Counting of Finite Fuzzy Subsets with Applications to Fuzzy Recognition and Selection Strategies

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Abstract

The counting of fuzzy subsets of a finite set is of great interest in both practical and theoretical contexts in Mathematics. We have used some counting techniques such as the principle of Inclusion-Exclusion and the Möbius Inversion to enumerate the fuzzy subsets of a finite set satisfying different conditions. These two techniques are interdependent with the Möbius inversion generalizing the principle of Inclusion-Exclusion. The enumeration is carried out each time we redefine new conditions on the set. In this study one of our aims is the recognition and identification of fuzzy subsets with same features, characteristics or conditions. To facilitate such a study, we use some ideas such as the Hamming distance, mid-point between two fuzzy subsets and cardinality of fuzzy subsets. Finally we introduce the fuzzy scanner of elements of a finite set. This is used to identify elements and fuzzy subsets of a set. The scanning process of identification and recognition facilitates the choice of entities with specified properties. We develop a procedure of selection under the fuzzy environment. This allows us a framework to resolve conflicting issues in the market place.

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Keywords: Finite Fuzzy Set; Möbius function; Zeta function; Zeta matrix; Set Partition; Dominance order.

Contents

ILLUSTRATIONS OF FIGURES AND TABLES

NOTATIONS

$\mathbb{Z}$, integers

PIE, The Principle of Inclusion-Exclusion

$S(n,k)$, Stirling Number of the second kind

$\mathcal{F}(\mathcal{X},\mathcal{K})$, Function from set X to K.

$(N_\mu \geq \alpha)$, Number of functions such that at least one element of X has a membership value greater than α.

μ_A, fuzzy subset A

$\overline{\mu_A}$, complement of fuzzy subset μ_A

$\mu_A(x)$, the membership value of element x by fuzzy subset μ_A

$\mathcal{F}(\mathcal{X})$ or I^X, family of fuzzy subsets in X

$\mu \cap \lambda$, $\mu \wedge \lambda$, intersection of μ and λ

$\mu \cup \lambda$, $\mu \vee \lambda$, union of μ and λ

$|\mu|$, the cardinality of fuzzy subset μ

μ^α, the α-cut, the set $\{x \in X, \mu(x) \geq \alpha\}$

$\mu^{[a,b]}$, the interval-cut of μ

$(\mu_1 \cup \mu_2)^{[a,b]}$, the interval-cut of $\mu_1 \cup \mu_2$

$Card_\mu$, the set of distinct cardinalities of fuzzy subsets of X

$(N_\mu(|\ .\ |= \alpha))_n$, the number of fuzzy subsets of X of cardinality equal to α

$N_\mu \mu(x_i) = 0$, the number of fuzzy subsets of X for which $\mu(x_i) = 0$ for $i = 1, 2, \cdots n$ and $x_i \in X$

$(\mu^\alpha)^i$, the α- induced fuzzy subset

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PREFACE

The primary purpose of our study is the enumeration of different entities of fuzzy subsets of a finite set through the principle of inclusion-exclusion and the Möbius inversion. In the process we identify some pattern among the elements of the set or even among the fuzzy subsets of the finite set and count those entities having the same features. While we are enumerating and identifying these objects, we use the ideas such as cardinality, support and core of fuzzy subsets; distance and mid-point between two fuzzy subsets. We impose various conditions and count accordingly entities satisfying these conditions. Among the various types of distance, we have chosen to use the Hamming distance and the generalized relative Hamming distance between two fuzzy subsets as suggested by A. Kaufmann, [20]. With this in mind we have defined the ordinary subset nearest to a fuzzy subset and the index of fuzziness.
It is again around the notion of Hamming distance that we have built our first, second and third criteria for identification and recognition of fuzzy subsets.The identification of an element or a fuzzy subset of a set can be achieved using procedures of selection under fuzzy environment. We limit ourselves in finding the majority, plurality and Borda winners. This selection under fuzzy environment can also be obtained by using the idea of a fuzzy scanner.

In Chapter 1, we recall the well established Principle of Inclusion-Exclusion (PIE) in crisp set context. This principle expresses the function of a union of finitely many sets as an alternating sum of functions of their intersections and dually we express the function of an intersection of finitely many sets as an alternating sum of functions of their unions. We will use this principle in crisp context in two cases, namely, the counting of surjections from an m-element set M to a n-element set K and also the enumeration of ordinary functions from an m-element set M to a n-element set K. The dual of the principle of Inclusion-Exclusion was first proved in [45]. We use some of these techniques in subsequent chapters of this thesis.

In Chapter 2, we review some basic definitions pertaining to fuzzy subsets. This includes definition of fuzzy subsets; cardinality of a fuzzy subset, Hamming distance between fuzzy subsets and the notion of α-cut. After these general terms, we later apply the PIE in counting objects in the lattice of

fuzzy subsets of a finite set. We count either elements of a finite set or elements in the α-cuts of fuzzy subsets and the intersections or unions of fuzzy subsets. With regards to the α-cut, some of these elements may have absolute desirability or worth. On the other hand some may have no worth at all since their membership value is zero. Considering an $\alpha \in [0,1]$, we count as well the fuzzy subsets with regards to the size of their α-cut. Here we will count the fuzzy subsets of the same, larger or smaller α-cut. We will consider the cardinality of fuzzy subsets of an n-element set X and enumerate the fuzzy subsets with either the same cardinality, cardinality zero or cardinality n.
The counting of elements of $\mathcal{F}(\mathcal{X})$, the lattice of fuzzy subsets, is done with regards to some imposed conditions that elements must satisfy. Each time we change the conditions, new identification and therefore new counting of elements satisfying the condition is done.
Towards the end of Chapter 2, we will discuss some other applications of the PIE. In one case we consider $\alpha_i \in [0,1]$ and check using the PIE that the intersections and unions of the pull-back of membership functions $\bigcap_{i=1}^{n} \mu^{-1}(\alpha_i)$ and $\bigcup_{i=1}^{n} \mu^{-1}(\alpha_i)$ is actually a partition of the set X.

In Chapter 3 we first review some facts related to the Möbius functions and Möbius inversion and then establish in the process the Möbius function and inversion in the lattice $\mathcal{F}(\mathcal{X})$ of fuzzy subsets of a finite set X.
The Möbius inversion (MI) is actually a generalization of the principle of Inclusion-exclusion (PIE). So we find it relevant here to extend our discussion of PIE in $\mathcal{F}(\mathcal{X})$ by studying the Möbius Inversion in $\mathcal{F}(\mathcal{X})$.

In Chapter 4 we talk about the identification process or the pattern recognition. Here we try to identify some common patterns among the elements and also among the fuzzy subsets of a finite set. Using the Hamming distance and the notion of cardinality of fuzzy subsets, we define the idea of fuzzy subset nearest to a fuzzy subset. We also look at the number of fuzzy subsets between two fuzzy subsets and the number of fuzzy subsets mid-point between two fuzzy subsets. Considering a fuzzy subset as a vector $(\mu_1, \mu_2, \cdots, \mu_n) \in [0,1]^n$, we are able to define the fuzzy mid-point between two fuzzy subsets. This fuzzy mid-point between two fuzzy subsets is not always unique, but is a collection

of possibilities to choose from. Because fuzzy subsets allow many mid-points or alternatives to choose from between two chosen ones, then we can use them to model a solution for some day to day situations. We have in mind some conflicting situations such as water and electricity distribution in a community. Distribution and usage of water and electricity are issues of conflict in one community. Different stakeholders do not agree on how these commodities should equitably be distributed. Business, industry and farms think they bring food and livelihood to the community and therefore should be given sufficient attention by the municipal council. They also believe that they are being over-taxed while the same amount of energy and water is supplied to hospitals, streets and schools for free. Other people on the other hand might think that they do not have money to pay for services and they might say that industries and farms are responsible for water pollution and therefore should not be supplied with much needed water. How do we solve such kind issues? We are going to use finite fuzzy subsets to model and find solutions to these types of problems.
In this chapter we suggest a new way of representing an element of a set taking into account its membership values to the fuzzy subsets of the set. It will be shown that this unique way of representing the characteristics of an element, which we will call fuzzy $bar-code$, is important. A tool which is able to read the element's details, as well as identify an element of a set by using its fuzzy $bar-code$ will be introduced and would be used in the process of fuzzy recognition.

In Chapter 5 we introduce a procedure of election activity under fuzzy environment. The election in this context is actually a selection of a desired outcome among many alternatives. While the selection by itself is not vague, we use vague or fuzzy descriptors to achieve our goal of selection. Here we will talk about fuzzy $plurality\ winner$, fuzzy $Borda\ winner$ and fuzzy $majority\ winner$.
We are not concerned about the election as the one in political arena. We are interested in the selection of the best possible choice among a myriad of possibilities. This election is envisaged as a solution in case of conflict where the best option possible is chosen among many in an environment where terms are defined in a vague way.

Chapter 6 gives a summary of the work done in the thesis and recommendations for future research and as such can be regarded as a conclusion chapter of the thesis.

Chapter 1

The Principle of inclusion-exclusion

1.1 Introduction

We will start our study by introducing the famous principle of inclusion-exclusion. This is a counting technique which consists of over-counting and under-counting elements of a set, including some extra elements but later on excluding the ones which have been taken twice, three times... and so on. Another appropriate name to this principle is the sieve technique.
The Principle of Inclusion and Exclusion (PIE) as a counting technique has been used for enumerating crisp subsets of a set. In the following Chapter we extend this counting tool to the set of fuzzy subsets of a finite set X. Much work was done in [45] Here we define some new facts, state and improve the most interesting results in this regard.
We want naturally to define the principle of Inclusion and Exclusion. We will show with some examples how this principle is used practically.

1.2 The idea of PIE.

Initially the Principle of Inclusion and Exclusion (PIE) was expressed as in the theorem below:

Theorem 1.2.1 Let $A_1, A_2, \cdots A_n$ be subsets of a finite set X, Then the number of elements of X in the union of n subsets $A_i\, 1 \leq i \leq n$ is given as in the expression below

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} | A_i | - \sum_{1\leq i<j} | A_i \cap A_j | + \sum_{1\leq i<j<k} | A_i \cap A_j \cap A_k |$$

$$+ \cdots + (-1)^{n-1} | A_1 \cap A_2 \cap \cdots \cap A_n |$$

Alternatively, taking complements in X, the theorem can also be expressed as follows:

$$\left|X \setminus \bigcup_{i=1}^{n} A_i\right| =| X | - \Sigma_{i=1}^{n} | A_i | + \Sigma | A_i \cap A_j | + \cdots + (-1)^n | A_1 \cap A_2 \cap \cdots A_n | .$$

Using the duality principle, we can express the above theorem as in the following statement: If $A_1, A_2, \cdots A_n$ are subsets of a finite set X, Then

$$\left|\bigcap_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} | A_i | - \sum_{1\leq i<j} | A_i \cup A_j | + \sum_{1\leq i<j<k} | A_i \cup A_j \cup A_k |$$

$$+ \cdots + (-1)^{n-1} | A_1 \cup A_2 \cup \cdots \cup A_n | .$$

Proof : We prove the theorem by induction on n.
From $| A \cup B | = | A | + | B | - | A \cap B |$, we draw that

$$| A \cap B | =| A | + | B | - | A \cup B | .$$

Thus for $n = 2$ the formula is valid. Now suppose the formula is true for the intersection of n subsets $A_1, A_2, \cdots A_n$ of S. That is:
$| A_1 \cap A_2 \cdots \cap A_n |= \sum_{i=1}^{n} | A_i | - \sum_{1\leq i<j} | A_i \cup A_j | + \sum_{1\leq i<j<k} | A_i \cup A_j \cup A_k | + \cdots + (-1)^{n-1} | A_1 \cup A_2 \cup \cdots \cup A_n |$.
Then we prove that the theorem is true for $n + 1$. We wish to show that
$| A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1} |= \sum_{i=1}^{n+1} | A_i | - \sum_{1\leq i\leq j\leq n+1} | A_i \cup A_j | + \cdots + \cdots + (-1)^n | A_1 \cup A_2 \cup \cdots \cup A_{n+1} |$ is true.
Let $A = A_1 \cap A_2 \cap \cdots \cap A_n$, then the left side of the above expression becomes

$| A \cap A_{n+1} |$ which gives rise to the following equality
$| A \cap A_{n+1} |=| A | + | A_{n+1} | - | A \cup A_{n+1} |$ as seen above in the case of two (2) subsets.
From the above equality, we can express $| A | + | A_{n+1} |$ as :

$$| A | + | A_{n+1} |= \sum_{i=1}^{n+1} | A_i | - \sum_{1\leq i<j\leq n} | A_i \cup A_j | + \sum_{1\leq i<j<k\leq n} | A_i \cup A_j \cup A_k |+ \cdots + (-1)^{n-1} | A_1 \cup A_2 \cup \cdots \cup A_n | \tag{1.2.1}$$

Again $| A \cup A_{n+1} |$ can be expressed as
$| A \cup A_{n+1} |=| (A_1 \cup A_{n+1}) \cap \cdots \cap (A_n \cup A_{n+1}) |$ by using the distributive law of union.

Now if we set that $A \cup A_{n+1} = A'$ and $A_i \cup A_{n+1} = A'_i$ for each i, we further obtain, again by inductive hypothesis,
$| A' | = | A'_1 \cap \cdots \cap A'_n | = \sum_{i=1}^{n} | A'_i | - \sum_{1\leq i<j\leq n} | A'_i \cup A'_j$
$| + \sum_{1\leq i<j<k\leq n} | A'_i \cup A'_j \cup A'_k | + \cdots + (-1)^{n-1} | A'_1 \cup A'_2 \cup \cdots \cup A'_n |$.
The rule of set union allows us to write $A'_{i_1} \cup A'_{i_2} \cup \cdots \cup A'_{i_k}$ as
$A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} \cup A_{n+1}$ for any subset of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ where $1 \leq k \leq n$. Therefore:

$$| A \cup A_{n+1} |= \sum_{i=1}^{n} | A_i \cup A_{n+1} | - \sum_{1\leq i<j\leq n} | A_i \cup A_j \cup A_{n+1} | + \sum_{1\leq i<j<k\leq n} | A_i \cup A_j \cup A_k \cup A_{n+1} | + \cdots + (-1)^{n-1} | A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1} | \tag{1.2.2}$$

Now subtracting each side of the equation 1.2.2 from the corresponding side of the equation 1.2.1, we see that the inductive process on n is valid. This completes the proof. $\square$ [45].

We consider below in the following subsections some applications of the PIE. We see how the principle is used in concrete cases. We will apply this in two incidents: the counting of surjections from a set M to a set K and the general counting of functions from any set M to an ordered set K.

1.3 The enumeration of surjections

Let M and K be an m-element set and a k-element set respectively. A function $f : M \to K$ is a surjection if each element of K is image of some element x of M. The number of functions from M to K is k^m. This number is easily used to compute the number of surjections from M to K when K has only few elements. For example let us consider K to be $\emptyset, \{a\}, \{a_1, a_2\}$ respectively. If $K = \emptyset$, there is only one function, namely the empty function from M to K. If $K = \{a\}$, again there is only one function from M to K, but this function is a surjection. If $K = \{a_1, a_2\}$: There are 2^m functions from M to K. Of these, one only skips a_1 and one only skips a_2. This means two of the 2^m functions are not surjections. Therefore $2^m - 2$ are surjections.
Generally consider that $K = \{a_1, a_2, \cdots a_k\}$. Let $p_1, p_2, \cdots, p_k$ be the properties that $a_1, a_2, \cdots a_k$ are not in the range of the function respectively. Also let $p_1^{'}, p_2^{'}, \cdots, p_k^{'}$ be the properties that $a_1, a_2, \cdots a_k$ are in the range of the function respectively.
We denote by $N(p_i)$ and $N(p_i^{'})$ the number of functions that do not have a_i in their range and the number of functions that do have a_i in their range respectively for $i = 1, 2$. Also we denote by $N(p_i p_j)$ and $N(p_i^{'} p_j^{'})$ the number of functions that do not have both a_i and a_j in their range and the number of functions that do have both a_i and a_j in their range respectively.
Using the PIE to enumerate functions that do have every element of K in their range we can write :$N(p_1^{'} p_2^{'} \cdots p_m^{'}) = N - [N(p_1) + N(p_2) + \cdots + N(p_k)] + [N(p_1p_2) + \cdots + N(p_ip_j + \cdots + N(p_{k-1}p_k)] - [N(p_1p_2p_3) + \cdots + N(p_ip_jp_k) + \cdots + [N(p_1p_2 \cdots p_k)]$, where N is k^n. For each a_i not in the range there are $k-1$ choices for the value of the function at each element of the domain. Therefore there are $(k-1)^m$ functions. That is to mean that there are $\binom{k}{1}(k-1)^m$ functions skipping one element of K. If two elements a_i and a_j are not in the range, then there are $k-2$ choices for the value of the function at each element of the domain. That means there are $(k-2)^m$ functions that skip any two elements of K. There are $\binom{k}{2}(k-2)^m$ functions skipping any two elements of K. Since any function skipping two elements of K, skips at least one element of K, these functions are counted among the $N(p_1^{'} p_2^{'} \cdots p_m^{'})$. We continue this way until none of the $a_i \forall i$ is in the range of the functions. That

is to mean that $N(p_1^{'}p_2^{'}\cdots p_m^{'}) = 0$, in which case there are no such functions. Thus the number of surjections for an m-elements set M to a k-elements-set is $k^m - \begin{pmatrix} k \\ 1 \end{pmatrix}.(k-1)^m + \begin{pmatrix} k \\ 2 \end{pmatrix}.(k-2)^m + \begin{pmatrix} k \\ 3 \end{pmatrix}.(k-3)^m + \cdots \pm \begin{pmatrix} k \\ k-1 \end{pmatrix}.1^m$
When the above number is multiplied by $\frac{1}{k}!$, that is $\frac{1}{k!}.(k^m - \begin{pmatrix} k \\ 1 \end{pmatrix}.(k-1)^m + \cdots \pm \begin{pmatrix} k \\ k-1 \end{pmatrix}.1^m)$, the resulting number is called Stirling Number of the second kind and is denoted $S(m,k)$.

Now we wish to tackle the more general case of counting the number of functions from a set M to K. This time we consider an order in the set K.

1.4 Counting of ordinary functions

In the above example the range set K was not ordered in any fashion. But it is useful in our further work to have an ordering and more particularly a total ordering on K. So we impose that K is a totally ordered set with $a_1 \leq a_2 \leq \cdots \leq a_k$. We could count, using PIE, the number f of functions from X to K, each satisfying a property that all the elements of the range of the function exceed a certain specified element of K. With that specification, we associate a number p which is the number of elements exceeding the specified element. We consider functions of a finite set X with values in a finite set $K \subset [0,1]$. Assume $|X| = n$ and $|K| = k$. As seen earlier, there are k^n possible functions of X in $\mathcal{F}(\mathcal{X},\mathcal{K})$. Consider $\alpha \in K$.
In this section we count, among the k^n functions, those which are such that no element of X has a value in M that exceeds α. Let us use the notation $(N_\mu \geq \alpha)$ for the number of functions which are such that at least one element of X has a value that is equal to or exceeds α. Now the number of functions sought is given by the expression in the following lemma.

Proposition 1.4.1 $(N_\mu \geq \alpha) = k^n - \sum_{i=0}^{n} C(n,i)(-1)^{n-i}k^i p^{n-i}$

Proof: Let $\alpha \in K$ and $\mid K \mid= k$, while $\mid X \mid= n$.
Since K is finite and ordered with the usual ordering in $\mathbb{Z}$, and M has the minimum element denoted by l; and the maximum elements denoted here as h; we observe that $l \leq \alpha \leq h$.
Let us now find the number of functions (fuzzy subsets) such that no element of X has a membership value (image) equal to or exceeding α.
This problem is the same as that of finding the number of functions (fuzzy subsets) from X with (membership) values in M skipping $\mid [\alpha, h] \mid$ values in K.
Call $\mid [\alpha, h] \mid= p$. Since $\mid K \mid= k$, there are $k - p$ values in K not exceeding α. Therefore there are $(k - p)^n$ functions (fuzzy subsets)from X with (membership) values in M with no image (membership) exceeding α out of a total of k^n functions.
This means there are $k^n - (k - p)^n$ functions (fuzzy subsets) with an image (membership value) greater or equal to α.
This also means $(N_\mu \geq \alpha) = k^n - (k - p)^n \; = p^n$.
Expanding $k^n - (k - p)^n$ we get:
$k^n - (k - p)^n = k^n - [\sum_{i=0}^{n} C(n, i).k^i.(-p)^{n-i}\,]$
and so $(N_\mu \geq \alpha) = k^n - \sum_{i=0}^{n} C(n, i)(-1)^{n-i} k^i p^{n-i}.\square$ The two examples mentioned above are typical to show how the PIE is used. In day to day life many other situations require this counting technique. In the literature, there are many other examples that are mentioned.

In the next chapter we deal with the lattice of fuzzy subsets of a finite set, denoted by $\mathcal{F}(\mathcal{X})$. We will recall the definition of the concept of fuzzy subset, recall also the properties and operations on fuzzy subsets. Later we will explore and devise some ways of counting fuzzy subsets of a set or counting of elements of the set using the counting technique of PIE. Our discussions will center mostly around the notions of cardinality, α-cuts, Hamming distance, etc...

Chapter 2

Applications of PIE in enumerating elements and fuzzy subsets of the finite set

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2.1 Fuzzy subsets of a finite set.

In this section we recall the basic definitions, notations and operations for fuzzy subsets of a finite non-empty set with regards to the principle of Inclusion-Exclusion. Research on fuzzy subsets has been underway for over 40 years now. It is therefore impossible to cover all aspects in this field nor is necessary to look into all aspects since we are only interested in the applications of PIE to a collection of fuzzy subsets of a finite set. We merely aim to provide a summary of the basic concepts central to the study of fuzzy subsets and refer to various excellent textbooks available in the literature. See [20].

2.1.1 Definition of a fuzzy subset

Let X be a nonempty set. A fuzzy subset A of X is characterized by a membership function: $\mu_A : X \longrightarrow [\,0, 1\,] = I$ such that the number $\mu_A(x)$ in the unit interval I is interpreted as the degree of membership of element x to

the fuzzy subset A, for each $x \in X$. This number $\mu_A(x)$ is also called the *membership value* of element x to the fuzzy subset μ_A. The set X is referred to as the universe of discourse.
Every $\{0,1\}$ -valued fuzzy subset with membership function taking only either 0 or 1 is called a crisp subset, that is just a subset of X in the usual sense of the term. This means each object x of X either belongs to A when the degree of membership is 1 or does not belong to the subset A (membership-0) of X. Therefore we can identify a subset A with its characteristic function $\chi_A : X \to I$ such that

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise, that is, } x \notin A \end{cases}$$

for all $x \in X$.
Unlike the crisp subset, a fuzzy subset expresses the degree to which an element belongs to the fuzzy subset. Hence the extended characteristic function of A, $\chi_A : X \to I$, of a genuine fuzzy subset A that is not crisp is allowed to have values strictly between 0 and 1, which denotes the degree of membership, that is partial membership of an element in a given set.
Let X be a non-empty set and μ_A a fuzzy subset of X. We call μ_A weakly empty fuzzy set of X if
$\mu_A(x) < 0.5, \forall x \in X$.
The fuzzy subset A is completely determined by the set of tuples: $A = \{(x, \mu(x))\}, x \in X\}$ provided either X is finite or countably infinite. For example,

Example 2.1.1 .
Let $X = \{x_1, x_2, x_3, x_4,\}$ such that $\mu(x_1) = 0.2$, $\mu(x_2) = 0$, $\mu(x_3) = 0.3$, $\mu(x_4) = 0.8$. Therefore $\mu_A = \{(x_1; 0.2), (x_2; 0), (x_3; 0.3), (x_4; 0.8)\}$ is a fuzzy subset of X with a four-tuple representation.
The set such as $M = \{0.2; 0; 0.3; 0.8\}$ in the above case is called the membership set of the fuzzy subset μ_A.
Consider all the fuzzy subsets of set X with memberships in the set $M \subseteq I$. These are elements of $\mathcal{F}(\mathcal{X})$. We are imposing some conditions on the set of membership values M restricting the fuzzy subsets to a certain order for convenience.

2.1.2 Family of fuzzy subsets of a set.

Throughout the remainder of this thesis, $X = \{x_1, x_2, \cdots, x_n\}$ is a finite set with $1 \leq n$ elements and all fuzzy subsets μ of X take n membership values not all necessarily distinct and hence take m values with $1 \leq m \leq n$. The membership values in the interval $I = [0, 1]$ are taken to be uniformly spaced, with the usual ordering given by $M_m = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$.
This uniform choice of values in M_m does not affect the counting of fuzzy subsets with special property and also is in line with preferential equality discussed elsewhere, [28].
The family of all fuzzy subsets in X is denoted by $\mathcal{F}(\mathcal{X})$ or I^X
Here it is useful to have the notation $\mid X \mid$ to stand for the cardinality of a set X. In general if $\mid X \mid = n$ and $\mid M \mid = m$, then there are m^n possible fuzzy subsets in total which is $\mid \mathcal{F}(\mathcal{X}) \mid = m^n$.
$\mathcal{F}(\mathcal{X})$ is finite if both X and M are finite.
Note that the set $\mathcal{P}(\mathcal{X})$ of crisp subsets of X has 2^n elements.

Example 2.1.2 .
Refer to the diagram attached to the thesis.

$X = \{x_1, x_2, x_3\}$ and $M = \{0, \frac{1}{2}, 1\}$. There are $27 = 3^3$ distinct fuzzy subsets as members in $\mathcal{F}(\mathcal{X})$ among which $2^3 = 8$ are crisp subsets of X.
The fuzzy subsets of a set have more properties than their counterparts, the crisp subsets. It is therefore interesting to study how the usual operations such as inclusion, intersection, union of crisp subsets extend to fuzzy subsets.

2.1.3 Operations on Fuzzy Subsets.

In this section we extend some of the operations of crisp set theory such as inclusion, intersection, union and complement. These extensions are done in such as way that the extended operations restricted to two-valued subsets, namely crisp subsets, coincide with the usual operations.

1°. Inclusion.

Let μ , $\lambda \in \mathcal{F}(\mathcal{X})$ be two fuzzy subsets of X. μ is said to be *included* in λ (or, equivalently, μ is *contained* in λ, or μ is smaller than or equal to λ) if and only if $\forall x \in X, \quad \mu(x) \leq \lambda(x)$. We express the containment or inclusion also as *domination* or *dominance* in the sense that λ dominates μ if and only if $\forall x \in X, \quad \lambda(x) \geq \mu(x)$.

Thus, clearly the set $(\mathcal{F}(\mathcal{X}), \leq)$ is a partially ordered set. This means that the relation $\leq$ defined on $\mathcal{F}(\mathcal{X})$ is reflexive, anti-symmetric and transitive since the ordering is the point-wise usual ordering of real numbers. Thus we may view the partial orders containment and dominance as dual to each other on the set of all fuzzy subsets.

Let μ and λ be two fuzzy subsets of a set X. Then μ is said to be *equal* to λ ($\mu = \lambda$) if and only if $\mu \leq \lambda$ and $\lambda \leq \mu$.
We say that λ is strictly contained in μ written as $\lambda < \mu$ in the sense that there is at least one $x \in X$ for which $\lambda(x) < \mu(x)$.

2°. Intersection.

Let μ and λ be two fuzzy subsets of a set X. The *intersection* of μ and λ is a fuzzy subset γ defined as:
$\gamma(x) = (\mu \cap \lambda)(x) = \min(\mu(x), \lambda(x)) = \mu(x) \wedge \lambda(x)\ \forall\ x$ in X. The fuzzy subset γ is sometimes denoted by $\mu \wedge \lambda$.
Two fuzzy subsets μ and λ are said to be *disjoint* if and only if $(\mu \wedge \lambda)(x) = 0$ for all $x \in X$. That means wherever $\lambda(x) \neq 0$, then $\mu(x) = 0$ and whenever $\mu(x) \neq 0$, then $\lambda(x) = 0$. The intersection of μ and λ is the " largest" fuzzy subset which is contained in both μ and λ.

3°. Union.

The *union* of μ and λ is a fuzzy subset γ defined as: $\gamma(x) = (\mu \cup \lambda)(x) = \max\ (\mu(x), \lambda(x)) = \mu(x) \vee \lambda(x)\ \forall\ x$ in X. The fuzzy subset union γ can be denoted as $\mu \vee \lambda$. The union of μ and λ is the smallest fuzzy subset that contains both μ and λ.

4°. Complementation.

Let μ and λ be two fuzzy subsets of a set X. The two fuzzy subsets are said to be complementary if $\forall x \in X; \lambda(x) = 1 - \mu(x)$ or $\mu(x) = 1 - \lambda(x)$. In case λ is complement of μ, we write $\lambda = \overline{\mu}$.

Remark 2.1.3
1. A fuzzy subset μ has a unique complement $\overline{\mu}$ by definition of the complement of a fuzzy subset. In fact if we assume that λ and η were two complements of μ, then by definition of complement, $\lambda(x) = 1 - \mu(x)$ and $\eta(x) = 1 - \mu(x)$. This says $\lambda(x) = \eta(x)$ and means that μ has only one complement.
2. In general the intersection of a fuzzy subset and its complement is not the empty fuzzy subset. Suppose μ is a fuzzy subset which is not a crisp subset. Let $\overline{\mu}$ be its complement. Now assume $\forall x \in X, min(\mu(x), \overline{\mu}(x)) = min(\mu(x), (1 - \mu(x)) = 0$. Then for every x either $\mu(x) = 0$ or $1 - \mu(x) = 0$. But since μ is not a crisp subset, there is at least one $x \in X$ such that $0 < \mu(x) < 1$. This implies $1 - \mu(x)$ is also strictly between 1 and 0. This is a contradiction.
So we can conclude the intersection of a fuzzy subset and its complement is the empty fuzzy set if and only if the fuzzy subset is a crisp subset.

The following is an example of a fuzzy set whose intersection with its complement is not empty fuzzy set.

Example 2.1.4 .
Consider a fuzzy subset μ of a finite set $X = \{x_1; x_2, x_3, x_4, x_5, x_6\}$ such that:
$\mu = \{(x_1/0.13); (x_2/0.61); (x_3/0); (x_4/0); (x_5/1); (x_6/0.03)\}$ and $\overline{\mu} = \{x_1/0.87); (x_2/0, 39); (x_3/1); (x_4/$
Then $\mu \cap \overline{\mu} = \{(x_1/0.13); (x_2/0.39); (x_3/0); (x_4/0); (x_5/0); (x_6/0.03)\}$ which is not empty.

Note 2.1.5
1. The complementation used for fuzzy subsets is not the same as the complementation of Boolean lattice. The only time these two coincide is when the membership set $M = \{0, 1\}$.
2.The lattice $(\mathcal{F}(\mathcal{X}), \leq)$ is distributive but not complementary,that is, a vector lattice and not a Boolean lattice.
3. We observe that the union of a fuzzy subset and its complement is not the universal set X.

Example 2.1.6 .
Consider again the $\mu = \{(x_1/0.13); (x_2/0.61); (x_3/0); (x_4/0); (x_5/1); (x_6/0.03)\}$

and $\overline{\mu} = \{(x_1/0.87); (x_2/0,39); (x_3/1); (x_4/1); (x_5/0); (x_6/0.97)\}$ as in the example above.
$\mu \cup \overline{\mu} = \{(x_1/.87); (x_2/0.61); (x_3/1); (x_4/1); (x_5/1); (x_6/0.97)\}$ which does not represent X. In addition $\forall \mu, \lambda \in (\mathcal{F(X)}, \leq)$, $\mu \wedge \lambda$; $\mu \vee \lambda$ exist in $(F(X), \leq)$ as we have defined above. Therefore $(\mathcal{F(X)}, \leq)$ is a lattice.
Furthermore this lattice is bounded because $\forall \mu \in (\mathcal{F(X)}, \leq)$, $\exists \lambda$ such that $\lambda \leq \mu$. This λ is defined as the empty fuzzy subset of X such that $\chi_{\emptyset}(x) = 0 \ \forall x \in X$ and $\forall \mu \in (\mathcal{F(X)}, \leq)$, $\exists \lambda'$ such that $\mu \leq \lambda'$. This λ' is the fuzzy subset χ_X.

5°. Difference.

Let μ and λ be two fuzzy subsets of a set X, with $\overline{\mu}$ the complement of μ. The fuzzy subset difference of λ and μ, noted $(\lambda - \mu)$ is defined as $\lambda \cap \overline{\mu}$.

Note 2.1.7

Throughout the remaining part of this thesis we will represent a fuzzy subset $\mu = \{(x_1, \mu(x_1), (x_2, \mu(x_2)), \cdots (x_n, \mu(x_n)\}$ of X simply by writing their respective membership values with a particular ordering of the elements of the set X as $\mu(x_1)\mu(x_2)\cdots\mu(x_n)$. This is done in order to simplify the identification of fuzzy subsets. For instance $\frac{1}{2}01$ means the fuzzy subset

$$\mu(x_i) = \begin{cases} \frac{1}{2}, & \text{if i = 1} \\ 0, & \text{if i = 2} \\ 1, & \text{1 if i = 3} \end{cases}$$

for all $x_i \in X$. [20]
The above is a simplification of Kaufmann's notation $\{(x_1|\frac{1}{2}); (x_2|0); (x_3|1)\}$.

2.1.4 Cardinality of a fuzzy subset

Let X be a set. The cardinality of a subset $A \subseteq X$ is the number of elements of X contained in A. The cardinality of a crisp set is a natural number except when the set is empty in which case we assign zero. That is the cardinality of the empty set is zero.
Let X be a n-elements set and let M be a subset of $[0, 1]$ to be defined suitably

in the context of discussions, for instance see chapter 4 or 5 or in this chapter section 2. We define below the cardinality of a fuzzy set. It was first defined by Kaufmann in [20].
Many other definitions of cardinality have emerged. In this thesis we will use this one suggested by Kaufmann.

Definition 2.1.8 Suppose μ is a fuzzy subset of X. Then the *cardinality* of the fuzzy subset μ of X, denoted $\mid \mu \mid$ is defined as $\displaystyle\sum_{1}^{n} \mu(x_i) \; \forall \; x_i \in X$.
We normally restrict the membership values of μ to the subset M of I.

This number is not necessarily a natural number. It is a sum of some real numbers in the interval $[0, 1]$. In practice we only allow membership values to be rational numbers or more useful numbers such as $\frac{1}{n}$ or $\frac{m}{n}$ for $1 \leq m \leq n$. Therefore the cardinality for all practical purposes is a manageable rational number being the sum of a finite number of rational numbers.
The cardinality of a finite fuzzy subset is finite. Generally the cardinality of a fuzzy set of an infinite set is infinite according to our definition above. But by assigning suitable membership values to certain elements of an infinite fuzzy subset, we can make the cardinality of an infinite fuzzy subset to be finite. The cardinality of a crisp subset A of X coincides with the concept of the cardinality of a fuzzy subset when we assign to each element of A the membership value 1 and assign 0 to others elements of X not in A.

