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STUDIES IN FUZZY GROUPS

by

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ABSTRACT

In this thesis we first extend the notion of fuzzy normality to the notion of normality of a fuzzy subgroup in another fuzzy group. This leads to the study of normal series of fuzzy subgroups, and this study includes solvable and nilpotent fuzzy groups, and the fuzzy version of the Jordan–Hölder Theorem.

Furthermore we use the notion of normality to study products and direct products of fuzzy subgroups. We present a notion of fuzzy isomorphism which enables us to state and prove the three well–known isomorphism theorems and the fact that the internal direct product of two normal fuzzy subgroups is isomorphic to the external direct product of the same fuzzy subgroups.

A brief discussion on fuzzy subgroups generated by fuzzy subsets is also presented, and this leads to the fuzzy version of the Basis Theorem. Finally, the notion of direct product enables us to study decomposable and indecomposable fuzzy subgroups, and this study includes the fuzzy version of the Remak–Krull–Schmidt Theorem.

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KEY WORDS :

Fuzzy subgroup, normal fuzzy subgroup, level subgroup, fuzzy isomorphism, fuzzy point, homomorphism, fuzzy product, fuzzy direct product, solvable, nilpotent and fuzzy quotient.

CONTENTS

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PREFACE

In [12] Rosenfeld introduced the notion of fuzzy subgroup of a group. He proved that a homomorphic pre–image of a fuzzy subgroup is always a fuzzy subgroup, and a homomorphic image of a fuzzy subgroup that has a certain condition (sup property) is always a fuzzy subgroup. Since then we proved that a homomorphic image of any fuzzy subgroup is always a fuzzy subgroup, and this proof is included in this thesis. Subsequently, we have discovered that in [30] Eroğlu also proved that a homomorphic image of a fuzzy subgroup is a fuzzy subgroup. A similar result was also obtained by Kumar in [35].

In [2] Das introduced the notion of a level subgroup of a fuzzy subgroup. He characterized fuzzy subgroups of finite groups by their level subgroups. Further characterizations of fuzzy subgroups were established by Bhattacharya in [4]. In [57] Mashinchi and Zahedi corrected a Theorem of [2].

In [7] Bhattacharya and Mukherjee introduced the notions of fuzzy normality and a fuzzy coset. In this thesis we develop more general notions of fuzzy normality and cosets. This general notion of fuzzy normality is characterized by level subgroups. (We also generalize the notion of a fuzzy coset given in [7] using the notion of fuzzy point given in [20]). We then prove that if μ is a fuzzy normal subgroup, then the supremum of its fuzzy cosets is a fuzzy subgroup.

In [7] Bhattacharya and Mukherjee presented a definition of a fuzzy quotient group. In [11] we used this definition to prove analogues of the first and the third isomorphism theorems. However, the fuzzy version of the second isomorphism theorem fails to hold. In this thesis we have modified the definition of a fuzzy quotient group given by Foster in [3], (this makes it possible for us) to prove a fuzzy version of the second isomorphism theorem.

In [43] Mukherjee and Bhattacharya introduced a notion of a fuzzy normalizer of a fuzzy subgroup. This fuzzy normalizer is not a fuzzy set, it is a crisp group in which the fuzzy subgroup is fuzzy normal. In this thesis we define a fuzzy normalizer $N(\mu)$ of a fuzzy subgroup μ such that $N(\mu)$ is a fuzzy subgroup in which μ is fuzzy normal. In [43] and [50] two different notions of fuzzy Abelian were defined. In [11] we defined a fuzzy subgroup μ to be fuzzy Abelian if each nonzero level subgroup of μ is Abelian. It is now clear that this is a weaker notion of fuzzy Abelian than that given in [50]. In this thesis we use the weaker notion of [11].

In [1] Bhattacharya and Mukherjee introduced the concept of fuzzy solvable. We deduce from this definition that a fuzzy solvable fuzzy subgroup is necessarily fuzzy normal. It is desirable that a fuzzy subgroup be fuzzy solvable without necessarily being fuzzy normal, since in the crisp case a subgroup H of a group $\mathscr{G}$ can be solvable without being normal in $\mathscr{G}$. In this thesis we give a definition of fuzzy normality which will ensure that a fuzzy solvable fuzzy subgroup need not be fuzzy normal. In [11] we introduced a notion of fuzzy isomorphism. In this thesis we improve on it.

