Stone spectra of von Neumann algebras and foundations of quantum theory

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Einleitung

Eine Art Lied

Lass die Schlange warten unter
ihrem Unkraut
und das Schreiben
sei von Worten, sacht und schnell, scharf
zu treffen, still zu warten,
schlaflos.

-durch Metapher zu versöhnen
die Menschen und die Steine.
Verfass'. (Ideen gibt's
nur in den Dingen) Erfind'!
Steinbrech, meine Blume, sie spaltet
Fels.[1]

William Carlos Williams

Ideen gibt's nur in den Dingen – es könnte kein knapperes einleitendes Motto für eine Arbeit in Mathematischer Physik geben. Nicht alle Mathematik hat physikalische Bedeutung und Auswirkung, doch nahmen ganz offensichtlich viele der schönsten und erfolgreichsten mathematischen Entwicklungen ihren Ausgang von physikalischen Betrachtungen, vom Betrachten der *Dinge*.

In seinem Gedicht spricht W. C. Williams wohl von der Arbeit eines Dichters, oder genauer davon, was er als die Aufgabe eines Dichters sieht, doch ist es verblüffend, wie nahe er der Beschreibung der Aufgabe eines Physikers and Mathematikers kommt (zumindest eines Mathematikers mit Interesse an Physik): ausgehend von den Dingen gewinne die Ideen durch Komponieren und Erfinden. Dies ist die Herausforderung, der sich sowohl ein Künstler als auch ein Wissenschaftler in der Arbeit mit ihrem Material stellen müssen, und offenbar ist mehr gemeint als die Methode der Induktion: "Erfind'!" unterscheidet sich deutlich von Newtons "hypotheses non fingo"; die Ideen wohnen den Dingen inne, aber sie sind nicht dasselbe wie die Dinge, noch können sie einfach aus ihnen gelesen werden.

[1] Übersetzung vom Verfasser. Original siehe Kapitel 1, "Introduction".

Andererseits (überflüssig, dies Mathematikern zu erklären,) gibt es eine lange und fruchtbare Tradition in der Wissenschaft, bei der das Augenmerk nicht auf der realen Welt liegt, sondern auf der guten *Form*. In der Mathematik war dies immer eine der wichtigsten treibenden Kräfte, aber diese Haltung ist auch der modernen Physik nicht fremd. Ein Grund hierfür ist, dass unsere jetzigen physikalischen Theorien zu erfolgreich sind. Das gilt in dem Sinn, dass es praktisch keine Experimente gibt, die die Grenzen dieser Theorien aufzeigen. Dabei handelt es sich um die Allgemeine Relativität zur Beschreibung der Gravitation und um das Standardmodell der Teilchenphysik, das die übrigen drei fundamentalen Kräfte beschreibt. Heute versucht man in der Grundlagenphysik, diese extrem genauen Theorien zu überwinden und sie in eine größere Theorie einzuschließen, die ominöse Theorie der Quantengravitation oder gar Theorie für Alles. Da es zum ersten Mal in der Geschichte der Physik keine experimentellen Vorgaben gibt, muss man sich anderen Kriterien wie guter Form, Schönheit und struktureller Reichhaltigkeit zuwenden. Natürlich haben diese immer eine große Rolle gespielt, aber heute sind sie gleichsam zur einzigen Richtschnur geworden.

In der Physik bleibt das Ergebnis recht ernüchternd, trotz harter Arbeit sind die Fortschritte klein. Es gibt einige faszinierende mathematische Ableger aus der theoretischen Physik, aber die Gemeinde, die an Physik jenseits des Standardmodells arbeitet, hat bisher nur eine sehr geringe Anzahl physikalischer Vorhersagen hervorgebracht, die in absehbarer Zukunft experimentell überprüfbar sein werden.

Betrachtet man diese Zugänge zu einer weiterführenden Theorie, drängt sich der Eindruck auf, dass das Problem viel eher in der Quantentheorie liegt als in der Relativitätstheorie. Ist eine Theorie der Quantengravitation das ultimative Ziel in der Grundlagenphysik, dann müssen wir sicher die Quantentheorie wesentlich besser verstehen. Wie Fuchs angemerkt hat [Fuc03], gab es nie Konferenzen zur Bedeutung und Interpretation der Relativitätstheorie, jedoch eine unaufhörliche Reihe solcher Konferenzen zur Quantentheorie.

Der hier gewählte Zugang besteht darin, wieder einige elementarere Punkte der Quantentheorie zu betrachten. Es mag sogar schwieriger sein, Fortschritte beim Verständnis dieser elementaren Fragen zu erzielen, als eine spekulative Theorie der Quantengravitation zu entwickeln, doch haben wir so wenigstens die Chance zu wissen, wann wir falsch liegen. Ausgehend von mathematischen Konzepten, die von der Physik unabhängig sind, wird ein neuer mathematischer Blick auf einige der elementaren Begriffe der Quantentheorie entwickelt. Die hier dargestellten Betrachtungen fußen nicht auf irgendwelchen unüberprüften physikalischen Annahmen oder vorläufigen Theorien, sondern auf der Mathematik, die allen erfolgreichen und etablierten Quantentheorien zugrundeliegt: Hilbertraum-Techniken und Operator-Algebren. Ein weiterer wichtiger mathematischer Bestandteil ist die Verbandstheorie. Es liegt anscheinend eine Menge mathematischer Struktur in der Quantentheorie verborgen, die bisher nicht betrachtet worden ist. Hier stellen wir keine gänzlich neue Theorie dar (wir haben keine), sondern präsentieren einige mathematische Resultate, die schließlich helfen mögen, ein verbessertes Bild der Quantentheorie zu entwickeln.

Die mathematischen Begriffe, auf denen diese Arbeit basiert −Stonesche Spektren und observable Funktionen−, wurden von de Groote entwickelt. Seine Arbeit ist bis heute größtenteils unveröffentlicht, was es erforderlich macht, die Resultate hier in angemessener Breite darzustellen. In Kapitel 2, "Foundations - Stone spectra and observable functions", wird die Theorie in dem Umfang entwickelt, den wir in den nachfolgenden Kapiteln brauchen. Die Ergebnisse in Kapitel 2 stammen von de Groote. Die einzigen Ausnahmen bilden Abschnitt 2.6, "Generalization to categories", von P. Krallmann und Unterabschnitt 2.10.2, "The Stone spectrum of a finite direct sum of von Neumann algebras", der vom Verfasser stammt. Kapitel 4, "First applications to physics", wurde vom Verfasser ausgehend von de Grootes Anregungen entwickelt. Die Hauptergebnisse findet man in Kapitel 3, "Stone spectra of finite von Neumann algebras", insbesondere in den Abschnitten 3.2, "The Stone spectrum of a type I_n von Neumann algebra", und 3.3, "The Stone spectrum of a type II_1 factor". Kapitel 5, "The Kochen-Specker theorem", enthält die weiteren Hauptergebnisse. Es kann weitgehend unabhängig gelesen werden, die Beweise verwenden vornehmlich funktionalanalytische Methoden. Die physikalische Bedeutung des Kochen-Specker-Theorems und seiner Verallgemeinerung wird in Sektion 5.1 ausführlich dargelegt. Kapitel 5 ist eng verwandt mit einem Artikel, der im Web verfügbar ist und zur Veröffentlichung im *International Journal of Theoretical Physics* angenommen wurde [Doe04].

Um auf die Eingangsbemerkungen zurückzukommen, sei erwähnt, dass diese Arbeit versucht, die Kluft zwischen reiner Suche nach guter Form und Betrachtungen der realen, physikalischen Welt zu überbrücken. Zumindest das in Kapitel 5 bewiesene verallgemeinerte Kochen-Specker-Theorem hat unmittelbare Bedeutung für die Modellbildung in der Physik, weil es einen wesentlichen Unterschied zwischen klassischen und Quantentheorien aufzeigt: es gibt kein Modell der Quantentheorie, so dass alle Observablen gleichzeitig einen Wert haben. Das gibt uns ein wenig Einsicht in die Natur der "Quantendinge".

"Es scheint eine der grundlegenden Eigenschaften der Natur zu sein, dass grundlegende physikalische Gesetze in Formen von großer Schönheit und Kraft beschrieben sind... Im Lauf der Zeit wird zunehmend klarer, dass die Regeln, die der Mathematiker interessant findet, dieselben sind, die die Natur gewählt hat."

P. A. M. Dirac

Zusammenfassung des Inhalts

Kapitel 1, Einleitung

Kapitel 1 der Arbeit entspricht obigem nicht nummerierten Einleitungskapitel. (Zur Benennung: Kapitel entspricht chapter, Abschnitt entspricht section und Unterabschnitt subsection).

Kapitel 2, Grundlagen - Stonesche Spektren und observable Funktionen

Das gesamte Kapitel 2, "Foundations - Stone spectra and observable functions" dient der Darstellung der von de Groote entwickelten Grundlagen der Theorie, auf denen der Rest der Arbeit basiert. Wichtige Referenzen sind der Artikel [deG01] sowie, insbesondere ab Abschnitt 2.7, die kommende Veröffentlichung [deG05]. Wir stellen nicht alle Details dar und verweisen ausdrücklich auf diese Arbeiten.

Nach Festlegung einiger Notationen und Konventionen in Abschnitt 2.1 wird in Abschnitt 2.2 die garbentheoretische Motivation erläutert, die ursprünglich zur Definition des Stoneschen Spektrums eines Verbands führte: Während es leicht möglich ist, den Begriff einer **Prägarbe** auf Verbände zu verallgemeinern (Def. 38), existieren auf vielen interessanten Verbänden keine vollständigen Prägarben, d.h. **Garben**, siehe z.B. Thm. 40 für den in der Quantentheorie wichtigen Verband $\mathbb{L}(\mathcal{H})$ der abgeschlossenen Unterräume des Hilbertraums $\mathcal{H}$. Um dieses Problem zu umgehen, will man die sog. *Garbifizierung* [MacMoe92, ConDeG94], die einer Prägarbe eine Garbe zuordnet, verallgemeinern. Die Garbifizierung beruht auf einer Keimbildung. Dabei stellt sich insbesondere die Frage, wie man in einem Verband geeignet *lokalisieren* kann (was in einem topologischen Raum kein Problem darstellt).

In Abschnitt 2.3 werden zunächst die bekannten Definitionen eines Verbands und eines Verbandsmorphismus mit entsprechenden Beispielen gegeben. Es folgen die Definitionen eines Punktes in einem Verband, Def. 7, und insbesondere die eines Quasipunkts in einem Verband, Def. 12:

Definition *Eine Untermenge $\mathfrak{B}$ eines (mindestens σ-vollständigen) Verbandes $\mathbb{L}$ heißt ein **Quasipunkt von** $\mathbb{L}$, wenn $\mathfrak{B}$ folgende Eigenschaften hat:*

(i) $0 \notin \mathfrak{B}$,

(ii) $\forall a, b \in \mathfrak{B}\ \exists c \in \mathfrak{B} : c \leq a \wedge b$,

(iii) $\mathfrak{B}$ *ist maximal bezüglich (i) und (ii).*

Dieser Begriff ist zentral für alle weiteren Betrachtungen. Quasipunkte sind maximale Filterbasen und maximale duale Ideale in dem Verband $\mathbb{L}$. Es handelt sich um eine direkte Verallgemeinerung der von Stone [Sto36] betrachteten maximalen dualen Ideale in *distributiven* Verbänden. Abschnitt 2.3 enthält drei Beispiele von Verbänden und ihren zugehörigen Quasipunkten.

In Abschnitt 2.4 wird der Raum $\mathcal{Q}(\mathbb{L})$ mit einer ebenfalls durch Stone inspirierten Topologie versehen, wobei die Mengen von der Form $\mathcal{Q}_a(\mathbb{L}) := \{\mathfrak{B} \in \mathcal{Q}(\mathbb{L}) \mid a \in \mathfrak{B}\}$ eine Basis der Topologie bilden. Diese Mengen sind auch abgeschlossen, und $\mathcal{Q}(\mathbb{L})$, genannt das **Stonesche Spektrum von** $\mathbb{L}$, wird mit dieser Topologie zu einem null-dimensionalen, vollständig regulären Hausdorffraum (Lemma 31 und Rem. 32). **Atomare Quasipunkte** werden definiert als isolierte Punkte des Stoneschen Spektrums (Def. 33).

Abschnitt 2.5 kommt auf die Garbentheorie zurück und zeigt, dass die Quasipunkte ein geeignetes Werkzeug zur Lokalisierung in einem Verband darstellen, das über eine Äquivalenzklassenbildung die Definition von *Keimen* und *Halmen* erlaubt (Def. 41). Damit gelingt die angestrebte Verallgemeinerung der bisher nur für Prägarben auf topologischen Räumen definierten Garbifizierung. Einer Prägarbe $\mathcal{P}$ auf einem Verband $\mathbb{L}$ wird eine Garbe auf dem Stoneschen Spektrum $\mathcal{Q}(\mathbb{L})$ zugeordnet (Def. 43).

In Abschnitt 2.6, der auf P. Krallmanns Diplomarbeit [Kra04] basiert, wird kurz gezeigt, wie sich Quasipunkte und Stonesche Spektren auf kleine Kategorien verallgemeinern lassen. Das in Kapitel 5 wichtige Beispiel der Kategorie $\mathfrak{A}(\mathcal{R})$ der abelschen Unteralgebren einer von Neumann-Algebra $\mathcal{R}$ wird eingeführt und atomare Quasipunkte von $\mathfrak{A}(\mathcal{R})$ werden klassifiziert.

Beginnend mit den Abschnitten 2.7 und 2.8 konzentrieren wir uns im Folgenden auf den Projektionenverband $\mathcal{P}(\mathcal{R})$ einer von Neumann-Algebra $\mathcal{R}$. Die Abschnitte 2.7 und 2.8 basieren auf de Grootes kommender Veröffentlichung [deG05] und referieren lediglich einige der dort ausführlich dargestellten Ergebnisse. Zur Notation: das Stonesche Spektrum $\mathcal{Q}(\mathcal{P}(\mathcal{R}))$ wird geschrieben als $\mathcal{Q}(\mathcal{R})$, ein Quasipunkt von $\mathcal{R}$ ist ein Quasipunkt von $\mathcal{P}(\mathcal{R})$. Grundlegend ist Def. 56:

Definition *Sei $A \in \mathcal{R}_{sa}$ ein selbstadjungierter Operator in der von Neumann-Algebra $\mathcal{R}$ und sei $E^A = (E^A_\lambda)_{\lambda\in\mathbb{R}}$ die Spektralschar von A. Die Funktion*

$$f_A : \mathcal{Q}(\mathcal{R}) \longrightarrow \mathbb{R},$$

gegeben durch

$$f_A(\mathfrak{B}) := \inf\{\lambda \in \mathbb{R} \mid E^A_\lambda \in \mathfrak{B}\},$$

*heißt **die zu A gehörige observable Funktion**.*

Da die Spektralschar E^A eindeutig aus der observablen Funktion f_A rekonstruiert werden kann, ist die Abbildung $A \mapsto f_A$ injektiv. Observable Funktionen haben einige interessante Eigenschaften: es gilt $\operatorname{im} f_A = \operatorname{sp} A$ (Thm. 57) und $f_A : \mathcal{Q}(\mathcal{R}) \to \mathbb{R}$ ist stetig (Thm. 63). Andererseits sind nur dann *alle* beschränkten reellwertigen stetigen Funktionen auf $\mathcal{Q}(\mathcal{R})$ observabel, wenn $\mathcal{R}$ abelsch ist (Thm. 65).

In Unterabschnitt 2.7.2 wird zunächst der Definitionsbereich einer observablen Funktion vom Stoneschen Spektrum $\mathcal{Q}(\mathcal{R})$ auf den Raum $\mathcal{D}(\mathcal{R})$ aller dualen Ideale in $\mathcal{P}(\mathcal{R})$

erweitert. $f_A : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$ ist nach oben halbstetig (Prop. 69) und hat die **Durchschnittseigenschaft** (Prop. 68): Sei $(\mathcal{I}_j)_{j\in J}$ eine Familie in $\mathcal{D}(\mathcal{R})$. Dann gilt

$$f_A(\bigcap_{j\in J} \mathcal{I}_j) = \sup_{j\in J} f_A(\mathcal{I}_j).$$

Eine **abstrakte observable Funktion** wird definiert als nach oben halbstetige Funktionen $f : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$ mit der Durchschnittseigenschaft (Def. 71). Man kann zeigen, dass es zu jeder abstrakten observablen Funktion f einen selbstadjungierten Operator $A \in \mathcal{R}_{sa}$ gibt, so dass $f_A = f$ gilt (Thm. 73). Die Begriffe der observablen Funktion und der abstrakten observablen Funktion fallen also zusammen.

Der folgende Abschnitt 2.8 versammelt Ergebnisse zu abelschen von Neumann-Algebren $\mathcal{M}$, z.T. gesehen als Unteralgebren einer nicht-abelschen von Neumann-Algebra $\mathcal{R}$. Ein **Boolescher Sektor** wird definiert als ein maximaler distributiver Unterverband $\mathbb{B}$ des Projektionenverbands $\mathcal{P}(\mathcal{R})$ einer von Neumann-Algebra $\mathcal{R}$ (Def. 74). Ein **Boolescher Quasipunkt** $\beta \subset \mathcal{P}(\mathcal{R})$ hat die Eigenschaften eines Quasipunkts von $\mathcal{P}(\mathcal{R})$ und erfüllt zusätzlich die Bedingung, dass die in β enthaltenen Projektionen vertauschen (Def. 76). Offensichtlich ist jeder Boolesche Quasipunkt von $\mathcal{P}(\mathcal{R})$ in einem Quasipunkt enthalten. Ein Boolescher Quasipunkt ist ein Quasipunkt in der von einem Booleschen Sektor $\mathbb{B}$ erzeugten maximal abelschen Unteralgebra $\mathcal{M}(\mathbb{B})$ von $\mathcal{R}$ (Prop. 75 und Rem. 78).

Unterabschnitt 2.8.1 enthält einen wichtigen Satz, der bereits in [deG01] bewiesen ist:

Theorem *Sei $\mathcal{R}$ eine von Neumann-Algebra, $\mathbb{B}$ ein Boolescher Sektor von $\mathcal{P}(\mathcal{R})$, $\mathcal{M}(\mathbb{B})$ die maximal abelsche von Neumann-Unteralgebra, die von $\mathbb{B}$ erzeugt wird. Dann ist das Stonesche Spektrum $\mathcal{Q}(\mathbb{B})$ homöomorph zum Gelfand-Spektrum $\Omega(\mathcal{M}(\mathbb{B}))$ von $\mathcal{M}(\mathbb{B})$.*

Wenn $\mathcal{R}$ abelsch ist, also $\mathbb{B} = \mathcal{P}(\mathcal{R})$ gilt, dann sind Gelfand- und Stone-Spektrum von $\mathcal{R}$ homöomorph. Für eine beliebige von Neumann-Algebra $\mathcal{R}$ ist das Stonesche Spektrum $\mathcal{Q}(\mathcal{R})$ eine Verallgemeinerung des Gelfand-Spektrums.

Nachfolgend wird in 2.8.2 der Beweis skizziert, dass für eine abelsche von Neumann-Algebra $\mathcal{M}$ die Abbildung $\mathcal{M}_{sa} \to C(\mathcal{Q}(\mathcal{M}), \mathbb{R})$, $A \mapsto f_A$ die Einschränkung der Gelfand-Transformation auf $\mathcal{M}_{sa}$ ist (Thm. 83). Damit ist klar, dass für allgemeines $\mathcal{R}$ die Abbildung $A \mapsto f_A$ eine Verallgemeinerung der Gelfand-Transformation darstellt.

In 2.8.3 wird gezeigt, dass eine abelsche von Neumann-Algebra $\mathcal{M}$ als maximal abelsche von Neumann-Unteralgebra von $\mathcal{L}(\mathcal{K})$ dargestellt werden kann für einen geeigneten Hilbertraum $\mathcal{K}$. 2.8.4 enthält den Beweis, dass es zu jedem kompakten Hausdorffraum X einen Hilbertraum $\mathcal{H}$, einen Booleschen Sektor $\mathbb{B} \subseteq \mathcal{L}(\mathcal{H})$ und eine stetige identifizierende surjektive Abbildung $\pi : \mathcal{Q}(\mathbb{B}) \to X$ gibt.

In Abschnitt 2.9 wird aufgezeigt, in welcher Weise klassische und Quantenobservablen strukturell gleichartig sind: bisher hatten wir gesehen, dass selbstadjungierte Operatoren, also Quantenobservablen, als Spektralscharen und als observable Funktionen aufgefasst werden können. Umgekehrt wird jetzt der Begriff der Spektralschar auf beliebige Verbände verallgemeinert (Def. 89) und die von einer solchen Spektralschar induzierte Funktion definiert (Def. 91). Die Spektralscharen, die stetige Funktionen induzieren, werden charakterisiert (Def. 93 und Thm. 96). Klassische Observablen, hier aufgefasst als stetige Funktionen, lassen sich also auch durch Spektralscharen darstellen.

Schließlich enthält Abschnitt 2.10 einige weitere einfache Resultate zu Stoneschen Spektren, und zwar zum Stoneschen Spektrum von $\mathcal{L}(\mathcal{H})$, zur endlichen direkten Summe von von Neumann-Algebren und zur Wirkung der unitären Gruppe $\mathcal{U}(\mathcal{R})$ auf $\mathcal{Q}(\mathcal{R})$.

Kapitel 3, Stonesche Spektren endlicher von Neumann-Algebren

In diesem Kapitel wird die Struktur der Quasipunkte *endlicher* von Neumann-Algebren, genauer Typ-I_n-Algebren (Abschnitt 3.2) und Typ-II_1-Faktoren (Abschnitt 3.3), betrachtet. Da Quasipunkte mittels des Zornschen Lemmas definiert sind und damit unübersichtlich, braucht man zusätzliche Strukturen der jeweiligen von Neumann-Algebra, um die Eigenschaften der Quasipunkte und des Stoneschen Spektrums aufzuklären. Einleitend werden kurz einige grundlegende Fakten zur Klassifikation von von Neumann-Algebren dargestellt. Insbesondere sind endliche von Neumann-Algebren solche, bei denen die Identität I eine endliche Projektion ist.

Die Ergebnisse in Abschnitt 3.1 gelten für beliebige von Neumann-Algebren (und nicht nur für endliche). Zu Beginn wird der Begriff eines **abelschen Quasipunktes** eingeführt (Def. 106): ein Quasipunkt heißt abelsch, wenn er eine abelsche Projektion enthält. Sei $\mathfrak{B} \in \mathcal{Q}_E(\mathcal{R})$ ein Quasipunkt, der E enthält. Der E-**Stamm** $\mathfrak{B}_E$ von $\mathfrak{B}$ ist definiert als $\mathfrak{B}_E := \{F \in \mathfrak{B} \mid F \leq E\}$ (Def. 107). $\mathfrak{B}_E$ legt $\mathfrak{B}$ eindeutig fest (Lemma 108). Ist $\theta \in \mathcal{R}$ eine partielle Isometrie mit $\theta^*\theta = E$ und ist $\mathfrak{B}$ ein Quasipunkt, der E enthält, dann ist $\theta(\mathfrak{B}_E) := \{\theta F \theta^* \mid F \in \mathfrak{B}_E\}$ der $\theta E \theta^*$-Stamm eines Quasipunkts. Der davon induzierte Quasipunkt wird notiert als $\theta_{\mathcal{Q}}(\mathfrak{B})$.

Seien $\mathfrak{B}$ ein abelscher Quasipunkt und $\mathcal{C} = \mathcal{C}(\mathcal{R})$ das Zentrum von $\mathcal{R}$. $\mathcal{C}$ ist selbst eine (abelsche) von Neumann-Algebra. Es wird gezeigt, dass $\mathfrak{B} \cap \mathcal{C}$ ein Quasipunkt von $\mathcal{C}$ ist, dabei spielt Prop. 5.5.5 aus [KadRinI97] die zentrale Rolle. Man kann weiter zeigen, dass die Abbildung

$$\zeta : \mathcal{Q}^{ab}(\mathcal{R}) \longrightarrow \mathcal{Q}(\mathcal{C}), \quad \mathfrak{B} \longmapsto \mathfrak{B} \cap \mathcal{C}$$

von den abelschen Quasipunkten auf die Quasipunkte des Zentrums eine (surjektive) Abbildung ist, für die $\zeta(\mathfrak{B}) = \zeta(\mathfrak{B}')$ genau dann gilt, wenn es eine partielle Isometrie θ gibt, so dass $\theta_{\mathcal{Q}}(\mathfrak{B}) = \mathfrak{B}'$ ist (Thm. 115).

In Abschnitt 3.2 werden von Neumann-Algebren vom Typ I_n betrachtet. Eine solche Algebra ist bekanntlich von der Form $\mathcal{R} \simeq \mathbb{M}_n(\mathcal{A})$, wobei $\mathcal{A} := \mathcal{C}(\mathcal{R})$ das Zentrum von $\mathcal{R}$ ist. Diese Matrixstruktur nutzen wir im Folgenden aus, wobei $\mathbb{M}_n(\mathcal{A})$ auf den freien rechten *Hilbertmodul* $\mathcal{A}^n$ über $\mathcal{A}$ wirkt. Ziel ist es, am Ende des Kapitels eine Äquivalenzrelation einzuführen, die $\mathcal{A}$ zu einem Körper macht und $\mathcal{A}^n$ zu einem n-dimensionalen Vektorraum, so dass man die Ergebnisse zu Quasipunkten des Verbands $\mathbb{L}(V)$ der abgeschlossenen Unterräume eines endlichdimensionalen Vektorraums V benutzen kann. Ein solcher Quasipunkt ist immer von der Form $\mathfrak{B}_{\mathbb{C}x} := \{U \in \mathbb{L}(V) \mid \mathbb{C}x \subseteq U\}$, also durch eine Gerade $\mathbb{C}x$ in V bestimmt. Die Projektion $P_{\mathbb{C}x}$ von V auf $\mathbb{C}x$ ist eine abelsche Projektion in $\mathcal{L}(V)$. Daher werden im Folgenden Projektionen auf "Geraden" in $\mathcal{A}^n$ betrachtet.

In Unterabschnitt 3.2.1 werden zunächst die wichtigsten Grundlagen zu Hilbertmoduln dargestellt. Insbesondere ist das $\mathcal{A}$-wertige Produkt $(\_|\_) : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}$, $(a,b) \mapsto \sum_{k=1}^n a_k^* b_k$ linear in der *zweiten* Variablen. Die Projektionen in $\mathbb{M}_n(\mathcal{A})$ werden identifiziert (Lemma 116). Anschließend werden Projektionen $E_a : \mathcal{A}^n \to \mathcal{A}^n$ auf "Geraden" der Form $a\mathcal{A}$ eingeführt und ihre Eigenschaften untersucht (Lemma 117 bis Rem. 120). Dabei ist $a \in \mathcal{A}^n$ so gewählt, dass $(a|a) \in \mathcal{A}$ eine Projektion ist, und damit definiert man

$$E_a : \mathcal{A}^n \longrightarrow \mathcal{A}^n, \quad b \longmapsto a(a|b).$$

Diese Projektionen sind abelsch (Lemma 117). Es handelt sich um Spezialfälle der sogenannten *ket-bra-Operatoren* zwischen Hilbertmoduln.

Unterabschnitt 3.2.2 ist ein technischer Abschnitt, in dem zunächst gezeigt wird, wie sich ein beliebiges Element $a \in \mathcal{A}^n \backslash \{0\}$ zu $\widetilde{a}$ normieren lässt, so dass $(\widetilde{a}|\widetilde{a})$ eine Projektion in $\mathcal{A}$ ist und $E_{\widetilde{a}}$ die (abelsche) Projektion von $\mathcal{A}^n$ auf $a\mathcal{A} = \widetilde{a}\mathcal{A}$. Dabei sind einige topologische Feinheiten zu beachten (Lemma 122, Prop. 123). Anschließend werden *projektive Untermoduln* $M = P\mathcal{A}^n$ betrachtet, wobei $P \in \mathbb{M}_n(\mathcal{A})$ eine Projektion ist. Der **Träger von** M ist definiert als

$$S(M) := \overline{\{\beta \in \Omega \mid \exists a \in M : a(\beta) \neq 0\}},$$

wobei Ω das Gelfand-Spektrum von $\mathcal{A}$ ist. Ist M endlich erzeugt, so gibt es ein $a \in M$ mit $S(M) = S(a) := \overline{\{\beta \in \Omega \mid a(\beta) \neq 0\}}$ (Cor. 128). Dieses Resultat wird anschließend verwendet, um Folgendes zu zeigen: die endlich erzeugten projektiven Untermoduln $M \subseteq \mathcal{A}^n$, für die $End_{\mathcal{A}}(M)$, die Menge der $\mathcal{A}$-linearen Abbildungen von M in sich, *abelsch* ist, sind genau von der Form $a\mathcal{A}$ (Prop. 130). Beim Beweis spielt außerdem die Beziehung $End^0_{\mathcal{A}}(M) = P\mathbb{M}_n(\mathcal{A})P$ (Lemma 2.18 aus [GVF01]) eine Rolle, wobei $End^0_{\mathcal{A}}(M)$ die *$\mathcal{A}$-kompakten Operatoren* sind.

All das dient der Vorbereitung des kommenden Unterabschnitts, wo Quasipunkte von $\mathcal{R} \simeq \mathbb{M}_n(\mathcal{A})$ als Familien projektiver Untermoduln von $\mathcal{A}^n$ statt als Familien von Projektionen betrachtet werden.

In 3.2.3 wird schließlich die anfangs erwähnte Äquivalenzrelation auf $\mathcal{A}^n$ definiert: $a, b \in \mathcal{A}^n$ heißen **äquivalent im Quasipunkt** $\beta \in \mathcal{Q}(\mathcal{A})$, wenn es ein $p \in \beta$ gibt, so dass $pa = pb$ gilt (Def. 131). Damit wird $[\mathcal{A}]_\beta$ zu einem Körper und $[\mathcal{A}^n]_\beta$ zu einem n-dimensionalen Vektorraum über $[\mathcal{A}]_\beta$ (Thm. 132). Anschließend zeigt man, dass $[M]_\beta \neq 0$ ist für $M = P\mathcal{A}^n \in \mathfrak{B}$ und dass $\mathfrak{B}_\beta := \{[M]_\beta \mid M \in \mathfrak{B}\}$ eine Filterbasis im Verband der abgeschlossenen Unterräume von $[\mathcal{A}^n]$ ist (Rem. 135). Sei $\widetilde{\mathfrak{B}}_\beta$ ein Quasipunkt, der $\mathfrak{B}_\beta$ umfasst. $\widetilde{\mathfrak{B}}_\beta$ enthält genau eine Gerade $[a_0]_\beta[\mathcal{A}]_\beta$, weil $[\mathcal{A}^n]_\beta$ ein endlichdimensionaler Vektorraum ist, dessen Quasipunkte wie eingangs erwähnt immer genau eine Gerade enthalten. Es bleibt zu zeigen, dass die Äquivalenzrelation nicht zu grob ist und tatsächlich auch $a_0\mathcal{A} \subseteq M$ gilt für alle $M \in \mathfrak{B}$. Man erhält Thm. 136:

Theorem *Alle Quasipunkte von $\mathbb{M}_n(\mathcal{A})$ sind abelsch.*

Aus der oben definierten Abbildung $\zeta : \mathcal{Q}^{ab}(\mathcal{R}) \longrightarrow \mathcal{Q}(\mathcal{C})$, $\mathfrak{B} \longmapsto \mathfrak{B} \cap \mathcal{C}$ von den abelschen Quasipunkten der von Neumann-Algebra auf die Quasipunkte des Zentrums und der Tatsache, dass sich in einer endlichen Algebra partielle Isometrien durch unitäre Operatoren ersetzen lassen, ergibt sich als zentrales Ergebnis dieses Abschnitts eine Charakterisierung der Wirkung der unitären Gruppe $\mathcal{U}(\mathcal{R})$ auf $\mathcal{Q}(\mathcal{R})$: jeder Quasipunkt $\beta \in \mathcal{Q}(\mathcal{C})$ des Zentrums entspricht einem Orbit dieser Wirkung (Thm. 138).

Das Stonesche Spektrum eines Faktors vom Typ II_1 wird in Abschnitt 3.3 betrachtet. Das wesentliche Werkzeug hierbei ist die **Spur** in Form der normierten **Dimensionsfunktion** $\Delta : \mathcal{P}(\mathcal{R}) \to [0,1]$. Diese Abbildung ist surjektiv. Zunächst wird gezeigt, dass ein Quasipunkt $\mathfrak{B}$ eines Faktors vom Typ II_1 keine minimalen Projektionen enthält, d.h., jede Projektion $E \in \mathfrak{B}$ hat stets eine Unterprojektion $F < E$ in $\mathfrak{B}$ (Lemma 140). Anschließend wird eine Metrik auf $\mathcal{Q}(\mathcal{R})$ eingeführt, die eine Topologie erzeugt, die feiner als die Stonesche Topologie ist (Prop. 141). Umgekehrt gilt das nicht, also stimmen die beiden Topologien nicht überein.

Prop. 142 zeigt, dass ein Quasipunkt eines Faktors vom Typ II_1 Projektionen aller Dimensionen im reellen Intervall $]0,1]$ enthält. Darüber hinaus enthält auch der Durchschnitt zweier Quasipunkte Projektionen aller Dimensionen (Prop. 148), und dieser Durchschnitt ist dicht in $\mathcal{P}(\mathcal{R})$ (Prop. 149). Die genannten Ergebnisse zeigen, dass das Stonesche Spektrum eine feine Invariante der Algebra ist, dessen Struktur sich mit Hilfe der Dimensionsfunktion nur teilweise auflösen lässt.

In Unterabschnitt 3.3.1 wird gezeigt, wie sich die Situation bei Booleschen Quasipunkten eines Typ-II_1-Faktors mittels zweier Resultate von Kadison klären lässt. Insbesondere enthält auch ein Boolescher Quasipunkt $\beta \subset \mathcal{P}(\mathcal{R})$ Projektionen aller Dimensionen im Intervall $]0,1]$ (Prop. 156), was über mehrere Zwischenschritte gezeigt wird. β ist dicht im Projektionenverband der von β erzeugten maximal abelschen Unteralgebra $\mathcal{M}$ des Typ-II_1-Faktors $\mathcal{R}$ (Prop. 157).

Kapitel 4, Erste Anwendungen in der Physik

In diesem kurzen Kapitel werden einige Anwendungen von Stoneschen Spektren und observablen Funktionen in der Physik dargestellt. Nach Klärung der verwendeten Konventionen und physikalischen Sprechweisen in Abschnitt 4.1 wird in 4.2 der *harmonische Oszillator* betrachtet, das heißt, das durch den Hamilton-Operator $H = \frac{1}{2}(p^2 + q^2)$ bestimmte physikalische System. Ausgehend vom (als bekannt vorausgesetzten) Spektrum $\operatorname{sp} H = \{n + \frac{1}{2} \mid n = 0, 1, 2, ...\}$ und der Spektralschar von H wird die zugehörige observable Funktion f_H bestimmt und ihr Werteverhalten auf $\mathcal{Q}(\mathcal{L}(\mathcal{H}))$ betrachtet. Sei $(e_n)_{n \in \mathbb{N}}$ die Orthonormalbasis von $\mathcal{H}$, die aus Eigenvektoren von H besteht, wobei e_n der Eigenvektor zum Eigenwert $n + \frac{1}{2}$ ist, sei $x \in \mathcal{H}$, $x = \sum_{j=1}^{n} a_j e_{i_j}$, und sei $\mathfrak{B}_{\mathbb{C}x} := \{P \in \mathcal{P}(\mathcal{L}(\mathcal{H})) \mid P_{\mathbb{C}x} \leq P\}$

der (atomare) Quasipunkt über der Geraden $\mathbb{C}x$. Dann gilt

$$f_H(\mathfrak{B}_{\mathbb{C}x}) = \max(i_1, ..., i_n) + \frac{1}{2}.$$

In Abschnitt 4.3 werden zwei verschiedene Möglichkeiten dargestellt, quantenmechanische Erwartungswerte auszudrücken. Ein solcher Erwartungswert ist ein Ausdruck der Form $\langle Ax, x\rangle$ oder allgemeiner $tr(\rho A)$, wobei $A \in \mathcal{R}_{sa}$ eine Observable ist und $P_{\mathbb{C}x}$ bzw. ρ den *Zustand* des physikalischen Systems beschreiben. Die betrachtete von Neumann-Algebra ist $\mathcal{L}(\mathcal{H})$. Die erste Möglichkeit besteht darin, eine bestimmte Prägarbe auf $\mathcal{P}(\mathcal{R})$ zu betrachten, die jeder Projektion E den Raum $\mathcal{L}(E\mathcal{H})$ zuordnet und deren Einschränkungsabbildung die Einschränkung von Operatoren ist,

$$\rho^E_F : \mathcal{L}(E\mathcal{H}) \longrightarrow \mathcal{L}(F\mathcal{H}), \quad A \longmapsto FAF.$$

Der Keim dieser Prägarbe im atomaren Quasipunkt $\mathfrak{B}_{\mathbb{C}x}$ ist gerade $\langle Ax, x\rangle$. Die zweite Möglichkeit wird in 4.3.2 beschrieben. Es wird benutzt, dass $f_A(\mathfrak{B}_{\mathbb{C}x})$ die kleinste obere Schranke ist, so dass für alle $\mu \geq f_A(\mathfrak{B}_{\mathbb{C}x})$ gilt $E^A_\lambda x = x$. Daher lässt sich im gewöhnlichen Integralausdruck für einen Erwartungswert $\langle Ax, x\rangle$ die obere Schranke $|A|$ durch $f_A(\mathfrak{B}_{\mathbb{C}x})$ ersetzen:

$$\langle Ax, x\rangle = tr(\rho_{\mathbb{C}x} A) = \int_{-|A|}^{f_A(\mathfrak{B}_{\mathbb{C}x})} \lambda \; d\left\langle E^A_\lambda x, x\right\rangle.$$

Abschnitt 4.4 zeigt, wie sich die Zeitentwicklung der observablen Funktionen, vermittelt durch eine unitäre Einparameter-Gruppe, auch durch die Wirkung derselben unitären Gruppe auf das Stonesche Spektrum ausdrücken lässt. Im Fall $\mathcal{R} = \mathcal{L}(\mathcal{H})$ lässt sich das als Wechsel zwischen dem *Heisenberg*- und dem *Schrödingerbild* auffassen.

Kapitel 5, Das Kochen-Specker-Theorem

Das abschließende Kapitel behandelt das Kochen-Specker-Theorem, das ausschließt, dass die Quantentheorie eine Phasenraumformulierung besitzt. Der hypothetische Phasenraum wird allgemein als Raum der *versteckten Zustände* bezeichnet. Eng damit verknüpft ist die Frage nach der Existenz sog. **Bewertungsfunktionen** $v : \mathcal{R}_{sa} \to \mathbb{R}$, die jeder Observablen $A \in \mathcal{R}_{sa}$ einen Wert $v(A) \in \mathrm{sp}\, A$ zuweisen, und zwar in der Art, dass funktionale Relationen zwischen den Operatoren erhalten bleiben (s.u.). Aus der Nicht-Existenz von Bewertungsfunktionen folgt das Kochen-Specker-Theorem. Abschnitt 5.1 enthält eine ausführliche Darstellung des mathematischen Problems und seiner physikalischen Bedeutung.

Das Kochen-Specker-Theorem wurde ursprünglich 1967 auf kombinatorischem Wege für von Neumann-Algebren der Form $\mathcal{R} = \mathcal{L}(\mathcal{H})$, also Faktoren vom Typ I_n, bewiesen [KocSpe67]. Die von Neumann-Algebra $\mathcal{R}$ ist dabei die Observablenalgebra des betrachteten physikalischen Systems. Wir wählen einen funktionalanalytischen Zugang und zeigen erstmals, dass das Kochen-Specker-Theorem allgemein für alle von Neumann-Algebren ohne Summanden vom Typ I_1 und I_2 gilt.

In Abschnitt 5.2 werden zunächst der bekannte Begriff eines *Wahrscheinlichkeitsmaßes* auf dem Projektionenverband $\mathcal{P}(\mathcal{R})$ einer von Neumann-Algebra $\mathcal{R}$ (Def. 159) und der Satz von Gleason angeführt (Thm. 160, [Gle57]), der die Wahrscheinlichkeitsmaße auf $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ charakterisiert. Es folgt die Definition einer Bewertungsfunktion (Def. 161):

Definition *Sei $\mathcal{H}$ ein separabler Hilbertraum, $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ eine von Neumann-Algebra. Eine **Bewertungsfunktion** ist eine Abbildung $v : \mathcal{R}_{sa} \to \mathbb{R}$, so dass folgende Bedingungen erfüllt sind:*
(a) $v(A) \in \operatorname{sp} A$ (die Spektrums-Regel) und
(b) für alle Borelfunktionen $f : \mathbb{R} \to \mathbb{R}$ gilt $v(f(A)) = f(v(A))$ (das FUNC-Prinzip).

Die zentrale Idee besteht darin zu zeigen, dass eine Bewertungsfunktion v in kanonischer Weise einen Charakter $v'|_{\mathcal{M}}$, also einen reinen Zustand, für jede abelsche Unteralgebra $\mathcal{M}$ von $\mathcal{R}$ induziert (Lemmata 163, 164). $v' : \mathcal{R} \to \mathbb{C}$ wird damit zu einem **Quasi-Zustand** im Sinn von Aarnes' Definition (Def. 165 und Lemma 166), und dieser Quasi-Zustand ist sogar ein Zustand der Algebra $\mathcal{R}$, wie man im Folgenden mit dem Satz von Gleason zeigt. Lemma 169 zeigt, dass ein normaler Zustand ϕ von $\mathcal{R}$ durch Einschränkung auf $\mathcal{R}_{sa}$ niemals eine Bewertungsfunktion induziert, wenn $\mathcal{R}$ nicht vom Typ I_1 ist. (Typ-I_1-Algebren sind abelsch.)

Unterabschnitt 5.2.2 behandelt Faktoren vom Typ I_n, $n \geq 3$. Mit Hilfe eines Resultats zu Booleschen Sektoren (Thm. 170, [deG01]) und dem Satz von Gleason wird gezeigt, dass der von einer Bewertungsfunktion $v : \mathcal{R}_{sa} \to \mathbb{R}$ induzierte Zustand v' für endliches n normal ist und im Widerspruch zur Spektrums-Regel gewissen Projektionen $P \in \mathcal{P}(\mathcal{R})$ weder 0 noch 1 zuweist. (Auch mit Lemma 169 erhält man einen Widerspruch). Der Fall $n = \infty$ folgt leicht; für Typ-I_n-Faktoren gibt es somit keine Bewertungsfunktion. Damit ist das "klassische" Kochen-Specker-Theorem neu bewiesen.

In 5.2.3 werden beliebige von Neumann-Algebren ohne Typ-I_2-Summanden behandelt. Um zu zeigen, dass eine Bewertungsfunktion auch in dieser allgemeinen Situation einen Zustand v' von $\mathcal{R}$ induziert, braucht man das Gleason-Christensen-Yeadon-Theorem (Thm. 172). Anders als für Typ-I_n-Faktoren ist der Zustand v' im allgemeinen nicht normal, man kann aber zeigen, dass v' ein *multiplikativer* Zustand ist (Lemma 173, [Ham93]). Multiplikative Zustände existieren nur auf abelschen, also Typ-I_1-Algebren (Lemmata 175, 176), also konzentriert sich der von einer Bewertungsfunktion induzierte Zustand auf den Typ-I_1-Summanden (wenn vorhanden) einer beliebigen von Neumann-Algebra $\mathcal{R}$ (Lemma 177). Es folgt das verallgemeinerte Kochen-Specker-Theorem (Thm. 178):

Theorem *Sei $\mathcal{R}$ eine von Neumann-Algebra ohne Typ-I_2-Summanden. Wenn $\mathcal{R}$ keinen Typ-I_1-Summanden besitzt, dann gilt das verallgemeinerte Kochen-Specker-Theorem (wie in der Einleitung, Abschnitt 5.1, beschrieben). Hat $\mathcal{R}$ einen Typ-I_1-Summanden, dann gibt es einen Raum versteckter Zustände wie in der Einleitung beschrieben, aber lediglich für den trivialen, abelschen Teil von $\mathcal{R}$.*

Isham, Butterfield und Hamilton haben in einer Reihe von Artikeln [IshBut98, IshBut99, HIB00, IshBut02] das (klassische) Kochen-Specker-Theorem umformuliert und dabei *Prä-*

garben über einer Kategorie verwendet (Def. 179). Das Kochen-Specker-Theorem entspricht der Aussage, dass bestimmte Prägarben keine globalen Schnitte besitzen, wie man aus dem FUNC-Prinzip ableitet. In Abschnitt 5.3 betrachten wir die sog. **spektrale Prägarbe** (Def. 181) über der Kategorie $\mathfrak{A}(\mathcal{R})$ der abelschen Unteralgebren von $\mathcal{R}$, was einen Vorschlag von Isham, Butterfield und Hamilton verallgemeinert. Die Tatsache, dass diese Prägarbe keinen globalen Schnitt hat, steht in enger Verbindung mit den in Lemma 164 gegebenen Argumenten.

Eine zweite Prägarbe über $\mathfrak{A}(\mathcal{R})$, die **Zustands-Prägarbe** (Def. 182), ist mit Hilfe Stonescher Spektren statt Gelfand-Spektren formuliert. Diese Prägarbe besitzt zwar globale Schnitte –jeder Zustand von $\mathcal{R}$ induziert einen–, aber keine Schnitte von der Art, wie sie von einer Bewertungsfunktion stammen. Da eine Bewertungsfunktion einen reinen Zustand auf jedem $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$ induziert (Lemma 164), müsste ein Schnitt dieser Prägarbe ausschließlich aus Punktmaßen auf den Stoneschen Spektren $\mathcal{Q}(\mathcal{M})$ der abelschen Unteralgebren $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$ bestehen. Das verallgemeinerte Kochen-Specker-Theorem schließt diese Möglichkeit aus.

Es folgt in Abschnitt 5.4 eine abschließende Diskussion. Tatsächlich ist sogar mehr gezeigt als das verallgemeinerte Kochen-Specker-Theorem: die Tatsache, dass es für von Neumann-Algebren ohne Summanden vom Typ I_2 und I_1 keine Bewertungsfunktionen gibt, bedeutet, dass kein realistisches Modell der Quantentheorie möglich ist, bei dem alle Observablen $A \in \mathcal{R}_{sa}$ gleichzeitig einen definierten Wert besitzen, so dass funktionale Relationen zwischen den Operatoren gemäß dem FUNC-Prinzip erhalten bleiben.

1. Introduction

A Sort of a Song

Let the snake wait under
his weed
and the writing
be of words, slow and quick, sharp
to strike, quiet to wait,
sleepless.

-through metaphor to reconcile
the people and the stones.
Compose. (No ideas
but in things) Invent!
Saxifrage is my flower that splits
the rocks.