Example 2.1.9 .
$\mu : \{(x_1; 0.8), (x_2; 0.7), (x_3; 0.5), x_4; 1)\}$

$$\mid \mu \mid = \sum_{1}^{4} \mu(x_i) = 0.8 + 0.7 + 0.5 + 1 = 3$$

Example 2.1.10 .
Consider the set $\mathbb{N} = \{1, 2, 3, \cdots\}$ of natural numbers. If we assign to the first 2000 members of the set the membership $\frac{1}{2}$ and assign 0 as membership value to the remaining elements of the set, then the fuzzy subset obtained in this fashion has cardinality of 1000, which is finite.
On the other hand the cardinality of the fuzzy set $\mu : \mathbb{N} \longrightarrow [0, 1]$ with the

assignment $\mu(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$ is infinite since it is equal to $\sum_{1}^{\infty} \frac{1}{n}$.

Proposition 2.1.11 Let μ_1 and μ_2 be two fuzzy subsets of a set $X = \{x_1, x_2, \cdots, x_n\}$ with cardinalities $|\mu_1|$ and $|\mu_2|$ respectively. The cardinality of the fuzzy subset union of μ_1 and μ_2 is obtained as follows:

$$|\mu_1 \vee \mu_2| \;=\; |\mu_1| + |\mu_2| \;-\; |\mu_1 \wedge \mu_2|$$

Proof: It is clear that for any two real numbers a and b such that $0 \leq a \leq 1$ and $0 \leq b \leq 1$ we have $a + b = \min(a, b) + \max(a, b)$. Based on this equality for each $i = 1, 2, \cdots, n,$ we have

$$|\mu_1(x_i)| + |\mu_2(x_i)| = \min(|\mu_1(x_i)|, |\mu_2(x_i)|) + \max(|\mu_1(x_i)|, |\mu_2(x_i)|)$$

Summing up the value of each term on either side of the above equality from $i = 1$ through to $i = n$, we get

$$|\mu_1| + |\mu_2| \;=\; \min(|\mu_1|, |\mu_2|) \;+\; \max(|\mu_1|, |\mu_2|)$$

from which we get the required equality of the proposition. $\square$.

2.1.5 The Hamming distance

The notion of distance is quite an interesting one. We wish to define it with regards to the notion of fuzzy subsets. Later on we will attempt to explore the idea of the counting of fuzzy subsets of a finite set X, at a distance from a fixed one.

The distance "d" between two fuzzy subsets μ and λ of a set X can be expressed in two different ways: A linear distance, also called Generalized Hamming distance expressed as:

$d_1(\mu, \lambda) = \sum_{i=1}^{n} \mid \mu(x_i) - \lambda(x_i) \mid$

and a quadratic distance or Euclidean distance expressed as

$d_2(\mu, \lambda) = \sqrt{\sum_{i=1}^{n} (\mu(x_i) - \lambda(x_i))^2}$ [20].

The expression $\sum_{i=1}^{n}(\mu(x_i) - \lambda(x_i))^2$ is called the euclidean norm.

One can verify that these two definitions of distance satisfy the necessary conditions of distance. That is to say that for any three fuzzy subsets μ, λ and α of X:
1.$d(\mu, \lambda) \geq 0$
2.$d(\mu, \lambda) = d(\lambda, \mu)$
3.$d(\mu, \alpha) \leq d(\mu, \lambda) + d(\lambda, \alpha)$
In addition to these three conditions, $0 \leq d_1(\mu, \lambda) \leq n$ for any two fuzzy subsets of a n-element set X.
In fact $0 \leq d(\mu, \lambda)$ by definition of distance. The second inequality is justified by definition of $Card_\mu$ dealt with later in Chap 2.

Example 2.1.12 . Consider $\mu = \{(x_1/0.87); (x_2/0.39); (x_3/1); (x_4/1); (x_5/0); (x_6/0.97)\}$ and $\lambda = \{(x_1/0.2); (x_2/0); (x_3/0); (x_4/0.6); (x_5/0.8); (x'_6 1)\}$. Then $d(\mu, \lambda) = \mid 0.87 - 0.2 \mid + \mid 0.39 - 0 \mid + \mid 1 - 0 \mid + \mid 1 - 0.6 \mid + \mid 0 - 0.8 \mid + \mid 0.97 - 1 \mid = 0.67 + 0.39 + 1 + 0.4 + 0.8 + 0.03 = 3.29$
To end this section we wish to ask a question which a search in google revealed that few people are interested in the counting of fuzzy subsets of a finite set at a distance from fixed one.
Let μ be a fuzzy subset a finite set X. How many fuzzy subsets are at a distance d from a fixed fuzzy subset μ ?
We attempt answering the question by first considering the case of two fuzzy subsets λ and γ, both at the distance d from μ. How are λ and γ related ?
Using linear distance formula we can write that:
$d_1(\mu, \lambda) = \text{d} = d_1(\mu, \gamma)$ and that $d_1(\lambda, \gamma) \leq d_1(\lambda, \mu) + d_1(\mu, \gamma)$ or $d_1(\lambda, \gamma) \leq 2d_1(\mu, \gamma)$. This means that γ is such that $d_1(\lambda, \gamma) \leq 2d_1(\mu, \lambda)$.

2.1.6 The α- cut

The α - cut or also called α- level set of a fuzzy subset μ of a set X, is a crisp subset of X denoted by μ^α where $\mu^\alpha = \{x \in X/ \ \mu(x) \geq \alpha\} \forall \alpha \ \in [0, 1]$.
μ^α can also be defined as $\mu^{-1}([\alpha, 1])$ [49]. It is the weak α-cut.
The complementary of μ^α denoted here as μ_α can be defined as $\{x \in X/ \ \mu(x) < \alpha\} \forall \alpha \ \in [0, 1]$.

After definition and discussion on operations concerning fuzzy subsets, we wish to find out how the PIE can be applied in the set of fuzzy subsets of a finite set. As it is the case for crisp sets, we need to set properties in the set of fuzzy subsets. Later we count the subsets of X which satisfy or not satisfy these properties. In this process we will be able to count elements of the set X whose membership values satisfy certain conditions.

Let $\mathcal{F}(\mathcal{X})$ be the set of all possible fuzzy subsets of a finite set X with membership in a finite set M. These fuzzy subsets have many interesting properties or characteristics. In this study we capture some of these properties or patterns which are common to many of these fuzzy subsets as well as those properties common to the elements of the set X. Later we count the fuzzy subsets and elements of X with common properties. We sometimes come across elements of $\mathcal{F}(\mathcal{X})$ or elements of X falling neither in this category nor in the other. These are outliers; they seem not have common properties. Most of these can be found in [45]. We have improved some in this project.
In this chapter we do some 19 different enumerations. These enumerations include fuzzy subsets with regard to elements of set X or we count elements of X in relation to their membership values to the fuzzy subsets of set X.
Because these properties describe either elements or fuzzy subsets, we will use them in the future chapter when we draw a list of attributes in our fuzzy pattern recognition. In the next paragraph we count elements of X having a membership degree α to n fuzzy subsets of X. This, in fact, expresses the number of elements in a subset of the α-cut of the intersection of n fuzzy subsets of the finite set X.

2.2 Enumeration of elements of the set X with regard to their membership values.

2.2.1 Elements of a set belonging to n fuzzy subsets to a degree at least α.

Let X be a finite set and we consider n fuzzy subsets $\mu_1, \mu_2, \cdots, \mu_n$ of X. Let us also consider one specific membership value $\alpha \in [0, 1]$.
We wish to enumerate the elements of X with regards to their membership

value $\alpha \in [0,1]$ to the n fuzzy subsets. The number of elements of X belonging to each one of the n fuzzy subsets of X at a degree of membership at least $\alpha \in [0,1]$ is given by the following propositions.
We first study the case of two fuzzy subsets μ_1 and μ_2. Later on we will generalize the concept to n distinct fuzzy subsets.

Theorem 2.2.1 If μ_1 and μ_2 are two fuzzy subsets of a set X, then
(1): $(\mu_1 \wedge \mu_2)^\alpha = \mu_1^\alpha \cap \mu_2^\alpha$ and
(2): $(\mu_1 \vee \mu_2)^\alpha = \mu_1^\alpha \cup \mu_2^\alpha$

Proof : Let $A = (\mu_1 \wedge \mu_2)^\alpha$ and $B = \mu_1^\alpha \cap \mu_2^\alpha$ be two subsets of X. We show that $A = B$ by showing $A \subseteq B$ and $B \subseteq A$.
Let $a \in A$, that is $a \in (\mu_1 \wedge \mu_2)^\alpha$, then $(\mu_1 \wedge \mu_2)(a) \geq \alpha$. That means $\mu_1(a) \wedge \mu_2(a) \geq \alpha$. It follows that both $\mu_1(a) \geq \alpha$ and $\mu_2(a) \geq \alpha$. In other words it means $a \in \mu_1^\alpha$ and $a \in \mu_2^\alpha$. That is to say $a \in \mu_1^\alpha \cap \mu_2^\alpha = B$, therefore $A \subseteq B$.
Conversely let $x \in B = \mu_1^\alpha \cap \mu_2^\alpha$; then $x \in \mu_1^\alpha$ and $x \in \mu_2^\alpha$. That is to say that $\mu_1^\alpha(x) \geq \alpha$ and $\mu_2^\alpha(x) \geq \alpha$ or that $\mu_1(x) \wedge \mu_2(x) \geq \alpha$ which also means $(\mu_1 \wedge \mu_2)(x) \geq \alpha$. This expresses the fact that $x \in (\mu_1 \wedge \mu_2)^\alpha = A$ and that $B \subseteq A$. We conclude therefore that $A = B$. That is $(\mu_1 \wedge \mu_2)^\alpha = \mu_1^\alpha \cap \mu_2^\alpha$
Similarly it is clear to prove that $(\mu_1 \vee \mu_2)^\alpha = \mu_1^\alpha \cup \mu_2^\alpha$.□.
Generally we have the following statements:

Proposition 2.2.2 Suppose μ_i for i equals to $1, 2, \cdots, n$ are fuzzy subsets of X and α is a value in $[0,1]$. Then

$$| (\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n)^\alpha | = | \mu_1^\alpha \cap \mu_2^\alpha \cap \cdots \cap \mu_n^\alpha | = \sum_{i=1}^{n} | \mu_i^\alpha | - \sum_{1 \leq i \leq j \leq n} | \mu_i^\alpha \cup \mu_j^\alpha | + \sum_{1 \leq i \leq j \leq k \leq n} | \mu_i^\alpha \cup \mu_j^\alpha \cup \mu_k^\alpha | + \cdots + (-1)^{n-1} | \mu_1^\alpha \cup \mu_2^\alpha \cup \cdots \cup \mu_n^\alpha |$$

Proof : The α-cut of the fuzzy intersection of fuzzy subsets is the crisp intersection of their α-cuts. Again the α-cut of the fuzzy union of fuzzy subsets is the crisp union of their α-cuts. These are subsets of X. Then using the result of theorem 1.2 in the context of the set X and its subsets, the α-cuts help us complete the proof of the proposition.□
The above proposition and its dual express the way of counting elements of X

in the α-cut of the intersection and the α-cut of the union of n fuzzy subsets of X. These elements $x \in X$ are such that $[\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n](x) \geq \alpha$ and $[\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n](x) \geq \alpha$ for an $\alpha \in [0, 1]$.

2.2.2 Elements of a set in the α-cut of union of fuzzy subsets [45]

In this subsection we are talking about elements of X in the α-cut of the union $\bigcup_{i=1}^{n} \mu_i$. We express the number of such elements of the set X as in the following proposition which is the dual of the above proposition 2.2.1.

Proposition 2.2.3 The number of elements of X that belong to at least one of the fuzzy subsets $\mu_1, \mu_2, \cdots, \mu_n$ of X to a degree at least α is

$$| (\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n^\alpha) | = \sum_{i=1}^{n} | \mu_i^\alpha | - \sum_{1 \leq i \leq j \leq n} | \mu_i^\alpha \cap \mu_j^\alpha | + \sum_{1 \leq i \leq j \leq k \leq n} | \mu_i^\alpha \cap \mu_j^\alpha \cap \mu_k^\alpha | + \cdots + (-1)^{n-1} | \mu_1^\alpha \cap \mu_2^\alpha \cap \cdots \cap \mu_n^\alpha | \text{ [26]}$$

Now if $x \in \mu^\alpha$, then $x \in \mu^0$ and therefore $x \in \bigcup_{i=1}^{n} \mu_i^0$.

We note that the above formula can be rewritten in terms of α-cuts of union of fuzzy subsets as

$$| (\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n)^\alpha | = | \mu_1^\alpha \cap \mu_2^\alpha \cap \cdots \cap \mu_n^\alpha | = \sum_{i=1}^{n} | \mu_i^\alpha | - \sum_{1 \leq i \leq j \leq n} | (\mu_i \vee \mu_j)^\alpha | + \sum_{1 \leq i \leq j \leq k \leq n} | (\mu_i \vee \mu_j \vee \mu_k)^\alpha | + \cdots + (-1)^{n-1} | (\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n)^\alpha | \quad (2.2.1)$$

and

$$| (\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n)^\alpha | = | \mu_1^\alpha \cup \mu_2^\alpha \cup \cdots \cup \mu_n^\alpha | = \sum_{i=1}^{n} | \mu_i^\alpha | - \sum_{1 \leq i \leq j \leq n} | (\mu_i \wedge \mu_j)^\alpha | + \sum_{1 \leq i \leq j \leq k \leq n} | (\mu_i \wedge \mu_j \wedge \mu_k)^\alpha | + \cdots + (-1)^{n-1} | (\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n)^\alpha | . \quad (2.2.2)$$

2.2.3 Elements of a set in the intersection of the α-cuts of fuzzy subsets.

Now we consider n fuzzy subsets of X and an equal number n of real numbers in $[0, 1]$.
Here we enumerate elements of X using n fuzzy subsets and n membership values in $[0, 1]$. Consider n fuzzy subsets $\mu_1\, \mu_2, \cdots \mu_n$ of X and $\alpha_1,\ \alpha_2,\ \cdots \alpha_n \in I$. We intend to count the elements of X which are simultaneously such that $\mu_1(x) \geq \alpha_1, \quad \mu_2(x) \geq \alpha_2, \cdots, \mu_n(x) \geq \alpha_n$. In other words we count the elements of X such that $x \in \mu_1^{\alpha_1}$, $x \in \mu_2^{\alpha_2}, \cdots x \in \mu_n^{\alpha_n}$. Then the subset of such elements of X is given by $\mu_1^{\alpha_1} \cap \mu_2^{\alpha_2} \cap \cdots \cap \mu_n^{\alpha_n}$. Their number is denoted by $\mid \cap_{i=1}^{n} \mu_i^{\alpha_i} \mid$ and can be expressed using the PIE as in the following proposition.

Proposition 2.2.4 $\mid \cap_{i=1}^{n} \mu_i^{\alpha_i} \mid = \sum_{i=1}^{n} \mid \mu_i^{\alpha_i} \mid - \sum_{1 \leq i \leq j \leq n} \mid \mu_i^{\alpha_i} \cup \mu_j^{\alpha_j} \mid + \cdots + (-1)^n \mid \mu_1^{\alpha_1} \cup \cdots \cup \mu_n^{\alpha_n} \mid$.
And dually we write:

$$\mid \cup_{i=1}^{n} \mu_i^{\alpha_i} \mid = \sum_{i=1}^{n} \mid \mu_i^{\alpha_i} \mid - \sum_{1 \leq i \leq j \leq n} \mid \mu_i^{\alpha_i} \cap \mu_j^{\alpha_j} \mid + \cdots + (-1)^n \mid \mu_1^{\alpha_1} \cap \cdots \cap \mu_n^{\alpha_n} \mid.$$

This proposition follows proposition 2.2.1. The proof is similar.
Assume now that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Then we get $\mid \cap_{i=1}^{n} \mu_i^{\alpha_i} \mid = \mu_n^{\alpha_n}$ and on the other hand if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, we obtain that $\mid \cup_{i=1}^{n} \mu_i^{\alpha_i} \mid = \mu_1^{\alpha_1}$.

2.2.4 Elements of a set at minimum and maximum membership value.

We consider in the following subsections two types of enumerations. In the first case we take n fuzzy subsets of a set X and enumerate elements of X having a specified minimum degree of membership say α and with a specified maximum degree of membership say β. Later we consider n fuzzy subsets and two special fixed membership values, namely 0 and 1. Here we count elements of the finite set X having membership value 1 as well as elements of X having membership values greater than 0.
Consider n fuzzy subsets and two specified membership values.

We enumerate elements of X that belong to fuzzy subsets of X with minimum degree of membership α and a maximum degree membership β. Obviously α has to be smaller than or equal to β, that is $\alpha \leq \beta$. Suppose $\mu \in \mathcal{F}(\mathcal{X})$ is a given fuzzy subset of X. We want to enumerate the elements of X which satisfy the conditions $\mu(x) \geq \alpha$ and $\mu(x) < \beta$ simultaneously. That means the elements $x \in X$ must satisfy $\alpha \leq \mu(x) < \beta$. We introduce the following simple and natural notation which is very useful later.

$$\mu^{\alpha\beta} = \{x \in X : \alpha \leq \mu(x) < \beta\}.$$

Therefore there are two properties that these elements of X must have.
The two properties are:
1. Having $\mu(x) \geq \alpha$,
2. Having $\mu(x) < \beta$.
Consider a fuzzy subset μ of X. The number of elements of X belonging to μ at a minimum degree α and at a maximum degree β is $\mid \mu^{\alpha} \setminus \mu^{\beta} \mid$. It is clear that these elements are in μ^{α} since $\mu(x) \geq \alpha$. They do not belong to μ^{β} since $\mu(x) < \beta$. In brief, they are in $\mu^{\alpha} \setminus \mu^{\beta}$. Therefore their number is $\mid \mu^{\alpha\beta} \mid = \mid \mu^{\alpha} \setminus \mu^{\beta} \mid$.
It is necessary to mention that α is not equal to β. Otherwise if $\alpha = \beta$, then the number of elements sought would simply be $\mid \mu^{-1}(\alpha) \mid$.
Let us introduce another useful notation $\mid \mu^{\alpha} \mid$ and $\mid \mu^{\alpha'} \mid$ as the number of element in μ^{α} and the number of elements not in μ^{α}, respectively. $\mid \mu^{\alpha'} \cap \mu^{\beta'} \mid$ is the number of elements neither in μ^{α} nor in μ^{β}.
We recall that μ^{α} and $\mu^{\alpha'}$ are crisp complementary subsets of X. We can use the PIE on the set X to enumerate the elements of minimum and maximum membership values as in the following statement.

Lemma 2.2.5 . Let μ be a fuzzy subset of X with $\mid \mu^{\alpha} \mid$ and $\mid \mu^{\alpha'} \mid$ as defined above. The number of elements of X such that $\alpha \leq \mu(x) < \beta$ can be obtained by solving

$$|\mu^{\alpha\beta}| = \mid X \mid - (\mid \mu^{\alpha'} \mid + \mid \mu^{\beta} \mid) + \mid \mu^{\alpha'} \cap \mu^{\beta} \mid \tag{2.2.3}$$

where $\mu^{\alpha\beta}$ denotes the set of elements such that $\alpha \leq \mu(x) < \beta$.

Proof : By using PIE in X to enumerate elements enjoying two properties, we subtract from $\mid X \mid$ the number of those elements x without one property at the time, then add those without both properties simultaneously. We have this result:

$$|\mu^{\alpha\beta}| = | X | - (| \mu^{\alpha'} | + | \mu^{\beta} |) + | \mu^{\alpha'} \cap \mu^{\beta} |$$

Now since $\alpha \leq \beta$, $\mu^{\beta} \subseteq \mu^{\alpha}$ and $\mu^{\alpha} \cap \mu^{\alpha'} = \emptyset$ are all true, then $\mu^{\beta} \cap \mu^{\alpha'} = \emptyset$. Therefore $| \mu^{\alpha'} \cap \mu^{\beta} | = 0$. This says that the number $| X | - (| \mu^{\alpha'} | + | \mu^{\beta} |)$ is equal to $| \mu^{\alpha} \setminus \mu^{\beta} | = |\mu^{\alpha\beta}|$. This completes the proof.$\square$.
By using the PIE we are also able to determine the elements of X in the complement of $\mu^{\alpha\beta}$.

Example 2.2.6 .
Refer to the diagram attached to the thesis.
For $X = \{x_1, x_2, x_3\}$; $n = 3$ and $M = \{0, \frac{1}{2}, 1\}$, $\mu_i = 1\frac{1}{2}\frac{1}{2}$ Consider $\alpha = \frac{1}{2}$ and $\beta = 1$.
We have $| \mu_i^{\alpha} | = 3$; therefore $| \mu_i^{\alpha'} | = 0$; while $| \mu^{\beta} | = 1$
Therefore $3 - [0 + 1] + 0 = 2$.
Now assume that $\mu_1 , \mu_2 , \cdots , \mu_n$ are n given fuzzy subsets of X such that $\forall i$, $\alpha \leq \mu_i(x) < \beta$. We are now interested in enumerating elements of X which are such that $\alpha \leq \cap_{i=1}^{n} \mu_i(x) < \beta$. Similarly it is interesting to enumerate the elements of X such that $\alpha \leq \cup_{i=1}^{n} \mu_i(x) < \beta$. First we take up the case of intersection. Before we state the proposition, it is useful to observe that $\{x \in X : \alpha \leq \cap_{i=1}^{n} \mu_i(x) \leq \beta\} = (\wedge_{i=1}^{n} \mu_i)^{\alpha\beta}$.

Proposition 2.2.7 Let $\mu_1 , \mu_2 , \cdots , \mu_n$ be fuzzy subsets of X and $\forall i$, $\alpha \leq \mu_i(x) < \beta$. Then the number of elements of X such that $\alpha \leq \cap_{i=1}^{n} \mu_i(x) < \beta$ is

$$(\wedge_{i=1}^{n} \mu_i)^{\alpha\beta} = | X | - (| \cap_{i=1}^{n} \mu^{\alpha'} | + | \cap_{i=1}^{n} \mu^{\beta} |), \tag{2.2.4}$$

which can be expressed using PIE as

$$|(\wedge_{i=1}^{n} \mu_i)^{\alpha\beta}| = |X| - \sum_{i=1}^{n} (|\mu_i^{\alpha'}| + |\mu_i^{\beta}|) + \sum_{1 \leq i \leq j \leq n} (|(\mu_i^{\alpha'} \cup \mu_j^{\alpha'}| + |\mu_i^{\beta} \cup \mu_j^{\beta}|) + \cdots$$

Proof: Set $\gamma = \cap_{i=1}^{n} \mu_i$. Then this takes us back to the case of a given $\mu = \gamma$ dealt with earlier in Lemma 2.2.5. Generally we can use Proposition 2.2.1 to expand $| \cap_{i=1}^{n} \mu^{\alpha'} |$ and $| \cap_{i=1}^{n} \mu^{\beta} |$ as follows:

$$| \cap_{i=1}^{n} \mu^{\alpha'} | = \sum_{i=1}^{n} | \mu_i^{\alpha'} | - \sum_{1 \leq i \leq j \leq n} | \mu_i^{\alpha'} \cup \mu_j^{\alpha'} | + \sum_{1 \leq i \leq j \leq k \leq n} | \mu_i^{\alpha'} \cup \mu_j^{\alpha'} \cup \mu_k^{\alpha'} | +$$

$\cdots + (-1)^{n-1} \mid \mu_1^{\alpha'} \cup \mu_2^{\alpha'} \cup \cdots \cup \mu_n^{\alpha'} \mid$
and $\mid \cap_{i=1}^{n} \mu^{\beta} \mid = \sum_{i=1}^{n} \mid \mu_i^{\beta} \mid - \sum_{1 \le i \le j \le n} \mid \mu_i^{\beta} \cup \mu_j^{\beta} \mid + \sum_{1 \le i \le j \le k \le n} \mid \mu_i^{\beta} \cup \mu_j^{\beta} \cup \mu_k^{\beta} \mid +$
$\cdots + (-1)^{n-1} \mid \mu_1^{\alpha} \cup \mu_2^{\alpha} \cup \cdots \cup \mu_n^{\beta} \mid$
Now collecting and regrouping the terms carefully of the α-cuts and β-cuts of each of n fuzzy subsets corresponding to indices, we can rewrite the sum in 2.2.4 as:
$\mid X \mid -(\mid \cap_{i=1}^{n} \mu^{\alpha'} \mid + \mid \cap_{i=1}^{n} \mu^{\beta}) \mid =$
$|X| - \sum_{i=1}^{n} (\mid \mu_i^{\alpha'} \mid + \mid \mu_i^{\beta} \mid) + \sum_{1 \le i \le j \le n} (\mid \mu_i^{\alpha'} \cup \mu_j^{\alpha'} \mid + \mid \mu_i^{\beta} \cup \mu_j^{\beta} \mid) + \cdots$ This completes the proof.

Similarly we can express $\eta = \cup_{i=1}^{n} \mu_i$ in terms of α-cuts of individual fuzzy subsets and their intersections using PIE.

If we consider the set of all m^n fuzzy subsets of X then $\gamma = \cap_{i=1}^{n} \mu_i$ would be the fuzzy subset $\emptyset$ which takes the membership value zero for all $x \in X$. Hence if $\alpha = 0$ and $\beta > 1$ then every $x \in X$ satisfies the property $x \in (\gamma)^{\alpha\beta}$. Hence $\mid (\gamma)^{\alpha\beta} \mid = \mid X \mid$. With the same consideration $\eta = \cup_{i=1}^{n} \mu_i$ would be the fuzzy subset X and $\mid X \mid$ would be the number of elements sought provided $\beta = 1$ and $\alpha < 1$.
Suppose we have k subsets $[\alpha_1, \beta_1], [\alpha_2, \beta_2], \cdots [\alpha_k, \beta_k]$ of $[0, 1]$. Suppose also that these intervals do not intersect. We can generalize 4.2.3 to enumerate elements of X that are such that $\forall j,\ \alpha_j \le \cap_{i=1}^{n} \mu_i(x) \le \beta_j$ as follows:

$$|X| - \sum_{i=1}^{n} (\sum_{j=1}^{k} (\mid \mu_i^{\alpha_j'} \mid + \mid \mu_i^{\beta_j} \mid)) + \cdots$$

Illustration
Refer to the diagram attached to the thesis.
Consider $X = \{x_1, x_2, x_3\}$ $\alpha = \frac{1}{2}$, $\mu_1 = 1\frac{1}{2}\frac{1}{2}$, $\mu_2 = \frac{1}{2}01$, $\mu_3 = 0\frac{1}{2}\frac{1}{2}$
$\mid \mu_1^{\alpha} \cup \mu_2^{\alpha} \cup \mu_3^{\alpha} \mid = \{\mid \mu_1^{\alpha} \mid + \mid \mu_2^{\alpha} \mid + \mid \mu_3^{\alpha} \mid\} - \{\mid \mu_1^{\alpha} \cap \mu_2^{\alpha} \mid + \mid \mu_1^{\alpha} \cap \mu_3^{\alpha} \mid + \mid \mu_2^{\alpha} \cap \mu_3^{\alpha} \mid\} + \mid \mu_1^{\alpha} \cap \mu_2^{\alpha} \cap \mu_3^{\alpha} \mid$.
$= \{3 + 2 + 2\} - \{2 + 2 + 1\} + 1$
$7 - 5 + 1 = 3$
The three elements of X are either in μ_1^{α} , in μ_2^{α} or even in μ_3^{α}.

Again concerning union we can write

Proposition 2.2.8 Let $\mu_1, \mu_2, \cdots, \mu_n$ be fuzzy subsets of X and $\forall i,\ \alpha \leq \mu_i(x) < \beta$. Then the number of elements of X such that $\alpha \leq \cup_{i=1}^{n}\mu_i(x) < \beta$ is

$$(\vee_{i=1}^{n}\mu_i(x))_{\alpha}^{\beta} = \mid X \mid - (\mid \cup_{i=1}^{n}\mu^{\alpha'} \mid + \mid \cup_{i=1}^{n}\mu^{\beta} \mid), \qquad (2.2.5)$$

which can be expressed using PIE as

$$|(\vee_{i=1}^{n}\mu_i)_{\alpha}^{\beta}| = |X| - \sum_{i=1}^{n}(|\mu_i^{\alpha'}| + |\mu_i^{\beta}|) + \sum_{1\leq i\leq j\leq n}(|(\mu_i^{\alpha'} \cap \mu_j^{\alpha'}| + |\mu_i^{\beta} \cap \mu_j^{\beta}|) + \cdots.$$

Definition. The *interval*-cut of a fuzzy subset.

Concerning the two values a and b in the unit interval I such that $a \leq b$, we can obtain different intervals such as $[a, b]$, $[a, b)$, $(a, b]$, (a, b) and from there determine elements of set X whose membership values fall in each of these intervals in I.

Let us consider for argument sake the interval $[a, b] \subseteq I$, the elements of X whose membership values for a fuzzy subset μ of X are in $[a, b]$ are described by $\{x \in X, a \leq \mu(x) \leq b\}$. This interval can be considered as an α-cut of μ, which we will call here as the *interval*-cut of μ.

Using this interval we can define another fuzzy subset of X in the following way:

$$\mu^{[a,b]}(x) = \begin{cases} 1, & \text{if } a \leq \mu(x) \leq b \\ 0, & \text{if } otherwise. \end{cases}$$

The fuzzy subset obtained in this manner would be called the fuzzy subset *induced* by both μ and the interval $[a, b]$.

For each fuzzy subset $\mu \in \mathcal{F(X)}$ and for the interval $[a, b]$, we have a fuzzy induced fuzzy subset.

In this manner we now know how many elements of X have their membership values to μ between a and b. This number is $\mid \mu^{[a,b]} \mid$, the cardinality of the induced fuzzy subset. The fuzzy subset induced and its cardinality give a partition of X for each chosen interval. We are now able to enumerate members of each family of the partition obtained.

Proposition 2.2.9 Consider the interval $[a, b] \subseteq I$. Let μ_1 and μ_2 be two fuzzy subsets of X. If $\mu_1 = \mu_2$, then $\mu_1^{[a,b]} = \mu_2^{[a,b]}$. The converse is not valid.

Example 2.2.10 . Consider $\mu_=111\frac{1}{2}$ and $\frac{1}{2}111$. Consider also $[a, b] = [\frac{1}{4}, 1]$. Then $\mu_1^{[a,b]} = 1111 = \mu_2^{[a,b]} = 1111$ but $\mu_1 \neq \mu_2$.

Proposition 2.2.11 If $\mu_1 \leq \mu_2$, then $\mu_1^{[a,b]} \leq \mu_2^{[a,b]}$.

In fact if $\mu_1^{[a,b]}(x) = 1$, then $a \leq \mu_1(x) \leq b$. But $\mu_1 \leq \mu_2$; so we have $a \leq \mu_1(x) \leq \mu_2(x) \leq b$ which means $\mu_2^{[a,b]}(x) = 1$. If on the other hand $\mu_1^{[a,b]}(x) = 0$, we have also $\mu_2^{[a,b]}(x) = 0$. To prove this we assume $\mu_1^{[a,b]}(x) = 0$. That is $\mu_1(x) \leq a$ and $\mu_1(x) \geq b$. But because $\mu_1 \leq \mu_2$, we have $\mu_2(x) \geq b$ and $\mu_2(x) \leq a$. If $\mu_2(x) \geq a$, then the interval $[a, b]$ would become $[b, 1]$.
Now we consider the union and intersection of the induced fuzzy sets. Consider $\mu_1^{[a,b]}$ and $\mu_2^{[a,b]}$. If $a \leq \mu_1(x) \leq b$, $(\mu_1 \cup \mu_2)^{[a,b]}(x) = 1$ and $(\mu_1 \cap \mu_2)^{[a,b]}(x) = 0$ if $\mu_2(x) \notin [a, b]$.

$$(\mu_1 \cup \mu_2)^{[a,b]}(x) = \begin{cases} 1, & \text{if either } a \leq \mu_1(x) \leq b \text{ or } a \leq \mu_2(x) \leq b \\ 0, & \text{if } \textit{otherwise} \text{ that is, neither } a \leq \mu_1(x) \leq b \text{ nor } a \leq \mu_2(x) \leq b. \end{cases}$$

and

$$(\mu_1 \cap \mu_2)^{[a,b]}(x) = \begin{cases} 1, & \text{if } a \leq \mu_1(x) \leq b \text{ and } a \leq \mu_2(x) \leq b \\ 0, & \text{if } \textit{otherwise}. \end{cases}$$

2.2.5 Elements with *no worth* to a fuzzy subset and Element with *absolute* worth.

Consider n fuzzy subsets of X and two special numbers $\alpha = 0$ and $\beta = 1$ in $[0, 1]$. We recall the definition of the *support*, as well as that of the *core*, of a fuzzy subset written as $supp\mu$ and $core\mu$ respectively.
$supp\mu = \{x \in X : \mu(x) > 0\}$ and $core\mu = \{x \in X : \mu(x) = 1\}$. Beside these definitions, we will use $|\mu_i|_s$ to represent the cardinality of the *support* of μ_i

and $|\mu_i|_c$ to represent the cardinality of the *core* of μ_i.
Clearly $core\mu \subseteq support\,\mu$.
Consider a set X as before and n fuzzy subsets of X. Each element of X in $supp\mu$ has a membership value which is a degree of desirability to the fuzzy subset. We also note that any element of X not in $supp\mu$ has membership value *zero*.
Now the number of elements of X with no *worth to* μ_i, that is elements with membership value *zero* to a fuzzy subset μ_i, denoted here by $|\overline{\mu}_i|_s$ is $|X| - |\mu_i|_s$.
Each element of X in $core\mu$ has absolute membership value which we call absolute desirability or absolute worth to the fuzzy subset.
If we consider the n fuzzy subsets, then the number of elements with *no worth* at all to any of the n fuzzy subsets is obtained as stated in this following proposition.

Proposition 2.2.12 $| \cap_{i=1}^{n} \overline{\mu}_i|_s = |X| - \sum_{i=1}^{n} |\mu_i|_s + \sum_{1 \leq i \leq j \leq n} |\mu_i \cup \mu_j|_s + \cdots + (-1)^n |\mu_1 \cup \mu_2 \cup \cdots \cup |\mu_n|_s$

Consider a set X, and elements $x, y \in X$ and two fuzzy subsets μ_1 and μ_2 of X. We state without proof the following proposition.

Proposition 2.2.13 If $x \in core\mu_1$ and $y \in core\mu_2$,
1. $\{x, y\} \subseteq support(\mu_1 \cup \mu_2)$.
2. $\{x, y\} \subseteq core\mu_1 \cup core\mu_2$.