In [49] Ray has also given a notion of fuzzy isomorphism. It is easy to check that our notion of fuzzy isomorphism is stronger than Ray's. Our version allows us to prove fuzzy counterparts of the isomorphism theorems.

In [14] Sherwood introduced the concept of the external direct product of fuzzy subgroups. In [11] we introduced the concept of an internal direct product and then proved that the internal and the external direct products are isomorphic when the fuzzy subgroups are fuzzy normal.

In [9] Murali studied fuzzy congruence relations. In [10] we proved that normal fuzzy subgroups and fuzzy congruence relations determine each other in a group–theoretic sense. This study of fuzzy congruence relations is included here. Fuzzy congruence relations have also been studied by, *inter alia*, Sidky and Ghanim in [52].

In Chapter 1 we first present preliminaries. Then we characterize fuzzy normality in several ways. We introduce the notion of a fuzzy subgroup being normal in another fuzzy subgroup. This notion is further characterized by level subgroups. Our definition of fuzzy normality ensures that any fuzzy subgroup is normal in itself as in the crisp case. We also present a general notion of a fuzzy coset. We then show that the supremum (fuzzy union) of all fuzzy cosets of a fuzzy normal subgroup is also a fuzzy subgroup which is analogous to the crisp case. The fuzzy union of fuzzy cosets of a fuzzy subgroup μ can then be regarded as a fuzzy quotient group $\mathscr{G}/\mu$, where $\mathscr{G}$ is the underlying group. However, this notion of quotient does not produce expected results and so we do not pursue it. Instead we present a more suitable notion of a quotient group. This is a generalization of Foster's quotient in [3]. Fuzzy isomorphism is also presented in Chapter 1. We end Chapter 1 with the notion of a product of two fuzzy subgroups of the same group, (see Zadeh [15] and Makamba [11]).

In Chapter 2 we first study fuzzy subgroups generated by fuzzy subsets. This study includes the notions of a fuzzy normalizer and a fuzzy centre of a given fuzzy subgroup. We end Chapter 2 by linking the notion of fuzzy normality given in [7] to the notion of fuzzy congruence relations.

Chapter 3 is a further study of direct products of fuzzy subgroups. In Section 3.1 we show that if μ and ν are normal fuzzy subgroups, then $\mu\nu$ is the smallest fuzzy subgroup of $\mathscr{G}$ containing both μ and ν provided $\mu(e) = \nu(e)$, where e is the identity element of the underlying group $\mathscr{G}$. We also establish analogues of the Dedekind and Modular laws. Hence the lattice of fuzzy normal subgroups on $\mathscr{G}$ is a modular lattice. When $\mathscr{G}$ is finite, we show that $\mu\nu$ is the internal direct product if and only if every nonzero level subgroup of $\mu\nu$ is an internal direct product of the corresponding level subgroups of μ and ν. We also show that the internal and the external direct products of fuzzy normal subgroups are isomorphic. We end the chapter by stating and proving analogues of the three well–known isomorphism theorems in group theory. We also show, by means of an example, that the second isomorphism theorem need not hold if we use the quotient of Mukherjee and Bhattacharya introduced in [1].

In Chapter 4 we discuss cyclic fuzzy subgroups. We show that every finite Abelian fuzzy subgroup can be decomposed into a direct product of cyclic p–fuzzy subgroups. (This is an analogue of the Basis Theorem in group theory). In [50] Sidky and Mishref have also worked with cyclic fuzzy subgroup. Our definition of a cyclic fuzzy subgroup does not require the zero–level subgroup to be cyclic as is the case in [50]. We end Chapter 4 with a notion of dimension of a fuzzy subgroup which is also a fuzzy vector space over the field $\mathbb{Z}_p$, where p is a prime number. This notion of dimension is related to Lowen's notion of dimension given in [16] in the sense that both notions use the crisp dimension of the support of the fuzzy subgroup and also the range of the fuzzy subgroup.

Chapter 5 is an extension of Chapter 4. In Section 5.1 we discuss decomposable and indecomposable fuzzy subgroups. We state and prove an analogue of the well–known Remak–Krull–Schmidt Theorem in group theory. Before proving this theorem, we discuss equivalent fuzzy subgroups. This leads us to define the length of a fuzzy subgroup : see [61]. The Remak–Krull–Schmidt Theorem then holds for a fuzzy subgroup of finite length. In Section 5.2 we study normal series of fuzzy subgroups. Our definition of a normal series is such that if $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ is a normal series, then each μ_i is normal in μ_{i-1} but need not be normal in the whole group $\mathscr{G}$ as is the case in [1]. We end Chapter 5 by stating and proving analogues of the Zassenhaus lemma and the Jordan–Hölder Theorem.

