the 'why' of the procedures has central meaning" (Givvin et al., 2009, p. 43). Emmis encouraged and emphasised the development of a deep understanding of theorems and angle rules in a logical, linear fashion. He also demonstrated the mathematical links, connections and relationships between different geometric concepts, such as the interrelationships of the properties of a circle and its constituent concepts (refer to lesson 1, vignette 1, and lines 3 to 11).

7.3.1.1.5. Sann (Teacher 5)

Sann's mathematical discourse displayed conceptual understanding through the ways in which she explained concepts and procedures, posed questions, interpreted students' answers and used representations. There was some variation in that she developed mathematical concepts by using representations and modelling, making conceptual links or references to interesting contexts that students could relate to (refer to lesson video 1, vignette 1 and lines 3 and 13). Sann explained mathematical terminologies by using real life terms, contexts, tactile experiences, aesthetic representations and other physical manipulatives as well as metaphors. Special qualities of appropriate manipulatives or metaphors as described and summarised by Clements (1999, p. 51) show that:

> Good physical manipulatives are those that are meaningful to students; provide control and flexibility to students. Also they have characteristics that are consistent with, or mirror cognitive and mathematical structures and assist the student in making connections between various pieces and types of [mathematical concepts] knowledge.

7.3.1.2. The development of students' procedural fluency (PF)

All the mathematics teachers emphasised their students' PF by means of exposition, questioning and reflection on problem solving and the way students worked out problems using different solution strategies. From my observations, it seemed that the development of students' PF was a central concern for the teachers because of their experiences and the perceived needs of their students. In almost every lesson video, computational practice and discussion of the meaning of the solving procedure or steps constituted a central part of the lesson.

It was observed that PF and computations were more firmly entrenched than other strands of MP namely, SC and AR. A close examination of the vignettes revealed that they all encouraged development of the PF of their students inclusively with other mathematical foci such as CU and PD. Sometimes they addressed procedural skills with a focus on AR or SC. Lesson observations showed that the teachers placed a greater emphasis on PF whilst repeatedly probing students to recall, reproduce and use formulae and rules accurately, efficiently, appropriately and flexibly. All five mathematics teachers used their knowledge of procedures in different contexts (Kilpatrick et al., 2001). For example, teachers drew on their own PF skills when they solved tasks and connected various representations. Their PF also played an important role in developing students' SC. This aspect of their practice reverberated with Kilpatrick et al.'s (2001) claim that without sufficient procedural fluency, teachers have trouble deepening their students' understanding of mathematical ideas or enhancing their ability to solve mathematical problems. Likewise, Anthony and Walshaw (2009, p. 25) argue that "teachers with in-depth knowledge have clear ideas about how to build procedural proficiency and how to extend and challenge students' ideas".

Each teacher also addressed the development of students' PF differently, as discussed in sections 7.3.1.2.1 to 7.3.1.2.5 below.

7.3.1.2.1. Demis (Teacher 1)

Demis juxtaposed her procedural practice with considerable attention to mathematical procedures. Procedures for solving a problem or steps in solving mathematical tasks were a strong focus in her practice. She maximised opportunities to develop appropriate, efficient and fluent procedures in all three observed lessons. For example, she explicitly explained what, when and how a mathematical rule is used or applied (refers to lesson videos 1 and 2). In this way, she involved students in an activity that required them to cut out a rectangle and a circle on the two square grid papers (graph sheets) to construct combined shapes. By discovering and using mathematical procedures accurately through the handling of concrete manipulatives, the fluency with which students carried out procedural tasks was significantly enhanced. Kilpatrick et al. (2001) contend that PF for effective teaching means that teachers are able to perform teaching tasks which teach the concepts and underlying procedures.

7.3.1.2.2. Jisa (Teacher 2)

The development of students' PF appeared to be a major theme in Jisa's lessons, where she emphasised mathematical procedures and correct computations. From the three lessons observed, her teaching was dominated by explaining procedures and providing algorithms to solve the mathematical problems at hand. Jisa' method for promoting students' understanding and acquisition of skills, procedures and techniques was to encourage them to use mathematical formulae accurately and appropriately. Jisa also asked students to explain and defend their answers or methods for solving problems. This practice aligns with findings that students should be provided with an opportunity to clarify and justify the meanings of their solutions (Kilpatrick et al., 2001; Tshabalala, 2012).

7.3.1.2.3. Ndara (Teacher 3)

In the lessons observed, Ndara also displayed a procedural approach to teaching loci and angles in circle geometry (refer to lesson video 1, vignette 2 and lesson video 3, vignette 2). He addressed the development of students' procedural skills to deepen their CU. For example, during the first lesson on loci terms, he offered mathematically accurate explanations of the procedures for constructing the locus of points and concepts building on the relationships between lines and angles students measured, encouraging them as he did so to use mathematical formulae, procedures and techniques accurately and correctly. Where the teaching of PF was most prominent, Ndara showed evidence of rich connections of PF and CU as he was able to move from algorithms to the conceptually meaningful phase. Rittle-Johnson and Alibali (1999; 2001) argue that the two forms of knowledge do not develop independently, and view conceptual knowledge as coming before procedural knowledge.

7.3.1.2.4. Emmis (Teacher 4)

Emmis' practice regarding procedural knowledge and the development of mathematical skills contrasted positively with that of other teachers. Specifically, Emmis made definite attempts to encourage students to use mathematical procedures accurately and appropriately and thus to realise the relevance of mathematical techniques. He emphasised rules and procedures as well as concepts through demanding hands-on activities and complex mathematical tasks. Hence, his instruction gave students the opportunity to discover mathematical procedures and techniques (refer to lesson video 2, vignette 2 and lines 8 to 38). Across the three lesson videos on circle geometry, Emmis addressed the development of students' PF in a variety of ways. Firstly, he took students through the material step by step in a clear and direct way to encourage them to use mathematical techniques and procedures accurately, correctly and appropriately(refer to lesson video 1, vignette 1 and lines 21 to 28). Secondly, he encouraged multiple procedures in solving problems to deepen students' PF and understanding of geometric concepts. Researchers argue that effective teachers should possess good understanding of both conceptual knowledge and procedural knowledge of mathematics to be able to provide students with clear explanations of underlying concepts and procedures (Hiebert and Lefevre, 1986; Hill et al., 2008).

7.3.1.2.5. Sann (Teacher 5)

Sann' practice also showed abundant evidence of PF, which played a crucial role in developing students' CU, SC and AR. Sann developed PF as she encouraged students to use their SC to come up with their own solution strategies and choose among effective mathematical procedures. Significantly, whilst emphasising PF in the use of rules and formulae, she also encouraged her students to articulate and justify their mathematical ideas (refer to lesson video 1, vignette 2 and 3). While Sann was explaining ideas and procedures, and how these procedures should be used, she simultaneously engaged students in the underlying mathematical concepts they were studying. This is consistent with Star (2005), who views conceptual and procedural knowledge as co-existent and equally important for the development of mathematical proficiency.

7.3.1.3. The development of students' strategic competence (SC)

In my observation of teachers' practice, there was little evidence of teachers supporting the development of students' strategic thinking or competence. On the few occasions where SC was addressed, it was managed explicitly or implicitly through students' engagement with the solution of non-routine tasks. For example, the formulation of angle theorems or devising of working plans. As problem solving is central to SC, the teachers whose lessons addressed this strand frequently engaged students with non-routine mathematical tasks that elicited multiple strategies for solving problems. Through these tasks, teachers encouraged students to solve problems collaboratively and to come up with their own solution strategies. In addition, representations were a key aspect of teachers' practice across the lesson videos. For example, teachers used multiple representations to enable students to investigate, analyse, recognise, describe and represent important different geometric concepts. Through this approach, they enabled their students to work confidently with various mathematical concepts and their relationships to estimate, calculate and check solutions. These included mapping graphical, symbolic representations as well as algebraic notation, and concrete pictures or diagrams to enable students to see the interrelatedness of the mathematical concepts they were learning. In the study it was clear that teachers used different strategies to encourage students to learn the content effectively. For example, teachers used effective questions to assess students' understanding of concepts, and to get them to think more deeply about different solution strategies.

To get a more direct sense of the **uniqueness** of these teachers, I look at features of their teaching with regard to pedagogy and instructional approach.

7.3.1.3.1. Demis (Teacher 1)

Demis' teaching approach showed ample evidence of SC. She was able to articulate her general strategy, thereby intersecting her content knowledge with pedagogical techniques. Demis demonstrated a sound understanding of PCK as she explained in the interview, demonstrating the mathematical connections and relationships between taught concepts. Two additional themes emerged from Demis' teaching approach. Firstly, she showed respect for her students and engaged them by treating teaching as a collaborative process, valuing students' thoughts and ideas in the pedagogical relationship. Thus her teaching can be characterised as encouraging *partnership* in which the teacher plays the part of a co-learner or critical friend to students (Mennin, 2010). Secondly, her teaching approach showed that SC was prioritised, with considerable attention paid to heuristics and mathematical reasoning.

In this situation, how did Demis develop her students' SC? Demis' pedagogy was also marked by a flexible use of *questioning* to support the development of students' thinking, coupled with a creative selection of tasks for problem solving. Across all three geometry lessons, she provided students with practice that allowed evaluation of multiple solving strategies. Demis also encouraged students' engagement with the solution of non-routine tasks, while simultaneously emphasising the conceptual development and procedural fluency of her students. For example, she invited students to formulate mathematical procedures or derive working procedures for determining the perimeter and area of combined shapes (refer to lesson video 2, vignette 2, and lines 12 to 16). The literature suggests that effective teachers provide students with multiple representations to develop both their CU and their computational flexibility (Anthony and Walshaw, 2009). As she addressed CU and PF, Demis focused on mathematical reasoning and problem solving. Ball and Bass (2003) affirm that the use of both procedural and conceptual knowledge is effective in creating opportunities for improving students' SC.

7.3.1.3.2. Jisa (Teacher 2)

Jisa demonstrated a combination of conceptual and procedural knowledge in her approach to lesson planning for geometry. For example, she prepared lessons by designing posters, placards and group activities. She also asked students to share their solving methods or work out exercises on the board, and to assist each other when working in small groups or pairs. However, a careful analysis of Jisa' teaching approach and pedagogy showed that there was no difference in the interactive patterns in the three lessons. In almost every lesson, there was an emphasis on rules and procedures as well as concepts and process. That is, Jisa engaged students in purely procedural and conceptual practice at the expense of strategic competence and adaptive reasoning.

7.3.1.3.3. Ndara (Teacher 3)

In contrast, Ndara' mathematics classroom displayed unique characteristics. In particular, he worked collaboratively with the students, guiding them to solutions and explaining concepts. He also engaged students in tasks that allowed multiple solving strategies and evaluation of those strategies. Hence, Ndara' general approach to teaching can be characterised as both procedurally and conceptually oriented. Each lesson observed showed a strong emphasis on mathematical procedures together with explanation of the relationships and mathematical connections between concepts being learned, as well as what, how and why procedures are applied. Moreover, in most lessons, Ndara used worksheets selected from previous examination papers to supplement exercises in the textbook.

7.3.1.3.4. Emmis (Teacher 4)

Emmis's teaching approach contrasted sharply with that of other teachers as he emphasised the importance of mathematical meanings of concepts that are useful to students through investigation and the correct application of mathematical techniques. For example, the derivation of mathematical procedures as well as accurate explanations of angle theorems in circle geometry (refer to lesson 1, vignette 1) engaged students to deepen their CU. More importantly, the mathematical discourses in Emmis' class appeared to be focused more on the acquisition of underlying mathematical concepts and procedures. The analysis of Emmis' lessons showed that he predominantly adopted a conceptual knowledge approach to teaching geometry with a substantial mathematical focus on teaching technical abilities and rules, while emphasising mathematical concepts and problem solving (refer to lesson video 1 and 2). As students worked on problem solving, he encouraged them to explore mathematical formulae and procedures on their own and construct their own mathematical understanding.

Leikin and Rota (2006) note that the success of this practice depends heavily on the teacher's knowledge and skills to strengthen and broaden conjecture, exploration and investigative procedures. With regard to the pedagogy, Emmis often planned carefully, representing mathematical ideas in various ways to ensure better comprehension of mathematical concepts, ideas and procedures.

7.3.1.3.5. Sann (Teacher 5)

From my observation, there was ample evidence of SC in Sann' teaching. She revealed a sound understanding of her own PCK. For example, she used the strategies of whole class discussion, oral questioning, real life contexts and practical examples familiar to students as well as checking and marking homework assignments. One distinguishing feature of her instruction is that she used bodily actions as visual representations to elaborate on and provide accurate explanations of concepts to deepen students' mathematical understanding. This showed that Sann was not only able to bring explanations and illustrations of mathematical ideas into the world of the physical, but she literally used her own body to make students realise angle properties (refer to lesson video 1, vignette 1). This finding resonates with Kilpatrick et al. (2001) who urge teachers to continue using physical manipulatives to model concepts in representational and abstract forms.

In a different incident, Sann explained a variety of solving strategies to allow students to evaluate different ideas and solution strategies (refers to lesson video 2, vignette 2, and line 22). Another distinctive feature of her practice was that making mathematics fun, interesting and concrete for her students was a priority for Sann. Uniquely, this led to lessons where mathematics was presented as "mathematics all around us".

7.3.1.4. The development of students' adaptive reasoning (AR)

In my observation, there was not much evidence of AR in the teaching of the participants. I thus argue that the development of students' mathematical reasoning, conjecturing and higher order thinking presented a challenge in practice.

Though this strand was a minor focus in many lesson videos, it was interesting to note that some teachers provided students with situations and activities that required mathematical reasoning. In my observation, the teachers addressed students' AR in particular when they questioned the "why" of mathematical procedures to trigger constructive thinking processes. In many cases, teachers used their AR to help students navigate facts, procedures, concepts and solution methods, and to see that they all fitted together in some way. This means they provided students with opportunities, encouragement and assistance to engage in thinking, reasoning and sense making (Kilpatrick et al., 2001).

It was also interesting to observe that the ‘teachers’ use of (and reflection on) different problem solving strategies and analysis of students’ mathematical thinking or reasoning were on-going in many lessons. Teachers constantly monitored students’ work and progress to gauge how effective particular learning tasks were in helping students to master key concepts and develop mathematical proficiency. All teachers’ mathematical ideas, interpretations, reasoning and thoughts were evident in the mathematical discourses in those classrooms. There were, however, **differences** between teachers’ practice, as discussed in sections 7.3.1.4.1 to 7.3.1.4.5.

7.3.1.4.1. Demis (Teacher 1)

One notable difference between Demis and other participating teachers is the extent to which she provided situations and activities that required students to justify their solution strategies. However, such tasks were observed in only two lessons. For example, in the second lesson on measurement of plane shapes, Demis asked students to make their mathematical thinking, ideas and argument clear. Her questions included “why do you think they make it a broken line” (line 1), “what is the practical reason” (line 3), or “why there is a difference in your answers” (line 13) (vignette 1). Using these methods, Demis’ questions required students to explain why a procedure worked or why a step or method was correct or valid. Moreover, students were consistently encouraged to justify their claims (refer to lesson video 3, vignette 4). This showed that Demis placed a stronger emphasis on reasoning and higher order thinking. This is in line with Ball and Bass (2003) who argue that effective mathematics teachers provide students with opportunities, encouragement and assistance to engage in thinking, reasoning and make sense of mathematical ideas.

7.3.1.4.2. Jisa (Teacher 2)

In contrast, Jisa seemed to ignore potential opportunities to engage students with mathematical reasoning. She did not ask questions to elicit students’ mathematical thinking about ideas. As discussed earlier, the teaching of mathematical concepts and procedures was given most time in Jisa’ lessons, with higher level mathematical thinking and reasoning getting the least attention. Jisa presented mathematical tasks in ways that enabled students primarily to learn mathematical procedures and compute right answers. This is not in line with Kilpatrick et al.'s (2001) argument that mathematical proficiency is the evidence of mathematical reasoning.

7.3.1.4.3. Ndara (Teacher 3)

In the third lesson on grade 11 circle geometry, Ndara addressed students' development and articulation of justification (refer to lesson video 3, vignette 4, and lines 53 and 57). For example, in this lesson Ndara encouraged reflection, asked students to justify why angles were equal, and how they got certain answers. However, classroom discourse patterns undermined the opportunities students had to explore concepts and to appreciate that doing mathematics is conjecturing, reasoning about solutions and methods and justifying ideas (Kilpatrick et al., 2001). At best, Ndara required students to understand definitions and the mathematical concepts underlying mathematical procedures to use mathematical formulae, procedures and techniques correctly, accurately and efficiently.

7.3.1.4.4. Emmis (Teacher 4)

Emmis' practice showed evidence of mathematical thinking and reasoning and opportunities for students to gain a more profound understanding of concepts. Emmis engaged with students to think deductively on many occasions. Classroom activities used inductive or deductive reasoning, that is, empirical mathematical generalisations or verification in geometrical contexts, such as constructing and measuring angles of several shapes in order to generalise or to verify angle rules or theorems in circle geometry (refer to lesson video 1, vignette 3, and lines 33 to 37). In this way, he provided conceptually rich tasks that required students to make mathematical generalisations or theorems and conjectures from the specifics of particular tasks and problems. Emmis valued deductive reasoning as an important mathematical process as he often led students to make conjectures and to verify mathematical concepts and rules, pressing them to think mathematically. As Jaworski (1994) says, an effective teacher offers challenges to stimulate mathematical thinking and to promote critical reflection on mathematical understanding.

7.3.1.4.5. Sann (Teacher 5)

A notable difference between Sann and the other teachers is that she addressed students' development of mathematical reasoning and argumentation in all three lessons. She presented mathematical tasks that were grounded in mathematical reasoning such as conjecturing, justification and the explanation of mathematical ideas and solution strategies through a whole class discussion (refer to lesson video 1, vignette 3). Questions posed were: "why did you divide by 12" (line 8); "why is it the same" and "what can you tell me about $4m$ and l" (line 13 and 15). In this manner, she asked students to justify their ideas and make their thinking explicit instead of simply pressing for correct answers. From a pedagogical standpoint, Kilpatrick et al. (2001) comment that practices which engage learners in a range of highly demanding mathematical tasks, such as reasoning and generalising, may be effective in assisting learners to see geometry as meaningful, purposeful and sensible in their life.

7.3.1.5. The development of students' productive disposition (PD)

In terms of the Kilpatrick et al.'s (2001) assertion that a PD "develops when the other strands do" (p. 131), there was abundant evidence indicating that all five mathematics teachers addressed the development of their students' PD in observable ways. PD was a lesson objective in almost every lesson. For example, teachers' PD was evident from the lesson videos in terms of their passion and enthusiasm as well as the homework assignments that they gave to their students. All five teachers gave a homework assignment at the end of the lesson, which they then went through at the beginning of the following lesson using the problems given in the homework task to connect learners with the new concepts and move them from one step of the lesson to another. This routine practice helped teachers to check on the students' prior and existing mathematical understanding, allowed for more practice and suggested how they could move out of one strand of mathematical proficiency to another. In addition, the majority of teachers had good rapport with their learners, promoting PD not just through kindness and patience but also through attempting to provide interesting hands-on activities. Lewis (2007) notes that PD is a character trait in the service of mathematics teaching that is encouraged, not a mathematical process or objective in its own right.

As the teachers developed other substantive MP strands of their pedagogy, there was also a more focussed consideration of the broader mathematical identities of the students. The five teachers regularly made efforts to consider and address their students' mathematical productive disposition, attitudes, interests and ideas in various ways. These were apparent in the lesson videos, and will be discussed next (sections 7.3.1.5.1 to 7.3.1.5.5). My observation with respect to this strand is consistently positive.

7.3.1.5.1. Demis (Teacher 1)

Demis promoted the development of students' mathematical PD by making the lessons interesting with hands-on activities, concrete manipulatives and other real life contexts. She encouraged them by saying making mistakes in the mathematics classroom were acceptable (refer to lesson video 1, vignette 3). Demis provided immediate and positive feedback so that students could see the worth of the lessons. She also promoted a friendly relationship with her students.

7.3.1.5.2. Jisa (Teacher 2)

Jisa tried to address the development of PD through extra mathematics lessons and explanations of concepts after school hours and during weekends. In addition, a respectful relationship was evident between Jisa and her students. Order was kept at all times.

7.3.1.5.3. Ndara (Teacher 3)

Ndara promoted the development of his students' PD by making the lessons interesting with hands-on activities, applying procedures and making explicit what students could expect in the examinations (refer to lesson video 2, vignette 2). He also affirmed students' responses by repeating their utterances and expected them to work hard and be enthusiastic (lines 43 and 47).

7.3.1.5.4. Emmis (Teacher 4)

Across the three lessons, Emmis promoted the development of students' PD by making the lessons interesting with conceptually rich mathematical and hands-on activities, in order to encourage students to make sense of abstract mathematical concepts and ideas in circle geometry (refer to lesson video 1, vignette 4). Further, the climate in Emmis' mathematics classroom was positive: the lessons were enjoyable, respectful and serene.

7.3.1.5.5. Sann (Teacher 5)

The special qualities in Sann' lessons were derived from the informal, partner-like relationship between her and the students. Sann promoted development of PD through real life contexts and genuine practical examples, demonstrating how mathematics relates to architecture and design and including anecdotes, in order to create mathematically productive confidence among students (refer to lesson video 1, vignette 4). She also showed respect for students' ideas and contributions, joked from time to time, and the students were happy with her kinaesthetic explanations of concepts.

7.3.2. Reflections on insights gleaned from enactivism

The teachers all used embodied mathematical experiences, mathematical discourse and interaction to encourage students to think mathematically. While recognising alternative perspectives in the social dimension of classroom learning, I found the enactivist position worthwhile in expressing what I observed in these classrooms. Mathematics teachers, in this study, developed students' construction of mathematical meaning and understanding at four levels:

1. Autopoesis – inter-subjectivity of teachers and students as co-participants in the mathematics classroom contexts,
2. Co-emergence – construction of mathematical knowledge with understanding as collective participation in complex classroom system,
3. Sharing effective teaching practices through the research process – the researcher drawing inter-subjectivity with participants as co-researchers, and
4. Embodiment and structural determinism – construction of mathematical knowledge or understanding in the classroom, inter-subjectivity of teachers and students.