The usefulness of counting the number of elements with *no worth* is that if a great number of elements of X are *worthless* with regards to one or many fuzzy subset, then we need to set up the fuzzy subsets differently.
As for the $core\mu$, we may denote by $|\overline{\mu}_i|_c$ the set of elements of X with *no absolute desirability* to the fuzzy subset μ.
Therefore the number of elements with *no absolute desirability* to the n fuzzy subsets is given by the proposition,

Proposition 2.2.14 $| \cap_{i=1}^{n} \overline{\mu}_i|_c = |X| - \sum_{i=1}^{n} |\mu_i|_c + \sum_{1 \leq i \leq j \leq n} |\mu_i \cup \mu_j|_c + \cdots + (-1)^n |\mu_1 \cup \mu_2 \cup \cdots \cup |\mu_n|_c$

In this case knowing the number of elements with *absolute desirability* to the n fuzzy subsets is necessary because if their number is $|X|$, then no good selection was done and thus new setting up of fuzzy conditions is required .

Consider the set $\mathcal{F}(\mathcal{X})$ of the possible fuzzy subsets of X. Given a number in $Card_\mu$; we want to find a way of counting the fuzzy subsets of X with a common pattern that is: Their cardinality is same.

2.2.6 Similar elements and Similar fuzzy subsets of X.

Consider a fuzzy subset μ of X and two elements x and y of the set X such that $\mu(x) = \mu(y)$.
The elements x and y in this case are said to be *similar* with respect to μ.
Let x and y be two similar elements of X. Then there exists a fuzzy subset μ of X such that $\mu(x) = \mu(y)$.
Again, if there exists μ such that $\mu(x) = \mu(y)$, then x and y are similar. We establish the following proposition to illustrate the above statement.

The similarity of elements of X with respect to fuzzy subsets of X is an equivalence in X. When two elements x and y are similar, we can say in other words that there exists $\alpha \in [0, 1]$ such that $\{x, y\} \subseteq \mu^\alpha$. This means that μ^α has a minimum of two members which are the two similar elements of X.

Now suppose there are two or more fuzzy subsets $\mu_1, \cdots, \mu_k$ such that $\mu_1(x) = \mu_1(y) \;\cdots, \mu_k(x) = \mu_k(y)$. This means on the one hand that x and y are similar with respect to these k fuzzy subsets. Therefore $\bigcap_{i=1}^{k} \mu_i(\alpha)$ has at least two elements. It also means on the other hand that the fuzzy subsets $\mu_1, \cdots, \mu_k$ are also similar to one another with respect to the two elements x and y.

We can show that the similarity of elements is an equivalence in X, while the similarity of fuzzy subsets is an equivalence in $\mathcal{F}(\mathcal{X})$.
It is clear in this context that similar elements of set X or similar fuzzy subsets of X are important.
If two elements are similar with respect to a fuzzy subset, then they are interchangeable. It is like taking a generic when the prescribed medication is not

available in pharmacy. It is also the case when substituting an injured player in a soccer field by another player among many available in the team, who has the same style and approach of the game.

Example 2.2.15 . Let $|X| = n$ and $|M| = m$ as usual. Given two *similar* elements x_1 and x_2 of X. With respect to these two elements, there are at least three fuzzy subsets *similar* to one another, namely the fuzzy subsets $\mu_1 = 0, 0 \cdots 0$, $\mu_2 = \frac{1}{,} m - 1, \frac{1}{m-1}, \cdots, \frac{1}{m-1}$ and $\mu_3 = 1, 1, \cdots, 1$. How many other can we enumerate in $\mathcal{F(X)}$?
If we extend the similarity of two fuzzy subsets to all elements of set X, then the fuzzy subsets are therefore equal.

A more comprehensive definition of similarity of fuzzy subsets is this one found in [29]. Two fuzzy subsets μ and λ are similar if they maintain the same relative degrees of membership values with respect to any two elements x and y. As said earlier, the similarity of fuzzy subsets is an equivalence in $\mathcal{F(X)}$ and can also be defined as follows:

$\mu \equiv \lambda$ if and only if $\forall x, y \in X$
1.$\mu(x) > \mu(y)$ if and only if $\lambda(x) > \lambda(y)$
2.$\mu(x) = 1$ if and only if $\lambda(x) = 1$
3.$\mu(x) = 0$ if and only $\lambda(y) = 0$
Assume $\mu \equiv \lambda$. For each $\alpha \in [0, 1]$ and for each $x \in \mu^\alpha$, that is $\mu(x) \geq \alpha$, there must be an $\beta \in [0, 1]$ such that $\lambda(x) \geq \beta$, that is $x \in \lambda^\beta$. In short $\mu^\alpha \subseteq \lambda^\beta$. Because $\lambda \equiv \mu$, we can state that $\lambda^\beta \subseteq \mu^\alpha$, and we write $\mu^\alpha = \lambda^\beta$. In summary we say that if $\mu \equiv \lambda$,therefore there exists an α and a β in I such that $\mu^\alpha = \lambda^\beta$. As a result of this equivalence the following propositions are justified.

Proposition 2.2.16 If $\mu \equiv \lambda$, then $Core(\mu) = Core(\lambda)$ and $Supp(\mu) = Supp(\lambda)$.

Proof: Let $x \in Core(\mu)$, then $\mu(x) = 1$. Then for any $y \in X$, $\mu(x) \geq \mu(y)$. That means $\lambda(x) \geq \lambda(y)\ \forall x, y \ \in X$ since $\mu \equiv \lambda$. Should $x \notin Core(\lambda)$, that also mean that $\lambda(x) < 1$. But because $\lambda \equiv \mu$, that leads to $\mu(x) < 1$. That is contrary to our first assumption. That means that $Core(\mu) \subseteq Core(\lambda)$.

In the same manner we prove that $Core(\lambda) \subseteq Core(\mu)$. This prove for all x that $Core(\mu) = Core(\lambda)$.
In fact the condition $\mu(x) = 1$ if and only if $\lambda(x) = 1$ means clearly that $x \in Core(\mu)$ if and only if $x \in Core(\lambda)$ and conversely.
Now we let $x \in Supp(\mu)$,and let $\mu(x) > 0$. Since $\mu \equiv \lambda$, we have $\lambda(x) > 0$, which means $x \in Supp(\lambda)$, concluding that $Supp(\mu) \subseteq Supp(\lambda)$. Indeed we also conclude that $Supp(\lambda) \subseteq Supp(\mu)$ so that $Supp(\mu) = Supp(\lambda)$.
Generally the condition $\mu(x) = 0$ if and only if $\lambda(x) = 0$ also means that if $y \in X$ such that $\mu(y) \geq \mu(x) = 0$ then $\lambda(y) \geq \lambda(x) = 0$. This clearly says that $Supp(\mu) = Supp(\lambda)$.

Proposition 2.2.17 If $\mu \equiv \lambda$, then $\mid Im(\mu) \mid = \mid Im(\lambda) \mid$.

Proof: Let $x \in X$. Define a function f such that $f(\mu(x)) = \lambda(x)$. Indeed each elemen x has one membership value $\mu(x)$ which is mapped through f to the only membership value $\lambda(x)$. This means $f(\mu(x)) = \lambda(x)$ is unique for each element x. Now let us consider $f(\mu(x)) = f(\mu(y))$. That is $\lambda(x) = \lambda(y)$ by definition of f. But $\lambda \equiv \mu$ and $\mu \equiv \lambda$. Therefore $\mu(x) = \mu(y)$. The function f is one-to-one. Let $z \in Im(\lambda)$. This means $\exists x \; z = \lambda(x)$ such that $\mu(x)$ as well as $f(\mu(x))$ exist and are unique in each case. f is now onto and therefore a bijection so that $\mid Im(\mu) \mid = \mid Im(\lambda) \mid$.

Proposition 2.2.18 1. If $\mu_1 \equiv \mu_2$, then any fuzzy subset dominating both μ_1 and μ_2 is equivalent to them.
2. If $\mu_1 \equiv \mu_2$, then any fuzzy subset dominated by both μ_1 and μ_2 is equivalent to them.

Proof 1. Let μ be such that $\mu \geq \mu_1$ and $\mu \geq \mu_2$. For x and y in X, $\mu \geq \mu_1$ means there exists $k, 0 \leq k \leq m - 1$ such that $\mu_1(x) + k = \mu(x)$ and if $\mu_1(x) \geq \mu_1(y)$ then $\mu_1(x) + k \geq \mu_1(y) + k$ which also means that $\mu(x) \geq \mu(y)$ since $\mu_1(x) + k = \mu(x)$. Thenμ is equivalent to μ_1 and μ_2. 2. Let $\mu_1 \equiv \mu_2$ and $\mu \leq \mu_1$. We have $\mu(x) + k = \mu_1(x)$. If again $\mu_1(x) \leq \mu_1(y)$, then $\mu_1(x) + k \leq \mu_(y) + k$. Therefore $\mu(x) \leq \mu(y)$ and μ is equivalent to μ_1. Similarly we show that μ is also equivalent to μ_2.

In summary $\mu_1 \equiv \mu \equiv \mu_2$.
In a special case the above proposition will be valid if we use $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$

as the following statement:

Proposition 2.2.19 $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are equivalent to μ_1 and μ_2.

Example 2.2.20 .
Consider two fuzzy subsets $\mu = 1\frac{1}{2}1$ and $\lambda = \frac{1}{2}1\frac{1}{2}$. Clearly $Im(\mu) = Im(\lambda) = \{\frac{1}{2}, 1\}$ but $\mu \not\equiv \lambda$.

Proposition 2.2.21 $Im(\mu) = Im(\lambda)$ does not necessarily mean $\mu \equiv \lambda$.

2.3 Enumeration of fuzzy subsets of X

2.3.1 Fuzzy subsets of a given cardinality.

We know that each fuzzy subset of the set X has a cardinality. If we take one given cardinality, how many fuzzy subsets of the set have this cardinality?
In the next paragraph we wish to enumerate the fuzzy subsets of X having a specified cardinality. We will first consider the number of elements in $Card_\mu$, which is the set of all possible distinct cardinalities of fuzzy subsets of X with membership values in M.
Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set with n elements. All fuzzy subsets μ of X take n membership values not necessarily distinct in M whose sum is the cardinality of the fuzzy subset.
We set the membership values in the unit interval I to be uniformly spaced, with the usual ordering given by $M_m = \{0, \frac{1}{m-1}, \frac{2}{m-1} \cdots, \frac{m-1}{m-1} = 1\}$. This uniform choice of values in M_m does not affect the counting of fuzzy subsets with specified property and also is in line with preferential equality discussed in [28].
We restrict ourselves to fuzzy subsets of X taking values in M_m and denote the set of all such fuzzy subsets by M_m^X. If we denote by $Card_\mu$ the set of all possible distinct cardinalities of fuzzy subsets of X with membership values in M, then $Card_\mu$ is in fact made up of elements which are sums of elements of M_m. So we have:

$$Card_\mu = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1}, \frac{m}{m-1}, \cdots, 2, \cdots, 3, \cdots, n\}.$$

The top element of $Card_\mu$ is n while the bottom element is 0. The set $(Card_\mu, \leq)$ is totally ordered with the usual order in $\mathbb{R}$.
We observe that for a fixed $M \subset I$, the cardinalities of fuzzy subsets depend on the number of elements of the set X. The number of elements in X influences the cardinality of the fuzzy subsets of X since it is actually the sum of membership values of all elements of X to a given fuzzy subset.
We can also say that the quality of elements (that is),
$(no\, desirability$, $absolute\, desirability$, $etc.\cdot)$ has an impact on the cardinality of the fuzzy subset.
The more *no worth* elements to a fuzzy subset, the smaller its cardinality. On the other hand the more *absolute desirable* elements to a fuzzy subset, the greater the cardinality. This is also true for elements in $Card_\mu$.
Let us denote by $Card_{\mu_{X_n}}$ the set of all possible distinct cardinalities of fuzzy subsets of $X_n = \{x_1, x_2, \cdots, x_n\}$ with membership values in M. Subsequent to the above notation we will denote by $Card_{\mu_{X_{n+1}}}$ the set of all possible distinct cardinalities of fuzzy subsets of $X_{n+1} = \{x_1, x_2, \cdots, x_n, x_{n+1}\}$ with membership values in M and $Card_{\mu_\emptyset}$ will denote the set of cardinalities of fuzzy subset of the empty set $\emptyset$.
We first determine the number of elements in $Card_\mu$,the set of possible cardinalities of fuzzy subsets of X.
In the following proposition we determine the number of elements in $Card\mu$, the set of cardinalities of fuzzy subsets of a n-element set X with membership values in a m-element set $M \subset I$.

Proposition 2.3.1 Let the sets X and M be such that $\mid X \mid = n$ and $\mid M \mid = m$, then $\mid Card_\mu \mid = (m-1)n + 1$.

Proof: Let m be any number. This implies as per our definition of M that $M = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$. Now if $X = \emptyset$, $n = 0$ then $Card_\mu = 1$ as seen before. Suppose $n = 1$. Then $Card_\mu = m = (m-1)1+1$. This is evident since the only one element of X would have m possible distinct membership values in M.
Suppose $n = k,$ and $Card_\mu = (m-1)k + 1$. We observe that for $n = k+1$,

$Card_{\mu_{X_{n+1}}}$ is obtained by including into $Card_{\mu_{X_n}} = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1}, \cdots n\}$, $(m-1)$ additional members. These numbers are $n + \frac{1}{m-1}, n + \frac{2}{m-1}, \cdots, n + \frac{m-1}{m-1} = n+1$. Therefore $|Card_{\mu_{X_{n+1}}}| = |Card_{\mu_{X_n}}| + (m-1)$. That is $(m-1)n + 1 + (m-1) = m(n+1) - (n+1) + 1$. This confirms that for $n = k+1$, $|Card_{\mu_{X_{n+1}}}| = m(n+1) - (n+1) + 1$. $\square$.
We are now able to determine the number of fuzzy subsets of a finite set X, each having cardinality $\alpha \in Card_\mu$.

2.3.2 Number of fuzzy subsets of X with cardinalities equal to α.

The cardinality of a fuzzy subset is a number α such that $0 \leq \alpha \leq n$. We consider therefore two cases.
1. $\alpha \in M$ and
2. $\alpha \notin M$.
If $\alpha \in M$, then $0 \leq \alpha \leq 1$ by definition of M. The number of fuzzy subsets of X with cardinality α is the number of fuzzy subsets with cardinality in the set $M_m = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\} \subseteq Card_\mu$. Now there is 1 fuzzy subset of cardinality $0 \, \forall n$, $|X| = n$. There are m fuzzy subsets of cardinality $\frac{1}{m-1}$. This is obtained when only one membership value is $\frac{1}{m-1}$ while the remaining $(n-1)$ are 0.
There are $\begin{pmatrix} m \\ 1 \end{pmatrix} = \frac{m!}{1!(m-1)!}$ crisp subsets of cardinality 1 [45], [8]. This case enumerates only the fuzzy subsets where one membership value is 1, while the $(n-1)$ all are 0. There are m fuzzy subsets with cardinality $\frac{2}{m-1}$ where one membership value is $\frac{2}{m-1}$ while the $(n-1)$ are each 0.There are $\begin{pmatrix} m \\ 2 \end{pmatrix} = \frac{m!}{2!(m-2)!}$ fuzzy subsets of cardinality $\frac{2}{m-1}$ among those of cardinality $\frac{2}{m-1}$ in M. In this case any 2 membership values out of the n are each $\frac{1}{m-1}$ while the remaining $(n-2)$ are all 0. We carry on the count until we get cardinality $\frac{m-2}{m-1}$.

We consider this time the case where $\alpha \notin M$. That is $\frac{m}{m-1} \leq \alpha \leq n$.
It is obvious in this case that $n \geq 1$. The cardinality $\frac{m}{m-1}$ can be obtained if either each membership characterizing the fuzzy subset is $\frac{1}{m-1}$ [Note here

that there is only one such case] , or any fashion that makes a partition of the natural n with a denominator being $(m-1)$.

Example 2.3.2 . Consider $n = 4$ and $m = 4$. The different ways of obtaining the cardinality $\frac{4}{3}$ are:
1. Each membership is $\frac{1}{3}$ and we have the fuzzy subset $\mu = \frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3}$. As previously said , there is only one such fuzzy subset.
2. Two membership values are each $\frac{2}{3}$ while the remaining two are 0 each. There are 6 such fuzzy subsets.
3. Any combination which sums to $\frac{4}{3}$ of the numbers $\frac{0}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$. There are 23 such fuzzy subsets with cardinality $\frac{4}{3}$.

Proposition 2.3.3 Let the set X be of cardinality n, M be a m-elements set and α a number in the unit interval I. Then the number of fuzzy subsets of X of cardinality equal to α, denoted here as: $(N_\mu(\mid . \mid= \alpha))_n$ is given by the expression: $(N_\mu(\mid . \mid= \alpha))_n = \sum_{i \in M}(N_\mu(\mid . \mid= \alpha - i))_{n-1}$.

Proof: The left of the above equation is an expression counting the number of fuzzy subsets of the set X, that is $X = \{x_1, x_2, x_3, \cdots, x_n\}$, but the expression on the right of the equation counts the fuzzy subsets of the set $X' = \{x_1, x_2, x_3, \cdots, x_{n-1}\} = X \setminus \{x_n\}$.
Any fuzzy subset μ of set X is obtained from a fuzzy subset μ' of set X' by associating with the missing element x_n the difference between $\mid \mu \mid$ and $\mid \mu' \mid$, if indeed this difference belongs to M. That ends the proof of the proposition. $\Box$.
The above proposition focuses on the number of fuzzy subsets with cardinality $\alpha \in I$, the unit interval. Now we can extend the proposition to enumerate the fuzzy subsets with cardinality any number $p \geq 1$ in $\alpha \in Card_\mu$. We express the new result in the following proposition.

Proposition 2.3.4 Let the cardinalities of X and M be n and m respectively. The number of fuzzy subsets of X with cardinality $p \in Card_\mu$ is equal to the sum of the number of fuzzy subsets of $X \setminus \{x_i\} \forall x_i \in X$ with cardinality p and the number of fuzzy subsets of $X \setminus \{x_i\} \forall x_i \in X$ with cardinality $p - \frac{1}{m-1}$.

The above proposition can be illustrated by the following table of $N_\mu(|\ .\ |= p)$. We call this table the *Pascal Rectangle* since it is rectangular and also its entries are obtained in a way similar to the process in the *Pascal triangle*.

n/p	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
1	1	1	1	0	0	0	0	0	0
2	1	2	3	2	1	0	0	0	0
3	1	3	6	7	6	3	1	0	0
4	1	4	10	16	19	16	10	4	1

The entries in each line n represent the number of fuzzy subsets of a set with n elements and with cardinality the number on the column p. In short, the entry on line n and column p is the number of fuzzy subsets of a n-element set having cardinality p. This number is obtained when we sum the three entries on line $n-1$ and on columns $p-\frac{0}{2}$, $p-\frac{1}{2}$, $p-\frac{2}{2}$.
It is clear in this context that the number of fuzzy subsets of a n-element set with cardinality $p = n$ is 1. This is so because the membership of each of the n elements of the set would be 1 which sums to n.
The number of fuzzy subsets of a n-element set with cardinality $p > n$ is 0. Here even if each of the n elements would have maximum membership value 1, their sum would not exceed n. Therefore there is no such fuzzy subset.

Example 2.3.5 .
Refer to the diagram attached to the thesis.
Check that when $n = 3$ and $M = \{0, \frac{1}{2}, 1\}$, the number of fuzzy subsets of cardinality , say $p = 1\frac{1}{2}$ is 7.
This number is the sum of the number of fuzzy subsets when $n = 2$ and with cardinalities $p = 1\frac{1}{2} - 0 = 1\frac{1}{2}$; $\quad p = 1\frac{1}{2} - \frac{1}{2} = 1$; $\quad p = 1\frac{1}{2} - 1 = \frac{1}{2}$ as stated in the above proposition.
We write: $7 = 2 + 3 + 2$.
Consider an element $x \in X$. We are interested to count all fuzzy subsets of X with the following common pattern: The membership value for a given $x \in X$ is the same for all fuzzy subsets considered.

2.3.3 The counting of fuzzy subsets of X when membership values for a given element x of X are equal.

Let $x_1 \in X$. We wish to consider all fuzzy subsets μ_i of X which are such that the membership values $\mu_i(x_1)$ $\forall i$ of element x_1 to each fuzzy subset is the same value $\alpha \in M$. Assume that there are m elements in M and n elements in X. Then there m^n fuzzy subsets of X with values in M.
In the following proposition we enumerate the fuzzy subsets of X that have the same membership value for a given $x_1 \in X$.

Proposition 2.3.6 1. There are $(m^n/m) = m^{(n-1)}$ fuzzy subsets μ satisfying $\mu(x_1) = \alpha$ in M. 2. There are $m^{(n-1)}$ fuzzy subsets for each of the n elements of X.

Proof: Let fix us $x_i \in X$ for which the image $\mu(x_i)$ is also fixed in M. Then there are $(n-1)$ elements in X free to take any value in the m-element set M. The challenge we have is to find the number of functions from $X \setminus \{x_i\}$ to M. This number is $m^{(n-1)}$.

Let us consider $\alpha \in M$. We define the relation $\mathcal{R}$ in $\mathcal{F}(\mathcal{X})$ such that $\mu_1 \mathcal{R} \mu_{\in}$ if $\mu_1(x_i) = \mu_2(x_i) \forall x_{\in} X$. We realize that $\mathcal{R}$ defined above is an equivalence relation in $\mathcal{F}(\mathcal{X})$.
For any $x_i \in X$, the family Π_i of elements of $\mathcal{F}(\mathcal{X})$ is an equivalence class. Therefore $\{\Pi_i, \forall x_i \in X\}$ is a partition of $\mathcal{F}(\mathcal{X})$ which we will call the α-partition. From its definition we can tell that there are n such α-partitions of $\mathcal{F}(\mathcal{X})$.
If we consider another $\beta \in M$, we also obtain a partition.
We may be interested in counting the fuzzy subsets of X where any two elements x_i and x_j of X have the same membership value. We can also extend our counting to find the fuzzy subsets of X where two or more selected elements of X have same membership value.
When an element $x \in X$ is such that $\mu_1(x) = \mu_2(x)$, we say that x is stable or invariant with regards to these fuzzy subsets. Now restating the proposition above, we say that for every $\alpha \in M$, there are $m^{(n-1)}$ fuzzy subsets for which any element x of X is stable.
An element x_i that has the same membership value to the fuzzy subsets $\mu_1, \cdots, \mu_j$ is said to be stable/invariant to these fuzzy subsets.

Now we fix two distinct elements of X, that is two elements x_1 and x_2 such that their membership values to each fuzzy subset $\mu_i(x_1)\ \forall i$ remain unchanged. Here two cases can be envisaged. The first case is for the two chosen elements of X to have the same image, that is $\mu_i(x_1) = \mu_i(x_2)\ \forall i$. The second case is that each of the elements has a fixed image in M, with $\mu_i(x_1) \neq \mu_i(x_2)\ \forall i$.

Proposition 2.3.7 Consider $\alpha \in M$ and let x_1 and x_2 be two given elements of X. The number of fuzzy subsets for which the two elements are simultaneously stable is: m^{n-2}.

Proposition 2.3.8 If x is invariant to $\mu_1, \cdots \mu_j$, then
1. x is invariant to $\mu_i \cup \mu_j\ \forall i,\ j$ and
2. x is invariant to $\mu_i \cap \mu_j \forall i,\ j$.

Knowing which element of X is invariant to some fuzzy subsets of X is important. Imagine x to be a fruit or any other commodity a shop has ordered. Wouldn't it not be useful to know which fruit would keep its status should some properties of fuzzy nature change.
In other words identifying elements of a set which keep the same membership value to a number of fuzzy subsets is important. In this way for instance banks customers are partitioned according to some given fuzzy properties such as *monthly income above* $R20,000$; *age about* 30 to 40 *years old*; with some *higher learning education*; with a *total monthly expenditure* in excess of ... etc...

A person who falls under all of these categories might be the most sought customer a bank would like to engage with.

2.3.4 Fuzzy subsets of many similar elements of X.

Consider the set $X = \{x_1, x_2, \cdots, x_n\}$ and the set $\mathcal{F}(\mathcal{X})$ of all possible fuzzy subsets of set X. Our interest is to find out how many fuzzy subsets of X have the highest number of similar elements of X. This counting does not include

the obvious fuzzy subsets $\mu_1 = 11\cdots1$, $\mu_2 = 00\cdots0$ and $\mu_3 = \frac{1}{m-1}\frac{1}{m-1}\cdots\frac{1}{m-1}$ each with n similar elements. That is the set X is the set of similar elements for these three fuzzy subsets. How many fuzzy subsets have $n-1$, $n-2$ and so on similar elements?
For $n = 3$ and $m = 3$, there are 18 fuzzy subsets with 2 similar elements such as $11\frac{1}{2}$, 110, 100, $\frac{1}{2}$.
Let $|X| = n$ and $|M| = m$. In the following proposition we enumerate the fuzzy subsets of X with $(n-1)$ similar elements.

Proposition 2.3.9 The number of fuzzy subsets of X with membership values in M with $(n-1)$ similar elements is $n.m(m-1)$.

Proof. If $(n-1)$ elements of X are similar of membership value say α in M, then the only element left in X has $(m-1)$ values to choose in M. Since there are $|X| = n$, we have therefore $n(m-1)$ for α. Since $|M| = m$, there are finally $n.m(m-1)$ different fuzzy subsets where $(n-1)$ elements of X are similar.□

We consider fuzzy subsets of X and their α-cuts. We want to enumerate fuzzy subsets with the smallest α-cut and the fuzzy subsets with the larger α-cut. For these α-cuts,which are subsets of X, we consider the *inclusion* of subsets as a tool to determine their size. Indeed we exclude in our search the obvious cases of $\emptyset$ and X.

2.3.5 Fuzzy subsets of X of smaller or larger α-cut.

Consider $\alpha \in M \subseteq I$. If $\alpha = 0$, then $\mu^0 \supseteq supp\mu$. For each α, $0 \leq \alpha$; so that $\mu^\alpha \subseteq \mu^0$. If $\alpha = 1$ then $\mu^1 = core\mu$ and there is one fuzzy subset, namely $\mu = 11\cdots1$ whose α-cut is X. Any other fuzzy subset where only one element of X has membership value 1 has a small α-cut which is $\{x_i\}, \forall 1 \leq i \leq n$. What value of α other than 0 makes the larger α- cut in X? Which value of α generates the smaller α- cut in X?
Let $\alpha \in [0,1]$; If $\alpha = 0$, there is one subset of X equal to 0-cut. It is the greatest α-cut.

2.3.6 Fuzzy subsets of X of same α- cut.

Consider $\alpha \in I$. We want to consider the fuzzy subsets of X that have the same α-cut for a given α. For clarity we may realize for instance that $\mu_1 = \frac{1}{2}11,\ \mu_2 = 11\frac{1}{2},\ \mu_3 = 1\frac{1}{2}1$ have the same $\frac{1}{2}$-cut.
Let x be an element in the common α-cut of these fuzzy subsets μ_i. Then $\mu_i(x) \geq \alpha\, \forall i$ and $\cap_{i=1}^{n} \mu_i(x) \geq \alpha$. It is also clear that $\cup_{i=1}^{n} \mu_i(x) \geq \alpha$.
We state the following proposition without proof.

Proposition 2.3.10 Let μ be a fuzzy subset of X and X_i as its α_μ-cut. For any other fuzzy subset $\lambda \geq \mu$ the α_μ-cut is a subset of α_λ-cut.

Concerning the α-cuts, we can enumerate different other types of fuzzy subsets. We have in mind :
-the fuzzy subsets of empty α-cut for a chosen $\alpha \in M$.
-the fuzzy subsets of α-cut equal to X,
-the fuzzy subsets of α-cut a single element $x_i \in X$.

2.3.7 Fuzzy subsets of constant membership value.

Let $X = \{x_1, x_2, \cdots x_n\}$. We want to count the fuzzy subsets μ of X where $\mu(x_1) = \mu(x_2) = \cdots \mu(x_n)$. The membership value of all elements is a single value say $\alpha \in I$. Consider that single value α, then the α-cut for each of these fuzzy subsets is X. If $\alpha = 0$, the support of μ is $\emptyset$, the core is also $\emptyset$. Now if $\alpha = 1$, the support is X, the core is also X.
For any other value $\gamma \in I$ such that $\gamma > \alpha$, then the γ-cut is empty set; that is $\mu^\gamma = \emptyset$.

Example 2.3.11 . Two of such fuzzy subsets are $\mu_1 = 00\cdots 0,\ \mu_2 = 11\cdots 1$.

How many fuzzy subsets of this nature are there in $\mathcal{F}(\mathcal{X})$?.
Consider $\mathcal{F}(\mathcal{X})$ and the set M of all membership values. The size of M determines the number of fuzzy subsets of constant membership value. This number is $\mid M \mid$.
These kinds of fuzzy subsets are special. We refer to our example of fuzzy

subset of marks awarded to students in their mathematics test. A case of a fuzzy subset of constant value informs us that all students got the same mark. Something may have happened during the test. Either the test was so difficult and everyone got say 0; very easy and all got say 1. This requires an new setting of conditions to be able to partition X in a way that we can describe the behavior of its elements.

2.3.8 Fuzzy subsets with $supp\,\mu$ equal X.

The fuzzy subsets considered in this case are such that every element $x_i \in X$ has a membership value greater than 0. In other words this means no element of X has membership $\mu(x_i) = 0$. We introduce the notation $N_\mu \mu(x_i) = 0$ for the number of fuzzy subsets of X for which $\mu(x_i) = 0$ for $i = 1, 2, \cdots n$ and $x_i \in X$.
Now using PIE we have the following proposition :

Proposition 2.3.12 The number of fuzzy subsets μ_j with $supp\,\mu_j$ equals X, denoted here as $N_\mu(supp\,\mu = X) = m^n - [\sum_{i=1}^{n}(N_\mu \mu(x_i) = 0] + [\sum_{\mu} \mu(x_i) = 0, \mu(x_j) = 0] - [N_\mu \mu(x_i) = 0, \mu(x_j) = 0, \mu(x_k) = 0] + \cdots + (-1)^{n+1} N_\mu \mu(x_i) = 0, \mu(x_2) = 0 \cdots \mu(x_n) = 0$

Example 2.3.13 . Any fuzzy subset of constant value 1 is of $supp\mu = X$.

2.3.9 Fuzzy subsets with $core\,\mu$ equal X.

The number of fuzzy subsets with $core\,\mu$ equals X can only be one. This is the only fuzzy subset such that each element $x_i \in X$ has membership value 1 to the fuzzy subset.

Example 2.3.14 . The fuzzy subset of constant value 1 is the only fuzzy subset of $core\,\mu = X$.

2.3.10 Fuzzy subsets with $supp\,\mu$ equal $\emptyset$.

In this case there is only one fuzzy subset of X with $supp_\mu$ equal to $\emptyset$. This is the fuzzy subset of X where each element $x_i \in X$ has membership value *zero* to the fuzzy subset.

2.3.11 Fuzzy subsets with $core\,\mu$ equal $\emptyset$.

Let us first introduce some useful notations. We denote by $N_\mu[core\,\mu = \emptyset]$ the number of fuzzy subsets of X with $core\,\mu = \emptyset$ and by $N_\mu\mu(x_i) = 1$ the number of fuzzy subset of X for which the element $x_i \in X$ has membership value 1. By $[N_\mu\mu(x_i) = 1, \mu(x_j) = 1]$ we denote the number of fuzzy subsets of X for which the elements x_i and x_j simultaneously have membership value 1. The expression giving the number of fuzzy subsets of X with $core\,\mu = \emptyset$ is given by the following proposition:

Proposition 2.3.15 $N_\mu[core\,\mu = \emptyset] =$

$$m^n - [\sum_{i=1}^{n} N_\mu\mu(x_i) = 1] + [\sum_{i=1,j=1}^{n} N_\mu\mu(x_i) = 1, \mu(x_j) = 1] +$$

$$\cdots + (-1)^{n+1}[\sum_{i=1}^{n} N_\mu\mu(x_i) = 1, \mu(x_2) = 1 \cdots \mu(x_n) = 1].$$

2.3.12 Subsets of distinct membership values.

We consider now the fuzzy subsets $\mu = \mu(x_1)\mu(x_2)\cdots\mu(x_n)$ of an n-set X with membership values in an m-set M where the values $\mu(x_1), \mu(x_2), \cdots, \mu(x_n)$ are distinct from one another. If we set $n \geq m$, then all fuzzy subsets have at least one membership value repeating itself. We therefore consider the cases $n = m$ and $n \leq m$. The element x_1 has m choices of values in M; the next element x_2 of X has $m-1$ choices in M; x_3 has $m-2$ choices etc...
The number of fuzzy subsets with all membership values distinct from one another is therefore $m! = m(m-1)(m-2)\cdots(m-m+1)$.
When $n = m$, all these fuzzy subsets have the same cardinality.

2.4 Some Other Applications of PIE in $\mathcal{F}(\mathcal{X})$.

2.4.1 Union of the sets $\mu^{-1}(\alpha)$

Consider a fuzzy subset μ of set X with distinct membership values $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $M \subset [0,1]$. It is clear that $\{\mu^{-1}(\alpha_i)\}_{i=1}^{n}$ is a partition of X and that $(\sum_{i=1}^{n} \mid \mu^{-1}(\alpha_i) \mid = \mid X \mid$ while $\mid \bigcap_{i=1}^{n} \mu^{-1}(\alpha_i) \mid = \emptyset$.

We define in X a relation $\mathcal{R}$ such that two elements x_i and x_j are in relation $\mathcal{R}$ and we write $x_i \mathcal{R} x_j$ if and only if $\mu(x_i) = \mu(x_j)$. In other words we may say that $\forall \alpha \in [0,1]$ $x_i \mathcal{R} x_j$ if and only if $x_i \in \mu^{-1}(\alpha)$ and $x_j \in \mu^{-1}(\alpha)$ This relation $\mathcal{R}$ is an equivalence on X. The equivalence classes are the $\mu^{-1}(\alpha_i)$ $\forall i$. The cardinality of $M \subset [0,1]$ is the number of equivalence classes in X.
We express $\mid (\bigcup_{i=1}^{n} \mu^{-1}(\alpha_i) \mid$ as in the proposition below:

Proposition 2.4.1 Let μ be a fuzzy subset of set X with membership values $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $M \subset [0,1]$, then:
$\mid (\bigcup_{i=1}^{n} \mu^{-1}(\alpha_i) \mid = (\sum_{i=1}^{n} \mid \mu^{-1}(\alpha_i) \mid) - \sum_{1 \leq i < j} \mid \mu^{-1}(\alpha_i) \cap \mu^{-1}(\alpha_j \mid + \sum_{1 \leq i < j < k} \mid \mu^{-1}(\alpha_i) \cap \mu^{-1}(\alpha_j) \cap \mu^{-1}(\alpha_k) \mid + \cdots + (-1)^{n-1} \mid \mu^{-1}(\alpha_1) \cap \mu^{-1}(\alpha_2) \cap \cdots \cap \mu^{-1}(\alpha_n) \mid$
and dually
$\mid (\bigcap_{i=1}^{n} \mu^{-1}(\alpha_i)) \mid = (\sum_{i=1}^{n} \mid \mu^{-1}(\alpha_i) \mid) - \sum_{1 \leq i < j} \mid \mu^{-1}(\alpha_i) \cup \mu^{-1}(\alpha_j) \mid + \sum_{1 \leq i < j < k} \mid \mu^{-1}(\alpha_i) \cup \mu^{-1}(\alpha_j) \cup \mu^{-1}(\alpha_k) \mid + \cdots + (-1)^{n-1} \mid \mu^{-1}(\alpha_1) \cup \mu^{-1}(\alpha_2) \cup \cdots \cup \mu^{-1}(\alpha_n) \mid$.