William Carlos Williams

No ideas but in things – there could be no more concise motivational motto for a work in mathematical physics. Not all of mathematics has physical implications and meaning, yet obviously many of the most beautiful and most successful mathematical developments started from physical considerations, from looking at *things*. In his poem, W. C. Williams most probably speaks of a poet's work, or more precisely of what he sees as a poet's task, but it is striking how close he gets to describing the task of a physicist and mathematician (at least a mathematician interested in physics): starting from things, extract the ideas by composition and invention. This is the challenge both an artist and a scientist have to meet when working with their material, and it obviously does not simply mean the method of induction: "Invent!" is quite different from Newton's "hypotheses non fingo"; the ideas are *in* the things, but they are not the same as the things, nor can they simply be read from them.

On the other hand, (needless to explain this to mathematicians,) there is a long and fruitful tradition in science where the focus does not lie on the real world but on good *form*. In mathematics, this has always been one of the main driving forces, but this attitude is not alien to modern physics, too. One reason of course is that our current physical theories are too successful in the sense that there are practically no experiments showing the limits of these theories, which are General Relativity, describing gravity, and the Standard Model of particle physics, describing the other three fundamental forces. Today, fundamental physics tries to transcend these extremely accurate theories and incorporate them into a bigger theory, the ominous theory of quantum gravity or even the theory of everything. Since for the first time ever in physics, there is no experimental input, one has to appeal to other criteria like good form, beauty and structural richness. Of course, they have always played a major role, but today they are practically the only guiding lights.

In physics the results are somewhat sobering, despite hard work, progression has been slow. There have been some intriguing spin-offs from theoretical physics to mathematics, but the community working on physics beyond the Standard Model has produced only a very small number of physical predictions that could be tested experimentally in the foreseeable future.

When looking at these approaches, one cannot escape the impression that the problem lies within quantum theory rather than relativity. If a theory of quantum gravity is the ultimate goal in foundational physics, then we will surely have to understand quantum theory much better. As Fuchs remarked [Fuc03], there have never been conferences on the meaning and interpretation of relativity theory, but an unceasing series of such conferences on quantum theory.

The approach we take here is to go back to some of the more basic issues of quantum theory. It may even be harder to make progress in understanding basic issues in quantum mechanics than developing a speculative theory of quantum gravity, but at least we have some chance of knowing when we are going wrong. Starting from mathematical concepts which are independent of physics, a new mathematical view on some of the basic notions of quantum theory is developed. The considerations presented here are not based on any untested physical assumptions or tentative models, but on the mathematics underlying all successful and well-established quantum theories: Hilbert space techniques and operator algebras. Another important mathematical ingredient is lattice theory. There seems to be a lot of mathematical structure hidden in quantum theory that has not been considered up to now. We do not present a whole new theory (we have none), but show some mathematical results that may help to develop a revised picture of quantum theory sometime.

The mathematical notions on which this work is based, Stone spectra and observable functions, were developed by de Groote. His work is largely unpublished to date, which forces us to lay down the results in sensible detail. In chapter 2, "Foundations - Stone spectra and observable functions", the theory is developed to the extent that we will need in the following chapters. The results in chapter 2 are de Groote's. The only exceptions are section 2.6, "Generalization to categories", which is due to P. Krallmann, and subsec. 2.10.2, "The Stone spectrum of a finite direct sum of von Neumann algebras", which is by the author. Chapter 4, "First applications to physics", has been developed by the author starting from de Groote's suggestions. The main results can be found in chapter 3, "Stone spectra of finite von Neumann algebras", especially sections 3.2, "The Stone spectrum of a type I_n von Neumann algebra", and 3.3, "The Stone spectrum of a type II_1

factor". Chapter 5, "The Kochen-Specker theorem", contains the other main results. It can be read quite independently, the proofs mainly use functional analytic methods. The physical meaning of the Kochen-Specker theorem and its generalization is laid down in detail in section 5.1. Chapter 5 is closely related to an article that is available on the web and which has been accepted for publication by the *International Journal of Theoretical Physics* [Doe04].

Coming back to the introductory remarks, we should mention that this work tries to bridge the gap between pure search of good form and considerations concerning the real, physical world. At least, the generalized Kochen-Specker theorem proved in chapter 5 is of direct relevance for physical model making, since it shows a significant difference between classical and quantum theories: there is no model of quantum theory such that all the observables have a value at the same time. This gives some insight into the nature of "quantum things".

"It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of great beauty and power... As time goes on it becomes increasingly evident that the rules that the mathematician finds interesting are the same as those that Nature has chosen."

P. A. M. Dirac

2. Foundations – Stone spectra and observable functions

2.1. Notations and conventions

All Hilbert spaces $\mathcal{H}$ used here are complex and separable. All von Neumann and C^*-algebras are unital and are represented as subalgebras of some $\mathcal{L}(\mathcal{H})$, the algebra of bounded operators on an appropriate separable Hilbert space $\mathcal{H}$ such that the unit I of the algebra is the identity operator on $\mathcal{H}$. Projection means orthogonal projection onto a closed subspace of $\mathcal{H}$, i.e. a projection P is an idempotent, self-adjoint operator. All projections except the zero projection 0 are of norm 1 and all projections except 0 and the identity 1 have spectrum $\{0, 1\}$. $\mathcal{P}(\mathcal{R})$ denotes the lattice of projections of some von Neumann algebra $\mathcal{R}$, $\mathcal{P}_0(\mathcal{R}) := \mathcal{P}(\mathcal{R}) \backslash \{0\}$ and $\mathcal{C}(\mathcal{R})$ is the center of $\mathcal{R}$. We will write $\mathcal{C}$ if no confusion can arise. The group of unitary operators of a von Neumann algebra $\mathcal{R}$ is denoted by $\mathcal{U}(\mathcal{R})$, and we define $\mathcal{U}(\mathcal{H}) := \mathcal{U}(\mathcal{L}(\mathcal{H}))$. The real linear space of self-adjoint operators of a von Neumann algebra $\mathcal{R}$ is denoted by $\mathcal{R}_{sa}$.

2.1.1. Table of symbols

We merely list some notations used in this work which might be not completely customary:

$\amalg$	disjoint union
$\simeq$	bijection
$\mathcal{R}'$	commutant (centralisator) of the algebra $\mathcal{R}$
$\operatorname{sp} \mathcal{M}$	spectrum of the abelian (C^*- or von Neumann) algebra $\mathcal{M}$
$\operatorname{sp} A$	spectrum of the operator A
$\mathcal{T}(X)$	topology of X
$E^\perp := I - E$	complement of the projection E

2.2. Motivation

The theory of Stone spectra of lattices started from sheaf-theoretic considerations ([deG01]). The motivation came from physics: in order to reveal structural similarities between classical and quantum theoretical observables, de Groote used lattice- and sheaf-theoretic methods. This involved the introduction of presheaves on a lattice and lead to the question of how to define sheaves from these presheaves.

To do so, the notion of a point of a topological space must be generalized. A first ansatz is the definition of a *point of a lattice* $\mathbb{L}$ (Def. 7), formalizing the idea that a point in a topological space is characterized by the set of its open neighbourhoods, which can be phrased exclusively in lattice-theoretic terms. While this works reasonably well for topological spaces (provided that the topology is not too wild), it turns out that many important lattices have no points at all. The most prominent example is the lattice of closed subspaces of Hilbert space.

Instead, one can define the so-called *quasipoints* of a lattice (Def. 12), equip the set $\mathcal{Q}(\mathbb{L})$ of quasipoints with a canonical topology to obtain the *Stone spectrum* of $\mathbb{L}$ and define sheaves from presheaves on $\mathbb{L}$ by a generalization of the well-known process of sheafification (see [MacMoe92, ConDeG94]). Such a sheaf is defined not directly on $\mathbb{L}$, but on $\mathcal{Q}(\mathbb{L})$, and the sheafification involves a localizing procedure in a lattice. The "points" at which we will localize are the quasipoints.

The theory of sheaves on $\mathcal{Q}(\mathbb{L})$ is not really developed in this work (nor is it developed far at all), but since this is the starting point of the whole theory, we will present these constructions here. In subsection 4.3.1, "Expectation values as germs", we will employ the sheaf-theoretic constructions in a simple example. On the other hand, we will of course make extensive use of quasipoints and Stone spectra in the rest of this work.

A classical (physical) observable f is a real-valued function on the phase space of the physical system, either measurable, continuous or smooth, depending on the chosen formulation of the physical theory. The set of classical observables may be subject to symmetries, resulting in invariance properties of the functions. For example, in a radial-symmetric situation a classical observable would be a function $f(r)$ depending on the radius r only.

In contrast to that, a quantum theoretical observable A is a self-adjoint operator on a Hilbert space. (For more details, see sec. 4.1.) Possible symmetries are encoded in the von Neumann algebra $\mathcal{R}$ in which A is contained. (More precisely, the commutant $\mathcal{R}'$ of $\mathcal{R}$ encodes the symmetries of the quantum system described by $\mathcal{R}$.) In order to show that classical and quantum theoretical observables are not as different as it seems, we will now start to give the relevant lattice- and sheaf-theoretic definitions.

2.3. Points and quasipoints of a lattice

Definition 1 *A* ***lattice*** *is a partially ordered set* $(\mathbb{L}, \leq)$ *such that any two elements* $a, b \in \mathbb{L}$ *have a* ***maximum*** *(or* ***join****)* $a \vee b \in \mathbb{L}$ *and a* ***minimum*** *(or* ***meet****)* $a \wedge b \in \mathbb{L}$*. Moreover,* $\mathbb{L}$ *has a* ***zero element*** $0_{\mathbb{L}}$ *such that* $0_{\mathbb{L}} \leq a$ *for all* $a \in \mathbb{L}$ *and a* ***unit element*** $1_{\mathbb{L}}$ *such that* $a \leq 1_{\mathbb{L}}$ *for all* $a \in \mathbb{L}$*. Let* $\mathfrak{m}$ *be an infinite cardinal number.*

The lattice $\mathbb{L}$ *is called* $\mathfrak{m}$***-complete*** *if every family* $(a_i)_{i \in I}$ *has a maximum* $\bigvee_{i \in I} a_i$ *and a minimum* $\bigwedge_{i \in I} a_i$ *in* $\mathbb{L}$*, provided* $\#I \leq \mathfrak{m}$ *holds (*$\#I$ *denotes the cardinality of* I*). A lattice* $\mathbb{L}$ *is simply called* ***complete*** *if every family* $(a_i)_{i \in I}$ *in* $\mathbb{L}$ *(without any restriction on the cardinality of* I*) has a maximum and a minimum in* $\mathbb{L}$*.*

A lattice $\mathbb{L}$ *is called* ***distributive*** *if the two distributive laws*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

hold for all elements $a, b, c \in \mathbb{L}$.

If $\mathbb{L}$ *is equipped with an orthocomplement operation* ${}^c : \mathbb{L} \to \mathbb{L}$ *such that* $a^{cc} = a$, $a^c \vee a = 1_{\mathbb{L}}$, $a^c \wedge a = 0_{\mathbb{L}}$ *and* $(a \wedge b)^c = a^c \vee b^c$ *for all* $a, b \in \mathbb{L}$, *then* $\mathbb{L}$ *is called* ***orthocomplemented****. If, moreover,* $\mathbb{L}$ *is distributive, it is called a* ***Boolean lattice*** *(or* ***Boolean algebra****).*

The maximum $\bigvee_{i \in I} a_i$ is characterized by the following universal property:

(i) $\forall j \in I : a_j \leq \bigvee_{i \in I} a_i$,

(ii) $\forall c \in \mathbb{L} : ((\forall i \in I : a_i \leq c) \Longrightarrow \bigvee_{i \in I} a_i \leq c)$.

The maximum $\bigvee_{i \in I} a_i$ is the smallest element of the lattice larger than or equal to all the a_i. The universal property characterizing the minimum $\bigwedge_{i \in I} a_i$ is

(i) $\forall j \in I : a_j \geq \bigwedge_{i \in I} a_i$,

(ii) $\forall c \in \mathbb{L} : ((\forall i \in I : a_i \geq c) \Longrightarrow \bigwedge_{i \in I} a_i \geq c)$.

The minimum is the largest element of the lattice smaller than or equal to all the a_i.

Notation 2 *If we speak of a lattice in this work, we will always mean a lattice that is* σ*-complete, i.e.* $\aleph_0$*-complete, at least. When no confusion can arise, we will write* 0 *and* 1 *(instead of* $0_{\mathbb{L}}$ *and* $1_{\mathbb{L}}$*) for the zero element and the unit element, respectively.*

We will now introduce some of the basic examples of lattices:

Example 3 *Let* M *be a topological space, and let* $\mathcal{T}(M)$ *be the topology of* M*, i.e. the set of all open subsets of* M*.* $\mathcal{T}(M)$ *can be made into a complete distributive lattice: the maximum of a family* $(U_i)_{i \in I}$ *of open subsets* U_i *of* M *is given by*

$$\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i,$$

the minimum is

$$\bigwedge_{i \in I} U_i = \operatorname{int}(\bigcap_{i \in I} U_i),$$

where $\operatorname{int} N$ *denotes the interior of a subset* N *of* M*.*

Example 4 *Let* M *be a topological space, and let* $\mathcal{B}(M)$ *be the set of Borel subsets of* M*. Equipped with the usual set theoretic operations,* $\mathcal{B}(M)$ *is a distributive* $\aleph_0$*-complete Boolean lattice which is called the* σ***-algebra of Borel subsets of*** M*.*

Example 5 *Let $\mathcal{H}$ be a Hilbert space, and let $\mathbb{L}(\mathcal{H})$ be the set of closed subspaces of $\mathcal{H}$. If we define lattice operations by*

$$\begin{aligned} U \wedge V &:= U \cap V, \\ U \vee V &:= \overline{U+V}, \\ U^{\perp} &:= \text{orthogonal complement of } U \text{ in } \mathcal{H}, \end{aligned}$$

then $\mathbb{L}(\mathcal{H})$ becomes a complete lattice. In contrast to the examples above, $\mathbb{L}(\mathcal{H})$ is highly non-distributive.
$\mathbb{L}(\mathcal{H})$ plays a major role in the foundations of quantum theory, as first suggested by Birkhoff and von Neumann in their seminal article [BirvNeu36], where they regarded $\mathbb{L}(\mathcal{H})$ as representing "quantum logic" in contrast to the classical "Boolean logic". $\mathbb{L}(\mathcal{H})$ is called the ***quantum lattice****.*

Definition 6 *Let $\mathbb{L}_1$, $\mathbb{L}_2$ be lattices. A mapping $\phi : \mathbb{L}_1 \to \mathbb{L}_2$ is called a* ***lattice morphism*** *if*

(i) $\phi(0_{\mathbb{L}_1}) = 0_{\mathbb{L}_2}$,

(ii) $\phi(1_{\mathbb{L}_1}) = 1_{\mathbb{L}_2}$,

(iii) $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$,

(iv) $\phi(a \vee b) = \phi(a) \vee \phi(b)$

for all $a, b \in \mathbb{L}_1$. If $\mathbb{L}_1$ and $\mathbb{L}_2$ are $\mathfrak{m}$-complete, a lattice morphism $\phi : \mathbb{L}_1 \to \mathbb{L}_2$ is called ***left-continuous*** *if for every family $(a_i)_{i \in I}$ in $\mathbb{L}_1$ such that $\#I \leq \mathfrak{m}$, it holds that*

$$\bigvee_{i \in I} \phi(a_i) = \phi(\bigvee_{i \in I} a_i).$$

ϕ is called ***right-continuous*** *if*

$$\bigwedge_{i \in I} \phi(a_i) = \phi(\bigwedge_{i \in I} a_i)$$

holds for any such family. ϕ is called ***continuous*** *if ϕ is both left- and right-continuous.*

Let X and Y be topological spaces. The elements of Y correspond bijectively to the constant mappings $f : X \to Y$. The constant mappings $f : X \to Y$ correspond via the inverse image morphism

$$\phi : V \longmapsto f^{-1}(V) \quad (V \in \mathcal{T}(Y))$$

to the left-continuous lattice morphisms

$$\phi : \mathcal{T}(Y) \longrightarrow \mathcal{T}(X)$$

with the property

$$\forall V \in \mathcal{T}(Y) : \phi(V) \in \{\varnothing, X\} = \{0_{\mathcal{T}(X)}, 1_{\mathcal{T}(X)}\}.$$

The set

$$\mathfrak{p} := \{V \in \mathcal{T}(Y) \mid \phi(V) = X\}$$

obviously has the following properties:

(i) $\varnothing \notin \mathfrak{p}$,

(ii) $V, W \in \mathfrak{p} \Longrightarrow V \cap W \in \mathfrak{p}$,

(iii) $V \in \mathfrak{p},\ W \in \mathcal{T}(Y),\ V \subseteq W \Longrightarrow W \in \mathfrak{p}$,

(iv) If $(V_i)_{i \in I}$ is a family in $\mathcal{T}(Y)$ such that $\bigcup_{i \in I} V_i \in \mathfrak{p}$, then $V_j \in \mathfrak{p}$ for at least one $j \in I$.

This can be made into a definition, giving our first ansatz for a generalization of points of a topological space:

Definition 7 *Let $\mathbb{L}$ be an $\mathfrak{m}$-complete lattice. A non-empty subset $\mathfrak{p} \subseteq \mathbb{L}$ is called a* ***point of*** *$\mathbb{L}$ if*

(i) $0 \notin \mathfrak{p}$,

(ii) $a, b \in \mathfrak{p} \Longrightarrow a \wedge b \in \mathfrak{p}$,

(iii) $a \in \mathfrak{p},\ b \in \mathbb{L},\ a \leq b \Longrightarrow b \in \mathfrak{p}$,

(iv) If $(a_i)_{i \in I}$ is a family in $\mathbb{L}$ with $\#I \leq \mathfrak{m}$ such that $\bigvee_{i \in I} a_i \in \mathfrak{p}$ holds, then $a_j \in \mathfrak{p}$ for at least one $j \in I$.

Example 8 *Let M be a non-empty set and $\mathbb{L} \subseteq Pot(M)$ an $\mathfrak{m}$-complete lattice such that $0_{\mathbb{L}} = \varnothing$, $1_{\mathbb{L}} = M$ and for all index sets I with $\#I \leq \mathfrak{m}$,*

$$\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i.$$

Then for each $x \in M$,

$$\mathfrak{p}_x := \{U \in \mathbb{L} \mid x \in U\}$$

is a point of $\mathbb{L}$.

What about the converse? If we take the topology $\mathcal{T}(X)$ of some topological space X as our lattice $\mathbb{L}$ and choose some $x \in X$, we would expect that there is a point $\mathfrak{p}_x$ of the lattice $\mathcal{T}(X)$ that consists of all open neighbourhoods of x. On the other hand, each point $\mathfrak{p}$ of $\mathcal{T}(X)$ should be of that form, at least if the topology is not too wild. Indeed, we make the mild assumption that X is a *regular* topological space, i.e. for each $x \in X$ and each closed set $A \subset X$ not containing x, there are disjoint open sets U, V such that $x \in U$ and $A \subseteq V$. Then we have

Theorem 9 *Let X be a regular topological space. A non-empty subset $\mathfrak{p} \subseteq \mathcal{T}(X)$ is a point of $\mathcal{T}(X)$ if and only if there is some $x \in X$ such that $\mathfrak{p}$ is the set of open neighbourhoods of x. x is determined uniquely by $\mathfrak{p}$.*

The proof is straightforward and can be found in [deG05].

It turns out that not all lattices $\mathbb{L}$ have points. We will show this for the case of the lattice $\mathbb{L}(\mathcal{H})$ of closed subspaces of a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} > 1$. The fact that a lattice $\mathbb{L}$ has no points does *not* depend on the non-distributivity of $\mathbb{L}$. For example, one can show that the distributive lattice $\mathcal{T}_r(X)$ of regular open subsets of some topological space X (having some nice topological properties) has no points.

Theorem 10 *Let $\mathcal{H}$ be a Hilbert space with* $\dim \mathcal{H} > 1$. *Then* $\mathbb{L}(\mathcal{H})$ *has no points.*

Proof. Let $\mathfrak{p} \subseteq \mathbb{L}(\mathcal{H})$ be a point, and let $(e_i)_{i \in I}$ be an orthonormal basis of $\mathcal{H}$. Then

$$\bigvee_{i \in I} \mathbb{C}e_i = \mathcal{H} \in \mathfrak{p},$$

and so there is some $i_0 \in I$ such that $\mathbb{C}e_i \in \mathfrak{p}$. Let $U \in \mathbb{L}(\mathcal{H})$ be such that neither U nor $U^\perp$ contain the line $\mathbb{C}e_i$. Then

$$\mathbb{C}e_i \cap U = 0 = \mathbb{C}e_i \cap U^\perp,$$

so $U, U^\perp \notin \mathfrak{p}$. Since $U \vee U^\perp = \mathcal{H} \in \mathfrak{p}$, this contradicts property (4) in Def. 7. ∎

Since some interesting lattices, in particular the quantum lattice $\mathbb{L}(\mathcal{H})$, have no points, we need another notion. One possibility −almost suggesting itself− is to generalize what M. H. Stone did in his seminal work [Sto36] on Boolean σ-algebras: one can take maximal filter bases in $\mathbb{L}$ (that turn out to be maximal dual ideals also, see below). This also fits in perfectly well with our sheaf-theoretic motivation, see section 2.5.

Definition 11 *Let $\mathbb{L}$ be a lattice. A non-empty subset $B \subseteq \mathbb{L}$ is called a* ***filter base*** *if*

(i) $0 \notin B$,

(ii) $\forall a, b \in B \; \exists c \in B : c \leq a \wedge b$.

Instead of regarding the vast set of all filter bases in a lattice $\mathbb{L}$, we use Zorn's lemma and only take the *maximal* filter bases:

Definition 12 *A subset $\mathfrak{B}$ of a lattice $\mathbb{L}$ is called a* ***quasipoint of*** *$\mathbb{L}$ if $\mathfrak{B}$ has the following properties:*

(i) $0 \notin \mathfrak{B}$,

(ii) $\forall a, b \in \mathfrak{B} \; \exists c \in \mathfrak{B} : c \leq a \wedge b$,

(iii) $\mathfrak{B}$ *is maximal with respect to (i) and (ii).*

Remark 13 *Each filter base $B \subseteq \mathbb{L}$ is contained in some quasipoint $\mathfrak{B}$ of $\mathbb{L}$.*

Proof. Let $(B_i)_{i \in I}$ be an ascending chain of filter bases in $\mathbb{L}$. Then $\bigcup_{i \in I} B_i$ is a filter base. According to Zorn's lemma, the set of filter bases that contain the given filter base B has a maximal element. ∎

Remark 14 *Let $\mathfrak{B}$ be a quasipoint of the lattice $\mathbb{L}$. Then*

$$\forall a \in \mathfrak{B} \; \forall b \in \mathbb{L} : (a \leq b \Longrightarrow b \in \mathfrak{B}).$$

In particular, $\forall a, b \in \mathfrak{B} : a \wedge b \in \mathfrak{B}$.

Proof. $\mathfrak{B} \cup \{b\}$ is a filter base in $\mathbb{L}$: since $a \leq b$, we have

$$a \wedge c \leq b \wedge c$$

for all $c \in \mathfrak{B}$, and therefore

$$\exists d \in \mathfrak{B} : d \leq a \wedge c \leq b \wedge c.$$

Thus $\mathfrak{B} = \mathfrak{B} \cup \{b\}$ from the maximality of $\mathfrak{B}$, i.e. $b \in \mathfrak{B}$. ■

This remark shows that a quasipoint of $\mathbb{L}$ is a **maximal dual ideal** in $\mathbb{L}$ (see [Bir73] and Def. 66 below).

Examples of quasipoints

We will now give three different examples hinting at the usefulness of the new notion of quasipoints: the first example is topological, using the topology of a locally compact Hausdorff space as our lattice. The second example is "logical-topological", namely a Boolean σ-algebra. This makes the connection with the classical works of M. S. Stone from the 1930s, see [Sto36] and also [Bir73]. The third example is the quantum lattice $\mathbb{L}(\mathcal{H})$ of closed subspaces of Hilbert space. We will identify this lattice with the projection lattice of the von Neumann algebra $\mathcal{L}(\mathcal{H})$.

(1) Let M be a locally compact Hausdorff space, $\mathbb{L} := \mathcal{T}(M)$. A quasipoint $\mathfrak{B} \subseteq \mathcal{T}(M)$ either contains a U_0 such that $\overline{U_0}$ is compact, or there is no such $U_0 \in \mathfrak{B}$. This alternative corresponds to two different types of quasipoints: assume that there is a $U_0 \in \mathfrak{B}$ such that $\overline{U_0}$ is compact. Then

$$\bigcap_{U \in \mathfrak{B}} \overline{U} \neq \varnothing,$$

since otherwise we would have $\bigcap_{U \in \mathfrak{B}} \overline{U \cap U_0} = \varnothing$. So from compactness of $\overline{U_0}$, there would be elements $U_1, ...U_n \in \mathfrak{B}$ such that

$$\bigcap_{i=1}^{n} \overline{U_i \cap U_0} = \varnothing,$$

and so we would have $U_0 \cap U_1 \cap ... \cap U_n = \varnothing$, contradicting the defining properties of a filter base. $\mathfrak{B}$ is a quasipoint, so every open neighbourhood V of $x \in \bigcap_{U \in \mathfrak{B}} \overline{U}$ must belong to $\mathfrak{B}$, since $V \cap U \neq \varnothing$ for all $U \in \mathfrak{B}$ and therefore $\mathfrak{B} \cup \{V \cap U \mid U \in \mathfrak{B}\}$ is a filter base containing V (since $M \in \mathfrak{B}$). From maximality of $\mathfrak{B}$, $V \in \mathfrak{B}$. M is Hausdorff, so $\bigcap_{U \in \mathfrak{B}} \overline{U}$ consists of a single element.

If a quasipoint $\mathfrak{B}$ of $\mathcal{T}(M)$ contains a relatively compact element U_0, then it is called **bounded**, otherwise **unbounded**.

Let $\mathfrak{B}$ be a bounded quasipoint of $\mathcal{T}(M)$, and let $pt(\mathfrak{B}) := \bigcap_{U \in \mathfrak{B}} \overline{U} \in M$. $\mathfrak{B}$ is called a **quasipoint over** $x \in M$ if $x = pt(\mathfrak{B})$.

Remark 15 *Let $\mathfrak{B}$ be an unbounded quasipoint. Then $M\backslash K \in \mathfrak{B}$ for all compact subsets $K \subseteq M$. On the other hand, if $M\backslash K \in \mathfrak{B}$ for every compact subset $K \subseteq M$, then $\mathfrak{B}$ is unbounded.*

Proof. Let $M\backslash K \notin \mathfrak{B}$ for some compact subset K of M. Then from maximality there is a $U \in \mathfrak{B}$ such that $(M\backslash K) \cap U = \varnothing$. Therefore, $U \subseteq K$, that is, U is relatively compact. The converse is clear. ■

From this, one easily sees:

Remark 16 *Let $M_\infty := M \amalg \{\infty\}$ be the one-point-compactification of the locally compact space M. Then the unbounded quasipoints of $\mathcal{T}(M)$ are the quasipoints of $\mathcal{T}(M_\infty)$ over ∞.*

(2) Let $\mathcal{B}$ be a Boolean σ-algebra, that is, a σ-complete complemented distributive lattice. The complement of $A \in \mathcal{B}$ is denoted by A^c. We will show:

Lemma 17 *A filter base $\mathfrak{B} \subseteq \mathcal{B}$ is a quasipoint if and only if*

$$\forall A \in \mathcal{B} : A \in \mathcal{B} \text{ or } A^c \in \mathcal{B}.$$

Proof. Let $\mathfrak{B} \subseteq \mathcal{B}$ be a filter base such that $A \notin \mathfrak{B} \Rightarrow A^c \in \mathfrak{B}$. Let $\mathfrak{B}_0$ be a quasipoint of $\mathcal{B}$ containing $\mathfrak{B}$. If $B \in \mathfrak{B}_0\backslash\mathfrak{B}$, then $B^c \in \mathfrak{B}$ and hence $0 = B \wedge B^c \in \mathfrak{B}_0$, which is a contradiction.

On the other hand, if $\mathfrak{B} \subseteq \mathcal{B}$ is a quasipoint, then from $A, A^c \notin \mathfrak{B}$ follows

$$\exists B, C \in \mathfrak{B} : A \wedge B = A^c \wedge C = 0,$$

so

$$\begin{aligned} B \wedge C &= (A \vee A^c) \wedge B \wedge C \\ &= (A \wedge B \wedge C) \vee (A^c \wedge B \wedge C) = 0, \end{aligned}$$

contradicting $B \wedge C \in \mathfrak{B}$. ■

From this, we immediately get:

Remark 18 *Each point $\mathfrak{p}$ of $\mathcal{B}$ is a quasipoint.*

Proof. Let $\mathfrak{p}$ be a point of $\mathcal{B}$. We have $1 = A \vee A^c \in \mathfrak{p}$, so $A \in \mathfrak{p}$ or $A^c \in \mathfrak{p}$ from the defining condition (4) of points of a lattice (cf. Def. 7). ■

Definition 19 *A subset J of a lattice $\mathbb{L}$ is called an **ideal in** $\mathbb{L}$ if*

(i) $a, b \in J \Longrightarrow a \vee b \in J$,

(ii) $a \in J,\ b \in \mathbb{L} \Longrightarrow a \wedge b \in J$.

*J is called a **proper ideal** if $1 \notin J$. An ideal J is called a σ-**ideal** if $\bigvee_n a_n \in J$ holds for every sequence $(a_n)_{\in \mathbb{N}}$ in J.*

If J is an ideal in a Boolean algebra $\mathcal{B}$, then an equivalence relation on $\mathcal{B}$ is defined by

$$A \equiv B \bmod J :\Longleftrightarrow \exists N \in J : A \vee N = B \vee N.$$

It is easy to see that $\mathcal{B}/J$ is a Boolean algebra. If J is a σ-ideal, then $\mathcal{B}/J$ is a Boolean σ-algebra, see [Sik64]. The quasipoints of the quotient $\mathcal{B}/J$ can be characterized in the following way:

Theorem 20 *Let $\mathcal{B}$ be a Boolean σ-algebra, J a σ-ideal in $\mathcal{B}$, and let $\pi : \mathcal{B} \to \mathcal{B}/J$ be the projection onto the quotient. $\mathfrak{B} \subseteq \mathcal{B}/J$ is a quasipoint if and only if $\pi^{-1}(\mathfrak{B}) \subseteq \mathcal{B}$ is a quasipoint of $\mathcal{B}$ with $\pi^{-1}(\mathfrak{B}) \cap J = \varnothing$.*

We will omit the proof. A Boolean σ-algebra need not have points in general:

Theorem 21 *Let M be a T_1-space fulfilling the first countability axiom and the Lindelöf condition (for example, a metric space with a countable base). Let J be a σ-ideal of the Boolean σ-algebra $\mathcal{B}(M)$ of Borel subsets of M containing all atoms of $\mathcal{B}(M)$. Then the Boolean σ-algebra $\mathcal{B} := \mathcal{B}(M)/J$ has no points.*

This proof is omitted here, too, since we will not need the result in the rest of this work. From this theorem, we obtain:

Corollary 22 *Let M be a complete metric space with countable base. Then the lattice $\mathcal{T}_r(M)$ of regular open subsets of M has no points.*

(3) Let $\mathbb{L}(\mathcal{H})$ be the lattice of closed subspaces of the Hilbert space $\mathcal{H}$. There are two types of quasipoints $\mathfrak{B}$:

(i) $\mathfrak{B}$ contains a finite-dimensional subspace $U_0 \subseteq \mathcal{H}$.

(ii) $\mathfrak{B}$ contains no finite-dimensional subspace.

In the first case, we have

Remark 23 *If a quasipoint $\mathfrak{B} \subseteq \mathbb{L}(\mathcal{H})$ contains a finite-dimensional element, then there is a unique one-dimensional subspace $\mathbb{C}x_0 \subseteq \mathcal{H}$ such that*

$$\mathfrak{B} = \{U \in \mathbb{L}(\mathcal{H}) \mid \mathbb{C}x_0 \subseteq U\}.$$

Conversely, for each one-dimensional subspace $\mathbb{C}x \subseteq \mathcal{H}$, the set

$$\mathfrak{B}_{\mathbb{C}x} := \{U \in \mathbb{L}(\mathcal{H}) \mid \mathbb{C}x \subseteq U\}$$

is a quasipoint of $\mathbb{L}(\mathcal{H})$.

Proof. Let $U_0 \in \mathfrak{B}$ be n-dimensional, $n < \infty$. Then $U \cap U_0 \neq 0$ for all $U \in \mathfrak{B}$ and therefore $\{U \cap U_0 \mid U \in \mathfrak{B}\}$ contains an element V_0 of minimal (positive and finite) dimension. Then $V_0 \subseteq U$ for all $U \in \mathfrak{B}$ and so $\dim V_0 = 1$ from the maximality of $\mathfrak{B}$. Since $\mathbb{C}x \cap \mathbb{C}y = 0$ if $\mathbb{C}x \neq \mathbb{C}y$, this one-dimensional space is determined uniquely by $\mathfrak{B}$. The converse statement is clear. ∎

If a quasipoint $\mathfrak{B} \subseteq \mathbb{L}(\mathcal{H})$ contains no finite-dimensional element, then we have the following

Remark 24 *A quasipoint $\mathfrak{B} \subseteq \mathbb{L}(\mathcal{H})$ has no finite-dimensional element if and only if $\mathfrak{B}$ contains* every *closed subspace $U \in \mathbb{L}(\mathcal{H})$ of finite codimension.*

Proof. Let $\mathfrak{B}$ contain every closed subspace of finite codimension, and let $U \in \mathbb{L}(\mathcal{H})$ be finite-dimensional. Then $U^\perp \in \mathfrak{B}$ and hence $U \notin \mathfrak{B}$, since $U \cap U^\perp = 0$.

Conversely, let $V \in \mathbb{L}(\mathcal{H})$ be of finite codimension and $V \notin \mathfrak{B}$. Then there is some $U \in \mathfrak{B}$ such that $U \cap V = 0$. Let $P_{V^\perp} : \mathcal{H} \to V^\perp$ be the orthogonal projection onto $V^\perp$. Since $U \cap V = 0$, $P_{V^\perp}|_U$ is injective. $V^\perp$ is finite-dimensional, so U must be finite-dimensional. ∎

A quasipoint $\mathfrak{B} \subseteq \mathbb{L}(\mathcal{H})$ which contains no finite-dimensional element is called a **continuous quasipoint**.

Remark 25 *The quantum lattice $\mathbb{L}(\mathcal{H})$ can be identified with the projection lattice $\mathcal{P}(\mathcal{H}) := \mathcal{P}(\mathcal{L}(\mathcal{H}))$ of the von Neumann algebra $\mathcal{L}(\mathcal{H})$ via the bijection*

$$\begin{aligned} p : \mathbb{L}(\mathcal{H}) &\longrightarrow \mathcal{P}(\mathcal{H}) \\ U &\longmapsto P_U, \end{aligned}$$

where P_U is the orthogonal projection onto the closed subspace U. If $\mathcal{H}$ is infinite-dimensional, $\mathcal{L}(\mathcal{H})$ is a factor of type I_∞ whose quasipoints are (roughly) characterized by the above remarks. We will come back to this important example on several occasions.

2.4. The Stone spectrum of a lattice

2.4.1. Some topological notions

For the convenience of the reader, some well-known topological notions and some relations between them are listed here. We follow Bourbaki [BToI66, BToII89].

Definition 26 *A topological space X is called* ***totally disconnected*** *if the connected component of each point $x \in X$ consists of the point alone.*

A discrete space is totally disconnected, but not every totally disconnected space is discrete. For example, the rational line is totally disconnected, but not discrete.

Definition 27 *A topological space X is called* ***zero-dimensional*** *if it is Hausdorff and if every point $x \in X$ has a fundamental system of neighbourhoods which are both open and closed.*

A zero-dimensional space X is totally disconnected, since the connected component of a point x is contained in all the sets containing x which are both open and closed, and the intersection of these sets is just $\{x\}$ if X is zero-dimensional.

Definition 28 *A topological space X is called* ***extremely disconnected*** *if for each open set $U \in \mathcal{T}(X)$ the closure $\overline{U}$ of U is open.*

Every extremely disconnected regular space has an open base composed of closed-open sets (namely the closures of open sets), hence is zero-dimensional (cf. V.46.VI in [Kur68]). The Gelfand spectrum of an abelian von Neumann algebra is an extremely disconnected compact Hausdorff space (see Thm. 5.2.1 in [KadRinI97]). This gives the most important examples of extremely disconnected spaces.

An extremely disconnected compact Hausdorff space is called **stonean** or a **Stone space** [TakI02]. A Stone space Ω is called **hyperstonean** if there are sufficiently many positive normal measures, that is, if for any nonzero positive $f \in C(\Omega, \mathbb{R})$, there exists a positive normal measure μ with $\mu(f) \neq 0$. We cite Thm. III.1.18 from [TakI02] in parts:

Theorem 29 *For an (abstract) abelian C^*-algebra A with spectrum Ω, the following statements are equivalent:*

(i) Ω is hyperstonean.

(ii) A admits a faithful representation $\{\pi, \mathcal{H}\}$ such that $\pi(A)$ is an abelian von Neumann algebra on $\mathcal{H}$.

2.4.2. Definition and topological properties of the Stone spectrum

Let $\mathbb{L}$ be a lattice, and let $\mathcal{Q}(\mathbb{L})$ denote the set of its quasipoints. $\mathcal{Q}(\mathbb{L})$ is equipped with a natural topology: for $a \in \mathbb{L}$, let

$$\mathcal{Q}_a(\mathbb{L}) := \{\mathfrak{B} \in \mathcal{Q}(\mathbb{L}) \mid a \in \mathfrak{B}\}.$$

Obviously, we have

$$\mathcal{Q}_{a \wedge b}(\mathbb{L}) = \mathcal{Q}_a(\mathbb{L}) \cap \mathcal{Q}_b(\mathbb{L}),$$

so $\{\mathcal{Q}_a(\mathbb{L}) \mid a \in \mathbb{L}\}$ is a base for a topology on $\mathbb{L}$. The topology given by the sets $\mathcal{Q}_a(\mathbb{L})$ is a direct generalization to arbitrary lattices of the Stone topology on $\mathcal{Q}(\mathcal{B})$, the set of quasipoints of a Boolean algebra $\mathcal{B}$ [Sto36]. (Of course, Stone did not speak of quasipoints but of maximal dual ideals.)

Definition 30 *The set $\mathcal{Q}(\mathbb{L})$ of quasipoints of a lattice $\mathbb{L}$, equipped with the topology given by the sets $\mathcal{Q}_a(\mathbb{L})$ defined above, is called the* ***Stone spectrum of the lattice*** *$\mathbb{L}$.*

Lemma 31 *The topological space $\mathcal{Q}(\mathbb{L})$ is zero-dimensional, in particular it is totally disconnected.*

Proof. Let $a \in \mathbb{L}$. $\mathcal{Q}_a(\mathbb{L})$ is open by definition. Let $\mathfrak{B} \notin \mathcal{Q}_a(\mathbb{L})$, i.e. $a \notin \mathfrak{B}$. Then there is some $b \in \mathfrak{B}$ with $a \wedge b = 0$ and it follows

$$\mathcal{Q}_b(\mathbb{L}) \cap \mathcal{Q}_a(\mathbb{L}) = \mathcal{Q}_{a\wedge b}(\mathbb{L}) = \varnothing,$$

that is, $\mathfrak{B} \in \mathcal{Q}_b(\mathbb{L}) \subseteq \mathcal{Q}(\mathbb{L})\backslash\mathcal{Q}_a(\mathbb{L})$. So the complement of $\mathcal{Q}_a(\mathbb{L})$ is open and hence $\mathcal{Q}_a(\mathbb{L})$ is closed. ■

It will be shown in [deG05] that if $\mathbb{L}$ is a completely distributive lattice (i.e. $a\wedge(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \wedge b_i)$ holds for all $a \in \mathbb{L}$ and all families $(b_i)_{i\in I} \subseteq \mathbb{L}$), then $\mathcal{Q}(\mathbb{L})$ is extremely disconnected.

Remark 32 *The Stone spectrum $\mathcal{Q}(\mathbb{L})$ of a lattice $\mathbb{L}$ is a completely regular Hausdorff space.*

Proof. This follows immediately from the fact that the sets $\mathcal{Q}_a(\mathbb{L})$ are open and closed, so their characteristic functions are continuous. ■

Definition 33 *A quasipoint is called an* ***atomic quasipoint*** *if it is an isolated point of the Stone spectrum. The set of atomic quasipoints of a lattice $\mathbb{L}$ is denoted by $\mathcal{Q}_a(\mathbb{L})$.*

Example 34 *We saw that for the quantum lattice $\mathbb{L}(\mathcal{H})$, there are two types of quasipoints, those containing no finite-dimensional subspace $U \subseteq \mathcal{H}$ (continuous quasipoints) and those containing a finite-dimensional subspace. The latter are atomic quasipoints: remark 23 shows that each of these quasipoints is of the form*

$$\mathfrak{B}_{\mathbb{C}x} = \{U \in \mathbb{L}(\mathcal{H}) \mid \mathbb{C}x \subseteq U\}.$$

Obviously, the closed-open set $\mathcal{Q}_{\mathbb{C}x}(\mathbb{L})$ contains the quasipoint $\mathfrak{B}_{\mathbb{C}x}$ as its only element.

Using the identification of $\mathbb{L}(\mathcal{H})$ with the projection lattice $\mathcal{P}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$ (see Rem. 25), an atomic quasipoint of $\mathcal{L}(\mathcal{H})$ is given by

$$\mathfrak{B}_{\mathbb{C}x} := \{P \in \mathcal{P}(\mathcal{H}) \mid P_{\mathbb{C}x} \leq P\}.$$

This will be used quite often, especially in chapter 4.

Remark 35 *If $\mathcal{H}$ is finite-dimensional, clearly there are no continuous quasipoints in $\mathcal{P}(\mathcal{H})$, and we have*

$$\mathcal{Q}(\mathcal{P}(\mathcal{H})) = \{\mathfrak{B}_{\mathbb{C}x} \mid x \in S^1(\mathcal{H})\},$$

where $S^1(\mathcal{H})$ denotes the unit sphere in Hilbert space. For finite-dimensional $\mathcal{H}$, $\mathcal{P}(\mathcal{H}) \simeq \mathbb{L}(\mathcal{H})$ is the projection lattice of a type I_n factor, where $n = \dim\mathcal{H}$. This type I_n factor simply is represented as $\mathbb{M}_n(\mathbb{C})$, the $n \times n$ complex matrices acting on $\mathcal{H}$.

Of course, the inner product of $\mathcal{H}$ plays no role here, but only the linear structure, so we have also characterized the quasipoints of the lattice of subspaces of a vector space.

Let $U_0 \in \mathbb{L}$ and $\mathbb{L}_{U_0} = \{U \in \mathbb{L} \mid U \leq U_0\}$. $\mathbb{L}_{U_0}$ is an ideal in $\mathbb{L}$, since $\mathbb{L}_{U_0} = \{U \wedge U_0 \mid U \in \mathbb{L}\}$.

Lemma 36 *$\mathcal{Q}(\mathbb{L}_{U_0})$ is homeomorphic to $\mathcal{Q}_{U_0}(\mathbb{L})$.*

Proof. Let $\mathfrak{B} \in \mathcal{Q}_{U_0}(\mathbb{L})$. Then $\mathfrak{B}_{U_0} := \{V \wedge U_0 \mid V \in \mathfrak{B}\}$ is closed under taking minima. (In sec. 3.1, $\mathfrak{B}_{U_0}$ will be called the U_0-*trunk* of $\mathfrak{B}$). Let $V \leq U_0$ and $V \notin \mathfrak{B}_{U_0}$. Then $V \notin \mathfrak{B}$ and so $V \wedge W = 0$ for some $W \in \mathfrak{B}$. It follows

$$U_0 \wedge W \wedge U_0 \wedge V = 0,$$

therefore $\mathfrak{B}_{U_0}$ is maximal, i.e. a quasipoint of $\mathbb{L}_{U_0}$. We obtain a mapping

$$\begin{aligned} \varphi : \mathcal{Q}_{U_0}(\mathbb{L}) &\longrightarrow \mathcal{Q}(\mathbb{L}_{U_0}) \\ \mathfrak{B} &\longmapsto \mathfrak{B}_{U_0}. \end{aligned}$$

φ is bijective: let $\mathfrak{B}$, $\mathfrak{B}' \in \mathcal{Q}_{U_0}(\mathbb{L})$ with $\mathfrak{B}_{U_0} = \mathfrak{B}'_{U_0}$. Assume that $\mathfrak{B} \neq \mathfrak{B}'$. Then there are $U \in \mathfrak{B}$, $U' \in \mathfrak{B}'$ with $U \wedge U' = 0$ and thus $U \wedge U_0 \wedge U' \wedge U_0 = 0$, contradicting $U \wedge U_0$, $U' \wedge U_0 \in \mathfrak{B}_{U_0}$.

Let $\mathfrak{B}^{U_0} \in \mathcal{Q}(\mathbb{L}_{U_0})$. Then

$$\mathfrak{B} := \{V \in \mathbb{L} \mid \exists U \in \mathfrak{B}^{U_0} : U \leq V\}$$

is a quasipoint of $\mathbb{L}$ which contains U_0. Obviously, $\mathfrak{B}_{U_0} = \mathfrak{B}^{U_0}$. Let $V_0 \leq U_0$. Then

$$\varphi(\mathcal{Q}_{V_0}(\mathbb{L})) = \mathcal{Q}_{V_0}(\mathbb{L}_{U_0}),$$

thus φ is a homeomorphism. ■

Theorem 37 *Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} > 1$. Then the Stone spectrum $\mathcal{Q}(\mathbb{L}(\mathcal{H}))$ is not compact. If $\mathcal{H}$ is infinite-dimensional, then $\mathcal{Q}(\mathbb{L}(\mathcal{H}))$ is not even locally compact.*

Proof. The non-compactness of $\mathcal{Q}(\mathbb{L}(\mathcal{H}))$ can be inferred from the following observation:

$$\mathcal{Q}(\mathbb{L}(\mathcal{H})) \neq \bigcup_{U \in J} \mathcal{Q}_U(\mathbb{L}(\mathcal{H}))$$

holds for all finite sets $J \subseteq \mathbb{L}(\mathcal{H}) \backslash \{\mathcal{H}\}$: assume that

$$\mathcal{Q}(\mathbb{L}(\mathcal{H})) = \bigcup_{k=1}^{n} \mathcal{Q}_{U_k}(\mathbb{L}(\mathcal{H}))$$

with $U_1, ..., U_n \in \mathbb{L}(\mathcal{H}) \backslash \{\mathcal{H}\}$. Then $\mathcal{H} = \bigcup_{k=1}^{n} U_k$, since for $x \in \mathcal{H} \backslash \{0\}$, one has $\mathfrak{B}_{\mathbb{C}x} \in \mathcal{Q}_{U_k}(\mathbb{L}(\mathcal{H}))$ for some k and so $\mathbb{C}x \subseteq U_k$. But then we must have $\mathcal{H} = U_i$ for some i. (From Baire's category theorem it follows that the same argument holds for countable J, too. Thus $\mathcal{Q}(\mathbb{L}(\mathcal{H}))$ does not fulfill the Lindelöf condition.)