I thus begin by discussing the four key ideas of enactivist theory in relation to effective teaching and learning of mathematics.

7.3.2.1. Autopoesis of mathematical communication

In my observation, there was evidence of autopoesis in the mathematical communication as well as evidence of high levels of mathematical thinking and challenges within teachers' classroom routine. Significant features of interactions involved teachers seeking inter-subjectivity with students, trying to understand the students' thinking processes and offering appropriate challenges (Jaworski, 1994). In autopoetic terms, the complexity of the teaching task aided in creating a sophisticated social environment through which mathematical meaning and understanding were developed (McMillan, 2004). From my observation, teachers attempted to create classroom situations through which learners could construct mathematical understanding individually, collaboratively or interactively. Evidence of teaching processes which fostered an autopoetic interpretation was provided by teachers' asking the question "why", to develop mathematical thinking in students. Such questioning allowed a variety of discussions/discourses to develop during the learning process.

7.3.2.2. Autopoesis of teachers and teachers/students as co-participants

One critical condition for effective learning, from an enactivist perspective, is a space in which the agents (teachers and students) interact with each other (Davis and Sumara, 1997, 2006). I observed teachers and students working together to solve problems. When teachers enacted autopoesis, connecting students to their instructional practices and with mathematical concepts, the learning was at its height. Working with students as co-participants, in this case, was not just limited to the teachers themselves, but also extended to the presentation of the content in response to students' ideas and responses.

7.3.2.3. Co-emergence: construction of mathematical knowledge as collective participation in complex systems

It was evident from the analysis that teachers emphasised effective instructions structured in ways that included the co-emergence of mathematical ideas, embodied cognition and students actively co-constructing mathematical knowledge along with acquisition of procedural and problem solving skills. Thus, learning is inseparable from fully embodied nets of on-going action, social relations and history in complex systems (Reid, 1996). As the teachers and students were working together on different tasks/problems, their mathematical ideas, thoughts and actions co-emerged as the collective practices of a classroom community (Davis and Sumara, 1997). The literature suggests that mathematical concepts and procedures learned without shared and embodied understanding tend to be error prone, are easily forgotten and do not transfer easily to novel problem types (Davis and Simmt, 2003, 2006).

My own understanding of effective practice through the lens of co-emergence was brought into focus when I engaged teachers as co-researchers in critical reflection on their own teaching practice. All five teachers seemed to value discussions in which they shared their experiences and effective practices. This highlighted their willingness to engage in critical reflection together and emphasised the importance of inter-subjectivity. The stimulated recall analysis session, for example, was a collective activity in which the teachers participated, with mathematical objects mediating their actions and dialogue, as they together defined effective teaching practice, and a shared understanding emerged. As each teacher contributed, he/she changed the interaction and the emerging object. The relational understanding of effective practice among them changed as the looping-back effect changed the contributor's subject position within the collective activity. This was a valuable example of *mutual specification* (Varela et al., 1991), the fundamental dynamic of systems continually engaging in joint action and interaction.

7.3.2.4. Embodiment and structural determinism

The preceding chapter discussed how teachers conceptualised and operationalised their mathematical knowledge for teaching according to the notion of embodiment. This suggests that teachers' knowledge of mathematics was inseparable from their knowledge of how mathematics is taught and their own teaching practice. Under scrutiny, teachers revealed their mathematical knowledge for teaching as dynamic and embodied, and thus not a distinct domain of knowledge to be mastered by individuals. It occurs in contexts that involve the students, the mathematics content and the teacher (Kilpatrick et al., 2001). Hence an understanding of how these components might be engaged in productive interaction was an important aspect of this study.

From the observation, a key element of effective teaching practice was a focus on "knowing" instead of "knowledge", captured in the enactivist slogan "all doing is knowing and all-knowing is doing" (Maturana and Varela, 1992, p. 26). This emphasis on doing in relation to embodied cognition underpinned the teachers' practice. In this study, doing mathematics was a core characteristic of solving mathematical problems, and provided a platform to apply the enactivist notion of embodiment in learning environments that use mathematical discourses. Problem solving was a social practice through which teachers encouraged their students to learn algorithms and procedures for solving mathematical tasks.

7.3.3. Reflections on the interview findings

The interview findings suggest that the pedagogical practices of the teachers align well with Kilpatrick et al.'s (2001) model. Furthermore, the results also pointed to a good alignment with the principles of enactivism. The analysis of teachers' responses from interviews identified broad themes of characteristics of effective teaching, which were categorised as either external or internal factors. The *external factors* are factors enabling effective teaching that are outside the teachers' control, i.e. opportunities presented by the external environment. These include school management and leadership and efficient organisation, teacher collegiality, collaboration and teamwork, and effective links with the broader community. The *internal factors* are key factors enabling effective teaching within teachers' control. These include teachers' educational background and experience, commitment, hard work, discipline and passion for mathematics. The following additional internal factors were identified: a conceptual approach to teaching, structured and well-planned lessons and high expectations (of themselves as teachers and of the children in terms of cognitive tasks), the ability to adapt the curriculum to suit learning needs, the use of high level content, the purposeful assessment of learning, a supportive and conducive learning environment; and the use of graded homework assignments to enhance understanding of mathematical concepts.

The themes revealed two aspects which helped students construct a deeper understanding of mathematical concepts: first, accurate explanations, and secondly, comprehensible, appropriate definitions. The teachers also believed conceptually rich focused mathematical tasks as well as concrete manipulatives were helpful tools for the effective teaching and learning of mathematics. It was apparent to the teachers that discovery or investigation tasks helped their students to grasp mathematical concepts with greater ease. Teachers further reported incorporating real world examples into their lessons to foster the engagement of students with concepts that might interest them and to make abstract concepts more comprehensible. Effective teachers used examples as a "hook to engage students and to provide a bridge between what students already know and the more abstract concepts" (Weiss et al., 2003, p. 88). However, the comprehension and representation of abstract mathematical ideas seemed to be a problem for the teachers. It is therefore advisable that mathematical reasoning and investigation linking concepts to real life situations should be used as much as possible.

The interview data suggests that there is a need in Namibia to create collaborative deliberation similar to what I did in this study to allow mathematics teachers to participate in group discussion and share experiences and ideas about effective teaching and learning approaches. Through participation in these discussions, teachers would gain insights into the knowledge and experience of colleagues and expand their repertoire of teaching approaches for individual strands of MP. The lesson study method adopted by Japanese teachers (Suh, 2007) maintains that teachers stand to gain from such collaborative reflections.

7.3.3.1. Key factors/characteristics that contribute to teachers' effectiveness and success

Here I describe key enabling factors of teacher effectiveness as identified by the participants in the interviews. These factors should not be regarded as independent of each other, and I draw attention to links which may help to provide a better understanding of possible mechanisms of teacher effectiveness. While the list is not complete, it provides a useful framework for understanding effective teaching practice and improving the effectiveness of the learning environment.

7.3.3.1.1. Effective school leadership and management

The interview data analysis across the five schools indicated effective school leadership and management as the key factor necessary to bring about change in many factors that affect teacher effectiveness. Stephens (2009) suggests two characteristics of successful school leadership which apply here. They are: involving other staff members in decision-making and professional authority in the process of teaching and learning. Effective school leaders are essential for effective teaching, and are usually engaged, firm, motivated, purposeful, proactive and aiming to achieve at the highest level.

7.3.3.1.2. Clear and shared vision and goals

One of the important findings from the interviews pertained to shared vision. The data suggests that teachers are more effective when they have goals based on a common understanding of the aims and values of the school and their students' needs. The teachers put this into practice through consistent and collaborative partnerships and decision-making. Studies that focus on effective schools emphasise the importance of teamwork in developing effective, mutual relationships to achieve the shared vision (Sammons et al., 1995).

7.3.3.1.3. Collegiality, teamwork and collaboration

Another finding was teachers' collegiality and collaboration in the form of interaction among them. This collaborative culture manifests as willingness to share ideas and effective practice, observe each other's lessons and give feedback, learn from each other and work together to improve their use of strategies, and/or teaching and learning programmes.

7.3.3.1.4. High academic standards and expectations among teachers and students

The interview data showed evidence that effective teachers believe all students can learn and that they can teach them effectively. The data also revealed a high expectation of student performance, particularly among teachers but also students and parents, as one of the most important requirement for teachers' effectiveness. Teachers set challenging standards and were dedicated to supporting every student achieve those standards. Equally, all students were engaged in rigorous mathematical tasks in which high standards of performance are clear and consistently attained.

7.3.3.1.5. Emphasis on conceptual teaching of core knowledge

Another significant finding was how teachers enabled students to achieve understanding along with skill mastery for any mathematical topic. In the interviews, teachers indicated that they used multiple representations and real world contexts to promote students' critical thinking and conceptual understanding. When teaching conceptually there is more emphasis on the reasoning behind the process and on understanding how and why the problem works, not just finding an answer (Kilpatrick et al., 2001).

7.3.3.1.6. Teachers' Pedagogical Content Knowledge (PCK)

Teacher PCK, as a repertoire of teaching strategies used when directing the classroom interactions, was underscored throughout this research. Teachers' PCK opened up possibilities for students to see mathematics as accessible, worthwhile, doable and achievable. The literature places PCK as an essential prerequisite for effective mathematics instruction (Shulman, 1986, 1987). The teachers revealed a deep and connected mathematical knowledge that spanned their instructional practices. Teachers used their PCK dynamically, given the confines of the mathematics curriculum and how they viewed themselves within the broader educational landscape. One notable feature of teachers' instructional practice was that they used their PCK in conceptual and procedural teaching to encourage students to use mathematical procedures correctly, appropriately and efficiently. Thus, teachers such as Demis, Emmis and Sann who demonstrated a solid PCK emerged as remarkable examples. It is important to acknowledge that teachers' pedagogy was influenced by their personal and educational experiences and MP as well as the complexity of the circumstances in their classrooms and schools. The data from this study supported previous findings that having high PCK within a certain domain is a necessary, but not a sufficient condition for developing students' MP (Ball et al., 2008).

7.3.4. Reflections on the conceptual and theoretical frameworks adopted for this study

In the following sections I evaluate the conceptual and theoretical frameworks used in this study. This evaluation is done to determine the extent to which both the Kilpatrick et al.'s (2001) model and enactivism enabled me to answer the research questions and achieve the research goal.

7.3.4.1. Reflections on Kilpatrick et al.'s (2001) model

Applying Kilpatrick et al.'s (2001) five-component model of teaching for MP as an analytical frame for the teaching of geometry posed several challenges. First, since the five strands of the Kilpatrick framework are interconnected it was very difficult to deal with any component separately, and inevitably a large amount of overlap occurred between the categories. Secondly, dealing with the components of MP one by one in a linear fashion (as I did with CU, PF, SC, AR and PD in my analysis) was problematic since these strands occur simultaneously. This was especially challenging when applying the five interwoven strands to teaching practice.

Furthermore, Kilpatrick et al.'s (2001) model was "developed from the findings of studies in quite different contexts, and while it was based on sound practice and substantial research evidence" (Groth and Berger, 2006, p. 811), there was no guarantee that it would be appropriate in the context of Namibian schools. However, the data analysis and the evaluation indicated that a number of the pedagogical dimensions of the model were working well and were applicable to the Namibian teachers and the learners in the participating schools. Indeed, as Kilpatrick et al.'s (2001) model is gaining international prominence, I believe that these instructional practices that develop conceptual understanding, problem solving and thinking skills should be reinforced in order to facilitate the development of MP in students and improve teaching proficiency.

7.3.4.1.1. My contribution to Kilpatrick et al.'s (2001) model

This section summarises the contributions I have made to understanding the terms *effectiveness* and *mathematical teaching proficiency* or *competence*. In Chapter Two, I raised issues and concepts related to effectiveness and teaching proficiency as theorised by Kilpatrick et al. (2001) and others. Generally, this study supports the work of these authors because I have been able to extend their work, both with empirical knowledge from the field and with theoretical knowledge. In other words, this study found that teachers' teaching practice in the mathematics classroom supported the effectiveness of the Kilpatrick et al. (2001) framework in facilitating the development of MP in students. My understanding of Namibian teacher effectiveness and teaching proficiency is thus essentially the same as that of Kilpatrick et al. (2001).

I now present a ten-principle framework of strategies for effective teaching. This framework arises from the participating teachers' practice in Namibian schools and is embedded in the Kilpatrick et al.'s (2001) model that framed this study. The 10 principles support and extend Kilpatrick et al.'s (2001) work and are intended to assist effectiveness by enabling teachers to plan and conduct lessons in a structured manner so that successful teaching and learning will take place. They are not intended as individual principles for separate consideration, but should be viewed as interwoven and interdependent threads in the teaching of mathematics. Similarly, these principles or strategies are situation-specific. The ten principles that underlie effective teaching are summarised in Figure 7.1.

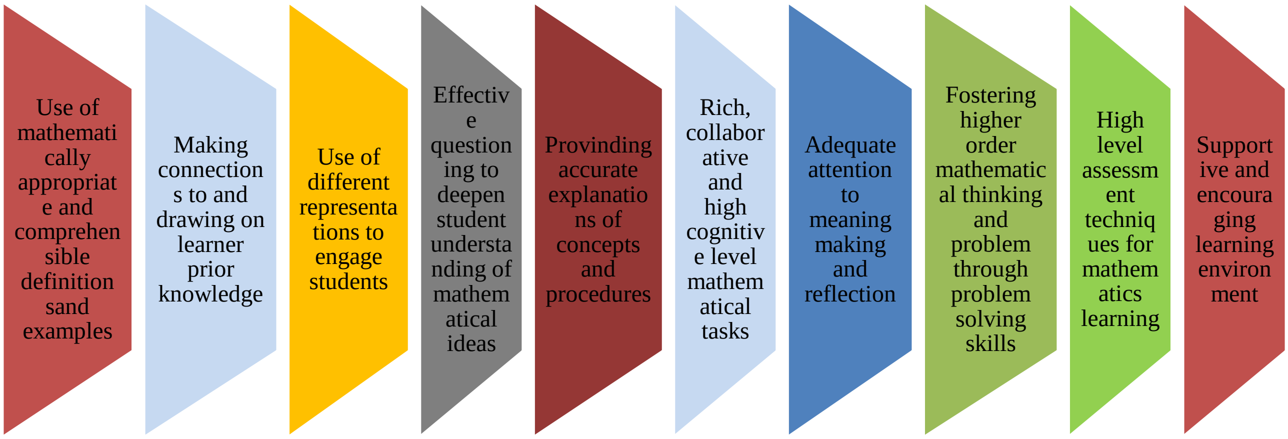


Figure 7.1: Ten principle framework of instructional strategies of mathematical teaching proficiency

This framework of ten principles is not intended to be exhaustive or prescriptive. Instead, it provides a structure for teachers to:

- ❖ plan and think about effective instructional activities and learning experiences in a way that enables the students to develop mathematical proficiency (MP);
- ❖ gain insights into the students' mathematical thinking (AR) as lessons progress, and
- ❖ influence students' interest, productive disposition, view of the subject's significance, self-perception and perseverance.

In the next sections I elaborate on each principle involved in the new framework.

Use of mathematically appropriate and comprehensible definitions and examples

Teachers' instructional practices suggest that they followed particular patterns in their introduction of new concepts and that there was some consistency in their instructional patterns. Here, teachers gave mathematically relevant and practical examples, comprehensible definitions and explanations of key mathematical concepts and terminologies, and then followed up with more practice. They used real life practical examples to exemplify the concepts and definitions to prove the conjectures and mathematical theorems or techniques. Appropriate mathematical definitions were central to CU, AR and proving practice.

Making connections to and drawing on learners' prior knowledge

All the teachers started instruction by building on what their students know mathematically and experientially, while creating and connecting students with appropriate stories, metaphors, analogies and other representations that both contextualised and established a rationale for effective learning. Clarke and Clarke (2004) recommend that effective teachers build connections from prior lessons and experiences and use that knowledge effectively to inform learning.

Use of different representations to engage students

In many mathematical tasks and lessons, effective teachers used interesting models and multiple representations to enhance students' conceptual understanding, procedural fluency and strategic competence, and develop both adaptive reasoning and productive disposition. Representations and other manipulatives were used to illustrate key mathematical ideas and principles and to engage students with underlying concepts and procedures. Sullivan (2011) describes mathematical tasks using representations as "one that is explicitly-focused on experiences that engage students in developing and consolidating their mathematical understanding" (p. 33).

Effective questioning to deepen students understanding of mathematical ideas

During my observation I saw that effective teaching was characterised by a flexible use of high order questioning and whole class discussion, coupled with a creative selection of mathematical tasks. Much of the teachers' discourse was permeated with *procedural questions* soliciting the procedures for solving a problem. Research that focuses on successful teaching reveals that effective teachers ask a lot of questions and involve students in whole class discussion to monitor students' understanding of the concepts and procedures being taught (Webb et al., 2008).

Secondly, teachers asked *conceptual questions* that required students to explain and justify their answers or methods for solving problems. Examples of these questions were "why" and "how" questions, i.e. "why are the two angles equal" or "how did you get that answer".

Providing accurate explanations of concepts and procedures

Another finding was that teachers gave accurate and comprehensible explanations of both concepts and procedures, which promoted students' conceptual understanding and procedural fluency. Explanations of concepts were generally concerned with the why of mathematical procedures or ideas and the interrelationships among mathematical concepts.

Teachers also provided comprehensible explanations of the procedures for solving mathematical tasks and problems under discussion. These explanations were concerned with how procedures should be used in order to encourage students to apply mathematical formulae, procedures and techniques accurately, flexibly, appropriately and correctly. In addition, teachers provided algorithms to help solve mathematical problems. According to Wilson et al. (2005), giving and receiving explanations may widen students' thinking skills, and verbalising can help students structure their thought.

Rich, collaborative and high cognitive level mathematical tasks

One of the most important findings in this study lies in the concept of a "well-selected mathematical task". The data showed the importance of developing students' proficiency and ability to think mathematically through demanding high level and conceptually rich focused tasks. The data from lesson videos revealed that participating teachers employed well selected tasks to enable students to practice and understand procedural skills and gain mathematical power, which Kilpatrick et al. (2001) refer to as MP. According to Kilpatrick et al. (2001), learning mathematics is not only about applications of rules and procedures, but also about acquiring the ability to reason mathematically through the completion of cognitively demanding tasks. I was interested in noting that the teachers engaged their students in tasks that required complex forms of thinking and reasoning to develop students' PF, SC as well as AR. Ball and Bass (2003) argue that knowing procedures and mathematical ideas as just routine or mere facts without mathematical reasoning is not enough. Tasks provided students with opportunities to make conjectures, to validate theorems or rules and to justify ideas and findings.

Another aspect of critical importance for effective teaching, from an enactivist viewpoint, is to enable some levels of mathematical thinking or diversity in an attempt to promote novel learning and students' responses (Davis and Sumara, 2006). This aspect was realised in this study through multiple mathematical tasks that were both conceptually rich and flexible. Through such rich mathematical tasks, teachers were able to involve their students in exploring underlying concepts and drawing on their prior experience.

Adequate attention to meaning making and reflection

Teachers involved their students in developing mathematical meanings without necessarily giving the students definitions of concepts and solving procedures. Teachers supported students in developing deeper understandings of mathematical ideas through discussion and interaction with their peers in small groups. Teachers further encouraged mathematical sense making through argumentation, justification and theory formulation (theorems) and informal generalisations. I noted that teachers spent sufficient time on concepts to encourage their students to critically reflect on and make sense of key ideas.

Teachers also provided students with real problem situations and high cognitive level tasks situated in real world contexts that included appropriate sense making, so that students could understand the fundamental mathematical concepts and ideas underlying procedures, problem solving and investigations. Instead of providing solution strategies, teachers encouraged multiple approaches and allowed time for reflection. Students' SC, which is the ability to formulate mathematical problems, represent them and solve them (Kilpatrick et al., 2001, p. 124), was developed through appropriate use and exploration of procedures with connections to mathematics tasks of higher order.

Fostering higher order thinking and problem solving skills

The analysis indicated that teachers involved their students in higher level tasks to enable them to think mathematically about given practical situations and concepts. Problem solving instruction as well as an investigative approach were significantly effective for promoting students' CU, SC and AR. Teachers carefully planned and selected instructional activities that required students to engage in mathematical thinking and problem solving. Students' PD, beliefs in diligence and one's own efficacy, were also promoted when students interacted with the teachers and their peers.

High level assessment techniques for mathematics learning

Another aspect of teachers' practice that appeared to enhance the effective teaching and learning of mathematics was the use of high level assessment techniques. This was both overt and hidden. In the interviews, teachers indicated that they introduced unique techniques of assessing students' higher order mathematical thinking skills and capabilities alongside the external examination demands. This means for every test and exercise given to students, teachers used higher level assessment tasks that conformed to the external examination question papers to assess students' MP, understanding and ideas. In the interviews, teachers also acknowledged that they used more open-ended and thoughtful questions such as "why", "explain or justify" throughout their mathematics lessons to facilitate learners' learning, growth and understanding.

Supportive and encouraging learning environment

It was worth noting that in many mathematics classrooms, effective teachers provided a learning environment that was more democratic, supportive, encouraging and partner-like than Kilpatrick et al.'s (2001) model. The data appeared to indicate that teachers were able to maintain a supportive learning environment within their mathematics classrooms.