The latter theorem is preceded by notions of a maximal normal fuzzy subgroup and a maximal chain of normal fuzzy subgroups.

Chapter 6 is a further study of normal series of fuzzy subgroups. In Section 6.1 we study solvable fuzzy subgroups. Our definition of solvability differs from the definition of solvability given in [1] in that our notions of fuzzy normal and quotient are weaker than those given there, but our notion of fuzzy Abelian is stronger than the notion of fuzzy Abelian given in [1]. This makes the two notions of fuzzy solvability completely different. We also show, by means of examples, that if in our definition of a solvable series we replace our quotient by the quotient given in [1], then some of the crisp results on solvability have no fuzzy analogues. In crisp group theory every nilpotent group is solvable. In Section 6.3 we study nilpotent fuzzy subgroups and establish some analogues of the results on nilpotent groups.

CHAPTER 1

FUZZY NORMALITY, FUZZY QUOTIENT, FUZZY ISOMORPHISM AND PRODUCTS OF FUZZY SUBGROUPS.

INTRODUCTION

In [12] Rosenfeld proved that a homomorphic image of a fuzzy subgroup which has the sup property is a fuzzy subgroup. Since then we managed to prove that a homomorphic image of any fuzzy subgroup is always a fuzzy subgroup. The proof is included in this chapter. Subsequently we discovered that in [30] Eroğlu also proved that a homomorphic image of a fuzzy subgroup is a fuzzy subgroup. A similar result was obtained by Kumar in [35]. In Proposition 1.1.5 we characterize the notion of fuzzy normality given in [7] in several ways. In [4] Bhattacharya used the notion of a level subgroup introduced by Das in [2] to characterize fuzzy subgroups by their level subgroups. In [7] it is shown that a fuzzy subgroup is fuzzy normal if and only if all its level subgroups are normal subgroups of the underlying group. We show in this chapter that if the support of the fuzzy subgroup is normal in the underlying group, the fuzzy subgroup need not be fuzzy normal. It is obvious that fuzzy normality of a fuzzy subgroup implies normality of its support. The notion of fuzzy normality given in [7] is not general enough to allow us to say that a fuzzy subgroup is normal in another fuzzy subgroup. Hence in this chapter we generalize the notion of fuzzy normality given in [7]. This general fuzzy normality is further characterized by level subgroups. Our definition of fuzzy normality ensures that any fuzzy subgroup is normal in itself as in the crisp case.

We have also generalized the notion of a fuzzy coset given in [7]. It is then easily shown that the supremum of all fuzzy cosets of a normal fuzzy subgroup ν in a fuzzy subgroup μ is also a fuzzy subgroup. This is analogous to the crisp result which asserts that the set of all cosets of a normal subgroup is a group. The supremum of fuzzy cosets can then be regarded as a fuzzy quotient μ/ν. However, this gets more complicated, and we have not pursued it. Instead we present a quotient which is a modified version of Foster's definition of a fuzzy quotient group given in [3]. Our definition of a fuzzy quotient then allows us to prove analogues of the three isomorphism theorems in Chapter 3 and the Zassenhaus lemma in Chapter 5. In [7] Bhattacharya and Mukherjee introduced a notion of a fuzzy quotient group. In [11] we used this definition to prove analogues of the first and the third isomorphism theorems.

However, using the quotient given in [7], the second isomorphism theorem does not hold.

In crisp group theory it is true that $\mathcal{G} / \mathcal{G}$ is isomorphic to {e} for every group $\mathcal{G}$, where e is the identity element in $\mathcal{G}$. This result does have a fuzzy analogue if we use our notion of a fuzzy quotient. However, if we use the quotient given in [7], the above result has no fuzzy analogue. For this and other reasons mentioned earlier, in this thesis we do not use the quotient given in [7].