Proof: For the fact that $\{\mu^{-1}(\alpha_i)\}_{i=1}^{n}$ is a partition of X; each intersection $\mu^{-1}(\alpha_i) \cap \mu^{-1}(\alpha_j) = \emptyset$ for any two i, j, and furthermore it is clear that $\mid (\bigcup_{i=1}^{n} \mu^{-1}(\alpha_i) \mid = (\sum_{i=1}^{n} \mid \mu^{-1}(\alpha_i) \mid) = \mid X \mid$.
Similarly if we express each union of the kind $\mid \mu^{-1}(\alpha_i) \cup \mu^{-1}(\alpha_j) \mid$ or even of the type $\mid \mu^{-1}(\alpha_i) \cup \mu^{-1}(\alpha_j) \cup \mu^{-1}(\alpha_k) \mid$ found in $\mid (\bigcap_{i=1}^{n} \mu^{-1}(\alpha_i)) \mid$ as

$| \mu^{-1}(\alpha_i) | + | \mu^{-1}(\alpha_j) | - | \mu^{-1}(\alpha_i) \cap \mu^{-1}(\alpha_j) |$, then each term on the right side will cancel each other. As a result we will be able to write that $| \bigcap_{i=1}^{n} \mu^{-1}(\alpha_i) | = 0$. This confirms the fact that $\bigcap_{i=1}^{n} \mu^{-1}(\alpha_i) = \emptyset$ since the $\mu^{-1}(\alpha_i)$ $\forall i$ make a partition of X.$\square$.

Now if we assume that the membership values $\alpha_1, \alpha_2, \cdots, \alpha_n$ are such that $\alpha_1 \leq \alpha_2 \leq \cdots, \leq \alpha_n$. Then $\mu^{\alpha_n} \subseteq \mu^{\alpha_{n-1}} \subseteq \cdots \subseteq \mu^{\alpha_2} \subseteq \mu^{\alpha_1}$. These μ^{α_i} $\forall\, i, 1 \leq i \leq n$ are subsets of X. We can therefore apply the PIE on these finite subsets in this manner:

$$| \bigcap_{i=1}^{n} \mu^{\alpha_i} | = \sum_{i=1}^{n} | \mu^{\alpha_i} | - \sum_{1 \leq i < j} | \mu^{\alpha_i} \cup \mu^{\alpha_j} | + \sum_{i<j<k} | \mu^{\alpha_i} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k} |$$

$$\cdots (-1)^n | \mu^{\alpha_1} \cup \mu^{\alpha_2} \cup \cdots \cup \mu^{\alpha_n} |$$

Now, if $\alpha_1 \leq \alpha_j \leq \alpha_k$ then $\mu^{\alpha_1} \cup \mu^{\alpha_j} = \mu^{\alpha_1}$ and $\mu^{\alpha_1} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k} = \mu^{\alpha_1}$. Therefore μ^{α_1} appears $(n-1) = \binom{n-1}{1}$ times in expressions of the type $\mu^{\alpha_1} \cup \mu^{\alpha_j}$, $\binom{n-1}{2}$ in expressions of the form $\mu^{\alpha_1} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k}$ and so on. Finally because of the alternating signs, the sum
$\binom{n-1}{0} - \binom{n-1}{1} + \binom{n-1}{2} \cdots (-1)^{n-1} \binom{n-1}{n-1} = 0.$
See [21]. Therefore the term $| \mu^{\alpha_1} |$ will actually vanish in the right hand side of 2.4.1.
Similarly μ^{α_2} appears $\binom{n-2}{1} = (n-2)$ times in expression of the form $\mu^{\alpha_2} \cup \mu^{\alpha_j}$; it appears $\binom{n-2}{2}$ in expressions of the type $\mu^{\alpha_2} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k}$. As before we conclude that the term $|\mu^{\alpha_2}|$ vanishes in the right hand side of 2.4.1. We use this reasoning for each term of the type μ^{α_i} for $1 \leq i \leq n$ repeatedly until the term μ^{α_n} which is contained in each union $\mu^{\alpha_i} \cup \mu^{\alpha_n}$ $\forall i$ as per our assumption $\alpha_1 \leq \alpha_2 \leq \cdots, \leq \alpha_n$ remains. Therefore it appears only once in

the sum $\mid \bigcap_{i=1}^{n} \mu^{\alpha_i} \mid = \sum_{i=1}^{n} \mid \mu^{\alpha_i} \mid - \sum_{1 \leq i \leq j} \mid \mu^{\alpha_i} \cup \mu^{\alpha_j} \mid + \sum_{i \leq j \leq k} \mid \mu^{\alpha_i} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k} \mid$ $\cdots (-1)^n \mid \mu^{\alpha_1} \cup \mu^{\alpha_2} \cup \cdots \cup \mu^{\alpha_n} \mid$. In conclusion the right hand side of the expression 2.4.1 which solves $\mid \bigcap_{i=1}^{n} \mu^{\alpha_i} \mid$ is made up of only one term $\mid \mu^{\alpha_n} \mid$. This confirms the already known fact that $\mid \bigcap_{i=1}^{n} \mu^{\alpha_i} \mid = \mid \mu^{\alpha_n} \mid$.

2.4.2 Union of subsets of $\mu(X)$

Let μ be a fuzzy subset of X with membership values in set M such that $\mu(X) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset I$. Now we take finitely many subsets $J_1, J_2, \cdots, J_k$ of $\mu(X)$ with $k \leq n$ which are not necessary disjointed subsets of $\mu(X)$.
We denote by J_i^C the complement of J_i in $\mu(X)$. We consider the set $(J_1 \cap J_2 \cap \cdots \cap J_k)^C$ which is the same as the set $(J_1^C \cup J_2^C \cup \cdots \cup J_k^C)$. In addition we also consider

$$(J_1 \cup J_2 \cup \cdots \cup J_k)^C = (J_1^C \cap J_2^C \cap \cdots \cap J_k^C)$$

.
Now applying the PIE on these sets to find the number of elements of M in the union and intersection of these subsets of $\mu(X)$ respectively, we get:

$$\mid (J_1^C \cup J_2^C \cup \cdots \cup J_k^C) \mid = \mid \mu(X) \mid - \sum_{i=1}^{k} \mid J_i \mid + \sum_{1 \leq i \leq j \leq k} \mid J_i \cap J_j \mid + \ldots$$

and

$$\mid (J_1^C \cap J_2^C \cap \cdots \cap J_k^C) \mid = \mid \mu(X) \mid - \sum_{i=1}^{k} \mid J_i \mid + \sum_{1 \leq i \leq j \leq k} \mid J_i \cup J_j \mid + \ldots$$

We are interested to enumerate in the set X the number $\mid \bigcup_{\alpha \in J_i^C}^{k} \mu^{-1}(\alpha) \mid$. This number is obtained as follows [$\mid (\bigcup_{\alpha \in J_i^C}^{k} \mu^{-1}(\alpha) \mid = \mid X \mid - \sum_{\alpha \in \cup_{i=1}}^{k} J_i \mid \mu^{-1}(\alpha) \mid$
]

Let $\mathcal{F}(\mathcal{X})$ be the lattice of fuzzy subsets of an n-element set X. We recall that $M = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$. Now the set of cardinalities of the fuzzy subsets of the set X with membership values in M which is denoted here by $Card_\mu$ is defined as $Card_\mu = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1}, \frac{m}{m-1}, \cdots, n\}$ [45].
Let the fuzzy subsets λ_1 and λ_2 of X be represented as $\lambda_1 = \lambda_1(x_1)\lambda_1(x_2)\cdots\lambda_1(x_n)$ and $\lambda_2 = \lambda_2(x_1)\lambda_2(x_2)\cdots\lambda_2(x_n)$ [20]. Assume that $\lambda_1 \le \lambda_2$. This implies the following two statements:
1. $\lambda_1(x_1) \le \lambda_2(x_1)$, $\lambda_1(x_2) \le \lambda_2(x_2)$, $\cdots$, $\lambda_1(x_n) \le \lambda_2(x_n)$ and
2. $|\lambda_1| \le |\lambda_2|$; Which means that $\sum_{i=1}^{n} \lambda_1(x_i) \le \sum_{i=1}^{n} \lambda_2(x_i)$. This also implies that there exists a natural number t such that $0 \le t \le |X|(m-1)$ and $|\lambda_2| = |\lambda_1| + \frac{t}{m-1}$ [45]. We justify the existence of the natural number t from the nature of members of the set $Card_\mu$ as defined above.

2.5 The α-cut

Theorem 2.5.1 Dominance and α-cut
1. A fuzzy subset μ_1 of X is greater than another fuzzy subset μ_2 of X if each α-cut of μ_1 contains the α-cut of μ_2.
2. A fuzzy subset μ_1 of X is equal to another fuzzy subset μ_2 of X if each α-cut of μ_1 is equal to the α-cut of μ_2.
3. A fuzzy subset μ_1 of X is smaller than another fuzzy subset μ_2 of X if each α-cut of μ_1 is contained by the α-cut of μ_2. [10]

Proof: Let $\mu_1 \ge \mu_2$. Then $\forall x \in X$, $\mu_1(x) \ge \mu_2(x)$. Show that any α-cut of μ_1 contains the α-cut of μ_2 that is $\mu_2^\alpha \subseteq \mu_1^\alpha$.
Now consider $\alpha \in I = [0,1]$, such that $x \in \mu_2^\alpha$, that is $\mu_2(x) \ge \alpha$ which also means by transitivity of $\le$, for the same α, $\alpha \le \mu_2(x) \le \mu_1(x)$, as per our hypothesis $\mu_1(x) \ge \mu_2(x)$. Therefore $\mu_1(x) \ge \alpha$ or that $x \in \mu_1^\alpha$. Which concludes that $\mu_2^\alpha \subseteq \mu_1^\alpha$.
Again, let $\mu_1 = \mu_2$. That is to say $\forall x \in X$, $\mu_1(x) = \mu_2(x)$. We show that any α-cut of μ_1 is an α-cut of μ_2. Consider $x \in \mu_1^\alpha$. That is $\mu_1(x) \ge \alpha$. By the equality $\mu_1(x) = \mu_2(x)$ we write that $\mu_2(x) \ge \alpha$ or $x \in \mu_2^\alpha$ and conclude that $\mu_1^\alpha \subset \mu_2^\alpha$. In the same way we show that $\mu_2^\alpha \subset \mu_1^\alpha$ and prove that $\mu_2^\alpha = \mu_1^\alpha$.

Finally Let $\mu_1 < \mu_2$. Then $\forall x \in X,\ \mu_1(x) < \mu_2(x)$. Now if $x \in \mu_1^\alpha$, then $\alpha \leq \mu_1(x)$. By transitivity $\alpha \leq \mu_2(x)$ and therefore $x \in \mu_2^\alpha$. The α-cut of μ_1 is contained in the α-cut of μ_2.

Example 2.5.2 . Let us consider the marks of a group of 10 students in a mathematics test to be a fuzzy subset μ. Let also $\mu(x_i) = \frac{k}{100}$ be the actual mark of a student x_i for the test. These marks lie indeed in the interval $[0, 1]$. If we consider an $\alpha \in [0, 1]$, say $\alpha = .5$, then $\mu^{.5}$ would be the students among the 10 who had a minimum of $\frac{50}{100}$ pass while $\mu^{.75}$ is the subset of students who got at least a distinction to the test. We realize that the higher the value of α, the higher the quality of result. Thus the α-cut helps stream elements of the set in order to increase confidence in the choice of elements of the universe of discourse.
Let μ and γ be two fuzzy subsets. We know that $\mu^\alpha = \{x \in X : \mu(x) \geq \alpha\}$, we define $\mu_\alpha = \{x \in X : \mu(x) < \alpha\}$. μ_α is a crisp set and is called the complement of μ^α in X. Other authors called μ_α the inverse α-cut. [39] [48] [22]
The properties of fuzzy inclusion and equality are complement α-cut worthy. Again, the complement α-cuts preserve some kind of monotonicity and every fuzzy subset can be represented by the family of all its complement α- cuts. The next proposition clarifies this idea.

Proposition 2.5.3 Let μ^α and γ^α be the α-cuts of two fuzzy subsets μ and γ of a finite set X; with μ_α and γ_α their respective complements in X.
1. $\mu_{\alpha=1} = X$
2. If $\mu \leq \gamma$ then $\gamma_\alpha \subseteq \mu_\alpha$.
3. If $\mu = \gamma$ then $\mu_\alpha = \gamma_\alpha$.
4. For $\mu \in \mathcal{F}(\mathcal{X})$ and $\alpha, \beta \in [0, 1]$, if $\alpha \leq \beta$, then $\mu_\alpha \subseteq \mu_\beta$.
5. Any fuzzy subset μ can be represented by: $\mu = \bigcup_{\alpha \in [0,1]} \alpha.\mu_\alpha$ where $\alpha.\mu_\alpha$ is such that $(\alpha.\mu_\alpha)(x) = \alpha.\mu_\alpha(x),\ x \in X$.

Proof: 1. It is clear that $\forall x \in X, \forall \mu \in \mathcal{F}(\mathcal{X}),\ \mu(x) \leq 1$.
2. Let $\mu \leq \gamma$, then for any $x \in X$, $\mu(x) \leq \gamma(x)$. Now if $\gamma(x) \leq \alpha$, then $\mu(x) \leq \alpha$ as well.
3. Let $\mu = \gamma$; clearly $\mu(x) = \gamma(x)$ and if $\mu(x) \leq \alpha$, then $\gamma(x) \leq \alpha$.
4. Let $x \in \mu_\alpha$, then $\mu(x) < \alpha$. Since $\alpha \leq \beta$, we have also $\mu(x) < \beta$ which means $x \in \mu_\beta$.

2.5.1 The complement α-cut decomposition.

For any $\alpha \in [0,1]$ and from the definitions of μ^α and μ_α we can now partition a set X into two subsets. We can state that $X = \mu^\alpha \cup \mu_\alpha$ and $\mu^\alpha \cap \mu_\alpha = \emptyset$. This is a mere decomposition of set X.

2.5.2 The α-induced fuzzy subset and its complement.

Let $\alpha \in [0,1]$ and $\mu \in \mathcal{F}(\mathcal{X})$. We note by $(\mu^\alpha)^i$ the fuzzy subset obtained by :

$$(\mu^\alpha)^i(x) = \begin{cases} \mu(x), & \text{if } \mu(x) \geq \alpha \\ 0, & \text{if } \mathit{otherwise}. \end{cases}$$

This fuzzy subset $(\mu^\alpha)^i = \{(x, (\mu^\alpha)^i(x))|\ x \in X\}$ is called the α- induced fuzzy subset. As we can see, the α-cut only tell us the elements of X with membership value greater or equal to α. The α- induced fuzzy subset informs us of both the members of X with membership value below α and especially those with membership values above α coupled with their actual membership values.
We are now able to define the complement of the α- induced fuzzy subset denoted here by $(\mu_\alpha)^i$, as

$$(\mu_\alpha)^i(x) = \begin{cases} \mu(x), & \text{if } \mu(x) < \alpha \\ 0, & \text{if } \mathit{otherwise}. \end{cases}$$

With regards to the α-induced fuzzy subset and its complement we can show that for any fuzzy subset μ:
$\mu = (\mu^\alpha)^i \cup (\mu_\alpha)^i$ and $(\mu^\alpha)^i \cap (\mu_\alpha)^i = \emptyset$
In fact $\forall x \in X$, if $\mu(x) \geq \alpha$, then $(\mu^\alpha)^i(x) = \mu(x)$, and $(\mu_\alpha)^i(x) = 0$ such that $(\mu^\alpha)^i(x) \vee (\mu_\alpha)^i(x) = \mu(x) \vee 0 = \mu(x)$ while $(\mu^\alpha)^i(x) \wedge (\mu_\alpha)^i(x) = 0$. On the other hand, if $\mu(x) < \alpha$, then $(\mu^\alpha)^i(x) = 0$, and $(\mu_\alpha)^i(x) = \mu(x)$. Therefore: $(\mu^\alpha)^i(x) \vee (\mu_\alpha)^i(x) = \mu(x) \vee 0 = \mu(x)$ while $(\mu^\alpha)^i(x) \wedge (\mu_\alpha)^i(x) = 0$.

2.6 Equivalences in $\mathcal{F}(\mathcal{X})$.

There are many instances in which we would like to consider certain elements of a set to be the same. When we consider for instance all multiples of 2 (two)

in $\mathbb{R}$ to be the same, this is a generalization of the notion of equality. This generalization is well expressed in equivalence relations.
In the following paragraphs, we establish some equivalence relations in the set of fuzzy subsets of a finite set X.

2.6.1 Equivalence $\mathcal{R}_1$

Let us define in $\mathcal{F}(\mathcal{X})$ the relation $\mathcal{R}_1$ such that $\lambda_1 \mathcal{R}_1 \lambda_2$ if and only if there exists an integer t such that $0 \leq |t| \leq (m-1)$ and $|\lambda_2| = |\lambda_1| + \frac{t}{m-1}$.

Theorem 2.6.1 The relation $\mathcal{R}_1$ defined on $\mathcal{F}(\mathcal{X})$ as above is an equivalence on $\mathcal{F}(\mathcal{X})$.

1. It is clear that $\forall \lambda \in \mathcal{F}(\mathcal{X})$, $\lambda \mathcal{R}_1 \lambda$. In fact There exist $t = 0$ such that $|\lambda| = |\lambda| + \frac{0}{m-1}$. That is $\mathcal{R}_1$ is reflective.
2. Suppose λ_i and λ_j be two fuzzy subsets of X such that $\lambda_i \mathcal{R}_1 \lambda_j$. Then there exists $t \in \mathbb{Z}$ such that $|\lambda_j| = |\lambda_i| + \frac{t}{m-1}$. This means also for the same t that there is $r = -t$ such that $|\lambda_i| = |\lambda_j| + \frac{r}{m-1}$. Thus $\lambda_j \mathcal{R}_1 \lambda_i$ and therefore $\mathcal{R}$ is symmetric.
3. Let us assume now that $\lambda_i \mathcal{R}_1 \lambda_j$ and $\lambda_j \mathcal{R}_1 \lambda_p$. This means respectively that there is $t \in \mathbb{Z}$ such that $|\lambda_j| = |\lambda_i| + \frac{t}{m-1}$ and $r\, in\, \mathbb{Z}$ such that $|\lambda_p| = |\lambda_j| + \frac{r}{m-1}$. There exists $s = (t+r) \in \mathbb{Z}$ such that $|\lambda_p| = |\lambda_i| + \frac{s}{m-1}$. Thus $\lambda_i \mathcal{R}_1 \lambda_p$. The relation $\mathcal{R}_1$ is therefore transitive. Hence the relation $\mathcal{R}_1$ is an equivalence in $\mathcal{F}(\mathcal{X})$.

By definition, $\mathcal{R}_1$ implies dominance. Therefore each equivalence class has a top element from which can be deduced naturally each member of the class. Let $\mu_j = \mu_j(x_1)\mu_j(x_2)\cdots\mu_j(x_n)$ be the top element. The members of the class just below it are of the form: $\mu_j(x_1) - \frac{1}{m-1}\mu_j(x_2)\cdots\mu_j(x_n)$; $\mu_j(x_1)\mu_j(x_2) - \frac{1}{m-1}\mu_j(x_3)\cdots\mu_j(x_n)$; $\cdots \mu_j(x_1)\mu_j(x_2)\cdots\mu_j(x_n) - \frac{1}{m-1}$ Below these ones are the fuzzy subsets which are such that the membership values of two elements only at a time are each $\frac{1}{m-1}$ less than the respective membership values of the two elements in μ_j while the remaining elements' membership values are unchanged. The hunt of these fuzzy subsets will continue until the membership values of all elements are $\frac{1}{m-1}$ less than those of each element in

μ_j.
Now we wish to enumerate the elements in an equivalence class of $\mathcal{R}_1$. In how many ways the fuzzy subset $\mu_j = \mu_j(x_1)\mu_j(x_2)\cdots\mu_j(x_n)$ top of an equivalence class can be transformed in such a way that we get fuzzy subsets below it where the membership value of one element of X at a time, that of two elements at a time, $\cdots$, and finally the membership values of all elements of X are $\frac{1}{m-1}$ less than those in the equivalent positions in μ_j respectively.

Example 2.6.2 .
Refer to the diagram attached to the thesis.
Consider $\mathcal{F}(\mathcal{X})$ the set of all possible fuzzy subsets of $X = \{x_1, x_2, x_3\}$ with membership values in $M = \{0, \frac{1}{2}, 1\}$. One typical equivalence class is
$\{11\frac{1}{2}$, [110, $1\frac{1}{2}\frac{1}{2}$, $\frac{1}{2}1\frac{1}{2}$], [$1\frac{1}{2}0$, $\frac{1}{2}10$, $\frac{1}{2}\frac{1}{2}\frac{1}{2}$], $\frac{1}{2}\frac{1}{2}0\}$

It is clear that $11\frac{1}{2}$ is the top of the class from which 110 and $1\frac{1}{2}\frac{1}{2}$ are obtained each with one membership at a time $\frac{1}{3-1}$ less than that of $11\frac{1}{2}$. Again each of the fuzzy subsets $1\frac{1}{2}0$, $\frac{1}{2}10$, $\frac{1}{2}\frac{1}{2}\frac{1}{2}$] is such that two membership values are $\frac{1}{3-1}$ less than those in the top, that is $11\frac{1}{2}$ while $\frac{1}{2}\frac{1}{2}0$ being below every member of the class has three membership values each $\frac{1}{3-1}$ less than those in $11\frac{1}{2}$.

2.6.2 Equivalence $\mathcal{R}_2$

Let μ be a fuzzy subset of X. Consider two other fuzzy subsets λ_1 and λ_2 of a finite set X. Let us define in $\mathcal{F}(\mathcal{X})$ the relation $\mathcal{R}_2$ such that $\lambda_1\mathcal{R}_2\lambda_2$ if and only if $d(\mu, \lambda_1) = d(\mu, \lambda_2)$, where $d(\mu, \lambda)$ is the Hamming distance between a given fuzzy subset μ and any fuzzy subset λ in $\mathcal{F}(\mathcal{X})$.
1. Clearly $\lambda\mathcal{R}_2\lambda\ \forall\,\lambda \in \mathcal{F}(\mathcal{X})$.
2. Now if $\lambda_1\mathcal{R}_2\lambda_2$, it is evident that $\lambda_2\mathcal{R}_2\lambda_1$.
3. It is obvious that if $\lambda_1\mathcal{R}_2\lambda_2$ and $\lambda_2\mathcal{R}_2\lambda_3$, then $\lambda_1\mathcal{R}_2\lambda_3$.

Theorem 2.6.3 The relation $\mathcal{R}_2$ defined on $\mathcal{F}(\mathcal{X})$ such that $\lambda_1\mathcal{R}_2\lambda_2$ if and only if $d(\mu, \lambda_1) = d(\mu, \lambda_2)$, where $d(\mu, \lambda)$ is the Hamming distance between a given fuzzy subset μ and any fuzzy subset λ is an equivalence on $\mathcal{F}(\mathcal{X})$ and the set of fuzzy subsets at a distance d from the given fuzzy subset μ is an equivalence class.

This theorem says that the fuzzy subsets of X can be classified in term of their distance from a given fuzzy subset μ of X.

Example 2.6.4 .
If we consider $\mathcal{F}(\mathcal{X})$ the set of all possible fuzzy subsets of $X = \{x_1, x_2, x_3\}$ with membership values in $M = \{0, \frac{1}{2}, 1\}$ and take $\mu = 111$, then the fuzzy subsets 110, $1\frac{1}{2}\frac{1}{2}$, 101, $\frac{1}{2}1\frac{1}{2}$, $\frac{1}{2}\frac{1}{2}1$, 011 are in one class characterized by the distance of each of them from $\mu = 111$ being 1. Another interesting classification of fuzzy subsets is that of considering the cardinality of fuzzy subsets. It is shown in the following paragraph that fuzzy subsets of same cardinality form a class.

2.6.3 Equivalence $\mathcal{R}_3$

Let us define in $\mathcal{F}(\mathcal{X})$ the relation $\mathcal{R}_3$ such that two fuzzy subsets μ_1 and μ_2 are in relation $\mathcal{R}_3$ and we write $\mu_1 \mathcal{R}_3 \mu_2$ if and only if $\mid \mu_1 \mid = \mid \mu_2 \mid$.
Properties of the relation $\mathcal{R}_3$.
1. Reflexivity: $\forall \mu \in \mathcal{F}(\mathcal{X})$, $\mu_1\ \mathcal{R}_3\ \mu_1$ since $\mid \mu_1 \mid = \mid \mu_1 \mid$.
2. Symmetry: $\forall \mu_1, \mu_2 \in \mathcal{F}(\mathcal{X})$, If $\mu_1\ \mathcal{R}_3\ \mu_2$, then $\mu_2\ \mathcal{R}_3\ \mu_1$. In fact if $\mid \mu_1 \mid = \mid \mu_2 \mid$ then $\mid \mu_2 \mid = \mid \mu_1 \mid$, i.e. $\mu_2\ \mathcal{R}_3\ \mu_1$.
3. Transitivity: $\forall \mu_1, \mu_2, \mu_3 \in \mathcal{F}(\mathcal{X})$, If $\mu_1\ \mathcal{R}_3\ \mu_2$ and $\mu_2\ \mathcal{R}_3\ \mu_3$, then $\mu_1\ \mathcal{R}_3\ \mu_3$. In fact if $\mid \mu_1 \mid = \mid \mu_2 \mid$ and $\mid \mu_2 \mid = \mid \mu_3 \mid$, then $\mid \mu_1 \mid = \mid \mu_3 \mid$ which means $\mu_1\ \mathcal{R}_3\ \mu_3$.
From the above, we can establish the following theorem which says:

Theorem 2.6.5 The relation $\mathcal{R}_3$ defined above by $\mu_1 \mathcal{R}_3 \mu_2$ if and only if $\mid \mu_1 \mid = \mid \mu_2 \mid$ is an equivalence relation in the set $\mathcal{F}(\mathcal{X})$.

In the following theorem, we show that fuzzy subsets of same cardinality are all located at the same distance from a given fuzzy subset μ of set X.

Theorem 2.6.6 Theorem 2.6.3 and Theorem 2.6.5 imply that for any two fuzzy subsets μ_1 and μ_2 of a finite set X, $|\mu_1| = |\mu_2|$ if and only if:

$d(\mu, \mu_1) = d(\mu, \mu_2)$.

Proof. Let μ_1 and μ_2 be any two fuzzy subsets of a finite set X such that $|\mu_1| = |\mu_2|$. That is $\mu_1(x_1)+\mu_1(x_2)+\cdots+\mu_1(x_n) = \mu_2(x_1)+\mu_2(x_2+\cdots+\mu_2(x_n)$ and therefore $|\mu(x_1)-\mu_1(x_1)|+|\mu(x_2)-\mu_1(x_2)|+\cdots+|\mu(x_n)-\mu_1(x_n)| = |\mu(x_1)-\mu_2(x_1)|+|\mu(x_2)-\mu_2(x_2)|+\cdots+|\mu(x_n)-\mu_2(x_n)|$. This means $d(\mu, \mu_1) = d(\mu, \mu_2)$. Now assume $d(\mu, \mu_1) = d(\mu, \mu_2)$. Then $|\mu(x_1)-\mu_1(x_1)|+|\mu(x_2)-\mu_1(x_2)|+\cdots+|\mu(x_n)-\mu_1(x_n)| = |\mu(x_1)-\mu_2(x_1)|+|\mu(x_2)-\mu_2(x_2)|+\cdots+|\mu(x_n)-\mu_2(x_n)|$. Or $(|\mu_1(x_1)-\mu(x_1)|+|\mu(x_1)-\mu_2(x_1)|)+(|\mu_1(x_2)-\mu(x_2)|+|\mu(x_2)-\mu_2(x_2)|)+\cdots+(|\mu_1(x_n)-\mu(x_n)|+|\mu(x_n)-\mu_2(x_n)|) \geq |\mu_1(x_1)-\mu_2(x_1)|+|\mu_1(x_2)-\mu_2(x_2)|+\cdots+|\mu_1(x_n)-\mu_2(x_n)|$

Now if we consider the theorem 2.6.6, we can write the number of equivalence class as well as the number of members in an equivalence class.
We will state and improve some results found in [45] in this regard.
Let $|X| = n$, $|M| = m$ and $|\ Card_\mu\ |$ be the set of cardinalities of fuzzy subsets, while $(N_\mu(|\ .\ |= \alpha))$ is the number of fuzzy subsets of cardinality α; then

Proposition 2.6.7 $|\ Card_\mu\ | = \ (m-1)n+1.$ and
$(N_\mu(|\ .\ |= \alpha))_n = \sum_{i \in M}(N_\mu(|\ .\ |= \alpha - i))_{n-1}.$

Proof: On the left of the above equation we are counting the number of fuzzy subsets of set $X = \{x_1, x_2, x_3, \cdots, x_n\}$, while on the right of the equation we count the fuzzy subsets of the set $X' = \{x_1, x_2, x_3, \cdots, x_{n-1}\} = X \setminus \{x_n\}$.
Any fuzzy subset μ of set X is obtained from a fuzzy subset μ' of set X' by associating with element x_n the difference between $|\ \mu\ |$ and $|\ \mu'\ |$, if this difference belongs to M, which proves the proposition. $\square$.

The above proposition can be extended to include every $\alpha \in Card_\mu$. We express the new result in the following proposition.

Proposition 2.6.8 Let the cardinalities of X and M be n and m respectively. The number of fuzzy subsets of X with cardinality $p \in Card_\mu$ is equal to the sum of the number of fuzzy subsets of $X \setminus \{x_i\} \forall x_i \in X$ with cardinality p and the number of fuzzy subsets of $X \setminus \{x_i\} \forall x_i \in X$ with cardinality $p - \frac{1}{m-1}$.

The above proposition can be illustrated by the following table of $N_\mu(| \, . \, |= p)$. We call this table *Pascal Rectangle* since it is rectangular and the entries are obtained in a fashion similar to the process in the *Pascal triangle*.

n/p	0	$\frac{1}{m-1}$	$\frac{2}{m-1}$	$\frac{3}{m-1}$	$\cdots$	$\frac{m-1}{m-1}$	$\cdots$	n
1	1	1	1	1	$\cdots$	1	$\cdots$	0
2	1	2	3	3	$\cdots$	3	$\cdots$	0
3	1	3	6	8	$\cdots$	9	$\cdots$	0
$\vdots$	$\cdots$	$\vdots$	10	17	$\cdots$			

From this table we can deduce some interesting results which are recorded under the following two propositions.

Proposition 2.6.9 Let $Card_\mu$ be the set of cardinalities of fuzzy subsets of a set X with membership values in M and $N_\mu(| \, . \, |= p)$ be defined as above. Consider $p \in Card_\mu$. If $|X| = 1$ then
1. $(N_\mu(| \, . \, |= p)) = 1$ if $p \leq 1$ and
2. $(N_\mu(| \, . \, |= p)) = 0$ if $p > 1$.

Proof: For part (1) we refer to p. 61 of [45]. Now for part (2): Suppose $p > 1$ and $|X| = 1$. The only element of X can only have a membership $p \leq 1$, but not $p > 1$. Therefore $(N_\mu(| \, . \, |= p) = 0$ in such case.

Proposition 2.6.10 The number of fuzzy subsets of X with cardinality $p \in Card_\mu$ is equal to the sum of the number of fuzzy subsets of $X - \{x_i\}$, $\forall x_i \in X$ with cardinality p and the number of fuzzy subsets of $X - \{x_i\} \forall x_i \in X$ with cardinality $p - \frac{1}{m-1}$.
$(N_\mu(| \, . \, |= p))_{inX} = (N_\mu(| \, . \, |= p))_{inX-\{x_i\}} + (N_\mu(| \, . \, |= p - \frac{1}{m-1}))_{inX-\{x_i\}}$

Our focus now is to enumerate the fuzzy subsets of X of cardinality p for which some $k \leq n$ given elements of X have non-zero membership values. This means $X = X_1 \cup X_2$ with $X_1 = \{x_1, x_2, \cdots x_k\}$ and $X_2 = \{x_{k+1}, \cdots x_n\}$. It is clear that $|X_1| = k$, while $|X_2| = n - k$. Let us consider a fuzzy subset μ of X_1, $|\mu| = \alpha > 0$.
By our previous paragraph we count:
$r_1 = (N_\mu(| \, . \, |= \alpha))_k = \sum_{i \in M \setminus \{0\}} (N_\mu(| \, . \, |= \alpha - i))_{k-1}$ fuzzy subsets of X_1 of car-

dinality α.

Consider also the set $X_2 = X \setminus X_1$ with $(n-k)$ elements. Any fuzzy subset of X_2 has cardinality $(p-\alpha)$. Therefore there are as per the same reason $r_2 = (N_\mu(| \ . \ |= \alpha))_{n-k} = \sum_{i \in M} (N_\mu(| \ . \ |= p - \alpha - i))_{n-1}$ fuzzy subsets of X_2 of cardinality $(p-\alpha)$. Now each fuzzy subset of X is actually made of one from X_1 and one from X_2. For each fuzzy subset of X_1 there are r_2 choices of fuzzy subsets of X_2 to combine with. This in short means that for every subset of X we have $\sum_{0<\alpha<p} r_1.r_2$.