Let $\mathcal{H}$ be infinite-dimensional and let $\mathfrak{B}$ be a continuous quasipoint. Assume that there is a compact neighbourhood of $\mathfrak{B}$. This neighbourhood must contain a neighbourhood of $\mathfrak{B}$ of the form $\mathcal{Q}_U(\mathbb{L}(\mathcal{H}))$. Since $\mathcal{Q}_U(\mathbb{L}(\mathcal{H}))$ is closed, it also is compact. $\mathfrak{B}$ is continuous, so $U \in \mathfrak{B}$ must be infinite-dimensional. According to the previous lemma, $\mathcal{Q}_U(\mathbb{L}(\mathcal{H}))$ is homeomorphic to $\mathcal{Q}(\mathbb{L}(U))$, so $\mathcal{Q}(\mathbb{L}(U))$ is compact, which contradicts the first part of the proof. Therefore, $\mathfrak{B}$ has no compact neighbourhood. ■

2.5. Presheaves and sheaves

We now have the necessary preparations to treat presheaves and especially sheaves on a lattice.

Definition 38 *Let $\mathbb{L}$ be a lattice. A* ***presheaf on the lattice*** $\mathbb{L}$ *is a family* $\mathcal{P} = (\mathcal{P}_a, \rho^a_b)_{a,b\in\mathbb{L}\backslash\{0_\mathbb{L}\},b\leq a}$ *of sets* $\mathcal{P}_a$ *and mappings*

$$\rho^a_b : \mathcal{P}_a \longrightarrow \mathcal{P}_b$$

for $b \leq a$ *such that*

(i) $\rho^a_a = id_{\mathcal{P}_a}$,

(ii) $\rho^a_c = \rho^b_c \circ \rho^a_b$ *for* $c \leq b \leq a$.

If $\mathbb{L}$ *is* $\mathfrak{m}$*-complete, then the presheaf* $\mathcal{P}$ *is called* $\mathfrak{m}$***-complete*** *if the following condition holds: if for* $a \in \mathbb{L}$, $a = \bigvee_{i\in I} a_i$ *such that* $\#I \leq \mathfrak{m}$, *and for* $f_i \in \mathcal{P}_{a_i}$ *given such that for all* $i, j \in I$ *with* $a_i \wedge a_j \neq 0_\mathbb{L}$,

$$\rho^{a_i}_{a_i\wedge a_j}(f_i) = \rho^{a_j}_{a_i\wedge a_j}(f_j) \tag{$*$}$$

holds, then there is a unique $f \in \mathcal{P}_a$ *such that* $\rho^a_{a_i}(f) = f_i$ *for all* $i \in I$. *If* ($*$) *holds without any restriction on the cardinality of* I, *then* $\mathcal{P}$ *is called* ***complete***.

This definition generalizes the usual notion of a presheaf, which is defined on the distributive lattice of open subsets of some topological space. As usual, the definition of completeness amounts to the requirement that local data (the f_i) can be glued together in a unique manner to give a globally defined object (namely f).

Example 39 *Let* $\mathcal{H}$ *be some Hilbert space, and for* $U \in \mathbb{L}(\mathcal{H})\backslash\{0\}$ *let* $\mathcal{L}(U)$ *be the algebra of bounded operators* $U \to U$. *For* $V \subseteq U$ *in* $\mathbb{L}(\mathcal{H})\backslash\{0\}$, *let*

$$\begin{aligned}\rho^U_V : \mathcal{L}(U) &\longrightarrow \mathcal{L}(V)\\ T &\longmapsto \rho^U_V(T) := P_V T P_V.\end{aligned}$$

Here P_V *means the projection onto to closed subspace* V. *Trivially,* $\rho^U_U = id_{\mathcal{L}(U)}$, *and for* $W \subseteq V \subseteq U$ *we have*

$$\begin{aligned}\rho^U_W(T) &= P_W T P_W\\ &= (P_W P_V T) P_W\\ &= P_W (P_V T P_V) P_W\\ &= \rho^V_W(\rho^U_V(T)),\end{aligned}$$

so $(\mathcal{L}(U), \rho^U_V)_{V\subseteq U}$ *is a presheaf of von Neumann algebras on* $\mathbb{L}(\mathcal{H})$.

Although there are many presheaves on $\mathbb{L}(\mathcal{H})$, it turns out that there are no non-trivial *complete* presheaves:

Theorem 40 *Let $\mathcal{H}$ be a Hilbert space of dimension greater than* 1*, and let $\mathcal{P} = (\mathcal{P}_U, \rho^U_V)_{V\subseteq U}$ be a complete presheaf on $\mathbb{L}(\mathcal{H})$. Then*

$$\forall U \in \mathbb{L}(\mathcal{H})\backslash\{0\} : \#\mathcal{P}_U = 1,$$

that is, $\mathcal{P}$ is trivial.

Proof. Let $U \in \mathbb{L}(\mathcal{H})\backslash\{0\}$. Then $U = \bigvee_{\mathbb{C}x\subseteq U} \mathbb{C}x$. Since $\mathbb{C}x \cap \mathbb{C}y = 0$ for $\mathbb{C}x \neq \mathbb{C}y$, the family $(\mathcal{P}_{\mathbb{C}x})_{\mathbb{C}x\subseteq U}$ trivially fulfills the compatibility condition $(*)$ used in the definition of completeness. So for every family $(f_{\mathbb{C}x})_{\mathbb{C}x\subseteq U}$ of elements $f_{\mathbb{C}x} \in \mathcal{P}_{\mathbb{C}x}$, there is exactly one $f \in \mathcal{P}_U$ such that $\rho^U_{\mathbb{C}x}(f) = f_{\mathbb{C}x}$ holds for all $\mathbb{C}x \subseteq U$. Therefore, we get a bijection

$$\mathcal{P}_U \simeq \prod_{\mathbb{C}x\subseteq U} \mathcal{P}_{\mathbb{C}x},$$

hence we only have to show that each $\mathcal{P}_{\mathbb{C}x}$ has only one element. Let $\mathbb{C}e_1$, $\mathbb{C}e_2$ be two different lines in $\mathcal{H}$, $U := \mathbb{C}e_1 + \mathbb{C}e_2 \in \mathbb{L}(\mathcal{H})$, $\mathbb{C}x \subseteq U$ such that $\mathbb{C}x \notin \{\mathbb{C}e_1, \mathbb{C}e_2\}$. Then $U = \mathbb{C}e_1 \vee \mathbb{C}e_2 = \mathbb{C}x \vee \mathbb{C}e_1 \vee \mathbb{C}e_2$ and so

$$\mathcal{P}_U \simeq \mathcal{P}_{\mathbb{C}e_1} \times \mathcal{P}_{\mathbb{C}e_2} \simeq \mathcal{P}_{\mathbb{C}x} \times \mathcal{P}_{\mathbb{C}e_1} \times \mathcal{P}_{\mathbb{C}e_2}.$$

Let us assume that $\mathcal{P}_{\mathbb{C}x}$ contains more than one element. Let f_x, $g_x \subseteq \mathcal{P}_{\mathbb{C}x}$ be two different elements, and let $f_{e_k} \in \mathcal{P}_{\mathbb{C}_{e_k}}$ $(k = 1, 2)$ be fixed. There is exactly one $f \in \mathcal{P}_U$ such that

$$\begin{aligned} \rho^U_{\mathbb{C}x}(f) &= f_x, \\ \rho^U_{\mathbb{C}e_k}(f) &= f_{e_k} \quad (k = 1, 2), \end{aligned}$$

and exactly one $g \in \mathcal{P}_U$ such that

$$\begin{aligned} \rho^U_{\mathbb{C}x}(g) &= g_x, \\ \rho^U_{\mathbb{C}e_k}(g) &= f_{e_k} \quad (k = 1, 2). \end{aligned}$$

Since $U = \mathbb{C}e_1 \vee \mathbb{C}e_2$, we must have $f = g$ and accordingly $f_x = g_x$. It immediately follows $\#\mathcal{P}_{\mathbb{C}x} = 1$ for all $\mathbb{C}x \in \mathcal{H}$ and thus $\#\mathcal{P}_U = 1$ for all $U \in \mathbb{L}(\mathcal{H})\backslash\{0\}$. ■

As mentioned in the introduction of this chapter, in sheaf theory on topological spaces there is a way of associating a **sheaf**, that is, a complete presheaf, to a presheaf, the so-called *sheafification* ([MacMoe92, ConDeG94]). We will mimic this procedure to obtain a sheaf from a presheaf on a lattice.

The central aspect of the construction of a sheaf from a presheaf is the definition of *germs* and *stalks*, which in a topological space X are defined *at some point* $x \in X$. Thm. 10 shows that for the important example of the quantum lattice $\mathbb{L}(\mathcal{H})$ there are no points, so we cannot use points in a lattice to localize and to define germs and stalks of a presheaf on a lattice in general. (There are many other lattices that have no points.)

In the usual case of a presheaf $\mathcal{P} = (\mathcal{P}_U, \rho^U_V)_{\mathcal{T}(X)}$ on a topological space X, the stalk of $\mathcal{P}$ at a point x of the topological space X is the *inductive* (or *direct*) limit

$$\mathcal{P}_x = \varinjlim_{\mathcal{T}(X)_x} \mathcal{P}_U.$$

Now, to define an inductive limit we do not need a lattice-theoretical point $\mathcal{T}(X)_x$, but only an index set I filtered to the left. Put differently, it is enough to have a *filter base* B in a lattice $\mathbb{L}$. We will use maximal filter bases in $\mathbb{L}$, that is, quasipoints as a substitute for the points of a topological space:

Definition 41 *Let $\mathcal{P} = (\mathcal{P}_a, \rho^b_a)_{a \leq b}$ be a presheaf on a lattice $\mathbb{L}$. $f \in \mathcal{P}_a$ is called* ***equivalent to*** *$g \in \mathcal{P}_b$* ***at the quasipoint*** *$\mathfrak{B} \in \mathcal{Q}_{a\wedge b}(\mathbb{L})$ if there is some $c \in \mathfrak{B}$ such that*

$$c \leq a \wedge b \text{ and } \rho^a_c(f) = \rho^b_c(g).$$

If f and g are equivalent at $\mathfrak{B}$, we write $f \sim_{\mathfrak{B}} g$. $\sim_{\mathfrak{B}}$ is easily seen to be an equivalence relation. The equivalence class of $f \in \mathcal{P}_a$ at the quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathbb{L})$ is denoted by $[f]_{\mathfrak{B}}$ and is called the ***germ of*** *f* ***at*** *$\mathfrak{B}$. (Of course, we must have $\mathfrak{B} \in \mathcal{Q}_a(\mathbb{L})$.)*

Let $\mathfrak{B} \in \mathcal{Q}_a(\mathbb{L})$, $\iota_a : \mathcal{P}_a \hookrightarrow \coprod_{b\in\mathfrak{B}} \mathcal{P}_b$ the canonical injection and let $\pi_{\mathfrak{B}} : \coprod_{b\in\mathfrak{B}} \mathcal{P}_b \to (\coprod_{b\in\mathfrak{B}} \mathcal{P}_b)/\sim_{\mathfrak{B}}$ be the projection onto the equivalence class of $\sim_{\mathfrak{B}}$. Composition gives a canonical mapping

$$\rho^a_{\mathfrak{B}} := \pi_{\mathfrak{B}} \circ \iota_a : \mathcal{P}_a \longrightarrow \mathcal{P}_{\mathfrak{B}}.$$

Here

$$\mathcal{P}_{\mathfrak{B}} := (\coprod_{b\in\mathfrak{B}} \mathcal{P}_b)/\sim_{\mathfrak{B}}$$

is the inductive limit $\varinjlim_{b\in\mathfrak{B}} \mathcal{P}_b$ and is called the **stalk of $\mathcal{P}$ at $\mathfrak{B}$**. $\rho^a_{\mathfrak{B}}(f)$ is another notation for the germ $[f]_{\mathfrak{B}}$ of f at $\mathfrak{B}$ (see also [ConDeG94]).

Let

$$\mathcal{E}(\mathcal{P}) := \coprod_{\mathfrak{B}\in\mathcal{Q}(\mathbb{L})} \mathcal{P}_{\mathfrak{B}}$$

and

$$\pi_{\mathcal{P}} : \mathcal{E}(\mathcal{P}) \longrightarrow \mathcal{Q}(\mathbb{L})$$

the projection defined by $\pi_{\mathcal{P}}(\mathcal{P}_{\mathfrak{B}}) := \{\mathfrak{B}\}$. $\mathcal{E}(\mathcal{P})$ can be equipped with a topology such that $\pi_{\mathcal{P}}$ is a local homeomorphism: for $a \in \mathbb{L}$ and $f \in \mathcal{P}_a$ let

$$\mathcal{O}_{f,a} := \{\rho^a_{\mathfrak{B}}(f) \mid \mathfrak{B} \in \mathcal{Q}_a(\mathbb{L})\}.$$

It is easy to see that $\{\mathcal{O}_{f,a} \mid f \in \mathcal{P}_a, a \in \mathbb{L}\}$ is a base for a topology on $\mathcal{E}(\mathcal{P})$. The projection $\pi_{\mathcal{P}}$ is a local homeomorphism, since $\mathcal{O}_{f,a}$ is mapped bijectively onto $\mathcal{Q}_a(\mathbb{L})$.

Definition 42 *Let $\mathcal{P}$ be a presheaf on a lattice $\mathbb{L}$. $\mathcal{E}(\mathcal{P})$, together with the topology defined by $\{\mathcal{O}_{f,a} \mid f \in \mathcal{P}_a, a \in \mathbb{L}\}$, is called the* ***etale space of*** *$\mathcal{P}$* ***over*** *$\mathcal{Q}(\mathbb{L})$.*

If $\mathcal{P}$ is a presheaf of modules or algebras, then the algebraic operations can be transferred fiberwise to the etale space $\mathcal{E}(\mathcal{P})$. For example, addition gives a mapping from the fiberwise product

$$\mathcal{E}(\mathcal{P}) \circ \mathcal{E}(\mathcal{P}) := \{(\alpha, \beta) \in \mathcal{E}(\mathcal{P}) \times \mathcal{E}(\mathcal{P}) \mid \pi_{\mathcal{P}}(\alpha) = \pi_{\mathcal{P}}(\beta)\}$$

to $\mathcal{E}(\mathcal{P})$ defined as follows: let $f \in \mathcal{P}_a$, $g \in \mathcal{P}_b$ be such that

$$\alpha = \rho^a_{\pi_{\mathcal{P}}(\alpha)}(f), \quad \beta = \rho^b_{\pi_{\mathcal{P}}(\beta)}(g),$$

and let $c \in \pi_{\mathcal{P}}(\alpha)$ be some element such that $c \leq a \wedge b$. Then

$$\alpha + \beta := \rho^c_{\pi_{\mathcal{P}}(\alpha)}(\rho^a_c(f) + \rho^b_c(g))$$

is a well-defined element of $\mathcal{E}(\mathcal{P})$. It is routine to check that the algebraic operations

$$\begin{aligned} \mathcal{E}(\mathcal{P}) \circ \mathcal{E}(\mathcal{P}) &\longrightarrow \mathcal{E}(\mathcal{P}) \\ (\alpha, \beta) &\longmapsto \alpha - \beta \end{aligned}$$

(and $(\alpha, \beta) \mapsto \alpha\beta$ if $\mathcal{P}$ is a presheaf of algebras) and the scalar multiplication

$$\begin{aligned} \mathcal{E}(\mathcal{P}) &\longrightarrow \mathcal{E}(\mathcal{P}) \\ \alpha &\longmapsto r\alpha \end{aligned}$$

are continuous.

From the etale space $\mathcal{E}(\mathcal{P})$ over $\mathcal{Q}(\mathbb{L})$ we obtain a complete presheaf $\mathcal{P}^{\mathcal{Q}}$, that is, a sheaf on the topological space $\mathcal{Q}(\mathbb{L})$ by

$$\mathcal{P}^{\mathcal{Q}}(\mathcal{V}) := \Gamma(\mathcal{V}, \mathcal{E}(\mathcal{P})),$$

where $\mathcal{V} \subseteq \mathcal{Q}(\mathbb{L})$ is an open set and $\Gamma(\mathcal{V}, \mathcal{E}(\mathcal{P}))$ is the set of continuous sections of $\pi_{\mathcal{P}}$ over $\mathcal{V}$, i.e. the set of continuous mappings $s_{\mathcal{V}} : \mathcal{V} \to \mathcal{E}(\mathcal{P})$ such that $\pi_{\mathcal{P}} \circ s_{\mathcal{V}} = id_{\mathcal{V}}$. If $\mathcal{P}$ is a presheaf of modules, then $\Gamma(\mathcal{V}, \mathcal{E}(\mathcal{P}))$ is a module, too. This completes our generalization of the well-known process of sheafification of a presheaf. In contrast to the topological case, the sheaf $\mathcal{P}^{\mathcal{Q}}$ is not defined directly on the "space" the presheaf $\mathcal{P}$ is defined on, namely the lattice $\mathbb{L}$, but on the Stone spectrum $\mathcal{Q}(\mathbb{L})$ of the lattice. The sheafification involves a localization procedure, and this localization happens at the quasipoints $\mathfrak{B} \in \mathcal{Q}(\mathbb{L})$.

Definition 43 *The complete presheaf $\mathcal{P}^{\mathcal{Q}}$ on the Stone spectrum $\mathcal{Q}(\mathbb{L})$ is called the **sheaf associated to the presheaf $\mathcal{P}$ on $\mathbb{L}$**.*

Remark 44 *There is a highly developed theory of sheaves and generalizations of topological spaces in the form of topos theory, see e.g. [MacMoe92]. In contrast to the constructions here, those in topos theory are based on* distributive *lattices. This also holds for the advanced construction of* locales. *Moreover, there is no comparable localization procedure on a lattice in topos theory, where in some sense the role of points is downplayed and the role of neighbourhoods is emphasized and strongly generalized (see Ch. 3 in [MacMoe92]).*

2.6. Generalization to categories

Quasipoints and Stone spectra only need the minimum operation $\wedge$ in a lattice $\mathbb{L}$ for their definition, so their generalization to $\wedge$-semilattices is obvious. In her diploma thesis, P. Krallmann has started to examine a much stronger generalization, namely to categories [Kra04]. We will only give some definitions and basic results here in order to show how the concepts of quasipoints and Stone spectra can be generalized to a category-theoretical setting. The proofs run parallel to the case of lattices. Finally, an important example is introduced, the category of abelian subalgebras of a von Neumann algebra $\mathcal{R}$.

Definition 45 *Let $\mathcal{C}$ be a small category with an initial object 0 and a terminal object 1. A* ***filter base in the category*** *$\mathcal{C}$ is a non-empty subset $\mathcal{F} \subseteq \mathcal{C}$ of objects in $\mathcal{C}$ such that*

(i) F contains no initial object,

(ii) for all $A, B \in \mathcal{F}$, there is a $C \in \mathcal{F}$ and monomorphisms $f : C \to A$, $g : C \to B$.

Using Zorn's lemma, we obtain maximal filter bases again:

Definition 46 *Let $\mathcal{C}$ be a small category. A* ***quasipoint of*** *$\mathcal{C}$ is a non-empty subset $\mathfrak{B} \subseteq \mathcal{C}$ of objects in $\mathcal{C}$ such that*

(i) $\mathfrak{B}$ contains no initial object,

(ii) for all $A, B \in \mathfrak{B}$, there is some $C \in \mathfrak{B}$ and monos $f : C \to A$, $g : C \to B$,

(iii) $\mathfrak{B}$ is maximal with respect to (i) and (ii).

The set of quasipoints of the category $\mathcal{C}$ will be denoted by $\mathcal{Q}(\mathcal{C})$.

Lemma 47 *Let $\mathcal{C}$ be a small category, $\mathfrak{B} \in \mathcal{Q}(\mathcal{C})$ a quasipoint and $A \in \mathfrak{B}$. If $f : A \to B$ is a monomorphism in $\mathcal{C}$ such that B is no initial object, then $B \in \mathfrak{B}$.*

Proof. Let $\mathfrak{B}' := \mathfrak{B} \cup \{B\}$. Then $\mathfrak{B}'$ contains no initial object. Let $C \in \mathfrak{B}$. For A and C, there is some $D \in \mathfrak{B}$ and monos $g : D \to C$, $h : D \to A$, so there are monos $g : D \to C$ and $f \circ h : D \to B$. Thus $\mathfrak{B}'$ is a filter base, and from the maximality of $\mathfrak{B}$, we have $\mathfrak{B}' = \mathfrak{B}$, i.e. $B \in \mathfrak{B}$. ■

We define a topology on $\mathcal{Q}(\mathcal{C})$ just as before: let $A \in \mathcal{C}$ be a non-initial object, and let

$$\mathcal{Q}_A(\mathcal{C}) := \{\mathfrak{B} \in \mathcal{Q}(\mathcal{C}) \mid A \in \mathfrak{B}\}.$$

Then

$$\mathcal{Q}_A(\mathcal{C}) \cap \mathcal{Q}_B(\mathcal{C}) = \{\mathfrak{B} \in \mathcal{Q}(\mathcal{C}) \mid A, B \in \mathfrak{B}\}.$$

Let $\mathfrak{B}_0 \in \mathcal{Q}_A(\mathcal{C}) \cap \mathcal{Q}_B(\mathcal{C})$ and $C \in \mathfrak{B}_0$ be such that there are monos $f : C \to A$, $g : C \to B$. Then $\mathcal{Q}_C(\mathcal{C}) \subseteq \mathcal{Q}_A(\mathcal{C}) \cap \mathcal{Q}_B(\mathcal{C})$, and since $\mathfrak{B}_0 \in \mathcal{Q}_A(\mathcal{C}) \cap \mathcal{Q}_B(\mathcal{C})$ was chosen arbitrarily, it follows that $\mathcal{Q}_A(\mathcal{C}) \cap \mathcal{Q}_B(\mathcal{C})$ is the union of sets of the form $\mathcal{Q}_C(\mathcal{C})$. We have proven

Lemma 48 *The sets $\mathcal{Q}_A(\mathcal{C}) \subseteq \mathcal{Q}(\mathcal{C})$ defined above form the base of a topology on $\mathcal{Q}(\mathcal{C})$.*

Definition 49 *$\mathcal{Q}(\mathcal{C})$, equipped with the topology induced by the sets $\mathcal{Q}_A(\mathcal{C})$, is called the **Stone spectrum** of the small category $\mathcal{C}$.*

Lemma 50 *The Stone spectrum $\mathcal{Q}(\mathcal{C})$ of a small category $\mathcal{C}$ is a zero-dimensional topological space, in particular, it is totally disconnected.*

Proof. $\mathcal{Q}_A(\mathcal{C})$ is open per definition. Let $\mathfrak{B} \notin \mathcal{Q}_A(\mathcal{C})$, that is, $A \notin \mathfrak{B}$. Then there is some $B \in \mathfrak{B}$ such that there is no $E \in \mathfrak{B}$ with monos $f : E \to A$, $g : E \to B$, which implies

$$\mathcal{Q}_A(\mathcal{C}) \cap \mathcal{Q}_B(\mathcal{C}) = \varnothing,$$

i.e. $\mathcal{Q}_B(\mathcal{C}) \subseteq \mathcal{Q}(\mathcal{C}) \backslash \mathcal{Q}_A(\mathcal{C})$. Hence, the complement of $\mathcal{Q}_A(\mathcal{C})$ is open and $\mathcal{Q}_A(\mathcal{C})$ is closed. ■

Remark 51 *The Stone spectrum $\mathcal{Q}(\mathcal{C})$ is Hausdorff and completely regular, since the sets $\mathcal{Q}_A(\mathcal{C})$ are closed-open and hence their characteristic functions are continuous.*

We now turn to an example category that will play some role in this work, especially in ch. 5, "The Kochen-Specker theorem":

Definition 52 *Let $\mathcal{R}$ be a von Neumann algebra. Define a category $\mathfrak{A}(\mathcal{R})$ by:*

(i) the objects of $\mathfrak{A}(\mathcal{R})$ are the unital abelian von Neumann subalgebras $\mathcal{M}$ of $\mathcal{R}$ such that $I_{\mathcal{M}} = I_{\mathcal{R}}$,

(ii) there is a morphism $\mathcal{M}_1 \to \mathcal{M}_2$ whenever $\mathcal{M}_1 \subseteq \mathcal{M}_2$, and the morphism simply is the inclusion $\iota_{\mathcal{M}_1\mathcal{M}_2}$.

P. Krallmann has explored some features of this category and its Stone spectrum [Kra04] (in fact, Krallmann considered the case $\mathcal{R} = \mathcal{L}(\mathcal{H})$). Inclusion induces a partial order on the objects of $\mathfrak{A}(\mathcal{R})$. A $\wedge$-semilattice structure is defined by

$$\forall \mathcal{A}, \mathcal{B} \in \mathfrak{A}(\mathcal{R}) : \mathcal{A} \wedge \mathcal{B} := \mathcal{A} \cap \mathcal{B}.$$

Obviously, $\mathcal{A} \leq \mathcal{B}$ if and only if $\mathcal{A} \wedge \mathcal{B} = \mathcal{A}$. In contrast to this, there is no analogous $\vee$-operation since (the algebra generated by) $\mathcal{A} \cup \mathcal{B}$ is not abelian in general.

The simplest non-trivial abelian subalgebras of a von Neumann algebra $\mathcal{R}$ are of the form $\mathcal{A}(P) := \mathrm{lin}_{\mathbb{C}}\{P, I\} = \mathbb{C}I + \mathbb{C}P$, where $P \in \mathcal{P}(\mathcal{R})$, $P \neq 0, I$.

Clearly, there are quasipoints $\mathfrak{B} \in \mathcal{Q}(\mathfrak{A}(\mathcal{R}))$ of the form

$$\mathfrak{B}_P = \{\mathcal{A} \in \mathfrak{A}(\mathcal{R}) \mid \mathcal{A}(P) \subseteq \mathcal{A}\}.$$

One can show that these are precisely the atomic quasipoints of $\mathfrak{A}(\mathcal{R})$:

Lemma 53 *For all $\mathcal{A}, \mathcal{B} \in \mathfrak{A}(\mathcal{R})$, it holds that*

$$\mathcal{Q}_{\mathcal{A}}(\mathfrak{A}(\mathcal{R})) = \mathcal{Q}_{\mathcal{B}}(\mathfrak{A}(\mathcal{R})) \Longleftrightarrow \mathcal{A} = \mathcal{B}.$$

Proof. If $\mathcal{Q}_{\mathcal{A}}(\mathfrak{A}(\mathcal{R})) = \mathcal{Q}_{\mathcal{B}}(\mathfrak{A}(\mathcal{R}))$, then

$$\begin{aligned}\mathcal{Q}_{\mathcal{A}}(\mathfrak{A}(\mathcal{R})) &= \mathcal{Q}_{\mathcal{A}}(\mathfrak{A}(\mathcal{R})) \cap \mathcal{Q}_{\mathcal{B}}(\mathfrak{A}(\mathcal{R})) \\ &= \mathcal{Q}_{\mathcal{A}\cap\mathcal{B}}(\mathfrak{A}(\mathcal{R})).\end{aligned}$$

Hence, it suffices to show that from $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{Q}_{\mathcal{A}}(\mathfrak{A}(\mathcal{R})) = \mathcal{Q}_{\mathcal{B}}(\mathfrak{A}(\mathcal{R}))$, it follows that $\mathcal{A} = \mathcal{B}$. Assume that $\mathcal{A} \neq \mathcal{B}$. Then there is a projection $P \in \mathcal{B} \backslash \mathcal{A}$ and thus $\mathrm{lin}_{\mathbb{C}}\{P, I\} \cap \mathcal{A} = \mathbb{C}I$. Let $\mathfrak{B} \in \mathcal{Q}_{\mathcal{A}}(\mathfrak{A}(\mathcal{R})) = \mathcal{Q}_{\mathcal{B}}(\mathfrak{A}(\mathcal{R}))$ be a quasipoint which contains $\mathrm{lin}_{\mathbb{C}}\{P, I\}$. Then $\mathcal{B} \in \mathfrak{B}$ and $\mathcal{A} \in \mathfrak{B}$, but $\mathrm{lin}_{\mathbb{C}}\{P, I\} \cap \mathcal{A} = \mathbb{C}I$, which is a contradiction. ■

Proposition 54 $\mathfrak{B} \in \mathcal{Q}(\mathfrak{A}(\mathcal{R}))$ *is atomic if and only if there is some* $P \in \mathcal{P}(\mathcal{R})$, $P \neq 0, I$ *such that* $\mathfrak{B} = \mathfrak{B}_P$.

Proof. If there is some projection P such that $\mathfrak{B} = \mathfrak{B}_P$, then $Q_{\mathcal{A}(P)}(\mathfrak{A}(\mathcal{R})) = \{\mathfrak{B}\}$, i.e. $\mathfrak{B}$ is atomic.

Conversely, let $\mathfrak{B}$ be atomic and $\mathcal{A}_0 \in \mathfrak{B}$ with $\mathcal{Q}_{\mathcal{A}_0}(\mathfrak{A}(\mathcal{R})) = \{\mathfrak{B}\}$. Then for all $\mathcal{A} \in \mathfrak{B}$, $\mathfrak{B} \in \mathcal{Q}_{\mathcal{A}\cap\mathcal{A}_0}(\mathfrak{A}(\mathcal{R}))$ holds, so $\mathcal{Q}_{\mathcal{A}\cap\mathcal{A}_0}(\mathfrak{A}(\mathcal{R})) = \mathcal{Q}_{\mathcal{A}_0}(\mathfrak{A}(\mathcal{R}))$ for all $\mathcal{A} \in \mathfrak{B}$. The lemma above implies $\mathcal{A} \cap \mathcal{A}_0 = \mathcal{A}_0$, so $\mathcal{A}_0 \subseteq \mathcal{A}$. Thus $\mathcal{A}_0$ is a minimal element in $\mathfrak{B}$ and hence of the form $\mathcal{A}_0 = \mathrm{lin}_{\mathbb{C}}\{P, I\}$ for some non-trivial projection $P \in \mathcal{P}(\mathcal{R})$. ■

From this, a quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathfrak{A}(\mathcal{R}))$ is non-atomic if and only if $\bigcap_{\mathcal{A}\in\mathfrak{B}} \mathcal{A} = \mathbb{C}I$.

As shown above, an atomic quasipoint of the lattice $\mathbb{L}(\mathcal{H}) \simeq \mathcal{P}(\mathcal{L}(\mathcal{H}))$ is of the form $\mathfrak{B}_{\mathbb{C}x} = \{U \in \mathbb{L}(\mathcal{H}) \mid \mathbb{C}x \subseteq U\}$ (see example 34), so it corresponds to the projection $P_{\mathbb{C}x}$ onto the *one-dimensional* subspace $\mathbb{C}x \subseteq \mathcal{H}$. Compare this to the atomic quasipoints of the category $\mathfrak{A}(\mathcal{R})$: these are of the form $\mathfrak{B} = \mathfrak{B}_P$, where P is some arbitrary non-trivial projection in $\mathcal{P}(\mathcal{R})$, not necessarily projecting onto a one-dimensional subspace. So, even for the case $\mathcal{R} = \mathcal{L}(\mathcal{H})$, there is no simple relation between $\mathcal{Q}(\mathcal{R})$ and $\mathcal{Q}(\mathfrak{A}(\mathcal{R}))$. If $\mathcal{H}$ is infinite-dimensional, then there are non-atomic quasipoints $\mathfrak{B} \in \mathcal{Q}(\mathfrak{A}(\mathcal{L}(\mathcal{H})))$, as Krallmann shows in her diploma thesis by giving two examples. Thus, $\mathcal{Q}(\mathfrak{A}(\mathcal{L}(\mathcal{H})))$ is not discrete if $\dim \mathcal{H} = \infty$.

2.7. Observable functions

In this section, we will sketch the theory of the so-called observable functions. The results presented here are among the main achievements of de Groote's work and will appear in a more complete form soon [deG05]. In this section, in the related subsections 2.8.1, "Stone spectrum and Gelfand spectrum of an abelian von Neumann algebra", 2.8.2, "Observable functions and the Gelfand representation" and in section 2.9, "Classical and quantum observables", we will give all the necessary definitions and results, but only sketch some of the proofs. Instead, the reader is referred to de Groote's forthcoming work [deG05]. Some results can also be found in [deG01].

Observable functions are continuous real-valued functions on the Stone spectrum $\mathcal{Q}(\mathcal{R})$ of a von Neumann algebra $\mathcal{R}$. An observable function can either be obtained from a self-adjoint operator $A \in \mathcal{R}_{sa}$ or characterized intrinsically. The two views coincide (Thm. 73).

Mathematically, the mapping $A \mapsto f_A$ sending $A \in \mathcal{R}_{sa}$ to its observable function f_A amounts to a generalization of the Gelfand transform to arbitrary −in particular, non-abelian− von Neumann algebras. The Stone spectrum takes the role of a generalized Gelfand spectrum, while the observable functions are generalized Gelfand transforms. This will become clear in subsections 2.8.1 and 2.8.2, where it is shown that for *abelian* algebras we get back the Gelfand spectrum and transforms indeed.

Physically, a new view on observables is developed, since in quantum theory, the self-adjoint operators $A \in \mathcal{R}_{sa}$ are regarded as observables of a physical system. In section 2.9 below, we will show how this new mathematical picture can unify classical and quantum observables.

Observable functions will be used in applications in chapter 4. They constitute a major part of the theory, also motivating the interest in the classification of Stone spectra of (finite) von Neumann algebras (ch. 3).

2.7.1. Definition and basic properties

In this subsection, let $A \in \mathcal{R}_{sa}$ be a self-adjoint operator in a von Neumann algebra $\mathcal{R}$. In particular, A is bounded. Let $E^A = (E^A_\lambda)_{\lambda\in\mathbb{R}}$ be the spectral family of A. $\operatorname{sp} A$ denotes the spectrum of A. As is well known, the projection lattice $\mathcal{P}(\mathcal{R})$ of a von Neumann algebra $\mathcal{R}$ is complete. From now on, we will use the following

Notation 55 *Let $\mathcal{R}$ be a von Neumann algebra. A quasipoint of $\mathcal{R}$ means a quasipoint of $\mathcal{P}(\mathcal{R})$. Instead of $\mathcal{Q}(\mathcal{P}(\mathcal{R}))$, we will write $\mathcal{Q}(\mathcal{R})$ for the Stone spectrum of $\mathcal{P}(\mathcal{R})$. Accordingly, we will speak of the Stone spectrum of $\mathcal{R}$ instead of the Stone spectrum of $\mathcal{P}(\mathcal{R})$.*

The central definition is

Definition 56 *Let $A \in \mathcal{R}_{sa}$, and let $E^A = (E^A_\lambda)_{\lambda\in\mathbb{R}}$ be the spectral family of A. The function*

$$f_A : \mathcal{Q}(\mathcal{R}) \longrightarrow \mathbb{R},$$

defined by

$$f_A(\mathfrak{B}) := \inf\{\lambda \in \mathbb{R} \mid E^A_\lambda \in \mathfrak{B}\}$$

is called the ***observable function corresponding to*** *A.*

An important feature of observable functions is the following:

Theorem 57 *Let $A \in \mathcal{R}_{sa}$, and let $f_A : \mathcal{Q}(\mathcal{R}) \to \mathbb{R}$ be the observable function corresponding to A. Then*

$$\operatorname{im} f_A = \operatorname{sp} A.$$

Sketch of proof: The spectrum $\operatorname{sp} A$ of A consists of all $\lambda \in \mathbb{R}$ such that the spectral family E^A of A is non-constant on every neighbourhood of λ. It follows easily that $\operatorname{im} f_A \subseteq \operatorname{sp} A$. In order to show $\operatorname{sp} A \subseteq \operatorname{im} f_A$, for each $\lambda_0 \in \operatorname{sp} A$ one has to find a quasipoint $\mathfrak{B}_{\lambda_0}$ such that $f_A(\mathfrak{B}_{\lambda_0}) = \lambda_0$. Given the spectral family E^A, one can either find a decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_0 = \lim_{n\to\infty} \lambda_n$ and $E^A_{\lambda_{n+1}} < E^A_{\lambda_n}$ for all n or $E^A_\lambda < E^A_{\lambda_0}$ holds for all $\lambda < \lambda_0$ (these cases are not excluding each other). In both cases, one can easily find a quasipoint $\mathfrak{B}_{\lambda_0}$ with the required property $f_A(\mathfrak{B}_{\lambda_0}) = \lambda_0$. ■

Example 58 *Let $P \in \mathcal{P}(\mathcal{R})$ be a projection. The spectral family E^P of P is*

$$E^P_\lambda = \begin{cases} 0 & \text{for } \lambda < 0 \\ I - P & \text{for } 0 \leq \lambda < 1 \\ I & \text{for } 1 \leq \lambda \end{cases} .$$

Let $\chi_{\mathcal{Q}_{I-P}(\mathcal{R})}$ denote the characteristic function of the closed-open set $\mathcal{Q}_{I-P}(\mathcal{R})$. The observable function f_P of P is given by

$$f_P = 1 - \chi_{\mathcal{Q}_{I-P}(\mathcal{R})}.$$

In particular, f_P is a continuous function.

We will now show that the observable function f_A of a finite real-linear combination $A = \sum_{k=1}^n a_k P_k$ is continuous.

Lemma 59 *If $A \in \mathcal{R}_{sa}$ and $a \in \mathbb{R}$, then $f_{A+aI} = a + f_A$.*

The proof is simple and can be found in [deG05].

Let $A = \sum_{j=1}^n a_j P_j$ for pairwise orthogonal projections $P_1, ..., P_n$ and non-zero real numbers $a_1 < ... < a_n$, and choose $a > 0$ such that $a_j - a < 0$ for all $j = 1, ..., n$. A straightforward calculation shows that the observable function f_{A-aI} of the "translated" operator $A - aI$ is of the form

$$f_{A-aI} = \sum_{k=1}^{n+1} b_k \chi_{\mathcal{Q}_{Q_1+...+Q_k}(\mathcal{R}) \setminus \mathcal{Q}_{Q_1+...+Q_{k-1}}(\mathcal{R})}$$

for orthogonal projections Q_i ($i = 1, ...k+1$, $Q_0 := 0$) with sum I. This result, together with Lemma 59, proves that the observable function f_A of a finite real-linear combination $A := \sum_{k=1}^n a_k P_k$ with pairwise orthogonal P_k is a step function:

Proposition 60 *Let $P_1, P_2, \ldots, P_n \in \mathcal{P}(\mathcal{R})$ be pairwise orthogonal projections and $A := \sum_{k=1}^n a_k P_k$ with real coefficients $a_1, a_2, \ldots, a_n$. Then, setting $P_0 := 0$,*

$$f_A = \sum_{k=1}^{n} a_k \chi_{\mathcal{Q}_{P_1+...+P_k}(\mathcal{R}) \setminus \mathcal{Q}_{P_1+...+P_{k-1}}(\mathcal{R})}.$$

Therefore, f_A is continuous.

For details, see [deG05]. We note here that as a corollary of Lemma 17, we obtain

Lemma 61 *Let $\mathcal{M}$ be an abelian von Neumann algebra, and let $(P_j)_{j\in J}$ be a finite orthogonal family of projections in $\mathcal{M}$ such that $\sum_{j\in J} P_j = I$. Then*

$$\mathcal{Q}(\mathcal{M}) = \bigcup_{j\in J} \mathcal{Q}_{P_j}(\mathcal{M}).$$

If J is infinite, it may happen that $\bigcup_{j\in J} \mathcal{Q}_{P_j}(\mathcal{M})$ is only dense in $\mathcal{Q}(\mathcal{M})$.

Corollary 62 *If $\mathcal{M}$ is an abelian von Neumann algebra and $A = \sum_{k=1}^{n} a_k P_k$ is as in Prop. 60, then*

$$f_A = \sum_{k=1}^{n} a_k \chi_{\mathcal{Q}_{P_k}(\mathcal{M})}.$$

Proof. If $\mathcal{M}$ is abelian, then the projection lattice $\mathcal{P}(\mathcal{M})$ is distributive, so according to the above lemma (and Lemma 36),

$$\mathcal{Q}_{P_1+\ldots+P_k}(\mathcal{M}) = \bigcup_{j=1}^{k} \mathcal{Q}_{P_j}(\mathcal{M}).$$

■

We will now use the step functions of Prop. 60 to uniformly approximate an arbitrary observable function f_A in order to show that f_A is continuous:

Theorem 63 *Let $A \in \mathcal{R}_{sa}$. Then the observable function $f_A : \mathcal{Q}(\mathcal{R}) \to \mathbb{R}$ is continuous.*

Sketch of proof: Using a partition of the spectrum $\operatorname{sp} A$ of A with a width smaller than $\varepsilon > 0$, the operator A can be approximated as

$$A_\varepsilon := \sum_{k=1}^{n} \lambda_k^* (E^A_{\lambda_k} - E^A_{\lambda_{k-1}}) := \sum_{k=1}^{n} \lambda_k^* F_k,$$

where $F_k := E^A_{\lambda_k} - E^A_{\lambda_{k-1}}$. Then, by Prop. 60,

$$\begin{aligned} f_{A_\varepsilon} &= \sum_{k=1}^{n} \lambda_k^* \chi_{\mathcal{Q}_{F_1+\ldots+F_k}(\mathcal{R}) \backslash \mathcal{Q}_{F_1+\ldots+F_{k-1}}(\mathcal{R})} \\ &= \sum_{k=1}^{n} \lambda_k^* \chi_{\mathcal{Q}_{E^A_{\lambda_k}}(\mathcal{R}) \backslash \mathcal{Q}_{E^A_{\lambda_{k-1}}}(\mathcal{R})}. \end{aligned}$$

Each $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ is contained in exactly one set $\mathcal{Q}_{E^A_{\lambda_k}}(\mathcal{R}) \backslash \mathcal{Q}_{E^A_{\lambda_{k-1}}}(\mathcal{R})$ for some appropriate k, so $f_{A_\varepsilon}(\mathfrak{B}) = \lambda_k^*$ and

$$f_A(\mathfrak{B}) = \inf\{\lambda |\ E^A_\lambda \in \mathfrak{B}\} \in [\lambda_{k-1}, \lambda_k].$$

This implies $|f_A(\mathfrak{B}) - f_{A_\varepsilon}(\mathfrak{B})| < \varepsilon$. Since $\mathfrak{B}$ is arbitrary, we have

$$|f_A - f_{A_\varepsilon}|_\infty \leq \varepsilon,$$

so f_A is continuous. ■

Definition 64 *Let $\mathcal{R}$ be a von Neumann algebra. The set of observable functions $\mathcal{Q}(\mathcal{R}) \to \mathbb{R}$ is denoted by $\mathcal{O}(\mathcal{R})$.*

Thm. 63 shows that $\mathcal{O}(\mathcal{R})$ is a subset of $C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$, the algebra of bounded continuous functions $\mathcal{Q}(\mathcal{R}) \to \mathbb{R}$. Let $\mathfrak{B}, \mathfrak{B}' \in \mathcal{Q}(\mathcal{R})$ be two distinct quasipoints, and let $P \in \mathfrak{B}$ be a projection which is not contained in $\mathfrak{B}'$. Then $f_{I-P}(\mathfrak{B}) = 1 - \chi_{\mathcal{Q}_P(\mathcal{R})}(\mathfrak{B}) = 0$ and $f_{I-P}(\mathfrak{B}') = 1$, so $\mathcal{O}(\mathcal{R})$ separates the points of $\mathcal{Q}(\mathcal{R})$. Taking $A = 0 \in \mathcal{R}$, Lemma 59 shows that $\mathcal{O}(\mathcal{R})$ contains the constant functions.

If $\mathcal{R}$ is non-abelian, the pointwise defined multiplication on $\mathcal{O}(\mathcal{R})$ obviously cannot correspond to the multiplication in $\mathcal{R}$. Moreover, $\mathcal{O}(\mathcal{R})$ is neither an algebra nor even a linear space with respect to the pointwise defined algebraic operations in general:

Theorem 65 *Let $\mathcal{R}$ be a von Neumann algebra and let $\mathcal{O}(\mathcal{R})$ be the set of observable functions on $\mathcal{Q}(\mathcal{R})$. Then*

$$\mathcal{O}(\mathcal{R}) = C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$$

if and only if $\mathcal{R}$ is abelian.

For the proof, see [deG05].

2.7.2. Abstract characterization of observable functions

In this subsection, we will extend the domain of definition of observable functions from the Stone spectrum $\mathcal{Q}(\mathcal{R})$ to $\mathcal{D}(\mathcal{R})$, the set of *dual ideals* of $\mathcal{P}(\mathcal{R})$. We will then present some properties of these observable functions. Making those properties into a definition, *abstract observable functions* are introduced.

Definition 66 *Let $\mathbb{L}$ be a complete lattice. A nonempty subset $\mathcal{I} \subseteq \mathbb{L}$ is called a* ***dual ideal*** *if it has the following properties:*

(i) $0 \notin \mathcal{I}$,

(ii) $a, b \in \mathcal{I} \implies a \wedge b \in \mathcal{I}$,

(iii) if $a \in \mathcal{I}$ and $a \leq b$, then $b \in \mathcal{I}$.

Let $\mathcal{D}(\mathbb{L})$ denote the set of dual ideals of $\mathbb{L}$. Let $a \in \mathbb{L}\backslash\{0\}$. The ***principal dual ideal generated by*** *a is*

$$H_a := \{b \in \mathbb{L} \mid b \geq a\}.$$

A maximal dual ideal is a quasipoint of $\mathbb{L}$, so obviously $\mathcal{Q}(\mathcal{R}) \subseteq \mathcal{D}(\mathcal{R})$. For $a \in \mathbb{L}$, let

$$\mathcal{D}_a(\mathbb{L}) := \{\mathcal{I} \in \mathcal{D}(\mathbb{L}) \mid a \in \mathcal{I}\}.$$

Since $\mathcal{D}_{a \wedge b}(\mathbb{L}) = \mathcal{D}_a(\mathbb{L}) \cap \mathcal{D}_b(\mathbb{L})$, $\{\mathcal{D}_a(\mathbb{L}) \mid a \in \mathbb{L}\}$ is the base of a topology on $\mathcal{D}(\mathbb{L})$. The Stone spectrum $\mathcal{Q}(\mathbb{L})$ is dense in $\mathcal{D}(\mathbb{L})$ with respect to this topology. $\mathcal{D}(\mathbb{L})$ is *not a Hausdorff space* in general.