In [43] Mukherjee and Bhattacharya introduced a definition of fuzzy Abelian. However, that definition was retracted since it was equivalent to fuzzy normal. Another definition of fuzzy Abelian was introduced by the same authors in [1]. This definition of fuzzy Abelian is still not acceptable to us since it implies that any fuzzy subgroup μ of a group $\mathcal{G}$ satisfying $\mu(e) \gneq \mu(x)$ for all $x \in \mathcal{G} \setminus \{e\}$ is necessarily fuzzy Abelian. We feel this is a very weak condition for fuzzy Abelian. Also this definition is equivalent to saying that the $\mu(e)$ – level subgroup of μ is Abelian. In [11] we defined μ to be fuzzy Abelian iff each nonzero level subgroup of μ is Abelian. This is the definition that we will use in this thesis. In Section 1.2 we present a definition of fuzzy isomorphism which is a modified version of fuzzy isomorphism given in [11]. If two fuzzy subgroups μ and ν are fuzzy isomorphic, it is easily shown that each level subgroup of μ is isomorphic to some level subgroup of ν. We end this chapter with an introductory discussion of a product of two fuzzy subgroups. This is part of the work done in [11].

REMARK :

Subsequent to our definitions of fuzzy normal and fuzzy coset, we have discovered that these two definitions coincide with those of Malik, Mordeson and Nair in [59].

1.1 : PRELIMINARIES

Definition [15] : Let $\mathcal{G}$ be a set. A *fuzzy subset* of $\mathcal{G}$ is a mapping $\mu : \mathcal{G} \longrightarrow [0,1]$. If μ and ν are fuzzy subsets of $\mathcal{G}$ such that $\mu(x) \leq \nu(x)$ for all $x \in \mathcal{G}$, we write $\mu \leq \nu$ or $\mu \subseteq \nu$ and say that μ is contained in ν or μ is a fuzzy subset of ν.

DEFINITION : 1.1.1 [12]

Let $\mathcal{G}$ be a group. A map $\mu : \mathcal{G} \longrightarrow [0,1]$ is called a *fuzzy subgroup* of $\mathcal{G}$ if

(i) $\mu(xy) \geq \min(\mu(x), \mu(y))$ for all $x, y \in \mathcal{G}$;

(ii) $\mu(x) = \mu(x^{-1})$ for all $x \in \mathcal{G}$.

NOTE : In this thesis the letter $\mathcal{G}$ will always denote a group, unless specified otherwise, and the letter e will always denote the identity element of $\mathcal{G}$. If μ is a fuzzy subgroup of $\mathcal{G}$, we'll always assume that $\mu(e) > 0$. If μ and ν are fuzzy subgroups of $\mathcal{G}$ mentioned in a theorem, proposition, definition or example, we'll always assume that $\mu(e) = \nu(e)$, although this is not always necessary. We are not necessarily assuming that $\mu(e) = 1$.

DEFINITION : 1.1.2 [7]

Let μ be a fuzzy subgroup of $\mathcal{G}$. μ is called *fuzzy normal* in $\mathcal{G}$ if $\mu(a^{-1}xa) \geq \mu(x)$ for all $a, x \in \mathcal{G}$. We also say that μ is a normal fuzzy subgroup of $\mathcal{G}$.

The support of μ is the set $\text{supp}\ \mu = \{x \in \mathcal{G} : \mu(x) > 0\}$. It is clear that supp μ is a subgroup of $\mathcal{G}$ whenever μ is a fuzzy subgroup of $\mathcal{G}$. If μ is fuzzy normal, then supp μ is a normal subgroup of $\mathcal{G}$. But the converse is not true, see Example 1.1.6.

DEFINITION : 1.1.3 [2]

Let μ be a fuzzy subgroup of $\mathcal{G}$, $0 \leq \alpha \leq \mu(e)$.
Let $\mu^{\alpha} = \mu^{-1}[\alpha,1] = \{x \in \mathcal{G} : \mu(x) \geq \alpha\}$. Then μ^{α} is a subgroup of $\mathcal{G}$, called the *level subgroup* of μ corresponding to α.

PROPOSITION : 1.1.4

Let μ be a fuzzy subgroup of $\mathcal{G}$. Then $\text{supp}\ \mu = \cup\{\mu^{\alpha} : 0 < \alpha \leq \mu(e)\}$.

PROOF :

Straightforward.