Example 2.6.11 .
Refer to the diagram attached to the thesis.
Let $X = \{x_1, x_2, x_3\}$ $M = \{0, \frac{1}{2}, 1\}$ so that $Card_\mu = \{0, 1, 2, 3, \cdots n\}$. We consider $X_1 = \{x_1, x_2\}$ and $X_2 = \{x_3\}$. Take $p = 3$ so that $0 < \alpha < 3$. That means $\alpha = \frac{1}{2}; 1; 1\frac{1}{2}; 2; 2\frac{1}{2}$.
If $\alpha = \frac{1}{2}$; $(N_\mu(|.| = \frac{1}{2}))_{X_1} = 2$, $(N_\mu(|.| = (3 - \frac{1}{2}))_{X_2} = 0$
If $\alpha = 1$; $(N_\mu(|.| = 1))_{X_1} = 3$, $(N_\mu(|.| = (3 - 1)))_{X_2} = 0$
If $\alpha = 1\frac{1}{2}$; $(N_\mu(|.| = 1\frac{1}{2})_{X_1} = 2$, $N_\mu(|.| = (3 - 1\frac{1}{2}))_{X_2} = 0$
If $\alpha = 2$; $(N_\mu(|.| = 2)_{X_1} = 1$, $N_\mu(|.| = (3 - 2))_{X_2} = 1$.
If $\alpha = 2\frac{1}{2}$; $(N_\mu(|.| = 2\frac{1}{2})_{X_1} = 2$, $N_\mu(|.| = (3 - 2\frac{1}{2}))_{X_2} = 1$.
Therefore the number of fuzzy subset of cardinality $p = 3$ when two elements of X have non-zero membership value is $2.0 + 3.0 + 2.0 + 1.1 = 1$.
When $p = 2\frac{1}{2}; k = 2; 0 < \alpha < 2\frac{1}{2}$, the number of fuzzy subsets of cardinality $2\frac{1}{2}$ with $k = 2$ elements of X have non-zero membership is 3. Those of cardinality $p =$ when only one (1) element of X has a non-zero membership are 5.

2.7 Partition of set X

The importance of partitioning or clustering of data is well documented in the literature. Various areas such as taxonomy, medecine, geology, business, image processing use this process extensively.
In databank marketing, the bank tries to subdivide its customers into segments

or clusters which are homogeneous with respect to the needs of the customer in a segment. Now they can offer special products only to segments which have a high demand for the product. This is called customer segmentation. Using fuzzy clustering allows us to classify marginal customers, dynamic changes of the customers can be identified via changes of degree of membership of customers to cluster.
We refer to [12] [11] [4] [18] [38] [50]
Consider a non-empty set X, a fuzzy subset μ_1 and a real number $\alpha \in [0, 1]$. We can partition the set X into the two following subsets in the following manner:
$X^{\alpha}_{\mu_1} = \{x \in X; \mu_1(x) > \alpha\}$.
$X_{\alpha_{\mu_1}} = \{x \in X; \mu_1(x) \leq \alpha\}$ such that $X = X^{\alpha}_{\mu_1} \cup X_{\alpha_{\mu_1}}$.
If $\mu_1 \leq \mu_2$ we have the following proposition concerning their α-cuts.

Proposition 2.7.1 1. $X^{\alpha}_{\mu_1} \subseteq X_{\alpha_{\mu_2}}$, and
2. $X_{\alpha_{\mu_1}} \subseteq X_{\alpha_{\mu_2}} \cup X^{\alpha}_{\mu_2}$.

Proof: 1. Let $\mu_1 \leq \mu_2$, if $\mu_1(x) > \alpha$ or better $x \in X^{\alpha}_{\mu_1}$ then $\mu_2(x) > \mu(x) > \alpha$ which means that $x \in X^{\alpha}_{\mu_2}$.
2. Now if $x \in X_{\alpha_{\mu_1}}$, then $\mu_1(x) \leq \alpha$. We may have either $x \in X_{\alpha_{\mu_2}}$, that is $\mu_2(x) \leq \alpha$ or $\mu_2(x) \geq \alpha$, which means $x \in X^{\alpha}_{\mu_2}$. In both cases we have $X_{\alpha_{\mu_2}} \cup X^{\alpha}_{\mu_2}$.

Proposition 2.7.2 1. $X^{\alpha}_{\mu_1 \vee \mu_2} = X^{\alpha}_{\mu_1} \cup X^{\alpha}_{\mu_2}$
2. $X_{\alpha_{\mu_1 \wedge \mu_2}} \subseteq X_{\alpha_{\mu_1}} \cup X_{\alpha_{\mu_2}}$.

Proof: The proof is similar to the above proposition.

2.7.1 Fuzzy Partition

. **Definition of Fuzzy Partition**
Several concepts for defining fuzzy partitions of the universe X have been proposed in the literature.
One definition uses the idea of covering of X and the pair-wise disjointedness property suggested by E. H. Ruspini in [38]. He was among the first to propose the generalization of fuzzy partition. The other way of defining fuzzy partition is to use the set operations of T-norm and T-conorm S, which

replace the usual axioms for a partition by appropriate formulas. Note that $S(a,b) = 1 - T(1-a, 1-b)$. [13]
We will use the Ruspini definition.

The following definition is more or less established throughout the literature. A system P of fuzzy subsets of X is called a partition of X if the following properties are satisfied.
1. $\forall \mu \in P$, there is some $x \in X$, $\mu(x) = 1$.
2. $\forall x \in X$ there exists exactly one $\mu \in P$, $\mu(x) = 1$.
3. If $\mu, \lambda \in P$ such that $\mu(x) = \lambda(y) = 1$, then $\mu(y) = \lambda(x)$.[19]
There have been various extension of properties defining the fuzzy equivalence in the literature and therefore there are various extension of the concept of fuzzy partition.

Suppose that every object x_j from a set $X = \{x_1, x_2, \cdots, x_n\}$ should be placed into k classes $\{C_1, C_2, \cdots, C_k\}$ with $2 \leq k < n$. This is possible since each element of X can partially belong to each fuzzy class.
Let us set a $k \times n$- matrix $U = \{u_{ij}\} \cdot, 1 \leq i \leq k\,,\ 1 \leq j \leq n$, where u_{ij} represents the degree of membership of element x_j in μ_i. If $u_{ij} = \{0, 1\}$, then the matrix $U = \{u_{ij}\}$ represents a *hard* or *crisp* partition of X. But in contrary if $u_{ij} \in [0, 1]$ we get this time a fuzzy partition of X . Let $P = \{\mu_1, \mu_2, \cdots, \mu_k\}$ be a family of fuzzy subsets of X. For P to be a partition of X, each element of X must have a different membership value in each of these fuzzy subsets of the family.
Let $x \in X$, and P a *hard* partition of X. Then this partition P is possible if $\displaystyle\sum_i^k \mu_i(x) = 1$ and $\displaystyle\sum_j^n \mu_i(x_j)$ >0 [7][9]. These two conditions are valid for fuzzy partitions as well because every hard partition is also a fuzzy partition.[33]
The first condition is justified by the fact that x belongs to X, The union of members of P must be X. That is $\bigcup_{\mu_i \in P} \mu_i(x) = X$ the maximum membership value of an element x to a fuzzy subset of P is 1. The second condition also is justified since each $\mu(x) > 0$ and if $\displaystyle\sum_j^n \mu_i(x_j) = 0$, then this would mean each μ_i is empty therefore not part of any partition.

A hard, resp. fuzzy, partition is called non-degenerate if and only if $\sum_{i}^{k} u_{ij} \geq 1$ for row-wise sums and $\sum_{j}^{n} u_{ij} \geq 1$ for column-wise sums hold.

We will therefore simply use the term partition if no confusion is expected.

The condition $\sum_{i}^{k} \mu_i(x) = 1$ actually means that the sum of the membership values of any element $x \in X$ equals 1. This however has an outlier in the data. If one element of X has membership value *zero* in all the fuzzy subsets; how would the sum of its membership values be equal to 1? In which fuzzy subset this element is assigned?

In their paper [13], the authors introduce the concept of redundancy for families of fuzzy subsets. It may happen that one block of a fuzzy partition is fully or partly a subset of the union of the other blocks. That means that the information contained in this block is somewhat redundant as it is fully or partly available from the other blocks. Hence the need for introducing the redundancy of a family of fuzzy subsets of the universe of discourse to measure the degree to which some blocks of the family form a subset of the union of the others. It is therefore useful to look for partition with minimal redundancy, in order to simplify fuzzy systems for classification and control.

From the definition of a partition set above, we can draw some conclusions which constitute the following propositions.

Proposition 2.7.3 Two members of a partition P of X are either non-overlapping or identical.

Proof. Let's assume that $\mu \equiv \lambda$ as in [29]. That is $\mu(x) \geq \mu(y) \Leftrightarrow \lambda(x) \geq \lambda(y)$. With condition $\mu(x) = \lambda(y) = 1$ for $x, y \in X$, then $\mu(y) = \lambda(x)$. We have $1 \geq \mu(y) \Leftrightarrow \lambda(x) \geq 1$. This is possible only if $\mu(y) = \lambda(x) = 1$. Therefore $\mu = \lambda$.

Proposition 2.7.4 Let P be a partition of X. If there exists a fuzzy subset $\mu_i \in P$ such that $\mu_i(x) = 1$, then μ_i is unique.

Proof:In fact, imagine there were μ_1 , $\mu_2 \in P$ such that $\mu_1(x) = \mu_2(x) = 1$. This means that for only these two fuzzy subsets of P, $\sum_i^k \mu_i(x) = 2$. This is contrary to the condition $\sum_i^k \mu_i(x) = 1$ stated above.

Proposition 2.7.5 For $\mu, \lambda \in P$; $\mu(x) \cap \lambda(x) \leq 0,5$ and $\bigcup_{\mu \in P} \mu = X$.

Proof: If $x \in X$ assume $\mu(x) \cap \lambda(x) \nleq 0,5$ for $\mu, \lambda \in P$. Then $\mu(x) \cap \lambda(x) > 0,5$ and therefore $\mu(x) = \lambda(x) = 1$. This contradicts the uniqueness of μ such that $\mu(x) = 1$. In short $\mu(x) \cap \lambda(x) \leq 0,5$. The last part of the proposition stands by the definition of partition. Two fuzzy subsets μ and λ satisfying the above proposition are said to be $weak-\ separated$ fuzzy subsets.
Let P be a partition of X. The condition $\mu(x) \cap \lambda(x) \leq 0,5$ can be made strict, that is: $\mu(x) \cap \lambda(x) < 0,5$.

Proposition 2.7.6 If $x \in X$ and $\mu \in P$ such that $\mu(x) \geq 0.5$, then $\mu(x) = 1$'

Proof: Assume that $\mu(x) \geq 0.5$ and $\mu(x) \neq 1$. By definition of partition there exists $\lambda \in P$ with $\mu \neq \lambda$ such that $\lambda(x) = 1$. This means $(\mu \cap \lambda)(x) \geq 0.5$ which contradicts the proposition above. Therefore $\mu(x) = 1$.
Let $\mathcal{P}(\mathcal{X})$ be the set of all partitions of the set X; for any pair of partitions $P_i = \{\mu_i, i \in I\}$ and $P_j = \{\lambda_j, j \in J\}$ of X; $P_1 \leq P_2$ if and only if for every $i \in I$, $\mu_i \leq \lambda_j$ for some $j \in J$.
Therefore $\mathcal{P}(\mathcal{X})$ is a poset. The partition consisting of X alone is the greatest element while the partition $P = (\{x\}, x \in X)$ is the least element of the poset.

A family P of fuzzy subsets is called S-H fuzzy family if and only if $\forall \mu, \lambda \in P$ $\mu \cap \lambda \geq \lambda(a)$ with $\mu(a) = 1$. For a family of fuzzy subsets to be a S-H partition, not a single fuzzy subset of the family must be weakly empty fuzzy subset. By definition of fuzzy partition there must be at least an x of X for which the membership $\mu(x) = 1$. [43]

2.7.2 The α- cut of a Partition

For a given $\alpha \in [0,1]$, any fuzzy subset can be approximated by its α-cut by writing that:

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \mu^\alpha \\ 0, & \text{if } x \notin \mu^\alpha. \end{cases}$$

[7] [9] [6]

This indeed will give rise to a matrix that represents a partition of X when all the fuzzy subsets of a partition as well as their respective α-cuts are considered. This matrix will be called the α-cut of the partition.
This α-cut of a partition is characterized by a $k \times m$ matrix where entries μ_{ij} are either 1 if $\forall 1 \leq i \leq k$, $\forall 1 \leq j \leq m$, $\mu_i(x) \geq \alpha$ or 0 otherwise.

One naturally defined unique set partition of X associated with each fuzzy subset of X is obtained if we consider $\forall \alpha_i \in [0,1]$ the subset of X described by $\{x \in X : \ \mu^{-1}(\alpha_i) = x\}$. The family $\{\mu^{-1}(\alpha_i) = x\}$ $1 \leq i \leq m$ if $Im_\mu(X) = \{\alpha_1, \alpha_2, \cdots \alpha_m\}$ is a partition of X. This partition is called the *kernel of* μ.

If $\mu^{-1}(0) = X$, then the support of μ, denoted $Supp\mu$ is $\emptyset$. This means that $\forall x \in X$ $\mu(x) \ngtr 0$. With regards to $\alpha = 0$; the 0-cut of any partition of X will have all entries equal to 1. On the other hand, if $\mu^{-1}(0) = \emptyset$, then the support of μ is X.
Let us illustrate this with an example.

Example 2.7.7 .
Consider the partition $P_1 = \{\{x_1\}, \{x_2, x_3\}\}$ of $X = \{x_1, x_2, x_3\}$. This may be obtained if we use alternatively the fuzzy subsets $\mu_1 = 1\frac{1}{2}\frac{1}{2}$, $\mu_2 = \frac{1}{2}11$, $\mu_3 = 0\frac{1}{2}\frac{1}{2}$, $\mu_4 = 100$. The partition $P_2 = \{\{x_1, x_2, x_3\}\}$ is generated by $\{111,\ \frac{1}{2}\frac{1}{2}\frac{1}{2},\ 000\}$.
The partition $P_3 = \{\{x_1, x_2\},\ \{x_3\}\}$ is generated by the fuzzy subsets
$11\frac{1}{2}$ 110 $\frac{1}{2}\frac{1}{2}0$, $\frac{1}{2}\frac{1}{2}1$, 001, $00\frac{1}{2}$.
$P_4 = \{\{x_1, x_3\},\ \{x_2\}\}$ is generated by $1\frac{1}{2}1$, 101, $0\frac{1}{2}0$, $\frac{1}{2}0\frac{1}{2}$.
$P_5 = \{\{x_1\}.\ \{x_2\},\ \{x_3\}\}$ is generated by $0\frac{1}{2}1$, $01\frac{1}{2}$, $\frac{1}{2}01$, $\frac{1}{2}10$, $10\frac{1}{2}$, $1\frac{1}{2}0$.

It is clear from this context that two fuzzy subsets of a set may generate the same *kernel*- partition of the set. We may ask the following question. How may fuzzy subsets of the set generate the same partition. Subsequent to this question we may even find out the number of $Kernel$-partition we get in a n-element set X with membership values in a m-element set.
The equivalence of fuzzy subsets has been extensively studied. We want to look in the next section into the equivalence of fuzzy partitions.

2.7.3 Equivalent Fuzzy Partitions.

Let A and B be two fuzzy partitions, each of which is represented by its α-cut matrix.

1° Definition. A and B are said to be α-equivalent and we write $A =_\alpha B$ if their α-cut matrices A^α and B^α are equal.

2° The $=_\alpha$-equivalence.
The relation $=_\alpha$ is indeed an equivalence relation in the set of all partitions of set X.
Consider two matrices A^α and B^α with associated fuzzy subsets $\mu_i, i \in I$ and $\lambda_j, j \in J$. We can write that $A^\alpha \subseteq B^\alpha$ if and only if $\forall \alpha \in [0,1], \forall x \in X, \mu_{ij}(x) \leq \lambda_{ij}(x)$.
$A^\alpha = B^\alpha \; \forall\, \alpha \in [0,1]$ if and only if $A = B$.
Let A be a fuzzy partition. We consider the $r \times n$-matrix, $1 \leq i \leq r$ and $1 \leq j \leq n$ characterized by:

$$\mu_i(x) = \begin{cases} 1, & \text{if } x_j \in Supp_{\mu_i} \\ 0, & \text{if otherwise. That is } x_j \notin Supp_{\mu_i}. \end{cases}.$$

This matrix is called the *Support matrix* of the partition A denoted $Supp_A$.
Now if two k-partitions A and B are such that $Supp_A = Supp_B$, then for each fuzzy subset $\mu_i \in A$ and $\gamma \in B$, if $x \in Supp_{\mu_i}$ then $x \in Supp_{\gamma_j}$. Similarly if $x \notin Supp_{\mu_i}$ then $x \notin Supp_{\gamma_j}$. We note here that $x \in Supp_{\mu_i}$ and $x \in Supp_{\gamma_j}$ does necessarily mean that $\mu_i = \gamma_j$. Otherwise equality of the μ_i and γ_j would mean that $A = B$.
In the next section we extend the notion of Hamming Distance between two

fuzzy subsets to two fuzzy partitions. The distance between partition is an indication of their being equivalent or not.

2.7.4 Distance between two Fuzzy Partitions of X.

The idea of distance between two fuzzy partitions of X is to measure the extend to which two fuzzy partitions of X are close or far from one another for them to be α-equivalent. Therefore this distance will involve naturally their α-cuts.
We will use the *Hamming* distance between the fuzzy matrices A^α and B^α, for a given $\alpha \in [0,1]$ as follows: $H(A^\alpha, B^\alpha) = \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}|\mu_{ij} - \gamma_{ij}|$. [2] [7]
By definition of distance, we have $0 \leq H(A^\alpha, B^\alpha) \leq 1$;
$H(A^\alpha, B^\alpha) = H(B^\alpha, A^\alpha)$ and $H(A^\alpha, B^\alpha) = 0$ if and only if $A^\alpha = B^\alpha\ \forall\ \alpha \in [0,1]$. Now
$H(A^\alpha, B^\alpha) = 0$ if and only if $A =_\alpha B$. This means they are equivalent.
$H(A^\alpha, B^\alpha) = \frac{1}{2}$ then A and B are said to be nearly equivalent.
$H(A^\alpha, B^\alpha) = 1$ then A and B are said to be totally not equivalent.

2.7.5 The Complement of a Partition

Let A be a fuzzy partition of X. The fuzzy subsets of this family each of which has a complement. Let A be represented by its α-cut matrix A^α. The following discussion illustrates the way in which the family of fuzzy subsets constituting a fuzzy partition relates to the family of their fuzzy complements constituting the complement of the fuzzy partition.

Consider an $\alpha \in [0,1]$ with $\alpha \geq 0.5$,and a fuzzy subset $\mu \in A$. If $\mu(x) \geq \alpha$, then $1 - \mu(x) \ngeq \alpha$. On the other hand if $\mu(x) < \alpha$, then $1 - \mu(x) \geq \alpha$. We consider now $\alpha < 0.5$. If $\mu(x) \geq \alpha$ and $\mu(x) \leq 0.5$, that is $\alpha \leq \mu(x) \leq 0.5$, then $1 - \mu(x) \geq 0.5$. And if $\mu(x) < \alpha$ then $1 - \mu(x) \ngeq \alpha$. This clearly means that the α-cut representing the family A^c is deduced from the one representing A when 1 is replaced by 0 and 0 by 1. We write the α-cut representing the complement as $A^{1-\alpha}$.
We conclude that the fuzzy partition A and its fuzzy complement A^c are to-

tally not equivalent since $H(A^\alpha, A^{1-\alpha}) = 1$.

2.7.6 The Union of two Partitions.

Let A and B be two partitions of k fuzzy subsets of X represented by their respective α-cut matrices. We are interested to find out whether or not the family obtained by taking the union $\mu \cup \gamma$ such that $\forall \mu_i \, \gamma_i,\ 1 \leq i \leq k,\ 1 \leq j \leq n\ \mu_i(x_j) \vee \gamma_i(x_j)$ is also a partition of X.

Consider $\alpha \in [0,1]$, $x \in X$. We have these three situations:
1. If $x \in \mu_i^\alpha$ and $x \in \gamma_i^\alpha$, then $x \in (\mu_i \cup \gamma_i)^\alpha$.
2. If x belongs either to μ_i^α or to γ_i^α, then still $x \in (\mu_i \cup \gamma_i)^\alpha$.
3. If x belongs neither to μ_i^α nor to γ_i^α, then indeed $x \notin (\mu_i \cup \gamma_i)^\alpha$

From the above we can also write that:
If $\mu^{-1}(0) = X$, then the support of μ denoted $Supp\mu$ is $\emptyset$. This means that $\forall x \in X \mu(x) \ngtr 0$. With regards to $\alpha = 0$ and $\forall \mu_i$ we realize that the 0-cut of any partition of X will have all entries equal to 1. On the other hand if $\mu^{-1}(0) = \emptyset$, then the support of μ is X.
Now we choose $\alpha = 1$. If no element x of X has $\mu_i(x) = 1 \forall i$, the $Core_{\mu_i} = \emptyset$; therefore the α-cut partition has all entries equal 0.
If an entry in the α-cut partition matrix of A or B is 1,then it is 1 in the α-cut of the union $A \cup B$ since $1 \vee 1 = 1 \vee 0 = 1$. With this consideration we can say that the union of two partitions is also a partition. It is represented by the $C = A^\alpha \vee B^\alpha$.

2.7.7 Intersection of two Fuzzy Partitions.

Let $A = \{\mu_1, \mu_2, \cdots, \mu_n\}$ and $B = \{\lambda_1, \lambda_2, \cdots \lambda_n\}$ be two partitions of X with A^α and B^α their respective α-cut matrices. We define $A \cap B = \{\mu_i \cap \lambda_i \forall i \in I\}$. We also consider $C = A^\alpha \cap B^\alpha$ such that each entry of C is $\mu_{ij} \wedge \lambda_{ij}$.
If an entry is 0 in one of the α-cut matrices, then it is 0 in the α-cut of the intersection $A \cap B$ since $0 \wedge 0 = 0 \wedge 1 = 0$.
The intersection of two partitions is also a partition represented by the $C^\alpha = A^\alpha \wedge B^\alpha$
We can state therefore that the partially ordered set of all partitions of a set

is a lattice. In this lattice, the greatest lower bound of A and B is easy to describe. Each $x \in X$ belongs to one of the A_i and one of the B_j, Therefore it belongs to $A_i \cap B_j$. Either an $A_i \cap B_j$ is empty or else it has no element in common with any of the $A_i \cap B_j$ in the fuzzy sense. Thus the nonempty fuzzy subsets $A_r \cap B_s$ form the classes of a partition of X. In any partition below both A and B, each block is a subset of one of the A_i and subset of one of the B_j. Thus each block is a subset of $A_i \cap B_j$.
Therefore the partition whose blocks are the nonempty intersections of $A_i^{'s}$ and $B_j^{'s}$ is the greatest lower bound for A and B.
It turns out that for finite lattices we can use the *meets* to describe the *joins*. If we consider this time two partitions A and B represented by A^{α_1} and B^{α_2} such that $\alpha_1 < \alpha_2$; we know that $\mu^{\alpha_2} \subset \mu^{\alpha_1}$ and if $x \in \mu^{\alpha_2}$, then $x \in \mu^{\alpha_1}$. This means that

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \mu^{\alpha_2} \\ 0, & \text{if } x \notin \mu^{\alpha_2}. \end{cases}$$

This actually means that the entries in the α-cut matrix of the intersection are those of the B_2^{α}.
In crisp set there is a correspondence between the set of equivalence classes and the set of partitions of the same set. In the next section we want to check the existence of a match between fuzzy binary equivalence relation and fuzzy partition.

2.8 Fuzzy Equivalence Relation and fuzzy partition

In any crisp set X, there is a one-to-one relation between an equivalence relation R and partition of the set. Every equivalence relation in a set has the result of partitioning the set into classes of elements xRy and conversely. So enumerating partitions X is possible if we can enumerate equivalence relations in the set X.
The number of partitions of an n-element set X into k parts is $S(n, k)$, well known as Stirling numbers of the second kind. Concerning these numbers the following are true:
1. $S(n, 1) = 1$,

2. $S(n,n) = 1$,
3. $S(n,k) = 0$ unless $1 \leq k \leq n$
4. $S(n,k) = S(n-1,k-1) + k.S(n-1,k)$
5. The number of functions from an n-element set onto a k-element set is $k!.S(n,k)$.

As it is in crisp case there have been attempts to match the binary equivalence and fuzzy partition.[5][42][44]
R. Mesiar in [19] has established a bijection between the set $\mathcal{E}(\mathcal{X})$ of all equivalence relations on set X and $\mathcal{P}(\mathcal{X})$ of all partitions on set X in this fashion:

Let $P_X \in \mathcal{P}(\mathcal{X})$, and let $R \in \mathcal{E}(\mathcal{X})$, $\forall x \in X$, $[x] = \{y \in X;\ (x,y) \in R\}$. If we write $P_X = \{[x]; / x \in X\}$, it is clear that $P_X \in \mathcal{P}(\mathcal{X})$. Again if $P_X \in \mathcal{P}(\mathcal{X})$, then the set $E_p = \{(x,y) \in X^2 / \{x,y\} \in U$ for some $U \in P_X\} \in \mathcal{E}(\mathcal{X})$.
This is a one-to-one relation such that $xRy \Leftrightarrow x, y \in U$ for some $U \in P$.
Let R be a fuzzy equivalence relation on an non-empty set X and $x \in X$, the set of elements $B(x) = \{y \in X; \mu_R(y,x) \geq 0.5\}$ is called the set of elements of X with strong bond with x.
The fuzzy equivalence class determined by $x \in X$, denoted $[x]$, is defined as: $[x] = \{y \in X, (y, \mu_R(y))\}$ where

$$\mu_R(y) = \begin{cases} 1, & \text{if } y \in B(x) \\ \mu_R(x,y), & \text{otherwise} \end{cases}$$

$[X] = \{[x], x \in X\}$ is the set all equivalence classes.
Every fuzzy equivalence relation induces a crisp partition in each of its α-cuts.
The fuzzy clustering problem can be viewed as the problem of identifying an appropriate fuzzy equivalence relation on a given data.

Example 2.8.1 . Consider the set $X = \{x_1, x_2, x_3, x_4\}$ and the fuzzy relation R defined in X^2 and represented by the table below:

R	x_1	x_2	x_3	x_4
x_1	1	0.8	0.3	0.2
x_2	0.7	1	0.4	0.3
x_3	0.3	0.2	1	0.1
x_4	0.3	0.3	0.2	1

We are now able to determine the equivalence classes of each element of X as follows:
$[x_1] = \{(x_1, 1), (x_2, 1), (x_3, 0.3), (x_4, o.2)\}$
$[x_2] = \{(x_1, 1), (x_2, 1), (x_3, 0.4), (x_4, 0.3)\}$
$[x_3] = \{(x_1, 0.3), (x_2, 0.2), (x_3, 1), (x_4, 0.1)\}$
$[x_4] = \{(x_1, 0.3), (x_2, 0.3), (x_3, 0.2), (x_4, 1)\}$

We can write the partition $P = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ such that
$\mu_1 = \{(x_1, 1), (x_2, 0.8), (x_3, 0.3), (x_4, 0.2)\}$
$\mu_2 = \{(x_1, 0.7), (x_2, 1), (x_3, 0.4), (x_4, 0.3)\}$
$\mu_3 = \{(x_1, 0.3), (x_2, 0.2), (x_3, 1), (x_4, 0.1)\}$
$\mu_4 = \{(x_1, 0.3), (x_2, 0.3), (x_3, 0.2), (x_4, 1)\}$

One of the important enumeration principles was developed in Chapter 1: That is the principle of inclusion and exclusion.
A similar basic principle called the Möbius inversion will be the object of our discussion in the next Chapter. This principle was thoroughly developed in the 1960's by G. C. Rota. His formulation of the general method of the Möbius inversion in number theory was all-inclusive and unified most of the results and has had far-reaching applications to partially ordered sets.
In our case we want to apply this Möbius inversion in the poset of fuzzy subsets of a finite set.
We will first recall the Möbius inversion in general and later define the principle in a specific context of fuzzy subsets of a finite set.

Chapter 3

Möbius function and Möbius inversion formula

3.1 Introduction

An important topic in Combinatorics is the study of Möbius functions and their application to inversion formulae for counting functions. The earlier form of Möbius function dealt with number theoretic considerations. Möbius inversion is an over counting-under counting, or sieve procedure. We keep track of the over and undercount by indexing with the elements of a partially ordered set which classically was the subsets of a finite set. The Möbius inversion formula of number theory as given in Hardy and Wright (1960) indexes functions with the set of positive integers under the divisibility order.
The classical PIE is a special case (Feller (1968), Ryser (1963) of inversion problem.
The statement of the general Möbius inversion formula was first given independently by Weisner (1935) and Philip Hall (1936). In a fundamental paper on Möbius functions, Rota (1964) showed the importance of the theory in Combinatorics. He noted the relationship between Möbius inversion and the principle of Inclusion-Exclusion. Both principles are based on over counting and under counting entities which are functions for Möbius inversion and elements of a set for the principle of Inclusion-Exclusion .

3.1.1 The Incidence function, Incidence algebra.

Let X be a locally finite partially ordered set. A function θ: $X \times X \to \mathbf{Z}$ is an *incidence function* provided that for $x, y \in X$ if $\theta(x, y) \neq 0$ then $x \leq y \in X$. In others words $\theta(x, y) = 0$ if $x \not\leq y$ in X. [17] [37]
The set of all such functions is denoted by $\mathcal{I}(\mathcal{X})$
Scalar multiples and sums of incidence functions are also incidence functions.
The product of incidence functions $\theta, \epsilon \in \mathcal{I}(\mathcal{X})$ is defined by

$$(\theta\epsilon)(x, y) = \sum_{z \in X} \theta(x, z)\epsilon(z, y). \tag{3.1.1}$$

By our assumption of X being locally finite, the above sum has finitely many nonzero terms.
This gives $\mathcal{I}(\mathcal{X})$ the structure of an associative algebra over $\mathbf{Z}$, called the *incidence algebra of* of X. The identity elements in $\mathcal{I}(\mathcal{X})$ is the Kronecker delta function:

$$\delta(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases}$$

3.1.2 The Zeta function and the Zeta matrix.

The Zeta function of a poset is an incidence function on $X \times X$ defined by:

$$\zeta(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise} \end{cases}$$

The matrix associated with the Zeta function is called the *Zeta matrix* and is denoted by Z_{ij}. It is a square matrix whose row and whose columns are labeled by the members of the poset.
The entries of the Zeta matrix are either 0 or 1 as follows:

$$\zeta(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

When we read across row (i) of a Zeta matrix, the presence of a 1 means that *the column label is greater or equal to* i. Similarly, reading down column j, each occurrence of a 1 means that *the row label is less or equal to* j.

The property of reflexivity is satisfied since the Zeta matrix has 1 along its diagonal. The antisymmetry is satisfied since $Z(a,b)$ and $Z(b,a)$ cannot be both 1. The transitivity is satisfied if $Z(a,b) = 1$ when $\exists x$ such that $Z(a,x) = 1$ and $Z(x,b) = 1$.
The elements of the poset are arranged in a way consistent to the poset "ordering" so that the Zeta matrix will be an upper triangular matrix.
From this the Zeta matrix is invertible since its determinant is 1 and its diagonal element is 1.
Its inverse, called the Möbius matrix M_{ij} is therefore an upper triangular matrix.
The inverse of the Zeta function is the Möbius function μ of the poset. In other words, μ satisfies

$$\sum_{x \le z \le y} \mu(x,z) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

In particular $\mu(x,x) = 1$ for all x. Moreover, if we know $\mu(x,z)$ for $x \le z \le y$, then we can calculate

$$\mu(x,y) = -\sum_{x \le z \le y} \mu(x,z). \tag{3.1.2}$$

In particular the values of the Möbius function are all integers.

Example 3.1.1 . Find the Zeta matrix of the poset$(\mathcal{P}(\mathcal{X}), \subseteq)$ if $X = \{1,2,3\}$
Find the Möbius matrix of the poset.
We refer to table 1 , page 132 .

3.1.3 Definition of the Möbius function.

Let $(X, \le)$ be a locally finite poset. Then there exists a unique function μ: $X \times X \to \mathbf{Z}$, called Möbius function, such that $\mu(x,y) \ne 0$ if $x \le y$ and such that whenever $f, g \in \mathcal{I}(\mathcal{X})$ the following conditions are equivalent:

$$(a)\ g(x,y) = \sum_{x \le z \le y} f(x,y);$$

$$(b)\ f(x,y) = \sum_{x \leq z \leq y} g(x,z)\mu(z,y)$$

Example 3.1.2 . Consider the power-set of X ordered by "inclusion".
Let$f$ and g be two functions from $\mathcal{P}(\mathcal{X})$ to the real (or complex) numbers, and
$f(X) = \sum_{S \subseteq X} g(S)$.

Clearly $f(\phi) = g(\phi)$ so that $g(\phi) = f(\phi)$.
If $S = \{a\}$, then $f(S) = g(S) + g(\phi)$ so that $g(\{a\}) = f(\{a\}) - f(\phi)$
If $S = \{a,b\}$, then $f(S) = g(S) + g(\{a\}) + g(\{b\}) + g(\phi)$ so that
$g(\{a,b\}) = f(\{a,b\}) - f(\{a\}) - f(\{b\}) + f(\phi)$.
if $S = \{a,b,c\}$, then $f(S) = g(S) + g(\{a\}) + g(\{b\}) + g(\{c\}) + g(\{a,b\}) + g(\{a,c\}) + g(\{b,c\}) + g(\phi)$ so that
$g(\{a,b,c\}) = f(S) + f(\{a,b\}) + f(\{a,c\}) + f(\{b,c\}) - f(\{a\}) - f(\{b\}) - f(\{c\}) + f(\phi)$.
Further experimenting leads to the following result [27]

$$g(X) = \sum_{S \subseteq X} (-1)^{|S|} f(S)$$

This implies:

$$\mu(S,A) = \begin{cases} (-1)^{|A|-|S|} for S \subseteq A & \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.1.3 . Number D_n of derangements of n elements. Let $\mathcal{P}(\mathcal{X})$ be the lattice of subsets of an n-element set X. Consider $X = \{1, \cdots n\}$. Consider also the number $D_n(S)$ of permutations of X which fix every element in S and no element of $X \backslash S$. At this point we realize that $D_n(\emptyset) = D_n$. We denote by $F_n(S)$ the number of permutations which fix each element in S. The $F_n(S)$ and $D_n(S)$ are counting functions in $\mathcal{P}(\mathcal{X})$. We have:
$F_n(S) = \sum_{S \subseteq T} D_n(T)$ such that:
$D_n(S) = \sum_{T} \mu_{\mathcal{P}(\mathcal{X})}(S,T) F_n(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} F_n(T)$.