Now let $\mathbb{L}$ be the projection lattice $\mathcal{P}(\mathcal{R})$ of a von Neumann algebra $\mathcal{R}$ again. We extend the domain of definition of the observable function f_A:

Definition 67 *Let $A \in \mathcal{R}_{sa}$ be a self-adjoint operator, E^A the spectral family of A and f_A the observable function of A. f_A is extended to a function $\mathcal{D}(\mathcal{R}) \to \mathbb{R}$ on the space $\mathcal{D}(\mathcal{R})$ of dual ideals of $\mathcal{P}(\mathcal{R})$ (denoted by f_A again):*

$$\forall \mathcal{I} \in \mathcal{D}(\mathcal{R}) : \ f_A(\mathcal{I}) := \inf\{\lambda \mid E^A_\lambda \in \mathcal{I}\}.$$

The main property of observable functions that will serve as a defining condition of abstract observable functions below (Def. 71) is the **intersection property**:

Proposition 68 *Let $(\mathcal{I}_j)_{j\in J}$ be a family in $\mathcal{D}(\mathcal{R})$. Then*

$$f_A(\bigcap_{j\in J} \mathcal{I}_j) = \sup_{j\in J} f_A(\mathcal{I}_j).$$

Sketch of proof: The dual ideal $\mathcal{I} := \bigcap_{j\in J} \mathcal{I}_j$ is contained in all the $\mathcal{I}_j$, so $\sup_j f_A(\mathcal{I}_j) \leq f_A(\mathcal{I})$. It is easy to see that for $\varepsilon > 0$, there is some $\mathcal{I}_{j_0}$ such that $f_A(\mathcal{I}) - \varepsilon < f_A(\mathcal{I}_{j_0}) \leq \sup_j f_A(\mathcal{I}_j)$, which implies $f_A(\mathcal{I}) \leq \sup_j f_A(\mathcal{I}_j)$, since ε is arbitrary. ■

In the following, we merely collect some results concerning observable functions. Proofs can be found in [deG05].

Proposition 69 *Let $A \in \mathcal{R}_{sa}$. The observable function $f_A : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$ is upper semi-continuous, but not continuous in general.*

Proposition 70 *For any function $f : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$, the following two properties are equivalent:*

(i) f is upper semicontinuous and decreasing (i.e. $\mathcal{I}_1 \subseteq \mathcal{I}_2 \implies f(\mathcal{I}_2) \leq f(\mathcal{I}_1)$).

(ii) $\forall\, \mathcal{I} \in \mathcal{D}(\mathcal{R}) : \ f(\mathcal{I}) = \inf\{f(H_P) |\ P \in \mathcal{I}\}$.

As a corollary of the results above, one obtains $f_A(\mathcal{D}(\mathcal{R})) = \operatorname{sp} A$ for all $A \in \mathcal{R}_{sa}$.

Definition 71 *A function $f : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$ is called an* ***abstract observable function*** *if it is upper semicontinuous and satisfies the* intersection condition

$$f(\bigcap_{j\in J} \mathcal{I}_j) = \sup_{j\in J} f(\mathcal{I}_j)$$

for all families $(\mathcal{I}_j)_{j\in J}$ in $\mathcal{D}(\mathcal{R})$.

The intersection property implies that an abstract observable function is decreasing, so by Prop. 70, the definition of an abstract observable function can be reformulated:

Remark 72 *$f : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$ is an observable function if and only if the following two properties hold for f:*

(i) $\forall\, \mathcal{I} \in \mathcal{D}(\mathcal{R}) : f(\mathcal{I}) = \inf\{f(H_P) |\ P \in \mathcal{I}\}$,

(ii) $f(\bigcap_{j\in J}\mathcal{I}_j)=\sup_{j\in J} f(\mathcal{I}_j)$ *for all families* $(\mathcal{I}_j)_{j\in J}$ *in* $\mathcal{D}(\mathcal{R})$*.*

Let $\lambda \in \operatorname{im} f$. The intersection property implies that the preimage $\overset{-1}{f}(\lambda) \subseteq \mathcal{D}(\mathcal{R})$ has a minimal element $\mathcal{I}_\lambda$, given by

$$\mathcal{I}_\lambda = \bigcap \{\mathcal{I} \in \mathcal{D}(\mathcal{R}) |\ f(\mathcal{I}) = \lambda\}.$$

One can show that the spectral family E^A of $A \in \mathcal{R}_{sa}$ can be reconstructed from the observable function f_A by setting $E^A_\lambda := \inf \mathcal{I}_\lambda$ for all $\lambda \in \operatorname{im} f = \operatorname{sp} A$. Moreover, an observable function $f_A : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$ is uniquely determined by its restriction to $\mathcal{Q}(\mathcal{R})$, and a selfadjoint operator $A \in \mathcal{R}$ is uniquely determined by its observable function f_A.

The next theorem is central, since it shows that the two notions of observable function and abstract observable function coincide. We will only sketch the proof here and again refer to [deG05] for details.

Theorem 73 *Let* $f : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$ *be an abstract observable function. Then there is a unique* $A \in \mathcal{R}_{sa}$ *such that* $f = f_A$*.*

Sketch of proof: The proof proceeds in three steps. *First*, from the abstract observable function f an increasing family $(E_\lambda)_{\lambda\in\operatorname{im} f}$ in $\mathcal{P}(\mathcal{R})$ is constructed: let $\lambda \in \operatorname{im} f$, and let $\mathcal{I}_\lambda \in \mathcal{D}(\mathcal{R})$ be the smallest dual ideal such that $f(\mathcal{I}_\lambda) = \lambda$. The results mentioned above suggest the definition

$$E_\lambda := \inf \mathcal{I}_\lambda.$$

It is easy to see that $(E_\lambda)_{\lambda\in\operatorname{im} f}$ is an increasing family. Then it is shown that f is monotonely continuous, i.e. if $(\mathcal{I}_j)_{j\in J}$ is an increasing net in $\mathcal{D}(\mathcal{R})$, then we have

$$f(\bigcup_{j\in J} \mathcal{I}_j) = \lim_j f(\mathcal{I}_j).$$

Next, one shows that the image $\operatorname{im} f$ of the abstract observable function f is compact.

In the *second step*, $(E_\lambda)_{\lambda\in\operatorname{im} f}$ must be extended to a spectral family $E^f := (E_\lambda)_{\lambda\in\mathbb{R}}$. Of course, on the one hand we have to guarantee that the spectrum of the selfadjoint operator A corresponding to E^f coincides with $\operatorname{im} f$. On the other hand, E^f must be continuous from the right. To achieve this, for $\lambda \notin \operatorname{im} f$, let

$$S_\lambda := \{\mu \in \operatorname{im} f \mid \mu < \lambda\}.$$

Then define

$$E_\lambda := \begin{cases} 0 & \text{if } S_\lambda = \varnothing \\ E_{sup\ S_\lambda} & \text{otherwise.} \end{cases}$$

Note that $f(\{I\}) = \max\ \operatorname{im} f$ and that $\mathcal{I}_{f(\{I\})} = \{I\}$. It can easily be shown that E^f is continuous from the right, that is, $E_\lambda = \bigwedge_{\mu>\lambda} E_\mu$ holds for all $\lambda \in \mathbb{R}$.

In the *third step*, one has to show that the selfadjoint operator $A \in \mathcal{R}$ corresponding to the spectral family E^f has observable function $f_A = f$ and that A is uniquely determined by f: while it is obvious from the definition of E^f that $sp(A) \subseteq \operatorname{im} f$ holds, the reverse inclusion requires a little work. The uniqueness of A then simply follows from the fact that a self-adjoint operator A is uniquely determined by its observable function f_A: if $A, B \in \mathcal{R}_{sa}$ such that $f_A = f = f_B$, then $A = B$. This completes the proof. ■

Having shown this, one can simply speak of observable functions (instead of "concrete" and "abstract" ones).

2.8. Abelian von Neumann algebras, Boolean quasipoints and sectors

In this section, we will consider abelian von Neumann algebras, their quasipoints and Stone spectra. It is often interesting to regard an abelian algebra $\mathcal{M}$ as a subalgebra of a larger, non-abelian von Neumann algebra $\mathcal{R}$. (An important example in physics is the Kochen-Specker theorem, see ch. 5.) We will see that the notions of Boolean quasipoints and Boolean sectors are adapted to this situation.

The relation between the Stone spectrum $\mathcal{Q}(\mathcal{M})$ of an abelian von Neumann algebra and its Gelfand spectrum $\Omega(\mathcal{M})$ is shown in subsec. 2.8.1; these spaces are homeomorphic. In subsec. 2.8.2, the proof is sketched that the observable functions are the restrictions of the Gelfand transforms to the self-adjoint part $\mathcal{M}_{sa}$ of our abelian algebra $\mathcal{M}$. In subsec. 2.8.3, it is shown that every abelian von Neumann algebra $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ can be seen as a *maximal abelian* von Neumann algebra on an appropriate Hilbert space $\mathcal{K}$. Finally, in subsec. 2.8.4, a surprisingly simple relation between compact Hausdorff spaces and Stone spectra of distributive projection lattices is presented.

We start with the basic definitions and results:

Definition 74 *Let $\mathcal{R}$ be a von Neumann algebra. A maximal distributive sublattice $\mathbb{B}$ of $\mathcal{P}(\mathcal{R})$ is called a* ***Boolean sector of*** *$\mathcal{P}(\mathcal{R})$ (and of $\mathcal{R}$).*

Proposition 75 *The maximal abelian subalgebras of a von Neumann algebra $\mathcal{R}$ are in one-to-one correspondence with the Boolean sectors of the projection lattice $\mathcal{P}(\mathcal{R})$ of $\mathcal{R}$.*

Proof. Let $\mathbb{B}$ be a Boolean sector of $\mathcal{P}(\mathcal{R})$, and let $\mathbb{B}'' := \{P \mid P \in \mathbb{B}\}''$ be the abelian von Neumann algebra generated by $\mathbb{B}$. Let $A \in \mathcal{R}$ be an operator commuting with all the elements of $\mathbb{B}''$. Then all spectral projections $E^A(S)$ (where $S \subseteq \mathbb{R}$ is a Borel set) of A commute with $\mathbb{B}''$, in particular also with $\mathbb{B}$. Since $\mathbb{B}$ is a maximal distributive sublattice, all the $E^A(S)$ are contained in $\mathbb{B}$ and hence A is contained in $\mathbb{B}''$, that is, $W^*(\mathbb{B}) := \mathbb{B}''$ is a maximal abelian subalgebra of $\mathcal{R}$.

Conversely, let $\mathcal{A} \subseteq \mathcal{R}$ be a maximal abelian subalgebra of $\mathcal{R}$ with projection lattice $\mathcal{P}(\mathcal{A})$. Let $P \in \mathcal{R}$ be a projection commuting with all elements of $\mathcal{P}(\mathcal{A})$. Then P commutes with all $A \in \mathcal{A}$, so from maximality of $\mathcal{A}$ it follows that $P \in \mathcal{A}$ holds and hence $P \in \mathcal{P}(\mathcal{A})$. Thus $\mathcal{P}(\mathcal{A})$ is a maximal distributive sublattice of $\mathcal{P}(\mathcal{R})$, that is, a Boolean sector. ■

We will use the notation $W^*(\mathbb{B})$ for the abelian von Neumann algebra $\mathbb{B}''$ generated by a Boolean sector $\mathbb{B}$.

Definition 76 *Let $\mathcal{R}$ be a von Neumann algebra. A subset $\beta \subseteq \mathcal{P}(\mathcal{R})$ is called a* ***Boolean quasipoint*** *of $\mathcal{P}(\mathcal{R})$ if*

(i) $0 \notin \beta$,

(ii) $\forall P, Q \in \beta \ \exists R \in \beta : R \leq P \wedge Q$,

(iii) $\forall P, Q \in \beta : PQ = QP$,

(iv) β is maximal with respect to the properties (i)-(iii).

By property (iii), the orthocomplemented sublattice of $\mathcal{P}(\mathcal{R})$ generated by a Boolean quasipoint β is distributive. A Boolean quasipoint of $\mathcal{R}$ means a Boolean quasipoint of $\mathcal{P}(\mathcal{R})$.

Lemma 77 *Let $\mathcal{R}$ be a von Neumann algebra, β a Boolean quasipoint of $\mathcal{R}$. Then β generates a maximal abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$.*

Proof. $\mathcal{M} := \{P \mid P \in \beta\}''$ is an abelian von Neumann subalgebra of $\mathcal{R}$. Let $\mathbb{B}$ be a Boolean sector containing β (and hence $\mathcal{P}(\mathcal{M})$). Assume that $\mathcal{M}$ is not maximal, then there is a projection $Q \in \mathbb{B}$ such that Q, $I - Q \notin \mathcal{P}(\mathcal{M})$. From maximality, the Boolean quasipoint β contains either Q or $I - Q$ (see Lemma 17), so $Q \in \mathcal{M} = \{P \mid P \in \beta\}''$, contradicting our assumption. ■

By the last lemma and Prop. 75, the sublattice of $\mathcal{P}(\mathcal{R})$ generated by a Boolean quasipoint β is a Boolean sector $\mathbb{B} \subseteq \mathcal{P}(\mathcal{R})$.

Remark 78 *Let $\mathcal{R}$ be a von Neumann algebra, $\mathbb{B} \subseteq \mathcal{P}(\mathcal{R})$ a Boolean sector, and let $\beta \subseteq \mathbb{B}$. β is a Boolean quasipoint of $\mathcal{P}(\mathcal{R})$ if and only if it is a quasipoint of $\mathcal{M}(\mathbb{B}) := \{P \mid P \in \mathbb{B}\}''$.*

Proof. Obviously, a Boolean quasipoint of $\mathcal{R}$ that is contained in $\mathbb{B}$ is a quasipoint of $\mathcal{M}(\mathbb{B})$. Conversely, let β be a quasipoint of $\mathcal{M}(\mathbb{B})$. Assume that β is not a Boolean quasipoint of $\mathcal{R}$. Then there is a Boolean quasipoint $\widetilde{\beta}$ of $\mathcal{R}$ which contains β. Let $P_0 \in \widetilde{\beta} \backslash \beta$. From the maximality of β, $P_0 \notin \mathbb{B}$ and there is some $P_1 \in \mathbb{B}$ not commuting with P_0. Since β is a quasipoint of the *distributive* lattice $\mathbb{B}$, either $P_1 \in \beta$ or $P_1^{\perp} \in \beta$. It follows $P_1 \in \widetilde{\beta}$ or $P_1^{\perp} \in \widetilde{\beta}$. In both cases, P_0 must commute with P_1, since both are contained in the same Boolean quasipoint $\widetilde{\beta}$, which gives the contradiction. ■

The name "Boolean sector" is justified by the following

Proposition 79 *Each Boolean quasipoint β of $\mathcal{R}$ is contained in exactly one Boolean sector of $\mathcal{R}$.*

Proof. Let β be a Boolean quasipoint, and let $\mathbb{B}_1$, $\mathbb{B}_2$ be Boolean sectors such that $\beta \subseteq \mathbb{B}_1 \cap \mathbb{B}_2$. Let $P \in \mathbb{B}_1$. Then $P \in \beta$ or $P^\perp \in \beta$ and hence $P \in \mathbb{B}_2$ or $P^\perp \in \mathbb{B}_2$. Since $\mathbb{B}_2$ is a Boolean lattice, $P \in \mathbb{B}_2$ follows. The argument is symmetric. ■

Example 80 *Let $\mathcal{R} = \mathcal{L}(\mathcal{H})$. It is well known that there are three different types of maximal abelian von Neumann subalgebras $\mathcal{M}_a$, $\mathcal{M}_c$ and $\mathcal{M}_m^n$ of $\mathcal{L}(\mathcal{H})$ (see Thm. 9.4.1 in [KadRinII97]):*

(i) there is an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ such that $\mathcal{M}_a$ is the algebra generated by the projections P_{e_n},

(ii) $\mathcal{M}_c$ is unitarily equivalent to $L_\infty(]0,1[,\mathcal{S},\mu)$, where μ is the Lebesgue measure on the σ-algebra $\mathcal{S}$ of Borel subsets of $]0,1[$.

(iii) $\mathcal{M}_m^n$ is of the form $\mathcal{M}_a \oplus \mathcal{M}_c$. Here, $\mathcal{M}_a$ is defined on an n-dimensional Hilbert space $\mathcal{H}_a$ $(1 \leq n \leq \aleph_0)$.

The corresponding Boolean sectors of $\mathcal{P}(\mathcal{H})$, which are simply the projection lattices of the maximal abelian algebras (see Prop. 75), are called (i) ***purely atomic****, (ii)* ***purely continuous*** *and (iii)* ***mixed of type*** *n, respectively. (Compare the constructions in subsec. 2.8.3 below.)*

It is obvious that each Boolean quasipoint β, being a filter base in $\mathcal{P}(\mathcal{R})$, is contained in some quasipoint $\mathfrak{B}$ of $\mathcal{R}$. On the other hand, it is not clear if every quasipoint of $\mathcal{R}$ contains a Boolean quasipoint, not even for the case $\mathcal{R} = \mathcal{L}(\mathcal{H})$: if $\mathfrak{B}_{\mathbb{C}x} \in \mathcal{Q}(\mathcal{R})$ is an atomic quasipoint, then there are Boolean quasipoints contained in $\mathfrak{B}_{\mathbb{C}x}$, but for a continuous quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$, this remains to be clarified.

2.8.1. Stone spectrum and Gelfand spectrum of an abelian von Neumann algebra

Thm. 81 is one of the most important results of the whole theory, showing that the Stone spectrum and the Gelfand spectrum of an abelian von Neumann (sub)algebra $\mathcal{M}$ are homeomorphic.

Theorem 81 *Let $\mathcal{R}$ be a von Neumann algebra, $\mathbb{B}$ a Boolean sector of $\mathcal{P}(\mathcal{R})$, $\mathcal{M}(\mathbb{B})$ the maximal abelian von Neumann subalgebra generated by $\mathbb{B}$. Then the Stone spectrum $\mathcal{Q}(\mathbb{B})$ is homeomorphic to the Gelfand spectrum $\Omega(\mathcal{M}(\mathbb{B}))$ of $\mathcal{M}(\mathbb{B})$.*

The proof can be found in [deG01], Thm. 5.2. The C^*-algebra $C^*(\mathbb{B})$ in Thm. 5.2 is the von Neumann algebra $\mathcal{M}(\mathbb{B})$. Thm. 5.2 is formulated for the von Neumann algebra $\mathcal{R} = \mathcal{L}(\mathcal{H})$, but actually the proof works for arbitrary von Neumann algebras. The final clause in Thm. 5.2 ("With respect to this homeomorphism the strongly continuous characters correspond to the atomic quasipoints in $\mathbb{B}$") only holds for $\mathcal{R} = \mathcal{L}(\mathcal{H})$.

Sketch of proof: Let Ω denote the Gelfand spectrum of $\mathcal{M}(\mathbb{B})$, and let $\tau \in \Omega$ be a character. Then $\beta_\tau := \{P \in \mathbb{B}\backslash\{0\} \mid \tau(P) = 1\}$ is a quasipoint of $\mathbb{B}$. Conversely, let $\beta \in \mathcal{Q}(\mathcal{M}(\mathbb{B}))$. The mapping $\tau_\beta : \mathbb{B} \to \{0, 1\}$ given by

$$\tau_\beta(P) := \begin{cases} 1 & \text{if } P \in \beta \\ 0 & \text{if } P \notin \beta. \end{cases}$$

can be linearly extended to a continuous linear functional on $\mathrm{lin}_{\mathbb{C}}\,\mathbb{B}$, so it can further be extended to a character $\tau_\beta \in \Omega$ of $\mathcal{M}(\mathbb{B})$, and by construction, $\beta_{\tau_\beta} = \beta$. Continuity of $\tau \mapsto \beta_\tau$ follows easily. Since Ω and $\mathcal{Q}(\mathcal{M}(\mathbb{B}))$ are compact, the mapping $\tau \mapsto \beta_\tau$ is a homeomorphism. ■

A refined version of the proof will appear in [deG05].

2.8.2. Observable functions and the Gelfand representation

In this subsection, we will sketch a proof of the following: the mapping $A \mapsto f_A$ from an abelian von Neumann algebra $\mathcal{M}$ onto $C_b(\mathcal{Q}(\mathcal{M}), \mathbb{R})$ is the Gelfand transformation of $\mathcal{M}$ up to the isomorphism $C(\mathcal{Q}(\mathcal{M}), \mathbb{R}) \to C(\Omega(\mathcal{M}), \mathbb{R})$. (We have $C_b(\mathcal{Q}(\mathcal{M}), \mathbb{R}) = C(\mathcal{Q}(\mathcal{M}), \mathbb{R})$ since $\mathcal{P}(\mathcal{M})$ is distributive, so $\mathcal{Q}(\mathcal{M}) = \mathcal{Q}(\mathcal{P}(\mathcal{M}))$ is compact from Stone's theorem.) For details, see [deG05]. A closely related proof can already be found in [deG01], Thm. 6.3.

Let $A \in \mathrm{lin}_{\mathbb{C}}\,\mathcal{P}(\mathcal{M})$ and consider some orthogonal representation $A = \sum_{j=1}^{m} b_j P_j$. Then

$$\mathcal{F}_{\mathcal{M}}(A) := \sum_{j=1}^{m} b_j \chi_{\mathcal{Q}_{P_j}(\mathcal{M})}$$

is the observable function of A on $\mathcal{Q}(\mathcal{M})$ by Cor. 62. It is easy to see that $\mathcal{F}_{\mathcal{M}}(A)$ is well defined. Thus, we have a mapping

$$\mathcal{F}_{\mathcal{M}} : lin_{\mathbb{C}}\mathcal{P}(\mathcal{M}) \to C(\mathcal{Q}(\mathcal{M})).$$

Proposition 82 $\mathcal{F}_{\mathcal{M}} : lin_{\mathbb{C}}\mathcal{P}(\mathcal{M}) \to C(\mathcal{Q}(\mathcal{M}))$ *is an isometric homomorphism of algebras.*

Sketch of proof: Let $A, B \in \mathrm{lin}_{\mathbb{C}}\,\mathcal{P}(\mathcal{M})$. A straightforward calculation gives an orthogonal representation of $A + B$. Inserting this into $\mathcal{F}_{\mathcal{M}}$, we easily see that $\mathcal{F}_{\mathcal{M}}$ is $\mathbb{C}$-linear. Multiplicativity can be shown in a similar manner. The important point is isometry: for this, let $A = \sum_{j=1}^{m} b_j P_j$ be an orthogonal representation of $A \in lin_{\mathbb{C}}\mathcal{P}(\mathcal{M})$. Then

$$|A| = \max_{j \le m} |b_j|$$

and

$$|\sum_{j=1}^{m} b_j \chi_{\mathcal{Q}_{P_j}(\mathcal{M})}|_\infty = \max_{j \le m} |b_j|,$$

since the sets $\mathcal{Q}_{P_j}(\mathcal{M})$ are pairwise disjoint (see Lemmas 17 and 61), so $\mathcal{F}_{\mathcal{M}}$ is isometric indeed. ■

From the Stone-Weierstraß theorem, it follows that $\mathrm{lin}_{\mathbb{C}}\{\chi_{\mathcal{Q}_P(\mathcal{M})} \mid P \in \mathcal{P}(\mathcal{M})\}$ is dense in $C(\mathcal{Q}(\mathcal{M}))$, so $\mathcal{F}_{\mathcal{M}}$ can be extended to an isometric $*$-isomorphism from $\mathcal{M}$ to $C(\mathcal{Q}(\mathcal{M}))$ in a unique way. The extension is surjective, too, and will also be denoted by $\mathcal{F}_{\mathcal{M}}$. In order to show that $\mathcal{F}_{\mathcal{M}} : \mathcal{M} \to C(\mathcal{Q}(\mathcal{M}))$ is the Gelfand transformation of the abelian von Neumann algebra $\mathcal{M}$, one must consider the evaluation

$$\begin{aligned} \varepsilon_\beta : C(\mathcal{Q}(\mathcal{M})) &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \varphi(\beta) \end{aligned}$$

at the quasipoint $\beta \in \mathcal{Q}(\mathcal{M})$. For all $P \in \mathcal{P}(\mathcal{M})$, one obtains $(\varepsilon_\beta \circ \mathcal{F}_{\mathcal{M}})(P) = \tau_\beta(P)$, so $\varepsilon_\beta \circ \mathcal{F}_{\mathcal{M}} = \tau_\beta$ holds on a dense part of $\mathcal{M}$ by linear extension. (For the definition and properties of τ_β, see Thm. 81). Since τ_β is continuous, equality holds on all of $\mathcal{M}$.

Now regard the Gelfand transformation

$$\Gamma : \mathcal{M} \to C(\Omega(\mathcal{M})), \quad A \mapsto \hat{A},$$

which is defined by

$$\forall\, \tau \in \Omega(\mathcal{M}) : \ \hat{A}(\tau) := \tau(A).$$

The homeomorphism $\theta : \beta \mapsto \tau_\beta$ from $\mathcal{Q}(\mathcal{M})$ onto $\Omega(\mathcal{M})$ induces a $*$-isomorphism

$$\begin{aligned} \theta^* : C(\Omega(\mathcal{M})) &\longrightarrow C(\mathcal{Q}(\mathcal{M})) \\ \varphi &\longmapsto \varphi \circ \theta. \end{aligned}$$

Since

$$\theta^*(\hat{A})(\beta) = \hat{A}(\theta(\beta)) = \hat{A}(\tau_\beta) = \tau_\beta(A) = \varepsilon_\beta(\mathcal{F}_{\mathcal{M}}(A)) = \mathcal{F}_{\mathcal{M}}(A)(\beta)$$

holds for all $A \in \mathcal{M}$ and all $\beta \in \mathcal{Q}(\mathcal{M})$, we obtain

$$\mathcal{F}_{\mathcal{M}} = \theta^* \circ \Gamma,$$

i.e., $\mathcal{F}_{\mathcal{M}}$ and the Gelfand transformation of $\mathcal{M}$ conincide up to the isomorphism
$C(\mathcal{Q}(\mathcal{M}), \mathbb{R}) \to C(\Omega(\mathcal{M}), \mathbb{R})$.

Theorem 83 *Let $\mathcal{M}$ be an abelian von Neumann algebra. Then the mapping $A \mapsto f_A$ from $\mathcal{M}$ onto $C(\mathcal{Q}(\mathcal{M}), \mathbb{R})$ is the restriction of the Gelfand transformation to $\mathcal{M}_{sa}$.*

Proof. With the above results in mind, we only have to show that $f_A = \mathcal{F}_{\mathcal{M}}(A)$ holds for *all* $A \in \mathcal{M}_{sa}$ (and not just for the finite linear combinations $A \in \mathrm{lin}_{\mathbb{C}}\, \mathcal{P}(\mathcal{M})$, for which $\mathcal{F}_{\mathcal{M}}(A)$ is the observable function of A by Cor. 62). Let A be an arbitrary element of $\mathcal{M}_{sa}$. The proof of theorem 63 shows that f_A is the uniform limit of observable functions f_B with $B \in lin_{\mathbb{R}}\mathcal{P}(\mathcal{M})$. So, by definition, $\mathcal{F}_{\mathcal{M}}(A)$ is the uniform limit of functions $\mathcal{F}_{\mathcal{M}}(B)$ with $B \in lin_{\mathbb{R}}\mathcal{P}(\mathcal{M})$ and $f_A = \mathcal{F}_{\mathcal{M}}(A)$. ■

Since every operator $B \in \mathcal{M}$ has a unique decomposition

$$B = A_1 + iA_a,$$

where A_1, A_2 are self-adjoint operators in $\mathcal{M}$, the mapping

$$\begin{aligned} \mathcal{M}_{sa} &\longrightarrow C(\mathcal{Q}(\mathcal{M}), \mathbb{R}) \\ A &\longmapsto f_A \end{aligned}$$

can be extended canonically to a mapping

$$\begin{aligned} \mathcal{M} &\longrightarrow C(\mathcal{Q}(\mathcal{M}), \mathbb{C}) \\ B &\longmapsto f_{A_1} + i f_{A_2}, \end{aligned}$$

where $B = A_1 + iA_2$ ($A_1, A_2 \in \mathcal{M}_{sa}$) is the decompostion of B mentioned above. Using the identification of Stone spectrum and Gelfand spectrum, this mapping is the Gelfand transformation of $\mathcal{M}$.

2.8.3. "Economic" representation of an abelian von Neumann algebra

In this subsection, we will show how an abelian von Neumann algebra $\mathcal{M}$ can be represented as a maximal abelian subalgebra of $\mathcal{L}(\mathcal{K})$ for some appropriate Hilbert space $\mathcal{K}$. This result is included as a nice application of the Stone spectrum $\mathcal{Q}(\mathcal{M})$ to well-known structures (compare sections 9.4, 9.3 in [KadRinII97] and section III.1 in [TakI02]).

Let Ω be a Stone space (in the usual sense, that is, a compact, extremely disconnected Hausdorff space), $\Omega = \Omega_1 \amalg \Omega_2$ a partition of Ω into closed-open sets Ω_1, Ω_2. Then

$$C(\Omega) \simeq C(\Omega_1) \times C(\Omega_2).$$

This isomorphy is simply given by

$$C(\Omega) \ni f \longmapsto (f|_{\Omega_1}, f|_{\Omega_2}).$$

Since Ω_k is closed-open, this mapping also is surjective. It is isometric since

$$|f|_\infty = \max(|f|_{\Omega_1}|_\infty, |f|_{\Omega_2}|_\infty).$$

A canonical partition of the Stone space Ω is

$$\Omega = \Omega_d \amalg \Omega_c,$$

where Ω_d is the closure of the set of isolated points of Ω. Ω_d is closed-open, since Ω is a Stone space.

Let Ω be hyperstonean, that is, it is stonean and there are sufficiently many positive normal measures on Ω (see subsec. 2.4.1).

$C(\Omega_d)$ can be seen as a subalgebra of $C(\Omega)$ via the embedding

$$\begin{aligned} C(\Omega_d) &\hookrightarrow C(\Omega) \\ f &\longmapsto f_0, \\ f_0(\omega) &:= \begin{cases} f(\omega) & \text{for } \omega \in (\Omega_d) \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

analogously for $C(\Omega_c)$. Hence each normal measure on Ω induces normal measures on Ω_d and Ω_c. Therefore, these spaces are hyperstonean, i.e. $C(\Omega), C(\Omega_d)$ and $C(\Omega_c)$ are abelian von Neumann algebras.

According to Thm. III.1.22 in [TakI02], $C(\Omega_c)$ is isomorphic to $L^\infty(]0,1[,\mu)$, where μ is the Lebesgue measure on $]0,1[$.

Lemma 84 *Let Ω be a Stone space, $\mathcal{CO}(\Omega)$ the lattice of closed-open subsets of Ω. Then*

$$\mathcal{Q}(\mathcal{CO}(\Omega)) \simeq \Omega,$$

i.e. the Stone spectrum *of the lattice $\mathcal{CO}(\Omega)$ of closed-open subsets of the Stone* space Ω *is homeomorphic to Ω itself.*

Proof. Let $\mathfrak{B}$ be a quasipoint $\mathcal{CO}(\Omega)$. Then $\bigcap_{U\in\mathfrak{B}} U$ is not empty, since all U are closed and Ω is compact. $\bigcap_{U\in\mathfrak{B}} U$ consists of exactly one point $\omega_{\mathfrak{B}} \in \Omega$, and so, from maximality, $\mathfrak{B}$ is the set of closed-open neighbourhoods of $\omega_{\mathfrak{B}}$. Therefore, we obtain a bijection

$$pt : \mathcal{Q}(\mathcal{CO}(\Omega)) \longrightarrow \Omega.$$

For this bijection, obviously

$$pt(\mathcal{Q}_U(\mathcal{CO}(\Omega))) = U,$$

so pt is a homeomorphism. ∎

Remark 85 *It holds that $C(\Omega_d) \simeq W^*(\mathbb{B})$, where $\mathbb{B}$ is a purely atomic Boolean sector of $\mathcal{P}(\mathcal{H})$ for some appropriate Hilbert space $\mathcal{H}$.*

Proof. We will show that there is some Hilbert space $\mathcal{H}_d$ such that $C(\Omega_d) \hookrightarrow \mathcal{L}(\mathcal{H}_d)$ (where $I_{C(\Omega_d)} = id_{\mathcal{H}_d}$) and such that the isolated points of Ω_d correspond to the minimal projections in $\mathcal{L}(\mathcal{H}_d)$: let D be a discrete open dense subset of Ω_d. If D is finite, then $\Omega_d = D$ and we can set $\mathcal{H}_d = \mathbb{C}^{\#D}$.

If D is infinite, let $\mathcal{H}_d$ be a $\#D$-dimensional Hilbert space, and let $(e_\delta)_{\delta\in D}$ be an orthonormal basis of $\mathcal{H}_d$. Let

$$\mathbb{B} := \{P_U \in \mathcal{P}(\mathcal{H}_d) \mid \forall \delta \in D : e_\delta \in U \cup U^\perp\}.$$

$\mathbb{B}$ is a distributive lattice: let $P_U, P_V \in \mathbb{B}$. Then for all $\delta \in D$, one has:

$$\begin{aligned} P_U P_V e_\delta &= \begin{cases} 0 & \text{if } e_\delta \in V^\perp \\ P_U e_\delta & \text{if } e_\delta \in V \end{cases} \\ &= \begin{cases} 0 & \text{if } e_\delta \in U^\perp \cup V^\perp \\ e_\delta & \text{if } e_\delta \in U \cap V \end{cases} \\ &= P_V P_U e_\delta, \end{aligned}$$

so $P_U P_V = P_V P_U$.

Let $P_W \in \mathcal{L}(\mathcal{H}_d)$ and $P_U P_W = P_W P_U$ for all $P_U \in \mathbb{B}$. Let $\delta \in D$ and $e_\delta \notin W$. If $e_\delta \in U$, then $P_U P_W e_\delta = P_W e_\delta$, so $P_W e_\delta \in U$. If we choose $P_U = P_{\mathbb{C}e_\delta}$ (projections of this form clearly belong to $\mathbb{B}$), then

$$P_W e_\delta = \lambda e_\delta$$

for some $\lambda \in \mathbb{C}$, so $P_W e_\delta = 0$, since $e_\delta \neq W$. Therefore, $e_\delta \in W^\perp$, and $\mathbb{B}$ is maximal, i.e. a purely atomic Boolean sector.

To $P_U \in \mathbb{B}$ corresponds a function $p_U \in C(\Omega_d)$, defined by

$$\begin{aligned} p_U &:= \sup\{\chi_{\{\delta\}} \mid e_\delta \in P_U\} \\ &= \bigvee_{\delta : e_\delta \in U} \chi_{\{\delta\}}. \end{aligned}$$

Since a characteristic function on Ω_d corresponds to a projection in $C(\Omega_d)$, we obtain a bijection

$$\tau : \mathbb{B} \longrightarrow \mathcal{P}(C(\Omega_d)).$$

τ is a lattice isomorphism, since

$$\begin{aligned} \tau(P_{U^\perp}) &= \sup\{\chi_{\{\delta\}} \mid e_\delta \notin U\} \\ &= \sup\{1 - \chi_{\{\delta\}} \mid e_\delta \in U\} \\ &= 1 - \tau(P_U) \end{aligned}$$

and

$$P_U, P_V \in \mathbb{B}, P_U \subseteq P_V \Longrightarrow \tau(P_U) \leq \tau(P_V).$$

Therefore, τ induces a homeomorphism

$$\tau_* : \mathcal{Q}(\mathbb{B}) \longrightarrow \mathcal{Q}(\mathcal{P}(C(\Omega_d))).$$

According to Lemma 84, the homeomorphism τ_* can be regarded as a homeomorphism

$$\tau_* : \mathcal{Q}(\mathbb{B}) \longrightarrow \Omega_d.$$

It immediately follows that $W^*(\mathbb{B})$ is isomorpic to $C(\Omega_d)$. ■

Summing up, we have proven:

Proposition 86 *Let $\mathcal{M} \simeq C(\Omega)$ be an abelian von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Then there is a separable Hilbert space $\mathcal{H}_d$ and a Boolean sector $\mathbb{B} \subseteq \mathcal{L}(\mathcal{H}_d)$ such that $\mathcal{M}$ is $*$-isomorphic to $L^\infty(]0,1[,\mu) \oplus W^*(\mathbb{B}) \subseteq \mathcal{L}(L^2(]0,1[,\mu) \oplus \mathcal{H}_d)$, where μ is the Lebesgue measure. In particular, $\mathcal{M}$ is maximal abelian in $\mathcal{L}(\mathcal{K})$ for an appropriate Hilbert space $\mathcal{K} = L^2(]0,1[,\mu) \oplus \mathcal{H}_d$. The isomorphy class of $\mathcal{M}$ is determined by the cardinality of the set of isolated points of the Gelfand spectrum of $\mathcal{M}$.*

2.8.4. On the universality of the Stone spectra $\mathcal{Q}(\mathbb{B})$

Despite their thorny topological properties, Stone spectra have a nice relationship to compact Hausdorff spaces: we will show that for every compact Hausdorff space X, there is a Hilbert space $\mathcal{H}$, a Boolean sector $\mathbb{B} \subseteq \mathcal{L}(\mathcal{H})$ and a continuous identifying surjective mapping $\pi : \mathcal{Q}(\mathbb{B}) \to X$. If Stone spectra are to play some role in the foundations of quantum theory, this mapping may find some interpretation as a coarse-graining map from the Stone spectrum $\mathcal{Q}(\mathbb{B})$ to the "classical" space X. Of course, a lot remains to be done before this could really become a genuine physical interpretation. But for now, at least topologically, the Stone spectra turn out not to be too exotic. A compact Hausdorff space is the continuous image of a Stone spectrum of some *distributive* lattice $\mathbb{B}$.

In this subsection, let $\mathcal{B}$ be a commutative C^*-algebra with unit I, and let $\mathcal{A} \subseteq \mathcal{B}$ be a C^*-subalgebra containing the unit of $\mathcal{B}$. We assume $\dim \mathcal{A} > 1$. The following lemma and the ensuing arguments are well-known and are only included for the sake of completeness.

Lemma 87 *Every maximal ideal in $\mathcal{A}$ is contained in some maximal ideal in $\mathcal{B}$.*

Proof. Let $\mathfrak{m} \subseteq \mathcal{A}$ be a maximal ideal, and let $\mathcal{B} = C(X)$ in the Gelfand representation. Let us assume that there is no maximal ideal $\mathfrak{M}$ in $C(X)$ containing $\mathfrak{m}$. Then for each $x \in X$, there is an $a_x \in \mathfrak{m}$ such that $a_x(x) \neq 0$. Since $\mathfrak{m}$ is self-adjoint, $a_x \in \mathfrak{m}$ can be chosen such that $a_x(x) > 0$. From continuity of a_x, $a_x(y) > 0$ holds on some neighbourhood $U_x \subseteq X$ of x. Since X is compact, finitely many of such neighbourhoods suffice to cover X, that is, $X = U_{x_1} \cup ... \cup U_{x_n}$. Then

$$a := a_{x_1} + ... + a_{x_n} \in \mathfrak{m}.$$

Since the a_{x_i} can be chosen such that $a_{x_i} \geq 0$ on the whole of X, a is positive everywhere, so a is invertible in $C(X) = \mathcal{B}$. Then a is invertible in $\mathcal{A}$, too, and we have $\mathfrak{m} = \mathcal{A}$, which is a contradiction. ■

Let $\mathfrak{m}_{\mathcal{B}}$ be a maximal ideal in $\mathcal{B}$. Then

$$\mathfrak{m} := \mathfrak{m}_{\mathcal{B}} \cap \mathcal{A}$$

is a maximal ideal in $\mathcal{A}$: assume that $\mathfrak{m} = 0$. Let

$$\pi_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}/\mathfrak{m}_{\mathcal{B}} \simeq \mathbb{C}$$

be the canonical projection. $\pi_{\mathcal{B}}|_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$ has kernel $\mathfrak{m}_{\mathcal{B}} \cap \mathcal{A} = 0$, so $\dim A = 1$ follows. Since this was excluded in our assumptions, we have $\mathfrak{m} \neq 0$. $\mathfrak{m}_{\mathcal{B}}$ corresponds to a character τ of $\mathcal{B}$ by $\mathfrak{m}_{\mathcal{B}} = \ker \tau$. $\tau|_{\mathcal{A}}$ is a character of $\mathcal{A}$, since $\tau|_{\mathcal{A}} \neq 0$ from $I \in \mathcal{A}$, $\tau(I) = 1$. Moreover,

$$\begin{aligned} \ker \tau|_{\mathcal{A}} &= \{a \in \mathcal{A} \mid \tau(a) = 0\} \\ &= \ker \tau \cap \mathcal{A} \\ &= \mathfrak{m}_{\mathcal{B}} \cap \mathcal{A} \\ &= \mathfrak{m}, \end{aligned}$$

so $\mathfrak{m}$ is a maximal ideal in $\mathcal{A}$.

Therefore, we obtain a mapping

$$\begin{aligned} \widetilde{\pi} : \operatorname{sp} \mathcal{B} &\longrightarrow \operatorname{sp} \mathcal{A} \\ \mathfrak{m}_{\mathcal{B}} &\longmapsto \mathfrak{m}_{\mathcal{B}} \cap \mathcal{A}, \end{aligned}$$

and the above lemma shows that this mapping is surjective. Since $\widetilde{\pi}$ can be described by $\tau \mapsto \tau|_{\mathcal{A}}$, it follows from the definition of the Gelfand topology that $\widetilde{\pi}$ is continuous. Furthermore, the Gelfand topology is the finest topology on $\operatorname{sp}\mathcal{A}$ such that $\widetilde{\pi}$ is continuous. Since $\widetilde{\pi}$ is continuous, the quotient topology with respect to $\widetilde{\pi}$ could only be finer than the Gelfand topology on $\operatorname{sp}\mathcal{A}$. Since $\operatorname{sp}\mathcal{B}$ is compact, the quotient topology on $\operatorname{sp}\mathcal{A}$ is compact, too. But on a Hausdorff space, there is no properly finer compact topology than the Gelfand topology. We obtain:

Proposition 88 *For every compact Hausdorff space X, there is a Hilbert space $\mathcal{H}$, a Boolean sector $\mathbb{B} \subseteq \mathcal{L}(\mathcal{H})$ and a continuous identifying surjective map $\pi : \mathcal{Q}(\mathbb{B}) \to X$.*

Proof. $C(X)$ is isometrically isomorphic to some C^*-subalgebra $\mathcal{A}_X$ of $\mathcal{L}(\mathcal{H})$ for some appropriate Hilbert space $\mathcal{H}$ such that $I = id_{\mathcal{H}} \in \mathcal{A}_X$. Let $\mathcal{M}_X \subseteq \mathcal{L}(\mathcal{H})$ be a maximal abelian von Neumann subalgebra containing $\mathcal{A}_X$. Then $\mathcal{M}_X = W^*(\mathbb{B}) = \{P \mid P \in \mathbb{B}\}''$ for some Boolean sector $\mathbb{B} \subseteq \mathcal{L}(\mathcal{H})$. Let $\widetilde{\pi} : \operatorname{sp}\mathcal{M}_X \to \operatorname{sp}\mathcal{A}_X$ be the identifying continuous mapping described above, and let $\theta : \mathcal{Q}(\mathbb{B}) \to \operatorname{sp}\mathcal{M}_X$ and $\psi : \operatorname{sp}\mathcal{A}_X \to X$ be the "canonical" homeomorphisms. Then

$$\pi := \psi \circ \widetilde{\pi} \circ \theta$$

is the mapping we are searching for. ■

2.9. Classical and quantum observables

We will now show that classical observables and quantum observables, though very different objects mathematically in the usual formulation, can be treated in a coherent fashion. The unifying concepts are *spectral families* and (observable) functions defined from them. Again, only the main features will be presented. For details, see [deG01] and [deG05].

Remember that a classical observable is a real-valued function on a topological space, which is interpreted as the phase space or configuration space of a physical system. Depending on the context, one can regard smooth, continuous or measurable functions. We will concentrate on the continuous case here.

We saw in section 2.7, "Observable functions" and in subsection 2.8.2, "Observable functions and the Gelfand representation", how quantum observables, i.e. self-adjoint operators in a von Neumann algebra $\mathcal{R}$, can be seen as functions on the Stone spectrum of $\mathcal{R}$. Moreover, it was shown how these observable functions can be defined from the spectral families and how the Stone spectrum reduces to the Gelfand spectrum if $\mathcal{R}$ is abelian. Thus, quantum observables can either be seen as functions on a topological space or as spectral families.

The picture will be more complete if classical observables, i.e. functions, can be expressed as spectral families. There is an obvious generalization of spectral families to arbitrary complete lattices:

Definition 89 *Let $\mathbb{L}$ be a complete lattice. A **spectral family in** $\mathbb{L}$ is a mapping $\sigma : \mathbb{R} \to \mathbb{L}$ such that*

(i) $\sigma(\lambda) \leq \sigma(\mu)$ for $\lambda \leq \mu$,

(ii) $\sigma(\lambda) = \bigwedge_{\mu > \lambda} \sigma(\mu)$ for all $\lambda \in \mathbb{R}$,

(iii) $\bigwedge_{\lambda \in \mathbb{R}} \sigma(\lambda) = 0$, $\bigvee_{\lambda \in \mathbb{R}} \sigma(\lambda) = 1$.

Of course, we will define functions from spectral families in $\mathcal{T}(M)$, the topology of a topological space M, by using the same inf-construction as for observable functions (Def. 56). Before doing so, we must exclude the possibility of taking the infimum of an empty set, so we define

Definition 90 *Let M be a non-empty topological space, and let $\sigma : \mathbb{R} \to \mathcal{T}(M)$ be a spectral family in $\mathcal{T}(M)$. Then the **admissible domain of** σ is the set*

$$\mathcal{D}(\sigma) := \{x \in M \mid \exists \lambda \in \mathbb{R} : x \notin \sigma(\lambda)\}.$$

Clearly, we have $\mathcal{D}(\sigma) = M \backslash \bigcap_{\lambda \in \mathbb{R}} \sigma(\lambda)$. It is easy to see that $\mathcal{D}(\sigma)$ is dense in M.

Definition 91 *Let $\sigma : \mathbb{R} \to \mathcal{T}(M)$ be a spectral family with admissible domain $\mathcal{D}(\sigma)$. Then the function*

$$f_\sigma : \mathcal{D}(\sigma) \longrightarrow \mathbb{R},$$

defined by

$$\forall x \in \mathcal{D}(\sigma) : f_\sigma(x) := \inf\{\lambda \in \mathbb{R} \mid x \in \sigma(\lambda)\}$$

*is called the **function induced by** σ.*

A spectral family also induces a function on the set $\mathcal{D}(\mathcal{T}(M))$ of dual ideals in $\mathcal{T}(M)$ in the usual way:

$$\begin{aligned} f_\sigma : \mathcal{D}(\mathcal{T}(M)) &\longrightarrow \mathbb{R} \\ \mathcal{I} &\longmapsto \inf\{\lambda \in \mathbb{R} \mid \sigma(\lambda) \in \mathcal{I}\}, \end{aligned}$$

where $\mathcal{I}$ is a dual ideal such that $\varnothing \neq \mathcal{I} \cap \operatorname{im}(\sigma) \neq \operatorname{im}(\sigma)$. In particular, this condition is satisfied for **bounded** spectral families, i.e. those σ for which

$$\exists a, b \in \mathbb{R} : \sigma(\lambda) = 0 \text{ for all } \lambda < a, \quad \sigma(\lambda) = \mathbb{R} \text{ for all } \lambda > b$$

holds. If M is a locally compact Hausdorff space, it is easy to see that $f_\sigma : \mathcal{D}(\mathcal{T}(M)) \to \mathbb{R}$ is an extension of $f_\sigma : \mathcal{D}(\sigma) \to \mathbb{R}$, where an $x \in \mathcal{D}(\sigma)$ is identified with the point $\mathfrak{p}_x \in \mathcal{D}(\mathcal{T}(M))$ (see example 1 in section 2.3).