7.3.4.1.2. Reflections on ten principles of instructional strategies

Although these processes or principles have been discussed in separate sections, they are interwoven and interrelated. For example, teachers used appropriate multiple representations in order to provide accurate explanations of concepts, and to encourage students to solve problems. Where problem solving was concerned, the aim was to help students towards a systematic approach for identifying and organising relevant information, and gradually develops more general methods of solution. Teachers also used representations flexibly and appropriately alongside effective questioning in developing clear and accurate mathematical arguments. The interwoven nature of these processes is the rationale behind the current model of instructional strategies of mathematical proficiency. This model is still evolving, but it constitutes an initial attempt to represent effective teaching approaches and findings from the data analysis of this study.

7.3.4.2. Reflections on enactivism as a theoretical vantage lens

Enactivism worked well as a theoretical lens in this research. The four pedagogical elements of enactivism enhanced the Kilpatrick et al.'s (2001) framework for analysing teaching proficiency. The data analysis succeeded in highlighting their personal experiences and characteristics as effective teachers in the mathematics classroom as a complex system. This approach may work with prepared learning activities that allow for the lessons to move in a different direction in response to students' interactions. Teachers manifested all the concepts of enactivism, in their teaching as a group and in the interviews individually.

Enactivism (Maturana and Varela, 1992; Reid, 1996) provided powerful insights into the teaching and learning of geometry in this study and offered an alternative lens for analysing and interpreting effective teaching practice, and teachers' understanding of their own effectiveness. In particular, autopoesis offered the opportunity to articulate effective teaching practice in relation to teachers' mathematical interactions with students and their understanding of their own effectiveness. In autopoetic terms, all five teachers focused on self-organising systems, wherein they saw mathematical content knowledge, strong personal knowledge or cognition and students' understanding as co-emerging in the teaching and learning process. It is important to note that, in terms of my analysis and the current discussion, the enactivist interpretive framework does not deny the insights of constructivism, although it does offer a challenge to, for instance, the narrowness of its scope.

7.4. CONCLUSION TO THE CHAPTER

This chapter reflected on the teachers' experiences of teaching mathematics effectively and presented the key findings of the thesis. It explained how I arrived at the empirical findings presented in Chapter Four through Chapter Six. The final chapter consolidates the findings of this analysis with specific reference to the general objective of this study and seven guiding interrelated research sub-questions, as originally outlined in Chapter One.

CHAPTER EIGHT

CONSOLIDATION AND CONCLUSION

> Although much is known about effective instruction, many questions merit close study if teachers and researchers are to develop the kind of knowledge base needed to improve instruction (Kilpatrick et al., 2001, p. 357). I thus conclude this thesis with some core issues crucial to building the knowledge base on teaching and learning for mathematical proficiency.

8.1. INTRODUCTION

This final chapter of my research considers the extent to which the research questions introduced in Chapter One have been answered. It consolidates the findings with reference to the original research questions and within the context of the theoretical and methodological frameworks elaborated in Chapters Two and Three. To achieve this, I will first reflect on the purpose and objective of the study. Secondly, I will briefly remind the reader about the theoretical and conceptual frameworks. Thirdly, I will provide a summary outline of the research process. Fourthly, I will present an overview of the most significant research findings related to each of the seven guiding research questions. Fifthly, limitations will be pointed out. I will then identify the potential value of the study before concluding with the recommendations for further research.

8.2. REVIEW OF THE OBJECTIVE OF THE STUDY

The core goal of this research was to investigate selected effective secondary school teachers' geometry teaching in Namibia in order to gain insight into the teaching characteristics of proficient teachers. The main question that guided this investigation was:

> *What are the teaching proficiency characteristics of selected effective mathematics teachers?*

In seeking an answer to this key question, the following interrelated sub-questions were investigated.

(1) How do these proficiency characteristics inform teachers' classroom practice?

(2) What characteristics of teaching proficiency are similar or different across the teaching practices of the selected teachers?

(3) What are teachers' personal experiences and characteristics enabling them to be effective?

(4) What contextual factors shape these effective practices?

(5) What mathematical proficiency and pedagogical content knowledge of geometry do mathematics teachers who are considered effective have, and need to demonstrate, in solving scenario based geometry tasks?
(6) What factors outside Kilpatrick et al.'s (2001) analytical framework characterise effective teaching practice?

8.3. REVIEW OF THE THEORETICAL AND CONCEPTUAL FRAMEWORKS

This study drew on the Kilpatrick et al.'s (2001) model of mathematical proficiency (MP) and Maturana and Varela's (1992) enactivist worldview to elaborate on the notion of teaching for MP and embodied cognition. A conceptual framework deriving from Kilpatrick et al.'s (2001) model of teaching for MP guided both the data collection and data analysis processes. Kilpatrick et al.'s (2001) model consists of five interwoven strands of teaching for MP, and these have provided a core conceptual frame and categories for analysis in this research (see Chapter Two, sections 2.6.2.1.2 to 2.6.2.1.6).

Enactivist theory draws on complexity science (Davis and Simmt, 2003, 2006) and therefore enables examination of the intricacy of teaching and learning. The enactivist perspective of complexity and embodied cognition allowed me to include individual, situational and cultural aspects in the study of effective teaching practices. I used key enactivist ideas such as "autopoesis", "embodiment", "structural determinism or coupling" and "co-emergence" to complement my analysis of effective teachers' practices in Namibia. The teachers concerned created effective learning communities that promoted mutually engaging mathematical interactions. They gave students opportunities to think and reason mathematically (justifying, explaining, abstracting, problem solving).

As discussed in Chapter Five, I studied the effective practices of selected teachers. Using Kilpatrick et al.'s (2001) model and the concepts of enactivism enabled me to respond to the main research question, viz. *What are the teaching proficiency characteristics of selected effective mathematics teacher*? This question was answered as follows. All five teachers made efforts to facilitate students' engagement with different mathematical tasks and learning activities through effective learning processes that align with Kilpatrick et al.'s (2001) model. In so doing, they enhanced students' understanding of underlying mathematical ideas and procedures. Through high cognitive level tasks, teachers involved their students in doing mathematics in more flexible and accurate ways. More importantly, the notion of mathematical knowing and doing represented in these tasks supported the four concepts of enactivism, though differently emphasised by different teachers. For example, an interesting characteristic of teacher practice was a tendency to promote effective learning by representing the complexities of the everyday experiences, thus allowing their students to exhibit their embodied mathematical understanding. What was striking was the manner in which the five teachers fundamentally transformed the mathematical tasks by engaging students in doing mathematics with "investigations" into real world situations. However, a notable difference among these teachers' classrooms was the extent to which students actively participated in the mathematical discourses and the discussion of concepts and problem solving.

8.4. A BRIEF OUTLINE OF THE RESEARCH PROCESS AND METHODOLOGY

This research adopted a qualitative case study design. It was oriented in an interpretive paradigm using descriptive narrative analysis. The study aimed ultimately to develop a detailed description of the characteristics of effective teachers who teach mathematics in Namibian secondary schools. A multiple case studies approach was adopted to capture the complexity of participants' lived experiences and their own perceptions of their success and effectiveness. Employing a purposive sampling technique, the study involved five teachers from five secondary schools in five Namibian educational regions who were chosen for their reputation of being **effective teachers** of mathematics by their peers.

The study was structured into five phases of data collection. I collected qualitative data chiefly through biographical questionnaires (Phase II), a geometry scenario-based questionnaire (Phase III), classroom lesson observations (Phase IV), and open-ended and semi-structured interviews in the form of post-lesson reflection (Phase V) and stimulated recall analysis sessions (Phase VI) with the participating teachers. The process of data analysis commenced during data collection, and ran through four interrelated stages. Data analysis was tied up with the idea of multiple interpretations. This means that the same events were interpreted many times in different ways by the researcher (Reid, 1996). A detailed chronology of how this analysis was conducted is given in Chapter Three, sections 3.4.8.1 to 3.4.8.4.

8.5. A SUMMARY AND DISCUSSION OF KEY FINDINGS OF THIS STUDY

The main findings of this study are presented in relation to the seven research questions. The following sections recap the key findings relevant to the teachers' personal experiences and characteristics, perceptions of their effectiveness and contextual factors enabling effective practices (Questions 2, 4 and 5); teachers' responses to geometry questionnaire tasks (Question 6); their teaching practices (Questions 1 and 3), and the case study teachers' reflection on their own teaching practice (Question 7), respectively. The first section synthesises the findings in terms of Kilpatrick et al.'s (2001) model, while the last two sections analyse the findings in terms of Kilpatrick et al.'s (2001) model and an enactivist framework.

(3) *What are teachers' personal experiences and characteristics enabling them to be effective* and (4) *what are the contextual factors shaping these effective practices?*

These questions sought insights into teachers' personal experiences and characteristics, and into contextual factors that shaped their practice. The questions relate to Phases II, V and VI of the data collection and were achieved through the initial biographical questionnaire and intensive interviews with teachers. Overwhelmingly, the initial analysis of results showed that participants expressed confidence in their mathematical productive disposition and self-efficacy, declaring a high mathematical content and strong personal knowledge. Additionally, participants identified certain key themes in and characteristics of effective teaching. I divided these themes into contextual (or external) and internal factors shaping their practices, and enabling effectiveness. External factors included the influence of effective school management and leadership and efficient organisation; influence of teacher collegiality, collaboration and teamwork, and the effectiveness of links with the broader community. Enabling internal factors included having the core knowledge required and employing a conceptual approach to teaching; strategic competence, adaptive reasoning and production disposition. The teachers shared favourite teaching strategies which they felt contributed to their own success and effectiveness. These included promoting a favourable classroom environment that enabled students to explore deeper levels of mathematical knowing, thinking and understanding. In addition, the findings revealed that the social and cultural contexts of the five schools make it easy for teachers to operate in these environments.

The results further suggest that the five teachers appreciated the importance of community of practice, not only to share their experiences, but also to tap into best practices from other schools and teachers. This openness suggests that they were aware of their own mathematical self-efficacy and mathematical proficiency (see section 4.3.1).

(5) *What mathematical proficiency and pedagogical content knowledge of geometry do mathematics teachers who are considered effective have, and need to demonstrate, in solving scenario-based geometry tasks*?

This research question relates to Phase III, and the answers to it are reported in Chapter Four, where the results of the teachers' responses to the geometry scenario-based questionnaire were discussed (see section 4.2.2). From the perspective of mathematical understanding, teachers' responses showed evidence of the first four strands of MP. Despite what happened during lesson observations, teachers showed awareness of how they might promote CU, PF, SC and AR.

What are the teaching proficiency characteristics of selected effective mathematics teachers and *(2) what are the teaching proficiency characteristics that are similar and different across the teaching practices of the selected teachers*?

The analysis of Phase IV data highlighted that Namibian effective teachers' proficiency characteristics were shaped by a sound mathematical understanding of concepts and procedural skills. The practice of the five participants championed MP and confirmed that the various strands of MP are intertwined and interrelated (Kilpatrick et al., 2001). Teachers demonstrated efficient and appropriate use of procedures with examples for the students to grasp algorithms and techniques. This was good for developing students' CU. The teachers who addressed SC and AR flexibly moved students from concrete to more abstract representation in teaching geometry. They made effective connections between symbolic representation and concept definition, using symbols and models to explain both concepts and procedures. This research revealed that the selected teachers' teaching practices have the following features:

- the teachers addressed all five strands of MP but, generally speaking conceptual understanding (CU), procedural fluency (PF) and productive disposition (PD) were more regularly addressed and interwoven in classroom practice across all lessons than the other two strands (SC and AR). Though the prevalence of each strand was shown and discussed separately, this does not mean that each strand was developed independently from other strands;

- procedural teaching dominated teachers' practice and was addressed in almost every lesson. That is, the teachers addressed students' knowledge of procedures, when and how to use them suitably, and algorithmic skills to perform them flexibly, accurately and efficiently;
- the teachers addressed students' ability to comprehend mathematical concepts, ideas, operations and relations, and stressed knowing the importance of certain mathematical ideas and the contexts in which they are used;
- the teachers addressed students' ability to explain and justify mathematical ideas and how they obtained solutions to specific mathematical problems. Students were expected to explain, justify and think logically about problems and reflect upon the relationships among concepts, procedures and situations;
- the nature of the tasks students were given played an important role in supporting and facilitating the development of their proficiency in mathematics. Through conceptually rich focused tasks, the teachers gave their students opportunities for, and encouragement and assistance with mathematical thinking, reasoning and sense making in the mathematics classroom;
- the teachers' use of representations and models, a specific feature of conceptual teaching, to formulate and represent mathematical procedures and solve problems, helped students visualise mathematical concepts, situations and problems in different and more creative ways, and
- the teachers used their pedagogical content knowledge (PCK) to open up possibilities and learning experiences to enable students to view mathematics as accessible, and to build students' skills in abstraction and conjecturing.

From an enactivist perspective the participants found the following instructional practices effective when interacting with students in their mathematics classrooms:

- autopoesis, embodiment, co-emergence and structural determinism/coupling were part of teachers' embodied cognition and conceptual knowledge, and shaped effective teaching and learning as teachers and students worked together on tasks or problems;
- mathematical discussions and conversations between teachers and students working on mathematical tasks, helped develop students' conceptual understanding, mathematical thinking, reasoning and problem solving skills. That is, the teachers engaged students in mathematical argumentation, justification and reasoning to empower them to be proficient doers and users of mathematics;

- ❖ defining, explaining and developing concepts, representing procedures with models, using effective questions and problem solving (sufficient drill and practice to consolidate mathematical procedures), were all key elements of effective teachers' mathematical instructional practices;
- ❖ the teachers engaged students in conceptually rich focused mathematical activities, hands-on experiences and higher cognitive level tasks, that served to encourage them to regard mathematics as useful and practicable, and to see themselves as capable learners and doers of mathematics. Through these kinds of tasks, teachers developed students' understanding of key concepts. They also developed students' procedural fluency when they were solving tasks. In turn procedural fluency played an important role in developing students' strategic competence as they were choosing among effective procedures to solve challenging mathematical problems;
- ❖ the teachers drew students into a community of practice that promoted mutually engaging mathematical interactions and opportunities to mathematize. For example, judging, arguing about and justifying what constitutes an effective solution (Yackel, Cobb and Wood, 1991) are critical if students are to develop their intellectual autonomy, and thus their mathematical proficiency and power. It was this community of practice that promoted the development of students' productive mathematical disposition, and
- ❖ scaffolding through dynamic ways represented both teachers and students as active co-participants in the teaching and learning process.

(6) *What factors outside of Kilpatrick et al.'s (2001) analytical framework characterise effective teaching practice*? *In other words, what factors represent and affect effective teaching practice, and how might these factors be interpreted within the enactivist framework used in this study*? This research question related to Phases V and VI of the data collection, and asked: What other conclusions can be drawn with regard to the teachers' understanding of MP? The results indicated that the teachers concerned were instrumental in establishing effective classroom instructional practices or norms. Thus students depended largely on their teachers' effectiveness. The results showed that these teachers used bodily actions and well-designed embodied experiences to help students understand important mathematical concepts. They also aligned their real world experiences with abstract representations and conceptual models, helping students to become proficient in the world of mathematics.

8.6. LIMITATIONS OF THE STUDY

The limitations of this study pertain to the data collection and the applicability of its results. First, the instruments used to analyse teachers' perceptions of their effectiveness and current classroom instructional practices were broad, but not necessarily comprehensive. Secondly, the time taken to collect the data was over eight months. Kilpatrick et al. (2001) indicate that when one is exploring teacher classroom practice to deconstruct teaching proficiency, adequate time is required, especially if the researcher intends to share effective practice with other schools. Terwel as cited in Tshabalala (2012) reiterates that it took the Netherlands and United Kingdom 20 years to observe an improvement in mathematics achievement. Hence investigating the teachers' proficiency and effectiveness should have taken at least two years of interaction with the participants. Thirdly, it is inappropriate to generalise the main findings of this research to the broader context or the population of Namibia secondary school mathematics teachers as a whole. The situations reported in the participant regions may be different from those in other regions that I did not have access to, especially if such differences are due to the commitment, hard work and efforts of teachers in selecting worthwhile mathematical tasks and examples to influence how students come to view, develop, use and make sense of mathematics.

8.7. THE SIGNIFICANCE AND POTENTIAL VALUE OF THE STUDY

This thesis represents a complex journey towards an understanding of teachers' geometry teaching for mathematical proficiency in Namibian high schools. The study offers insights on how effective Namibian mathematics teachers teach for MP and discusses the characteristics of their teaching proficiency. It also highlights issues and tensions encountered in effective teaching. The study has certainly expanded my own understanding of effective mathematics teaching and the ways teachers' classroom practices develop MP. I believe that it contributes to a wider and more sophisticated understanding of effective mathematics teaching and its development, raising many questions and issues. It is hoped that the study will serve also to inform and advise teachers, teacher educators, curriculum designers and researchers. Should these people be encouraged by the work done in this study to reflect on and question their own perspectives and practices, I shall be well satisfied and cherish that the study has made a contribution to more than just my own personal learning.

As this study was fundamentally rooted in the notion of 'teaching for mathematical proficiency', Kilpatrick et al.'s (2001) framework proved very useful for my analysis. Yet the framework had its limitations and I made use of the principles of enactivism to extend my analyses. I found that the frameworks complemented each other well and indeed the contribution of these theories and their integration into my research design is a contribution made by the study. Specifically, Kilpatrick et al.'s (2001) model enabled me to understand the teaching of effective Namibian teachers in relation to their instructional practices, while enactivism allowed me to better understand effective teachers' mathematical embodiment and actions as they taught for MP.

8.8. IMPLICATIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

Before moving into the recommendations, I first consider several implications of the data analyses that contribute to a "common sense view" of mathematics and classroom practices that facilitate development of students' MP. While contemporary research in mathematics education (Clarke and Clarke, 2004; Hill et al., 2008; Stephens, 2009) is driven by Kilpatrick et al.'s (2001) model and the enactivist framework (Maturana and Varela, 1992), the common sense view appears to position MP as useful rather than acknowledging the complex yet essential pedagogic processes involved in mathematical actions.

Kilpatrick at el. (2001, p. 380) urge that as MP involves interconnecting strands, so too does teaching for MP. They also maintain that mathematical reasoning is the evidence of MP. My research suggests that it is appropriate for mathematics teachers to aim to develop students' MP according to the five interconnecting strands. In this way, teachers should develop the mathematical (geometrical) reasoning, thinking skills and strategic competence of all students. However, this is not a minor undertaking for many teachers. Obstacles include uncertainty about mathematical reasoning and about how to teach this skill, assessment activities which do not recognise students' oral powers or reasoning, practices that de-emphasise the development of deductive and inductive mathematical reasoning and communication skills for all students, a lack of appreciation of geometry as a vehicle for developing MP for higher achievers, and a lack of exemplars of effective teaching practice.

8.8.1. Implications of teaching for mathematical proficiency

The findings of this research have significant implications for effective teaching practices, development of MP and student learning. Anthony and Walshaw (2009) argue that “mathematics is the most international of all curriculum subject and mathematical understanding influences decision making in all areas of life, i.e. private, social and civil. Mathematics education is a key to increasing the post-school and citizenship opportunities of young people, BUT today, as in the past, many students struggle with mathematics and become disaffected as they continually encounter obstacles to engagement” (p. 6). It is imperative, therefore, that we understand what **effective mathematics teaching** is and what effective Namibian mathematics teachers do to break negative perceptions. I suggest any education system that does not place a focus on effective mathematics teaching, is in a way “refusing to acknowledge the teachers’ need to learn and grow, both in their subject knowledge and professionalism” (Miranda, 2009, p. 32). I also emphasise that an educational system in which teachers do not share effective practices and experiences is problematic. Teachers need to learn from each other and share experiences of effective teaching and learning. For example, focus group discussions of lesson videos and examining their own practice conducted by effective, skilful and knowledgeable teachers would provide teachers with helpful learning opportunities in this area.

Secondly, if an education system does not acknowledge teachers’ instructional practices; there may be conflict between what is intended in the curriculum and what actually happens in the classroom. My study characterised effective teaching practices and allowed selected effective mathematics teachers to share experiences and ideas on how to deal with concepts in geometry. The findings outlined in this thesis are of course not the only indicators of best or effective teaching practices. Any practice must be understood as nested within a larger network that includes the teacher, the students, the content, the context and the wider education system. Teachers will find that some practices are more applicable to their own needs and local circumstances than others. This thesis therefore begins a conversation which needs to continue among Namibian mathematics teachers and teacher educators.

There is little research or discourse in Namibia on the issues of teaching and learning certain mathematical concepts, especially on teachers developing innovative ways of working with students. Studies such as this therefore allow Namibia to enter and contribute to the broader discourse on teaching for mathematical proficiency. Drawing on a wide range of research and the empirical observation, this research describes effective pedagogical approaches that engage students and lead to their development into proficient students of mathematics. The aim of the study is to deepen the understanding of what the five selected effective mathematics teachers in Namibia do in the teaching of geometry and how they optimize opportunities for mathematics students to develop mathematical proficiency.

The following recommendations may help teachers understand this overall vision of teaching for mathematical proficiency, and for improving instructional practices in their own contexts. The first section contains recommendations for ways in which secondary school teachers could empower students to be proficient in mathematics. The second focuses on how the teaching and learning contexts may be reconstructed through new practices to bring about new kinds of agency for teachers, and therefore also for policy designers and teacher educators. Lastly, some recommendations for future research are made.

8.8.2. Recommendations for teachers

- **First**, teachers need to structure classroom activities so that all five strands are emphasised and synchronised. Kilpatrick et al. (2001) argue that MP cannot be achieved by focusing on one or two of the strands, but that development across all five strands is necessary because the strands interact and reinforce each other. It follows logically that learners who have the opportunity to develop all the strands of MP are more likely to become truly competent at each (p. 144). My research demonstrated strongly the strength and usefulness of Kilpatrick et al.'s (2001) position in this regard.
- **Secondly**, teachers need opportunities to analyse a variety of lessons in relation to the five strands of mathematical proficiency in order to understand key features of effective teaching practices, particularly teacher questioning and sense making focused on conceptual understanding, strategic competence and adaptive reasoning.
- **Thirdly**, workshops and other teacher development activities need to reflect the elements of effective teaching practices, with clear, explicit learning goals, a supportive but challenging learning environment and the means to ensure that teachers are developing understanding. The study findings and classroom observation suggest that teachers need expertise in helping students develop an understanding of the content, including knowing how students typically think about particular mathematical concepts, how to determine what a particular student or group of students is thinking about those concepts, and how the available instructional materials can be used to help students deepen their conceptual understanding.