If we consider that $|S| = k$, therefore there are $n-k$ elements not in S. The above sum can be rewritten as $\sum_{l=k}^{n}(n-l)!\binom{n-k}{l-k}(-1)^{l-k}$. If $k=0$, which means $S=\emptyset$, or better if $S=X$ the above sum gives the expression for D_n, that is:

$$D_n = \sum_{l=0}^{n}(n-l)!\binom{n}{l}(-1^l)$$
$$= n!(1 - \tfrac{1}{1!} + \tfrac{1}{2!} - \cdots + (-1)^n \tfrac{1}{n!})$$
$$= \tfrac{n!}{e}.$$

Example 3.1.4 .

Consider $f(n)$ and $g(n)$ two functions from the set of positive integers to the set of real (complex) numbers. Suppose for all $n \geq 0$, $f(n) = \sum_{i|n} g(i)$.

$g(1) = f(1)$
$f(2) = g(2) + g(1)$ so that $g(2) = f(2) - f(1)$
$f(3) = g(3) + g(1)$ so that $g(3) = f(3) - f(1)$
$f(4) = g(4) + g(2) + g(1)$ so that $g(4) = f(4) - f(2)$
$g(5) = f(5) - f(1)$
$g(6) = f(6) - f(3) + f(1)$
$g(12) = f(12) - f(4) + f(1)$

If we use this process to find $g(n)$ in terms of $f(i)$ for the i divisors of n, we will see that $\mu(x,y)$ is expressed by:

$$\mu(x,y) = \begin{cases} (-1)^s, & \text{if y/x is the product of s distinct primes} \\ 0, & \text{if x does not divide y or if y/x is not squarefree.} \end{cases}$$

3.1.4 Möbius inversion.

Let $(X; \leq)$ be a locally finite partially ordered set. Let there be a given function f in X. We define a summation function g in X such that $g(m) = \sum_{n \leq m} f(n)$.

This summation function is with respect to the given "ordering" in X and is therefore over all elements n of X such that $n \leq m$.
We are intending to solve the given function f in terms of the summation function g. Therefore we are to invert a system of linear equations. In other words solving f in terms of g is an inversion problem in the poset $(X, \leq)$.

Example 3.1.5 . Let the ordered set $(X, \leq)$ be the positive integers ordered by *divisibility*: $a \leq b$ means $a|b$ (a divises b).
Note that this poset is locally finite. Let $f(m)$ be a function defined $\forall m$ in the poset $(X, \leq)$. Now we define in $(X, \leq)$ a function h such that $h(m) = \sum_{n|m} f(n)$.
The summation is over all divisors n of m. We wish to invert the sum by solving for $f(m)$ in terms of h.

Example 3.1.6 . Let the set $\mathcal{P}(\mathcal{X})$ of subsets of a set X be ordered by *inclusion*. It is locally finite. Consider a given function $f(S)$ on $\mathcal{P}(\mathcal{X})$ $\forall \mathcal{S} \in \mathcal{P}(\mathcal{X})$. Define a summation $g(S) = \sum_{T \subseteq S} f(T) \quad \forall S, T \in \mathcal{P}(\mathcal{X})$.

3.1.5 The Classical Möbius inversion Formula (MIF) vs PIE.

The Möbius inversion and the PIE have something in common:
1. In each case we had a partially ordered set which is the set of positive integers with ordering $x \leq y$ if $x|y$ for the MI and the poset of power set of some given set X with the ordering $A \leq B$ if $A \subseteq B$.
2. In each case the ordering has the property that given any two elements in the set, there are finitely many other elements between them, (locally finite poset).
3. Each of the posets contains the smallest element.
4. In Möbius inversion, there is no mention of properties that some of elements would or would not satisfy; but instead we are interested in how the two functions f and g are related.

3.2 Möbius function, Möbius inversion in the lattice of fuzzy subsets of finite sets.

From now on we will denote the Möbius function of the poset exclusively by μ and will use the letter λ for fuzzy subset.
The Möbius function in the poset of positive integers ordered by divisibility and the Möbius function in the poset of subspaces of a finite vector space are a few cases where this function has been dealt with. We wish to study for an n-element set X, the Möbius function in the poset $(\mathcal{F}(\mathcal{X}), \leq)$ of fuzzy subsets of X, naturally ordered by: $\lambda_1 \leq \lambda_2 \Leftrightarrow \lambda_1(x, y) \leq \lambda_2(x, y)$., where λ_i are fuzzy subsets of X $\forall i$ $0 \leq i \leq n$.
The point-wise ordering in $\mathcal{F}(\mathcal{X})$ is also called dominance order.
Under the point-wise order, $\lambda_1 \leq \lambda_2$ means that for all $x \in X$ $\lambda_1(x) \leq \lambda_2(x)$
We will denote fuzzy subsets, as elements of $\mathcal{F}(\mathcal{X})$, by λ_1, λ_2 etc. using the subscripted λ's.
We also recall that $(\mathcal{F}(\mathcal{X}), \leq)$ is a distributive but not complementary lattice. Thus a vectorial lattice.
We note that $\forall \lambda \in \mathcal{F}(\mathcal{X})$ the set $\{\lambda_i \in \mathcal{F}(\mathcal{X})\lambda_\rangle \leq \lambda\}$ is finite. Therefore $\mathcal{F}(\mathcal{X})$ is locally finite.
Now let us define in $\mathcal{F}(\mathcal{X})^2$ two functions f and g such that.
$f(\lambda_i, \lambda_j) = \sum_{\lambda_i \leq \lambda \leq \lambda_j} g(\lambda_i, \lambda_j)$ for λ_i, λ, λ_j in $\mathcal{F}(\mathcal{X})$.
The above guarantees the existence of the Möbius function function μ of the poset $\mathcal{F}(\mathcal{X})$ such that by inversion we have:
$g(\lambda_i, \lambda_j) = \sum_{\lambda_i \leq \lambda \leq \lambda_j} \mu(\lambda, \lambda_j) f(\lambda_i, \lambda_j)$

3.3 Equations that describe the Möbius function in $\mathcal{F}(\mathcal{X})$

Proposition 3.3.1 The function μ is the unique function such that: (a) $\mu(\lambda, \lambda) = 1 \forall \lambda \in \mathcal{F}(\mathcal{X})$
(b) $\sum_{\lambda : \lambda_i \leq \lambda \leq \lambda_j} \mu(\lambda_i, \lambda) = 0 \forall \lambda_i < \lambda_j$

(c) $\mu(\lambda_i, \lambda_j) = 0$ if $\lambda_i \not\leq \lambda_j$.

We envisage to study the Möbius function and Möbius inversion in the Partition lattice of fuzzy subsets.

We recall that $M = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$
Let $\lambda_i = \lambda_i(x_1)\lambda_i(x_2)\cdots\lambda_i(x_n)$ and $\lambda_j = \lambda_j(x_1)\lambda_j(x_2)\cdots\lambda_j(x_n)$ be two fuzzy subsets of X.
With these notations the point-wise order $\lambda_1 \leq \lambda_2$ would mean that
$\lambda_i(x_1) \leq \lambda_j(x_1)$, $\lambda_i(x_2) \leq \lambda_j(x_2)$, $\cdots$, $\lambda_i(x_n) \leq \lambda_j(x_n)$.

Lemma 3.3.2 Let $\lambda_i = \lambda_i(x_1)\lambda_i(x_2)\cdots\lambda_i(x_n)$ and $\lambda_j = \lambda_j(x_1)\lambda_j(x_2)\cdots\lambda_j(x_n)$ be two fuzzy subsets of X, then:
1.$\lambda_i \leq \lambda_j$ results in $\mid \lambda_i \mid \leq \mid \lambda_j \mid$
2.$\mid \lambda_i \mid \leq \mid \lambda_j \mid$ does not necessary mean that $\lambda_i \leq \lambda_j$

Proof: 1. Let $\lambda_i = \lambda_i(x_1)\lambda_i(x_2)\cdots\lambda_i(x_n)$ and $\lambda_j = \lambda_j(x_1)\lambda_j(x_2)\cdots\lambda_j(x_n)$ be two fuzzy subsets such that $\lambda_i \leq \lambda_j$. Then for each $x_k, 1 \leq k \leq n$, $\lambda_i(x_k) \leq \lambda_j(x_k)$. Therefore $\mid \lambda_i \mid = \sum \lambda_i(x_k) \leq \sum \lambda_j(x_k) = \mid \lambda_j \mid$.
2. It is clear that the sum of membership values of two fuzzy subsets λ_i and λ_j being equal does not imply that each corresponding values for x_k is such that $\lambda_i(x_k) \leq \lambda_j(x_k)$. □
Let λ_j be an element of $\mathcal{F}(\mathcal{X})$, by abusing the notation, we can write, using the function f above that:
$f(\lambda_j) = \sum_{\lambda_i \leq \lambda_j} g(\lambda_j)$
Now solving for g in the above definition of f we can write that:
$g(\lambda_j) = \sum_{\lambda_i \leq \lambda_j} f(\lambda_i)\mu(\lambda_j)$.
That is $g(\lambda_j)$ is defined in terms of $f(\lambda_i)$ where $\lambda_i \leq \lambda_j$ and $\mid \lambda_i \mid \leq \mid \lambda_j \mid$
Recall also that $X = \{x_1, x_2, x_3, \cdots, x_n\}$
Now if $\lambda_i \leq \lambda_j$, then $\lambda_i(x_1) \leq \lambda_j(x_1)$, $\lambda_i(x_2) \leq \lambda_j(x_2)$, $\cdots$, $\lambda_i(x_n) \leq \lambda_j(x_n)$
In other words if $\lambda_i \leq \lambda_j$, then there exists t, with $0 \leq t \leq \mid X \mid$, such that $\mid \lambda_j \mid = (\mid \lambda_i \mid + \frac{t}{m-1})$.
To summarize, we express $g(\lambda_j)$ as the sum of the $f(\lambda_i)$ which are such that there exists t with $\mid \lambda_j \mid = (\mid \lambda_i \mid + \frac{t}{m-1})$. This statement is true in relation with the definition of $M = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$ and that of

$Card_\mu = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1}, \frac{m}{m-1}, \cdots, m\}$ as per our discussion in 4.6.1.
That is to say $\frac{t}{m-1} = \mid \lambda_j \mid - \mid \lambda_i \mid$,
If $t = 0$, then $\frac{t}{m-1} = 0$ and $\lambda_j = \lambda_i$.
If $t = 1$, then $\frac{t}{m-1} = \frac{1}{m-1}$ and the λ_i satisfying the order $\lambda_i \leq \lambda_j$ for a fixed λ_j are those λ_i in $\mathcal{F}(\mathcal{X})$ with the membership value of elements x_k of X to λ_i one at a time, $\frac{1}{m-1}$ less than the membership value of the corresponding elements x_k, one at a time to λ_j.
When $t = 2$, we consider the λ_i in $\mathcal{F}(\mathcal{X})$ with membership values of elements of X two at a time, $\frac{1}{m-1}$ less than the membership of corresponding elements to λ_j. We can vary t until we reach the value of t which makes each $\lambda_i(x_k)$ be $\frac{1}{m-1}$ less than $\lambda_j(x_k)$ i.e. all x_k in X are such that their membership values to λ_i are $\frac{1}{m-1}$ than their corresponding values to λ_j.

Proposition 3.3.3 Let M be defined as $M = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$. For a natural number t; $0 \leq t \leq \mid X \mid$ and a fixed λ_j in $\mathcal{F}(\mathcal{X})$ we have:
$g(\lambda_j) = \sum_{\lambda_i \leq \lambda_j} f(\lambda_i)\mu(\lambda_j) = \sum_{\lambda_i \leq \lambda_j \;\; 0 \leq t \leq |X|} (-1)^t f(\lambda_i)$
such that $\frac{t}{m-1} = \mid \lambda_j \mid - \mid \lambda_i \mid$ and $\mu(\lambda_j) = (-1)^t$ being the expression of the Möbius function in $\mathcal{F}(\mathcal{X})$.

Consider $\lambda_j = \lambda_j(x_1)\lambda_j(x_2)\cdots\lambda_j(x_n) = a_1 a_2 a_3 \cdots a_n$ then:
$g(\lambda_j) = f(a_1 a_2 a_3 \cdots a_n) - (f(a_1 - \frac{1}{m-1}; a_2, \cdots, a_n) + f(a_1; a_2 - \frac{1}{m-1}; a_3; \cdots, a_n) + \cdots + f(a_1; a_2; \cdot; a_n - \frac{1}{m-1})) + (f(a_1 - \frac{1}{m-1}; a_2 - \frac{1}{m-1}; a_3; \cdots; a_n) + \cdots, + f(a_1; a_2, \cdots, a_{n-1} - \frac{1}{m-1}; a_n - \frac{1}{m-1}) + \cdots + f(a_1 - \frac{1}{m-1}; a_2 - \frac{1}{m-1}, a_3 - \frac{1}{m-1}, \cdots, a_n - \frac{1}{m-1})$
The above can be written using the value of t; $0 \leq t \leq n = \mid X \mid$ as :
$g(\lambda_j) = (-1)^0 f(a_1; a_2, a_3, \cdots, a_n) - (-1)^1 (f(a_1 - \frac{1}{m-1}; a_2, \cdots, a_n) + f(a_1; a_2 - \frac{1}{m-1}; a_3; \cdots, a_n) + \cdots + f(a_1; a_2; \cdot; a_n - \frac{1}{m-1})) + (-1)^2 (f(a_1 - \frac{1}{m-1}; a_2 - \frac{1}{m-1}; a_3; \cdots; a_n) + \cdots, + f(a_1; a_2, \cdots, a_{n-1} - \frac{1}{m-1}; a_n - \frac{1}{m-1}) + \cdots + (-1)^n f(a_1 - \frac{1}{m-1}; a_2 - \frac{1}{m-1}, a_3 - \frac{1}{m-1}, \cdots, a_n - \frac{1}{m-1})$
The above statement is in line with P. Hall Lemma [14] which says :
If a and b are elements of the poset X, then

$$\mu_X(a, b) = \sum (-1)^{\mathcal{L}(\mathcal{C})}, \tag{3.3.3}$$

where the sum is over all chains C in X with minimal element a and maximal

element b.
The above can also be expressed as in [17]
Let X be a finite bounded poset. For each $j \geq 1$ l, let c_j denote the number of j-elements chains $C \subseteq X$ such that $C^{min} = \{0\}$ and $C^{max} = \{1\}$. Then

$$\mu(X) = c_1 - c_2 + c_3 - c_4 + \cdots \tag{3.3.4}$$

Let $\mathcal{F}(\mathcal{X})$ be the poset with elements $\lambda_i, \cdots \lambda_n$ and let the $n \times n$ matrix $Y_{\mathcal{F}(\mathcal{X})}$ be such that the ij-entry is equal to 1 if and only if $\lambda_i < \lambda_j$. Thus if we arrange the Zeta matrix as triangular, $Y_{\mathcal{F}(\mathcal{X})} = Z_{\mathcal{F}(\mathcal{X})} - I$. Therefore for $m \in \mathbb{N}$, the ij-entry in the matrix $Y^m_{\mathcal{F}(\mathcal{X})}$ is the number of chains of length m in $\mathcal{F}(\mathcal{X})$ with least element λ_i and maximal element λ_j.
We are left only to show that $\mathcal{L}(\mathcal{C})$ and t appearing in equation 3.3.3 and proposition 3.3.3 respectively are the same. Note that the Möbius function does not just alternate signs from + to -. Rather signs change in relation with t or $\mathcal{L}(\mathcal{C})$.

Let's illustrate this with an example.

Example 3.3.4 . Consider $X = \{x_1, x_2, x_3\}$, $M = \{0, \frac{1}{2}, 1\}$
Define in $\mathcal{F}(\mathcal{X})$ two functions f and g such that
$f(\lambda_j) = \sum_{\lambda_i \leq \lambda_j} g(\lambda_j)$
Then:
$f(000) = g(000)$ so that $g(000) = f(000)$ or $g(000) = (-1)^0 f(000)$
$f(\frac{1}{2}00) = g(000) + g(\frac{1}{2}00)$ so that
$g(\frac{1}{2}00) = f(\frac{1}{2}00) - f(000)$ or $g(\frac{1}{2}00) = (-1)^0 f(\frac{1}{2}00) + (-1)^1 f(000)$
$f(0\frac{1}{2}0) = g(0\frac{1}{2}0) + g(000)$ so that
$g(0\frac{1}{2}0) = f(0\frac{1}{2}0) - f(000)$ or
$g(0\frac{1}{2}0) = (-1)^0 f(0\frac{1}{2}0) + (-1)^1 f(000)$
$f(00\frac{1}{2}) = g(00\frac{1}{2}) + g(000)$ so that
$g(00\frac{1}{2}) = f(00\frac{1}{2}) - f(000)$or
$g(00\frac{1}{2}) = (-1)^0 f(00\frac{1}{2}) + (-1)^1 f(000)$
$\vdots$
$f(\frac{1}{2}\frac{1}{2}0) = g(\frac{1}{2}\frac{1}{2}0) + g(\frac{1}{2}00) + g(0\frac{1}{2}0) + g(000)$ so that
$g(\frac{1}{2}\frac{1}{2}0) = f((\frac{1}{2}\frac{1}{2}0) - [\, f(\frac{1}{2}00) + f(0\frac{1}{2}0)\,] + f(000)$ or

$g(\frac{1}{2}\frac{1}{2}0) = (-1)^0 f((\frac{1}{2}\frac{1}{2}0) + (-1)^1[\,f(\frac{1}{2}00) + f(0\frac{1}{2}0)\,] + (-1)^2 f(000)$

$\vdots$

$g(1\frac{1}{2}1) = (-1)^0 f(1\frac{1}{2}1) + (-1)^1[\,f(\frac{1}{2}\frac{1}{2}1) + f(101) + f(1\frac{1}{2}\frac{1}{2})] + (-1)^2[f(\frac{1}{2}01) + f(10\frac{1}{2}) + f(\frac{1}{2}\frac{1}{2}\frac{1}{2})\,] + (-1)^3 f(\frac{1}{2}0\frac{1}{2})$

$\vdots$

$g(11\frac{1}{2}) = (-1)^0 f(11\frac{1}{2}) + (-1)^1[\,f(110) + f(1\frac{1}{2}\frac{1}{2}) + f(\frac{1}{2}1\frac{1}{2})\,] + (-1)^2[f(1\frac{1}{2}0) + f(\frac{1}{2}\frac{1}{2}\frac{1}{2}) + f(\frac{1}{2}10)\,] + (-1)^3 f(\frac{1}{2}\frac{1}{2}0)$

$\vdots$

$g(111) = (-1)^0 f(111) + (-1)^1[f(\frac{1}{2}11) + f(1\frac{1}{2}1) + f(11\frac{1}{2})] + (-1)^2[f(\frac{1}{2}\frac{1}{2}1) + f(\frac{1}{2}1\frac{1}{2}) + f(1\frac{1}{2}\frac{1}{2})] + (-1)^3 f(\frac{1}{2}\frac{1}{2}\frac{1}{2})$.

3.3.1 REMARKS

1. Now solving in this fashion each $g(\lambda_j)$ of $\mathcal{F}(\mathcal{X})$ in terms of $f(\lambda_i)$, we are able to establish a table of μ matrix of the lattice $\mathcal{F}(\mathcal{X})$ with $X = \{x_1, x_2, x_3\}$ and $M = \{0, \frac{1}{2}, 1\}$. This table could have been obtained by inverting the ζ matrix of the lattice $\mathcal{F}(\mathcal{X})$.
2. The function μ does not depend on the order in which we list the elements of $\mathcal{F}(\mathcal{X})$ even though the μ matrix of the lattice $\mathcal{F}(\mathcal{X})$ depends on that order.
3. For any λ_j, if $t = 0$, then $\frac{t}{m-1} = 0$ and $\lambda_j = \lambda_i$. This means that $\mu(\lambda_j, \lambda_j) = (-1)^0 = 1$. In Table 2, at the end of the thesis, this fact is confirmed since the entries on the first diagonal are 1 throughout.
4. For any λ_j and $\lambda_i \leq \lambda_j$, $\sum_{\lambda;\lambda_i \leq \lambda \leq \lambda_j} \mu(\lambda_i, \lambda) = 0$. In Table 2 provided, the sum of entries in each row is 0.
5. It is clear that if if $\lambda_i \not\leq \lambda_j$, then there is no t with $0 \leq t \leq\mid X \mid$ such that $\mid \lambda_j \mid = (\mid \lambda_i \mid + \frac{t}{m-1})$. In which case $\mu(\lambda_i, \lambda_j) = 0$.

To establish the tables of μ matrix and ζ matrix for the lattice $\mathcal{F}(\mathcal{X})$ with $X = \{x_1, x_2, x_3\}$ and $M = \{0, \frac{1}{2}, 1\}$, we let $A = 000; B = \frac{1}{2}00; C = 0\frac{1}{2}0; D = 00\frac{1}{2}; E = 100; F = \frac{1}{2}\frac{1}{2}0; G = \frac{1}{2}0\frac{1}{2};\ H = 010; I = 0\frac{1}{2}\frac{1}{2}; J = 001; K = 1\frac{1}{2}0; L = 10\frac{1}{2}; M = \frac{1}{2}10; N = \frac{1}{2}\frac{1}{2}\frac{1}{2}; O = \frac{1}{2}01;$
$P = 01\frac{1}{2}; Q = 0\frac{1}{2}1; R = 110; S = 1\frac{1}{2}\frac{1}{2}; T = 101; U = \frac{1}{2}1\frac{1}{2}; V = \frac{1}{2}\frac{1}{2}1;$
$W = 011; X = 11\frac{1}{2}; Y = 1\frac{1}{2}1; Z = \frac{1}{2}11; Z_1 = 111$

The fuzzy subsets of X are arranged in such a way that the ζ matrix and its inverse the μ matrix are triangular with the diagonal constituted by the 1's. The sum of the entries in each line of the μ matrix satisfies the relation

$$\sum_{z\in[x,y]} \mu(x,z) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

which allows the Möbius function of a poset to be calculated recursively.
In other words, we start with $\mu(x,x) = 1 \forall x \in X$.
Now, if $x \leq y$ and we know the values of $\mu(x,z)$ for all $z \in [x,y] \setminus \{y\}$, then we have $\mu(x,y) = -\sum_{z\in[x,y]\setminus\{y\}} \mu(x,z)$.
In particular $\mu(x,y) = -1$ if y covers x and $\mu(x,y) = -\sum_{z\leq x\leq y} \mu(x,z)$.

TABLE 1

ζ	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	$Z1$
A	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
B	0	1	0	0	1	1	1	0	0	0	1	1	1	1	1	0	0	1	1	1	1	1	0	1	1	1	1
C	0	0	1	0	0	1	0	1	1	0	1	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	1
D	0	0	0	1	0	0	1	0	1	1	0	1	0	1	1	1	1	0	1	1	1	1	1	1	1	1	1
E	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	1	1	1	0	0	0	1	1	0	1
F	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	1	1	0	1	1	0	1	1	1	1
G	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	1	1	1	1	0	1	1	1	1
H	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	0	0	1	0	1	1	0	1	1
I	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	1	0	1	1	1	1	1	1	1
J	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	1	0	1	1	0	1	1	1
K	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	1
L	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	1	1	0	1
M	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	1	1
N	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	1	1	1
O	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	1	1
P	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	1
Q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	1	1	1
R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	1
S	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	1
T	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	0	1
U	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	1
V	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1
W	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1
X	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
Y	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
$Z1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

TABLE 2

μ	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	$Z1$
A	1	−1	−1	−1	0	1	1	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
B	0	1	0	0	−1	−1	−1	0	0	0	1	1	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0
C	0	0	1	0	0	−1	0	−1	−1	0	0	0	1	1	0	1	0	0	0	0	−1	0	0	0	0	0	0
D	0	0	0	1	0	0	−1	0	−1	−1	0	0	0	1	1	0	1	0	0	0	0	−1	0	0	0	0	0
E	0	0	0	0	1	0	0	0	0	0	−1	−1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
F	0	0	0	0	0	1	0	0	0	0	−1	0	−1	−1	0	0	0	1	1	0	1	0	0	−1	0	0	0
G	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	−1	0	0	0	1	1	0	1	0	0	−1	0	0
H	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0	0
I	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	−1	0	0	0	1	1	1	0	0	−1	0
J	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	0	0	0	0	1	0	0	0	0	0
K	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1	−1	0	0	0	0	1	0	0	0
L	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1	−1	0	0	0	0	1	0	0
M	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	0	−1	0	0	1	0	0	0
N	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	−1	0	1	1	1	−1
O	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	0	0	1	0	0
P	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	0	0	1	0
Q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	−1	0	0	1	0
R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	−1	0	0	0
S	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	−1	0	1
T	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	0
U	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1	0	−1	1
V	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1	−1	1
W	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1	0
X	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1
Y	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1
Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	−1
$Z1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Chapter 4

Identification Process or Pattern Recognition

The identification process emerges when it is impossible or even not practical to measure simultaneously all variables (members) of the system (set), while we wish to select one of them as a basis for making decisions or take actions. We need therefore to break a given system(set) into subsystems (subsets) that still preserve enough information about the overall system. This will help reduce the complexity of the system involved and therefore increase clarity concerning the system. It is easier to monitor several sets of variables than one large set of variables in a situation when decision concerning the set has to be made quickly.

Pattern recognition aims to discover structures that are within data in order to recognize pattern and classify objects. This also help to discover the existence of clusters in a large set of data, whose members display similarity to one another along some relevant dimensions. A pattern recognition system (PRS) [3] is an automatic system that aims at classifying the input pattern into a specific class. It proceeds into two successive tasks. The analysis (or description) that extracts the characteristics from the pattern being studied and the classification (or recognition) that enables us to recognize an object (or a pattern) by using some characteristics derived from the first task. There are four major methodologies in PRS, which are the statistical approach, the syntactical approach, the template matching approach and neural networks. In our study we have chosen to use none of these approaches. The most used of the four methods is the statistical approach which use some element of probability. The entities we want to observe are fuzzy subsets or elements having some fuzzy properties, whose behaviors are the kinds not to be projected. We will use rather attributes which are related to the nature of fuzzy subsets.

This will enable a machine to identify an object (element or even fuzzy subset) automatically without human interaction. It is natural to use fuzzy subsets since many of these patterns or attributes have vague boundaries or definitions.
In the next subsection, we establish among fuzzy subsets of a set X, criteria for identifying fuzzy subsets, based on the distance from a given fuzzy subset in relation to the cardinality of fuzzy subsets involved.

4.1 Distance and Cardinality: First Pattern Recognition

The relations $\mathcal{R}_2$ and $\mathcal{R}_3$ defined earlier in section 2.2, Theorem 2.6.3 and Theorem 2.6.5 are equivalent. The above theorems imply that enumerating fuzzy subsets having the same cardinality p is the same exercise as counting all fuzzy subsets located at a fixed distance d from a given fuzzy subset μ of the set. This means fuzzy subsets at a distance from a given one have the same cardinality. Conversely we may say that all fuzzy subsets of a set X having the same cardinality are all located at a distance d from a given fuzzy subset of X. This constitutes our first pattern recognition of fuzzy subsets of a finite set.

Theorem 4.1.1 Let μ_1 be a fuzzy subset of an n-element set X such that $|\mu_1| = p$. Let μ be the top fuzzy subset of X. Then $d(\mu, \mu_1) = n - p$.

Proof: It is clear that $|\mu| - |\mu_1| = n - p$. But $(\mu(x_1) - \mu_1(x_1)) + (\mu(x_2) - \mu_1(x_2)) + \cdots + (\mu(x_n) - \mu_1(x_n)) = |\mu| - |\mu_1| = d(\mu, \mu_1)$. Therefore $d(\mu, \mu_1) = n - p$

3°. Number of fuzzy subsets of X at a distance d from a given fuzzy subset μ of $\mathcal{F}(\mathcal{X})$, Second Pattern Recognition.

Let d be the Hamming distance between a given fuzzy subset μ and any other fuzzy subset μ_1. As seen above, the fuzzy subsets of X at a distance d

from μ have same cardinality as μ_1 does.

Elements in $\mathcal{F}(\mathcal{X})$ can be classified in terms of their distance (Hamming) from a given fuzzy subset μ of $\mathcal{F}(\mathcal{X})$.
In the next section we wish to answer the following questions:
1. How many fuzzy subsets are there at a distance d from a given fuzzy subset μ of $\mathcal{F}(\mathcal{X})$?
2. How many fuzzy subsets are there in $\mathcal{F}(\mathcal{X})$ with cardinality α?
3. Knowing the distance d from the given (chosen) fuzzy subset μ to another fuzzy subset μ_1 of X, what is the cardinality of the fuzzy subset μ_1 therefore?

Proposition 4.1.2 Let X be an n-element set and M be the set of membership values of fuzzy subsets of X. Let $\mathcal{R}$ be the equivalence relation defined on $\mathcal{F}(\mathcal{X})$ in Theorem 2.6.5 and Theorem 2.6.6. Consider $\alpha \in \mathbb{R}$. Then the number of elements in each equivalence class of $\mathcal{R}$ is the number $(N_\mu(\mid . \mid = \alpha))_n = \sum_{i\in M}(N_\mu(\mid . \mid = \alpha - i))_{n-1}$.

Proof: We know that an equivalence class in $\mathcal{F}(\mathcal{X})$ is made of elements with the same cardinality, by definition of $\mathcal{R}$. Now we know from 2.6.7 previous result the number of fuzzy subsets of X of cardinality α. This number denoted by $(N_\mu(\mid . \mid = \alpha))_n = \sum_{i\in M}(N_\mu(\mid . \mid = \alpha - i))_{n-1}$ is exactly the number of elements in a class of cardinality α, which is also the number of fuzzy subsets at a distance d from a given fuzzy subset μ of $\mathcal{F}(\mathcal{X})$.$\Box$
Since all members of an equivalence class have the same cardinality by definition of $\mathcal{R}$, we can determine for special classes, for instance equivalence classes containing crisp subsets, the number of elements in that equivalence class.
Let μ and μ_1 be the top fuzzy subset and any fuzzy subset of $\mathcal{F}(\mathcal{X})$. $\mid \mu \mid \geq \mid \mu_1 \mid$.
Because of the nature of members of $Card_\mu$, we are able to say that there exists $t \in \mathbb{Z}$ such that $\mid \mu \mid - \mid \mu_1 \mid = \frac{t}{m-1}$. The number $\frac{t}{m-1}$ is therefore the distance between μ and μ_1 if t exists. So if t exists, then knowing the distance d between the given (chosen) fuzzy subset μ to another fuzzy subset μ_1 of X, allows us to know the cardinality $(\mid \mu \mid - \frac{t}{m-1})$ of fuzzy subsets of the class of μ_1.
Given the distance from the top fuzzy subset of the lattice $\mathcal{F}(\mathcal{X})$ to any fuzzy subset of X informs us of the cardinality of the fuzzy subset concerned.

This constitutes our second criteria of fuzzy recognition.
In the next subsection, we tackle the issue of the relationship between the Hamming distance between fuzzy subsets and their cardinality.

4.2 Hamming distance and cardinality

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The counting of fuzzy subsets of a finite set is of great interest. We have used the well known counting techniques of Inclusion-Exclusion and that of Möbius Inversion to enumerate the fuzzy subsets of a finite set under some given conditions. In this study we develop, given a set X, a way of recognizing fuzzy subsets of X by using the tool of Hamming distance in relation to the cardinalities of the fuzzy subsets involved.
The Hamming distance "d" between two fuzzy subsets μ and λ of a set X is $d(\mu, \lambda) = \sum_{i=1}^{n} \mid \mu(x_i) - \lambda(x_i) \mid$. See [20]. Let μ be a fuzzy subset of X. Now let us define in $\mathcal{F}(\mathcal{X})$ the relation $\mathcal{R}_1$ such that $\lambda_1 \mathcal{R}_1 \lambda_2$ if and only if $d(\mu, \lambda_1) = d(\mu, \lambda_2)$, where $d(\mu, \lambda)$ is the Hamming distance between a given fuzzy subset μ of X and any other fuzzy subset λ in $\mathcal{F}(\mathcal{X})$.
1. Clearly $\lambda \mathcal{R}_1 \lambda \; \forall \, \lambda \in \mathcal{F}(\mathcal{X})$.
2. Now if $\lambda_1 \mathcal{R}_1 \lambda_2$, it is evident that $\lambda_2 \mathcal{R}_1 \lambda_1$.
3. It is obvious that if $\lambda_1 \mathcal{R}_1 \lambda_2$ and $\lambda_2 \mathcal{R}_1 \lambda_3$, then $\lambda_1 \mathcal{R}_1 \lambda_3$.

Theorem 4.2.1 The relation $\mathcal{R}_1$ defined on $\mathcal{F}(\mathcal{X})$ such that $\lambda_1 \mathcal{R}_1 \lambda_2$ if and only if $d(\mu, \lambda_1) = d(\mu, \lambda_2)$, where $d(\mu, \lambda)$ is the Hamming distance between a given fuzzy subset μ and any other fuzzy subset λ is an equivalence on $\mathcal{F}(\mathcal{X})$ and the set of fuzzy subsets at a distance d from a given fuzzy subset μ, is an equivalence class.

Elements in $\mathcal{F}(\mathcal{X})$ can be classified in term of their distance (Hamming) from a given fuzzy subset μ of $\mathcal{F}(\mathcal{X})$.
In the next section we wish to answer the following questions:
1. How many fuzzy subsets are there in $\mathcal{F}(\mathcal{X})$ with cardinality α?
2. How many fuzzy subsets are there at a distance d from a given fuzzy subset

μ of $\mathcal{F(X)}$?
3. Do fuzzy subsets at a distance d from a given fuzzy subset have the same cardinality?
4. Knowing the distance d from the given (chosen) fuzzy subset μ, what is the cardinality of the fuzzy subset therefore?
Let us also discuss three important ideas:

4.2.1 Ordinary subset nearest to a fuzzy subset

An ordinary subset of X is a crisp subset of X. Now given a fuzzy subset μ_A of X, we wish to characterize the crisp subset μ_{A_o} of set X, near μ_A with respect to the Hamming Distance.
The ordinary subset μ_{A_o} near μ_A is such that:
$\mu_{A_o}(x_i) = 0$ if $\mu_A(x_i) \leq 0.5$
$\mu_{A_O}(x_i) = 1$ if $\mu_A(x_i) > 0.5$ [20]

4.2.2 Properties

1.$(\mu_A \cap \mu_B)_o = \mu_{A_o} \cap \mu_{B_o}$
2.$(\mu_A \cup \mu_B)_o = \mu_{A_o} \cup \mu_{B_o}$
3.$\forall \in X; |\mu_A(x_i) - \mu_{A_o}(x_i)| = \mu_{A \cap \overline{A}}(x_i)$.