Let $\sigma : \mathbb{R} \to \mathcal{T}(M)$ be a spectral family. Then

$$R(\sigma) := \{\lambda \in \mathbb{R} \mid \sigma \text{ is constant on a neighbourhood of } \lambda\}$$

is called the **resolvent of** σ, the (closed) set $\operatorname{sp}\sigma := \mathbb{R} \backslash R(\sigma)$ is called the **spectrum of** σ. One can show (see [deG05])

Proposition 92 *Let $f_\sigma : \mathcal{D}(\sigma) \to \mathbb{R}$ be the function induced by the spectral family σ. Then* $\operatorname{sp}\sigma = \overline{\operatorname{im} f_\sigma}$.

Interestingly, it is possible to state a lattice-theoretic condition on a spectral family σ in $\mathcal{T}(M)$ that guarantees that the function f_σ induced by σ is continuous. Using the pseudo-complement $U^c := M\backslash\overline{U}$ $(U \in \mathcal{T}(M))$, we set

Definition 93 *A spectral family $\sigma : \mathbb{R} \to \mathcal{T}(M)$ is called* ***continuous*** *if for all $\lambda < \mu$, we have $\sigma(\lambda)^c \cup \sigma(\mu) = M$. Equivalently, for all $\lambda < \mu$, $\overline{\sigma(\lambda)} \subseteq \sigma(\mu)$.*

One can show that continuous spectral families have nice topological properties: $\mathcal{D}(\sigma)$ is an open (and dense) subset of M and for all $\lambda \in \mathbb{R}$, $\sigma(\lambda)$ is a regular open set, that is, $\sigma(\lambda)^{cc} = \sigma(\lambda)$.

Example 94 *Let M be a topological space. Consider the spectral family $\sigma : \mathbb{R} \to \mathcal{T}(M)$ defined by*

$$\sigma(\lambda) := \begin{cases} 0 & \text{for } \lambda < 0 \\ U & \text{for } 0 \leq \lambda < 1 \\ M & \text{for } \lambda > 1 \end{cases} .$$

(σ is defined in analogy to the spectral family of a projection $P \in \mathcal{P}(\mathcal{H})$.) σ is continuous if and only if U is closed and open: if U is closed and open, then $\overline{\sigma(\lambda)} \subseteq \sigma(\mu)$ holds for all $\lambda < \mu$, so σ is continuous.

Conversely, let σ be continuous. We have to show that U is closed-open, that is, $U = \overline{U}$. Since σ is continuous, we have $\overline{\sigma(\lambda)} \subseteq \sigma(\mu)$ for all $\lambda < \mu$. Since $\sigma(\lambda) = U$ for all $\lambda \in [0, 1[$, we have $\overline{U} \subseteq U$, i.e. $\overline{U} = U$.

Now consider some continuous function $f : M \to \mathbb{R}$. f induces a spectral family σ_f:

Definition 95 *Let M be a topological space, and let $f : M \to \mathbb{R}$ be a continuous function. The* ***spectral family*** *σ_f* ***induced by*** *f is defined by*

$$\forall\lambda \in \mathbb{R} : \sigma_f(\lambda) := \operatorname{int} \overset{-1}{f}(]-\infty, \lambda]).$$

The close relation between continuous functions and continuous spectral families is stated in the following

Theorem 96 *Let M be a topological space, $f : M \to \mathbb{R}$ a continuous function and σ_f the spectral family induced by f. Then the admissible domain $\mathcal{D}(\sigma)$ is M and the function f_{σ_f} induced by σ_f equals f. Conversely, if $\sigma : \mathbb{R} \to \mathcal{T}(M)$ is a continuous spectral family, then the function f_σ induced by σ is continuous and the induced spectral family σ_{f_σ} in $\mathcal{T}(\mathcal{D}(\sigma))$ is the restriction of σ to $\mathcal{D}(\sigma)$, that is,*

$$\forall\lambda \in \mathbb{R} : \sigma_{f_\sigma}(\lambda) = \sigma(\lambda) \cap \mathcal{D}(\sigma).$$

For the proof, see [deG05].

2.10. Some basic results

In this section, we collect some basic results concerning quasipoints and Stone spectra of von Neumann algebras. The lattices we consider will accordingly be the projection lattices of von Neumann algebras.

2.10.1. The Stone spectrum of $\mathcal{L}(\mathcal{H})$

Lemma 97 *Let $\mathfrak{B}_{\mathbb{C}x_0}$ be an atomic quasipoint of $\mathcal{L}(\mathcal{H})$ (see Def. 33 and Example 34), and let $(E_i) \subset \mathfrak{B}_{\mathbb{C}x_0}$ be a net converging strongly to E. Then $E \in \mathfrak{B}_{\mathbb{C}x_0}$.*

Proof. We have $x_0 = E_i x_0 \to E x_0$, so $P_{\mathbb{C}x_0} \leq E x_0$ and therefore $E \in \mathfrak{B}_{\mathbb{C}x_0}$. ∎

Notation 98 *The set of continuous quasipoints of $\mathcal{L}(\mathcal{H})$ (see example 3 at end of section 2.3) is denoted by $\mathcal{Q}_c(\mathcal{R})$.*

Lemma 99 *The set of projections $\bigcap_{\mathfrak{B} \in \mathcal{Q}_c(\mathcal{L}(\mathcal{H}))} \mathfrak{B}$, which is contained in every continuous quasipoint, is strongly dense in $\mathcal{P}(\mathcal{H}) = \mathcal{P}(\mathcal{L}(\mathcal{H}))$.*

Proof. Each continuous quasipoint contains all projections of finite codimension, so these projections are contained in the intersection $\bigcap_{\mathfrak{B} \in \mathcal{Q}_c(\mathcal{L}(\mathcal{H}))} \mathfrak{B}$. Let P_U be a projection such that $\dim U^{\perp} = \infty$, let M_1, M_2 be countable, disjoint subsets with $M_1 \cup M_2 \simeq \mathbb{N}$, and let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ such that

$$e_n \in \begin{cases} U & \text{for } n \in M_1 \\ U^{\perp} & \text{for } n \in M_2 \end{cases} .$$

Let $M_2 = \{n_m \mid m \in \mathbb{N}\}$ and

$$W_m = lin\{e_{n_0}, ..., e_{n_m}\}^{\perp} \supset U.$$

We have

$$\begin{aligned} \bigcap_m W_m &= (\bigvee_m lin\{e_{n_0}, ..., e_{n_m}\})^{\perp} \\ &= U^{\perp\perp} \\ &= U. \end{aligned}$$

W_m is a subspace of finite codimension, therefore P_{W_m} is contained in every continuous quasipoint and thus in the intersection $\bigcap_{\mathfrak{B} \in \mathcal{Q}_c(\mathcal{L}(\mathcal{H}))} \mathfrak{B}$. It follows that P_U is contained in the strong closure $\overline{\bigcap_{\mathfrak{B} \in \mathcal{Q}_c(\mathcal{L}(\mathcal{H}))} \mathfrak{B}}$. ∎

2.10.2. The Stone spectrum of a finite direct sum of von Neumann algebras

We will now regard the quasipoints of a finite direct sum of von Neumann algebras: Let $A = \{1, ...n\}$, and let $\mathcal{R}_a$ $(a \in A)$ be von Neumann algebras which are represented on

Hilbert spaces $\mathcal{H}_a$ ($a \in A$). As is well known, the direct sum $\mathcal{R} := \bigoplus_{a\in A}\mathcal{R}_a$ of the von Neumann algebras is a von Neumann algebra on the Hilbert space $\mathcal{H} := \bigoplus_{a\in A}\mathcal{H}_a$ with algebraic operations

$$\begin{aligned}(a_1 \oplus ... \oplus a_n) + (b_1 \oplus ... \oplus b_n) &= (a_1 + b_1) \oplus ... \oplus (a_n + b_n),\\ (a_1 \oplus ... \oplus a_n)\cdot(b_1 \oplus ... \oplus b_n) &= (a_1 b_1) \oplus ... \oplus (a_n b_n).\end{aligned}$$

In a similar manner, the minimum and maximum are defined summand-wise. Hence, obviously, $\mathcal{R}$ has quasipoints $\mathfrak{B}$ of the form

$$\mathfrak{B} = \mathcal{P}(\mathcal{R}_1) \oplus ... \oplus \mathcal{P}(\mathcal{R}_{i-1}) \oplus \mathfrak{B}_i \oplus \mathcal{P}(\mathcal{R}_{i+1}) \oplus ... \oplus \mathcal{P}(\mathcal{R}_n),$$

where $\mathfrak{B}_i$ is a quasipoint of the von Neumann algebra $\mathcal{R}_i$ ($i \in A$). Such a $\mathfrak{B}$ is a filter base, since $\mathfrak{B}_i$ is a filter base and the minimum is taken summand-wise. $\mathfrak{B}$ clearly is maximal, since $\mathfrak{B}_i$ is maximal and the other summands are the whole projection lattices $\mathcal{P}(\mathcal{R}_j)$ ($j \in A, j \neq i$). We have to show that *all* quasipoints of $\mathcal{R} = \bigoplus_{a\in A}\mathcal{R}_a$ are of this form.

Lemma 100 *Using the notation from above, it holds that all quasipoints of the von Neumann algebra $\mathcal{R} = \bigoplus_{a\in A}\mathcal{R}_a$ are of the form $\mathfrak{B} = \mathcal{P}(\mathcal{R}_1) \oplus ... \oplus \mathcal{P}(\mathcal{R}_{i-1}) \oplus \mathfrak{B}_i \oplus \mathcal{P}(\mathcal{R}_{i+1}) \oplus ... \oplus \mathcal{P}(\mathcal{R}_n)$ for some $i \in A = \{1, ..., n\}$.*

Proof. Let us assume that there is a quasipoint $\mathfrak{B}' = \mathcal{P}_1 \oplus ... \oplus \mathcal{P}_n$ ($\mathcal{P}_a \subseteq \mathcal{P}(\mathcal{R}_a)$) which is not of the given form. This means that while $\mathfrak{B}'$ is a filter base in $\mathcal{P}(\mathcal{R})$, none of the sets $\mathcal{P}_a$ ($a \in A$) is a filter base in the corresponding $\mathcal{P}(\mathcal{R}_a)$: if $\mathcal{P}_a$ was a filter base, then it would be a maximal filter base, since $\mathfrak{B}'$ is a maximal filter base, so $\mathcal{P}_a$ would be a quasipoint of $\mathfrak{B}_a$ of $\mathcal{R}_a$. Then, from maximality again, $\mathfrak{B}'$ would be of the form $\mathfrak{B}' = \mathcal{P}(\mathcal{R}_1) \oplus ... \oplus \mathcal{P}(\mathcal{R}_{a-1}) \oplus \mathfrak{B}_a \oplus \mathcal{P}(\mathcal{R}_{a+1}) \oplus ... \oplus \mathcal{P}(\mathcal{R}_n)$.

Since none of the sets $\mathcal{P}_a$ is a filter base, there are $2n$ elements $a_1^1 \oplus ... \oplus a_n^1, ..., a_1^{2n} \oplus ... \oplus a_n^{2n} \in \mathfrak{B}'$ (upper index numbers the elements, lower index runs through the index set A) such that

$$\begin{aligned}r_1 &:= (a_1^1 \oplus ... \oplus a_n^1) \wedge (a_1^2 \oplus ... \oplus a_n^2)\\ &= 0 \oplus (a_2^1 \wedge a_2^2) \oplus ... \oplus (a_n^1 \wedge a_n^2),\\ r_2 &:= (a_1^3 \oplus ... \oplus a_n^3) \wedge (a_1^4 \oplus ... \oplus a_n^4)\\ &= (a_1^3 \wedge a_1^4) \oplus 0 \oplus (a_3^3 \wedge a_3^4) \oplus ... \oplus (a_n^3 \wedge a_n^4),\\ &.\\ &.\\ &.\\ r_n &:= (a_1^{2n-1} \oplus ... \oplus a_n^{2n-1}) \wedge (a_1^{2n} \oplus ... \oplus a_n^{2n})\\ &= (a_1^{2n-1} \wedge a_1^{2n}) \oplus ... \oplus (a_{n-1}^{2n-1} \wedge a_{n-1}^{2n}) \oplus 0.\end{aligned}$$

So one has n elements r_i of $\mathcal{R}$ with the ith summand of r_i equal to 0. Taking the minimum

$$r_1 \wedge ... \wedge r_n = 0 \oplus 0 \oplus ... \oplus 0,$$

we obtain a contradiction, since $\mathfrak{B}'$ is a filter base (which does not contain $0 \oplus 0 \oplus ... \oplus 0$).
■

Remark 101 *The proof of Lemma 100 does not use any special features of the von Neumann algebra situation, so it can easily be generalized to the finite sum of lattices.*

In the proof of Lemma 100, we used the fact that in a filter base, the minimum of finitely many *elements is an element of the filter base again. That is why the proof cannot be used for an infinite sum of von Neumann algebras, since the minimum of infinitely many elements of a filter base can be* 0. *(A concrete example is given by the continuous quasipoints of* $\mathcal{L}(\mathcal{H})$. *It is sufficient to choose an orthonormal basis* $\{e_i\}_{i\in\mathbb{N}}$ *of* $\mathcal{H}$ *and to regard the countable set of projections* $A := \{(P_{\mathbb{C}e_i})^{\perp} \mid i \in \mathbb{N}\}$, *which is contained in every continuous quasipoint. For the minimum, we obtain* $\bigwedge_{P\in A} P = 0$.*)*

2.10.3. The action of the unitary group on the Stone spectrum $\mathcal{Q}(\mathcal{R})$

A unitary operator transforms a quasipoint in the obvious way:

Definition 102 *Let* $T \in \mathcal{U}(\mathcal{H})$ *be a unitary operator.* T *acts on* $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ *(*$\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$*) by*

$$T.\mathfrak{B} := \{TET^* \mid E \in \mathfrak{B}\}.$$

Lemma 103 $T.\mathfrak{B}$ *is a quasipoint of the von Neumann algebra* $T\mathcal{R}T^*$. *If* $T \in \mathcal{U}(\mathcal{R})$, *then* $T.\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$.

Proof. We have

$$T(E \wedge F)T^* \leq TET^* \wedge TFT^*.$$

Moreover,

$$T^*(TET^* \wedge TFT^*)T \leq E \wedge F,$$

so $T(E \wedge F)T^* = TET^* \wedge TFT^*$. Thus $T.\mathfrak{B}$ is a filter base and hence contained in some quasipoint $\mathfrak{B}' \in T\mathcal{R}T^*$. $T^*.\mathfrak{B}'$ also is a filter base. We have

$$T^*.(T.\mathfrak{B}) = \mathfrak{B} \subseteq T^*.\mathfrak{B}'.$$

From the maximality of $\mathfrak{B}$, equality holds. ■

3. Stone spectra of finite von Neumann algebras

Stone spectra of von Neumann algebras are of some interest mathematically as well as physically, as already should have become clear. In this chapter we will present our results on the structure of Stone spectra of *finite* von Neumann algebras.

The classification of von Neumann algebras goes back to the classical works of Murray and von Neumann [MurVNeu36, MurVNeu37, vNeu40, MurVNeu43]. It is based on the *comparison theory of projections* as can be found in chapter 6 of [KadRinII97]. We take some definitions from there.

As is well known, two projections $E, F \in \mathcal{P}(\mathcal{R})$ are called *equivalent relative to a von Neumann algebra* $\mathcal{R}$ if there is a partial isometry $\theta \in \mathcal{R}$ such that $\theta^*\theta = E$ and $\theta\theta^* = F$. Notation: $E \sim_{\mathcal{R}} F$ or simply $E \sim F$, if $\mathcal{R}$ is fixed. If E is equivalent to a subprojection of F, then we write $E \precsim F$ and call E *weaker than* F.

A projection E is called *infinite (relative to $\mathcal{R}$)* if it is equivalent to one of its subprojections, $E \sim E_0 < E$. Otherwise, E is called *finite*. If E is infinite and PE is either 0 or infinite for each central projection $P \in \mathcal{C}(\mathcal{R})$, then E is said to be *properly infinite*. A von Neumann algebra is said to be *finite* if the identity I is a finite projection.

A projection $E \in \mathcal{P}(\mathcal{R})$ is called *abelian* if $E\mathcal{R}E$ is abelian. Let $A \in \mathcal{R}$ be an operator. The *central carrier C_A of* A is the projection $I - P$, where P is the maximum of all central projections $P_a \in \mathcal{C}(\mathcal{R})$ such that $P_a A = 0$ (Def. 5.5.1 in [KadRinI97]). Obviously, $C_A \geq A$.

The possible combinations of these features lead to the classification of von Neumann algebras into three different types with further subtypes (Def. 6.5.1 in [KadRinII97]):

Definition 104 *A von Neumann algebra $\mathcal{R}$ is said to be* ***of type*** *I if it has an abelian projection with central carrier I –* ***of type*** *I_n if I is the sum of n equivalent abelian projections. If $\mathcal{R}$ has no non-zero abelian projections but has a finite projection with central carrier I, then $\mathcal{R}$ is said to be* ***of type*** *II –* ***of type*** *II_1 if I is finite and* ***of type*** *II_∞ if I is properly infinite. If $\mathcal{R}$ has no non-zero finite projections, $\mathcal{R}$ is said to be* ***of type*** *III.*

Each von Neumann algebra is a direct sum of von Neumann algebras of these types, as shown by the following (Thm. 6.5.2 of [KadRinII97]):

Theorem 105 (Type decomposition) *If $\mathcal{R}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, there are mutually orthogonal central projections P_n, n not exceeding* $\dim \mathcal{H}$, P_{c_1},

P_{c_∞}, and P_∞, with sum I, maximal with respect to the properties that $\mathcal{R}P_n$ is of type I_n or $P_n = 0$, $\mathcal{R}P_{c_1}$ is of type II_1 or $P_{c_1} = 0$, $\mathcal{R}P_{c_\infty}$ is of type II_∞ or $P_{c_\infty} = 0$, and $\mathcal{R}P_\infty$ is of type III or $P_\infty = 0$.

A **factor** $\mathcal{R}$ is a von Neumann algebra with trivial center, i.e. $\mathcal{C}(\mathcal{R}) = \mathbb{C}I$. As a corollary of the theorem above, we obtain that a factor $\mathcal{R}$ is either of type I_n, II_1, II_∞ or III.

As said above, we will only regard finite von Neumann algebras here. The elements of the Stone spectrum $\mathcal{Q}(\mathcal{R})$, the quasipoints, are defined using Zorn's lemma. As usual, it is not easy to get some intuition of such objects. Some extra structure of the von Neumann algebra is needed to clarify the properties of the quasipoints and the Stone spectrum. In the case of type I_n algebras (section 3.2), we will make use of the fact that such an algebra is of the form $\mathbb{M}_n(\mathcal{A})$, where $\mathcal{A}$ is the center of $\mathcal{R}$. Drawing on a result on abelian quasipoints (section 3.1), i.e. quasipoints containing an abelian projection, a fairly complete characterization of the Stone spectrum of a type I_n algebra is obtained. Type I_n algebras include all von Neumann algebras on finite-dimensional Hilbert spaces and all abelian von Neumann algebras. The latter are of those of type I_1. Note that while $\mathcal{R} \simeq \mathbb{M}_n(\mathcal{A})$ is given by "finite" $n \times n$-matrices, the center $\mathcal{A}$ of $\mathcal{R}$ may be represented on an infinite-dimensional Hilbert space.

Furthermore, we regard type II_1 factors, where the center-valued trace in the form of the dimension function $\Delta : \mathcal{P}(\mathcal{R}) \to [0,1]$ plays a major role. It turns out that quasipoints are quite large objects in the sense that each quasipoint contains projections of all dimensions and (even the intersection of two quasipoints) is strongly dense in the projection lattice $\mathcal{P}(\mathcal{R})$. Moreover, we can show that Boolean quasipoints of a type II_1 factor also contain projections of all dimensions.

3.1. Abelian quasipoints of von Neumann algebras

In this section, we will regard quasipoints containing an abelian projection. It will be shown that there is a close relationship between the abelian quasipoints of a von Neumann algebra $\mathcal{R}$ and the quasipoints of the center of $\mathcal{R}$. This result will be central to the classification of Stone spectra of type I_n von Neumann algebras.

Definition 106 *A quasipoint $\mathfrak{B} \subseteq \mathcal{P}(\mathcal{R})$ is called **abelian** if it contains an abelian projection $E \in \mathcal{R}$. The set of abelian quasipoints of a von Neumann algebra $\mathcal{R}$ is denoted by $\mathcal{Q}^{ab}(\mathcal{R})$.*

Definition 107 *The E-**trunk** $\mathfrak{B}_E$ ($E \in \mathfrak{B}$) of a quasipoint $\mathfrak{B}$ is the set*

$$\mathfrak{B}_E := \{F \in \mathfrak{B} \mid F \leq E\}.$$

Obviously, $\mathfrak{B}_E$ is a filter base.

Lemma 108 *The E-trunk $\mathfrak{B}_E$ uniquely determines the quasipoint $\mathfrak{B}$.*

Proof. Let $\mathfrak{B}_1, \mathfrak{B}_2$ be two quasipoints whose E-trunk is $\mathfrak{B}_E$. Let F be a projection in $\mathfrak{B}_1$. Then we have $E \wedge F \in \mathfrak{B}_E \subset \mathfrak{B}_2$. If a quasipoint contains a projection, it contains all larger projections, so $F \in \mathfrak{B}_2$ and $\mathfrak{B}_1 = \mathfrak{B}_2$ follows. ■

This lemma holds analogously for any lattice $\mathbb{L}$, since no features of the von Neumann algebra are used.

Definition 109 *Let $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ be a von Neumann algebra, $\mathfrak{B} \subset \mathcal{Q}_E(\mathcal{R})$ a quasipoint containing E and $\theta \in \mathcal{R}$ a partial isometry such that $E = \theta^*\theta$. We set*

$$\theta(\mathfrak{B}_E) := \{\theta F \theta^* \mid F \in \mathfrak{B}_E\}.$$

Lemma 110 *If $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ is a von Neumann algebra and $\theta \in \mathcal{R}$ is a partial isometry such that $E := \theta^*\theta$, then for all projections $P_U \in \mathcal{R}$ such that $P_U \leq E$ it holds that*

$$\theta P_U \theta^* = P_{\theta U}.$$

Proof. For $x \in U$, we have

$$\begin{aligned} \theta P_U \theta^* \theta x &= \theta P_U E x \\ &= \theta x \\ &= P_{\theta U} \theta x. \end{aligned}$$

If $y \in (\theta U)^\perp$, then $\theta^* y \in U^\perp$ and thus

$$\theta P_U \theta^* y = 0 = P_{\theta U} y.$$

■

From this lemma, we see that $\theta(\mathfrak{B}_E)$ is a filter base. It is then easy to show that $\theta(\mathfrak{B}_E)$ is the $\theta E \theta^*$-trunk of a quasipoint of $\mathcal{R}$ (given that $\theta^*\theta = E$).

Notation 111 *We will denote the quasipoint induced by $\theta E \theta^*$ by $\theta_{\mathcal{Q}}(\mathfrak{B}_E)$.*

Remark 112 *Since in general $\theta^*\theta \notin \mathfrak{B}$ for an arbitrary partial isometry θ and an arbitrary quasipoint $\mathfrak{B}$, we have no action of the set of partial isometries on the Stone spectrum $\mathcal{Q}(\mathcal{R})$. On the other hand, if θ is unitary, we can define an operation, see subsection 2.10.3.*

Lemma 113 *Let $\mathcal{R}$ be a von Neumann algebra, and let $\mathcal{C} := \mathcal{C}(\mathcal{R})$ be the center of $\mathcal{R}$. If $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ is a quasipoint of $\mathcal{R}$, then $\mathfrak{B} \cap \mathcal{C}$ is a quasipoint of $\mathcal{C}$.*

Proof. Obviously, $\mathfrak{B} \cap \mathcal{C}$ is a filter base in $\mathcal{P}(\mathcal{C})$. Let $\beta \in \mathcal{Q}(\mathcal{C})$ be a quasipoint that contains $\mathfrak{B} \cap \mathcal{C}$. Assume that there is some $C \in \beta \backslash (\mathfrak{B} \cap \mathcal{C})$. Then there is some $Q \in \mathfrak{B}$ such that $C \wedge Q = 0$. Since $\{C, Q, I\}''$ is abelian, we have $C \wedge Q = CQ$ and

$$Q = CQ + (I - C)Q = (I - C)Q,$$

which implies $(I - C) \in \mathfrak{B}$, since $I - C \geq Q$, hence $I - C \in \mathfrak{B} \cap \mathcal{C} \subset \beta$. But then we have $C, I - C \in \beta$, which is a contradiction. ■

It is clear that the same argument works for an *abelian* algebra $\mathcal{M}$ (instead of $\mathcal{R}$) and an *arbitrary* abelian subalgebra $\mathcal{N} \subseteq \mathcal{M}$:

Corollary 114 *Let $\mathcal{M}$ be an abelian von Neumann algebra, and let $\mathcal{N} \subseteq \mathcal{M}$ be an abelian subalgebra. If $\beta \in \mathcal{Q}(\mathcal{M})$ is a quasipoint of $\mathcal{M}$, then $\beta \cap \mathcal{N}$ is a quasipoint of $\mathcal{N}$.*

We define the mapping

$$\begin{aligned} \zeta : \mathcal{Q}(\mathcal{R}) &\longrightarrow \mathcal{Q}(\mathcal{C}) \\ \mathfrak{B} &\longmapsto \mathfrak{B} \cap \mathcal{C}. \end{aligned}$$

ζ is surjective, since every quasipoint $\beta \in \mathcal{Q}(\mathcal{C})$ of $\mathcal{C}$ (being a filter base in $\mathcal{P}(\mathcal{R})$) is contained in some quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$. Let $\mathcal{Q}_C(\mathcal{C})$ be an open set in $\mathcal{Q}(\mathcal{C})$. The inverse image $\overset{-1}{\zeta}(\mathcal{Q}_C(\mathcal{C}))$ of $\mathcal{Q}_C(\mathcal{C})$ is $\mathcal{Q}_C(\mathcal{R})$, so ζ is continuous.

We will now consider the mapping ζ in the context of abelian quasipoints. Let $\mathfrak{B} \in \mathcal{Q}^{ab}(\mathcal{R})$ be an abelian quasipoint, and let $E \in \mathfrak{B}$ be an abelian projection. Each $F \in \mathfrak{B}_E$ is a subprojection of the abelian projection E and hence of the form $F = QE$, where $Q \in \mathcal{R}$ is a central projection. Then $Q \in \mathfrak{B}$ holds, so $Q \in \mathfrak{B} \cap \mathcal{C}$. If, conversely, $Q \in \mathfrak{B} \cap \mathcal{C}$ holds, then $QE \in \mathfrak{B}_E$, therefore we have

$$\mathfrak{B}_E = \{QE \mid Q \in \mathfrak{B} \cap \mathcal{C}\}.$$

The mapping

$$\begin{aligned} \zeta_E : \mathcal{R}'E &\longrightarrow \mathcal{R}'C_E, \\ T'E &\longmapsto T'C_E \end{aligned}$$

is a $*$-isomorphism (see Prop. 5.5.5 in [KadRinI97]). Since

$$\begin{aligned} PE \wedge QE &= PQE = (P \wedge Q)E, \\ PE \vee QE &= (P+Q)E - PQE = (P \vee Q)E, \end{aligned}$$

$\zeta_E|_{\mathfrak{B}_E}$ is a lattice isomorphism from $\mathfrak{B}_E$ onto $(\mathfrak{B} \cap \mathcal{C})C_E$. $\mathfrak{B} \cap \mathcal{C}$ is a quasipoint of $\mathcal{P}(\mathcal{C})$, hence

$$\zeta_E(\mathfrak{B}_E) = (\mathfrak{B} \cap \mathcal{C})_{C_E}.$$

Let $\mathfrak{B}, \widetilde{\mathfrak{B}} \in \mathcal{Q}^{ab}(\mathcal{R})$ be abelian quasipoints such that

$$\beta := \mathfrak{B} \cap \mathcal{C} = \widetilde{\mathfrak{B}} \cap \mathcal{C}.$$

Let $E \in \mathfrak{B}, \widetilde{E} \in \widetilde{\mathfrak{B}}$ be abelian projections. Since $C_E, C_{\widetilde{E}} \in \beta$, $C_E C_{\widetilde{E}} \in \beta$ holds and $C_E C_{\widetilde{E}} E \in \mathfrak{B}, C_E C_{\widetilde{E}} \widetilde{E} \in \widetilde{\mathfrak{B}}$ are abelian projections with the same central carrier $C_E C_{\widetilde{E}}$. Hence, without loss of generality, one can assume $C_E = C_{\widetilde{E}}$. It follows that E and $\widetilde{E}$ are equivalent (see Prop. 6.4.6 in [KadRinII97]). Let $\theta \in \mathcal{R}$ be a partial isometry such that $\theta^*\theta = E, \theta\theta^* = \widetilde{E}$, therefore $\theta E \theta^* = \widetilde{E}$. It follows that

$$\begin{aligned} \theta \mathfrak{B}_E \theta^* &= \{\theta Q E \theta^* \mid Q \in \beta\} \\ &= \{Q \theta E \theta^* \mid Q \in \beta\} \\ &= \{Q\widetilde{E} \mid Q \in \beta\} \\ &= \mathfrak{B}_{\widetilde{E}}, \end{aligned}$$

so

$$\theta_{\mathcal{Q}}(\mathfrak{B}) = \widetilde{\mathfrak{B}}.$$

Conversely, let $\mathfrak{B}$,$\widetilde{\mathfrak{B}}$ be abelian quasipoints, and let $\theta \in \mathcal{R}$ be a partial isometry such that $E := \theta^*\theta \in \mathfrak{B}, \widetilde{E} := \theta\theta^* \in \widetilde{\mathfrak{B}}$. From this, $\theta_{\mathcal{Q}}(\mathfrak{B}) = \widetilde{\mathfrak{B}}$ as shown. Let $F \in \mathfrak{B}$ be abelian, $F \leq E$. Then θF is a partial isometry from $(\theta F)^*\theta F = FEF = F$ to $\theta F\theta^* \in \widetilde{\mathfrak{B}}$. Since $\theta F\theta^*$ is abelian, too, we can assume without loss of generality that E and $\widetilde{E}$ are abelian. From the definition of $\theta_{\mathcal{Q}}$, it follows that

$$\begin{aligned}\widetilde{\mathfrak{B}}_{\widetilde{E}} &= \theta\mathfrak{B}_E\theta^* \\ &= \{\theta QE\theta^* \mid Q \in \mathfrak{B} \cap \mathcal{C}\} \\ &= \{Q\widetilde{E} \mid Q \in \mathfrak{B} \cap \mathcal{C}\}\end{aligned}$$

holds, so

$$\{P\widetilde{E} \mid P \in \widetilde{\mathfrak{B}} \cap \mathcal{C}\} = \{Q\widetilde{E} \mid Q \in \mathfrak{B} \cap \mathcal{C}\},$$

and hence, since $C_E = C_{\widetilde{E}}$,

$$\begin{aligned}&\{PC_E \mid P \in \widetilde{\mathfrak{B}} \cap \mathcal{C}\} = \{QC_E \mid Q \in \mathfrak{B} \cap \mathcal{C}\} \\ \Longleftrightarrow\ &(\widetilde{\mathfrak{B}} \cap \mathcal{C})_{C_E} = (\mathfrak{B} \cap \mathcal{C})_{C_E},\end{aligned}$$

that is, $\widetilde{\mathfrak{B}} \cap \mathcal{C} = \mathfrak{B} \cap \mathcal{C}$. Summing up, it is proven that:

Theorem 115 *Let $\mathcal{R}$ be a von Neumann algebra with center $\mathcal{C}$. Then the mapping*

$$\begin{aligned}\zeta : \mathcal{Q}(\mathcal{R}) &\longrightarrow \mathcal{Q}(\mathcal{C}) \\ \mathfrak{B} &\longmapsto \mathfrak{B} \cap \mathcal{C}\end{aligned}$$

is surjective. If $\mathfrak{B}, \widetilde{\mathfrak{B}} \in \mathcal{Q}^{ab}(\mathcal{R})$ are two abelian quasipoints, then $\zeta(\mathfrak{B}) = \zeta(\widetilde{\mathfrak{B}})$ holds if and only if there is a partial isometry $\theta \in \mathcal{R}$ such that $\theta_{\mathcal{Q}}(\mathfrak{B}) = \widetilde{\mathfrak{B}}$.

3.2. The Stone spectrum of a type I_n von Neumann algebra

The Stone spectrum of a type I_n von Neumann algebra can be described in a fairly complete manner. Let $\mathcal{R}$ be such an algebra. We will show that every quasipoint $\mathfrak{B}$ of $\mathcal{R}$ is abelian, i.e. contains an abelian projection. In order to do so, we will use the fact that $\mathcal{R}$ is (isomorphic to) a $n \times n$*-matrix* algebra, albeit with entries from another von Neumann algebra, the center of $\mathcal{R}$. We regard $\mathcal{R}$ as acting on the Hilbert module $\mathcal{A}^n$, which generalizes the vector space $\mathbb{C}^n$. The abelian projections will be those projecting onto "lines" of the form $a\mathcal{A}$, where $\mathcal{A} := \mathcal{C}(\mathcal{R})$ is the center of $\mathcal{R}$. Of course, $\mathcal{A}$ is not a field and $\mathcal{A}^n$ is not a vector space, so we cannot use arguments for subspace lattices of finite-dimensional vector spaces directly (in which case every quasipoint is abelian). But we will introduce equivalence relations on $\mathcal{A}$ and $\mathcal{A}^n$ that turn them into a field and an n-dimensional vector space, respectively, and show that after taking equivalence classes, enough of the structure remains intact to allow the conclusion that every quasipoint of $\mathcal{R}$ is abelian. The intuition from linear algebra carries through. From Thm. 115, we know that the abelian quasipoints can be mapped to the quasipoints of the center of $\mathcal{R}$ via

$$\begin{aligned}\xi : \mathcal{Q}^{ab}(\mathcal{R}) &\longrightarrow \mathcal{Q}(\mathcal{C}),\\ \mathfrak{B} &\longmapsto \mathfrak{B} \cap \mathcal{C},\end{aligned}$$

where two quasipoints $\mathfrak{B}, \widetilde{\mathfrak{B}}$ are mapped to the same quasipoint of the center if and only if there is a partial isometry $\theta \in \mathcal{R}$ such that $\theta_{\mathcal{Q}}(\mathfrak{B}) = \widetilde{\mathfrak{B}}$. Using the fact that $\mathcal{R}$ is a finite algebra, we can replace partial isometries with unitary operators. This will allow us to specify the orbits of the unitary group $\mathcal{U}(\mathcal{R})$ acting on $\mathcal{Q}(\mathcal{R})$ (Thm. 138).

3.2.1. Hilbert modules and the projections E_a

It is well known that each type I_n von Neumann algebra $\mathcal{R}$ is $*$-isomorphic to $\mathbb{M}_n(\mathcal{A})$, the matrix algebra with entries from $\mathcal{A} = \mathcal{C}(\mathcal{R})$, the center of $\mathcal{R}$ (see Thm. 6.6.5 in [KadRinII97]). Let $\mathcal{A}^n$ be the free right module over $\mathcal{A}$ consisting of n copies of $\mathcal{A}$. Another common notation for $\mathbb{M}_n(\mathcal{A})$ is $End_{\mathcal{A}}(\mathcal{A}^n)$, the algebra of $\mathcal{A}$-linear endomorphisms of $\mathcal{A}^n$. $\mathbb{M}_n(\mathcal{A})$ acts on the Hilbert space $\widetilde{\mathcal{H}} := \bigoplus^n \mathcal{H}_{\mathcal{A}}$, the n-fold direct sum of $\mathcal{H}_{\mathcal{A}}$, which is the Hilbert space $\mathcal{A}$ acts on. We will not make use of $\widetilde{\mathcal{H}}$ and the representation of $\mathbb{M}_n(\mathcal{A})$ on it, because we will regard $\mathbb{M}_n(\mathcal{A})$ as an algebra that acts on the $\mathcal{A}$-module $\mathcal{A}^n$ from the left. Elements $a = (a_1 \oplus ... \oplus a_n)^t$ of $\mathcal{A}^n$ are regarded as column vectors. The operation of $\mathbb{M}_n(\mathcal{A})$ on $\mathcal{A}^n$ is a "matrix$\times$vector" operation. (Since $\mathcal{A}$ is commutative, $\mathcal{A}^n$ can be regarded as a left module as well. The chosen convention fits the natural structure of $\mathcal{A}^n$ as an $\mathbb{M}_n(\mathcal{A})$-$\mathcal{A}$-bimodule.)

$\mathcal{A}^n$ has a canonical basis with basis elements

$$e_j := (0 \oplus ... \oplus 0 \oplus \overset{\substack{j\\ \downarrow}}{1} \oplus 0 \oplus ... \oplus 0)^t,$$

where 1 is the unit of $\mathcal{A}$. With respect to this basis, $a \in \mathcal{A}^n$ is denoted as $a = (a_1 \oplus ... \oplus a_n)^t$. The sign of transposition will be omitted from now on.

There is an $\mathcal{A}$-valued product defined on $\mathcal{A}^n$ such that $\mathcal{A}^n$ becomes a **Hilbert-$\mathcal{A}$-module**. Since $\mathcal{A}^n$ is a right module, the inner product is $\mathcal{A}$-linear with respect to the *second* variable:

$$\begin{aligned}(a|b) = (a_1 \oplus ... \oplus a_n | b_1 \oplus ... \oplus b_n) &:= \sum_{k=1}^{n} a_k^* b_k,\\ (a|b\alpha) = (a|b)\alpha = \alpha(a|b) &= (a\alpha^*|b)\end{aligned}$$

for $a, b \in \mathcal{A}^n, \alpha \in \mathcal{A}$. In the second line the commutativity of $\mathcal{A}$ was employed. The inner product induces a norm on $\mathcal{A}^n$ by

$$|a| := |(a|a)|^{\frac{1}{2}},$$

where the norm on the right hand side is the norm on $\mathcal{A}$.

Let $\Omega := \mathcal{Q}(\mathcal{A})$ be the Stone spectrum of $\mathcal{A}$. Without loss of generality, we can assume $\mathcal{A} = C(\Omega)$. Let $(\Omega_1, ..., \Omega_n)$ be a partition of Ω into closed-open sets $\Omega_k \neq \varnothing$, and let $a_k := \chi_{\Omega_k}$. Then for all $\beta \in \Omega$,

$$\sum_k |a_k(\beta)|^2 = 1$$

and hence $|a_1 \oplus ... \oplus a_n| = 1$. Moreover, for $k \neq j$,

$$(a_k|a_j) = a_k^* a_j = 0,$$

that is, the a_k are pairwise orthogonal. But (the analogue of) Pythagoras' theorem does not hold, since for our example one obtains

$$|a_1 \oplus ... \oplus a_n|^2 = 1 < n = \sum_{k=1}^{n} |a_k|^2.$$

In general, operators on Hilbert modules are −different from those on Hilbert spaces− not (all) adjointable, which is due to the lack of self-duality of Hilbert modules, see for example [WeO93, p 240]. A mapping $T : E \to E$ from a Hilbert module E to itself is called **adjointable** if there is a mapping T^* such that

$$(Ta|b) = (a|T^*b)$$

for all $a, b \in E$. The mapping T^* is called the **adjoint of** T. One can show that if T is adjointable, then T^* is unique, $T^{**} = T$ and both T and T^* are module maps which are bounded with respect to the operator norm ([WeO93, Lemma 15.2.3]).

For our purpose, we have to characterize the projections in $\mathcal{R} \simeq \mathbb{M}_n(\mathcal{A})$.

Lemma 116 *The elements of* $\mathbb{M}_n(\mathcal{A})$ *are adjointable,* $T = T_{kj} \in \mathbb{M}_n(\mathcal{A})$ *has adjoint* $(T^*)_{kj} = T_{jk}^*$.

Proof. Let $a, b \in \mathcal{A}^n, T = T_{kj} \in \mathbb{M}_n(\mathcal{A})$. We have

$$\begin{aligned}
(Ta|b) &= \sum_{k,j} (T_{kj} a_j)^* b_k \\
&= \sum_{k,j} a_j^* T_{kj}^* b_k \\
&= \sum_{j,k} a_j^* (T_{jk})^* b_k \\
&= (a|T^* b).
\end{aligned}$$

■

The adjoint of $T \in \mathbb{M}_n(\mathcal{A})$ in the Hilbert module sense thus is the usual, Hilbert space adjoint of T. It follows that the projections of the von Neumann algebra $\mathbb{M}_n(\mathcal{A})$ are the projections of the algebra $\mathbb{B}(\mathcal{A}^n)$ of adjointable operators of the Hilbert module $\mathcal{A}^n$. (A projection P is $\mathcal{A}$-linear, so it is contained in $\mathbb{M}_n(\mathcal{A})$.)

We will now introduce projections E_a that map from $\mathcal{A}^n$ onto "lines" of the form $a\mathcal{A}$: let $a \in \mathcal{A}^n$ be such that $p := (a|a) \in \mathcal{A}$ is a projection. Then

$$\forall k \leq n : pa_k = a_k,$$

since for $\beta \in \mathcal{Q}(\mathcal{A})$ such that $a_k(\beta) \neq 0$, one obtains $p(\beta) = \sum_j a_j^* a_j \neq 0$ and hence $p(\beta) = 1$, since p is a projection. Here and in the following, the components $a_k \in \mathcal{A}$ of a are identified with their Gelfand transforms. We get

$$\forall \beta \in \mathcal{Q}(\mathcal{A}) \;\; \forall k \leq n : (a_k p)(\beta) = a_k(\beta)p(\beta) = a_k(\beta),$$

so $ap = a$.

Now define

$$\begin{aligned} E_a : \mathcal{A}^n &\longrightarrow \mathcal{A}^n, \\ b &\longmapsto a(a|b). \end{aligned}$$

For all $b, c \in \mathcal{A}^n$,

$$\begin{aligned} (E_a b|c) &= (a(a|b)|c) \\ &= (a|b)^*(a|c) \\ &= (b|a)(a|c) \\ &= (b|a(a|c)) \\ &= (b|E_a c), \end{aligned}$$

so we have $E_a^* = E_a$. Moreover, for all $b \in \mathcal{A}^n$,

$$\begin{aligned} E_a^2 b &= a(a|E_a b) \\ &= a(a|a(a|b)) \\ &= a(a|a)(a|b) \\ &= ap(a|b) \\ &= a(a|b) \\ &= E_a b, \end{aligned}$$

so $E_a^2 = E_a$, that is, E_a is a projection with $\operatorname{im} E_a \subseteq a\mathcal{A}$. In fact, equality holds: let $b = a\alpha \in a\mathcal{A}$. Then

$$a(a|b) = a(a|a)\alpha = a\alpha,$$

therefore $a\mathcal{A} \subseteq \operatorname{im} E_a$. The central carrier C_{E_a} of E_a is $I_n(a|a)$: obviously, $I_n(a|a)$ is a central projection, since $I_n\mathcal{A} \simeq \mathcal{A}$ is the center of $\mathbb{M}_n(\mathcal{A})$. It holds that

$$\begin{aligned} E_a b(a|a) &= a(a|b)(a|a) \\ &= a(a|a)(a|b) \\ &= a(a|b) \\ &= E_a b, \end{aligned}$$

so $C_{E_a} \leq I_n(a|a)$. Conversely, let $I_n q$ be a central projection such that $qE_a = E_a q = E_a$. Then for all $b \in \mathcal{A}^n$,

$$\begin{aligned} E_a b &= a(a|b) \\ &= qE_a b \\ &= qa(a|b) \\ &= a(a|qb) \\ &= E_a(qb). \end{aligned}$$

In particular, one obtains

$$\begin{aligned} & E_a(qa) = E_a a \\ \iff & a(a|qa) = a(a|a) = ap = a \\ \iff & a(a|a)q = a \\ \iff & aq = qa = a \\ \implies & q(a|a) = qp = (a|a) = p \\ \implies & p \leq q, \end{aligned}$$

therefore, the central carrier of E_a is $C_{E_a} = I_n p = I_n(a|a)$.