8.8.3. Recommendations for teacher educators and policy designers

- Policy makers and administrators need to work with teachers in establishing a joint coherent understanding regarding effective practice. Providing opportunities and incentives for teachers to deepen their understanding of the mathematical concepts they should teach and how these should be taught will have a positive influence on instruction. Only by assessing the most important knowledge and skills can we expect to realise the vision of mathematical proficiency as well as the goal of high quality instruction.

8.8.4. Recommendations for future research

First, this research was complex and there were perhaps a number of issues that I did not examine thoroughly as a result of space and time constraints. Kilpatrick et al.'s (2001) Strategic competence (SC) and Adaptive reasoning (AR) strands proved difficult to apply, and there is a need for rich illustrations of classroom dialogues that indicate how geometric ideas emerge in the teachers' and students' thinking, and what problems and questions help to elicit them. More research needs to be done here than I have conducted, for there is ample scope for Kilpatrick's ideas to be extended and adapted.

Secondly, I researched Kilpatrick et al.'s (2001) conceptualisation of effectiveness as it is, but it is worth remembering that these definitions were made for schools in the USA and Britain. Although my research in Namibian schools recognised many of the characteristics of effectiveness which appear in the literature (Clarke and Clarke, 2004; Kilpatrick et al., 2001), I think there are other characteristics that Kilpatrick et al. (2001) and other researchers have not addressed. These are contextual in nature; that is, they relate to the Namibian context and the particular interpretation of the context within specific schools. My research suggests that certain extensions to these theories of effectiveness or competence are required in the Namibian context. I thus argue that effective Namibian teachers are committed and hardworking, provide extra support for their learners, set high challenging academic standards and have varied and highly productive dispositions. For example, Demis said "*hard work*, *commitment* and *being personally effective*". This was reiterated by Jisa and Emmis, Ndara and Sann. They also mentioned *collegiality, team work* and *collaboration* and *sharing effective practices with other teachers within and outside their schools*. These are strong indicators of effectiveness, referencing interactions which do not necessary match the Kilpatrick et al.'s (2001) framework. Of course, different contexts would also present different cultural and teaching enablements and constraints, and would thus provide possibilities for examining how different kinds of interaction, teaching proficiency strands and challenges influence the development of mathematical proficiency in students.

8.9. A FINAL WORD OF PERSONAL REFLECTION ON THE RESEARCH PROCESS

In conclusion, I would like to comment on my research journey. The research journey was a valuable one. **First**, in terms of my personal development, I will cherish this project on *effective teaching practice of mathematics* as one of the most fruitful and amazing of all learning endeavours that I have undertaken in my academic career. Completing this PhD thesis provided me with the opportunity to know who I am as a researcher and what type of researcher I would like to become in the future. Mathematically, I confirmed my interest in geometry in particular. I gained fresh insights into the practices of my participants, which helped me to learn a lot about their proficiency in teaching mathematics. Their instructional decisions, teaching proficiency and the ways they enacted the curriculum critically shaped my understanding of effective teaching, and I am looking forward to implementing these effective practices in my own practice as a mathematics subject advisor in Namibia. Their practice has profoundly shaped my thinking of how to improve mathematics teaching in Namibia, including an understanding of their contexts in which effective pedagogy is situated. I was inspired by these beacons of excellence which is particularly important for nurturing learners' talent and development of mathematical proficiency in Namibian schools.

Secondly, this doctoral thesis provided me with the knowledge and skills to reconstruct and develop mathematical proficiency, explore the different teaching models that emerged from the study of the research literatures in mathematics education landscape and related disciplines, and write original research that represents my own contribution to the knowledge. In terms of the mathematics content, I was enriched by engaging in the teaching practice of the participants. The research methodologies I employed provided me with a valuable reflective platform to improve my own understanding of qualitative research design and planning. Further, I have learned to be better organised in terms of planning and time management.

Thirdly, it was also important that the teachers too learned from this research process. Their participation in this study and collaborative reflections on their own practice were invaluable. In particular, teachers expressed their own positive reactions to being part of the research. Demis indicated that "*for me even if I am doing a class visit in my school, I really always focus on learning something new*". Jisa said that "*I learned many competencies needed for teaching mathematics effectively*". Ndara asserted that "*the research project was a wonderful experience as I learned some new technologies, for example, using square grid papers (graph worksheet) and an overhead projector*". Emmis strongly appreciated being a participant in the process and said "*I saw different teaching styles that are really worth it. And so I really learned a lot and will be able to express all what others have said [in my teaching]*". Similarly, Sann described her experience as follows. "*I also followed a lot and want to try every teaching method or concept more with my learners. And it is also nice to see that there are teachers in the country that are excited about what they are doing and how they view different ways of teaching, and how their personalities come out in what they are doing. And the learners alike with respect to teaching, they get the concepts in the first place*".

Fourthly and lastly, if I were to do this research again, I would perhaps have a smaller sample and analyse their teaching practices in greater detail. I would perhaps choose to focus on one or two strands of Kilpatrick et al.'s (2001) model as I spent a lot of time on analysing a huge amount of collected data. My advice to other prospective researchers would be that they should be well organised and use time strategically in making sense of the huge sets of collected data.

Thank you again to my five participants and all those who have supported my work!

REFERENCES

Adler, J. (2005). Mathematics for teaching: What is it and why is it important that we talk about it? *Pythagoras*, *62*, 2-11.

Adler, J. (2009). A methodology for studying mathematics for teaching. *Researches en Didactique des Mathematiques*, *29*(1), 33-58.

Ally, N. (2011). *The promotion of mathematical proficiency in grade 6 mathematics classes from the Umgungundlovu district in Kwazulu-Natal*. Unpublished master's thesis, University of Kwazulu-Natal, Pietermaritzburg.

Andrade, A. D. (2009). Interpretive research aiming at theory building: Adopting and adapting the case study design. *The Qualitative Report*, *14*(1), 42-60.

Andrews, P. (2011). The teaching of linear equations: Comparing effective teachers from three high achieving European countries. In M. Pytlak, T. Rowland and E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education (CERME 7): Working Group 11: Comparative studies in mathematics education* (pp. 1555-1564). Rzeszów, Poland: University of Rzeszów.

Ang, W. H. (2008). Singapore's textbook experience 1965-97. Meeting the ends of curriculum change. In S. Lee, C. Goh, B. Fredricksen and J. Tan (Eds.), *Toward a better future: Education and training for economic developing in Singapore since 1965* (pp. 69-95). Washington, DC: World Bank.

Anthony, G. and **Walshaw, M**. (2009). *Effective pedagogy in mathematics. Educational practices series 19*. Brussels: International Academy of Education.

Askew, M., Rhodes, V., Brown, M., William, D., and **Johnson, D**. (1997). *Effective teachers of numeracy: Report of a study carried out for the teacher training agency*. London: King's College, University of London.

Atebe, H. U. (2008). *Students' van Hiele levels of geometry thought and conception in plane geometry: A collective case study of Nigeria and South Africa.* Unpublished doctoral dissertation Volume 1, Rhodes University, Grahamstown.

Ball, D., and **Bass, H.** (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin and D. Schifter (Eds.), *A research companion to principles and standards for school Mathematics*. Reston, V.A: NCTM.

Ball, D., and **Bass, H**. (2009). With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures. Paper prepared based on keynote address at the 43rd Jahrestagung fur Didaktik der mathematik in Oldenburg, March 1- 4. Germany.

Ball, D. and **Hill, H**. (2008). Measuring teacher quality in practice. In D. Gitomer (Ed.). Measuring issues and assessment for teaching quality (pp. 80-98). Thousand Oaks, California: Sage.

Ball, D. and **Hill, H.** (2009). The curious-and crucial case of mathematical knowledge for teaching. *Phi Delta Kappan, 91*(2), 68-71.

Ball, D., **Hill, H**., and **Bass, H**. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade and how do we decide? *Journal for Research in Mathematics Education, 39*(4), 372- 400.

Ball, D., **Bass, H**. and **Hill, H.** (2004). Knowing and using mathematical knowledge in teaching: Learning what matters. Papers presented at the 12th Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE), January 18-21, University of Kwa-Zulu Natal, Durban.

Ball, D. L., **Thames, M.H**., and **Phelps, G.** (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. Lester. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843-908). Reston, VA: NCTM.

Begg, A. (2000). *Enactivism: A personal interpretation*. Retrieved April 11, 2010 from: http://www.emr.vic.edu.au/Downloads/enactivism_andybegg.doc

Begg, A. (2002). *Challenging curriculum: Process and product. Conference of mathematics into the 21st century*. Sicily. Retrieved 11, 2013, from htt://math.unipa.it/~grim/siBegg.PDF

Bleeker, M., and **Goosen, L.** (2009). Understanding the barriers to learning senior phase geometry. In M. Schäfer and C. McNamara (Eds.): *Reclaiming knowledge in mathematics, science and technology education in South Africa. Proceedings of the seventeen annual meeting of the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE)*, January 19–22, Rhodes University, Grahamstown.

Brodie, K. (2009). *Contexts, resources and reform. Teaching mathematics reasoning in secondary school classroom.* New York, NY: Springer.

Brodie, K. (2010). *Teaching mathematical reasoning in secondary school classrooms*. New York, NY: Springer.

Butterfield, B. and **Chinnappan, M.** (2010). Walking the talk: Translation of mathematical content knowledge to practice. In L. Sparrow, B. Kissane and C. Hurst (Eds.): *Shaping the future of mathematics education* (Vol. 2). *Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 109-116). Freemantle, WA: MERGA.

Capra, F. (1996). *The web of life: A new scientific understanding of living systems*. New York: Doubleday.

Capra, F. (2002). *The hidden connections: Integrating the biological, cognitive and social dimensions of life into a science of sustainability*. New York: Doubleday.

Carnoy, M., **Chisholm, L., Addy, N., Arends, F., Baloyi, H., Irving, M. [et al.]**. (2011). *The process of learning in South Africa: The quality of mathematics teaching in North West Province*. Stanford University: Human Science Research Council.

Carpenter, P., **Franke, L**. and **Levi, L**. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Chen, L.L., **Benton, B**., **Cicatelli, E**. and **Yee, L**. (2004). Designing and implementing technology collaboration projects: Lessons learned. *TechTrends*, *48*(3), 46-51.

Chick, H., **Pham, T**. and **Baker, M**. (2006). Probing teachers' pedagogical content knowledge: Lessons from the case of the subtraction algorithm. In P. Grootenboer, R. Zevenbergen and M. Chinnappan (Eds.). *Identities, cultures and learning spaces: Proceedings of the 29th Annual Conference of Mathematics Education Research Group of Australasia* (pp. 139-146). Sydney: MERGA.

Choppin, J. M. (2007). Teacher-orchestrated classroom arguments. *Mathematics Teacher*, *101*(4), 306-310.

Clandinin, D. J. (2007). *Handbook of narrative inquiry: Mapping a methodology*. Thousand Oaks: Sage Publications.

Clarke, D. M. and **Clarke, B. A.** (2004). Mathematics teaching in grade K-2: Painting a picture of challenging, supportive and effective classrooms. In R.N. Rubenstein and G.W. Bright (Eds.). *Perspectives on the teaching of mathematics* (pp. 67-81): Reston, VA: NCTM.

Clements, D. (1999). 'Concrete' manipulative, concrete ideas. *Contemporary Issue in Early Childhood*, *1*(1), 45-60.

Clements, D. (2001). Teaching and learning geometry. In J. Kilpatrick (Ed), *Research companion to the NCTM standards for mathematics*. Reston, VA: NCTM.

Cobb, P., Wood, T., And **Yackel, E.** (1991). A constructivist approach to second grade mathematics. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 157-176). Dordrecht, The Netherlands: Kluwer.

Cohen, D. K., and **Ball, D. L**. (2001). Making change: Instruction and its improvement. *Kappan*.

Cohen, L., Manion, L., and **Morrison, K**. (2007). *Research methods in education (6th ed.).* London: Routledge.

Creswell, J. W. (2002). *Educational research: Planning, conducting and evaluating quantitative and qualitative research*. New Jersey: Merill.

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches.* Thousand Oaks Calif.: Sage Publications.

Davis, B. (1996). *Teaching mathematics: Towards a sound alternative*. New York: Garland Publishing.

Davis, B. (2004). *Inventions of teaching: A genealogy*. London: Lawrence Erlbaum.

Davis, B. (2006). Mathematics for teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics*, *61*(3), 293-319.

Davis, B. and **Simmt, E**. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education*, *34*(2), 137-167.

Davis, B. and **Simmt, E**. (2006). Mathematics for teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics*, *61*, 293-319.

Davis, B. and **Sumara, D**. (1997). Cognition, complexity and teacher education. *Harvard Education Review*, *67*(1), 105-125.

Davis, B. and **Sumara, D**. (2006). *Complexity and education: Inquiries into learning, teaching and research*. New York: Routledge.

Davis, B. and **Sumara, D**. (2007). Complexity science, ecology and enactivism. In J.L. Kincheloe and R.A. Horn Jr. (Eds.), *The Praeger handbook of education and psychology*, Volume 3 (pp. 463-473). London: Praeger.

Davis, B., **Sumara, D**. and **Kieren, T.** (1996). Cognition, co-emergence, curriculum. *Journal of Curriculum Studies*, *28*(2), 151-169.

Davis, B., **Sumara, D**. and **Luke-Kapler, R**. (2008*). Engaging minds: Changing teaching in complex times*. New York: Routledge.

Dimarco, S. (2010). *Mathematics in the middle: Shaping the proficiency footprint*. Unpublished doctoral dissertation. James Cook University. Retrieved February 10, 2013 from http://eprints.jcu.edu.ac/17217

Dittmar, F., Mendelsohn, J. and **Ward, V**. (2002). *The school cluster system in Namibia: Framework for quality education*. Windhoek: HANSA.

Douglas, K. (2009). *Sharpening our focus in measuring classroom instructions*. *Educational Researcher*, *38*(7), 518-521.

Ernest, P. (2002). Empowerment in mathematics education. *Philosophy of Mathematics Education Journal*, *15*, 3-13.

Fenwick, T. (2004). The practice-based learning of educators: A co-emergent perspective. *A Journal for the Scholar-Practitioner Leader*, *2* (4), 43-59.

Fenwick, T. (2009). Responsibility, complexity science and education: Dilemmas and uncertain responses. *Studies in Philosophy and Education*, *28*(2), 101-118.

Franke, M. **L**., **Kazemi, E**., and **Battey, D. S**. (2007). Mathematics teaching and classroom practices. In F.K. Lester (Ed.). *The second handbook of research on mathematics teaching and learning* (pp. 225 - 256). Charlotte, NC: Information Age.

French, D. (2004)**.** *Teaching and learning geometry*. London: Continuum International Publishing Group.

Gaoseb, N. (2009). Secondary school mathematics teachers' conception of teaching geometry. In M. Schäfer and C. McNamara (Eds.): Reclaiming knowledge in mathematics, science and technology education in South Africa. *Proceedings of the seventeen Annual Conference of the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE)*, January 19-22, Rhodes University, Grahamstown.

Gass, S. and **Mackey, A**. (2000). *Stimulated recall methodology in second language research*. Thousand Oaks: SAGE Publications.

Givvin, K., **Jacobs, J., Hollingsworth, H.** and **Hiebert, J.** (2009). What is effective mathematics teaching? International educators' judgements of mathematics lessons from TIMSS 1999 video study. In J. Cai, G. Kaiser, B. Perry and N.Y. Wong (Eds.), Effective mathematics teaching from teachers' perspectives. *National and Cross-National Studies*, 37-69.

Graven, M. H. (2002). *Mathematics teacher learning, communities of practice and the centrality of confidence*. Unpublished doctoral dissertation, University of the Witwatersrand, Johannesburg.

Groth, R. E. and **Bergner**, **J. A**. (2006). Teachers' perspectives on mathematics education research reports. *Teaching and Teacher Education, 23*(2007), 809-825.

Hamilton, M. A. (2005). *We grow in the shade of each other: A study of connectedness, empowerment and learning in the middle years of schooling*. Unpublished doctoral dissertation. Australia Catholic University, Victoria.

Hart, G. and **Picciotto, H**. (2001*). Zome geometry: Hands-on learning with zome*^{TM} *models*. Emeryville, CA: Key Curriculum Press.

Hiebert, J. and **Carpenter, P**. (1992). Learning and teaching with understanding. In D.W. Grouws (Ed.), *Handbook of research in teaching and learning of mathematics* (pp.65-97). New York: Macmillan.

Hiebert, J. and **Lefevre, P**. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Hill, H. and **Ball, D**. (2009). The curious - and crucial - case of mathematical knowledge for teaching. *Phi Delta Kappan, 91*(2), 68-71.

Hill, H., **Ball, D**. and **Schilling, S**. (2008). Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4), 372-400.

Hill, H., **Blunk, M., Chalambos, C., Lewis, J., Phelps, G., Sleep, L.** and **Ball, D**. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.

Holton, D. **L**. (2010). Constructivism + embodied cognition = enactivism: Theoretical and practical implications for conceptual change. Retrieved February 10, 2011 from: http://www.academia.edu/.../Constructivism_Embodied.Cognition_Enactivism

Howie, T. (2004). *Interpreting teachers' mathematical knowledge in the context of learner-centred practice in a selection of grade nine mathematics classrooms in South Africa*. Unpublished master's thesis, University of Witwatersrand, Johannesburg.

Hudgens-Haney, M. (2010). Review of on blindness: Letters between Bryan Magnee and Martin Milliqan, by Bryan Magee and Martin Milliqan. *Metapsychology Online Reviews*, *14*(9), 6-13.

Jaworski, B. (1994). *Investigating mathematics teaching: A constructive enquiry*. Washington, DC. The Falmer Press.

Jenkins, M. W. (2007). *Curriculum recontextualising using gardens for the health promotion in the life orientation learning areas of the senior phase*. Unpublished master's thesis, Rhodes University, Grahamstown.

Johnson, S. (2001). *Emergence: The connected lives of ants, brains, cities and software*. New York: Scribner.

Jones, K. (2002). Issues in the teaching and learning of geometry. In L. Haggarty (Ed), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp 121-139). London, Routledge Falmer.

Kamwi, K. K. (2001). *Teacher responses to the Namibian education reform: A case study of two Caprivi schools*. Unpublished master's thesis, Rhodes University, Grahamstown.

Kapenda, H. (2007). Learner-Centred approaches in math classes in Khomas region: Namibia cases. Paper presented at the 15th Annual SAARMSTE conference, Eduardo Mondlane University, Maputo.

Kazemi, E. and **Franke, M**. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7,* 203-235.

Kazemi, **E**. and **Hubbard, A**. (2008). New directions for the design and study of professional development: Attending to the co-evolution of teachers' participating across contexts. *Journal of Teacher Education*, *59*(5), 428-441.

Kennedy, M. (1997). Education reform and subject matter knowledge. *Journal of Research in Science Teaching*, *35*(3), 249-263.

Khan, T. and **Gray, R**. (2012). *Revisiting autopoesis: A research note and another narrative on accounting and sustainability*. Melbourne, RMIT University.

Kho, T. H. (2007). The model-drawing method with algebra. In P. Y. Lee (Ed.). *Teaching secondary school mathematics: A resource book* (pp. 393-412). Singapore: McGraw-Hill.

Kieren, T. E. (1996). Teaching mathematics (in-the-middle): Enactivist views on learning and teaching mathematics. A paper presented at the Queens/Gage Canadian National Mathematics Leadership Conference, Queens University, Kingston, Canada.

Kilpatrick, A., Swafford, J. and Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Koontz, T. Y. (2005). Instructional strategies for improving student achievement. In S. Wagner (Ed.), *PRIME: Prompt Intervention in Mathematics Education* (pp. 99-125). Ohio Resource Centre for Mathematics, Science and Reading, Ohio Department of Education.

Lakoff, G. and **Johnson, M**. (1999). *The philosophy in the flesh: The embodied mind and its challenge to western thought*. New York: University of Chicago Press.

Lankshear, C. and **Knobel, M**. (2004). *A handbook for teacher research: From design to implementation*. London: Open University.

Larson, M. (2002). *Essential characteristics of effective instructions*. Lincoln: Houghton Mifflin.

Lave, J. and **Wenger, E**. (1991). *Situated learning: Learning peripheral participation*. New York: Cambridge University Press.

LeCompte, M., **Millroy, W**. and **Preissle, J.** (1992). *The handbook of qualitative research in education*. San Diego, CA: Academic Press.

Leikin, R. and Rota, S. (2006). Learning through teaching: A case study on the development of mathematics teachers' proficiency in managing an inquiry-based classroom. *Mathematics Education Research Journal*, *18*(3), 44-68.

Lerman Stephen. July 2012. Personal communication.

Lewis, J. **M**. (2009). *Teaching as invisible work*. Unpublished doctoral dissertation, University of Michigan, California.

Li, Q., Clark, B. and **Winchester, I.** (2010). Instructional design and technology grounded in enactivism: A paradigm shift? *British Journal of Educational Technology*, *41*(3), 403-419.

Lincoln, Y. S. and **Guba, E.G.** (2003). Paradigmatic controversies, contradictions and emerging confluences. In N.K. Denzin and Y.S. Lincoln (Eds.), *The landscape of qualitative research: Theories and issues* (pp. 253-291). Thousand Oaks, CA: Sage.