Proof : 1. Let $x \in X$ such that $(\mu_A \cap \mu_B)_o(x) = 1$. This means that $\mu_A \cap \mu_B(x) > 0.5$ and $\mu_A(x) \wedge \mu_B(x) > 0.5$. As a result both $\mu_A(x) > 0.5$ and $\mu_B(x) > 0.5$ so that $\mu_{A_o}(x) = 1$ and $\mu_{B_o}(x) = 1$. We conclude that $(\mu_{A_o} \cap \mu_{B_o})(x) = 1$. If instead $(\mu_A \cap \mu_B)_o(x) = 0$, then $\mu_A \cap \mu_B(x) \leq 0.5$. Either $\mu_A(x) \leq 0.5$, $\mu_B(x) \leq 0.5$ or both $\mu_A \leq 0.5$ and $\mu_B \leq 0.5$ which results in either $\mu_{A_o}(x) = 0$ or $\mu_{B_o}(x) = 0$ which in turn means that $\mu_{A_o} \cap \mu_{B_o}(x) = 0$. The proof of part 2 is similar to this one. 3. If $\mu_A(x) > 0.5$, then $\mu_{A_o}(x) = 1$, and therefore $|\mu_A(x) - \mu_{A_o}(x)| = |\mu_{A_o}(x) - \mu_A(x)| \leq 0.5$. But since $\mu_A(x) > 0.5$, then $\mu_{\overline{A}}(x) < 0.5$ such that $\mu_{A \cap \overline{A}}(x) < 0.5$.
If on the other hand $\forall x \in X$, $\mu_A(x) \leq 0.5$, then $\mu_{A_o}(x) = 0$, therefore $|\mu_A(x) - \mu_{A_o}(x)| < 0.5$ and since $\mu_A(x) < 0.5$, we have $\mu_{\overline{A}}(x) \geq 0.5$ so that $\mu_{A \cap \overline{A}}(x) < 0.5$. It shows that for both cases $\mu_A(x) > 0.5$ or $\mu_A(x) \leq 0.5$, we have $\mu_{A \cap \overline{A}}(x) < 0.5$.

Consider the generalized relative Hamming distance between two fuzzy subsets μ_A and μ_B, given by the expression $\delta(\mu_A, \mu_B) = \frac{d(\mu_A,\mu_B)}{n}$. Since $d(\mu_A, \mu_B) \leq n$, we can write that $0 \leq \delta(\mu_A, \mu_B) \leq 1$.
If we consider the ordinary subset μ_{A_o} near μ_A, the number $\gamma(\mu_A) = \frac{2}{n}d(\mu_{A_o}, \mu_A)$ is called the index of fuzziness with respect to the generalized relative Hamming distance.
We have $0 \leq \gamma(\mu_A) \leq 1$. In fact since $n.\delta(\mu_A, \mu_B) = d(\mu_A, \mu_B)$, we can now show that $d(\mu_A, \mu_{A_o}) \leq \frac{n}{2}$ so that $0 \leq \delta(\mu_A, \mu_{A_o}) \leq \frac{1}{2}$.

4.3 Fuzzy subset nearest a Fuzzy subset

Throughout the remainder of this thesis $X = \{x_1, x_2, \cdots, x_n\}$ is a finite set with $1 \leq n$ elements and all fuzzy subsets μ of X take n membership values not all necessarily distinct and hence take m values with $1 \leq m \leq n$. The membership values in the interval $I = [0, 1]$ are taken to be uniformly spaced, with the usual ordering, given by $M_m = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$. This uniform choice of values in M_m does not affect the counting of fuzzy subsets with special property and also is in line with preferential equality discussed elsewhere, [28]. With the above consideration, we can now define the set $\mathcal{D}$ of Hamming distances between any two fuzzy subsets of X.
$\mathcal{D}= \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, n\}$.
With this definition in mind, we can see that the shortest Hamming distance between any two fuzzy subsets is therefore $\frac{1}{m-1}$. In the same manner we say that farthest distance between any two fuzzy subsets is n. This is justified by 4.2
. For a fuzzy subset $\mu(x_1)\mu(x_2)\mu(x_3)\cdots\mu(x_{n-1})\mu(x_n)$, any fuzzy subset such that:
$(\mu(x_1) - \frac{1}{m-1})\mu(x_2)\mu(x_3)\cdots\mu(x_n)$, $\mu(x_1)(\mu(x_2) - \frac{1}{m-1})\mu(x_3)\cdots\mu(x_n)$, $\cdots$ $\mu(x_1)\mu(x_2)\cdots\mu(x_{n-1})(\mu(x_n) - \frac{1}{m-1})$ are each fuzzy subset nearest to $\mu(x_1)\mu(x_2)\mu(x_3)\cdots\mu(x_n)$ because they are at the distance $\frac{1}{m-1}$ from $\mu(x_1)\mu(x_2)\mu(x_3)\cdots\mu(x_{n-1})\mu(x_n)$.

4.4 Fuzzy chain

Let μ_1 and μ_2 be two fuzzy subsets of set X. We define $[\mu_1, \mu_2]$ as $\{\mu \in \mathcal{F}(\mathcal{X}),$ $\mu_1 \leq \mu \leq \mu_2\}$.

This is indeed an interval, not in the sense of crisp sets. Let μ be a fuzzy subset of X, we also define a r-neighborhood of μ to be the set $\mathcal{B}_r$ of fuzzy subsets at the Hamming distance less than or equal to r from μ. We write that $\mathcal{B}_r = \{\mu_i \in \mathcal{F}(\mathcal{X})\ d(\mu, \mu_i) \leq r\}$.
If we consider any fuzzy subset μ, the shortest (Hamming) distance any other different fuzzy subset would be away from μ is therefore $\frac{1}{m-1}$. Surely the farthest any fuzzy subset would be from μ is n. This is justified since $0 \leq d(\mu_1, \mu_2) \leq n$.

4.5 Distance between a fuzzy subset and its complement.

Let μ be a fuzzy subset and $\overline{\mu}$ its complement. Then
$d(\mu, \overline{\mu})$
$= \sum_{i=1}^{n} \mid \mu(x_i) - \overline{\mu}(x_i) \mid$
$= |(1 - \mu(x_1)) - \mu(x_1)| + |(1 - \mu(x_2)) - \mu(x_2)| + \cdots + |\mu(x_n) - (1 - \mu(x_n))|$
$= |2\mu(x_1) - 1| + |2\mu(x_2) - 1| + \cdots + |2\mu(x_n) - 1|.$

$= |1 - 2\overline{\mu}(x_1)| + |1 - 2\overline{\mu}(x_2)| + \cdots |1 - 2\overline{\mu}(x_n)|.$
In summary we may state that:
$d(\mu, \overline{\mu}) = \sum_{i=1}^{n} |2\mu(x_i) - 1|$
$= \sum_{i=1}^{n} |1 - 2\overline{\mu}(x_i)|.$

We note that if $\mu = (\frac{1}{m-1}\frac{1}{m-1} \cdots \frac{1}{m-1})$, then $d(\mu, \overline{\mu}) = n.\frac{3-m}{m-1}$. In fact if $\mu(x) = \frac{1}{m-1}$, then $\overline{\mu}(x) = \frac{m-2}{m-1}\ \forall x \in X$ and each $|\mu(x) - \overline{\mu}(x)| = \frac{m-2}{m-1} - \frac{1}{m-1} = \frac{m-3}{m-1}\ \forall x \in X$; such that the sum of n terms equal each to $\frac{m-3}{m-1}$ is $n.(1 - 2\frac{m-2}{m-1}) = n.(\frac{m-1-2m+4}{m-1}) = n.\frac{3-m}{m-1}$.
In particular : if $n = m$ and $\mu = \frac{1}{2}\frac{1}{2} \cdots \frac{1}{2}$, then $d(\mu, \overline{\mu}) = 0$. Again if $\mu = 111 \cdots 1$, then $d(\mu, \overline{\mu}) = n$.
Now we consider two fuzzy subsets $\mu_1,\ \mu_2$ such that μ_2 dominates μ_1. We have

$\mu_1 \leq \mu_2$ and that $|\mu_1| \leq |\mu_2|$. That is to mean according to 2.6.1 $\exists t \in \mathbb{N}$ such that $|\mu_1| + \frac{t}{m-1} = |\mu_2|$
$d(\mu_1, \mu_2) = ||\mu_2| - |\mu_1||$.
If $t = 0$ then $\mu_1 = \mu_2$. that is $|\mu_1| = |\mu_2|$ and μ_1 and μ_2 are in same equivalence class.
If $t = 1$ then $|\mu_2| - |\mu_1| = \frac{1}{m-1}$
If $t = 2$ then $|\mu_2| - |\mu_1| = \frac{2}{m-1}$.
We consider here only the case $\mu_1 \leq \mu_2$ where $\mu_1(x_i)+\frac{1}{m-1} = \mu_2(x_i)\, \forall 1 \leq i \leq n$.
$\mu_2 = (\mu_1(x_1) + \frac{1}{m-1})(\mu_1(x_2) + \frac{1}{m-1}) \cdots \mu_1(x_n) + \frac{1}{m-1}$ such that
$2\mu_2 = (2\mu_1(x_1) + 2\frac{1}{m-1})(2\mu_1(x_2) + 2\frac{1}{m-1}) \cdots (2\mu_1(x_n) + 2\frac{1}{m-1})$.
We express $d(\mu_2, \overline{\mu}_2)$ in terms of μ_1 in line with 4.5 as :
$|(2\mu_1(x_1) + \frac{2-m+1}{m-1})| + |(2\mu_1(x_2) + \frac{2-m+1}{m-1})| \cdots + |(2\mu_1(x_n) + \frac{2-m+1}{m-1})|$ or
$|2\mu_1(x_1) + \frac{3-m}{m-1}| + |2\mu_1(x_2) + \frac{3-m}{m-1}| + \cdots |2\mu_1(x_n) + \frac{3-m}{m-1}|$ or
$2(\mu_1(x_1) + \mu_1(x_2) + \cdots + \mu_1(x_n)) + n.\frac{3-m}{m-1}$.

Example 4.5.1 . For $\mu_1 = \frac{1}{2}\frac{1}{2} \cdots \frac{1}{2}$, $\mu_2 = 11 \cdots 1$ and $\overline{\mu}_2 = 00 \cdots 0$, $d(\mu_2, \overline{\mu}_2) = n$. When expressed in terms of μ_1 the distance $d(\mu_2, \overline{\mu}_2) = 2(\frac{1}{2} + \frac{1}{2} + \cdots \frac{1}{2}) + n.\frac{3-m}{m-1} = n(1 + \frac{3-m}{m-1}) = \frac{2n}{m-1}$.

Proposition 4.5.2 . The Hamming distance between fuzzy subsets of the same equivalence class: Third Fuzzy Pattern Recognition.

Let the relation $\mathcal{R}_1$ be defined on $\mathcal{F}(\mathcal{X})$ such that $\lambda_1 \mathcal{R}_1 \lambda_2$ if and only if $d(\mu, \lambda_1) = d(\mu, \lambda_2)$., where $d(\mu, \lambda)$ is the Hamming distance between a given fuzzy subset μ and any other fuzzy subset λ. $\mathcal{R}$ is an equivalence on $\mathcal{F}(\mathcal{X})$ and the set of fuzzy subsets at a distance d from a given fuzzy subset μ is an equivalence class. The sum $\displaystyle\sum_{i=1}^{n} \mu(x_i) - \lambda(x_i) = 0$ if the fuzzy subsets μ and λ are in the same equivalence class.

The above proposition gives us the third way of recognizing fuzzy subsets of a set X. This means that the sum of the differences between membership values of elements to the fuzzy subsets of the same $\mathcal{R}$-equivalence class is *zero*. This principle is used in character recognition . The Hamming distance between two matrices characterizing two characters is zero if the two characters are the same.

4.6 Number of fuzzy subsets between two fuzzy subsets.

Consider two fuzzy subsets $\mu_1 = \mu_1(x)\mu_1(x_2)\cdots\mu_1(x_n)$ and
$\mu_2 = \mu_2(x_1)\mu_2(x_2)\cdots\mu_2(x_n)$ of X such that $\mu_1 \leq \mu_2$. How many fuzzy subsets are there between μ_1 and μ_2?
To achieve this counting we use the PIE by taking $|\mathcal{F}(\mathcal{X})|$ subtract the number of those such that $\mu_i(x_1) \leq \mu_1(x_1)$, $\mu_i(x_2) \leq \mu_1(x_2)$. To the number obtained so far, add those whose the memberships for any two elements of X are less than their respective memberships to μ_1 since these might have been subtracted twice. We will subtract or add accordingly until we reach the point where the membership values of all the elements to the fuzzy subsets are less than those of μ_1.

4.7 Number of fuzzy subsets at half-way between two fuzzy subsets.

B. Kosko introduced a fuzzy subset μ as a vector $(\mu_1, \mu_2, \cdots, \mu_n) \in [0,1]^n$ where the $\mu_i = \mu(x_i)$.
This is possible because of the bijection between the set $\mathcal{F}(\mathcal{X})$ and $[0,1]^n$.
Using Kosko's hypercube, we can identify a fuzzy subset with a point in a unit hypercube. A crisp set is a vertex of the hypercube. With this in mind we can therefore define terms such as *segment* joining two given fuzzy subsets; the set of mid-points between those two fuzzy subsets and the set of *equidistant* points from given points. There are more details in [34].
It is not always possible to find the fuzzy subset mid-point (half-way) between two fuzzy subsets. The fuzzy subset half-way between two fuzzy subsets, if it exist is not unique. There is an infinity of possibilities to choose from. This situation is totally different from Euclidean geometry, where for two given points, there is a unique mid-point.

Example 4.7.1 .

Let $X = \{x_1, x_2, x_3\}$ and $M = \{0, \frac{1}{2}, 1\}$ be the set of membership values of fuzzy subsets of X. There are two fuzzy subsets of X namely $1\frac{1}{2}1$ and $11\frac{1}{2}$ half-way between the fuzzy subsets 111 and $1\frac{1}{2}\frac{1}{2}$. On the other hand there are no fuzzy subset half-way between 111 and $1\frac{1}{2}0$.

Let us first take the case where $X = \{x_1, x_2\}$. Consider two fuzzy subsets μ and λ. The fuzzy segment between μ and λ can be described as:
$segment(\mu, \lambda) = \{\xi, d(\mu, \xi) + d(\xi, \lambda) = d(\mu, \lambda)\}$.
The set of fuzzy subsets equidistant from μ and λ denoted here as $Equid(\mu, \lambda)$, is therefore
$Equid(\mu, \lambda) = \{\xi, d(\mu, \xi) = d(\xi, \lambda\}$. With this consideration ξ is a mid-point fuzzy subset between μ and λ. We can also write that $d(\mu, \xi) = \frac{1}{2}d(\mu, \lambda)$. In other words $\xi(x_i) = \frac{1}{2}[\mu(x_i) + \lambda(x_i)]$ and
$d(\mu, \xi) = \sum_{i=1}^{n} \mid \mu(x_i) - \xi(x_i) \mid = \sum_{i=1}^{n} \frac{1}{2} \mid \mu(x_i) - \lambda(x_i) \mid = \frac{1}{2}d(\mu, \lambda)$. The mid-point fuzzy subset described in this fashion is termed *canonical mid−point fuzzy subset* between μ and λ and is denoted by $\xi = \frac{\mu+\lambda}{2}$. However, this mid-point fuzzy subset might not be unique. Let $\xi = (\frac{1}{2}, \frac{1}{2})$ be a mid-point fuzzy subset between $\mu = (0, 0)$ and $\lambda = (1, 1)$. Any other fuzzy subset $\gamma = (a_1, a_2)$ such that $a_1 + a_2 = 1$, $(1 - a_1) + (1 - a_2) = 1$ is also a mid-point fuzzy subset between $\mu = (0, 0)$ and $\lambda = (1, 1)$.

The set of mid-point fuzzy subsets between μ and λ, denoted here as $Mid(\mu, \lambda)$, is a subset of that of $Equid(\mu, \lambda)$. This inclusion is not always strict.

Let d be a metric on the set $\mathcal{F}(\mathcal{X})$, a mid-point between two fuzzy subsets μ and λ of $\mathcal{F}(\mathcal{X})$ is any fuzzy subset $\zeta \in \mathcal{F}(\mathcal{X})$ such that $d(\zeta, \mu) = d(\zeta, \lambda) = \frac{1}{2}d(\mu, \lambda)$. These mid-points depend on the distance d chosen. There are pairs of points without mid-point. There are cases where the mid-points between two fuzzy subsets are finitely many.

We consider these mid-points as *middle ways* or *compromises* between two situations described by the fuzzy subsets μ and λ.

Proposition 4.7.2 Let ζ be the mid-point between μ and λ.
Then $\mu_i \wedge \lambda_i \leq \zeta_i \leq \mu_i \vee \lambda_i \, \forall i$.

Proof: Consider $\zeta \in Mid(\mu, \lambda)$. Using the Hamming distance we may write: $\sum_{i=1}^{n} |\mu(x_i) - \zeta(x_i)| + |\lambda(x_i) - \zeta(x_i)| - |\mu(x_i) - \zeta(x_i)| = 0$. We know that each term in the above sum is positive. But since their sum is 0, then each of them is equal to 0. Therefore $\mu_i \wedge \lambda_i \leq \zeta_i \leq \mu_i \vee \lambda_i \, \forall i$.

Example 4.7.3 . Take $X = \{x_1, x_2, x_3\}$, $\mu = \frac{4}{10}\frac{3}{10}\frac{5}{10}$, $\lambda = \frac{8}{10}\frac{3}{10}\frac{3}{10}$.
Any fuzzy subset ζ mid-point between μ and λ is such that $\frac{4}{10} \leq \zeta_1 \leq \frac{8}{10}$; $\zeta_2 = \frac{3}{10}$; $\frac{3}{10} \leq \zeta_3 \leq \frac{5}{10}$.

4.8 Mid-point Illustration

Let us try to illustrate this in the actual life situation. Consider the affiliated members of the South African *COSATU* different Labour Unions meeting at the bargaining chamber to resolve issues pertaining to employment. It is clear that the working conditions of members forming this Union are not the same. It is therefore also clear that they do not have the same benefits and conditions of employment.
Consider the set of fuzzy subsets to be *COSATU* representatives one side and the representatives of employers one side. Each individual fuzzy subset as the member component of *COSATU*. When they come in such meetings they have grievances as workers, but each category of workers has his actual target to satisfy the members of the profession they represent in the meeting. After meeting with the government or employers representatives concerning salary or any other type of negotiation they may or may not reach agreement. We can use the fuzzy logic to understand this situation.

When the discussions end without agreement we compare this case to the situation of no unique mid-point between two fuzzy subsets was found. This case often leads to a strike action. But fortunately these negotiations will always come up with a certain kind of consensus, a range of possibilities which are solutions halfway between the government or employers' proposal and the unions expectations to choose from. This consensus or compromises may be multivariate, that is favourable to this union on this point, favourable to that union on another point. Each union may find that the agreement makes sense in their particular predicament. This situation represents the fact that the middle between two fuzzy subsets if it exists is not always unique. In many others instances fuzzy logic is used to tackle conflicts.

One of the most current problem is the water issue. In one community the needs of stake holders in terms of water usage might not be the same. A farmer may argue he produces food. He would say he therefore needs more water for his crops and animals. Meanwhile a school or hospital next door believe the farmer is spoiling this scarce commodity. Electricity is also one of the hot contention in our communities. Businesses have the feeling that amount of electrical energy sold to them is insignificant while its rate is exorbitant. They bring employment to the community and should be given preference. A lobby group on the other hand might want free electricity for street lights in order to lower the crime rate, arguing that this will protect the business as well as members of the community. In all these instances, what is the right amount either of water or electricity should the municipality allocate to farmers or business; how much to the general public, bearing in mind that these commodities are being produced at a very high cost.

In many of these instances fuzzy logic is used to tackle conflicts when we consider grievances as fuzzy subsets and suggested solutions in this case as multivariate mid-points of parties' expectations. We refer to this as fuzzy model of conflict resolution.

4.9 Alpha-cuts and size of a fuzzy subset: Fourth Fuzzy Pattern Recognition

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Let α be such that $0 \leq \alpha \leq 1$. We know that a fuzzy subset μ_1 of X is greater than another fuzzy subset μ_2 of X if all the α-cut of μ_1 contains the α-cut of μ_2. Now, given the set $\mathcal{F}(\mathcal{X})$ of all fuzzy subsets of set X and given α in the unit interval I. Any fuzzy subset greater than a given fuzzy subset μ_1

has its α-cut greater (containing) than that of μ_1.
That means that the size of the α-cut determines the size of the fuzzy subset.
The greater the $\alpha - cut$, the greater the fuzzy subset.
This will constitute another way of recognizing fuzzy subsets of a set. We will remember this as our fourth pattern recognition process.

4.10 Fuzzy subsets and their supports: Fifth Fuzzy Pattern Recognition

Proposition 4.10.1 Let μ_1 and μ_2 be two fuzzy subsets of a finite set X, and their respective supports denoted $Supp\mu_1$ and $Supp\mu_2$. If $\mu_1 \leq \mu_2$ then $Supp\mu_1 \subseteq Supp\mu_2$.

In fact let $x_i \in X$ such that $0 < \mu_1(x_i)$. This means $x_i \in Supp\mu_1$ and because $\mu_1 \leq \mu_2$, we have $0 < \mu_2(x_i)$ and $x_i \in Supp\mu_2$ too.
Now it is clear that $Supp\mu_1 = Supp\mu_2$ does not imply that $\mu_1 = \mu_2$. We will remember that the size of the Support of a fuzzy subset informs us of the size of fuzzy subset. We call this our fifth pattern recognition criteria.

4.11 Fuzzy subsets and their Cores: Sixth Fuzzy Pattern Recognition

We recall here that the *core* of a fuzzy subset denoted here as $Core\mu$ is defined as $\{x \in X, \mu(x) = 1\}$.

Proposition 4.11.1 Let μ_1 and μ_2 be two fuzzy subsets of a finite set X, and their respective cores denoted $Core\mu_1$ and $Core\mu_2$. If $\mu_1 \leq \mu_2$ then $Core\mu_1 \subseteq Core\mu_2$.

Let $x_i \in X$ such that $\mu_1(x_i) = 1$. Therefore $x_i \in Core\mu_1$. But $\mu_1 < \mu_2$ and since $\mu(x_i) = 1$, it is evident that $\mu_2(x_i) = 1$ also and therefore $x_i \in Core\mu_2$.
This means that $Core\mu_1 \subseteq Core\mu_2$.

4.12 Similarity between fuzzy subsets

Two fuzzy subsets μ and ν are similar if they maintain the same relative degree of membership values with respect to any two elements. That is to mean that μ and ν are similar if and only if the three conditions below are satisfied [31]
1)$\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$.
2) $\mu(x) = 1$ if and only if $\nu(x) = 1$.
3) $\mu(x) = 0$ if and only if $\nu(x) = 0$.
This relation defines an equivalence on set $\mathcal{F}(\mathcal{X})$ also noted as I^X. We use the notation $\mu \sim \nu$ to express that μ is similar to ν.

Characterization of equivalent fuzzy subsets:

Proposition 4.12.1 1. $\mu \sim \nu$ if $Im(\mu) = Im(\nu)$. The converse of this statement is not valid.
2. $\mu \sim \nu$ if and only if $\forall\, \alpha > 0 \,\exists\, \beta > 0\, such\, that\, \mu^\alpha = \nu^\beta$

4.13 Fuzzy scanner

Let X be an n-element set and $\mu_1, \mu_2, \cdots, \mu_m$ be m fuzzy subsets of set X. Each element of X belongs to each of the fuzzy subsets to a certain extent. That degree of extent is called membership value. In fact if $X = \{x_1, x_2, \cdots, x_n\}$, a fuzzy subset μ of X is an entity $(\mu(x_1)\mu(x_2), \cdots, \mu(x_n))$ where each $\mu(x_i)$ describes the membership value of element x_i to the fuzzy subset μ. We wish to rewrite this situation by collecting for each element x of X, its membership values with regards to all fuzzy subsets of X. This is how it will look like: $(\mu_1(x)\mu_2(x) \cdots \mu_r(x))$.
This will be called the $fuzzy\ bar-code$ of element $x \in X$. This is a new approach all together.
We can justify this way of writing by the fact that it makes it easier to pick up one element of X and determine its behaviour or attributes with regards or towards the properties enjoyed by members of set X.
With this bar-code, a fruit is described not in terms of its mass alone, as it is the case up to now, but in terms of its characteristics with regards to the fuzzy subsets. A mention concerning the time left before it expires can also be added. This can be a fraction of the time left out of the total time from

production date to expiry date.

One customer pays for the quality of the fruit he has picked from the shelf. Unlike the conflictive situation in operation up to now whereby one pays the same amount of money for a kilogram of either bruised, green or even yellow bananas because only the mass of the fruit counts.
For this purpose a gadget, kind of the scanner used extensively in the grocery shop may be handy. This new machine will extend the performance of the currently used machine because of the inclusion of fuzzy characteristics. Calibrated membership values of elements will be used to describe and identify objects (elements).

Each element of X therefore would have a tag, called $fuzzy\ bar-code$ which this scanner would have to read in order to identify the element. With this code a fruit or any other object is described not in terms of its mass alone but also in terms of its fuzzy characteristics, its membership values to the different fuzzy subsets of the set X.

One would pay for the quality of the fruits he has picked on the shelves. This kind of identification of the element will be called $fuzzy\ recognition\ identity$ (FRI). In this fashion we can determine at ease the element with $absolute\ desirability$; *no absolute desirability*, elements of X that are *worthless* to the fuzzy subsets of X respectively. Our aim is selecting elements of X, and also setting up fuzzy conditions which give us the desired output. This new approach will lead us not only to determine which element of X satisfies the properties we have chosen to monitor but also to determine which fuzzy subset of X is the most satisfactory with regards to the chosen properties.
With this fuzzy scanner we have a solution to a conflict in the market place. We term this as resolution of conflict under fuzzy environment.

Consider X as set of fruits of same kind and imagine that some fuzzy properties are defined in X, which the fruits $x_1, x_2, \cdots x_n$ in X may enjoy partially or totally. In this case it is possible to determine for instance the intensity of colour, the softness of the fruit's skin, to mention just a few characters, as a means for grading these fruits.
One other way of identifying these fruits would be to determine the sum of

their membership values appearing as bar-code on the tag attached to each fruit.

The number obtained in the process, which we call $worthness\, of the\, element$, might not be the best criteria of identification but it has more chance to reduce conflict in this matter.
It is possible that two elements (fruits in this instance) may have the same $worthness$; that is the sum of the numbers making the $bar - code$ is same.
Two $bar-codes$ $\mu_1(x)\mu_2(x)\cdots\mu_r(x)$ and $\mu_1(y)\mu_2(y)\cdots\mu_r(y)$ are equal if $\forall i, 1 \leq i \leq r, \mu_i(x) = \mu_i(y)$. Therefore $x = y$.
This means the $bar - code$ is unique for each element.

If the $bar - code$ of a certain element is given, how sure are we that this code represents a unique element of X. In other words are we able to detect a mistake on a given $bar - code$? What tool in our disposal would help discover an error and also tell at what level such error has occured.

Let us consider the elements of X together with their membership values to the respective fuzzy subsets of X. We can establish a matrix of elements of X versus their respective membership values to the fuzzy subsets of X. Let the following matrix represent the membership values of the n elements to the r fuzzy subsets.

$$\begin{array}{c} x_i/\mu_j \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccc} \mu_1 & \mu_2 & \cdots & \mu_r \end{array} \\ \begin{pmatrix} \mu_1(x_1) & \mu_2(x_1) & \cdots & \mu_r(x_1) \\ \mu_1(x_2) & \mu_2(x_2) & \cdots & \mu_r(x_2) \\ \vdots & \vdots & \cdots & \vdots \\ \mu_1(x_n) & \mu_2(x_n) & \cdots & \mu_r(x_n) \end{pmatrix} \end{array}$$

which is a $n.r$ matrix.

This matrix is called the $bar - code$ matrix. Every row of this matrix shows us the bar-codes of all elements of X. Every column of this matrix show us the fuzzy subsets of X. Reading the entries in one column gives us the information concerning a fuzzy subset and the membership values to the this fuzzy subset of the respective elements of X. Similarly reading from one row informs us of one element of X and the respective membership values of this element to the fuzzy subsets of X.

The matrix describes each individual element (commodity) x_i in term of the r fuzzy characteristics.
This matrix makes it easy to check for each fuzzy subset, elements in its $\alpha-cut$.
From this table we can quickly determine fuzzy subsets which are preferentially equal.
As it can easily be seen, the sum $(\sum_{i=1}^{r} \mu_i(x))$ of the entries on a row captures the sum of all membership values of element x. We call such sum the *fuzzy worthness of* x or *the Borda count of* x.
This number denoted here as W_x is such that $0 \leq W_x \leq r$.

If $W_x = r$, the element x is of absolute desirability. That means $\mu_i(x) = 1, \forall i$.
If on the other hand $W_x = 0$, then x is of *absolute worthless*. This means that the membership value of x to each fuzzy subset of X is 0.
Now if we take the sum of *worthness* of all elements of set X,that is $\sum_{j=1}^{n}\sum_{i=1}^{r} \mu_i(x_j)$, we then find the total *worthness* of the elements X. This number is of great importance if we are to compare two different sets of elements X and Y both subjected to the same type of fuzzy properties. We might be able to determine for instance, in the same period of the year, which farm between X and Y supplies our grocery shop with the acceptable set of fruits. That is which farm has the highest total *worthness*, if some fuzzy properties were already fixed.

When we consider the sum of the entries on one column, that is $\sum_{i=1}^{n} \mu(x_i)$.
This time we obtain the well-known cardinality of the fuzzy subset considered or the *worthness* of the fuzzy subset denoted here as W_μ. We also call this the *Borda count* of the fuzzy subset . If $W_\mu = 0$, then μ is the null fuzzy subset.
If $W_\mu = n$, then every element of X has absolute membership value to the

fuzzy subset.
Now comparing the sum of entries of the columns inform us of the fuzzy subset of highest cardinality. Now when we sum the cardinalities of all fuzzy subsets we obtain what we term as the *worthness of* $\mathcal{F}(\mathcal{X})$. We will realize that the *worthness of* X is actually the *worthness of* $\mathcal{F}(\mathcal{X})$. We write $W_X = W_{\mathcal{F}(\mathcal{X})}$.

In fact this equality is justified since the quality of elements of X is discovered with regards to their belonging to the fuzzy subsets. In the same fashion, fuzzy subsets are related to the kind of elements of X.
The similarity of elements seen earlier can be extended to their *worthness*. We can say that if *x is similar to y* then $W_x = W_y$.
Let us assume that x and y are similar with regard to some fuzzy subsets $\mu_1, \mu_2, \cdots, \mu_r$. That means that for any fuzzy subset μ_i , we have $\mu_i(x) = \mu_i(y)$. Therefore $(\sum_{i=1}^{r} \mu_i(x)) = (\sum_{i=1}^{r} \mu_i(y))$.
Now assume that $\mu_1, \mu_2, \cdots, \mu_r$ are similar to one another with respect to the element x. As seen earlier $\mu_1(x) = \mu_2(x) = \cdots = \mu_r(x) = \alpha$,$\alpha \in I$. Then $W_x = \alpha.r$.

Let x_i and x_j be two fruits with fuzzy $bar - codes$ $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ respectively. We may establish $a_i \vee b_i$ and $a_i \wedge b_i$. Therefore $(a_1 \vee b_1, a_2 \vee b_2, \cdots, a_n \vee b_n)$ is the fuzzy $bar - code$ which we call the *Major Ideal fruit* between x_i and x_j. It is *ideal* because it may not represent an actual fruit, rather it is a fruit the characteristics of which we admire. In the same manner $(a_1 \wedge b_1, a_2 \wedge b_2, \cdots a_n \wedge b_n)$ is the fuzzy bar-code of *the Minor Ideal fruit* between x_i and x_j.
These *ideal major* and *ideal minor* may or may not be members of the set of fruits. It is rather a technical terminology.
If we are given n fruits and m fuzzy subsets of a set; we can in the same manner find the *ideal major* or *ideal minor* of the n fruits which will have the following $fuzzy\ bar - codes$:
$(a_{11} \vee a_{21} \vee \cdots \vee a_{n1}, a_{12} \vee a_{22} \vee \cdots a_{n2}, \cdots a_{1m} \vee a_{2m} \cdots a_{nm})$ and $(a_{11} \wedge a_{21} \wedge \cdots \wedge a_{n1}, a_{12} \wedge a_{22} \wedge a_{n2} \cdots, a_{1m} \wedge a_{2m} \wedge \cdots a_{nm}$ respectively.
We know that each fruit has a membership value to each fuzzy subset. With this in mind and considering the $fuzzy\, bar - code$ $(a_1, a_2, \cdots, a_n)$ of fruit x_i; we

can compute $(1-a_1, 1-a_2, \cdots 1-a_n)$. We call this the $Fuzzy\ complement\ bar-code$ of fruit x_i.
Consider two $fuzzy\ bar-codes$ $(\mu_1(x)\mu_2(x)\cdots\mu_r(x))$ and $(\mu_1(y)\mu_2(y)\cdots\mu_r(y))$ of two elements x and y of X. Clearly these two entities are not always equal. Their being equal would mean that for each $1 \leq i \leq r,\ \mu_i(x) = \mu_i(y)$. This also means that for any two elements x and y and any two fuzzy subsets μ_i and μ_j, $\mu_i(x) = \mu_i(y)$ and $\mu_j(x) = \mu_j(y)$. There is only one such fuzzy subset where membership values of elements are equal to a single value $\alpha \in I$.
The $fuzzy\ bar - code$ of an element is therefore unique and each element has only one $fuzzy\ bar-code$. This confirms that the $fuzzy\ recognition\ approach$ is the best way of characterizing each individual element of the set X. This could be used for forensic analysis.

In the following example we want to apply fuzzy subsets to determine the best house among many. Some fuzzy characteristics are used to descibe the houses in the market place. But only one should satisfy the expectations of the buyer. If we set the number of houses to be large, then it becomes confusing to any buyer to determine the best house. Therefore we set a manageable number of properties for the houses. Each house has a membership value to the fuzzy subset representing the property.