PROPOSITION : 1.1.5

Let μ be a fuzzy subgroup of $\mathcal{G}$. Then the following are equivalent :

(i) $\mu(a^{-1}xa) \geq \mu(x)$ for all $x \in$ supp μ, $a \in \mathcal{G}$;

(ii) $\mu(a^{-1}xa) \geq \mu(x)$ for all $x, a \in \mathcal{G}$;

(iii) $\mu(x) = \mu(a^{-1}xa)$ for all $a, x \in \mathcal{G}$;

(iv) $\mu(ax) = \mu(xa)$ for all $a, x \in \mathcal{G}$;

(v) $\mu^{-1}[\alpha,1]$ is a normal subgroup of $\mathcal{G}$ for all $\alpha \in [0,\mu(e)]$;

(vi) $\mu^{-1}(\alpha,1]$ is a normal subgroup of $\mathcal{G}$ for all $\alpha \in [0,\mu(e)]$.

PROOF :

Straightforward. (See [10]).

The following example shows that if supp μ is normal in $\mathcal{G}$, μ need not be fuzzy normal :

EXAMPLE : 1.1.6

Let $\mathcal{G} = S_3 = \{e, a, a^2, b, ab, a^2b\}$, $b^2 = e = a^3$. S_3 is the symmetric group on 3 symbols.

Define $\mu : \mathcal{G} \longrightarrow [0,1]$ by $\mu(x) = \begin{cases} 1 & x = e \\ {}^1/_2 & x = b \\ {}^1/_3 & \text{otherwise}. \end{cases}$

$\mu(a^2) = {}^1/_3 \geq {}^1/_3 = \mu(a)$.

$\mu(ab) = {}^1/_3 \geq {}^1/_3 = \mu(a)$.

$\mu(a^2b) \geq \mu(a^2) \wedge \mu(b) = {}^1/_3$.

So μ is a fuzzy subgroup of $\mathcal{G}$ and supp $\mu = S_3$ is normal in $\mathcal{G}$.

$$\mu^{-1}[{}^1/_2, 1] = \{x \in \mathcal{G}: \mu(x) \geq {}^1/_2\} = \{e,b\}$$

is not normal in $\mathcal{G}$. Hence μ is not fuzzy normal.

DEFINITION : 1.1.7 [7]

Let ν be a fuzzy subgroup of $\mathcal{G}$ and let $x \in \mathcal{G}$. A *left fuzzy coset* of ν associated with x, denoted by $x\nu$, is a fuzzy subset of $\mathcal{G}$ defined by $(x\nu)(y) = \nu(x^{-1}y)\ \forall\ y \in \mathcal{G}$. A *right fuzzy coset* of ν is defined by $(\nu x)(y) = \nu(yx^{-1})$.

If ν is fuzzy normal, then the set $\mathcal{F}_{\nu} = \{x\nu : x \in \mathcal{G}\}$ is a group under the binary operation defined by $(x\nu)(y\nu) = (xy)\ \nu$, x, y $\in \mathcal{G}$. We also have $x\nu = \nu x$ for all $x \in \mathcal{G}$. (See [7], Proposition 4.3 and Theorem 4.5).

PROPOSITION : 1.1.8 [7]

Let μ be a fuzzy subgroup of $\mathcal{G}$. μ is fuzzy normal if and only if $x\mu = \mu x$ for all $x \in \mathcal{G}$.

PROOF :

Obvious.

DEFINITION : 1.1.9 [12]

Let $\mathcal{G}$ and $\mathcal{G}'$ be groups, and $f : \mathcal{G} \longrightarrow \mathcal{G}'$ a homomorphism. Let μ be a fuzzy subgroup of $\mathcal{G}$. The image of μ under f, $f(\mu)$, is a fuzzy subset of $f(\mathcal{G})$ defined by $f(\mu)(f(x)) = \sup\ \{\mu(y) : f(y) = f(x)\}$. Let $f(\mu)(y) = 0$ if $y \notin f(\mathcal{G})$. Then $f(\mu)$ becomes a fuzzy subset of $\mathcal{G}'$. In fact, $f(\mu)$ becomes a fuzzy subgroup of $\mathcal{G}'$:–

PROPOSITION 1.1.10

Let $\mathcal{G}$ and $\mathcal{G}'$ be groups, $f : \mathcal{G} \longrightarrow \mathcal{G}'$ a homomorphism and μ a fuzzy subgroup of $\mathcal{G}$, then $f(\mu)$ is a fuzzy subgroup of $\mathcal{G}'$.

PROOF :

Suppose first that f is onto $\mathcal{G}'$.