Lozano, M. D. (2005). Mathematics learning: Ideas from neuroscience and the enactivist approach to cognition. *For the Learning of Mathematics*, 25(3), 24-27.

Lucas, C. (2005). *Autopoiesis and co-evolution*. Retrieved November 26, 2010 from: http://wwwq=autopoesis+and+coevolution+(lucas,+2005)

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Matoti, S. (2011). Assessing the teaching efficacy beliefs of teacher trainees. In L. Paditz and A. Rogerson (Eds.), *Turning dreams into reality: Transformations and paradigm shifts in mathematics education* (pp. 223-228). Rhodes University, Grahamstown.

Maturana, H. (1987). Everything is said by an observer. In W. I. Thompson (Ed.). *Gaia: A way of knowing* (pp. 65-82). Hudson, NY: Lindisfarne Press.

Maturana, H. and **Varela, F.** (1987). *The cognitive theories of learning*. Boston: Shambhala Press.

Maturana, H. and **Varela, F**. (1992). *The tree of knowledge: The biological roots of human understanding*. Boston: Shambhala Press.

Maxwell, J. A. (2008). Qualitative research design: An interpretive approach. London: Sage.

McGraner, K., **VanDeyden, A**. and **Holdheide, L**. (2011). *Preparation of effective teachers in mathematics: A TQ connection issue paper on applying the innovation configuration to mathematics teacher preparation*. Washington, DC: National Comprehensive Centre for Teacher Quality.

McMillan, E. (2004). *Complexity, organisations and change*. London: Routledge.

Megan, S. (2008). Promoting student collaboration in a detracted, heterogeneous secondary mathematics classroom. *Journal of Mathematics Teacher Education, 11*(5), 349-371.

Mennin, S. (2010). Teaching, learning and health professions education. *JIAMSE, 20*(2), 162-165.

Merleu-Ponty, M. (1962). *Phenomenology of perception*. London: Routledge and Kegan Paul.

Merriam, S. B. (1988). *Case study research in education: A qualitative approach*. San Francisco, CA: Jossey-Bass Inc.

Merriam, S. B. (1998). *Qualitative research and case study application in education*. San Francisco, CA: Jossey-Bass Inc.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. San Francisco, CA: Jossey-Bass Inc.

Mills, G. E. (2003). *Action research: A guide for the teacher researcher* (2nd ed.). New Jersey: Merril Prentice Hall.

Miranda, H. (2004). *Emergence and evolution of the mathematics classroom discourses*. Unpublished master thesis. University of Alberta, Edmonton.

Miranda, H. (2009). *Enacting algebraic meanings: Educating mathematics teachers' awareness*. Unpublished doctoral dissertation. University of Alberta, Edmonton.

Moyer-Packenham, P.S. (2011). An initial examination of effect sizes for virtual manipulatives and other instructional treatments. In L. Paditz and A. Rogerson (Eds.). Turning dreams into reality: Transformations and paradigm shifts in mathematics education. *Proceedings of the 11th International Conference of the Mathematics Education into the 21st Century project* (pp.316-321). Rhodes University, Grahamstown.

Muir, T. (2010). Using video stimulated recall as a tool for reflecting on the teaching of mathematics. In L. Sparrow, B. Kissane and C. Hurst (Eds.): Shaping the future of mathematics education. *Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia*. Fremantle: MERGA.

Nambira, G., **Kapenda, L**., **Tjipueja, G**. and **Sichombe, B**. (2009). *Performance of learners in mathematics at upper primary phase in Okahandja district: examining reasons for low performance*. Okahandja: NIED.

Namibia. Ministry of Basic Education and Culture. (1993). *Toward education for all: A development brief education, culture and training*. Windhoek: Gamsberg Macmillan.

Namibia. Ministry of Education. (2005a). *National standards and performance indicators for schools in Namibia*. Windhoek: Government Printer.

Namibia. Ministry of Education. (2005b). *Guidelines for school principals*. Windhoek: Government Printer.

Namibia. Office of the President. (2004). *Namibia Vision 2030 document: Prosperity, harmony, peace and political stability*. Windhoek: Gamsberg Macmillan.

National Council of Teachers of Mathematics (NCTM). (1989*). Curriculum and education standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM). (1998). *Principals and standards of school mathematics: Discussion draft*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (NCTM). (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: NCTM.

National Mathematics Advisory Panel. (2008). *Foundations of success: The final report of the National Mathematics Advisory Panel*. Washington, DC: US Department of Education.

Niessen, T. (2007). *Emerging epistemology: Making sense of teaching practice*. ICO dissertation series. Rethimno: University of Maastricht.

Ontario. Ministry of Education. (2005). The Ontario curriculum policy document: A guide to effective instruction in mathematics, kindergarten to grade 6, Volume 1. Retrieved November 11, 2012 from: http://www.edu.gov.on

Opie. C. (2004). *Doing educational research*. London: Sage Publications.

Organisation for Economic Cooperation and Development (OECD). (2004). *Learning for tomorrow's world-First results from PISA 2003. A profile of student performance in mathematics*. Paris: OECD.

Patton, M. (2002). *Qualitative research and evaluation methods* (3rd ed.). Thousand Oaks, CA: Sage.

Peedo, M. (2003). *"Creating better understanding": A phenomenographic case study of adolescent conceptions of religious lessons at an international school in Finland*. Unpublished master's thesis. Oulu, University of Oulu.

Pillay, V. (2006). Mathematical knowledge for teaching functions. *Proceedings of the 14th Annual conference of the Southern African Association for Research in Mathematics, Science and Technology Education* (*SAARMSTE),* January 14-19, University of Pretoria.

Proulx, J. (2008). Elaborations on Maturana and Varela's theory of cognition and constructivism: Some introductory notes on how this perspective differentiates itself from constructivist theories. *Complicity: An International Journal of Complexity and Education, 5*(1), 11-26.

Proulx, J. (2009). Some direction and possibilities for enactivism and mathematics education research. In M. Tzekaki, M. Kaladrimidou and C. Sakonidis (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (PME) (pp. 1-7). Thessaloniki, Greece.

Reid, D. (1996). Enactivism as a methodology. In L. Puig and A. Gutiérrez (Eds.), *Proceedings of the Twentieth Annual Conference of the International Group for the Psychology of Mathematics Education* (PME) (pp. 203-210). Valencia, Spain.

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education*, *33,* 5-29.

Rittle-Johnson, B. and **Alibali, M**. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, *91*(1), 175-189.

Rittle-Johnson, B., **Siegler, R**. and **Alibali, M**. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346-362.

Rowland, T. (2000). The pragmatics of mathematics education: Vagueness in mathematical discourse. London: Falmer Press.

Rowland, T., Turner, F., Thwaites, A., and Huckstep, P. (2009). *Developing primary mathematics teaching*. London, Sage Publication.

Sammons, P., **Hillman, J**. and **Mortimore, P**. (1995). *Key characteristics of effective schools: A review of school effectiveness research*. London: Institute of Education.

Samson, D. A. (2011). *The heuristic significance of enacted visualisation*. Unpublished doctoral dissertation, Rhodes University, Grahamstown.

Sanni, R. **I**. (2009). *Teachers' task practices in relation to teachers' knowledge: a focus on secondary school mathematics teaching in Nigeria*. Unpublished doctoral dissertation, University of Witwatersrand, Johannesburg.

Schäfer, M. (2003). *The impact of learners' spatial capacity and worldviews on their spatial conceptualisation: A case study*. Unpublished doctoral dissertation, Curtin University of Technology, Perth.

Schäfer, M. (2012). Concept literacy in mathematics and science: Experience with the development and use of a multilingual resources book in Xhosa, Zulu, English and Afrikaans (pp. 512-516). Retrieved November 11, 2012, from: math.unipa.it-grim/21_project/Schafer512-516.pdf

Schifter, D. (2001). Learning to see the invisible: What skills and knowledge are needed to engage with students' mathematical ideas? In T. Wood, B. Nelson and J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 109 - 134). Mahwah, NJ: Lawrence Erlbaum.

Schoenfeld, A. H. (2002). A highly interactive discourse structure. In J. Brophy (Ed.), *Social constructivist teaching: Its affordances and constrains* (pp. 131-170). New York: Elsevier.

Schoenfeld, A. H. (2007). *What is mathematical proficiency and how can it be assessed*. *MSRI Publications*, *53*, 59-73.

Short, J., Singh, P., Yarrow, A. and **Millwater, J**. (2000). How does pre-service teacher plan, enact and reflect upon classroom teaching in order to achieve learning outcomes for students? An application of Bernstein's theoretical framework. Paper presented at the Annual Conference of the Australian Association for Research in Education, December 4-7, Sydney.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22.

Simmt, E. (2000). *Mathematical knowing in action*: A fully embodied interpretation. In E. Simmt, B. Davis and J. McLoughlin (Eds.), Proceedings of the 2000 Annual Meeting of the Canadian Mathematics Education Study Group (pp. 153-159). Montreal: Universite du Quebec a Montreal.

Simmt, Elaine. August 2012. Personal communication.

Somekh, B. and **Lewin, C**. (2005). *Research methods in the social sciences*. London: Sage.

Speer, W. R. (2011). Creating desirable difficulties to enhance mathematics learning. In L. Paditz and A. Rogerson (Eds.): Turning dreams into reality: Transformations and paradigm shifts in mathematics education. *Proceedings of the 11th International Conference of the Mathematics Education into the 21st Century project* (pp.316-321). Rhodes University, Grahamstown.

Spillane, J. P. and **Zeuli, J. S**. (1999). Reform and teaching: Exploring patterns of practice in the context of national and state mathematics reform. *Educational Evaluation and Policy Analysis*, *2*(1), 1-27.

Sproston, C. (2005). *When students negotiate: An action research case study of a year 8 English class in Catholic Secondary College in Regional Victoria*. Unpublished doctoral dissertation, Australian Catholic University, Victoria.

Stake, R. **E**. (1994). Case studies. In N.K. Denzin and Y.S. Lincoln (Eds.). *Handbook of qualitative research* (pp. 236-247). Thousand Oaks, CA: Sage.

Stake, R. **E**. (1995). *The art of case study research*. Thousand Oaks: Sage Publications.

Stake, R. **E**. (2000). Case studies. In N.K. Denzin and Y.S. Lincoln (Eds.). *Handbook of qualitative research* (pp. 435-454). Thousand Oaks, CA: Sage Publications.

Stake, R. **E**. (2006). *Multiple case study analysis*. New York: The Guildford Press.

Staples, M. (2008). Promoting student collaboration in a detracked, heterogenous secondary mathematics classroom. *Journal of Mathematics Teacher Education*, *11*(5), 349-371.

Star, J. (2005). Conceptualising procedural knowledge. *Journal for Research in Mathematics Education*, *36*, 404-411.

Stephanus, G. H. (2008). *Factors affecting grade 11 learners' performance in mathematics as perceived by teachers, learners and principals in selected secondary schools in the Omusati education region*. Unpublished master thesis, University of Namibia (UNAM), Windhoek.

Stephanus, G. **H**. and **Schäfer, M.** (2011). Exploring teaching proficiency using elements of enactivism as an analytical tool: A Namibian experience. In T. Mamiala and F. Kwayisi (Eds.): Mathematics, science and technology education-a bridge to the future.

Proceedings of the Nineteenth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE), January 18-21, North West University, Mafikeng.

Stephanus, G. **H**. and **Schäfer, M.** (2013). Teachers' perceptions of factors that contribute towards their effective practice: What the data tell us. In S. K. Kwofie, M. B. Ogunniyi, O. Amosun, K. R. Langenhoven and S. Dinie (Eds.): Making mathematics, science and technology education, socially and culturally relevant in Africa. *Proceedings of the Twenty first Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE)*, January 14-17, University of the Western Cape, Cape Town.

Stephens, M. (2009). Numeracy in practice: Teaching, learning and using mathematics. Retrieved February 23, 2012 from: http://www/edu.vic.gov.au/eduPaperNo.18.pdf

Suh, J. M. (2007). Tying it all together: Classroom practices that promote mathematical proficiency for all students. *Teaching Children Mathematics*, *14*(3), 163-169.

Sullivan, P. (2011). *Australian education review: Teaching mathematics: Using research informed strategies*. Victoria: Australian Council for Education Research, ACER Press.

Sullivan, P., **Clarke, D**. and **O'Shea, H.** (2010). Students' opinions about characteristics of their desired mathematics lessons. In L. Sparrow, B. Kissane and C. Hurst (Eds.). *Shaping the future of mathematics education*: *Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia*. Fremantle: MERGA.

Sumara, D. and **Davis, B.** (1997a). Enactivist theory and community learning: Toward a complexified understanding of action research. *Educational Action Research*, *5*(3), 403-422.

Sumara, D. and **Davis, B.** (1997b). Cognitive, complexity and teacher education. *Harvard Educational Review*, *67*, 105-126.

Suurtamm, C., Koch, M. J., and **Arden, A**. (2010). Teachers' assessment practices in mathematics: Classroom in the context of reform. *Assessment in Education: Principle, Policy and Practice*, *17*(4), 399-417.

Taylor, N. (2008). Equity, efficiency and the development of South African schools. In T. Townsend (Ed.). *International handbook of school effectiveness and improvement (pp. 145-154).* Dordrecht: Springer.

Thom, P. (2000). *Places around the table: A qualitative enactivist exploration of food practice in a family context*. Unpublished master's thesis, The University of British Columbia, Vancouver.

Thompson, P. W. (1992). Notations, conventions and constraints: Contributions of effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education*, *23*(2), 123-147.

Tshabalala, L. (2012). Exploring the promotion of mathematical proficiency in a grade five class. In D. Nampota and M. Kazima (Eds.), Mathematics, science and technology education: A key to sustainable development. *Proceedings of the Twentieth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology education (SAARMSTE)*, January 16-19, University of Malawi, Lilongwe.

Van de Walle, J. A. (2007). *Elementary and middle school mathematics: teaching developmentally* (6th ed.). Boston: Pearson Education, Inc.

Varela, F. Thompson E. and **Rosch, E**. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Venkat, H. (2010). Exploring the nature and coherence of mathematical work in South African mathematical literacy classrooms. *Research in Mathematics Education*, 12, 53-68.

Webb, N. M., Franke, M. L., Ing, M., Chan, A., De, T., Freund, D. and **Battey, D.** (2008). The role of teacher instructional practices in student collaboration. *Contemporary Educational Psychol*ogy, 33, 360-381.

Weiss, J. **R**., **Pasley, J. D**., **Smith, P. S**., **Banilower, E. R**. and **Heck, D. J**. (2003). *Looking inside the classroom: A study of K-12 mathematics and science education in the United States*. Chapel Hills, NC: Horizon Research, Inc.

Williams, P. J. (2011). Technology teacher training and the development of pedagogical content knowledge (PCK). A paper presented at the 19th Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE), January19-20, North West University, Mafikeng.

Wilson, P., **Thomas, J**. and **Stinson, D**. (2005). What constitutes good Mathematics teaching and how it develops: Nine high school teachers' perspectives? *Journal of Mathematics Teacher Education*, *8*, 83-111.

Wong, M. and Evans, D. (2007). Student conceptual understanding of equivalent fraction. *mathematics Essential Research, Essential Practice*, *2*, 824-833.

Yackel, E., **Cobb, P**. and **Wood, T**. (1991). Small group interactions as a source of learning opportunities in second grade mathematics. *Journal for Research in Mathematics Education*, *22*(5), 390-408.

Yin, R. **K**. (1984). *Case study research: Design and methods*. Beverly Hills, CA: Sage.

Yin, R. K. (2003). *Case study research: Design and methods*. Thousand Oaks: Sage Publications.

Yin, R. K. (2009). *Case study research: Design and methods (4th ed.).* London: Sage.

Zazu, Clayton. October 2010. Personal communication.

Appendix A: Letter seeking permission

Letter to the Permanent Secretary, Namibia. Ministry of Education

Gervasius H. Stephanus
PhD. Candidate (Rhodes University)
Grahamstown
Professional Development Centre
Education Department

Phone (SA): +27 46 603 8700, 083 3747785
Namibia (Outapi): +64 65251464, 081 3665647
Email: gervasius2006@yahoo.com

To: The Permanent Secretary
Ministry of Education
Government Park-Windhoek

Date: 21th April 2011

Subject: Request for permission to undertake a research on teaching proficiency in geometry of selected effective mathematics teachers in Namibia

Introduction

I am a registered full-time student pursuing a PhD degree in mathematics education at Rhodes University, Grahamstown, South Africa, and would soon start my fieldwork in Namibia. Hence, I hereby request for permission to undertake a research on teaching proficiency in geometry of selected effective Namibian mathematics teachers.

Justification for the request

As part of the requirements of my current dissertation, I am exploring geometry teaching practices of selected successful Namibian secondary school mathematics teachers. The aim of the study is primarily envisaged to better understand how these cohorts of Namibian teachers have been effective. The study is aimed at leveraging and improving Mathematics teachers' practices in Namibian secondary schools and beyond. Specifically, I am looking at teachers' classroom instructional practices in relation to teaching for mathematical proficiency. The aim is to explore, deconstruct and document effective secondary school teachers' classroom practices in relation to their teaching proficiency using an adapted model of proficiency developed by Kilpatrick, Swafford and Findell (2001) in the US. To this end, I have already obtained both my supervisor's and Rhodes University Higher Degree Committee's approval to conduct the fieldwork. The first phase entails planning and negotiating access to schools for actual data collection, and will commerce in May 2011. I anticipate completing the fieldwork (data gathering) by end of this year. Thus, this letter seeks: (a) your formal permission to allow me to meet with the regional Education Directors, Inspectors of Education and school principals whose selected mathematics teachers are involved, and (b) to grant me permission to carry out my research in their settings.

The NIED personnel, staff in the Faculty of Education at UNAM and Ministry of Education officials will assist in identifying and selecting teachers or schools. Given below is a preliminary sample consisting of thirteen (13) Namibian senior secondary schools:

1. ST. Boniface College (Kavango Region)
2. Otjiwarongo Secondary School (Otjozojupa Region)
3. ST. Pauls College (Khomas Region)
4. Oshigambo High School (Oshikoto Region)
5. Canisianum Roman High School (Omusati Region)

6. Negumbo Secondary School (Omusati Region)
7. Delta Secondary School (Khomas Region)
8. Windhoek Gymnasium Private School (Khomas Region)
9. Haimbili Haufiku Secondary School (Ohangwena Region)
10. Jan Mohr Secondary school (Khomas Region)
11. Walvis High School (Erongo Region)
12. Namib High School (Erongo Region)
13. P.K De Villiers Senior Secondary School (Karas Region)

With your approval, I intend to visit the above listed schools in their respective regions, meet with regional senior officials and articulate, verbally and in writing, the objective of my study. On the account of this explanation and teachers' willingness to participate in the study, five effective mathematics teachers from five different schools in five educational regions will be asked to volunteer and take part in the study. The research fieldwork procedures will include two questionnaires for the teachers, non-participant classroom observations (to be video-recorded) and a series of interviews, which will also be audio-taped for later analysis. Participating teachers will also be involved in the data analysis process as co-researchers.

Further, I am aware of the ethical considerations in human subject research and I have put in place a number of provisions to ensure good ethics and professionalism. For instance, I do not intend to obstruct and disrupt the school teaching programme nor manipulate any lesson activity in any way. Lessons will be observed the way they are being conducted. Nevertheless, other ethical considerations include:

- The lesson videos will only be used for research data analysis and teachers' post lesson observation interview purposes;
- Participating teachers' and/or schools' identities will be kept anonymous by using pseudonyms or fictitious names;
- The written informed consent will be obtained from both the principals and teachers concerned prior to the commencement of the actual data collection, and
- Very importantly, I will make the final report available to the ministry and all participating schools after its examination process is completed by Rhodes University.

I trust to embark on this project soon and would be visiting identified schools for introduction, acquaintance and establishing a rapport with the teachers involved. Furthermore, should there be anything, within reason, that your office may want to bring to my attention, please feel free to contact me. My contact addresses and telephone details are as above. I should appreciate it very much if the above request receives your kind attention.

Thanking you in advance.

Sincerely Yours in academia,

…………………………………………….
Gervasius H. Stephanus (PhD. Scholar)

Cc: (1) Mrs. E.N. D'Almeida (Deputy Director: NSFAF), (2) Regional Education Directors of selected schools, (3) Inspectors of Education of selected schools, (4) School Principals of selected teachers, (5) Selected mathematics teachers.

Appendix B: Letter of permission to conduct research in Namibian schools

REPUBLIC OF NAMIBIA

MINISTRY OF EDUCATION

Tel: 264 61 2933200
Fax: 264 61 2933922
E-mail: mshimho@mec.gov.na
Enquiries: MN Shimhopileni

Private Bag 13186
Windhoek
NAMIBIA
16 MAy2011

File: 11/2/1

Mr Gervasius H. Stephanus
P.O. Box 597
OUTAPI

Dear Mr Stephanus

RE: REQUEST FOR PERMISSION TO CONDUCT A RESEARCH AT SOME SECONDARY SCHOOLS IN EIGHT REGIONS

Your letter dated 21 April 2011 and signed on 9 May 2011, requesting permission to conduct a research at some secondary schools in eight Education Directorates has reference.

Kindly be informed that the Ministry does not have an objection to your request to conduct a research at those identified secondary schools in the regions concerned. Among the identified schools, there are private schools which do not fall directly under the jurisdiction of the Ministry. You are therefore kindly advised to seek permission directly from those schools.

With regard to government schools, you are advised to approach the Regional Council Offices, Directorates of Education, for permission to enter and carry out your study in the schools. It has been indicated that five teachers would get practically involved in the research activities, approval for their participation should be sought from their principals. Video-taping should be done on a voluntary basis, and if minors are to be involved, consent should also be obtained from their parents/guardians.

Kindly ensure that your research activities do not interfere with the normal school activities.

By copy of this letter the Regional Directors of Education are made aware of your request.