The E_a are of interest, because they are *abelian* projections:

Lemma 117 *E_a is an abelian projection from $\mathcal{A}^n$ onto $a\mathcal{A}$ with central carrier $I_n(a|a)$.*

Proof. It only remains to show that E_a is abelian. Let $A, B \in \mathbb{M}_n(\mathcal{A})$. Then it holds for all $b \in \mathcal{A}^n$ that

$$\begin{aligned} E_a A E_a B E_a b &= E_a A E_a (Ba)(a|b) \\ &= E_a(Aa)(a|Ba)(a|b) \\ &= a(a|Aa)(a|Ba)(a|b) \\ &= a(a|Ba)(a|Aa)(a|b) \\ &= E_a(Ba)(a|Aa)(a|b) \\ &= E_a B E_a (Aa)(a|b) \\ &= E_a B E_a A E_a b, \end{aligned}$$

so $E_a A E_a B E_a = E_a B E_a A E_a$. ■

E_a is a projection in $\mathbb{M}_n(\mathcal{A})$ if $(a|a)$ is a projection in $\mathcal{A}$. The converse is also true:

Remark 118 *Let $a \in \mathcal{A}^n$ be such that E_a is a projection. Then $(a|a) \in \mathcal{A}$ is a projection.*

Proof. From $E_a^2 = E_a$, $a(a|b) = a(a|E_a b) = a(a|a)(a|b)$ for all $b \in \mathcal{A}^n$. For $b = a$,

$$a(a|a) = a(a|a)^2.$$

This means that $(a|a) \in \{0,1\}$ holds on the support $supp\ a := \bigcup_{k\leq n} supp\ a_k = supp(a|a)$. If $\beta \in \Omega = \mathcal{Q}(\mathcal{A})$ is such that $(a|a)(\beta) \neq 0$, then $a_k(\beta) \neq 0$ holds for at least one $k \leq n$ and thus $(a|a)(\beta) = 1$. So $(a|a) = 1$ holds on $supp(a|a)$ and $(a|a)$ is a projection. ■

If $a_1, ..., a_n \in \mathcal{A}$ are projections and $a := \sum_k a_k e_k$, then E_a is a projection if and only if the a_k are pairwise orthogonal, because, according to its definition and the above remark, E_a is a projection if and only if $(a|a)$ is a projection. For $a := \sum_k a_k e_k$, we have $(a|a) = \sum_k a_k$, and this is a projection if and only if the a_k are pairwise orthogonal. Furthermore, $\sum_k a_k E_{e_k} = \sum_k E_{a_k e_k}$ (the a_k are projections again): for all $b \in \mathcal{A}^n$, it holds that

$$\begin{aligned}(\sum_k a_k E_{e_k})b &= \sum_k a_k e_k (e_k|b) \\ &= \sum_k a_k^2 e_k (e_k|b) \\ &= \sum_k a_k e_k (e_k|b a_k) \\ &= \sum_k a_k e_k (a_k e_k|b) \\ &= \sum_k E_{a_k e_k} b.\end{aligned}$$

Lemma 119 *$A := \sum_{k=1}^n a_k E_{e_k}$ is a projection if and only if all the a_k are projections. In this case, the central carrier of A is $C_A = I_n(\bigvee_k a_k)$.*

Proof. From $(e_j|e_k) = \delta_{jk} e_k$ we get

$$\begin{aligned}\forall c \in \mathcal{A}^n : E_{e_j} E_{e_k} c &= e_j (e_j|E_{e_k} c) \\ &= e_j (e_j|e_k)(e_k|c) \\ &= \delta_{jk} E_j c,\end{aligned}$$

so $E_{e_j} E_{e_k} = \delta_{jk} E_{e_j}$. $A := \sum_{k=1}^n a_k E_{e_k}$ is a projection if and only if $a_k^* = a_k$ holds for all $k \leq n$ and if $A^2 = A$. Since we have

$$\begin{aligned}A^2 &= (\sum_{k=1}^n a_k E_{e_k})^2 \\ &= \sum_{j,k=1}^n a_j a_k E_{e_j} E_{e_k} \\ &= \sum_{j,k=1}^n a_j a_k \delta_{jk} E_{e_j} \\ &= \sum_{j=1}^n a_j^2 E_{e_j},\end{aligned}$$

this holds if and only if all the $a_j \in \mathcal{A}$ are projections. $C_A = I_n(\bigvee_k a_k)$ holds then, obviously. ■

Remark 120 *With respect to the basis* $(e_1, ..., e_n)$ *of* $\mathcal{A}^n$*,* E_a *has the matrix*

$$(E_a)_{jk} = (a_j a_k^*)_{j,k \leq n}.$$

Proof. It holds that

$$\begin{aligned} E_a e_k &= a(a|e_k) \\ &= a \sum_j a_j^* \delta_{jk} \\ &= a a_k^* \\ &= (\sum_j a_j e_j) a_k^* \\ &= (\sum_j a_j a_k^*) e_j. \end{aligned}$$

■

The projections E_a are special cases of the so-called **ket-bra-operators** (see e.g. [GVF01, p 71]). These (and their symbolic Dirac notation) are defined as

$$\begin{aligned} |r\rangle\langle s| : E &\longrightarrow F, \\ b &\longmapsto r(s|b), \end{aligned}$$

where E and F are Hilbert modules over a C^*-algebra $\mathcal{A}$, $r \in F$ and $s \in E$. For $E = F$, we have $E_a = |a\rangle\langle a|$. There are some relations among the ket-bra-operators ($\alpha \in \mathcal{A}$):

$$\begin{aligned} r(s|b\alpha) &= r(s|b)\alpha, \\ |r\rangle\langle s|^* &= |s\rangle\langle r|, \\ |r\rangle\langle s| \circ |t\rangle\langle u| &= |r(t|s)\rangle\langle u| = |r\rangle\langle u(s|t)| \end{aligned}$$

hence the finite sums of ket-bra-operators from E to E form a self-adjoint algebra $End^{00}_{\mathcal{A}}(E)$ contained in $End_{\mathcal{A}}(E)$. The operators in $End^{00}_{\mathcal{A}}(E)$ are called operators **of $\mathcal{A}$-finite rank**. The norm closure $End^{0}_{\mathcal{A}}(E)$ of $End^{00}_{\mathcal{A}}(E)$ contains the so-called **$\mathcal{A}$-compact operators**. Clearly, $End^{0}_{\mathcal{A}}(\mathcal{A}^n) = \mathbb{M}_n(\mathcal{A})$ holds.

3.2.2. The modules $a\mathcal{A}$

We now turn to the examination of the modules $a\mathcal{A}$ onto which the E_a project. This subsection is quite technical. After showing how to "normalize" an arbitrary $a \in \mathcal{A}^n \backslash \{0\}$ to $\widetilde{a}$ such that $(\widetilde{a}|\widetilde{a})$ and $E_{\widetilde{a}}$ are projections with $a\mathcal{A} = \widetilde{a}\mathcal{A}$, we define the *support* $S(M)$ of a projective submodule $M \subseteq \mathcal{A}^n$ and show that there is an $a \in M$ such that $S(a) = S(M)$ if M is finitely generated. This is used in the proof that the finitely generated projective submodules of $\mathcal{A}^n$ for which $End_{\mathcal{A}}(M)$ is *abelian* are exactly those of the form $a\mathcal{A} = \widetilde{a}\mathcal{A}$. To show this, we also need the fact that $End^{0}_{\mathcal{A}}(P\mathcal{A}^n) = P\mathbb{M}_n(\mathcal{A})P$ holds (Lemma 2.18 in [GVF01]).

All this is a preparation for the following subsection, where quasipoints of $\mathcal{R} \simeq \mathbb{M}_n(\mathcal{A})$ are regarded as families of projective submodules of $\mathcal{A}^n$ rather than families of projections.

Lemma 121 *Let $a \in \mathcal{A}^n$. $a\mathcal{A}$ is a closed submodule of $\mathcal{A}^n$.*

Proof. Let $(a\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $a\mathcal{A}$ converging to $b \in \mathcal{A}^n$. As before, let $supp\, a := \bigcup_{k\leq n} supp\, a_k$, then $supp\, a = supp(a|a)$. Without loss of generality, one can assume that $\alpha_n(\beta) = 0$ holds for $\beta \neq supp\, a$, since the sequence $(a\alpha_n)$ remains unchanged by that. From

$$|a\alpha_n - a\alpha_m|^2 = |\alpha_n - \alpha_m|^2(a|a)$$

for all $n, m \in \mathbb{N}$ it follows that $(\alpha_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}$. Thus $(\alpha_n)_{n\in\mathbb{N}}$ converges to some $\alpha \in \mathcal{A}$ and we get

$$|a\alpha_n - a\alpha|^2 = |\alpha_n - \alpha|^2(a|a) \longrightarrow 0 \quad \text{for } n \longrightarrow \infty,$$

that is, $b = a\alpha \in a\mathcal{A}$. ■

Let $M \subseteq \mathcal{A}^n$ be some submodule. The **orthogonal complement** $M^\perp$ of M is given by (see [WeO93, p 248])

$$M^\perp := \{b \in \mathcal{A}^n \mid \forall a \in M : (b|a) = 0\}.$$

For $M = a\mathcal{A}$, we obtain

$$\begin{aligned}(a\mathcal{A})^\perp &= \{b \in \mathcal{A}^n \mid \forall \alpha \in \mathcal{A} : (b|a\alpha) = 0\} \\ &= \{b \in \mathcal{A}^n \mid \forall \alpha \in \mathcal{A} : (b|a)\alpha = 0\} \\ &= \{b \in \mathcal{A}^n \mid (b|a) = 0\}.\end{aligned}$$

Obviously, $M \cap M^\perp = 0$ holds for any submodule M. A submodule M is called **complementable** if $M \oplus M^\perp = \mathcal{A}^n$ holds. One can show that M is complementable if and only if it is the image of some projection (Cor. 15.3.9 in [WeO93]).

Subsequently, it will be demonstrated how to normalize an arbitrary $a \in \mathcal{A}^n\backslash\{0\}$ to $\widetilde{a}$ such that $(\widetilde{a}|\widetilde{a})$ is a projection in $\mathcal{A}$ and $E_{\widetilde{a}}$ is the projection in $\mathbb{M}_n(\mathcal{A})$ onto $a\mathcal{A} = \widetilde{a}\mathcal{A}$. Let $a \in \mathcal{A}^n\backslash\{0\}$ be such that $(a|a)$ is not a projection. For $n \in \mathbb{N}$, let

$$A_n := \{\beta \in \Omega \mid (a|a)(\beta) > \frac{1}{n}\}.$$

A_n is open and $\overline{A_n}$ is open and closed, since Ω is extremely disconnected. For an appropriate $n_0 \in \mathbb{N}$, $A_n \neq \varnothing$ holds for all $n \geq n_0$.

Let $\alpha_n \in \mathcal{A}$ be defined by

$$\alpha_n(\beta) := \begin{cases} (a|a)(\beta)^{-\frac{1}{2}} & \text{for } \beta \in \overline{A_n} \\ 0 & \text{for } \beta \notin \overline{A_n} \end{cases}.$$

Then $a\alpha_n \in a\mathcal{A}$ and

$$(a\alpha_n|a\alpha_n) = (a|a)\alpha_n^2 = \begin{cases} 1 & \text{on } \overline{A_n} \\ 0 & \text{on } \Omega\backslash\overline{A_n} \end{cases},$$

therefore $(a\alpha_n|a\alpha_n)$ is a projection in $\mathcal{A}$. Let $E_n := E_{a\alpha_n}$ be the projection onto $a\alpha_n\mathcal{A}$ given by $a\alpha_n$. According to the definition of α_n, $\alpha_n\mathcal{A}$ is the closed ideal

$$\alpha_n\mathcal{A} = \{\alpha \in \mathcal{A} \mid \mathit{supp}\,\alpha \subseteq \overline{A_n}\} = \chi_{\overline{A_n}}\mathcal{A}.$$

For these ideals it holds that

$$\forall n \in \mathbb{N} : \alpha_n\mathcal{A} \subseteq \alpha_{n+1}\mathcal{A}.$$

Moreover, with $S(a) := \mathit{supp}\,a \subseteq \Omega = \mathcal{Q}(\mathcal{A})$, it holds that $\overline{\bigcup_n \alpha_n\mathcal{A}} = \mathcal{A}\chi_{S(a)}$: the inclusion "$\subseteq$" is clear from $\overline{A_n} \subseteq S(a)$. Let $b \in \mathcal{A}^n$ be such that $S(b) \subseteq S(a)$, with no loss of generality $b \geq 0$. Then $(b\chi_{\overline{A_n}})_{n\geq n_0}$ is increasing monotonously. One gets

$$\sup_{n\in\mathbb{N}} b\chi_{\overline{A_n}} = b \sup_{n\in\mathbb{N}} \chi_{\overline{A_n}}.$$

From

$$\bigcup_n A_n = \{\beta \in \Omega \mid (a|a)(\beta) > 0\},$$

we have

$$\overline{\bigcup_n \overline{A_n}} = \overline{\bigcup_n A_n} = S(a)$$

and thus $\sup_n \chi_{\overline{A_n}} = \chi_{S(a)}$, therefore

$$b = \sup_n b\chi_{\overline{A_n}}.$$

According to Dini's theorem, $b\chi_{\overline{A_n}}$ converges uniformly to b, and we have shown that $\overline{\bigcup_n \alpha_n\mathcal{A}} = \mathcal{A}\chi_{S(a)}$ holds.

The central carrier of the projection E_n is $\chi_{\overline{A_n}}$. $E_n \leq E_{n+1}$ holds, since for all $b \in \mathcal{A}^n$,

$$\begin{aligned} E_nE_{n+1}b &= a\alpha_n(a\alpha_n|E_{a\alpha_{n+1}}b) \\ &= a\alpha_n(a\alpha_n|a\alpha_{n+1})(a\alpha_{n+1}|b) \\ &= a\alpha_n(a\alpha_{n+1}|a\alpha_{n+1})(a\alpha_n|b) \\ &= a\alpha_n\chi_{\overline{A_{n+1}}}(a\alpha_n|b) \\ &= a\alpha_n(a\alpha_n|b) \\ &= E_nb, \end{aligned}$$

where we used $S(a\alpha_n) = \overline{A_n} \subseteq \overline{A_{n+1}}$ in the penultimate step. Let $E := \bigvee_{n\in\mathbb{N}} E_n$. The image of E_n is $a\mathcal{A}\chi_{\overline{A_n}}$, so

$$\operatorname{im} E = a\mathcal{A}\chi_{S(a)} = a\mathcal{A}.$$

When defining E, one cannot simply assume the properties of the Hilbert space situation.

The sesquilinear form

$$\begin{aligned} \mathcal{A}^n \times \mathcal{A}^n &\longrightarrow \mathcal{A}, \\ (b, c) &\longmapsto (E_nb|c) \end{aligned}$$

can be written as

$$
\begin{aligned}
(E_n b|c) &= (a\alpha_n(a\alpha_n|b)|c) \\
&= \alpha_n^*(a|c)(a\alpha_n|b)^* \\
&= \alpha_n^*(a|c)(b|a)\alpha_n \\
&= \chi_{\overline{A_n}}(a(a|a)^{-\frac{1}{2}}|c)(b|a(a|a)^{-\frac{1}{2}}),
\end{aligned}
$$

and for $n \to \infty$, the right hand side converges to

$$
(Eb|c) := \chi_{S(a)}(a(a|a)^{-\frac{1}{2}}|b)(a(a|a)^{-\frac{1}{2}}|c).
$$

Here we use the fact that every bounded continuous function f on an open dense subset $G \subseteq \Omega$ of the Stone space Ω can be extended to a continuous function $\widetilde{f}$ on the whole of Ω (see Cor. III.1.8 in [TakI02]): the support $S(a) = \mathit{supp}\, a \subseteq \Omega$ is open and closed, $G(a) := \{\beta \in \Omega \mid a(\beta) \neq 0\}$ is open and dense in $S(A)$ and the mapping $a(a|a)^{-\frac{1}{2}} : G(a) \to \mathbb{C}^n$ is continuous und bounded, therefore it can be extended to a continuous mapping $S(a) \to \mathbb{C}^n$, which will also be denoted by $a(a|a)^{-\frac{1}{2}}$. Define

$$
\widetilde{a}(\beta) := \begin{cases} a(a|a)^{-\frac{1}{2}} & \text{for } \beta \in S(a) \\ 0 & \text{for } \beta \in \Omega \backslash S(a) \end{cases}
$$

and from this, $E_{\widetilde{a}}$. Then it holds for all $b, c \in \mathcal{A}^n$ that

$$
\begin{aligned}
(E_{\widetilde{a}} b|c) &= (\widetilde{a}|c)(\widetilde{a}|b) \\
&= \chi_{S(a)}(a(a|a)^{-\frac{1}{2}}|c)(a(a|a)^{-\frac{1}{2}}|b).
\end{aligned}
$$

Obviously, $E_{\widetilde{a}}$ is the same as the limit E of the projections E_n defined above. We showed both ways of the definition, because we will need the sets A_n for the following lemma.

Let $a \in \mathcal{A}^n \backslash \{0\}$, and let $\widetilde{a} \in \mathcal{A}^n$ be as defined above. Then

$$
(\widetilde{a}|\widetilde{a}) = (a|a)^{-1}(a|a) = 1
$$

holds on $G(a)$ and hence also on $S(a)$. Thus $(\widetilde{a}|\widetilde{a})$ is a projection in $\mathcal{A}$.

Lemma 122 *For the closed submodules, we have $\widetilde{a}\mathcal{A} = a\mathcal{A}$.*

Proof. From $\overline{\bigcup_n \mathcal{A}\chi_{\overline{A_n}}} = \mathcal{A}\chi_{S(a)}$, one gets $a\mathcal{A} = \overline{\bigcup_n a\mathcal{A}\chi_{\overline{A_n}}}$. Moreover, it holds that

$$
a\mathcal{A}\chi_{\overline{A_n}} = a(a|a)^{-\frac{1}{2}}\mathcal{A}\chi_{\overline{A_n}} = \widetilde{a}\mathcal{A}\chi_{\overline{A_n}}.
$$

It follows from this and $S(a) = S(\widetilde{a})$ that

$$
a\mathcal{A} = \overline{\bigcup_n a\mathcal{A}\chi_{\overline{A_n}}} = \overline{\bigcup_n \widetilde{a}\mathcal{A}\chi_{\overline{A_n}}} = \widetilde{a}\mathcal{A}.
$$

■

Of course, $\widetilde{a} = a$ if $(a|a)$ is a projection in $\mathcal{A}$, so one obtains

Proposition 123 *For every $a \in \mathcal{A}^n$, $E_{\widetilde{a}}$ is an abelian projection with image $a\mathcal{A}$.*

$E_{\widetilde{a}}$ is the unique projection from $\mathcal{A}^n$ onto $a\mathcal{A}$: let $Q : \mathcal{A}^n \to a\mathcal{A}$ be a projection. Then

$$\forall c \in (a\mathcal{A})^\perp : (Qc|Qc) = (Qc|c) = 0,$$

so $Q|_{(a\mathcal{A})^\perp} = 0$. Let $Qa = a\alpha$. From $Q^2 = Q$, $a\alpha^2 = a\alpha$, therefore $\alpha(\beta) \in \{0, 1\}$ on $S(a)$. Since $Q\mathcal{A}^n = a\mathcal{A}$, it follows that $\alpha = 1$ holds on $S(a)$, so $Qa = a$. Let $b \in \mathcal{A}^n, b = a\gamma + a'$ be such that $\gamma \in \mathcal{A}$ and $a' \in (a\mathcal{A})^\perp$. Such a decomposition exists, since $a\mathcal{A}$ is the image of a projection and hence a complementable submodule of $\mathcal{A}^n$. We have $Qb = a\gamma$ and

$$\begin{aligned} E_{\widetilde{a}}b &= \widetilde{a}(\widetilde{a}|b) \\ &= \widetilde{a}(\widetilde{a}|a)\gamma \\ &= \widetilde{a}(\widetilde{a}|(a|a)^{\frac{1}{2}}\widetilde{a})\gamma \\ &= \widetilde{a}(a|a)^{\frac{1}{2}}\gamma \\ &= a\gamma \\ &= Qb, \end{aligned}$$

so $Q = E_{\widetilde{a}}$.

Next, we will define the support of a submodule $M \subseteq \mathcal{A}^n$. For this, we will need a

Remark 124 *Let $U, V \subseteq \Omega$ be open. If $U \cap V = \varnothing$, then $\overline{U} \cap \overline{V} = \varnothing$.*

Proof. Let $\overline{U} \cap \overline{V} \neq \varnothing$ and $\beta \in \overline{U} \cap \overline{V}$. Since $\overline{U}$ is open, $\overline{U} \cap V \neq \varnothing$. Since V is open, $U \cap V \neq \varnothing$. ■

Let M be a *projective submodule* of $\mathcal{A}^n$, i.e. $M = P\mathcal{A}^n$ for some projection $P \in \mathbb{M}_n(\mathcal{A})$ (see p 89 in [GVF01]). Then M is finitely generated.

$$ann(M) := \{\alpha \in \mathcal{A} \mid M\alpha = 0\},$$

the **annihilator of** M, is a closed ideal in $\mathcal{A}$. Let M be finitely generated, $\{g_1, ..., g_r\} \subseteq M$ a system of generators. For $\alpha \in \mathcal{A}$, let $P(\alpha) := \{\beta \in \Omega \mid \alpha(\beta) \neq 0\}$ and for $a \in \mathcal{A}^n$ let $P(a) := \{\omega \in \Omega \mid a(\omega) \neq 0\}$. Then

$$\alpha a = 0 \Longleftrightarrow P(\alpha) \cap P(a) = \varnothing,$$

and thus, according to the above remark,

$$\alpha a = 0 \Longleftrightarrow S(\alpha) \cap S(a) = \varnothing.$$

Let

$$P(M) := \{\beta \in \Omega \mid \exists a \in M : \beta \in P(a)\}.$$

$S(M) := \overline{P(M)}$ is called the **support of** M. Since $\{g_1, ..., g_r\}$ is a system of generators of M, $P(M) = \bigcup_{k\le r} P(g_k)$ holds and hence $S(M) = \bigcup_{k\le r} S(g_k) = \bigcup_{a\in M} S(a)$. The set $S(M)$ is open and closed. $\alpha \in ann(M)$ holds if and only if $\alpha g_k = 0$ for all $k \le r$, therefore

$$\begin{aligned}\alpha \in ann(M) &\iff S(\alpha) \cap S(g_k) = \varnothing \quad (k \le r)\\ &\iff S(\alpha) \cap S(M) = \varnothing\\ &\iff S(\alpha) \subseteq \Omega \backslash S(M)\\ &\iff \alpha|_{S(M)} = 0.\end{aligned}$$

This shows:

Remark 125 *$ann(M)$ is the vanishing ideal of the closed-open set $S(M)$.*

Lemma 126 *$I_n \chi_{S(M)}$ is the central carrier of the projection $P_M \in \mathbb{M}_n(\mathcal{A})$ from $\mathcal{A}^n$ onto M.*

Proof. Since $\chi_{S(M)} = 1$ on $S(a)$ for all $a \in M$, we have $\chi_{S(M)} a = a$ for all $a \in M$ and hence $\chi_{S(M)} P_M = P_M$, i.e. $C_{P(M)} \le I_n \chi_{S(M)}$. Let $p \in \mathcal{A}$ be a projection such that $I_n p \ge P_M$. Then $pa = pP_M a = P_M a = a$ for all $a \in M$, so $p = 1$ on $S(a)$, that is, $p \ge \chi_{S(a)}$. It follows that $p \ge \chi_{S(M)}$. ■

Lemma 127 *Let M be a finitely generated, projective submodule of $\mathcal{A}^n$, and let $a, b \in M$. Then there is a $c \in M$ such that $S(a) \cup S(b) \subseteq S(c)$.*

Proof. Regard the decomposition $b = a\alpha + a'$ with $\alpha \in \mathcal{A}$ and $a' \in (a\mathcal{A})^\perp$. Such a decomposition always exists, since $a\mathcal{A}$ is a projective and hence complementable submodule of $\mathcal{A}^n$. Therefore each $b \in \mathcal{A}^n$ can be decomposed, in particular each $b \in M \subseteq \mathcal{A}^n$. We have $a' = b - a\alpha \in M$, so $a + a' \in M$ and

$$\begin{aligned}(b|b) &= \alpha^*\alpha(a|a) + (a'|a'),\\ (a + a'|a + a') &= (a|a) + (a'|a'),\end{aligned}$$

therefore

$$\begin{aligned}(a|a)(\beta) > 0 &\implies (a + a'|a + a')(\beta) > 0,\\ (b|b)(\beta) > 0 &\implies (a|a)(\beta) > 0 \text{ or } (a'|a')(\beta) > 0\\ &\implies (a + a'|a + a')(\beta) > 0.\end{aligned}$$

It follows that $S(a|a) := supp(a|a) \subseteq S(a + a'|a + a')$ and $S(b|b) \subseteq S(a + a'|a + a')$, that is, $S(a) \cup S(b) \subseteq S(a + a')$. ■

Corollary 128 *Let M be as before. Then there is an $a \in M$ such that $S(a) = S(M)$. a can be chosen such that $(a|a)$ is a projection.*

Proof. Let $u_1, ..., u_r \in M$ with $M = u_1\mathcal{A} + ... + u_r\mathcal{A}$. According to the lemma above, there is an $a \in M$ such that $S(M) = \bigcup_{k\le r} S(u_k) \subseteq S(a) \subseteq S(M)$, so $S(M) = S(a)$. Since $\widetilde{a}\mathcal{A} = a\mathcal{A} \subseteq M$, $\widetilde{a} \in M$, $(\widetilde{a}|\widetilde{a})$ is a projection and $S(\widetilde{a}) = S(a)$. ■

Lemma 129 *For ket-bra-operators $|b\rangle\langle a|, |v\rangle\langle u| \in \mathbb{M}_n(\mathcal{A})$ it holds that $|b\rangle\langle a| \circ |v\rangle\langle u| = (a|v)|b\rangle\langle u|$ $(a, b, u, v \in M \subseteq \mathcal{A}^n)$.*

Proof. Let $c \in M$. Then

$$\begin{aligned} |b\rangle\langle a| \circ |v\rangle\langle u|(c) &= b(a||v\rangle\langle u|(c)) \\ &= b(a|v(u|c)) \\ &= b(a|v)(u|c) \\ &= (a|v)|b\rangle\langle u|(c). \end{aligned}$$

■

Let $M = P\mathcal{A}^n$ be a projective submodule. Lemma 2.18 in [GVF01] tells us that $End^0_{\mathcal{A}}(M) = P\mathbb{M}_n(\mathcal{A})P$, hence if P is abelian, so is $End^0_{\mathcal{A}}(P\mathcal{A}^n)$ and, in particular, $End^{00}_{\mathcal{A}}(P\mathcal{A}^n)$ is abelian. We will now characterize the finitely generated projective submodules $M \subseteq \mathcal{A}^n$ for which $End^{00}_{\mathcal{A}}(M)$, the set of $\mathcal{A}$-linear mappings of $\mathcal{A}$-finite rank from M to itself, is abelian. Let M be such a module. Let $u_1, ..., u_r \in M$ be generators of M such that $(u_k|u_k) \in \mathcal{A}$ is a projection for all $k \leq r$. Moreover, let $a \in M$ be such that $(a|a)$ is a projection and $S(a) = S(M)$ holds. Then

$$|u_k\rangle\langle a| \circ |a\rangle\langle u_k| = (a|a)|u_k\rangle\langle u_k|$$

is a projection from M *onto* $(a|a)u_k\mathcal{A}$, and

$$|a\rangle\langle u_k| \circ |u_k\rangle\langle a| = (u_k|u_k)|a\rangle\langle a|$$

is a projection from M *onto* $(u_k|u_k)a\mathcal{A}$. Since $End^{00}_{\mathcal{A}}(M)$ is abelian by assumption, we get

$$\forall k \leq r : (a|a)u_k\mathcal{A} = (u_k|u_k)a\mathcal{A},$$

and since $S(a|a) = S(M)$, it holds that

$$\forall k \leq r : u_k\mathcal{A} = (u_k|u_k)a\mathcal{A} \subseteq a\mathcal{A}.$$

Thus it follows that

$$M = \sum_k u_k\mathcal{A} \subseteq a\mathcal{A} \subseteq M,$$

that is, $M = a\mathcal{A}$, so M is simply generated. Summing up, we have shown:

Proposition 130 *A projection $P \in \mathbb{M}_n(\mathcal{A}) = End_{\mathcal{A}}(\mathcal{A}^n)$ is abelian if and only if there is an $a \in \mathcal{A}^n$ such that $P = E_a$. Then $(a|a) \in \mathcal{A}$ is a projection and $I_n(a|a)$ is the central carrier of E_a. For $a \in \mathcal{A}^n$, $E_{\tilde{a}}$ is the unique projection onto the simply generated submodule $a\mathcal{A}$ of $\mathcal{A}^n$. This submodule is projective. $a\mathcal{A}$ is free if and only if $S(a) = \Omega = \mathcal{Q}(\mathcal{A})$, i.e. if $E_{\tilde{a}}$ has central carrier I_n.*

3.2.3. The equivalence relation on $\mathcal{A}^n$ and abelian quasipoints of $\mathcal{P}(\mathbb{M}_n(\mathcal{A}))$

In the following, we will regard quasipoints $\mathfrak{B}$ of $\mathcal{P}(\mathbb{M}_n(\mathcal{A}))$ as families of projective submodules of $\mathcal{A}^n$ rather than families of projections. The filter base property of quasipoints will allow us to define a certain notion of germs on $\mathcal{A}$ and $\mathcal{A}^n$ and reduce the situation to that of finite dimensional vector spaces. The results known from that simple part of the theory (see remark 35) easily show that each quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathbb{M}_n(\mathcal{A}))$ contains a submodule of the form $a_0\mathcal{A}$ and hence is abelian.

So let $\mathfrak{B} \in \mathcal{Q}(\mathbb{M}_n(\mathcal{A}))$ a quasipoint, regarded as a family of projective submodules M of $\mathcal{A}^n$. We have

$$P_{M\cap N} = P_M \wedge P_N,$$

if M, N as well as $M \cap N$ are algebraically finitely generated (see Sec. 15.4 of [WeO93]).

Let $P_M \wedge P_N$ be the minimum of the projections $P_M, P_N \in \mathbb{M}_n(\mathcal{A})$. Then

$$P_M \wedge P_N = P_K,$$

where K is the largest finitely generated closed submodule of $\mathcal{A}^n$ such that $K \subseteq M \cap N$ holds. We will use the notation $K = M \wedge N$.

As mentioned above, $I_n\mathcal{A}$ is the center of $\mathbb{M}_n(\mathcal{A})$.

$$\beta := \mathfrak{B} \cap I_n\mathcal{A}$$

is a quasipoint of $\mathcal{A}$, and it holds that

$$\beta \simeq \{\chi_{S(M)} \mid M \in \mathfrak{B}\}.$$

This can also be expressed in the following way:

$$\begin{aligned} &\forall M \in \mathfrak{B} : \beta \in \bigcup_{a\in M} S(a) \\ \Longleftrightarrow &\forall M \in \mathfrak{B} : \chi_{S(M)}(\beta) = 1. \end{aligned}$$

Notice the double role of β: on the one hand as an element of the Stone spectrum (that is, the Gelfand spectrum) $\mathcal{Q}(\mathcal{A})$ of the center $I_n\mathcal{A} \simeq \mathcal{A}$ of $\mathbb{M}_n(\mathcal{A})$, on the other hand as a collection of characteristic functions $\chi_{S(M)}$ on this Stone spectrum.

We now define an equivalence relation on $\mathcal{A}^n$ that amounts to taking germs:

Definition 131 *Let $n \in \mathbb{N}$. Two elements a, b of $\mathcal{A}^n$ are called* ***equivalent at the quasipoint*** *$\beta \in \mathcal{Q}(\mathcal{A})$, if there is a $p \in \beta$ such that $pa = pb$. Notation: $a \sim_\beta b$.*

$\sim_\beta$ really is an equivalence relation: symmetry and reflexivity are obvious. Let $a \sim_\beta b, b \sim_\beta c$, then there are $p, q \in \beta$ such that $pa = pb, qb = qc \Rightarrow pq \in \beta$ and $pqa = pqb = pqc$, therefore $a \sim_\beta c$. Let $[a]_\beta$ be the equivalence class of $a \in \mathcal{A}^n$, and let $[\mathcal{A}^n]_\beta := \{[a]_\beta \mid a \in \mathcal{A}^n\}$.

Theorem 132 *(i)* $[\mathcal{A}]_\beta$ *is a field,*
(ii) $[\mathcal{A}^n]_\beta$ *is an* n*-dimensional vector space over* $[\mathcal{A}]_\beta$.

Proof. Let $a, b \in \mathcal{A}^n$. Then

$$[a]_\beta + [b]_\beta := [a+b]_\beta$$

is well defined, and also, for $\alpha \in \mathcal{A}$,

$$[a]_\beta[\alpha]_\beta := [a\alpha]_\beta$$

is well defined: from $a \sim_\beta a', b \sim_\beta b', \alpha \sim_\beta \alpha'$ it follows that there exist $p, q, r \in \beta$ such that

$$\begin{aligned} &pa = pa', qb = qb', r\alpha = r\alpha' \\ \Longrightarrow &pq(a+b) = pqa + pqb = pqa' + pqb' = pq(a'+b') \end{aligned}$$

and

$$pr(a\alpha) = pa(r\alpha) = pa'(r\alpha') = pr(a'\alpha').$$

Furthermore,

$$[\alpha]_\beta[\gamma]_\beta := [\alpha\gamma]_\beta$$

defines a multiplication ($\alpha, \gamma \in \mathcal{A}$): it holds that

$$\begin{aligned} a \sim_\beta 0 &\Longleftrightarrow \exists p \in \beta : pa = 0 \\ &\Longleftrightarrow \exists p \in \beta : P(p) \cap P(a) = \varnothing \\ &\Longleftrightarrow \exists p \in \beta : S(p) \cap S(a) = \varnothing \\ &\Longleftrightarrow \beta \notin S(a). \end{aligned}$$

Let $\alpha \in \mathcal{A}$ be such that $[\alpha]_\beta \neq 0$. Then $\alpha^*\alpha \geq \varepsilon > 0$ holds on a closed-open neighbourhood W of β in $\Omega = \mathcal{Q}(\mathcal{A})$, since from $p\alpha \neq 0$ for all $p \in \beta$ and $p(\beta) = 1$, it follows that $\alpha(\omega) \neq 0$ holds on a neighbourhood of β. Thus $\chi_W\alpha$ is invertible on W and there is a $\gamma \in C(\Omega)$ such that $S(\gamma) = W$ and $\chi_W\alpha\gamma = \chi_W$, that is, $\alpha\gamma \sim_\beta 1$, so $[\alpha]_\beta[\gamma]_\beta = 1$. Since obviously the algebraic rules for multiplication and addition are fulfilled, it follows that $[\mathcal{A}]_\beta$ is a field and $[\mathcal{A}^n]_\beta$ is a vector space over $[\mathcal{A}]_\beta$. $([e_1]_\beta, ..., [e_n]_\beta)$ is a basis of $[\mathcal{A}^n]_\beta$: let $a \in \mathcal{A}^n, a = \sum_{k=1}^n e_k a_k$, then $[a]_\beta = \sum_k [e_k]_\beta [a_k]_\beta$; if $\sum_k [e_k]_\beta[\gamma_k]_\beta = 0$, then there is some $p \in \beta$ such that

$$0 = p\sum_k e_k\gamma_k = \sum_k e_k(p\gamma_k),$$

so $p\gamma_k = 0$ for all k, that is, $[\gamma_k]_\beta = 0$ for all k. Thus we have

$$\dim_{[\mathcal{A}]_\beta}[\mathcal{A}^n]_\beta = n.$$

■

Let $M \subseteq \mathcal{A}^n$ be a submodule. Then $[M]_\beta := \{[a]_\beta \mid a \in M\}$ is a subspace of $[\mathcal{A}^n]_\beta$. If $N \subseteq \mathcal{A}^n$ is another submodule, then

Lemma 133 $[M \cap N]_\beta = [M]_\beta \cap [N]_\beta$.

Proof. The inclusion "$\subseteq$" is trivial. Let $[a]_\beta \in [M]_\beta \cap [N]_\beta$. Then $[a]_\beta = [b]_\beta$ holds for some $b \in N$, so $pa = pb$ for a $p \in \beta$ and hence $pa \in M \cap N$. Since $p \sim_\beta 1$ in $\mathcal{A}$ (p is a projection), it follows that $[a]_\beta = [pa]_\beta \in [M \cap N]_\beta$. ∎

Corollary 134 $M, N \in \mathfrak{B} \Rightarrow [M \wedge N]_\beta \subseteq [M]_\beta \cap [N]_\beta$.

Proof. This follows from the lemma and $M \wedge N \subseteq M \cap N$. ∎

Let $M \in \mathfrak{B}$, then $[M]_\beta \neq 0$: assume that $[M]_\beta = 0$. Then

$$\begin{aligned} &\forall a \in M \quad \exists p_a \in \beta : p_a a = 0 \\ \Longrightarrow &\forall a \in M : \beta \notin S(a), \end{aligned}$$

but $\beta \in S(M) = \bigcup_{a \in M} S(a)$, since $M \in \mathfrak{B}$. Thus we get

Remark 135 $\mathfrak{B}_\beta := \{[M]_\beta \mid M \in \mathfrak{B}\} \subseteq \mathbb{L}([\mathcal{A}^n]_\beta)$ *is a filter base in the lattice of subspaces of* $[\mathcal{A}^n]_\beta$.

Proof. $0 \notin \mathfrak{B}_\beta$ and for $[M]_\beta, [N]_\beta \in \mathfrak{B}_\beta$, it holds that $[M]_\beta \cap [N]_\beta \supseteq [M \wedge N]_\beta \in \mathfrak{B}_\beta$. ∎

Let $\widetilde{\mathfrak{B}}_\beta$ be a quasipoint of $\mathbb{L}([\mathcal{A}^n]_\beta)$ containing $\mathfrak{B}_\beta$. There is exactly one line $[a_0]_\beta[\mathcal{A}]_\beta \in \widetilde{\mathfrak{B}}_\beta$ such that

$$\forall M \in \mathfrak{B} : [a_0]_\beta[\mathcal{A}]_\beta \subseteq [M]_\beta,$$

because the discussion concerning atomic quasipoints for the lattice of subspaces of finite dimensional vector spaces holds (see remark 35). Of course we have $a_0 \nsim_\beta 0$. It remains to show that $a_0\mathcal{A} \wedge M \neq 0$ holds for all $M \in \mathfrak{B}$. We have $P_{a_0\mathcal{A}} = E_{\widetilde{a_0}}$ and, since $p_M\widetilde{a}_0\mathcal{A} \subseteq M$ holds for some appropriate $p_M \in \beta$, we obtain

$$\begin{aligned} &E_{p_M\widetilde{a_0}} \leq P_M \\ \Longleftrightarrow &p_M E_{\widetilde{a_0}} \leq P_M. \end{aligned}$$

From this, it follows (since $a_0 \nsim 0$, we have $p_M\widetilde{a}_0 \neq 0$ for all $p_M \in \beta$):

$$P_M \wedge E_{\widetilde{a_0}} \geq p_M E_{\widetilde{a_0}} \wedge E_{\widetilde{a_0}} = p_M E_{\widetilde{a_0}} > 0,$$

so $P_M \wedge E_{\widetilde{a_0}} \neq 0$, that is, $a_0\mathcal{A} \wedge M \neq 0$. Since $\mathfrak{B}$ is a quasipoint, $a_0\mathcal{A} \in \mathfrak{B}$ holds from maximality. Summing up, we have shown:

Theorem 136 *All quasipoints of* $\mathbb{M}_n(\mathcal{A})$ *are abelian.*

The following remark clarifies the relation between several abelian projections in a single quasipoint:

Remark 137 *Let* $E_a, E_b \in \mathfrak{B}$ *be abelian projections. There is some* $r \in \mathcal{P}(\mathcal{A})$ *such that* $rE_a = rE_b$.

Proof. $E_c := E_a \wedge E_b$ is an abelian projection in $\mathfrak{B}$, so there are $p, q \in \mathcal{P}(\mathcal{A})$ such that $E_c = pE_a = qE_b$. Then $E_c = pE_a = p^2 E_a = pqE_b$ holds and also $E_c = pqE_a$. If $p \notin \beta$, then $1 - p \in \beta$ and thus

$$0 = (1-p)pE_a = (1-p)E_c \in \mathfrak{B},$$

contradicting $0 \notin \mathfrak{B}$. Hence $p \in \beta$ and also $q \in \beta$, so $pq \in \beta$ and

$$pqE_a = pqE_b.$$

■

If $\mathcal{R}$ is a type I_n algebra with trivial center, i.e. $\mathcal{R} \simeq \mathbb{M}_n(\mathbb{C}) \simeq \mathcal{L}(\mathbb{C}^n)$, then there are only atomic quasipoints (see remark 35 again). An atomic quasipoint is of the form

$$\mathfrak{B}_{\mathbb{C}x} := \{P \in \mathcal{P}(\mathcal{R}) \mid P_{\mathbb{C}x} \leq P\},$$

where $x \in \mathcal{H}\backslash\{0\}$. Of course, $P_{\mathbb{C}x}$ is an abelian projection, too. The Stone spectrum $\mathcal{Q}(\mathcal{R})$ is discrete in this case, since atomic quasipoints are isolated points of the Stone spectrum $\mathcal{Q}(\mathcal{R})$.

As a corollary of the results proved above, we obtain the main result of this section:

Theorem 138 *Let $\mathcal{R}$ be a type I_n von Neumann algebra. The quasipoints $\beta \in \mathcal{Q}(\mathcal{A})$ of $\mathcal{A} = \mathcal{C}(\mathcal{R})$ correspond bijectively to the orbits of the action of the unitary group $\mathcal{U}(\mathcal{R})$ on the Stone spectrum $\mathcal{Q}(\mathcal{R})$ of $\mathcal{R}$.*

Proof. All quasipoints of $\mathcal{R}$ are abelian (Thm. 136), i.e. we have $\mathcal{Q}(\mathcal{R}) = \mathcal{Q}^{ab}(\mathcal{R})$. Using this, we apply Thm. 115: the mapping

$$\begin{aligned} \zeta : \mathcal{Q}^{ab}(\mathcal{R}) &\longrightarrow \mathcal{Q}(\mathcal{A}) \\ \mathfrak{B} &\longmapsto \mathfrak{B} \cap \mathcal{A} \end{aligned}$$

is surjective. Since $\mathcal{R}$ is finite, we can replace partial isometries by unitary operators in Thm. 115. Therefore, $\zeta(\mathfrak{B}) = \zeta(\widetilde{\mathfrak{B}})$ holds if and only if there is a *unitary* $U \in \mathcal{R}$ such that $U.\mathfrak{B} = \widetilde{\mathfrak{B}}$ (see Def. 102). It follows that the quasipoints of $\mathcal{A} = \mathcal{C}(\mathcal{R})$ correspond bijectively to the orbits of the unitary group $\mathcal{U}(\mathcal{R})$ acting on $\mathcal{Q}(\mathcal{R})$. ■

3.3. The Stone spectrum of a type II_1 factor

The main tool when dealing with finite von Neumann algebras usually is the normalized center-valued **trace** $\tau : \mathcal{R} \to \mathcal{C}(\mathcal{R})$ (see for example Chapter 8 in [KadRinII97]). While we did not make use of the trace in our treatment of type I_n algebras, we will need it here when dealing with factors of type II_1. It turns out that some information about the quasipoints of a type II_1 factor can be obtained from the trace, but that there is more structure than the trace can “resolve”.

We will restrict the trace to the projections of $\mathcal{R}$ to obtain the center-valued **dimension function** $\Delta : \mathcal{P}(\mathcal{R}) \to \mathcal{C}(\mathcal{R})$. The basic facts about the dimension function on a finite von Neumann algebra are collected in Thm. 8.4.3 of [KadRinII97]. In particular, it holds that (1) Δ is surjective, (2) if $E, F \in \mathcal{P}(\mathcal{R})$, then $E \sim F$ if and only if $\Delta(E) = \Delta(F)$ and (3) $E \preceq F$ if and only if $\Delta(E) \leq \Delta(F)$. We will regard factors of type II_1 here, so Δ maps (surjectively) to the real interval $[0, 1]$. Our first lemma is quite trivial, but useful:

Lemma 139 *Let $\mathcal{R}$ be a type II_1 factor, $E \in \mathcal{P}(\mathcal{R})$ and $a \in]0, \Delta(E)[$. Then there is a projection $F \in \mathcal{P}(\mathcal{R})$ such that $\Delta(F) = a$ and $F < E$.*

Proof. Let $G \in \mathcal{P}(\mathcal{R})$ with $\Delta(G) = a$. $G \prec E$ follows, so there is some partial isometry θ such that $\theta G \theta^* =: F < E$. ■

Lemma 140 *In a quasipoint $\mathfrak{B}_0$ contained in the Stone spectrum of a type II_1 factor $\mathcal{R}$ there are no minimal elements, which is to say that each projection $E \in \mathfrak{B}_0$ has a subprojection $F < E$ contained in $\mathfrak{B}_0$.*

Proof. Let $E \in \mathfrak{B}_0$ be a projection that has no subprojection $F \in \mathfrak{B}_0$. Since $E \neq 0$, it holds that $\Delta(E) > 0$ and $E \leq G$ for all $G \in \mathfrak{B}_0$ ($E \wedge G \leq E$ for all $G \in \beta$, and equality holds by the assumption that E has no subprojection in β). The dimension function Δ maps surjectively from $\mathcal{P}(\mathcal{R})$ onto $[0, 1]$, so there is a projection $F' \in \mathcal{P}(\mathcal{R}), F' \neq 0$ such that $0 < \Delta(F') < \Delta(E)$. From this, $F' \prec E$ follows, therefore $F' \sim F < E \leq G$ for some $F \neq 0$ and all $G \in \mathfrak{B}_0$. So $F \in \mathfrak{B}_0, F < E$, contradicting the assumed minimality of E. ■

The dimension function can be used to define a metric on the Stone spectrum: let $\mathcal{R}$ be a factor of type II_1 with Stone spectrum $\mathcal{Q}(\mathcal{R})$, and let $\Delta : \mathcal{P}(\mathcal{R}) \to [0, 1]$ be the dimension function. We define a metric on $\mathcal{Q}(\mathcal{R}) \times \mathcal{Q}(\mathcal{R})$ by

$$\begin{aligned} d : \mathcal{Q}(\mathcal{R}) \times \mathcal{Q}(\mathcal{R}) &\longrightarrow [0, 1], \\ (\mathfrak{B}_1, \mathfrak{B}_2) &\longmapsto \sup\{\Delta(E) \mid E \in \mathfrak{B}_1 \delta \mathfrak{B}_2\}, \end{aligned}$$

where $\mathfrak{B}_1 \delta \mathfrak{B}_2$ denotes the *symmetric difference* of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, that is, the set of projections that are either contained in $\mathfrak{B}_1$ or in $\mathfrak{B}_2$, but not in both. If $d(\mathfrak{B}_1, \mathfrak{B}_2) = 0$ holds, then we must have $\mathfrak{B}_1 \delta \mathfrak{B}_2 = 0$ and hence $\mathfrak{B}_1 = \mathfrak{B}_2$.

Proposition 141 *The metric topology defined on $\mathcal{Q}(\mathcal{R})$ by d is finer than the Stone topology.*

Proof. Let $\mathfrak{B}_0 \in \mathcal{Q}_E(\mathcal{R})$ be a quasipoint containing E, and let $B_{\Delta(E)}(\mathfrak{B}_0)$ be an open ball neighbourhood of $\mathfrak{B}_0$ with radius $\Delta(E)$ in the metric topology. Let $\mathfrak{B}$ be a quasipoint contained in $B_{\Delta(E)}(\mathfrak{B}_0)$, then we have

$$\begin{aligned} &d(\mathfrak{B}_0, \mathfrak{B}) < \Delta(E) \\ &\Longrightarrow E \notin \mathfrak{B}_0 \delta \mathfrak{B} \\ &\Longrightarrow E \in \mathfrak{B}_0 \cap \mathfrak{B}, \end{aligned}$$

since $E \in \mathfrak{B}_0$. It follows $\mathfrak{B} \in \mathcal{Q}_E(R)$, therefore

$$B_{\Delta(E)}(\mathfrak{B}_0) \subseteq \mathcal{Q}_E(\mathcal{R}).$$

■

The converse of the above lemma does not hold, so the topologies do not coincide: let $\varepsilon < \frac{1}{2}$, and let $B_\varepsilon(\mathfrak{B}_0)$ be an open ball neighbourhood of $\mathfrak{B}_0$. We will show that there is no open neighbourhood $\mathcal{Q}_E(\mathcal{R})$ of $\mathfrak{B}_0$ in the Stone topology which is contained in $B_\varepsilon(\mathfrak{B}_0)$.

Since $\mathfrak{B}_0 \in \mathcal{Q}_E(\mathcal{R})$, we have $E \in \mathfrak{B}_0$. We first show that $\mathcal{Q}_E(\mathcal{R}) \subseteq B_\varepsilon(\mathfrak{B}_0)$ cannot be fulfilled for $E \in \mathfrak{B}_0$ such that $\Delta(E) > \varepsilon$: such an E has a decomposition into orthogonal subprojections $E_1 \in \mathfrak{B}_0, E_2 = E - E_1$ such that $\Delta(E_1) \geq \varepsilon$. Let $\mathfrak{B}_1 \in \mathcal{Q}_E(\mathcal{R})$ be a quasipoint containing E and E_2. Then $E_1 \in \mathfrak{B}_0 \delta \mathfrak{B}_1$ and therefore $d(\mathfrak{B}_0, \mathfrak{B}_1) \geq \Delta(E) > \varepsilon$, that is, $\mathcal{Q}_E(\mathcal{R}) \nsubseteq B_\varepsilon(\mathfrak{B}_0)$.