Example 4.13.1 . Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a set of houses to be sold and
E = {expensive, wooden, beautiful, cheap, in green surroundings, concrete, moderately, beautiful, by the road side} be a set of parameters related to these houses.
Let $\mu_1$and μ_2 be the fuzzy subsets describing the cost of the houses given by :
$\mu_{1(Cheap)} = \{h_1/1, h_2/0, h_3/1, h_4/.2, h_5/1, h_6/.2\}$.
$\mu_{2(expensive)} = \{h_1/0, h_2/1, h_3/.1, h_4/.9, h_5/.3, h_6/1\}$
Let μ_3 and μ_4 be the fuzzy subsets describing the attractiveness of the houses given by:
$\mu_{3(beautiful)} = \{h_1/1, h_2/.4h_3/1, h_4/.4, h_5/.6, h_6/.8\}$
$\mu_{4(moderately\,beautiful)} = \{h_1/.3, h_2/.7, h_3/.5, h_4.6, h_5/.2, h_6/.3\}$
Let μ_5 and μ_6 be the fuzzy subsets describing the physical traits of the houses given by:

$\mu_{5(wooden)} = \{h_1/.2, h_2/.3, h_3/1, h_4/1, h_5/1, h_6/0\}$
$\mu_{6(concrete)} = \{h_1/.7, h_2/.9, h_3/0, h_4/.1, h_5/.3, h_6/.8\}$.
Let μ_7 and μ_8 describe the characteristics of the place where the houses are located by: $\mu_{7(in\, green\, surrounding)} = \{h_1/1, h_2/.1, h_3/.5, h_4/.3, h_5.2, h_6/.3\}$
$\mu_{8(near\, the\, road\, side)} = \{h_1/.2, h_2/.7, h_3/.8, h_4/1, h_5/.5, h_6.9\}$.
Mr X is interested in buying a house on the basis of his choice of parameters beautiful, wooden, cheap, in green surroundings. This means he has to select a house available in U that qualifies all the parameters of his choice.
We draw the following $bar - code$ matrix to represent the situation:

$$\begin{array}{c} h_i/\mu_j \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 \end{array} \\ \begin{pmatrix} \mu_1(h_1) & \mu_2(h_1) & \mu_3(h_1) & \mu_4(h_1) & \mu_5(h_1) & \mu_6(h_1) & \mu_7(h_1) & \mu_8(h_1) \\ \mu_1(h_2) & \mu_2(h_2) & \mu_3(h_2) & \mu_4(h_2) & \mu_5(h_2) & \mu_6(h_2) & \mu_7(h_2) & \mu_8(h_2) \\ \mu_1(h_3) & \mu_2(h_3) & \mu_3(h_3) & \mu_4(h_3) & \mu_5(h_3) & \mu_6(h_3) & \mu_7(h_3) & \mu_8(h_3) \\ \mu_1(h_4) & \mu_2(h_4) & \mu_3(h_4) & \mu_4(h_4) & \mu_5(h_4) & \mu_6(h_4) & \mu_7(h_4) & \mu_8(h_4) \\ \mu_1(h_5) & \mu_2(h_5) & \mu_3(h_5) & \mu_4(h_5) & \mu_5(h_4) & \mu_6(h_5) & \mu_7(h_5) & \mu_8(h_5) \\ \mu_1(h_6) & \mu_2(h_6) & \mu_3(h_6) & \mu_4(h_6) & \mu_5(h_6) & \mu_6(h_6) & \mu_7(h_6) & \mu_8(h_6) \end{pmatrix} \end{array}$$

Using Mr X's choice, we have the following reduced $bar - code$ matrix:

$$\begin{array}{c} h_i/\mu_j \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{array} \begin{array}{c} \begin{array}{cccc} \mu_1 & \mu_3 & \mu_5 & \mu_7 \end{array} \\ \begin{pmatrix} 1 & 1 & .2 & 1 \\ 0 & .4 & .3 & .1 \\ 1 & 1 & 1 & .5 \\ .2 & .4 & 1 & .3 \\ 1 & .6 & 1 & .2 \\ .2 & .8 & 0 & .3 \end{pmatrix} \end{array}$$

We take the $(\mu_1(h_i) \wedge \mu_3(h_i) \wedge \mu_5(h_i) \wedge \mu_7(h_i)) \forall i, 1 \leq i \leq 6$. We get:
$\{h_1/.2, h_2/0, h_3/.5, h_4/.2, h_5/.2, h_6/0\}$.
This shows that h_3 is the best choice.
We realize that with regards to the choice of Mr X, the worthiness of the houses are as follow: $W_{h_1} = 3.2$, $W_{h_2} = .8$, $W_{h_3} = 3.5$, $W_{h_4} = 1.9$, $W_{h_5} = 2.8$,

$W_{h_6} = 1.3$. We can see that h_3 has the highest worthiness. It is the best choice.
With regards to the table above, the *ideal major* house would have the worthiness of 4 and the *ideal minor* house would have the worthiness of 0.5.

The solution of the above problem was obtained by Miya in [23]. We have suggested a different approach to the way to go about it and have come to the same conclusion as Miya. We call our method the *min/max* method. It is because we first search for the *min* of all the $min\mu_j(h_i)$ and later the *max* of the these results.
We will get the same result as that of Miya if we were to use yet another alternative method. This consists of comparing the *worthiness* of each element (house in this case) to the fuzzy subsets. The house with highest worthiness is indeed the best choice.

4.14 Operations in the set of Fuzzy Matrices

Matrices play an important role in different branches of science and technology. But due to the presence of various types of uncertainties, the traditional classical matrix and operations on matrices may not be sufficient for the precise description of the characteristics of any system, pattern, etc. Thus the following definition and operations are suitable only for fuzzy matrices. A fuzzy matrix (FM) A of order $m \times n$ is defined as $A = [< a_{ij}, \mu(a_{ij}) >]_{m\times n}$ where $\mu(a_{ij})$ is the membership value of the element a_i to the fuzzy subset μ_j in A. This means that a fuzzy matrix is a matrix of membership values of elements of a set to the fuzzy subsets of the set.

For simplicity, we write A as $A = [\mu(a_{ij})]_{m\times n}$. In a Boolean Fuzzy Matrix $A = [a_{ij}]_{m\times n}$, all elements are either 1 or 0.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be any two fuzzy matrices of order $m \times n$. We define the following operations in the set of all fuzzy matrices of order $m \times n$. For any two elements a_{ij} and b_{ij} of a fuzzy matrix:
1. $a_{ij} \vee b_{ij} = max(a_{ij}, b_{ij})$
2. $a_{ij} \wedge b_{ij} = min(a_{ij}, b_{ij})$

3. $a_{ij} \oplus b_{ij} = a_{ij} + b_{ij} - a_{ij}.b_{ij}$
4. $a_{ij} \odot b_{ij} = a_{ij}.b_{ij}$
5. $a_{ij} \ominus b_{ij} = a_{ij}$ if $a_{ij} > b_{ij}$ and $a_{ij} \ominus b_{ij} = 0$ if $a_{ij} \leq b_{ij}$.
Now for any two fuzzy matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of order $m.n$, we have the following operations. [2]
1. $A \wedge B = [a_{ij} \wedge b_{ij}]$
2. $A \vee B = [a_{ij} \vee b_{ij}]$.
3. $A' = [a_{ji}]$, with A' the transpose of A_{ij}.
4. $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all i, j.
5. For any two fuzzy matrices A and B, $min(A, B) = A$ if $A \leq B$ and $min(A, B) = B$ if $B \leq A$.
6. $A^c = [1 - a_{ij}]$ where A^c is the Matrix complement of A
7. $A \leq B$ if $\mu(a_{ij}) \leq \mu(b_{ij})$.
8. $A.B = [a_{ij}.b_{ij}]$
9. $A + B = [a_{ij} + b_{ij} - a_{ij}.b_{ij}]$
The properties 8 and 9 above are based on the algebraic product and sum of fuzzy subsets.(See [20]).
It is clear from this context that the following properties concerning fuzzy matrices are valid.

Properties
1. $A.B = B.A$ and $A + B = B + A$.
2. $(A.B).C = A.(B.C)$ and $(A + B) + C = A + (B + C)$
3. $A.[\emptyset] = [\emptyset]$ and $A + [\emptyset] = A$ where $[\emptyset]$ is the fuzzy matrix where entries are all 0.
The properties of idempotence and distributivity are not satisfied by the product and sum of matrices. That is:
4. $A.A \neq A$ and $A + A \neq A$
5. $A.(B + C) \neq (A.B) + (A.C)$ and $A + (B.C) \neq (A + B).(A + B)$
6. $A.A^c \neq [\emptyset]$ and $A + A^c \neq [1]$, where [1] is the fuzzy matrix where each entry is 1.
Proof. 1. Let A , B and C be fuzzy matrices: $A.B = [a_{ij}.b_{ij}] = [b_{ij}.a_{ij}] = B.A$. Likewise $A + B = [a_{ij} + b_{ij} - a_{ij}.b_{ij}] = [b_{ij} + a_{ij} - b_{ij}.a_{ij}] = B + A$. The matrix $(A.B).C = ([a_{ij}].[b_{ij}]).[c_{ij}] = [a_{ij}.([b_{ij}.[C_{ij}) = A.(B.C)$. In the same manner $(A+B)+C = [(a_{ij}+b_{ij}-a_{ij}.b_{ij})]+[c_{ij}] = (a_{ij}+b_{ij}-a_{ij}.b_{ij}+c_{ij})-[a_{ij}+b_{ij}-a_{ij}.b_{ij}].c_{ij} = a_{ij} + (b_{ij} + c_{ij} - b_{ij}c_{ij}) - a_{ij}(b_{ij} + c_{ij} - b_{ij}.c_{ij}) = A + (B + C)$. For

any fuzzy matrix A, $A.A = [a_{ij}^2]$ with $0 \leq a_{ij}^2 \leq a_{ij}$. The only way $A.A = A$ is for the a_{ij} to be either 0 or 1, which means A is a boolean matrix. In the same manner $A + A \neq A$ because $A + A = A$ would mean that $2a_{ij} - a_{ij}^2 = a_{ij}$ and $2 - a_{ij} = 1$ so that $a_{ij} = 1$. Once again the matrix A would be boolean matrix. Now consider the fuzzy matrix A. If $A.A^c = [\emptyset]$, then $a_{ij}.(1 - a_{ij}) = 0$ and either $a_{ij} = 0$ or $a_{ij} = 1$. Again if $a_{ij} + (1 - a_{ij}) - a_{ij}.(1 - a_{ij}) = 1$, a_{ij} would be equal to 0 or to 1. In both cases the matrix A would be boolean.
Based on our discussion regarding the fuzzy matrix, and considering two different sets X_1 and X_2 in which the same type of fuzzy properties are defined. Then the matrix $A \wedge B = [a_{ij} \wedge b_{ij}]$ describes the matrix of lower membership values the elements of X_1 and X_2 have with regards to the fuzzy subsets. It is like the lowest benchmark we would accept for these two sets of elements. On the other hand the matrix $A \vee B = [a_{ij} \vee b_{ij}]$ is the highest benchmark of the sets X_1 and X_2. We can write $A \wedge B = [a_{ij} \wedge b_{ij}] \leq A \leq A \vee B = [a_{ij} \vee b_{ij}]$, $\forall A$.

4.15 Hamming Distance between two fuzzy matrices.

The Hamming distance between two fuzzy matrices A and B, denoted $H(A, B)$ is a mapping from the set of fuzzy matrices $\mathcal{M}$ to $\mathbb{R}$ defined as $H(A, B) = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij} - b_{ij}|$. The Hamming Distance between matrices is a metric on $\mathcal{M}$. That is to say that it satisfies the necessary conditions of distance. For any three fuzzy matrices A, B and C of $\mathcal{M}$:
1.$H(A, B) \geq 0$
2.$H(A, B) = H(B, A)$
3.$H(A, B) \leq H(A, C) + H(C, B)$

4.15.1 Properties of Distance between two fuzzy matrices

For any two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order $m.n$,
1. $H(A, A) = [0]$
2. $H(A, A \wedge B) = H(B, A \vee B)$.
3. $H(A, A \vee B) = H(B, A \wedge B$.
4. $H(A, B) = H(A \vee B, A \wedge B)$.
Proof: 1. It is clear that $H(A, A) = [0]$ for any matrix A.
2. If $A \wedge B = A$, then $A \vee B = B$ and $H(A, A \wedge B) = H(A, A) = [0] = H(B, B)$. If in the contrary $A \wedge B = B$, then $A \vee B = A$ and property 2 will be justified. [0] in this case is the nil matrix of same order as A and B. The proofs for properties 3 and 4 follow the same procedure □.
Let $A = [a_{ij}]$ be a fuzzy matrix. We define by A^k the fuzzy matrix $[(a_{ij})^k]$ made of powers $(a_{ij})^k$ of (a_{ij}).
We know that each $0 \leq a_{ij} \leq 1$ such that $(a_{ij})^2 \leq (a_{ij})$. In fact we can still write that $A^{k+1} \leq A^k$ $k = 1, 2, \cdots$. Therefore we can establish the following property:
$H(A, A^k) \leq H(A, A^{k+1})$ $k = 1, 2, \cdots$.
Consider two boolean fuzzy matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of same order $m.n$. If we impose the condition that if $a_{ij} = 1$ then $b_{ij} = 0$ or if $a_{ij} = 0$ then $b_{ij} = 1$, we will be able to say that the maximum Hamming distance between these two fuzzy matrices is $m.n$. The minimum Hamming distance between the fuzzy matrices is 0.

Let us denote by I_n the identity fuzzy matrix of order n^2. For any fuzzy matrix A of order n^2:
1. $A \wedge I_n$ is the diagonal matrix $[a_{ii}]$.
2. $A \vee I_n$, and $H(A, I_n) = H(A, A_n^k)$
3. $H(I_n, I_n^c) = n^2$.

Proposition 4.15.1 Hamming distance between fuzzy matrices of same equivalence class.
The Hamming distance between two fuzzy matrices whose entries are fuzzy subsets of the same equivalence class is zero.

This proposition is our seventh fuzzy recognition procedure.

In character recognition, this proposition says that the Hamming distance between the matrices representing one single character is zero.
One character can still be written in two different fonts, yet it is still the same character. [1] [47]

Chapter 5

An Election Activity under Fuzzy Environment

A fuzzy subset has a kind of preference built into it. The fact that one element of X say x has higher membership value than y to the fuzzy subset μ can be interpreted here as x is preferred more than y. This preference leads naturally to an election, when putting elements of X on a list ranging them from the most preferred to least preferred. $X = \{x_1, x_2, \cdots, x_n\}$ would be in this case the set of alternatives that m different persons called agents (also membership values) have to grade. That is to show for each person his preference regarding the alternatives by indicating which alternative comes first, second and so on until the last. Now instead of anyone grading the alternatives' we could consider the fuzzy subsets and take into account the m distinct membership values each individual element of X has with respect to the m fuzzy subsets. The set of membership values being ordered with the same order as $\mathbb{R}$, grading of alternatives is done naturally by associating each alternative with its membership value to the respective fuzzy subsets.
An election activity would determine in this case which element of X has the highest membership value thus the best possible qualities with regards to the given fuzzy properties. That is the element of X with the highest membership value, to the fuzzy properties, many times more than any other element of X. The election would also determine the most popular fuzzy property. It is like looking among a number of fruits or any other commodities, the most wanted one or searching the quality the most sought among many by customers in a

particular grocery shop.

5.1 Fuzzy preference schedule

We can display the membership values of elements of X for each fuzzy subset μ_j on a column, arranging them in order of size from highest to lowest value. Such ranking or representation will be called $fuzzy\ preference\ schedule$.

Example 5.1.1 .

$$\begin{array}{c} \mu_j \\ \mu_j(x_2) \\ \mu_j(x_4) \\ \vdots \\ \mu_j(x_k) \\ \vdots \\ \mu_j(x_n) \end{array}$$

is a preference schedule telling us that for the fuzzy subset μ_j, $\mu_j(x_2) > \mu_j(x_4) > \cdots > \mu_j(x_p) > \cdots > mu_j(x_n)$. In other words, x_2 is the most preferred alternative, while x_n is the least preferred in relation to the fuzzy property μ_j.
Since we may have this situation expressed by several fuzzy subsets among the m given as in the above example, we count the number of times each element x_i is listed first.
Consider a fixed $x_i \in X$ and the fuzzy subsets μ_1, μ_2, $\cdots, \mu_m$. We can also arrange the $\mu_j(x_i), 1 \leq j \leq m$, in a preference schedule of fuzzy subsets as in the table below.

$$\begin{array}{c} x_i \\ \mu_p(x_i) \\ \mu_r(x_i) \\ \vdots \\ \mu_s(x_i) \\ \vdots \\ \mu_j(x_i) \end{array}$$

which means that for element $x_i \in X$, $\mu_p(x_i) > \mu_r(x_i) > \cdots \mu_s(x_i) > \mu_j(x_i)$. That is to say that μ_p is the most appropriate property enjoyed by x_i or that x_i has the highest membership value in μ_p.

Now if we draw for each fuzzy subset μ_j and for each element x_i their preference schedules with respect to the $x_i \in X$ and with regards to the μ_j respectively, we are able to establish the number of times each fuzzy subset or each element is ranked first.

5.2 Plurality Winner and Borda Winner

We say that x_i is the *fuzzy plurality winner* (fpw) of our election if $\forall j$, $1 \leq j \leq m$, $\mu_j(x_i)$ is ranked first more than any other membership value of the elements of X to the m fuzzy subsets. In the same manner a fuzzy subset μ_j is the plurality winner if $\forall x \in X$; $\mu_j(x)$ is ranked first more than any of the m fuzzy subsets of X.

Now if an element of X or a fuzzy subset of X is ranked first on over half of the preference schedules respectively, then it is declared the fuzzy *majority winner* (fmw) of the elements of X or the fuzzy *majority* winner of the fuzzy subsets of X respectively.

Suppose we find for each $x_i \in X$ the sum $\sum_{j=1}^{m} \mu_j(x_i)$. Then the ranking of these results in establishing a winner among the n elements of X, is what we will call the *Borda* Fuzzy *Winner* (BFW) of elements of X.

Similarly, if we find $\sum_{i=1}^{n} \mu_j(x_i)$ for $1 \leq j \leq m$ we obtain the *Borda* Fuzzy

$Winner$ of the fuzzy subsets of X. We observe in this case that the fuzzy subset with the highest cardinality is the BFW. We will illustrate this in the following example.

Example 5.2.1 .

$$\begin{array}{ccccc} x_i/\mu_j & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ x_1 & \mu_1(x_1) & \mu_2(x_1) & \mu_3(x_1) & \mu_4(x_1) \\ x_2 & \mu_1(x_2) & \mu_2(x_2) & \mu_3(x_2) & \mu_4(x_2) \\ x_3 & \mu_1(x_3) & \mu_2(x_3) & \mu_3(x_3) & \mu_4(x_3) \end{array}$$

$=$

$$\begin{array}{ccccc} x_i/\mu_j & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ x_1 & .7 & .2 & .9 & .8 \\ x_2 & .3 & .8 & .6 & .7 \\ x_3 & .5 & .6 & .7 & .4 \end{array}$$

Such that

$$\begin{array}{c} \mu_1 \\ \mu_1(x_1) \\ \mu_1(x_3) \\ \mu_1(x_2) \end{array}$$

$$\begin{array}{c} \mu_2 \\ \mu_2(x_2) \\ \mu_2(x_3) \\ \mu_2(x_1) \end{array}$$

$$\begin{array}{c} \mu_3 \\ \mu_3(x_1) \\ \mu_3(x_3) \\ \mu_3(x_2) \end{array}$$

$$\mu_4$$
$$\mu_4(x_2)$$
$$\mu_4(x_1)$$
$$\mu_4(x_3)$$

and

$$x_1$$
$$\mu_3(x_i)$$
$$\mu_4(x_i)$$
$$\mu_1(x_i)$$
$$\mu_2(x_i)$$

$$x_2$$
$$\mu_4(x_2)$$
$$\mu_2(x_2)$$
$$\mu_3(x_2)$$
$$\mu_1(x_2)$$

$$x_3$$
$$\mu_3(x_3)$$
$$\mu_2(x_3)$$
$$\mu_1(x_3)$$
$$\mu_4(x_3)$$

Such that

$$x_1$$
$$\uparrow$$
$$3$$

$$x_2$$
$$\uparrow$$
$$1$$

$$x_3$$
$$\uparrow$$
$$0$$

$$\mu_1$$
$$\uparrow$$
$$0$$

$$\mu_2$$
$$\uparrow$$
$$1$$

$$\mu_3$$
$$\uparrow$$
$$2$$

$$\mu_4$$
$$\uparrow$$
$$0$$

It is clear that x_1 and μ_3 are Fuzzy plurality winners of the elements of X and of the fuzzy subsets of X respectively.

$\sum_{j=1}^{4} \mu_j(x_1) \doteq 2.6$, $\sum_{j=1}^{4} \mu_j(x_2) = 2.4$, $\sum_{j=1}^{4} \mu_j(x_3) = 2.2$ while $\sum_{i=1}^{3} \mu_1(x_i) = 1.5$, $\sum_{i=1}^{3} \mu_2(x_i) = 1.6$, $\sum_{i=1}^{4} \mu_3(x_i) = 2.2$, $\sum_{i=1}^{3} \mu_4(x_i) = 1.9$. In this case x_1 and μ_3 are the Fuzzy Borda Winners of the elements of X and the fuzzy subsets of X respectively.

Example 5.2.2 .

Let us consider again the matrix

$$\begin{array}{c c} x_i/\mu_j & \begin{array}{cccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} .7 & .2 & .9 & .8 \\ .3 & .8 & .6 & .7 \\ .5 & .6 & .7 & .4 \end{pmatrix} \end{array}$$

where x_1 is 3 times first, 0 times second, 1 times third. x_2 is first 1 once, 1 time second, 2 time third, x_3 is 0 times first, 3 times second, 1 times third. The product of matrices

$$\begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$$

and

$$\begin{array}{cc} No/x_i & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} 1st \\ 2nd \\ 3rd \end{array} & \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} \end{array}$$

gives

$$\begin{pmatrix} 10 & 7 & 7 \end{pmatrix}$$

showing that x_1 is the Fuzzy Borda Winner.
By assigning 3 to first place, 2 to second place and finally 1 to third place; we could have obtained the same result if we would compute in this fashion:
$x_1 : 3(3) + 0(2) + (1) = 10$
$x_2 : 1(3) + 1(2) + 2(1) = 7$
$x_3 : 0(3) + 3(2) + 1(1) = 7$ showing as above that x_1 is the winner. Similarly we note that μ_1 is first 0 time, second 0 time, third 1 time, fourth 1 time. μ_2 is first 1 time, second 1 time, third 0 time, fourth 1 time. μ_3 is first 2 times, 0 time second, 1 time third and 0 time fourth. μ_4 is first 1 time, 1 time second, 0 time third and 1 time fourth.
The product of the matrices

$$\begin{pmatrix} 4 & 3 & 2 & 1 \end{pmatrix}$$

and

$$\begin{array}{cc} No/\mu_i & \begin{array}{cccc} \mu_1 & {}_2 & {}_3 & \mu_4 \end{array} \\ \begin{array}{c} 1st \\ 2nd \\ 3rd \\ 4th \end{array} & \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

gives

$$\begin{pmatrix} 3 & 8 & 10 & 8 \end{pmatrix}$$

which shows that μ_3 is the Fuzzy Borda Winner. The same result is obtained if we work out in the following fashion:
$\mu_1 : 0(4) + 0(3) + 1(2) + 1(1) = 3$
$\mu_2 : 1(4) + 1(3) + 0(2) + 0(1) = 7$
$\mu_3 : 2(4) + 0(3) + 1(2) + 0(1) = 10$
$\mu_4 : 1(4) + 1(3) + 0(2) + 1(1) = 8$
which also shows that μ_3 is actually the *Borda* winner of the family of the given fuzzy subsets.

5.3 Mean Borda Count

Consider a set $X = \{x_1, x_2, \cdots, x_n\}$ and fuzzy subsets $\mu_1, \mu_2, \cdots, \mu_r$ as above. We are able to find for each element x_i of X the sum $\sum_{j=1}^{r} \mu_j(x_i)$ of its membership values to the r fuzzy subsets of X. Now the number $\frac{1}{r}[\sum_{j=1}^{r} \mu_j(x_1)]$ gives us an average, which we wish to call the $Mean\,Borda\,Count$ of x_i.
This $Mean$ Borda Count can be considered when selecting elements of X.
It is clear that the average gives an indication of the actual Borda count of the element. The larger the Borda count, the larger is the Mean Borda Count.
In the same manner the number $\frac{1}{n}[\sum_{i=1}^{n} \mu_j(x_i)]$ will be called the $Mean\,Borda\;Count$ of the fuzzy subset μ_j. Even in this case the mean reflects the actual cardinality or Borda count of the fuzzy subset.

Alternative to our approach is this one suggested by Garcia in [15]. We will give a summary of their work and compare these two approaches by an example.
Consider a set $X = \{x_1, x_2, \cdots x_n\}$, $n \geq 2$, of alternatives and assume $m \geq 2$ agents express their preferences over the pairs in X in a linguistic manner.

Let $L = \{l_o, l_1, \cdots l_s\}, s \geq 2$, be a set of linguistic labels ranked as $l_o < l_1 < \cdots < l_s$. There is a label such that the rest of labels are defined around it symmetrically. This label represents the indifference.
This means that $s+1$ is odd and that $l_{\frac{s}{2}}$ is a central figure in L.
This also tells us there is a relation $R^k : X \times X \to L$ where $R^k(x_i, x_j) = r^k_{i,j}$ represents the level of preference of x_i over x_j.
If $R^k(x_i, x_j) = l_s$ then x_i is totally preferred over x_j. But if $R^k(x_i, x_j) = l_o$ then x_j is totally preferred over x_i. If $R^k(x_i, x_j) = l_{\frac{s}{2}}$, then there is indifference. These l_i must be added and the results compared.
This indeed introduces the properties such as commutativity, associativity, existence of neutral element (l_o) in a set $\mathcal{L}$ with $L \subset \mathcal{L}$. There is also an order in $\mathcal{L}$ so that the results can be compared.
The total Borda count is obtained by the function $\overline{r_k} : X \to L; \overline{r}_k(x_i) = \sum_{j=1}^{n} r_{ij^k}$.
For a given $x_i \in X$, x_i is a Borda winner if $\overline{r}_k(x_i) \geq \overline{r}_k(x_j) \, \forall x_j \in X$.
In our case we consider the membership function as a ranking operation. We define in X a relation R^k such that
$R^k(x_i, x_j) = \mu(x_i) - \mu(x_j)$. This number may be positive or negative. We obtain a $i \times i$ matrix where the entries are the $\mu(x_i) - \mu(x_j)$ for each fuzzy subset μ of X. The sum of entries in one row say X_i gives the Borda count for x_i with regards to μ_j. We do the same for each element with regards to each fuzzy subset. The total Borda count is the sum of the step Borda counts.
The Borda winner is the element with the highest count. We can get more details in [25] and [16].

Example 5.3.1 .

Let us consider once more the matrix

$$\begin{array}{cc} x_i/\mu_j & \begin{array}{cccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} .7 & .2 & .9 & .8 \\ .3 & .8 & .6 & .7 \\ .5 & .6 & .7 & .4 \end{pmatrix} \end{array}$$

With regards to μ_1, we have:

$$\begin{array}{cc} x_i/\mu_1 & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 0 & .4 & .2 \\ -.4 & 0 & -.2 \\ -.2 & .2 & 0 \end{pmatrix} \end{array}$$

such that $\overline{r}_k(x_1) = .6,\ \overline{r}_k(x_2) = -,6,\ \overline{r}_k(x_3) = 0$

With regards to μ_2, we have :

$$\begin{array}{cc} x_i/\mu_2 & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 0 & -.6 & -.4 \\ .6 & 0 & .2 \\ .4 & -.2 & 0 \end{pmatrix} \end{array}$$

Here $\overline{r}_k(x_2) = -.1$, $\overline{r}_k(x_2) = .8$, $\overline{r}_k(x_3) = .2$

Concerning μ_3, we have

$$\begin{array}{cc} x_i/\mu_3 & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 0 & .1 & .4 \\ -.1 & 0 & .3 \\ -.4 & -.3 & 0 \end{pmatrix} \end{array}$$

and $\overline{r}_k(x_1) = 1.4$, $\overline{r}_k(x_2) = .2$, $\overline{r}_k(x_3) = -.7$.
The total Borda count will therefore be: $[x_1 : .6{-}1{+}1.4 = 1]$ $[x_2 : -.6{+}.8{+}.2 = .4]$ $[x_3 : 0 + .2 - .7 = .5]$.
This confirms again that x_1 whose count is the highest is the Borda winner.
In the same manner we would obtain the Borda winner of the fuzzy subsets of X with regards to the elements of X.

For x_1 we have:

$$\begin{array}{c} x_i/\mu_j \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} \\ \begin{pmatrix} 0 & .5 & -.2 & -.1 \\ -.5 & 0 & -.7 & -.6 \\ .2 & .7 & 0 & .1 \\ .1 & .6 & -.1 & 0 \end{pmatrix} \end{array}$$

For x_2 we have:

$$\begin{array}{c} x_i/\mu_j \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} \\ \begin{pmatrix} 0 & -.5 & -.3 & -.4 \\ .5 & 0 & .2 & .1 \\ .3 & -.2 & 0 & -.1 \\ .1 & .6 & -.1 & 0 \end{pmatrix} \end{array}$$

For x_3 we have:

$$\begin{array}{c} x_i/\mu_j \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{array} \\ \begin{pmatrix} 0 & -.1 & -.2 & .1 \\ .1 & 0 & -.1 & .2 \\ .2 & .1 & 0 & .3 \\ -.1 & -.2 & -.3 & 0 \end{pmatrix} \end{array}$$

The respective Borda counts for the fuzzy subsets are $[\mu_1 : -1.2]$ $[\mu_2 : -.8]$ $[\mu_3 : .6]$.
Therefore μ_3 is the Borda winner as seen before.

Chapter 6

Concluding Remarks

The study of fuzzy subsets of a finite set is a broad one and leads to various ramifications. We have chosen in our case the identification and enumeration of fuzzy subsets that have some common patterns.

The usage of the fuzzy scanner, as we have hinted in this study, can be extended from mere data recognition of goods in a grocery shop context to that of useful detection and forensic analysis. The identification obtained in this fashion seems to be not equivocal. This therefore solves a situation of conflict in the market place. We also think, in this regard, of the volume of water that would be saved if this kind of scanner could be successfully used in regulating the amount of water in public toilets. The scanner would read for example the quantity or proportion of urea, in the urine and other waste and tell how much water is needed to flush a toilet that has been used.

In public toilets seen in airports for example, water is flushed in all urinals irrespective of the number of people who stood there to use the facility. Our scanner would only allow flushing in the urinal being used at the time. The process of election in a fuzzy environment led us to determine the winner in a context in which characteristics are defined in fuzzy terms. This is also helpful in determining the best candidate for a job; the best house to buy; the best fruit or farm per specific season of the year etc...

A method consisting of comparing the membership values of elements has led us to determine the Borda count of elements of set X. This procedure is quite different from the one we found in the literature. In this method, a relation is defined in the set X to rank the preferences of elements over others. We have modified the usual ranking function and used the function consisting of adding the differences between the membership values of elements to each fuzzy subset.

We have identified the α-cut matrix of the complement of a partition as well as that of the union and intersection of a partition. We will have in the future to find out if the complement of a partition, the union and intersection of two partitions is also a partition of the set.

It will be interesting also to study the features of the partition generated by the composition of two binary relations in a set X.
In chapter 1 we discussed the principle of inclusion and exclusion. A similar principle called the Möbius inversion was discussed later in chapter 3 especially concerning the lattice of fuzzy subsets. We wish to extend this study looking into the Möbius inversion in the lattice of partitions of a finite set.

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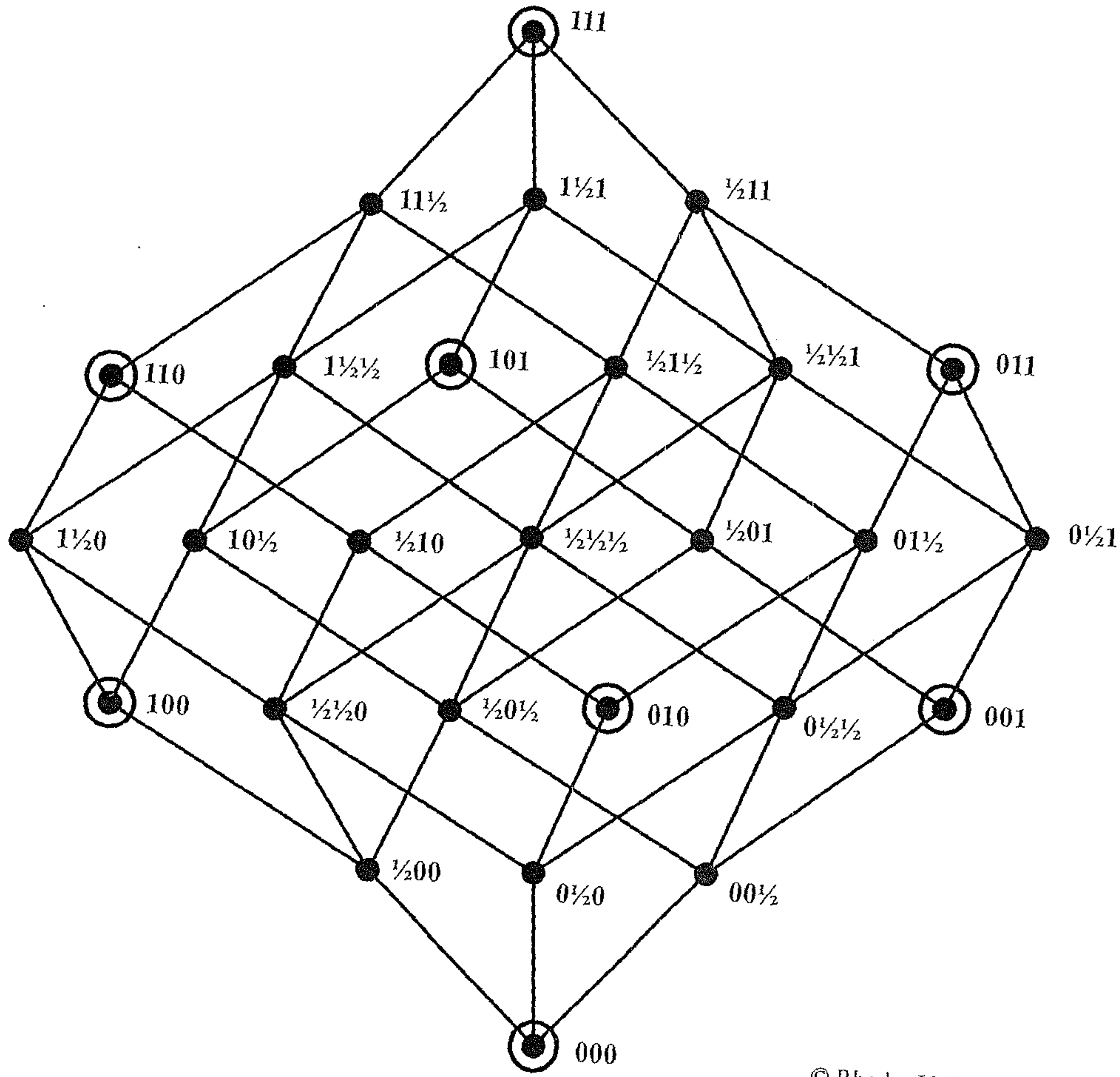
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