Yours faithfully

MINISTRY OF EDUCATION
Office of the Permanent Secretary
23 MAY 2011
Private Bag 13186, Windhoek, Namibia

………………
A Ilukena
PERMANENT SECRETARY

cc: Regional Directors: Erongo, Kavango, Karas, Khomas, Ohangwena, Oshikoto, Omusati, Otjozondjupa

Appendix C: Letter of Invitation

Gervasius H. Stephanus
PhD. Candidate (Rhodes University)
Grahamstown
Professional Development Centre
Education Department

Phone (SA): +27 46 603 8700, 083 3747785
Namibia (Outapi): +64 65251464, 081 3665647
Email: gervasius2006@yahoo.com

Subject: Information for participation in the PhD research project and request for participation consent

1. Introduction

I am a PhD. student in mathematics education in the Faculty of Education at Rhodes University, Grahamstown, South Africa. As you have been identified as a particularly successful mathematics teacher in Namibia, I would like to invite you to participate in an interesting research project. Given both the Ministry of Education Permanent Secretary and regional Directors' permission to conduct this study in your school, this correspondence serves to humbly request your participation in my research entitled *exploring teaching proficiency in geometry of selected effective mathematics teachers in Namibia*.

2. Background

This study is part of the requirements for the completion of my PhD degree in mathematics education. This PhD research project is primarily aimed at exploring geometry teaching practice of selected secondary school mathematics teachers who are perceived to be effective. The ultimate research objective is to analyse factors that contribute to effective practice. Specifically, I am looking at effective teachers' teaching for mathematical proficiency in relation to their classroom instructional practices. You have been selected as one of the effective "successful" mathematics teachers in Namibia, and I wish to invite you to participate in this research.

To accomplish the objectives of the research I will need to observe you in practice. The focus will be on classroom teaching of geometry in grade 10 to 12 classes. I therefore, humbly ask for your permission to allow me the opportunity to observe you as you engage in the teaching of geometry in one of your grade 10, 11 or 12 classes. I further request your permission to allow me to video-record your lessons, since this will provide us with a more comprehensive recording of the lessons for accurate analysis. You will be asked to participate in the data analysis as a co-researcher.

3. Description and Invitation

The study will take place throughout 2011 and part of 2012. The data collection involves completing two questionnaires (one on the participant's education background information and the other on insights into the geometry scenario-based items). You will be video-recorded as you go about teaching at most three (3) of your geometry lessons in a particular grade or class. You will be also involved in a number of interviews. The first is an individual post lesson reflection interview, during which we will reflect on each of your lessons in order to gain insights into your lessons and discuss in-depth evidences of your effective practices. The second interview will be a stimulated recall participative analysis session with other participating teachers at NIED, which involves the joint analysis of the video-recording of your lessons.

The ultimate research objective is not to evaluate your teaching or your compliance to the curriculum as a mathematics teacher, but rather to analyse and discuss factors that have made you the successful teacher you are. Your participation in this study will not affect your regular teaching in any way as observations are being planned in during normal class times, and interviews will be conducted at conveniently arranged times. The days for all of the activities will be negotiated with you. The data will be strictly confidential and only you, the researcher, my supervisor and other participating teachers will have access to it.

4. Risks and Benefits

There are no foreseeable risks involved in participating in the study. Although there is no financial compensation, your transport expenses to and from NIED inclusive of accommodation and meals will be catered for. Participating in this study will give you an opportunity to study and contribute to knowledge and understanding of effective teaching practices, and share your valuable experience with others. The study results will be used for leveraging and improving mathematics teachers' teaching practices within the Namibian education system and beyond.

5. Time involvement

I anticipate that you might take about thirty minutes to complete the questionnaire on the background information and two hours for the geometry questionnaire. The lesson observations will be conducted during normal class times. I also anticipate that the individual interviews might last for about an hour, and the stimulated recall analysis session about two days.

6. Participants' rights

Participating in this research is absolutely voluntary. That is, your participation is strictly optional and at your personal discretion. Hence, after acquainting yourself with this information, you may decide whether or not to take part in the study and give consent to that effect. You also have the right to withdraw from any part of the study at any time.

7. Video and audio recordings

All aspects of the study, including the results, will be strictly confidential. Upon completion of the project, all data collected will be archived and securely stored at Rhodes University for a maximum period of two years. Once tapes are no longer needed for the research purposes, they will all be discarded. The findings of my study will be communicated to you upon completion of my study.

However, should you have any concerns about your participation or the conduct of this research project, please feel free to contact me or my supervisor (see my contact details above or my research project supervisor Prof Marc Schäfer at M.Schafer@ru.ac.za).

8. Consent

Please complete, sign the attached consent form and return it to me at your earliest convenience. I will be happy to answer any questions or queries that you might have. Furthermore, if there is anything that I can do in return, please do not hesitate to inform me. Hoping for a favourable response!

9. Table 1: A detailed timeline summary of activities

Phase	Activity	Details	Time	Venue
1	Biographical questionnaire	Seeks information on your demographic and educational background as well as teaching experiences.	One hour	Your school
2	Geometry Scenario-based questionnaire	Includes scenario-based items and open-ended questions requiring you to respond to learners' comments and responses to specific geometrical problems frequently encountered in the teaching and learning of school geometry.	Two to three hours	Your school
3	Lesson Observations	Video filming three (3) geometry lessons	45 minutes each	Your school
4	Individual post lesson reflective Interviews	After each lesson in order to have an in-depth discussion and reflection on your teaching practice or lessons.	30 to 50 minutes	Your school
5	Lesson Video Analysis workshop	Analysis sessions of the video recordings of your actual teaching (lessons) together with other participating teachers.	Two days at most	NIED

Thanking you in advance while looking forward to learning from your effective practice.

Sincerely Yours in mathematics education,

………………………..
Gervasius H. Stephanus
(PhD. Scholar)

Appendix D: Participant consent form

I, …………………………….. (Print your name) of ……………………….. (School name) in the……………… (Region), agree to participate in the PhD research project entitled: *Exploring teaching proficiency in geometry of selected effective mathematics teachers in Namibia*.

In giving my consent I acknowledge that:

1. The procedures required for the project and the time involved have been spelled out to me, and any questions I had about this project have been answered to my satisfaction and expectation.
2. I have read the Information and Participant Consent Forms and have been given the opportunity to discuss the information and my involvement and level of participation in the research project with the researcher and other participating mathematics teachers, who will jointly analyse the recorded lesson videos with me.
3. I understand that I can withdraw from the study at any time at no cost without affecting my relationship with the researcher now and/or in the future.
4. I understand that the researcher will ask me to participate in completing two questionnaires, observing and recording three of my geometry lessons at most, involving me in both individual interviews and data analysis sessions, and that only the researcher, his supervisor and other teachers involved in this study will have access to these data. Hence, I agree to:

4.1 Participate in completing the first questionnaire on my educational background and teaching experiences.
YES [ ] **NO [ ] please tick**

4.2 Participate in completing the second questionnaire on geometry scenario-based task in order to share experiences and teaching practice with others.
YES [ ] **NO [ ] please tick**

4.3 Be observed and videotaped in my geometry lessons in which I, the teacher, will appear as part of the video text/clip.
YES [ ] **NO [ ] please tick**

4.4 Provide the researcher access to copies of any materials (i.e. class notes, worksheet and assessment tasks for students) that I might produce as part of the lessons assigned to the teaching of geometry.
YES [ ] **NO [ ] please tick**

4.5 Be involved in individual interviews with myself, the teacher, after each lesson in order to have an in depth discussion and reflection on my teaching practice.
YES [ ] **NO [ ] please tick**

4.6 Be tape-recorded in the interviews conducted with me, the teacher, with the purpose of providing an accurate record of the interviews for later analysis and interpretations.
YES [ ] **NO [ ] please tick**

4.7 Participate in analysis sessions of the video recordings of my actual teaching (lessons) in the mathematics classroom together with other participating teachers at NIED, for the purpose of reflecting, discussing and sharing video recordings to uncover what happened in the lessons and understand other teachers' perspective of their own teaching practices.
YES [ ] **NO [ ] please tick**

5. I understand that my involvement is strictly confidential and that no information about me or my school will be used in any way that reveals my identity or that of my school.

Name (participating teacher): ………………………………
Signed (participating teacher): …………………………… Date: …………………..
Name (principal): …………………………………………..
Signed (principal): ……………………………………… Date: …………………..

Appendix E: Biographical information questionnaire: Mathematics Teacher Profile

(a) Please indicate the school at which you are currently teaching and the region in which it is situated. School: .... Region: ....
(b) Indicate whether you are (Please tick.):
Female: .... Male: ....
(c) Please indicate your age category (Please tick.):
Below 25 yrs: ..., Between 25 and 30 yrs: ....., Between 30 and 40 yrs, above 40 yrs: ...
(d) What grades do you currently teach (Please tick.)
Mathematics Grade 8: ....
Mathematics Grade 9: ....
Mathematics Grade 10: ......
Mathematics Grade 11: ....
Mathematics Grade 12: ....

(e) For how long have you been teaching mathematics and at what grade level?
....
....

(f) Please record your educational qualifications below.
Academic qualifications (e.g. Grade 12 certificate, B. Sc, etc.):
……………………………………………………………………………………
……………………………………………………………………………………
Professional qualifications (e.g. BETD, BEd, Master of Education, etc.):
……………………………………………………………………………………
……………………………………………………………………………………

(g) Up to which level did you study mathematics (Please tick.)?
Grade 12 level: ....
College level: ....
University level: ....

(h) Was mathematics one of your major subjects during your post-secondary education?
Yes: ..., No: .... If your answer is no, what were your major subjects? ....

(i) Indicate the number of learners in the observed mathematics classroom(s):
…………..

(j) How would you describe yourself in terms of the subject content knowledge, reputation, and the passion that you bring to the teaching and learning of mathematics?
....
....

Appendix F: Demographic information: School Profile

School Principals

(a) Please indicate the name of the school and the region in which it is situated.
School: .... Region: ....
(b) Total number of the teachers at the school: ……………………………………
(c) Total number of the teachers teaching mathematics at the school: ……………
(d) Total number of the learners at the school: ……………………………………
(e) Total number of learners taking mathematics in the school: ………………....
(f) Type of the school (i.e. Government or Private): ……………………………
(g) School resident type (i.e. boarding school, etc.): ……………………………...
(h) School fund status per learner
N$…………………..per trimester
N$ ………………….per year

In 1-2 pages, prepare for me in writing a **contextual profile** of the (your) school that readers may have a sense of having been here in terms of the:

- School location
- School motto, mission or vision, etc.
- Learner admission criteria
- Resources and supports (either from the regional office or elsewhere)
- Performance ranking regionally and nationally
- Performance standard mathematically (examination results over the last three years)
- Beacons of excellence
- Reputation or status quo
- Typical features that makes it significantly different from other schools in the region and beyond (what you are doing that is different from other school)

- Anything that typify the school amongst other schools (in terms of school management/leadership function, school academic targets, policies, etc.).

Appendix G: Teacher interview protocol for geometry teaching (Semi-structures questions)

Observer: …………………………………   Date: ………………………….
School code: …………………   School name: ………………….
Teacher Name: …………………………   Teacher anonymous name: ……
Gender (circle):   Male or Female
Class (Grade level): …………..   Number of learners: ……………
Duration: Start: …………..   End: ……………………………
Lesson Topic: ……………………………………………………………………………….
Term (circle):   1   2   3   Year: …………….…………………..

NOTE: This is a confidential interview.
Remember to ask for permission to tape-record the interviews.

INTRODUCTORY COMMENTS

- ❖ Thank you so much for permitting me to sit in and observe your lesson(s).
- ❖ I have some questions I would like to ask related to this lesson (just to follow up some things with you).
- ❖ We might need to view and reflect on your lesson videos while engaging with the dialogue about your lesson(s) today.

REMINDERS

1. It is never possible to understand all that is going on in a lesson just from watching it.
   - ❖ I enjoy watching you teach. I want to understand from you how and why you teach as you do.
   - ❖ Your classroom is very different from the one I am used to. There is a lot I would like to talk to you about. (I.e. **I am not here to judge you or evaluate your lesson, nor am I here to check your compliance to the curriculum and/or comment on the quality of the lesson**).

2. If I ask anything that is not clear or sounds strange, please tell me so that I can try to make it clearer.

PART A: INDIVIDUAL INTERVIEW SCHEDULE

1. PRELIMINARY

I am interested in exploring your *effectiveness* in teaching mathematics, geometry in particular. Also, you have been selected or invited to participate in this study because you are one of the effective mathematics teachers in Namibia.

1.1 Please tell me: why do you think you are such an *effective teacher*?
(Why do you think you are getting such '*good mathematics results'*?)
1.2 What do you think makes you an *effective teacher*?
1.3 How, by way of examples, can you point to this in your classroom practice of this particular lesson?
1.4 Why, in your opinion, are there not more effective mathematics teachers?
1.5 Why do you think Grade 10 or 12 results in Namibia are relatively poor?

2. LESSON PRESENTATION AND DEVELOPMENT

For this interview, I would like you to walk me through the lesson, rather than me walking you through it.

2.1 In this/these lessons you were dealing with…… (topic/s)

What was the purpose of today's lesson?

2.2 What is your feeling (how do you feel) about how the lesson played out?
(In other words: was your lesson successful, and why?
What do you think the students gained from today's lesson?)

2.3 I have seen how you teach a few lessons. I have also observed that you … (describe the method of teaching I have observed).
Is this how you usually teach?
If not, what is different?
How do you usually teach? (Probe: what teacher thinks typifies his teaching).

3. USE OF RESOURCES FOR TEACHING AND LEARNING

3.1 How do you select the materials to use?

3.2 I noticed that you used … (materials) in your lesson.
Did this materials, facilities or resources have any influence on your choice of this lesson or how you taught it?

4 MEDIATION OF KNOWLEDGE (methods/teaching approach)

4.1 What is the *main method* that you use in your teaching, and why?
(Could you explain/articulate this method?)
Could you tell me briefly how you decide what approach/method to use in your teaching? (In other words, what do you draw on to develop lesson content and its delivery?)

4.2 What difficulties or challenges do you encounter in implementing this approach/method in your teaching?

4.3 To what extent, in your classroom practice, do you find yourself able to encourage students to generate ideas, questions, conjectures or propositions?
(Show/give me evidence of this from today's lesson).

4.4 I noticed in your lesson that you (describe how the teacher asks questions, give explanations or examples and gives feedback to students by referring to the videos).

- ❖ Why do you ask these kinds of questions?
- ❖ Why do you give these kinds of explanations?

5 TEACHING PROFICIENCY

I am interested in how you teach concepts that relate to this grade and subject or how you teach mathematics for understanding.

5.1 What was your intended strategy (plan) for this lesson to succeed?
(In other words: What was your basic approach to run this lesson effectively?)

5.2 Do you have a general view on how you teach concepts that relate to this grade and subject?

5.3 How do you teach for conceptual understanding?)
(In other words: Do you have a *general strategy* of teaching conceptually?)
(What did you do to develop learners' understanding of concepts?)
(Can you describe this in more detail)?

5.4 How do you teach towards developing students' mathematical thinking or reasoning?

5.5 Is there a relationship between conceptual and procedural knowledge?
(How do you explain that to me?)

5.6 What are the *teaching/mathematical competencies* do you think you need in teaching mathematics effectively?

5.7 During your mathematics lessons, how do you create and instil a passion for Mathematics in your learners?

5.8 There is a great deal of talk today in education about **teaching/mathematical proficiency**, where the teacher teaches for conceptual understanding, procedural fluency, mathematical reasoning, encourages pupils to talk more in class to each other, asks more questions, see the learner as an equal partner in the teaching/learning processes and so on. What are your views on this?

6 ENACTIVIST PERSPECTIVE

6.1 In your view, how does a learner learn (come to know) what you teach him/her?

6.2 While you are teaching:
How do you check, judge or decide whether your learners understand the concepts you teach them?
(Did your learners display an understanding of the concepts you taught them today? (How, give evidence please?)

6.3 Do you think it is important to interact with the learner?
How do you explain this to me?

6.4 Do you think it is important for the learner to interact with each other?
How did your today's lesson encourage a **collaborative approach** to learning among your pupils? (Give an example)

6.5 To what extent do you think that there is a **partnership** between you and the learners in your teaching?

6.6 To what extent is it important that a teacher is a role model to the learner?
(What does it mean to be a role model?)

6.7 How important do you think your **actions** are in your teaching?

6.8 In your opinion, was there a degree of "sense-making" of mathematical concepts within this lesson, was this appropriate for the needs of the pupils, or the purposes of the lesson?

6.9 Teaching or learning geometry means doing geometry. What does this mean to you?

6.10 What different learning opportunities do you offer your learners?
(Can you please elaborate on that?)

Appendix H: Teachers' geometry content knowledge questionnaire

Time: 2 – 3 hours

This geometry questionnaire is scenario-based and includes open-ended questions designed to explore your insights into key geometrical concepts frequently encountered in the teaching and learning of school geometry. The questionnaire is specifically aimed at exploring how you deal with key geometrical concepts in an educational context and establishing a sense of your geometrical expertise in your teaching.

In this questionnaire you are required to respond to learners' comments and responses to specific geometrical problems.

INSTRUCTIONS

- This questionnaire booklet consists of 14 pages (please check that your booklet is complete).

- Answer **all** the questions in the space provided by **clearly showing all your necessary working details**.
- Diagrams are not necessarily drawn to scale. However, where diagrams are supplied, please use them as you wish but do not erase whatever you have written, shown or drawn on them.
- You are welcome to use a calculator.
- Answers may be given in surd form or rounded off to one decimal digit, unless otherwise stated.

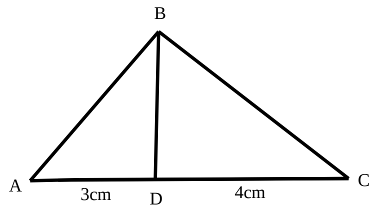
Table 1 Curriculum Content

Scenario	**NSSC (Grade 11 & 12) Mathematics curriculum/syllabus content section**
1	8(a) Trigonometry: Bearings and the trigonometrical ratios
2	8(a) Trigonometry: Bearings and the trigonometrical ratios
3	3(a) Perimeters, areas and volumes
4	4(d) Angle properties; 7(a) Graphs and Cartesian coordinates
5	3(a) Perimeters, areas and volumes
6	4(d) Angle properties; 4(e) Locus
7	3(a) Perimeters, areas, volumes; 4(a) Geometrical terms and relationships; 4(d) Angle properties
8	3(a) Perimeters, areas and volumes; 4(b) Geometrical constructions
9	7(a) Graphs and Cartesian coordinates
10	3(a) Perimeters, areas and volumes
11	4(b) Geometrical constructions; 7(a) Graphs and Cartesian coordinates
12	3(a) Perimeters, areas and volumes; 4(a) Geometrical terms and relationships

Scenario 1

Imagine that three learners in your class come to you with a mathematical argument. The argument is about how to find the length of BD (in simplest surd form) in the diagram below with the following information:

- Δ ABC is a right-angled triangle.
- The perpendicular line BD intersects AC at D.
- AD = 3cm and DC = 4cm.



One of the learners says the length of BD is the same as the length of CD.
Another one says BD is the same as AB.
The third student says BD is two times the area of ΔABC divided by seven.

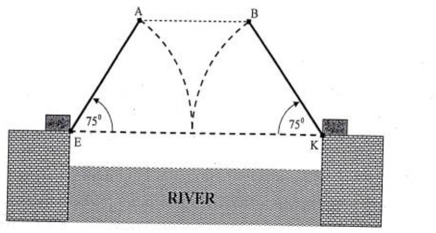
(a) How would you respond to each of the learner's responses?

(b) Clearly explain and illustrate one correct strategy to solve the problem (recognising that there may be more than one way to solve it).

Scenario 2

A group of students in your class regularly pose problems for one another to solve in class. One of them had brought a problem, which the others could not solve. The problem was to find/determine the length of AB (rounded to 2 decimal places) as shown in the diagram below.

The diagram illustrates a bridge positioned over a river. The bridge is made up of two arms AE and BK equal in length, such that AE = BK = 25 metres. The arms AE and BK rotate upward about E and K respectively until both make an angle of 75° with the horizontal EK in order to allow ships to pass through.



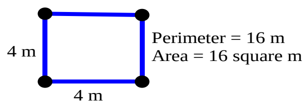
"It is not possible", said one of the students.
"Let's go to the teacher". They all agreed to come to you.

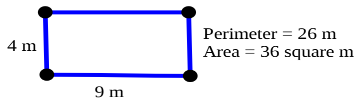
(a) Without providing the solution to the problem, how would you help the students to understand this problem by suggesting a number of "hints"?

(b) What is your solution to this problem?

Scenario 3

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told them. She explained that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you these pictures to prove what she is doing, and then asks you: *Is this always the case*?





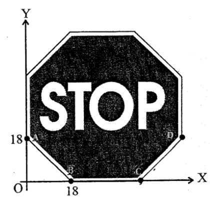
(a) How would you respond to this student?

(b) Explain in detail how you would plan and implement a set of lessons that deals with "this theory".

Scenario 4

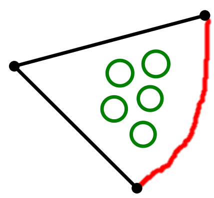
Half of your class find it difficult to internalise the concept of a linear equation. To help them understand this in teaching geometry, you choose a "regular octagon" stop sign, which is ubiquitous on the Namibian roads, and place it in a Cartesian plane as shown in the figure below.

How would you assist your students to understand the linear equation by finding the equation of line CD?
Clearly illustrates your steps.



Scenario 5

Imagine a sector of a circle with five small identical circular holes of diameter 1 cm, as shown in the figure below. The radius of the circle is 8.4 cm and the angle subtended at the centre is 60°. Your grade 10 students were to calculate the area of the sector (and justify their answers), and three students came with the initial thoughts below:



Student A: The area of the sector is the same, because it will cover exactly the same portion without the circular holes.