So we now assume $\Delta(E) \leq \varepsilon$. There is some projection $F < E$ in $\mathfrak{B}_0$. Let $F' := (I - E) + (E - F) = I - F$. Then we have $F \wedge F' = 0, E \wedge F \neq 0, E \wedge F' \neq 0$. Let $\mathfrak{B}_2 \in \mathcal{Q}_E(\mathcal{R})$ be a quasipoint that contains E and F'. Then $F' \in \mathfrak{B}_0 \delta \mathfrak{B}_2$ and

$$\begin{aligned} \Delta(F') &= \Delta(I - F) \\ &= 1 - \Delta(F) \\ &> 1 - \Delta(E) \\ &> \varepsilon, \end{aligned}$$

since $\Delta(E) \leq \varepsilon < \frac{1}{2}$. Summing up, it follows that $\mathcal{Q}_E(\mathcal{R}) \nsubseteq B_\varepsilon(\mathfrak{B}_0)$ holds for all $E \in \mathfrak{B}_0$ and $\varepsilon < \frac{1}{2}$. It remains an open question if the metric topology induced by d is the discrete topology or not.

Proposition 142 *Each quasipoint of a type II_1 factor contains projections of all dimensions in* $]0, 1]$.

Proof. We first show that a quasipoint always contains projections of arbitrarily small dimension: let $\mathfrak{B}_0$ be a quasipoint such that there exists some $\varepsilon > 0$ such that for all $E \in \mathfrak{B}_0$,

$$\Delta(E) > \varepsilon.$$

Let $E_0 := \bigwedge \mathfrak{B}_0 := \bigwedge\limits_{E \in \mathfrak{B}_0} E$. We must have $E_0 = 0$, since otherwise we would have a minimal element of $\mathfrak{B}_0$, contradicting Lemma 140. Therefore $\Delta(E_0) = 0$ holds. E_0 is the strong limit of the net $\mathfrak{B}_0$. The dimension function Δ is σ-weakly continuous. On the unit ball of $\mathcal{L}(\mathcal{H})$, which contains the projections, the σ-weak and the weak topology coincide (see [TakI02, Lemma II.2.5] for this and the following). Moreover, for nets $\{x_i\}$ consisting of elements from $\mathcal{L}(\mathcal{H})$, it holds

$$x_i \longrightarrow 0 \text{ in the strong topology} \Leftrightarrow x_i x_i^* \longrightarrow 0 \text{ in the weak topology.}$$

Since for projection operators, $P = P^*, PP^* = P$ hold, in our case the weak and the strong topology conincide, too. Accordingly, Δ can be regarded as a strongly continuous function on $\mathcal{P}(\mathcal{H})$ and in particular on $\mathfrak{B}_0$. Since $\Delta(E_0) = 0$ is the limit of the $\Delta(E)$, taken over the net $\mathfrak{B}_0$, we cannot have $\Delta(E) > \varepsilon$ for all $E \in \mathfrak{B}_0$ as assumed.

In a quasipoint $\mathfrak{B}_0$, there are projections of arbitrary dimension $a \in]0,1]$: let $E \in \mathfrak{B}_0$ be such that $\Delta(E) < a$. One chooses some $F \in \mathcal{P}(\mathcal{R})$ such that $\Delta(F) = a$. It follows $E \prec F$, so there is some partial isometry θ such that $\theta E \theta^* = F_1 < F$. Since $\mathcal{R}$ is finite, we can replace θ by a unitary U in $\mathcal{R}$, i.e.

$$UEU^* = F_1 < F$$
$$\Longleftrightarrow E = U^* F_1 U < U^* F U,$$

so $U^*FU \in \mathfrak{B}_0$, and we have $\Delta(U^*FU) = \Delta(F) = a$. ■

Lemma 143 *The E-trunk $\mathfrak{B}_E$ of a quasipoint $\mathfrak{B}$ of a type II_1 factor contains projections of all dimensions $a \in]0, \Delta(E)]$.*

Proof. We first show that $\mathfrak{B}_E$ contains projections of arbitrarily small dimension: let

$$\mu_0 = \inf\{\Delta(F) \mid F \in \mathfrak{B}_E\}.$$

Assume that $\mu_0 > 0$ holds. Let $G \in \mathfrak{B}$ be such that $\Delta(G) < \mu_0$. Then for $\varepsilon > 0$, there is some $F_\varepsilon \in \mathfrak{B}_E$ such that $\Delta(F_\varepsilon) < \mu_0 + \varepsilon$, and we have $G \wedge F_\varepsilon \in \mathfrak{B}_E$ and $\Delta(G \wedge F_\varepsilon) \leq \Delta(G) < \mu_0$, which contradicts our assumption.

Choose some $\lambda_0 \in]0, \Delta(E)]$. Let

$$E_1 := \bigvee \{F \in \mathfrak{B} \mid \Delta(F) \leq \lambda_0 \text{ and } F \leq E\},$$

then $\lambda_1 := \Delta(E_1) \leq \lambda_0$ and $E_1 \leq E$, so $E_1 \in \mathfrak{B}_E$. Assume that $\lambda_1 < \lambda_0$ holds. Choose some $G < E - E_1$ such that $\Delta(G) = \frac{1}{2}(\lambda_0 - \lambda_1)$. Then $E_1 + G \leq E$, $E_1 + G \in \mathfrak{B}_E$ and $\Delta(E_1 + G) = \lambda_1 + \frac{1}{2}(\lambda_0 - \lambda_1) > \lambda_1$, which gives the desired contradiction. ■

Corollary 144 *Let E be a projection in a quasipoint $\mathfrak{B}_0$, and let $a := \Delta(E)$. For each $b \in \mathbb{R}$, $0 < b < a$, there exists a subprojection $F \in \mathfrak{B}_0$, $F < E$, of dimension b.*

Definition 145 *Let $\mathfrak{B}$ be a quasipoint of a type II_1 factor. Let $0 < \lambda \leq 1$. The λ-**slice** $\mathfrak{B}(\lambda)$ of $\mathfrak{B}$ is the set of projections from $\mathfrak{B}$ that have dimension λ.*

Remark 146 *Let $\mathfrak{B}_1, \mathfrak{B}_2$ be quasipoints of a type II_1 factor such that there is some λ such that the λ-slices of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ coincide, $\mathfrak{B}_1(\lambda) = \mathfrak{B}_2(\lambda)$. Then*

$$\forall \mu \geq \lambda : \mathfrak{B}_1(\mu) = \mathfrak{B}_2(\mu).$$

Proof. Let $\mu > \lambda$, and let $E \in \mathfrak{B}_1(\mu)$. Since $\Delta(\mathfrak{B}_{1E}) =]0, \mu]$, there is some $F \in \mathfrak{B}_1(\lambda)$ such that $F \leq E$. Since $F \in \mathfrak{B}_2(\lambda)$, $E \in \mathfrak{B}_2$ holds, so $E \in \mathfrak{B}_2(\mu)$. This shows $\mathfrak{B}_1(\mu) \subseteq \mathfrak{B}_2(\mu)$. The argument is symmetric. ■

The following three results show that even the intersection of two quasipoints is "large":

Lemma 147 *Let $\mathfrak{B}_1, \mathfrak{B}_2$ be two quasipoints of a type II_1 factor. Then*

$$\inf\{\Delta(E) \mid E \in \mathfrak{B}_1 \cap \mathfrak{B}_2\} = 0.$$

Proof. Assume that $\mu_0 := \inf\{\Delta(E) \mid E \in \mathfrak{B}_1 \cap \mathfrak{B}_2\} > 0$. Then for all $F \in \mathfrak{B}_1$ such that $\Delta(F) < \mu_0$ it holds that $F \notin \mathfrak{B}_2$. For such an $F \in \mathfrak{B}_1$ let $\varepsilon > 0$ be such that $\Delta(F) + \varepsilon < \mu_0$. Then there is some $G \in \mathfrak{B}_2$ such that $F \wedge G = 0$ and $\Delta(G) < \varepsilon$. It follows that $\Delta(F \vee G) = \Delta(F) + \Delta(G) < \Delta(F) + \varepsilon < \mu_0$ and $F \vee G \in \mathfrak{B}_1 \cap \mathfrak{B}_2$ hold, which is a contradiction. ■

Proposition 148 *The intersection of two quasipoints of a type II_1 factor always contains projections of all dimensions $\lambda_0 \in]0,1]$.*

Proof. Let $\lambda_0 \in]0,1]$ be chosen arbitrarily. According to the above lemma, we have $\{E \in \mathfrak{B}_1 \cap \mathfrak{B}_2 \mid \Delta(E) \leq \lambda_0\} \neq \varnothing$, and we get

$$E_0 := \bigvee \{E \in \mathfrak{B}_1 \cap \mathfrak{B}_2 \mid \Delta(E) \leq \lambda_0\} \in \mathfrak{B}_1 \cap \mathfrak{B}_2.$$

Assume $\Delta(E_0) < \lambda_0$. Choose some ε such that $0 < \varepsilon < \lambda_0 - \Delta(E_0)$ and some $F \in \mathfrak{B}_1$ such that $\Delta(F) = \Delta(E_0) + \frac{\varepsilon}{2} (< \lambda_0)$. Then $F \notin \mathfrak{B}_2$. Let $G \in \mathfrak{B}_2$ such that $G \wedge F = 0$ and $\Delta(G) \leq \frac{\varepsilon}{2}$ (if we had $G \wedge F \neq 0$ for all $G \in \mathfrak{B}_2$, then $F \in \mathfrak{B}_1$ would hold). Then $F \vee G \in \mathfrak{B}_1 \cap \mathfrak{B}_2$, $\Delta(F \vee G) \leq \Delta(E_0) + \varepsilon < \lambda_0$ and $\Delta(F \vee G) > \Delta(E_0)$, giving a contradiction. ■

Proposition 149 *The intersection of two quasipoints $\mathfrak{B}_1, \mathfrak{B}_2$ of a type II_1 factor $\mathcal{R}$ is strongly dense in $\mathcal{P}(\mathcal{R})$.*

Proof. Let (E_n) be a net in $\mathfrak{B}_1 \cap \mathfrak{B}_2$ such that $E_n \to 0$. (Since $\Delta(\mathfrak{B}_1 \cap \mathfrak{B}_2) =]0,1]$ according to the proposition above, there is a net (E_n) such that $\Delta(E_n) \to 0$ and hence $E_n \to 0$.) Let $E \in \mathcal{P}(\mathcal{R})$, then $E \vee E_n \in \mathfrak{B}_1 \cap \mathfrak{B}_2$, and we have $E \vee E_n \to E$ in the strong topology, since $\mathcal{R}$ is finite. ■

This lemma depends on the finiteness of the algebra $\mathcal{R}$. To make this clearer, we give a counterexample for the *infinite* von Neumann algebra $\mathcal{L}(\mathcal{H})$, where $\mathcal{H} = L^2([0,1])$. Instead of a net converging strongly to 0, we use a sequence (E_n) strongly converging to I, given by the multiplication operators

$$E_n := \chi_{[0,1-\frac{1}{n}]}.$$

Let E be the projection onto the space of functions that are constant almost everywhere. Then we have

$$E_n \wedge E = 0,$$

so $E_n \wedge E \to 0 \neq E$.

Let $f \in L^2([0,1])$. Ef has some nice interpretation: it is the expectation value $\int f d\lambda$ of f. It holds that $(E\mathcal{H})^\perp = \{f \in \mathcal{H} \mid \int f d\lambda = 0\}$, therefore $f - \int f d\lambda \in (E\mathcal{H})^\perp$.

We saw that for factors $\mathcal{R}$ of type II_1, the dimension function helps to clarify some aspects of the structure of quasipoints. On the other hand, results like Props. 142 and 149 point to the fact that there is a lot more structure hidden in the quasipoints and their relations which cannot be brought to light by the dimension function alone. This means that the Stone spectrum of a type II_1 factor is a fine invariant of the algebra. While currently the methods do not seem at hand, it is a challenging task for the future to clarify more of its structure.

The next subsection concerns maximal abelian subalgebras of type II_1 factors and Boolean quasipoints.

3.3.1. Boolean quasipoints of a type II_1 factor

In this subsection, let $\mathcal{R}$ be a type II_1 factor and $\mathcal{M} \subset \mathcal{R}$ a maximal abelian von Neumann subalgebra of $\mathcal{R}$. We will show that a Boolean quasipoint β of $\mathcal{R}$ contains projections of all dimensions in $]0,1]$. In this sense, even a Boolean quasipoint β of a type II_1 factor is a large object. It is dense in the projection lattice $\mathcal{P}(\mathcal{M})$ of the maximal abelian von Neumann subalgebra $\mathcal{M} \subseteq \mathcal{R}$ generated by β.

The following result is by Kadison [Kad84, Prop. 3.13]:

Proposition 150 *Let $\mathcal{R}$ be a von Neumann algebra of type II_1, $\mathcal{M}$ be a maximal abelian subalgebra of $\mathcal{R}$, and Δ be the center-valued dimension function on $\mathcal{R}$. If H is an element in the center of $\mathcal{R}$ such that $0 \leq H \leq I$, then there is a projection E in $\mathcal{M}$ such that $\Delta(E) = H$.*

An easy corollary is [Kad84, Cor. 3.14]

Corollary 151 *Let $\mathcal{R}$ be a von Neumann algebra of type II_1, $\mathcal{M}$ be a maximal abelian subalgebra of $\mathcal{R}$, E be a projection in $\mathcal{M}$, and Δ be the center-valued dimension function on $\mathcal{R}$. If H is an element of the center $\mathcal{C}$ of $\mathcal{R}$ such that $0 \leq H \leq \Delta(E)$, then there is a projection F in $\mathcal{M}$ such that $\Delta(F) = H$ and $F \leq E$.*

If $\mathcal{R}$ is a type II_1 *factor*, the dimension function Δ maps $\mathcal{P}(\mathcal{M})$ surjectively onto $[0,1]$. Applying the corollary to a factor gives a result for maximal abelian subalgebras analogous to Lemma 139:

Corollary 152 *Let $\mathcal{R}$ be a type II_1 factor, $\mathcal{M}$ a maximal abelian subalgebra, $E \in \mathcal{P}(\mathcal{M})$ and $a \in]0, \Delta(E)[$. Then there is a projection $F \in \mathcal{P}(\mathcal{M})$ such that $\Delta(F) = a$ and $F < E$.*

Lemma 153 *A Boolean quasipoint β of $\mathcal{R}$ has no minimal elements, which is to say that each projection $E \in \beta$ has a subprojection $F < E$ contained in β.*

Proof. Let $E \in \beta$ be a projection that has no subprojection $F \in \beta$. Since $E \neq 0$, it holds that $\Delta(E) > 0$ and $E \leq G$ for all $G \in \beta$ ($E \wedge G \leq E$ for all $G \in \beta$, and equality holds by the assumption that E has no subprojection in β). According to Cor. 152, there is a non-zero projection $F \in \mathcal{M}$ such that $F < E \leq G$ for all $G \in \beta$. Since $F \wedge G = F \neq 0$ for all $G \in \beta$, we have $F \in \beta$, contradicting the assumed minimality of E. ■

In order to prove that a Boolean quasipoint β of $\mathcal{R}$ contains projections of all rational dimensions in $]0,1]$, we need [Kad84, Cor. 3.15]

Corollary 154 *If $\mathcal{R}$ is a von Neumann algebra of type II_1 and n is a positive integer, then each maximal abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$ contains n orthogonal equivalent projections with sum I.*

Lemma 155 *Each Boolean quasipoint β of a type II_1 factor $\mathcal{R}$ contains projections of all rational dimensions in $]0,1]$.*

Proof. Let $\frac{m}{n} \in]0,1]$ be a rational number, and let $\mathcal{M} \subset \mathcal{R}$ be the maximal abelian von Neumann algebra containing β. According to the above corollary, there are n orthogonal equivalent projections E_i, $i \leq n$ with $\sum_{i=1}^{n} E_i = I$ contained in $\mathcal{M}$. We regard β as a quasipoint of $\mathcal{M}$ (see Rem. 78). Since $\mathcal{P}(\mathcal{M})$ is a complete Boolean algebra, either $E_i \in \beta$ or $E_i^{\perp} \in \beta$, and $E_{i_0} \in \beta$ holds for exactly one $i_0 \leq n$, since $\bigwedge_{i=1}^{n} E_i^{\perp} = 0$. Choose $m-1$ projections $E_{j_1}, ..., E_{j_{m-1}}$ from the set $\{E_1, ...\widehat{E_{i_0}}, ..., E_n\}$ (E_{i_0} is omitted). The maximum $E := E_{i_0} \vee (\bigvee_{k=1}^{m-1} E_{j_k})$ is contained in β, since $E > E_{i_0}$, and we have $\dim E = \frac{m}{n}$. ■

Proposition 156 *A Boolean quasipoint β of a type II_1 factor $\mathcal{R}$ contains projections of all dimensions in $]0,1]$.*

Proof. Let $a \in]0,1]$, and let $\mathcal{M} \subset \mathcal{R}$ be the maximal abelian von Neumann subalgebra containing β. According to the lemma above, there is a (non-zero) projection $E \in \beta$ of rational dimension $b := \Delta(E)$ such that $0 < b < a$. We regard β as a quasipoint of $\mathcal{M}$. According to Cor. 152, the projection $I - E \in \mathcal{M}$ has a subprojection $F_1 \in \mathcal{M}$ with $\Delta(F_1) = a - b$ and $F_1 \wedge E = 0$. Let $F := E + F_1$, then $\Delta(F) = a$ and $F > E$, so $F \in \beta$. ■

Proposition 157 *A Boolean quasipoint β of a type II_1 factor $\mathcal{R}$ is strongly dense in the projection lattice $\mathcal{P}(\mathcal{M})$ of the maximal abelian von Neumann subalgebra $\mathcal{M}$ of $\mathcal{R}$ generated by β.*

Proof. Let (E_n) be a net in β such that $E_n \to 0$. (Since $\Delta(\beta) =]0,1]$, there is a net (E_n) such that $\Delta(E_n) \to 0$ and hence $E_n \to 0$.) Let $\mathcal{M}$ be the maximal abelian subalgebra of $\mathcal{R}$ generated by β (see Lemma 77), and let $E \in \mathcal{P}(\mathcal{M})$. Then $E \vee E_n \in \beta$, and we have $E \vee E_n \to E$ in the strong topology, since $\mathcal{R}$ is finite. ■

4. First applications to physics

This short chapter gives some applications of Stone spectra and observable function to physics.

4.1. Conventions

In this chapter and the following (ch. 5, “The Kochen-Specker theorem”), we will use some of the identifications and notations common in physics: in quantum theory, including quantum mechanics in the von Neumann representation, quantum field theory and quantum information theory, *observables* are represented by self-adjoint operators A in some von Neumann algebra $\mathcal{R}$, the *algebra of observables*. Like before, we assume that the algebra $\mathcal{R}$ is contained in $\mathcal{L}(\mathcal{H})$, the set of bounded linear operators on some separable Hilbert space $\mathcal{H}$. The set of observables $\mathcal{R}_{sa}$ forms a real linear space in the algebra $\mathcal{R}$.

When speaking of *states* of a von Neumann algebra $\mathcal{R}$ in this work (in particular in ch. 5, “The Kochen-Specker theorem”), we will mean the mathematical notion, i.e. positive linear functionals of norm 1 on $\mathcal{R}$, not necessarily normal.

If $\mathcal{R}$ is considered as the algebra of observables of a physical system, then the set of *physical states* is identified with the set of *normal states* of $\mathcal{R}$. A normal state $\phi : \mathcal{R}_{sa} \longrightarrow \mathbb{R}$ is of the form $\phi(\_) = tr(\rho\_)$ for some positive trace class operator ρ of trace 1, see Thm. 7.1.12 in [KadRinII97].

The positive trace class operator ρ determining a physical state $\phi(\_) = tr(\rho\_)$ often is identified with the state itself. In physics, ρ is called a *density matrix*. It is of the form

$$\rho = \sum_{i=1}^{\infty} b_i P_i,$$

where $0 \leq b_i \leq 1$ for all i, $\sum_{i=1}^{\infty} b_i = 1$ and all the P_i are projections onto pairwise orthogonal one-dimensional subspaces. Note that $\rho \notin \mathcal{R}$ in general. A *pure state* is an extreme point of the convex space of states. For $\mathcal{R} = \mathcal{L}(\mathcal{H})$, the pure states are exactly the projections onto one-dimensional subspaces, so ρ is interpreted as a *mixed state* if it is a non-trivial convex combination of such projections. (The choice $\mathcal{R} = \mathcal{L}(\mathcal{H})$ is common in quantum mechanics, but not so in quantum field theory.)

Moreover, if $A \in \mathcal{L}(\mathcal{H})_{sa}$ and $\rho_{\mathbb{C}x} = P_{\mathbb{C}x}$ is a pure state, then the expression $\langle Ax, x\rangle = tr(\rho_{\mathbb{C}x} A)$ is interpreted as the expectation value of the observable A if the physical system is in the pure state $\rho_{\mathbb{C}x}$.

A physical system has a *Hamilton operator*, which is an observable $H \in \mathcal{R}_{sa}$ ($\mathcal{R}$ is the algebra of observables of our system). The Hamilton operator is the observable of *energy* and determines the time evolution of the system.

4.2. The harmonic oscillator

The harmonic oscillator is the physical system with Hamilton operator $H = \frac{1}{2}(p^2 + q^2)$. (That is, if we fix the physical units in the right way.) We will neither bother to give any details about the Hilbert space $\mathcal{H}$ on which H acts, nor will we explain the *unbounded* momentum operator p and position operator q. This can be found in any textbook on quantum theory. We simply use the known facts about H, in particular, we have

$$\operatorname{sp} H = \{n + \frac{1}{2} \mid n = 0, 1, 2, ...\}.$$

This means that H is bounded from below, but not from above. The eigenspaces are known to be one-dimensional. Assume that we have an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H consisting of eigenvectors of H, e_n being the eigenvector corresponding to the eigenvalue $n + \frac{1}{2}$. (We use the convention $\mathbb{N} = \{0, 1, 2, ...\}$, in particular, $0 \in \mathbb{N}$.) Then the spectral family of H is defined by

$$E^H_\lambda := \bigvee_{n \leq \lambda - \frac{1}{2}} P_{\mathbb{C}e_n} \qquad (\lambda \in \mathbb{R}).$$

We go ahead to define the observable function of H (though we only considered observable functions of bounded operators up to now):

$$\begin{aligned} f_H(\mathfrak{B}) &:= \inf\{\lambda \in \mathbb{R} \mid E^H_\lambda \in \mathfrak{B}\} \\ &= \inf\{\lambda \in \mathbb{R} \mid \bigvee_{n \leq \lambda - \frac{1}{2}} P_{\mathbb{C}e_n} \in \mathfrak{B}\} \\ &= \inf\{\lambda \in \mathbb{R} \mid \bigvee_{n \leq \lambda} P_{\mathbb{C}e_n} \in \mathfrak{B}\} + \frac{1}{2}. \end{aligned}$$

(See also Lemma 59.) For simplicity, assume that the observable algebra $\mathcal{R}$ of our physical system is $\mathcal{L}(\mathcal{H})$.

We regard an atomic quasipoint $\mathfrak{B}_{\mathbb{C}x} \in \mathcal{Q}(\mathcal{L}(\mathcal{H}))$. If $x = \sum_{j=1}^n a_j e_{i_j}$ is a *finite* linear combination of basis vectors e_{i_j} with all coefficients a_j non-zero, then

$$P_{\mathbb{C}x} \leq \bigvee_{j=1}^{n} P_{\mathbb{C}e_{i_j}} \leq \bigvee_{k=0}^{\max(i_1,...,i_n)} P_{\mathbb{C}e_k},$$

from which follows

$$\bigvee_{k=0}^{\max(i_1,...,i_n)} P_{\mathbb{C}e_k} \in \mathfrak{B}_{\mathbb{C}x}.$$

Obviously, $\bigvee_{k=0}^{\max(i_1,...,i_n)} P_{\mathbb{C}e_k}$ is the smallest projection contained in E^H that is larger than or equal to $P_{\mathbb{C}x}$, so

$$f_H(\mathfrak{B}_{\mathbb{C}x}) = \max(i_1, ...i_n) + \frac{1}{2}.$$

For example, choose $x = e_j$. The smallest projection contained in E^H that is larger than or equal to $P_{\mathbb{C}e_j}$ is $\bigvee_{k=0}^{j} P_{\mathbb{C}e_k} = E_j^H$, so $f_H(\mathfrak{B}_{\mathbb{C}e_j}) = j + \frac{1}{2}$.

Clearly, it is sensible to use the convention $\inf \varnothing := \infty$. If $\mathfrak{B}_{\mathbb{C}y}$ is an atomic quasipoint such that $y = \sum_{j=1}^{\infty} b_j e_{i_j}$ with *infinitely* many non-zero coefficients b_j, then $f_H(\mathfrak{B}_{\mathbb{C}x}) = \infty$.

Since a continuous quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{L}(\mathcal{H}))$ contains no finite-dimensional projections at all, we have $f_H(\mathfrak{B}) = \infty$ for all continuous quasipoints. The fact that f_H attains the value ∞ is due to the unboundedness of H.

4.3. Expectation values

The techniques presented in this work offer two distinct ways to express the expectation value of an observable $A \in \mathcal{L}(\mathcal{H})_{sa}$ when the physical system under consideration is in the pure state $P_{\mathbb{C}x}$. The first one (see [deG01]) uses a certain presheaf on $\mathcal{P}(\mathcal{H}) = \mathcal{P}(\mathcal{L}(\mathcal{H}))$. The germs of this presheaf at atomic quasipoints are interpreted as expectation values.

The second alternative uses observable functions to express expectation values as integrals in a form similar to the usual expression.

4.3.1. Expectation values as germs

Let $\mathcal{R} = \mathcal{L}(\mathcal{H})$ and $\mathcal{P}_0 := \mathcal{P}(\mathcal{H})\backslash\{0\}$. On $\mathcal{P}_0$, a presheaf $\mathcal{P} = (\mathcal{P}_E, \rho_F^E)_{E,F\in\mathcal{P}_0}$ is defined in the following way: for $E \in \mathcal{P}_0$, let $\mathcal{P}_E := \mathcal{L}(E\mathcal{H})$, and if $F \leq E$, then the restriction function

$$\rho_F^E : \mathcal{L}(E\mathcal{H}) \longrightarrow \mathcal{L}(F\mathcal{H})$$

is given by

$$\rho_F^E(A) := FAF.$$

(Compare Example 39.)

Proposition 158 *Let $A \in \mathcal{P}_E$, and let $\mathfrak{B}_{\mathbb{C}x} \in \mathcal{Q}_E(\mathcal{L}(\mathcal{H}))$ ($x \in \mathcal{H}\backslash\{0\}$) be an atomic quasipoint of $\mathcal{L}(E\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$. Then the germ of A at $\mathfrak{B}_{\mathbb{C}x}$ is given by $\langle Ax, x\rangle$, where $|x| = 1$.*

Proof. If $A, B \in \mathcal{L}(E\mathcal{H})$, then $A \sim_{\mathfrak{B}_{\mathbb{C}x}} B$ if and only if $P_{\mathbb{C}x}AP_{\mathbb{C}x} = P_{\mathbb{C}x}BP_{\mathbb{C}x}$. If $|x| = 1$, then

$$\forall y \in \mathcal{H} : P_{\mathbb{C}x}AP_{\mathbb{C}x}y = \langle Ax, x\rangle \langle y, x\rangle x,$$

so $P_{\mathbb{C}x}AP_{\mathbb{C}x} = P_{\mathbb{C}x}BP_{\mathbb{C}x}$ if and only if $\langle Ax, x\rangle = \langle Bx, x\rangle$. ■

This proposition is a simple application of the presheaf- and sheaf-theoretic techniques developed in section 2.5.

4.3.2. Expectation values and observable functions

Let $\mathcal{R} = \mathcal{L}(\mathcal{H})$ again, and let $A \in \mathcal{R}_{sa}$. Usually, the expectation value of A when the physical system is in the pure state $\rho_{\mathbb{C}x} := P_{\mathbb{C}x}$ is expressed as

$$\langle Ax, x\rangle = tr(\rho_{\mathbb{C}x}A) = \int_{-|A|}^{|A|} \lambda \; d\left\langle E^A_\lambda x, x\right\rangle,$$

where $E^A = (E^A_\lambda)_{\lambda\in\mathbb{R}}$ is the spectral family of A. (If A is unbounded, the integration limits must be replaced by $-\infty$ and ∞, respectively.) Now, since we have

$$\begin{aligned} f_A(\mathfrak{B}_{\mathbb{C}x}) &= \inf\{\lambda \in \mathbb{R} \mid E^A_\lambda \in \mathfrak{B}_{\mathbb{C}x}\} \\ &= \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq E^A_\lambda\}, \end{aligned}$$

for the observable function f_A of A, we see that for all $\mu \geq f_A(\mathfrak{B}_{\mathbb{C}x})$, we have $P_{\mathbb{C}x} \leq E^A_\mu$. Hence, these values do not contribute to the value of the integral, and the upper integration limit can be replaced by $f_A(\mathfrak{B}_{\mathbb{C}x})$:

$$\langle Ax, x\rangle = tr(\rho_{\mathbb{C}x}A) = \int_{-|A|}^{f_A(\mathfrak{B}_{\mathbb{C}x})} \lambda \; d\left\langle E^A_\lambda x, x\right\rangle. \tag{$*$}$$

The corresponding expression for arbitrary density matrices follows from linearity of the trace.

The expression $(*)$ suggests an interpretation of the observable function f_A as some kind of *distribution function* for the observable A.

4.4. Time evolution

For a start, let $\mathcal{R} = \mathcal{L}(\mathcal{H})$ be the algebra of observables, $A \in \mathcal{L}(\mathcal{H})_{sa}$ an observable, $E^A = (E^A_\lambda)_{\lambda\in\mathbb{R}}$ its spectral family and $T \in \mathcal{U}(\mathcal{H})$ a unitary operator. The spectral family of TAT^* is given by

$$\begin{aligned} E^{TAT^*} : \mathbb{R} &\longrightarrow \mathcal{P}(\mathcal{H}) \\ \lambda &\longmapsto TE^A_\lambda T^*. \end{aligned}$$

Let $f_A : \mathcal{Q}(\mathcal{L}(\mathcal{H})) \to \mathbb{R}$ be the observable function of A. We restrict f_A to the atomic quasipoints and regard it is a function on projective Hilbert space $\mathbb{P}\mathcal{H}$:

$$\begin{aligned} \widetilde{f}_A : \mathbb{P}\mathcal{H} &\longrightarrow \mathbb{R} \\ \mathbb{C}x &\longmapsto \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq E^A_\lambda\}. \end{aligned}$$

This coincides with our previous definition, since $\widetilde{f}_A(\mathbb{C}x) = \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq E^A_\lambda\} = \inf\{\lambda \in \mathbb{R} \mid E^A_\lambda \in \mathfrak{B}_{\mathbb{C}x}\} = f_A(\mathfrak{B}_{\mathbb{C}x})$.

We have

$$\begin{aligned}\widetilde{f}_{TAT^*}(\mathbb{C}x) &= \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq E_\lambda^{TAT^*}\}\\ &= \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq TE_\lambda^{A}T^*\}\\ &= \inf\{\lambda \in \mathbb{R} \mid T^*P_{\mathbb{C}x}T \leq E_\lambda^{A}\}\\ &= \inf\{\lambda \in \mathbb{R} \mid P_{T^*\mathbb{C}x} \leq E_\lambda^{A}\}\\ &= \widetilde{f}_A(T^*\mathbb{C}x).\end{aligned}$$

Now, let G be a topological group and

$$\begin{aligned} T : G &\longrightarrow \mathcal{U}(\mathcal{H})\\ g &\longmapsto T_g \end{aligned}$$

a unitary representation of G on the Hilbert space $\mathcal{H}$. For every $A \in \mathcal{R}_{sa}$, we have

$$\forall \mathbb{C}x \in \mathbb{P}\mathcal{H}\ \forall g \in G : \widetilde{f}_{T_gAT_g^*}(\mathbb{C}x) = \widetilde{f}_A(T_g^*\mathbb{C}x). \tag{$*$}$$

In particular, if $G = \mathbb{R}$, interpreted as a time parameter, this is the **time evolution** of the observable function $\widetilde{f}_A$.

If quantum theory is formulated in such a manner that the observables change in time, one speaks of the **Heisenberg picture**. If we identify the line $\mathbb{C}x$ with the pure state $\rho_{\mathbb{C}x}$ as usual, then $(*)$ can be interpreted as the formula translating between the Heisenberg picture and the **Schrödinger picture**, in which the states change in time.

Now let $\mathcal{R}$ be an arbitrary von Neumann algebra, $A \in \mathcal{R}_{sa}$ with observable function f_A. In subsection 2.10.3, we had defined the action of the unitary group on the Stone spectrum $\mathcal{Q}(\mathcal{R})$. In particular, if $T \in \mathcal{U}(\mathcal{R})$ is a unitary operator and $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ is a quasipoint, then $T.\mathfrak{B} := \{TET^* \mid E \in \mathfrak{B}\} \in \mathcal{Q}(\mathcal{R})$ is a quasipoint of $\mathcal{R}$ again. Since $TFT^* \in \mathfrak{B} \Longleftrightarrow F = T^*(TFT^*)T \in T^*.\mathfrak{B}$ for $F \in \mathcal{P}(\mathcal{R})$, we obtain

$$\begin{aligned} f_{TAT^*}(\mathfrak{B}) &:= \inf\{\lambda \in \mathbb{R} \mid E_\lambda^{TAT^*} \in \mathfrak{B}\}\\ &= \inf\{\lambda \in \mathbb{R} \mid TE_\lambda^{A}T^* \in \mathfrak{B}\}\\ &= \inf\{\lambda \in \mathbb{R} \mid E_\lambda^{A} \in T^*.\mathfrak{B}\}\\ &= f_A(T^*.\mathfrak{B}\}. \end{aligned}$$

If we have a topological group G and a continuous representation T of G by unitary operators *in* $\mathcal{U}(\mathcal{R})$, we obtain

$$\forall \mathfrak{B} \in \mathcal{Q}(\mathcal{R})\ \forall g \in G : f_{T_gAT_g^*}(\mathfrak{B}) = f_A(T_g^*.\mathfrak{B}\}.$$

This is quite similar to the formula $(*)$ above, but quasipoints cannot be identified with states in general. Interestingly, time evolution of the observable functions can be expressed by the action of (a one-parameter family of) unitary operators on the Stone spectrum $\mathcal{Q}(\mathcal{R})$.

5. The Kochen-Specker theorem

"Ich warf allerlei Gedanken im Kopf herum
bis endlich folgender obenhin zu liegen kam."

Georg Christoph Lichtenberg

5.1. Introduction

Consider some physical system described by a von Neumann algebra $\mathcal{R}$, the algebra of observables. It is important for the interpretation of quantum theory to see if there is a possibility to assign a *value* to each observable. This means, we are looking for a mapping $v : \mathcal{R}_{sa} \to \mathbb{R}$ from the observables $\mathcal{R}_{sa}$ to the real numbers such that

(i) for $A \in \mathcal{R}_{sa}$, we have $v(A) \in \operatorname{sp} A$, i.e. each observable is assigned one element of its spectrum $\operatorname{sp} A$, and

(ii) for any two observables $A, B \in \mathcal{R}_{sa}$ such that $B = g(A)$ for some (Borel) function g, it holds that $v(B) = g(v(A))$, i.e. the value assigned to B is given as $g(v(A))$.

If this were possible, one could imagine to build some realistic model of the quantum world where all observables have definite values, like in classical mechanics.

The first condition, namely that each observable should be assigned one of its spectral values, is quite obvious. The second condition implements the fact that the observables are not all independent. In fact, for every abelian von Neumann algebra $\mathcal{M}$ (think of some abelian subalgebra of $\mathcal{R}$), there is a self-adjoint operator A generating $\mathcal{M}$, that is, $\mathcal{M} = \{A, I\}''$. This was already proved by von Neumann ([vNeu32]). For a modern reference, see Thm. III.1.21 in [TakI02]. Every operator $B \in \mathcal{M}$ is a Borel function of A, $B = g(A)$. One has $g(\operatorname{sp} A) \subseteq \operatorname{sp} B$ (see Ch. 5.2 in [KadRinI97]). If g is continuous or if $\mathcal{H}$ is finite-dimensional, equality holds. It is natural to demand that the spectral value b assigned to B is given as $g(a)$, where a is the value assigned to A. This condition is often called the *FUNC principle*.

Kochen and Specker started from a related question in their classical article [KocSpe67]: is there a space of *hidden states*? A hidden state ψ would be given by a probability measure μ_ψ on a generalized quantum mechanical phase space Ω such that an observable A is given as a mapping

$$f_A : \Omega \longrightarrow \mathbb{R},$$

a *hidden variable*. When the system is in the hidden state ψ and the observable A is measured, the probability to find a value r lying in the Borel set U is required to be

$$P_{A,\psi}(U) = \mu_\psi(f_A^{-1}(U)). \tag{1}$$

Moreover, the expectation value $E_\psi(A)$ of A when the system is in the hidden state ψ is required to be

$$E_\psi(A) = \int_\Omega f_A(\omega) d\mu_\psi(\omega).$$

Kochen and Specker demonstrate that it is trivial to construct such a generalized phase space Ω if functional relations between the observables are neglected, but the problem really starts when one takes these relations into account. If $B = g(A)$ for some observables $A, B \in \mathcal{R}_{sa}$ and a Borel function g, one should have

$$f_B = f_{g(A)} = g \circ f_A. \tag{2}$$

This simply translates the functional relation between the operators A and B into the corresponding relation between the hidden variables f_A and f_B. Since $B = g(A)$ can only be if A and B commute and since every abelian von Neumann algebra is generated by a self-adjoint operator, Kochen and Specker go on to introduce *partial algebras*, where algebraic relations are defined exclusively between observables that are *commeasurable.* If one regards a von Neumann algebra $\mathcal{R}$ as the algebra of observables, as we do here, $\mathcal{R}$ is a partial algebra in an obvious way: one just keeps the algebraic relations between commuting operators and neglects the algebraic relations between non-commuting operators, since at first sight (2) is a condition on commuting operators only and those are commeasurable. This point seems important, because a hidden variable no-go theorem by J. von Neumann ([vNeu32]) has been criticized (see e.g. [Bell66]) for the fact that von Neumann required additivity to be preserved even between non-commuting operators, which does not seem adequate in the light of (2). However, in fact (2) does *not* just pose conditions on commuting, but also on non-commuting operators. The reason is that typically an observable B is given as a function $B = g(A) = h(C)$ of non-commuting observables $A, C \in \mathcal{R}_{sa}$. We will see that in this way (2) becomes a very strong condition, ruling out hidden states models of the kind described above.

An abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$ is often called a *context* in the physics literature. The self-adjoint operators $\mathcal{M}_{sa}$ in a maximal context $\mathcal{M}$ form a maximal set of commeasurable observables. Condition (2) seems to be a condition within each context solely, but in fact it is a condition "across contexts", because each observable B typically is contained in many contexts.

The elements of the hypothetical generalized phase space Ω would be generalized pure states. In a slight abuse of language, Ω is also called the space of hidden states. If one assumes that there is some space Ω of hidden states such that (1) and (2) are satisfied and that there is an embedding $f : \mathcal{R}_{sa} \to \mathbb{R}^\Omega$ of the quantum mechanical observables into the mappings from Ω to $\mathbb{R}$, one would have a lot of valuations as described above, assigning a spectral value to each observable and preserving functional relations: every point $\omega \in \Omega$ defines such a valuation v by

$$v(A) := f_A(\omega).$$

Demonstrating that there are no such valuations (Kochen and Specker called them *prediction functions*) thus shows that there is no space Ω of hidden states as described above. More directly, the non-existence of valuation functions means that no realistic interpretation of quantum mechanics is possible which assumes that all the observables have definite values at the same time. Evaluating f_A at $\omega \in \Omega$ also gives the connection between condition (2) on hidden variables and condition (ii) on valuation functions (see the first paragraph of this section).

It is a funny fact that in spite of many references to it, there seems to be no single result called *the* Kochen-Specker theorem. Above, we tried to lay out (very roughly, admittedly) the train of thought in [KocSpe67], and it seems sensible to spell out the Kochen-Specker theorem as follows:

Kochen-Specker theorem: Let $\mathcal{R} \simeq \mathcal{L}(\mathcal{H}),\ \dim \mathcal{H} \geq 3$ be the algebra of observables of some quantum system ($\mathcal{R}$ is a type I_n factor, $n = \dim \mathcal{H}$). There is no space Ω of hidden states such that (1) and (2) are satisfied, i.e. there is no realistic phase space model of quantum theory assigning spectral values to all observables at once, preserving functional relations between them.

Replacing $\mathcal{L}(\mathcal{H})$ by a more general von Neumann algebra $\mathcal{R}$, we obtain the formulation of a **generalized Kochen-Specker theorem**. It is not obvious at first sight if the Kochen-Specker theorem holds for more general von Neumann algebras $\mathcal{R}$, since each $\mathcal{R}$ that is not a type I_n factor is properly contained in some $\mathcal{L}(\mathcal{H})$, and so there are less conditions (encoded in the FUNC principle) than for $\mathcal{L}(\mathcal{H})$ itself. This might lead to speculation if some hidden states, realistic model of quantum systems with an observable algebra $\mathcal{R}$ other than a type I_n factor exists. A necessary condition would be the existence of a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$.

To prove the non-existence of valuation functions, Kochen and Specker ([KocSpe67]) concentrate on the projections $\mathcal{P}(\mathcal{R})$ of $\mathcal{R}$. (In fact, only the case $\mathcal{R} = \mathcal{L}(\mathcal{H})$ is considered). The projections form a partial Boolean algebra. A valuation function v can assign 0 or 1 to a projection E, since $\operatorname{sp} E = \{0, 1\}$. Kochen and Specker examine if the partial Boolean algebra $\mathcal{P}(\mathcal{R})$ can be embedded into a Boolean algebra. The existence of such an embedding is a necessary condition for the existence of a valuation function. For the case $\mathcal{H} = \mathbb{R}^3, \mathcal{R} = \mathbb{M}_3(\mathbb{R})$, they construct a finitely generated subalgebra $D \subset \mathcal{R}$ that cannot be embedded into a Boolean algebra, thus showing that there is no valuation function in this case. Kochen and Specker use 117 vectors in their construction, corresponding to 117 projections onto one-dimensional subspaces. Later on, this number could be reduced to 33 by A. Peres and 31 by Conway and Kochen, see [Per93] and references therein. The proofs are combinatorial in nature, giving a counterexample.

In this chapter, we will use another approach. Let $\mathcal{R}$ be a von Neumann algebra. Assuming the existence of a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ (see Def. 161 below), we show that v induces a so-called *quasi-state* $v' : \mathcal{R} \to \mathbb{C}$ (see Def. 165) such that $v'|_{\mathcal{R}_{sa}} = v$. This quasi-state is a pure state of every abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$. Restricting v' to the projections $\mathcal{P}(\mathcal{R})$, we obtain a finitely additive probability measure on $\mathcal{P}(\mathcal{R})$.

If $\mathcal{R}$ is a type I_n factor ($n \in \{3, 4, ...\}$), Gleason's theorem ([Gle57]) shows that v' is a state of $\mathcal{R}$ of the form $v'(\_) = tr(\rho\_)$. But such a state does *not* assign 0 or 1 to every projection, hence we have a contradiction of one of the defining conditions of the valuation function v. The case of a type I_∞ factor can be treated easily.

For more general von Neumann algebras, another proof is presented. For the first time, von Neumann algebras other than the type I_n factors $\mathcal{L}(\mathcal{H})$ are treated at all; our results cover all von Neumann algebras. Using the Gleason-Christensen-Yeadon theorem (Thm. 172), a generalization of Gleason's theorem, we again show that v' is a state if $\mathcal{R}$ has no summand of type I_2. Hamhalter showed in [Ham93] that every finitely additive two-valued probability measure $\mu : \mathcal{P}(\mathcal{R}) \to \{0, 1\}$ gives rise to a *multiplicative* state of $\mathcal{R}$. Since $v'|_{\mathcal{P}(\mathcal{R})}$ is of this kind, a valuation function v induces a multiplicative state v'. If a von Neumann algebra $\mathcal{R}$ contains no summand of type I_1, then there are no multiplicative states of $\mathcal{R}$, so there is no valuation function for a von Neumann algebra $\mathcal{R}$ without summands of types I_1 and I_2. Thus, the generalized Kochen-Specker theorem holds for all von Neumann algebras $\mathcal{R}$ without summands of types I_1 and I_2.

In section 5.3, we give two different reformulations of the generalized Kochen-Specker theorem in the language of presheafs. For $\mathcal{R} = \mathcal{L}(\mathcal{H})$, this has been proposed by Isham, Butterfield and Hamilton ([IshBut98, IshBut99, HIB00, IshBut02]). They observed that the FUNC principle would mean that certain presheafs on small categories have global sections.

Our first presheaf formulation of the Kochen-Specker theorem, generalizing the category and presheaf chosen in [HIB00], is closely related to our first proof (subsections 5.2.1, 5.2.2). The second formulation uses another presheaf which does have global sections: each state of the von Neumann algebra $\mathcal{R}$ induces one. However, the Kochen-Specker theorem means that there are no global sections of the kind a valuation function would induce, giving a pure state of every abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$.

5.2. The new proofs

5.2.1. Valuation functions and quasi-states

Definition 159 *Let $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. A* ***finitely additive probability measure*** *μ is a mapping from $\mathcal{P}(\mathcal{R})$ to $\mathbb{R}$ such that*

(M1) $\forall E \in \mathcal{P}(\mathcal{R}) : 0 \leq \mu(E) \leq 1$ *and* $\mu(I) = 1$,

(M2) If $E, F \in \mathcal{P}(\mathcal{R})$ such that $EF = 0$, then $\mu(E \vee F) = \mu(E) + \mu(F)$.