Student B: It is the area of the sector minus the area of the five circular holes, because if you cut out the sector from a paper/cardboard, the circular holes will not be part of it.

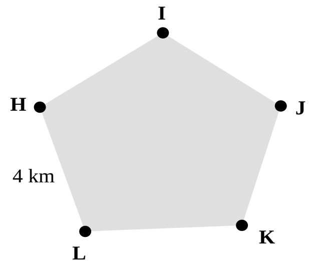
Student C: The area is the area of the sector plus the area of the circular holes, because they are all plane surface and the area is the total of all the areas.

How would you help each of these students so that they understand the correctness or incorrectness of their responses? Also provide the correct answer with a full explanation.

Scenario 6

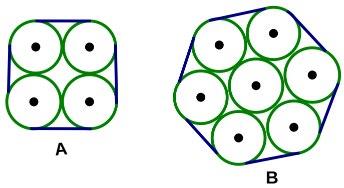
Imagine five village headmen in northern Namibia who want to install a water pump so that it is *equidistant* from all five villages. The five villages (H, I, J, K and L) form a regular pentagon, and the distance between adjacent villages is **4 km** as shown in the diagram below.

A learner from the house of one of the headmen suggests the distance from any village to the water pump is 3.4 km. How would you react to this suggestion? Is this learner correct?



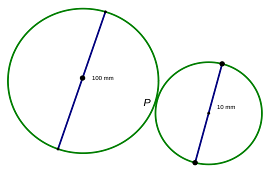
Scenario 7

The diameter of all the pencils in both A and B is **7 mm**. How would you help your learners to find the lengths of the elastic bands that hold the pencils together? For each diagram illustrates your strategies.



Scenario 8

Two friction wheels are of diameter 100 mm and 10 mm respectively. They touch at P and rotate without slipping [as shown below]. You asked your students to calculate the number of turns made by the small wheel when the large wheel rotates through 70°. But, none of your students was able to solve this problem. How would you explain the problem to your students (describe in detail your recommended strategies with a full explanation of each step)?

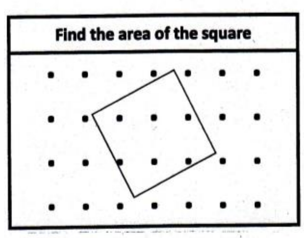


Scenario 9

The problem on finding the area of the square in the diagram below is highlighted as an exercise task in the NSSC mathematics textbooks. A group of learners from another mathematics teacher class in the school come to you with this mathematics dispute: "this problem has no right answer, unless you measure the spaces between the dots and then rotate the square".

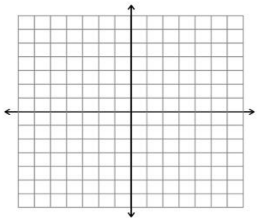
(a) How would you find the area of the square?

(b) How would you react to the learners' dispute?



Scenario 10

A quadrilateral has the following coordinates: M(-2; -1), A(0; 5), T(6; 3) and H(x; y), and is to be drawn on the Cartesian plane below. Point S is the midpoint of MT. The diagonals MT and HA of the shape MATH are perpendicular and bisect each other.



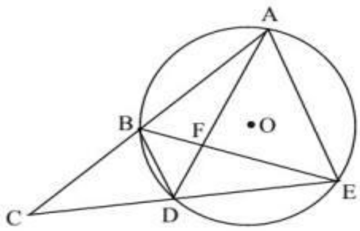
Student A suggests that the gradient of HA equals to - 2.
Student B suggests that MATH is a square.
Student C suggests that MATH is a kite.
Student D suggests that the coordinate of S is (2; 1).
Student E suggests that the equation of line MH equals to 3y-x = 5.

(a) Which of these suggestions is correct?

(b) Put yourself into the shoes of each student and suggest how each student arrived at his or her suggestion.

Scenario 11

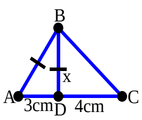
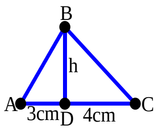
In the given diagram below, O is the centre of the circle, BD: DE: DA: AB = 2: 3: 4: 3. The line segments CBA, CDE, AD and BE are straight. Find angle DFE. Write down as many solutions to this problem as you can and provide a marking memo for each solution.

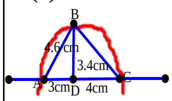
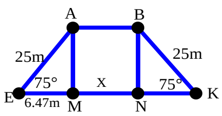
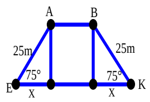
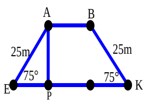
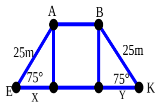


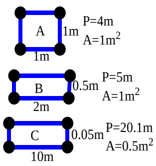
Scenario 12

A cylindrical tin full of engine oil has a diameter of 12 cm and a height 14 cm. The oil is poured into a rectangular tin 16 cm long and 11 cm wide. What is the depth of the oil in the rectangular tin? Prepare a marking scheme to assist other mathematics teachers in the region to assess their students' responses to this question in the end-year examinations. The marking scheme needs to reflect how you give credit for conceptual AND procedural understanding.

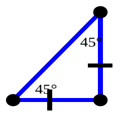
Appendix I: Teacher responses to the geometry content knowledge questionnaire

Scenario	Teachers' responses				
	T1	**T2**	**T3**	**T4**	**T5**
1 (a).	 BD≠CD because ∠DBC cannot be equal to ∠DCB. Because if they are equal then it will result into 45°. That will make ∠ABD 45° (if ∠ABC is the right angle). And then angle BAD is also 45°. But AD≠BD (AD=3cm and BD=4cm). BD≠AB: BD is a perpendicular line ∴ ∠BDA=90°. If BD=AB then∠BAD=∠BDA=90° (then no degrees left to form angle ABD).	BD≠CD, they are both on a right angle triangle. Also, BD≠AD, they are both on a right angle triangle. $SinQ = \frac{4}{x}, SinQ = \frac{3}{x}$	 Let the height BD be h. If ΔABC is right-angled as shown in the diagram, then by Pythagoras theorem: $AC^2 = AB^2 + BC^2$….(1) Considering that ΔADB and ΔCDB and also applying Pythagoras theorem to them we have: $AB^2 = AD^2 + DB^2$…(2) $AB^2 = 3^2 + h^2$ $AB = \sqrt{9 + h^2}$ Similarly: $BC^2 = DC^2 + BD^2$…(3)	I would honestly say I do not know. I would however invite them to investigate with me by drawing to scale ΔABC. I would use AC as diameter of a circle and BD as perpendicular to AC.	For the 1st and 2nd learner: the length of BD cannot be equal to CD or AD because if it equal to one of the sides and you start calculating the size of the angles of ΔABD and ΔBCD, you will find that 4cm is equal to 3cm according to the fact that if sides are equal, then the angles are also equal. 3rd learner: Area of ΔABC = $\frac{1}{2} \times b \times h$, which is $\frac{1}{2} \times 7 \times BD$. ∴ $2 \times Area = 7 \times BD \Rightarrow \frac{2 \times Area}{7} =$

	Area: ΔABC=$\frac{1}{2}base\times height$ = $\frac{1}{2}(7)(x)$=$\frac{7x}{2}$. The third student is correct: $\frac{2}{7}\left(\frac{7x}{2}\right)$ =x		$BC^2 = 4^2 + h^2$ $\therefore BC = \sqrt{16 + h^2}$. Substituting (2) and (3) in (1) we have: $7^2 = 9 + h^2 + 16 + h^2 \Rightarrow 49 = 25 + 2h^2$, $h = \sqrt{12}$ = $\sqrt{4\times 3}$ = $2\sqrt{3}$		BD. You are right in saying this, but the problem is that you do not have the area of the triangle, therefore you cannot complete the equation to calculate BD.
1 (b).	My method is: if BD=x, then in ΔABD, AB=$\sqrt{x^2+3^2}$, and in ΔBCD, BC=$\sqrt{x^2+4^2}$. In ΔABC (if ∠ABC=90°): $AB^2 + BC^2 = AC^2 \Rightarrow$ $x^2 + 9 + x^2 + 16 = 7^2$, $2x^2 + 25 = 49, x^2 = 12$, $x = \pm\sqrt{12}$ but length of side got positive value. ∴ $x = \sqrt{12}$ = $\sqrt{4\times 3}$ = $2\sqrt{3}$. BD= $2\sqrt{3}$ To test the third student: Area of ΔABC=$\frac{1}{2}$(x)(7)=$\frac{7x}{2}$. BD= $\frac{2\times(\frac{7x}{2})}{7}$ = x correct.		Using the value of h to get AB and CD: $AB = \sqrt{9 + (\sqrt{12})^2}$, ∴ $AB = \sqrt{21}$. Also, $CD = \sqrt{16 + (\sqrt{12})^2}$ ∴ $CD = \sqrt{28}$ From these values, it can be concluded that: (i) BD is not the same as CD, and (ii) BD is not the same as the AB. Area of ΔABC = $\frac{1}{2}b\times h, \Rightarrow$ $\frac{1}{2}\times 7\times\sqrt{12}$ = $3.5\sqrt{12}$. Area of ΔABC divided by 7 is: $\frac{3.5\sqrt{12}}{7}$ =$0.5\sqrt{12}$. Two times area of ΔABC divided by 7 is: $2\times 0.5\sqrt{12}$ = $\sqrt{12}$. ∴ Student 3 is correct (as proved above).	I would investigate as below as explained in 1(a).  Obviously BD is slightly less than 3.5 cm (radius) 1st: DC≠DB 2nd: AB≠BD 3rd: Area of ΔABC= $\frac{1}{2}\times 7\times 3.4$ = $11.9cm^2$, and $\frac{11.9\times 2}{7} = 3.4$, wow! The third student is correct.	Line BD cuts AC in the ratio 3:4 which means that it must also split angle B in the ratio 3:4 and $\angle CBD = \frac{4}{7}\times 90°$ = 51.4°. ∴ $\angle ABD = \frac{3}{7}\times 90°$ = 38.6°. From ΔBCD: $\frac{4}{BD} = tan51.4°$, $\frac{BD}{4} = \frac{1}{tan51.4°}$, $BD = 4(\frac{1}{tan51.4°})$ $BD = 3.19\ cm$
2 (a).	Work out the perpendicular height of A or B to show you if it is high enough. The distance EK will be $25\times 2 = 50m$. ∴ The width is correct.  	Hints: $AE + BK = EK, \Rightarrow$ $EK - Cos75°\ of\ AE - Cos75°of\ BK = AB.$	Divide the diagram as follows to create a rectangle in between and ask the learners to use trigonometry ratios.  	 I would first ask learners for the length of EK. And then ask them how far apart would A and B be when horizontal. If they realised that EK is actually 50 meters, then I would lead them through questions to get the distance AP (P being the foot of the perpendicular	 Draw construction lines. Find out how far E is to the south of A, work from EK and subtract values to get a part of EK=AB [that equals to AB].

<table>
<tr><td></td><td></td><td></td><td></td><td>line from A). They have to form a right-angled triangle and then use trigonometric ratios.</td><td></td></tr>
<tr><td>2(b).</td><td>AB=MN and EM=NK
$Cos75° = \frac{EM}{25}$,
$EM = 25 \times Cos75°$
$EM = 6.47\ m.$
$MX = 25 - 6.47 = 18.5$
If MX=XN, then
$MN = 18.5 \times 2$
$\therefore AB = 37.06m$</td><td>$Cos75° = \frac{x}{25}$,
$x = 6.4704 \times 2$
$x = 12.94m$
AB= $50 - 12.94$
$AB = 37.1m$</td><td>$x = 25Cos75°$,
Then, $2x = 2 \times 25Cos75°$
For both arms. Then the length of $AB = 50 - 50Cos75°$.
$\therefore AB = 37.06m$</td><td>$Cos75° = \frac{EP}{25}$,
$EP = 25Cos75°$.
$AB = 50 - 2 \times Cos75°$,
$AB = 37.06m$</td><td>$\frac{x}{25} = Cos75°$,
$x = 25Cos75°$,
$x = 6.47m.$
x and y will be the same lengths because ∠E=∠K and AE=BK.
$EK = 25 + 25 = 50m$
Subtract x and y from EK and then you will get AB:
$\therefore AB = 50 - 2(6.47)$
$AB = 37.06m$</td></tr>
<tr><td>3 (a).</td><td>No, not always. It depends on what type of figure and what are the dimensions. Start by doing an investigation. Help her to construct a table. Start with the original and then the lengths (high number) and width (low number) are to be more extreme values.
<table><tr><td>length</td><td>width</td><td>peri</td><td>Area</td></tr><tr><td>4</td><td>4</td><td>16</td><td>16</td></tr><tr><td>4</td><td>9</td><td>24</td><td>36</td></tr><tr><td>8</td><td>1</td><td>18</td><td>8</td></tr><tr><td>10</td><td>1</td><td>22</td><td>10</td></tr></table>
For the figure with the length of 8 and 10, and width of 1, the hypothesis is not correct.</td><td>I can prove that this learner is correct.</td><td>Area increases with sides
⇒ The perimeter increases with area.</td><td>Very interesting discovery, but is it always the case? I would give few more closed shapes with different perimeters.
E.g.


For A and B, areas are the same but the perimeters are different. And for C, perimeter increases but area decreases.</td><td>I would say to her that she is right.</td></tr>
<tr><td>3(b).</td><td>By investigating other numbers and tries to make a new hypothesis on the results. Depending on the grade this is a good introduction to the maxima and minima point in differentiation.</td><td>Make different shapes. Increase the sides and sometimes decreases measure, and measure [find] the perimeters and areas of these new shapes. Record and compare these ideas.</td><td>Should plan a lesson that covers geometrical constructions. E.g. the lesson can cover construction of similar triangles to prove that. If the ratio of sides is x:y which implies the ratio of the perimeter of the two similar triangles then, by calculation it should be proved that the ratio of the similar triangles' area will be</td><td>I would first suggest that learners investigate closed figures by keeping one of the sides (dimension) constant and then: ask them to draw conclusion, to see if the theory is true. I would then</td><td>I would get the learners to build shapes, from card to scale. They will then have to calculate the perimeter and area of each shape so that they can see for themselves that this theory is true. They must realise that if the perimeter increases the sides of the shape must</td></tr>
</table>

			$x^2 : y^2$. Two isosceles triangles with the following dimensions may be constructed. ΔABC with $AB = BC = 3cm, AC = 4cm$, $\angle A = \angle C = 50°$. ΔDEF with $DE = EF = 6cm, DF = 8cm$, $\angle D = \angle F = 50°$. Let learners calculate the area of the similar figures and relate to the perimeter.	ask them to vary both sides of the closed figures (quadrilaterals) by increasing one side and reducing the length of the other side (reducing into decimals), I ask them to draw conclusions whether the theory will not hold true.	increase and then the area will increase.
4.	(1) Remember the equation of a straight line: $y = mx + c$ and then find the value of m [gradient]. First determines the angle between CD and the x-axis. Angle of Octagon is $135°$. ∴ Angle between CD and x-axis is $180° - 135° = 45°$. In a $45°$ triangle, the base angles are equal, hence two sides are equal.  From left to right upwards: $m = \frac{\Delta y}{\Delta x} = 1$. Intercept of x-axis is 18+25.456 = 43.5. $y = mx + c$, $0 = 1(43.5) + c$, $c = -43.5$ $\therefore y = x - 43.5$	(0;18) and -1 $-1 = \frac{y-44}{x-0}$ $x = y - 44$ $\therefore x + 44 = y$	Given: $y = mx + c$, where $m = \frac{\Delta y}{\Delta x}$ and $c = y - intercept$ (1) Ask the learners to list all whole numbers coordinates which lie on the line CD (x_{1,y_1}); (x_{2,y_2}). (2) Ask the learners the changes in the x-coordinate and y-coordinates between two successive points, e.g. $\Delta x = x_2 - x_1$; $\Delta y = y_2 - y_1$. (3) Ask the learners to divide the corresponding changes. $\frac{\Delta y}{\Delta x}$. (4) Make them conclude from (3) that. $\frac{\Delta y}{\Delta x}$. Is constant. (5) Substitute the value of m from (3) in the equation $y = mx + c$. (6) Ask the learners to substitute the coordinates from (1) in the equation in (5) with m substituted, and (7) Make them conclude that C is constant and independent of the substituted points.	By extending the lines AB and CD, the y-intercept for CD would be 36. Gradient (m) of AB=-1. ∴ Gradient (m) of CD=1. The equation of line CD is $y = x + 36$ $y = x - 36$	(1) The length of AB=$\sqrt{18^2 + 18^2}$ = 25.5 units. Therefore, BC=25.5 units because the STOP sign is a regular Octagon. Thus, the coordinates of C (43.5; 0). And the coordinates of D (61.5; 18), because the same distance on the x-axis from 0 to B will be between the x-coordinates of C and D, D must also be at the same height as A, therefore the y-coordinate of D is also 18. Now use C(43.5; 0) and D(61.5; 18) to get the equation of CD. $y = mx + c$ $0 = m(43.5) + c$ $c = -43.5m$ $\therefore 18 = 61.5m + c$ $18 = 61.5m - 43.5m$ m=1 $\therefore c = -43.5m$ $c = -43.5$ (1) c=1 thus the equation of line CD is $y = x - 43.5$ (2) The equation of AB can be found by using A(0; 18) as your y-intercept (c) and B(18; 0) as

					another point on the line. $y = mx + c$ $0 = m(18) + 18$ $m = -1 \therefore$ $y = -x + 18.$ The line of reflection between AB and CD lies in the middle of BC. At B the x-coordinate is 18, the length of BC is the same as AB, you calculate it with Pythagoras: $AB = \sqrt{18^2 + 18^2}$ Thus the x-coordinate is 25.5. At C, C=18+25.5=43.5. AB and CD are perpendicular because they reflect each other. ∴ AB gradient ×CD gradient=-1. $-1 \times CD = -1$ CD gradient= 1. ∴ equation of CD using coordinates at C is: $y = mx + c$ $0 = 43.5(1) + c$ c=-43.5 $y = x - 43.5$
5.	Area sector (with holes) = $\frac{60°}{360°} \times \pi r^2 - 5(\pi r^2)$ =$\frac{60°}{360°} \times (3.142)(8.4)^2 - 5(3.142)(0.5)^2$ =36.95-3.93 =33.02cm^2 Student A: depending on what is asked, if the question states without the holes then you must subtract it. Student B: my method Student C: depending on what is asked, if the question states with holes then it is cut out and not added. Give example of added like square + half circle. Or square – half circle.	Student A is wrong because the open holes are not part of the sector. Student B is right. Student C is wrong. The answer is: $\frac{60°}{360°} \times \pi r^2 - 5(\pi r^2)$ $\frac{60°}{360°} \times 3.14(8.4)^2 - 5(3.14(1)^2 = 21.2$ cm^2	Definition of area: area is the space covered. (1) Ask learners to cut a similar shape to the one in the question from a paper and cover their table. (2) ask them the amount of space of the table covered, (3) now ask them to make circular holes on the paper, (4) make the learners conclude if the amount of space covered of the table is still the same with open holes on the paper, (5) ask the learners to suggest a way to find the space covered with open holes. Area of a circle is πr^2, area of a sector is: $\frac{\theta}{360°} \times \pi r^2$ $\frac{60°}{360°} \times (3.142)(8.4)^2$ =11.76π or 36.95cm^2	Student A is correct depending on what we are actually looking for. If it is just area in general, student A is correct. However, if it were for area of material of the sector then student A would be wrong. Student B, same as student A, is correct under the condition given, e.g. if it were for the area of the sector to be painted. Student C is	Area of the sector: $\frac{60°}{360°} \times \pi(8.4)^2$ =36.9cm^2 Area of the 5 holes: $5 \times \pi(1)^2$ =15.7cm^2 Subtract the area of the holes from the area of the sector: 36.9-15.7=21.2cm^2. Student A is right. It looks like it covers the whole area but it is not true because when you cut the holes in the sector you are taking some of the sector away. Thus the area will decrease. Student B is right because area refers to the material that I will use to cover the shape and the holes are not part

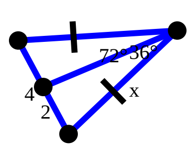
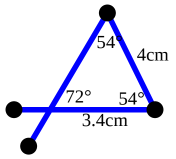
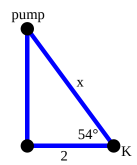
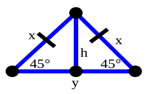
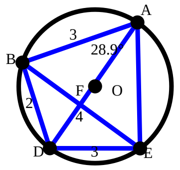
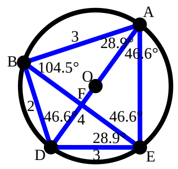
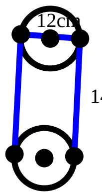
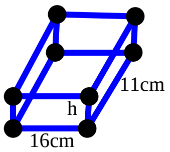
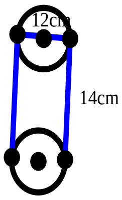
			(2dp) Area of small circle = πr^2 $=\pi(0.5)^2 = 0.25\pi cm^2$ Area of 5 small circle: $= 5 \times 0.25\pi cm^2 = 1.25\pi cm^2$ Area of sector with holes: $11.76\pi - 1.25\pi = 10.5\pi cm^2$	wrong, and I would use student A and B situations to explain why this student is wrong.	of the shape. Student C is not correct. If you cut a sector like this one from a paper and cut out the holes then we can see that the holes are not part of the area because area relates to the amount of material used to make the sector with its holes.
6.	 Centre angle of a pentagon is $\frac{360°}{5} = 72°$. Hence, $Sin\ 36° = \frac{2}{x}$ $x = \frac{2}{sin36°}$ = 3.4 (1dp). Yes the student is correct.	 Using the scale 1cm: 1km as shown in the diagram, the student is right.	Scale: 1km=1cm. The teacher should construct a pentagon as below and bisect all the interior angles of pentagon. Each interior angle of a pentagon is 108°. The point where all the angle bisectors meet is the point which is equidistant from all the points. The distance to the centre point can be measured and converted to actual distance using the scale. According to my construction which is subject to physical error, this learner is correct.	I would investigate by drawing to scale or simply calculating interior angle of a regular pentagon: $\frac{(5-2)180°}{5} = 108°$  $Cos54° = \frac{2}{x}$ $x = \frac{2}{Cos54°} = 3.4km$ The learner is correct.	I would say that the learner is right. You calculate the size of one angle of a regular pentagon: $\frac{180°\times(5-3)}{5} = 108°$ The line from each village to the water pump bisects each angle, and thus: $\frac{108°}{2} = 54°$. If you work in one triangle, the lines from the water pump to the centre of the distance between two villages are perpendicular. From this I use trigonometry to calculate the length (h): $Cos54° = \frac{2}{h}$, $\frac{1}{Cos54°} = \frac{h}{2}$, $h = 3.4km$
7.	Diagram A: Sector = $\frac{\theta}{360°} \times 2\pi r$ $= \frac{90°}{360°} \times 2(3.142)(3.5)$ $= 5.4985 \times 4 = 21.994$ $= 21.994 + 4(7)$ $\approx 50mm$ Diagram B: Angle of the sector is: $\frac{360°}{6} = 60°$ Perimeter = $6(7) + 6\left(\frac{60°}{360°}\right)(3.142)(3.5)^2$ $= 42 + 38.49$ $\approx 80.49mm$	Diagram A: I will just add the two diameters because the diameter is the same length in the whole circle. The length =14mm. Diagram B: I will add a half of the diameter to the diameter. This is because the length of the hexagon is equal to the diameter of one circle plus the radius of the circle. The length =10.5mm.	Diagram A: The diagram can be divided as above so as to make a rectangle and four sectors of circle which are equal and the four sectors make a full circle when put together. The learners can now be asked to calculate the perimeter of a (square) rectangle and the circumference of a full circle. Total perimeter = perimeter of (square) rectangle + circumference of a circle. $(7 \times 4) + \pi 7$ $= 28 + 7\pi$ $\approx 50mm$	Diagram A: I would find circumference of a quarter circle multiplying by 4. Since pencils are touching, the centres would be 7mm apart which is the length of the straight edges of the elastic band. The length of elastic band is: $\frac{\pi\times 7}{4} \times 4 + 7 \times 4$	Diagram A: At each “corner”, it is actually a sector of a circle; there are four sectors which also makes a circle again. From the centre of each pencil to the centre of the next pencil= 7mm and there are four sides. To calculate the length of the elastic band I would calculate the perimeter of the circle added to the four lengths equal to 7mm.

			Diagram B: In this diagram, lines are drawn to divide the shape into a hexagon. To get the angle of the shaded sectors in the diagram, a triangle ABC is considered. Since triangle ABC is isosceles, the base angles should be equal. Learners should calculate the angle at B using properties of the interior angles of a hexagon to get $120°$ and use properties of angles in triangle to get ∠BAC=∠BCA=30°. Learners should calculate the angle at point B which is vertically opposite the shaded sector to get: $120° - (30° + 30°) = 60°$. ∴ Each of the shaded sectors $= 60°$. The length of the elastic band= perimeter of hexagon+6×circumference of sector. Length= $7 \times 6 + 6 \times \frac{60°}{360°} \times \pi \times 7$ $l = 42 + 7\pi$ $\approx 64mm.$	=49.99mm. $\approx 50mm$ Diagram B	$2\pi(3.5) + 4(7) = 50mm$ to the nearest mm. Diagram B: There are 6 lengths of 7mm each, from the centre of one pencil to the next. There are also 6 (six) quarter circles thus $1\frac{1}{2}$ circles. Calculate the circumference of the $1\frac{1}{2}$circles and add 6 lengths of 7mm. $1\frac{1}{2}\left(2\pi\left(3\frac{1}{2}\right)\right) + 6(7) = 75mm$ to the nearest mm.
8.	The perimeter (Sector $70°$) Big wheel. P(Sector) = $\frac{70°}{360°} \times 2(3.142)(50°)$ $= 61.0944444$ P(small wheel) $= 2\pi r$ $= 2 \times 3.142 \times 5$ $= 31.42$ Turns= $\frac{61.44444}{31.42} = 1.9444$ Or $\frac{35}{18} = 1\frac{17}{18}$	These wheels have just rotated around the same angle. The difference is only that they cannot travel the same distance because they have different diameter. So the distance travelled by the small wheel is just the perimeter×the angle it rotates over $360°$. Arc length of $70°$ large wheel. Large wheel rotation: $\frac{70°}{360°} \times 2\pi \times 50 = 61.08mm$ Small wheel= $2\pi r$ $= 2 \times \pi \times 5 = 31.4mm$ $\therefore \frac{61.08}{31.4} = 1.9mm$	This problem can be put into day to day life experience of the learners by expressing them to a scenario whereby two people, a tall one and a short one walk at the same speed. If this is possible, let the learners decide who will have more steps. They should conclude that the short one. For a small wheel to stay at P, it should have a higher rate of rotation. Distance covered by the big wheel should be equal to distance covered by small wheel if it is to stay at P. distance covered by big wheel $= \frac{70°}{360°} \times$	Firstly, find the circumference of the two wheels. Circumference of A $= \pi \times 100 = 100\pi$ Circumference of B $= \pi \times 10 = 10\pi$ Anyway even by taking ratios of diameter: A is 10 times longer than B. Length of the sector A:	If the diameter of the big wheel =100mm, then the radius is 50mm. $\frac{70°}{360°}$ because the wheel only rotated 70°. $\frac{70°}{360°} \times 2\pi(50) = 61.1mm$ Circumference of the small circle: $2\pi(5) = 31.4$ Take the 61.1mm and divide it by the circumference of the small circle. $\frac{61.1}{31.4} = 1.95$ The small wheel turns 1.95 times.

		Only one complete turn.	πd $= \frac{70°}{360°} \times \pi \times 100$ Distance covered by small wheel in a complete revolution. $= \pi d$ $= \pi \times 10$ Let n be the number of turns $\Rightarrow$ distance covered in n turns: $= n\,(\pi d) = n(10\pi)$ $\Rightarrow n(10\pi) = \frac{700\pi}{36}$ $n = \frac{700\pi}{36\times 10\pi}$ $n = \frac{70}{36} \approx 1.94.$ This is 2 turns.	$\frac{70°}{360°} \times \pi \times 100 = 19\frac{4}{9}\pi$ ∴ Number of turns for B: $\frac{19\frac{4}{9}\pi}{10\pi} = 1\frac{17}{18}$ turns. (Only one complete turn).	
9(a).	Joining up the dots. Area = big rectangle minus 4 triangles. $A = (4 \times 3) - 4(1 \times 2)$ $A = 12 - 8 = 4units^2$	Measure the distance between 4 dots.	The G. Alexander Pick theorem can be applied which states the relationship between area, the number of dots on perimeter (p) and the number of dots inside the shape (i) as: $A = \frac{p}{2} + i - 1.$ Then according to Pick's theorem: $A = \frac{0}{2} + 6 - 1 = 5units^2$	I suppose it is to scale. Therefore I would measure the lengths of the sides.	 Divides the square into two isosceles triangles as above. $y = \sqrt{x^2 + x^2}$ $y = \sqrt{2}\,x$ …..(1) $Sin45° = \frac{x}{y}$ $ySin45° = x$...(2) Substitute (2) in (1) $y = \sqrt{2}(\sin 45°)$ $y = 1$ $1Sin45° = x$ $x = 0.707 \approx 0.8$ The area of the square=$(sin45°)^2$ $= 0.5\,units$
9(b).	If the question states estimate you can solve the problem like this.	These learners are correct because the square has the same distance as of 4 dots.	In order for the learners to understand the relationship between A, P and I, a dotted paper is used for the learner to construct different sizes of rectangles and squares then fill in the information under Area, P and i. | A | P | i | | | | | e.g. • • • • • • • • • • • • • • • **i=3; p=12, A=8units** The learner should be allowed time to figure out what the relationship between Area, P and I will be.	I would ask them for properties of a square and how to find the area of a square i.e. $A = side^2$ Then they would rather measure the side instead of the dots.	I would tell them that you must use the properties of a square in order to solve the problem. You need values to calculate an area and you must find the values. Also don't be afraid to use algebra, simultaneous equation to solve geometry problems.
10(a).	Students B and D suggestions are correct.	Students B and D suggestions are correct.	After sketching the quadrilateral using given coordinates, MT gradient $\frac{3-(-1)}{6-(-2)} =$	Student A is correct. $M_{MA} = 3,$ $M_{AT} = -\frac{1}{3},$	Student A: $y = mx + c$ $-3 = m(4) + 5$ $-2 = m,$ the

			$\frac{1}{2}$, the product of MT and HA= $m_2 \times (\frac{1}{2})$ $m_2 = -2$, thus HA gradient= -2. ∴ Student A is wrong. Length of AT: $\sqrt{(3-5)^2 + (6-0)^2}$ $AT = \sqrt{40}$, MATH is a square, hence student B is correct. Midpoint of MT $= \frac{1}{2}(x_1 + x_2; y_1 + y_2)$ Midpoint is (2; 1), thus student D is correct. Gradient of MH is $\frac{-3+1}{4+2} = -\frac{1}{3}$ and its equation is $3y + x = -5$. ∴ Student E is wrong in terms of the signs.	$M_{MA} \times M_{AT} = -1$, student B is correct. Student C is correct because a square is a kite. Student d is correct. Student E is wrong as M: 3(-2)-(-1)≠5 and H:3(-3)-(4)≠5 Student A, B, C and D are correct.	student could have done this $-8 = 4m$ to get $\frac{1}{2} = m$. ∴ Student A is not right, the gradient is -2. Student B: this student is right because the distances of MA and AT are equal and it says that the diagonals are perpendicular to each other and that they bisect each other. $MA = \sqrt{40} = 6.32$, $AT = \sqrt{40} = 6.32$. also $MT = \sqrt{80} = 8.94$ and $AS = \frac{1}{2}$MT=4.47 Student E: $y = mx + c$ M(-2;-1) ⇒ $-1 = -2m + c$ H(4;-3) ⇒ $-3 = 4m + c$ Solve simultaneous equation: $-1 = -2m + c$ $-3 = 4m + c$ $m = -\frac{1}{3}, c = 1\frac{2}{3}$ MH $3y + x = -5$ It is not true. $3y = 3 + 5$⇒ $y = \frac{1}{3}x + \frac{5}{3}$ The gradient is positive and the like must have a negative gradient.
10(b).	Student A: gradient of MT= $\frac{-1-3}{-2-6} = \frac{1}{2}$ because HA is perpendicular to MT but then it will be -2 not $-\frac{1}{2}$. $(m_1 \times m_2 = -1)$ Student B: gradient of MA= $\frac{-1-5}{-2-0} = 3$. Gradient of AT= $\frac{5-3}{0-6} = -\frac{1}{3}$. This shows that MA and AT are perpendicular. Distance adjacent sides: $MA = \sqrt{(-2-0)^2 + (-1-5)^2}$ ∴ $MA = \sqrt{40}$ $AT = \sqrt{(6-0)^2 + (3-5)^2}$ ∴ $AT = \sqrt{40}$. Adjacent	First, I find the gradient of MT then I make it inverse because it is perpendicular to HA. I also need to find the midpoint of MT. I then find the coordinates of H using the midpoint of MT.	Student A thinks that two lines are perpendicular, and then the known gradient of the line is multiplied by -1 to get the gradient of the other lines. Student B knows how to calculate the length of the line. Student C only remembers the general layout of a kite. Student D knows how to calculate the midpoint. Student E made mistakes in operational sign changes.	Student A arrives at answer by calculating or counting $\frac{\Delta y}{\Delta x}$ (gradient). Student B because diagonals intersect at 90° and bisect and sides are perpendicular. Student C just used intersection of diagonals at 90°. Student D arrives at answer by	Student C: the diagonals of a kite are perpendicular to each other and one of the diagonals gets bisected. Therefore the learner thought it could be a kite. Student D: if you get the vector of movement from M to T, it is $\binom{8}{4}$. If I must find the midpoint I only do half of the full movement, thus $\binom{4}{2}$. And if you start at M, you get

	sides are equal; hence student B is correct that MATH is a square. Student C: is not correct. MATH is not a Kite because diagonals MT and HA are perpendicular and **bisect each other** (in a kite, the diagonals are perpendicular but do not bisect each other). Student D is correct that S(2;1). This can be worked out like this: Midpoint of MT=($\frac{-2+6}{2};\frac{-1+3}{2}$) =(2;1). Student E not correct if $3y-x=5$. Worked out: M(-2;-1) and H(4;-3) Gradient MH $=\frac{-3-(-1)}{4-(-2)}=-\frac{1}{3}$. $-1=-\frac{1}{3}(-2)+c$, $c=-\frac{5}{3}$. ∴ $y=-\frac{1}{3}x-\frac{5}{3}$ ⇒$3y=x-5$			finding midpoint: $(\frac{x1+x2}{2};\frac{y1+y2}{2})$	to (2;1).
11.	 Triangle ABD=triangle EBD ∴ AD=EB=4 In triangle ABD, ∠BAD: CosBAD= $\frac{3^2+4^2-2^2}{2(3)(4)}$ BAD= $28.955°$ $BF^2=3^2+4^2-2(3)(4)Cos28.955°$ ∴BF=2 ∴ ΔBDF is equilateral triangle. Angle BFD= $60°$ ∴ angle DFE= $180°-60°=120°$		 Applying Cosine rule to get angle at D: ⇒ $3^2=2^2+4^2-2(2)(4)CosD$ (M1) $9=4+16-16CosD$ $CosD=\frac{11}{16}$ ∴ $D=46.6°$ (M1) ∠AEB=$46.6°$ (angle in the same segment with D). (M1) Using Cosine rule to get the angle at B: ⇒ $4^2=3^2+2^2-2(3)(2)CosB$ (M1) $CosB=\frac{-3}{12}$ $B=104.5°$ (M1) Using triangle properties: ∠BAD= $180°-(104.5°+46.6°)=28.9°$ (M1) ∠BED= $28.9°$ (angle in the same segment with angle A) (M1) Using the sine rule to get ∠DAE. $\frac{SinE}{4}=\frac{SinA}{3}$ ⇒ $SinA=\frac{3}{4}.Sin75.5°$ (M1) $A=46.6°$ ∴ Angle DFE=		

			$46.6° + 46.6°$ (M1)		
12.	  Volume of cylinder: $= \pi r^2 \times h$ (M1) $= 3.142(6)^2 \times 14$ (M1) $= 1583.568cm^3$ (M1) Volume of cuboids: $= l \times b \times h$ (M1) $= 16 \times 11 \times h$ (M1) $\therefore h = 8.9975 \approx 9cm$ (M1) Note: **Procedural knowledge** M2 $(3.142)(6)^2(14)$ A1 1583.568 **Conceptual knowledge** M2 $\frac{1583.568}{(16)(11)}$ or $\frac{1583.568}{176}$ A1 9cm	Volume of oil $= \pi r^2 h$ $= \pi \times 6^2 \times 14$ $= 1583.36cm^3$ (M2) Volume of rectangular tin $= lbh$ (M1) $1583.36 = 16 \times 11 \times h$ (M1) $h = 8.996$ $h \approx 9cm$ (M1)	 Volume of cylinder $= \pi r^2 h$ $= \pi(6)^2(14)$ $= 504\pi cm^3$ (M1) Volume of tin $= l \times w \times h$ $16 \times 11 \times h$ (M1) $h = \frac{504\pi}{16 \times 11}$ (M1) $h = 8.99$ cm (M1)	Volume of oil: $\pi \times 6^2 \times 14 = 504\pi$ For rectangular tin: $16 \times 11 \times depth = 504\pi$ $\Rightarrow depth = \frac{504\pi}{176}$ ≈ 8.996 cm	Cylindrical tin: diameter=12cm. $\therefore \quad r = \frac{12}{2} = 6cm$ (M1 if you see 6). Volume of the cylindrical tin $= \pi r^2 h$ $\pi(6)^2 \times 14 = 1583cm^3$ (M2). Volume of oil in a rectangular tin must also be 1583 (M1 for writing this in word or shown in the calculation process). Volume of a rectangular tin $= l \times b \times h$ $1583 = 16 \times 11 \times h$ (follow through) $1583 = 176h$ $8.99cm = h$ (M2 for follow through if the learner calculated the volume of the cylindrical tin wrongly but you can see that they have the correct method)

Appendix J: Stimulated recall analysis sessions (focus group interviews)

Good morning mathematicians

It is a privilege and great honour to be with you once again and continue working with such committed teachers in Namibia. It is indeed an honour to work with such a good community of mathematics teachers. Most, if not all of you, are much better in teaching mathematics than myself. Hence, I need your assistance in understanding your teaching strategies, success and your classroom actions and interactions with the learners in teaching mathematics (geometry) for understanding.

Colleagues, as I had informed you during my visits to your schools:

- The **task** today is very crucial and clear: to help me understand what makes you the effective mathematics teachers you are, why you are so successful in your teaching.
- The **aim** is thus to find out how and why you have been so effective in your teaching over the last three years or so.

Hence, for this joint analysis session, you are here to analyse your own practice and to help me understand why you have been doing well or maintaining that good performance over the past years. Are you really at the position of analysing your own teaching practice? I really want you to tell me why your lessons were successful. It is also important to understand that I will not take anything that we will do here for granted! That is, everything that you are going to say will contribute to the success of this project and, thus, I will treat it with care, respect and outmost confidentiality.

There are FIVE tasks that we are going to do:

Task 1: Member-check of the interview transcriptions

The first thing that I wish you to do is to member-check (analyse) the interview transcriptions and add any omissions to the transcripts, or delete any inaccuracies. I also wish you to extract common issues and differences that emerged from the discussion I had with each of you.

Task 2: Viewing and analysis of lesson videos

In this particular task we are going to use the videos and employ *narrative accounts* (stories and vignettes) as devices to analyse in-depth the nature of the quality of mathematics instructions across the five lessons or classrooms. The purpose is to provide snapshots of geometry teaching as they exist in mathematics classrooms in a variety of contexts in Namibia. These snapshots should include both the instruction that took place and the factors that shaped those instructions. Please select one lesson video from three videos that you think is a representative of your daily classroom teaching that we are going to have a discussion on.

In 1-2 pages, describe what happened in each selected lesson, *vicariously*, including enough rich details as possible to provide a rich and comprehensive account of the lesson. You may focus on high quality indicators of the lesson and include aspects such as:

- the focus (aim) of this lesson (e.g. the extent to which it addressed algorithms, vocabulary, mathematical concepts etc.);
- instructional materials used, if any;
- the nature (high quality indicators) of the lesson activities including teaching, class discussion, problem-solving, investigation etc.;
- the interactions with students or roles of the teacher in the intellectual work of the lesson (e.g. questioning, explanation, proposing conjectures, developing or applying strategies or procedures, and drawing, challenging or verifying conclusions), and
- the lesson’s impact on student understanding of mathematical concepts and procedures.

N.B: Consider what you (teacher) did (in that lesson) to engage students with mathematical or geometric concepts, how you created environment conducive to learning, how you ensured access for all students and what you did to help students make sense of the concepts or procedures. In other words, summarise why this lesson was taught or looked the way it did and how effective it worked (what influenced you to teach the way you did).

This description should stand on its own. Do not be concerned if you repeat information you have already provided elsewhere. You should also draw upon the interview I conducted with you.

Task 3: Nature of the quality of mathematics instruction

From your own practices as evidenced from your lesson videos, collectively:

1. What are most important **competencies** needed in teaching mathematics effectively or successful?
2. How do you construct effectiveness in your mathematics teaching?

3. How do those constructions of effectiveness in mathematics teaching inform your classroom practice?
4. What are the teaching proficiency characteristics that are **similar** and **different** across your teaching practices or lesson videos?

Task 4: Classroom Observation Protocol based on Kilpatrick et al.'s (2001) model (Putting it all together)

This "classroom Observation Protocol" calls for rating on the entwined teaching proficiency competencies: teacher's mathematical interactions with students to foster attainment of mathematical proficiency, nature of mathematical tasks and justifications, instructional routines and teacher's character traits in the service of mathematics as well as the classroom culture that facilitates the teaching-learning process. The protocol or rubric presented to you also indicates how the high quality lesson was structured and implemented in a manner which engages students with important geometric concepts that are likely to enhance their understanding of those concepts and develop capacity to do mathematics successfully.

Thus, in this exercise I would like you to use this tool in interpreting your teaching practices. You have now had the opportunity to watch/observe your lesson video and of others. You also explored what each of you was thinking as the lesson unfolded. For this task, I ask you to **put it all together**, highlighting the "story" of the lesson(s) and providing a "tag line" (teaching proficiency strand) that together communicate the *narrative account* that you provided earlier about the overall account of the lesson. I would also ask you to provide any additional information you would like to share about each lesson, and let me know how you think these lessons would make interesting vignettes.

Task 5: Reflection on the research process

I would like each of you to consider the following questions to wrap up the analysis sessions.

- What have you learned from this research process?
- How useful is the analytical tool/rubric (Kilpatrick framework) (**Strengths**)?
- Record what you feel are the **weaknesses** of the tool…something that the tool cannot capture/show e.g. teacher's smile on the face.

Thank you so much for your time, helpful ideas and contribution towards the success of this project. If I have any additional questions or need clarification, how and when is it best to contact you?

Appendix K: A note on lesson videos and interview CDs

An accompanying CD provides rich illustrations of classroom episodes or teachers' interactions with students, all clearly linked to the thesis. However, the appendix/addendums of lesson videos and transcripts will be available on request to the examiners. I decide not to append them due to the length of sessions and higher graphic volumes on all these lesson videos. The interview recordings and transcript CDs are also attached.