If in addition to (M2) one of the stronger conditions

(M2σ) $\mu(\bigvee_{n \in I} P_n) = \sum_{n \in I} \mu(P_n)$ *for every countable family $\{P_n\}_{n \in I}$ of orthogonal projections in $\mathcal{P}(\mathcal{R})$,*

(M2c) $\mu(\bigvee_{j\in J} P_j) = \sum_{j\in J} \mu(P_j)$ *for every family* $\{P_j\}_{j\in J}$ *of orthogonal projections in* $\mathcal{P}(\mathcal{R})$

holds, then μ *is called a* ***σ-additive (countably additive)*** *or a* ***completely additive probability measure****, respectively.*

Every *normal* state $\phi : \mathcal{R} \to \mathbb{C}$ is of the form $\phi(\_) = tr(\rho\_)$ for some positive trace class operator of trace 1, see Thm. 7.1.12 in [KadRinII97]. Such a normal state induces a completely additive probability measure by restriction to $\mathcal{P}(\mathcal{R})$. For type I factors, the converse is also true, as Gleason showed in his classical paper [Gle57]. For ease of reference, we cite Gleason's theorem:

Theorem 160 *(Gleason 1957) Let* $\mathcal{R}$ *be a type* I_n *factor,* $n \in \{3, 4, ...\}$*,* $\mathcal{R} \simeq \mathcal{L}(\mathcal{H})$*,* $\dim \mathcal{H} = n$*, and let* μ *be a finitely additive probability measure on* $\mathcal{P}(\mathcal{R})$*. There is some positive trace class operator* ρ *of trace* 1 *such that*

$$\forall E \in \mathcal{P}(\mathcal{R}) : \mu(E) = tr(\rho E). \tag{1}$$

If $\mathcal{H}$ *is infinite-dimensional and separable,* μ *is* σ*-additive and* $\mathcal{R}$ *is isomorphic to the type* I_∞ *factor* $\mathcal{L}(\mathcal{H})$*, then there is some positive trace class operator* ρ *of trace* 1 *such that (1) holds.*

If $\mathcal{H}$ *is an arbitrary infinite-dimensional Hilbert space (possibly non-separable),* μ *is completely additive and* $\mathcal{R}$ *is isomorphic to the type* I_∞ *factor* $\mathcal{L}(\mathcal{H})$*, then there is some positive trace class operator* ρ *of trace* 1 *such that (1) holds. (For this partial result, see Thm. 2.3 in [Mae89].)*

This classifies the probability measures on the projection lattices of type I factors. In particular, they all come from normal states of the form $tr(\rho\_)$.

From now on, we will assume that $\mathcal{H}$ is separable.

We now give the precise definition of a valuation function, which is the starting point for the proof of the Kochen-Specker theorem.

Definition 161 *Let* $\mathcal{H}$ *be a Hilbert space,* $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ *a von Neumann algebra. A* ***valuation function*** *is a mapping* $v : \mathcal{R}_{sa} \to \mathbb{R}$ *such that*
(a) $v(A) \in \operatorname{sp} A$ *and*
(b) for all Borel functions $f : \mathbb{R} \to \mathbb{R}$*, one has* $v(f(A)) = f(v(A))$*.*

Kochen and Specker call this a *prediction function*, see [KocSpe67]. $v(I) = 1$ and $v(0) = 0$ follow. Condition (a) is often called the **spectrum rule**, condition (b) is the **FUNC principle**.

Definition 162 *Let* $v : \mathcal{R}_{sa} \to \mathbb{R}$ *be a valuation function. We extend* v *in a canonical manner to a function*

$$\begin{aligned} v' : \mathcal{R} &\longrightarrow \mathbb{C}, \\ B = A_1 + iA_2 &\longmapsto v(A_1) + iv(A_2), \end{aligned}$$

where $B = A_1 + iA_2$ *is the unique decomposition of* B *into self-adjoint operators* $A_1, A_2 \in \mathcal{R}_{sa}$*.*

Obviously, $v'(A) = v(A)$ for a self-adjoint operator $A \in \mathcal{R}_{sa}$. This will be used throughout.

Lemma 163 *Let $g : \mathbb{R} \to \mathbb{C}$ be a Borel function, $g_r : \mathbb{R} \to \mathbb{R}$ its real part, $g_i : \mathbb{R} \to \mathbb{R}$ its imaginary part, $g = g_r + ig_i$. Thus g acts on $a \in \mathbb{R}$ as*

$$g(a) = g_r(a) + ig_i(a)$$

and on self-adjoint operators A as

$$g(A) = g_r(A) + ig_i(A).$$

Let $v : \mathcal{R}_{sa} \to \mathbb{R}$ be a valuation function, $v' : \mathcal{R} \to \mathbb{C}$ its extension. Then $v'(g(A)) = g(v'(A))$ holds for all self-adjoint operators $A \in \mathcal{R}_{sa}$.

Proof. One has

$$\begin{aligned} v'(g(A)) &= v'(g_r(A) + ig_i(A)) \\ &= v(g_r(A)) + iv(g_i(A)) \\ &= g_r(v(A)) + ig_i(v(A)) \\ &= g(v(A)) \\ &= g(v'(A)). \end{aligned}$$

■

Lemma 164 *If $v : \mathcal{R}_{sa} \to \mathbb{R}$ is a valuation function and $\mathcal{M} \subseteq \mathcal{R}$ is an abelian von Neumann subalgebra, then $v'|_{\mathcal{M}}$ is a character of $\mathcal{M}$. $v|_{\mathcal{M}_{sa}}$ is a real-valued, $\mathbb{R}$-homogeneous, linear functional.*

Proof. Let $A \in \mathcal{M}_{sa}$ be a self-adjoint operator that generates $\mathcal{M}$, that is, $\mathcal{M} = \{A, I\}''$ (see [TakI02, Prop. III.1.21]). All operators $B, C \in \mathcal{M}$ are Borel functions of A:

$$B = f(A), \quad C = g(A),$$

where $f, g : \mathbb{R} \to \mathbb{C}$ are Borel functions on $\operatorname{sp} A \subseteq \mathbb{R}$. Since $B + C \in \mathcal{M}$, there also is a Borel function $h : \mathbb{R} \to \mathbb{C}$ such that $B + C = f(A) + g(A) =: h(A)$ and hence

$$\begin{aligned} v'(B + C) &= v'(f(A) + g(A)) \\ &= v'(h(A)) \\ &= h(v'(A)) \\ &= f(v'(A)) + g(v'(A)) \\ &= v'(f(A)) + v'(g(A)) \\ &= v'(B) + v'(C). \end{aligned}$$

Analogously for $BC = CB$: there is a Borel function $j : \mathbb{R} \to \mathbb{C}$ such that $BC = f(A)g(A) =: j(A)$ and hence

$$\begin{aligned} v'(BC) &= v'(f(A)g(A)) \\ &= v'(j(A)) \\ &= j(v'(A)) \\ &= f(v'(A))g(v'(A)) \\ &= v'(f(A))v'(g(A)) \\ &= v'(B)v'(C). \end{aligned}$$

The $\mathbb{C}$-homogeneity of $v'|_{\mathcal{M}}$ is obvious. Let $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} v'(\alpha B) &= v'(\alpha f(A)) \\ &= v'(k(A)) \\ &= k(v'(A)) \\ &= \alpha f(v'(A)) \\ &= \alpha v'(f(A)) \\ &= \alpha v'(B), \end{aligned}$$

where $k := M_\alpha \circ f$. This shows that $v'|_{\mathcal{M}}$ is a character of $\mathcal{M}$. Restricting to $\mathcal{M}_{sa}$, one obtains a real-valued, $\mathbb{R}$-homogeneous, linear functional. ■

A character (multiplicative linear functional) of an abelian C^*-algebra $\mathcal{M}$ is a pure state of $\mathcal{M}$. So the above lemma shows that a valuation function induces a pure state on every abelian subalgebra $\mathcal{M} \subseteq \mathcal{R}$, which is exactly what one would expect from a physical point of view. Lemma 164 is closely related to the *sum rule* and the *product rule* first described in [Fin74], see also [Red87].

J. F. Aarnes has introduced the notion of a quasi-state on a C^*-algebra in his paper [Aar69]:

Definition 165 *Let $\mathcal{A}$ be a unital C^*-algebra. A* ***quasi-state*** *of $\mathcal{A}$ is a functional ρ satisfying the following three conditions:*

(i) For each $B \in \mathcal{A}_{sa}$, ρ is linear and positive on the abelian C^*-subalgebra $\mathcal{A}_B \subseteq \mathcal{A}$ generated by B and I.

(ii) If $C = A_1 + iA_2$ for self-adjoint $A_1, A_2 \in \mathcal{A}_{sa}$, then $\rho(C) = \rho(A_1) + i\rho(A_2)$.

(iii) $\rho(I) = 1$.

Lemma 166 *v' is a quasi-state.*

Proof. (i) v' is linear on every abelian subalgebra $\mathcal{M} \subseteq \mathcal{R}$. v' is positive on each such $\mathcal{M}$ (and on the whole of $\mathcal{R}$), since a positive operator B^*B is assigned some element of its spectrum, $v'(B^*B) = v(B^*B) \in sp(B^*B)$.

(ii) For $B = A_1 + iA_2$ ($A_1, A_2 \in \mathcal{R}_{sa}$) one has

$$v'(B) = v(A_1) + iv(A_2) = v'(A_1) + iv'(A_2),$$

the former because of the definition of v', the latter because $v'(A) = v(A)$ for $A \in \mathcal{R}_{sa}$.

(iii) $v'(I) = 1$ holds.

Thus v' is a quasi-state. ■

A quasi-state of an abelian von Neumann algebra is a state. This follows from the fact that an abelian von Neumann algebra on a separable Hilbert space is generated by an operator A ([TakI02, Prop. III.1.21]), a result we already used in Lemma 164.

It is easy to see that a quasi-state on an arbitrary von Neumann algebra $\mathcal{R}$, when restricted to the lattice of projections $\mathcal{P}(\mathcal{R})$, gives a finitely additive probability measure. We follow the proof given in [Mae89, Cor. 7.9]:

Lemma 167 *If ρ is a quasi-state of a von Neumann algebra $\mathcal{R}$, then $\rho|_{\mathcal{P}(\mathcal{R})}$ is a finitely additive probability measure $\mathcal{P}(\mathcal{R}) \to [0,1]$.*

Proof. For $E \in \mathcal{P}(\mathcal{R})$, we have $0 = \rho(0) \leq \rho(E) \leq \rho(I) = 1$, since ρ is positive on $\{E, I\}''$. If $EF = 0$ for $E, F \in \mathcal{P}(\mathcal{R})$, then $\mathcal{M} := \{E, F, I\}'' \subseteq \mathcal{R}$ is abelian and $\rho|_{\mathcal{M}}$ is a state, in particular, it is additive. Hence,

$$\rho(E \vee F) = \rho(E + F) = \rho(E) + \rho(F).$$

■

To clarify the relation between normal states and valuation functions, we will need the following fact (Lemma 6.5.6 in [KadRinII97]):

Lemma 168 *Let $\mathcal{R}$ be a von Neumann algebra with no central portion of type I (equivalently, with no non-zero abelian projections), and let $E \in \mathcal{P}(\mathcal{R})$. For each positive integer n, there are n equivalent orthogonal projections with sum E.*

Lemma 169 *Let $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ be a von Neumann algebra of type I_n, $n \in \{2, 3, ...\} \cup \{\infty\}$, or of type II or III, and let ϕ be a normal state of $\mathcal{R}$. There is a projection $E \in \mathcal{P}(\mathcal{R})$ such that $\phi(E) \notin \{0, 1\} = \operatorname{sp} E$.*

Proof. Since ϕ is a normal state, it is weakly continuous and of the form

$$\phi(\_) = tr(\rho\_)$$

for some positive trace class operator ρ of trace 1, see [KadRinII97, Thm. 7.1.12].

We assume that $\phi(E) = tr(\rho E) \in \{0, 1\}$ holds for all $E \in \mathcal{P}(\mathcal{R})$. Let $\{e_k\}_{k\in K}$ be an orthonormal basis of $\mathcal{H}$ that is adapted to E, that is, for all k, $e_k \in \operatorname{im} E \cup \operatorname{im}(I - E)$. Let $K_E := \{k \in K \mid e_k \in \operatorname{im} E\}$. If $tr(\rho E) = 1$, we have

$$\begin{aligned} 1 &= \sum_k \langle \rho e_k, E e_k \rangle \\ &= \sum_{k \in K_E} \langle \rho e_k, e_k \rangle . \end{aligned}$$

Since $\langle \rho e_k, e_k \rangle \geq 0$ for all $k \in K$ and $tr\rho = 1$, we see that $\langle \rho e_k, e_k \rangle = 0$ for all $k \in K \backslash K_E$, and hence $\rho e_k = 0$ for all $k \in K \backslash K_E$. Therefore, $\rho(I - E) = 0$, that is, $\rho = \rho E$ and

$$\rho E = \rho = \rho^* = E\rho.$$

If $tr(\rho E) = 0$, we have $tr(\rho(I - E)) = 1$, since $tr(\rho I) = 1 = tr(\rho E) + tr(\rho(I - E))$. It follows that $\rho(I-E) = (I-E)\rho$ and thus $\rho E = E\rho$ in this case, too. Since a von Neumann algebra is generated by its projections, we obtain

$$(\forall E \in \mathcal{P}(\mathcal{R}) : tr(\rho E) \in \{0, 1\}) \Longrightarrow \rho \in \mathcal{R}',$$

where $\mathcal{R}'$ is the commutant of $\mathcal{R}$. Now let $\theta \in \mathcal{R}$ be a partial isometry such that $\theta^*\theta = E$ and $F := \theta\theta^*$. One has

$$tr(\rho E) = tr(\rho\theta^*\theta) = tr(\theta\rho\theta^*) = tr(\theta\theta^*\rho) = tr(F\rho) = tr(\rho F),$$

so from $E \sim F$ it follows that $\phi(E) = tr(\rho E) = tr(\rho F) = \phi(F)$.

If $\mathcal{R}$ is of type I_n, $n \geq 2$, then the identity I is the sum of n equivalent abelian (orthogonal) projections E_j $(j = 1, ..., n)$. We have

$$1 = \phi(I) = \phi(\sum_{j=1}^{n} E_j) = \sum_{j=1}^{n} \phi(E_j),$$

so $\phi(E_j) = \frac{1}{n}$ $(j = 1, ..., n)$, which contradicts our assumption $\phi(E) \in \{0, 1\}$ for all $E \in \mathcal{P}(\mathcal{R})$.

If $\mathcal{R}$ is of type I_∞, we use the halving lemma (6.3.3 in [KadRinII97]) to show that there is a projection $F \in \mathcal{P}(\mathcal{R})$ such that $F \sim F^\perp := I - F$. If $\mathcal{R}$ is of type II or III, then we employ Lemma 168 for $E = I$ and $n = 2$ to obtain the same. We have

$$1 = \phi(I) = \phi(F + I - F) = \phi(F) + \phi(F^\perp),$$

so $\phi(F) = \phi(F^\perp) = \frac{1}{2}$, which contradicts our assumption. ■

This lemma means that restricting a normal state ϕ of a von Neumann algebra $\mathcal{R}$ that is not of type I_1 (that is, abelian) to $\mathcal{R}_{sa}$ can never give a valuation function. The proof of this lemma is based on the proof of Thm. 6.4 in [deG01] (Thm. 170 in our paper).

5.2.2. Type I_n factors

In this subsection, let $\mathcal{R}$ be a type I_n factor. Every state ρ of $\mathcal{R}$ induces a bounded positive Radon measure $\mu_\rho^{\mathbb{B}}$ of norm 1 on $\mathcal{Q}(\mathbb{B})$ by

$$\begin{aligned}\mu_\rho^{\mathbb{B}} : C(\mathcal{Q}(\mathbb{B})) &\longrightarrow \mathbb{C}\\ A &\longmapsto tr(\rho A).\end{aligned}$$

We will use the following result (Thm. 6.4 in [deG01]):

Theorem 170 *Let ρ be a state of $\mathcal{R}$, and let $\mathbb{B} \subseteq \mathcal{P}(\mathcal{R})$ be a Boolean sector. Then the Radon measure $\mu_\rho^{\mathbb{B}}$ on the Stone spectrum $\mathcal{Q}(\mathbb{B})$ is the point measure ε_{β_0} for some $\beta_0 \in \mathcal{Q}(\mathbb{B})$ if and only if there is an $x \in S^1(\mathcal{H})$ such that $\mathbb{C}x \in \mathbb{B}$, $\beta_0 = \beta_{\mathbb{C}x}$ and $\rho = P_{\mathbb{C}x}$. Here $\beta_{\mathbb{C}x}$ is the unique quasipoint containing $P_{\mathbb{C}x}$.*

Proposition 171 *Let $\mathcal{R}$ be a factor of type I_n, $n \in \{3, 4, ...\} \cup \{\infty\}$. There is no valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$.*

Proof. We first treat the case of finite n. As shown above, assuming the existence of a valuation function v, one has a quasi-state v' of $\mathcal{R}$, which is a pure state (that is, a character) of every abelian von Neumann subalgebra $\mathcal{M} \subseteq \mathcal{R}$. Lemma 167 shows that $v'|_{\mathcal{P}(\mathcal{R})}$ is a finitely additive probability measure. Gleason's theorem (Thm. 160) shows that $v'|_{\mathcal{P}(\mathcal{R})}$ comes from a state of $\mathcal{R} \simeq \mathcal{L}(\mathcal{H}_n)$, where $\mathcal{H}_n$ is an n-dimensional Hilbert space.

The state given by Gleason's theorem is of the form $tr(\rho\_)$, where ρ is a positive trace class operator of trace 1. According to the spectral theorem, every operator $A \in \mathcal{R}$ is the norm limit of complex linear combinations of projections, hence there is a unique possibility to extend the probability measure $v'|_{\mathcal{P}(\mathcal{R})}$ to a state (given by linearly extending $v'|_{\mathcal{P}(\mathcal{R})}$). Of course, this extension simply is v', so

$$v'(\_) = tr(\rho\_).$$

Let $\mathbb{B}$ be a Boolean sector of $\mathcal{P}(\mathcal{R})$, and let $\mathcal{M}(\mathbb{B})$ be the maximal abelian subalgebra of $\mathcal{R}$ generated by $\mathbb{B}$. Since $v'|_{\mathcal{M}(\mathbb{B})}$ is a pure state, it corresponds to exactly one element $\beta_0 \in \mathcal{Q}(\mathbb{B})$. v' induces a point measure on $\mathcal{M}(\mathbb{B}) = C(\mathcal{Q}(\mathbb{B}))$ in this way.

According to Thm. 170, ρ and β are of the form

$$\begin{aligned}\rho &= P_{\mathbb{C}x},\\ \beta &= \beta_{\mathbb{C}x},\end{aligned}$$

and thus

$$v'(\_) = tr(P_{\mathbb{C}x}\_).$$

This form does not depend on the chosen Boolean sector $\mathbb{B}$.

Now let $\mathbb{B}'$ be a different Boolean sector that does not contain the projections $P_{\mathbb{C}x}, I - P_{\mathbb{C}x}$. There is a projection $F' \in \mathbb{B}'$ such that

$$v'(F') = tr(P_{\mathbb{C}x}F') \notin \{0, 1\},$$

contradicting the defining condition $v'(E) = v(E) \in \operatorname{sp} E = \{0, 1\}$ $(E \in \mathcal{P}(\mathcal{R}))$ of a valuation function. This shows that there is no valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ for factors $\mathcal{R}$ of type I_n, $n \in \{3, 4, ...\}$, from which the Kochen-Specker theorem follows. Instead of referring to [deG01], we could have used Lemma 169.

Now let $\mathcal{R}$ be a type I_∞ factor, $\mathcal{R} \simeq \mathcal{L}(\mathcal{H})$ for an infinite-dimensional separable Hilbert space $\mathcal{H}$. $\mathcal{R}$ contains a subfactor of type I_n for every $n \in \{3, 4, ...\}$: Let $\mathcal{S}$ be a type I_n factor, $\mathcal{S} \simeq \mathcal{L}(\mathcal{H}_n)$. The separable Hilbert spaces $\mathcal{H}_n \otimes \mathcal{H}$ and $\mathcal{H}$ are isomorphic and will be identified. Embed $\mathcal{L}(\mathcal{H}_n)$ into $\mathcal{L}(\mathcal{H})$ via the mapping

$$\begin{aligned} \mathcal{L}(\mathcal{H}_n) &\longrightarrow \mathcal{L}(\mathcal{H}_n \otimes \mathcal{H}) \simeq \mathcal{L}(\mathcal{H}) \\ A &\longmapsto A \otimes I. \end{aligned}$$

This guarantees that the identity I_n of $\mathcal{L}(\mathcal{H}_n)$ is mapped to the identity I of $\mathcal{L}(\mathcal{H})$.

We assume that there is a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$. Restricting v to the self-adjoint part of a type I_n subfactor $\mathcal{S}$ of $\mathcal{R}$ gives a valuation function for $\mathcal{S}$. Since we saw that there is no such valuation function, there can be none for $\mathcal{R}_{sa}$. ■

5.2.3. Von Neumann algebras without type I_2 summand

The proof for the type I_n case proceeded in two steps: first, assuming that there is a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ for $\mathcal{R}$ a type I_n algebra, we showed that it induces a quasi-state $v' : \mathcal{R} \to \mathbb{C}$ in a canonical manner and thus a finitely additive probability measure $v'|_{\mathcal{P}(\mathcal{R})}$. In a second step, we used Gleason's theorem to see that v' is a *state* of $\mathcal{R}$ of the form $v'(\_) = tr(\rho\_)$, which cannot satisfy the defining conditions for a valuation function, because there are projections $F' \in \mathcal{P}(\mathcal{R})$ such that $v'(F') \notin \operatorname{sp} F' = \{0, 1\}$.

If we want to treat more general von Neumann algebras $\mathcal{R}$, we first must assure that the quasi-state v' is a state of $\mathcal{R}$. Since we know that $v'|_{\mathcal{P}(\mathcal{R})}$ is a finitely additive probability measure for an arbitrary von Neumann algebra $\mathcal{R}$ (Lemma 167), a generalization of Gleason's theorem is needed, showing that this probability measure comes from a state. There is a beautiful and detailed paper by S. Maeda [Mae89] on the generalizations of Gleason's theorem. Maeda is drawing on results by J. F. Aarnes [Aar69, Aar70], J. Gunson [Gun72], E. Christensen [Chr82, Chr85], F. J. Yeadon [Yea83, Yea84] and K. Saito [Sai85]. The proofs given in Maeda's paper are by no means trivial. The central point of course is to show that a quasi-state is linear on $\mathcal{R}_{sa}$ for *non-commuting* self-adjoint operators A, B. Maeda uses Gleason's theorem [Gle57] for the type I_n algebras. Types II and III require a lot more work. We cite the main result (Thm. 12.1 in [Mae89]):

Theorem 172 *(Christensen, Yeadon, Maeda et. al.) Let $\mathcal{R}$ be a von Neumann algebra without direct summand of type I_2, and let μ be a finitely additive probability measure on the complete orthomodular lattice $\mathcal{P}(\mathcal{R})$. μ can be extended to a state $\widehat{\mu}$ of $\mathcal{R}$, and moreover*

$$\forall E, F \in \mathcal{P}(\mathcal{R}) : |\mu(E) - \mu(F)| \leq ||E - F||.$$

It follows that the quasi-state v' is a state of $\mathcal{R}$ if the von Neumann algebra $\mathcal{R}$ has no summand of type I_2. However, the state v' is not normal necessarily, that is, it need not be of the form $v'(\_) = tr(\rho\_)$, so we cannot use the same argument as before. Instead, we will show that v' is a *multiplicative* state, using a result by J. Hamhalter ([Ham93]), and give a second proof of the Kochen-Specker theorem, valid for all von Neumann algebras without summands of types I_1 and I_2.

We will exploit the fact that $v'|_{\mathcal{P}(\mathcal{R})}$ is a *two-valued* measure, that is, $v'(E) \in \{0, 1\}$ for all $E \in \mathcal{P}(\mathcal{R})$. From now on, measure will always mean finitely additive probability measure. We cite Lemma 5.1 of [Ham93] with proof:

Lemma 173 *Let $\mathcal{R}$ be a von Neumann algebra without type I_2 summand. Every two-valued measure on $\mathcal{P}(\mathcal{R})$ can be extended to a multiplicative state of $\mathcal{R}$.*

Proof. Let μ be a two-valued measure on $\mathcal{P}(\mathcal{R})$. Using the Gleason-Christensen-Yeadon theorem (Thm. 172), we can extend μ to a state ϕ of $\mathcal{R}$. Let $\pi_\phi : \mathcal{R} \to \mathcal{H}_\phi$ be the GNS representation engendered by ϕ. Let x_ϕ be a unit cyclic vector of π_ϕ such that $\phi = \omega_{x_\phi} \circ \pi_\phi$, where ω_{x_ϕ} is the vector state given by x_ϕ. For every $E \in \mathcal{P}(\mathcal{R})$ we have $\mu(E) = \langle \pi_\phi(E) x_\phi, x_\phi \rangle$. We see that $\mu(E)$ is either 0 or 1. It follows that either $\pi_\phi(E)x_\phi = x_\phi$ or $\pi_\phi(E)x_\phi = 0$. Hence, $\mathcal{H}_\phi = \overline{\text{lin}}\{\pi_\phi(A)x_\phi \mid A \in \mathcal{R}\} = \overline{\text{lin}}\{\pi_\phi(E) \mid E \in \mathcal{P}(\mathcal{R})\} = \text{lin}\{x_\phi\}$, where lin means the linear span and $\overline{\text{lin}}$ its closure. Therefore, for every $A \in \mathcal{R}$ there is a complex number λ_A such that $\pi_\phi(A)x_\phi = \lambda_A x_\phi$. Obviously, $\lambda_{AB} = \lambda_A \lambda_B$ for $A, B \in \mathcal{R}$, and therefore

$$\begin{aligned} \phi(AB) &= \langle \pi_\phi(AB)x_\phi, x_\phi \rangle = \lambda_{AB} \\ &= \lambda_A \lambda_B = \langle \pi_\phi(A)x_\phi, x_\phi \rangle \langle \pi_\phi(B)x_\phi, x_\phi \rangle \\ &= \phi(A)\phi(B) \end{aligned}$$

for all $A, B \in \mathcal{R}$. ■

Corollary 174 *Let $\mathcal{R}$ be a von Neumann algebra without type I_2 summand. The state v' induced by a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ is multiplicative.*

We now give a short proof (in two lemmata) for the well-known fact that a von Neumann algebra $\mathcal{R}$ of fixed type has no multiplicative states unless $\mathcal{R}$ is of type I_1, that is, abelian. See also Thm. 5.3 in [Ham93].

Lemma 175 *Let $\mathcal{R}$ be a von Neumann algebra of type I_n, $n \geq 2$. There are no multiplicative states of $\mathcal{R}$.*

Proof. If $\mathcal{R}$ is of type I_n, then I is the sum of n equivalent abelian orthogonal projections E_j $(j = 1, ..., n)$. $E_1 \sim E_2$ means that there is a partial isometry $\theta \in \mathcal{R}$ such that $E_1 = \theta^*\theta$ and $E_2 = \theta\theta^*$. Let ϕ be a multiplicative state of $\mathcal{R}$. In particular, ϕ is a tracial state, i.e. $\phi(AB) = \phi(BA)$ for all $A, B \in \mathcal{R}$, hence

$$\phi(E_1) = \phi(\theta^*\theta) = \phi(\theta\theta^*) = \phi(E_2).$$

In the same manner, one obtains $\phi(E_1) = \phi(E_2) = \phi(E_3) = ... = \phi(E_n)$. But $\phi(E_1) \in \{0,1\}$, since ϕ is multiplicative, so

$$\phi(I) = \phi(\sum_{j=1}^{n} E_j) = \sum_{j=1}^{n} \phi(E_n) \in \{0, n\},$$

which is a contradiction. ■

Lemma 176 *Let $\mathcal{R}$ be a von Neumann algebra of type I_∞, II or III. There are no multiplicative states of $\mathcal{R}$.*

Proof. First regard the case that $\mathcal{R}$ is of type I_∞. Since $\mathcal{R}$ is properly infinite, we can use the halving lemma (Lemma 6.3.3 in [KadRinII97]) to show that there is a projection $F \in \mathcal{P}(\mathcal{R})$ such that $F \sim F^\perp := I - F$. For $\mathcal{R}$ a type II or III algebra, we use lemma 168 (choose $E = I$ and $n = 2$) to the same effect. $F \sim F^\perp$ means that there is a partial isometry $\theta \in \mathcal{R}$ such that $F = \theta^*\theta$ and $F^\perp = \theta\theta^*$. Let ϕ be a multiplicative state of $\mathcal{R}$, so

$$\phi(F) = \phi(\theta^*\theta) = \phi(\theta\theta^*) = \phi(F^\perp).$$

Since $\phi(F) \in \{0,1\}$, we have

$$\phi(I) = \phi(I - F + F) = \phi(F^\perp) + \phi(F) \in \{0, 2\},$$

which is a contradiction. ■

Now let $\mathcal{R}$ be an arbitrary von Neumann algebra without summand of type I_2. Let $P_{I_1} \in \mathcal{P}(\mathcal{R})$ be the maximal abelian central projection, P_I the maximal central projection such that $\mathcal{R}P_I$ is of type I, but has no central abelian portion, P_{II} the maximal central projection such that $\mathcal{R}P_{II}$ is of type II and P_{III} the maximal central projection such that $\mathcal{R}P_{III}$ is of type III. We have $I = P_{I_1} + P_I + P_{II} + P_{III}$ (see Thm. 6.5.2 in [KadRinII97]).

Every projection $E \in \mathcal{P}(\mathcal{R})$ can be written as $E = E_{I_1} + E_I + E_{II} + E_{III}$ for orthogonal projections $E_{I_1} \in \mathcal{R}P_{I_1}$, $E_I \in \mathcal{R}P_1$, $E_{II} \in \mathcal{R}P_{II}$ and $E_{III} \in \mathcal{R}P_{III}$. Let $v : \mathcal{R}_{sa} \to \mathbb{R}$ be a valuation function and let v' be the induced state of $\mathcal{R}$. Since $v'|_{\mathcal{P}(\mathcal{R})}$ is finitely additive, $v'|_{\mathcal{R}_{sa}} = v$ and $v(I) = 1 = v(P_{I_1}) + v(P_I) + v(P_{II}) + v(P_{III})$, exactly one term on the right hand side equals 1, the others are zero. Let P_x ($x \in \{I_1, I, II, III\}$) denote the central projection such that $v(P_x) = 1$. It follows that $v(E) = 0$ for all $E \leq P_y$ ($y \in \{I_1, I, II, III\}$) for all $y \neq x$, since $v'|_{\{E,P_y\}''}$ is positive. This means that the valuation function is concentrated at $\mathcal{R}P_x$ in the sense that $v(E) = 0$ for all projections E orthogonal to P_x.

$v|_{\mathcal{R}P_I}$ cannot be a valuation function for $(\mathcal{R}P_I)_{sa}$, since the induced state $(v|_{\mathcal{R}P_I})'$ on $\mathcal{R}P_I$ would be multiplicative, but $\mathcal{R}P_I$ is a sum of type I_n algebras, $n \in \{3, 4, ..., \infty\}$, and none of these algebras has a multiplicative state. Similarly, $v|_{\mathcal{R}P_{II}}$ cannot be a valuation function for $\mathcal{R}P_{II}$ and $v|_{\mathcal{R}P_{III}}$ cannot be a valuation function for $\mathcal{R}P_{III}$, because $\mathcal{R}P_{II}$ and $\mathcal{R}P_{III}$ have no multiplicative states. It follows that $v|_{\mathcal{R}P_{I_1}}$ is a valuation function for $(\mathcal{R}P_{I_1})_{sa}$ and the induced multiplicative state $(v|_{\mathcal{R}P_{I_1}})'$ equals v'. For the abelian part $\mathcal{R}P_{I_1}$, a "hidden state space" is given by the Gelfand spectrum $\Omega(\mathcal{R}P_{I_1})$, each element $\omega \in \Omega(\mathcal{R}P_{I_1})$ is a hidden pure state and induces a valuation function, assigning a spectral value to each $A \in \mathcal{R}P_{I_1}$ by evaluating $\omega(A)$, preserving functional relations. We have shown that only in this trivial situation one can have a valuation function. We obtain:

Lemma 177 *Let $\mathcal{R}$ be a von Neumann algebra without type I_2 summand, and let $P_{I_1} \in \mathcal{P}(\mathcal{R})$ be the maximal abelian central projection. There exists a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ if and only if $\mathcal{R}$ has a summand of type I_1, that is, $P_{I_1} \neq 0$. In this case, $v = v|_{\mathcal{R}P_{I_1}}$, and the valuation function v is completely trivial on the non-abelian part $\mathcal{R}(I - P_{I_1})$ of $\mathcal{R}$, $v|_{(\mathcal{R}(I-P_{I_1}))_{sa}} = 0$.*

Summing up, we have a generalized Kochen-Specker theorem:

Theorem 178 *Let $\mathcal{R}$ be a von Neumann algebra without type I_2 summand. If $\mathcal{R}$ has no type I_1 summand, then the generalized Kochen-Specker theorem (as described in the introduction, sec. 5.1) holds. If $\mathcal{R}$ has a type I_1 summand, then there is a hidden state space in the sense described in the introduction, but only for the trivial, abelian part $\mathcal{R}P_{I_1}$ of $\mathcal{R}$.*

5.3. The presheaf perspective

In a remarkable series of papers, C. J. Isham and J. Butterfield (with J. Hamilton as co-author of the third paper) have given several reformulations of the Kochen-Specker theorem ([IshBut98, IshBut99, HIB00, IshBut02]). They use the language of *presheafs on a category*:

Definition 179 *Let $\mathcal{C}$ be a small category. A* ***presheaf on*** $\mathcal{C}$ *is a covariant functor*

$$P : \mathcal{C}^{op} \longrightarrow \text{Set}\,.$$

The observation is that the FUNC principle, condition (b) in Def. 161, means that a certain square diagram commutes:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "v"'] & B \arrow[d, "v"] \\ v(A) \arrow[r, "g"'] & v(B) \end{tikzcd}$$

This diagram captures the situation $B = g(A)$ and $v(B) = v(g(A)) = g(v(A))$. Isham and Butterfield observe that such a diagram can be read as expressing that there is a *section* of a presheaf on a category:

Definition 180 *Let P be a presheaf on a small category $\mathcal{C}$. A* ***global section*** *s of P is a mapping $\mathcal{C} \to$ Set such that $s(a) \in P(a)$ for all $a \in \mathcal{C}$ and, whenever there is a morphism $\varphi : a \to b$ $(a, b \in \mathcal{C})$, the following diagram commutes:*

$$\begin{tikzcd} a \arrow[r, "\varphi"] \arrow[d, "s"'] & b \arrow[d, "s"] \\ s(a) & s(b) \arrow[l, "P(\varphi)"] \end{tikzcd}$$

Note that the the horizontal arrow at the bottom is reversed, because we are dealing with presheafs, that is, contravariant functors $\mathcal{C} \to \mathrm{Set}$.

There are several choices for the category and the presheaf that can be used to reformulate the Kochen-Specker theorem. We will generalize the proposal made in [HIB00]: Let $\mathfrak{A}(\mathcal{R})$ denote the category of unital abelian subalgebras of $\mathcal{R}$ (the unit of $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$ is the unit of $\mathcal{R}$, see Def. 52). A morphism $\iota_{\mathcal{MN}} : \mathcal{M} \to \mathcal{N}$ exists whenever $\mathcal{M} \subseteq \mathcal{N}$. Hamilton, Isham and Butterfield only regard the case $\mathcal{R} = \mathcal{L}(\mathcal{H})$ and denote this category by $\mathcal{V}$.

Definition 181 *(compare Def 2.3 in [HIB00]) The* ***spectral presheaf*** *over* $\mathfrak{A}(\mathcal{R})$ *is the contravariant functor* $\Sigma : \mathfrak{A}(\mathcal{R}) \to \mathrm{Set}$ *defined as follows:*

(i) On objects: $\Sigma(\mathcal{M}) := \Omega(\mathcal{M})$*, the Gelfand spectrum of* $\mathcal{M}$*.*

(ii) On morphisms: If $\iota_{\mathcal{MN}} : \mathcal{M} \to \mathcal{N}$ *is the inclusion, then* $\Sigma(\iota_{\mathcal{MN}}) : \Omega(\mathcal{N}) \to \Omega(\mathcal{M})$ *is defined by* $\Sigma(\iota_{\mathcal{MN}})(\omega) := \omega|_{\mathcal{M}}$*.*

If there was a global section s of Σ, the following diagram would commute:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\iota_{\mathcal{MN}}"] \arrow[d, "s"'] & \mathcal{N} \arrow[d, "s"] \\ \Omega(\mathcal{M}) & \Omega(\mathcal{N}) \arrow[l, "\Sigma(\iota_{\mathcal{MN}})"] \end{tikzcd}$$

For $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$, $s(\mathcal{M}) \in \Omega(\mathcal{M})$ and $s(\mathcal{M}) = \Sigma(\iota_{\mathcal{MN}})(s(\iota_{\mathcal{MN}}(\mathcal{M}))$, where $\iota_{\mathcal{MN}}(\mathcal{M})$ is the algebra $\mathcal{M}$ seen as part of $\mathcal{N}$ and $s(\iota_{\mathcal{MN}}(\mathcal{M})) \in \Omega(\mathcal{N})$. The commutativity of the diagram means that $s(\mathcal{M})$ is given as the restriction of $s(\iota_{\mathcal{MN}}(\mathcal{M}))$ to $\Omega(\mathcal{M}) \subseteq \Omega(\mathcal{N})$.

Such a choice of one element $s(\mathcal{M})$ of the Gelfand spectrum $\Omega(\mathcal{M})$ per abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$, compatible with the spectral presheaf mappings, that is, with restrictions $\Omega(\mathcal{N}) \to \Omega(\mathcal{M})$, would give a valuation function when restricted to the self-adjoint elements: for all $A \in \mathcal{M}_{sa}$, $s(\mathcal{M})(A) \in \operatorname{sp} A$ and $s(\mathcal{M})(f(A)) = f(s(\mathcal{M})(A))$. The generalized Kochen-Specker theorem (Thm. 178) hence shows that for von Neumann algebras $\mathcal{R}$ without summands of types I_1 and I_2, there is no global section of Σ.

In Lemma 164, we saw that having a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ would mean having a character $v'|_{\mathcal{M}}$ (an element of the Gelfand spectrum) for each abelian subalgebra $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$. It follows from the FUNC principle that these characters are subject to the same conditions as above: if $\mathcal{M} \subseteq \mathcal{N}$, then restricting $v'|_{\mathcal{N}}$ to $\Omega(\mathcal{M})$ must give $v'|_{\mathcal{M}}$ (which is not possible globally). This choice of a category and a presheaf thus brings the presheaf formulation of the Kochen-Specker theorem very close to our first proof.

There is a closely related formulation, using Stone spectra instead of Gelfand spectra:

Definition 182 *The **state presheaf** $\mathcal{M}^1$ on $\mathfrak{A}(\mathcal{R})$ is defined as follows:*

(i) On objects: $\mathcal{M}^1(\mathcal{M}) := \mathcal{M}^1(\mathcal{Q}(\mathcal{M}))$, the set of positive Radon measures of norm 1 on $\mathcal{Q}(\mathcal{M})$.

(ii) On morphisms: for $\mathcal{M}, \mathcal{N} \in \mathfrak{A}(\mathcal{R})$ such that $\mathcal{M} \subseteq \mathcal{N}$ let

$$\begin{aligned} p^{\mathcal{N}}_{\mathcal{M}} : \mathcal{M}^1(\mathcal{N}) &\longrightarrow \mathcal{M}^1(\mathcal{M}) \\ \mu_{\mathcal{N}} &\longmapsto p^{\mathcal{N}}_{\mathcal{M}}.\mu_{\mathcal{N}}, \end{aligned}$$

where $p^{\mathcal{N}}_{\mathcal{M}}.\mu_{\mathcal{N}}$ is the image measure defined by

$$(p^{\mathcal{N}}_{\mathcal{M}}.\mu_{\mathcal{N}})(U) := \mu_{\mathcal{N}}((p^{\mathcal{N}}_{\mathcal{M}})^{-1}(U)),$$

where $U \subseteq \mathcal{Q}(\mathcal{M})$ is a Borel set and $p^{\mathcal{N}}_{\mathcal{M}} : \mathcal{Q}(\mathcal{N}) \to \mathcal{Q}(\mathcal{M})$, $\beta \mapsto \beta \cap \mathcal{M}$ is the restriction map between the Stone spectra.

$\mathcal{M}^1$ really is a presheaf on $\mathfrak{A}(\mathcal{R})$, since obviously $p^{\mathcal{M}}_{\mathcal{M}} = id_{\mathcal{M}^1(\mathcal{M})}$ and since $p^{\mathcal{P}}_{\mathcal{M}} = p^{\mathcal{N}}_{\mathcal{M}} \circ p^{\mathcal{P}}_{\mathcal{N}}$ as mappings $\mathcal{Q}(\mathcal{P}) \to \mathcal{Q}(\mathcal{N}) \to \mathcal{Q}(\mathcal{M})$, the same holds for the mappings

$$\mathcal{M}^1(\mathcal{P}) \to \mathcal{M}^1(\mathcal{N}) \to \mathcal{M}^1(\mathcal{M}).$$

Let $\mathcal{R}$ have no type I_1 and I_2 summands. From the generalized Kochen-Specker theorem (Thm. 178) it follows that $\mathcal{M}^1$ has no global sections consisting entirely of point measures. The fact that $\mathcal{M}^1$ has no such global sections is equivalent to the generalized Kochen-Specker theorem, since a valuation function would induce a quasi-state v' (Lemma 166) such that $v'|_{\mathcal{M}}$ is a pure state for every $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$ (and hence gives a point measure on $\mathcal{Q}(\mathcal{M})$).

This presheaf formulation emphasizes the fact that a valuation function would give a *pure* state of every $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$. The presheaf $\mathcal{M}^1$ does have global sections (every state of $\mathcal{R}$ induces one, obviously), but it has *no global sections consisting entirely of point measures.*

5.4. Discussion

We have presented two functional analytic proofs for the fact that there are no valuation functions $v : \mathcal{R}_{sa} \to \mathbb{R}$ for $\mathcal{R}$ a von Neumann algebra. The first proof only uses Gleason's classical theorem (Thm. 160) and holds for $\mathcal{R}$ a type I_n factor, $n \geq 3$. The second proof depends on the Gleason-Christensen-Yeadon theorem (Thm. 172) and holds for von Neumann algebras $\mathcal{R}$ without summands of types I_1 and I_2. To the best of our knowledge, for the first time von Neumann algebras other than the type I_n factors $\mathcal{L}(\mathcal{H})$ have been treated. The generalized Kochen-Specker theorem follows: there is no hidden states model of quantum theory in the sense described in the introduction.

Both proofs are based on the fact that having a valuation function v would mean having a *state* v' of $\mathcal{R}$, which follows from Gleason's theorem, and this state has properties that lead to a contradiction. In the first proof, for type I_n factors ($n \geq 3$), the state is of the form $v'(\_) = tr(\rho\_)$, so there are projections $E \in \mathcal{P}(\mathcal{R})$ such that $v'(E) \notin \{0, 1\}$. The second proof, which is much more general, uses the fact that v' is a multiplicative state. Since there are no multiplicative states except on type I_1, that is, abelian, algebras, the Kochen-Specker theorem holds for all von Neumann algebras without summands of types I_1 and I_2. Type I_2 must be excluded since Gleason's theorem and the Gleason-Christensen-Yeadon theorem only hold if $\mathcal{R}$ has no type I_2 summand. It is known that every type I_2 algebra admits a two-valued measure and hence a valuation function, see Rem. 5.4 in [Ham93]. If $\mathcal{R}$ has a type I_1 summand, then there are valuation functions, but they are concentrated at the trivial, abelian part $\mathcal{R}P_{I_1}$ of $\mathcal{R}$, where $P_{I_1} \in \mathcal{P}(\mathcal{R})$ is the maximal abelian central projection.

The fact that the defining conditions of a valuation function v inevitably lead to a multiplicative state shows that these conditions are very strong. Indeed, although the FUNC principle only seems to pose conditions on commuting operators, this is not the case: v' is a state, i.e. it is additive on non-commuting operators also. In the physics literature, an abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$ is called a *context*. Of course, the contexts give nothing like a partition of $\mathcal{R}$ into abelian, "classical" parts, but are interwoven in an intricate manner, since an observable $A \in \mathcal{R}_{sa}$ typically is contained in many abelian subalgebras. The FUNC principle poses conditions within each context, but since typically $A = f(B) = g(C)$ for non-commuting observables B, C, it also poses conditions on non-commuting observables, across contexts. The presheaf formulations presented in section 5.3 clearly show that the Kochen-Specker theorem means that there is no state ϕ of $\mathcal{R}$ such that for all contexts $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$, the restriction $\phi|_{\mathcal{M}}$ is a pure state. This can also be expressed by saying that there are no dispersionless states. The article [Ham04] and the book [Ham03] (especially section 7.3) by J. Hamhalter are a valuable source of results on the dispersion of states, two-valued probability measures and hidden variables.

The Kochen-Specker theorem is little more than a corollary to Gleason's theorem, in a more general sense than worked out by Bell ([Bell66]). The fact that v' is a state comes from Gleason's theorem (or its generalization): a valuation function v defines a quasi-state v' in a canonical manner, and restricting the quasi-state v' to the projection lattice $\mathcal{P}(\mathcal{R})$ gives a finitely additive probability measure. Gleason's theorem shows that v' must be a state. The deep meaning of Gleason's theorem is that the simple, *lattice-theoretic* condition of finite additivity on each distributive sublattice, which is a condition on finite joins actually ($E + F = E \vee F$ for orthogonal projections E, F), suffices to guarantee additivity of the functional v' defined by linear extension of the probability measure (and taking the appropriate limit, see e.g. Ch. III.7 of [Mae89]). Of course, finite additivity on each distributive sublattice is a condition across distributive sublattices, since each projection E is contained in many distributive sublattices.

But the defining conditions of a valuation function v are even stronger: using the fact that $v(E) \in \operatorname{sp} E = \{0, 1\}$ ($E \in \mathcal{P}(\mathcal{R})$), we saw that the state v' is multiplicative, which is only possible if v is concentrated at the abelian part $\mathcal{R}P_{I_1}$ of $\mathcal{R}$. Thus, a valuation function and a hidden states model can only exist for the trivial, abelian situation. This generalizes to arbitrary von Neumann algebras a result found by J. D. Malley ([Mal04]). It also rebuts the critique of von Neumann's proof from 1932 ([vNeu32]). Von Neumann posed additivity conditions on non-commuting observables, which was strongly criticized by Bell ([Bell66]) as unphysical. Of course, the Gelfand representation of an abelian von Neumann algebra is a hidden states model, the Gelfand spectrum $\Omega(\mathcal{R})$ taking the role of the "hidden" state space.

We have shown more than the fact that there are no non-trivial hidden states models: each element ω of a hidden state space Ω would give a valuation function, as described in the introduction, but having a valuation function would not necessarily mean having a hidden states model. A valuation function would simply assign values to all observables in a manner consistent with the FUNC principle, which would be an important piece of a realistic quantum theory. Since we have ruled out this possibility, there are no such naïve realistic models of quantum theory.

Die Ameisen

In Hamburg lebten zwei Ameisen,
Die wollten nach Australien reisen.
Bei Altona auf der Chaussee
Da taten ihnen die Beine weh,
Und da verzichteten sie weise
Denn auf den letzten Teil der Reise.

So will man oft und kann doch nicht
Und leistet dann recht gern Verzicht.

Joachim Ringelnatz

A. Index and bibliography

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