Therefore $f(\mu)(f(x)) = \sup\limits_{f(a) = f(x)} \mu(a)$

It is clear that $f(\mu)(f(x)) = f(\mu)(f(x)^{-1})$.

We now show that $f(\mu)(f(x)f(y)) \geq f(\mu)(f(x)) \wedge f(\mu)(f(y))$.
Let $\alpha_1 = f(\mu)(f(x))$, $\alpha_2 = f(\mu)(f(y))$ and $\alpha = f(\mu)(f(x)f(y))$.

Assume α_1, α_2, $\alpha > 0$.

Now $\alpha_1 = \sup\{\mu(a): f(a) = f(x)\}$, $\alpha_2 = \sup\{\mu(a): f(a) = f(y)\}$,

$\alpha = \sup\{\mu(a): f(a) = f(xy)\}$.

Let $\epsilon > 0$ such that $\epsilon < \min(\alpha_1, \alpha_2, \alpha)$. Therefore there exist a_1, a_2, $f(a_1) = f(x)$ and $f(a_2) = f(y)$, such that $\alpha_1 - \epsilon < \mu(a_1)$ and $\alpha_2 - \epsilon < \mu(a_2)$.

Therefore $\alpha_1 - \epsilon \wedge \alpha_2 - \epsilon < \mu(a_1) \wedge \mu(a_2) \leq \mu(a_1a_2)$.

Since $f(a_1a_2) = f(xy)$, $\mu(a_1a_2) \leq \{\sup \mu(a) : f(a) = f(xy)\} = \alpha$.

Hence $\alpha_1 - \epsilon \wedge \alpha_2 - \epsilon < \alpha$. This is true for every $\epsilon \in (0, \min(\alpha_1,\alpha_2,\alpha))$. Therefore $\alpha_1 \wedge \alpha_2 \leq \alpha$.

If $\alpha = 0$, we claim that α_1 or α_2 is zero.

Suppose $\alpha_1,\alpha_2 > 0$, therefore $\mu(a_1)$, $\mu(a_2) > 0$ for some a_1,a_2 satisfying $f(a_1) = f(x)$ and $f(a_2) = f(y)$, hence $\sup_{f(a) = f(xy)} \mu(a) \geq \mu(a_1a_2) \geq \mu(a_1) \wedge \mu(a_2) > 0$.

Contradiction !

Therefore $f(\mu)$ is a fuzzy subgroup of $f(\mathcal{G})$.

Now suppose that f is not necessarily onto. Let $y_1 \in f(\mathcal{G})$ and $y_2 \notin f(\mathcal{G})$.

Therefore $y_1y_2 \notin f(\mathcal{G})$. Hence $f(\mu)(y_1y_2) = 0 \geq 0 = f(\mu)(y_1) \wedge f(\mu)(y_2)$.

Other cases are similarly proved. Therefore $f(\mu)$ is a fuzzy subgroup of $\mathcal{G}'$.

PROPOSITION 1.1.11

Let μ be a fuzzy normal subgroup of $\mathcal{G}$. Let $f: \mathcal{G} \longrightarrow \mathcal{G}'$ be a homomorphism where $\mathcal{G}'$ is a group. Then $f(\mu)$ is fuzzy normal in $f(\mathcal{G})$.

PROOF :

Straightforward.

In [1], P. Bhattacharya and N.P. Mukherjee introduced a notion of fuzzy Abelian. We feel that their notion of fuzzy Abelian is too weak since for example any fuzzy subgroup μ satisfying $\{x \in \mathcal{G} : \mu(x) = \mu(e)\} = \{e\}$ is necessarily fuzzy Abelian even if supp μ is not Abelian. Hence we introduce another notion of fuzzy Abelian which is strong enough to ensure that supp μ is also Abelian.

DEFINITION : 1.1.12

Let μ be a fuzzy subgroup of $\mathcal{G}$. μ is *fuzzy Abelian* if μ^t is Abelian for all $t \in (0,\mu(e)]$.

Whenever fuzzy Abelian is mentioned in this thesis, the version of Definition 1.1.12 should be assumed.

PROPOSITION : 1.1.13

μ is fuzzy Abelian if and only if supp μ is Abelian.

PROOF :

Suppose supp μ is Abelian. Now $\mu^t \subseteq$ supp μ for all $t \in (0,\mu(e)]$, and so μ^t is Abelian for all $t \in (0,\mu(e)]$. Hence μ is fuzzy Abelian.

Conversely, suppose μ^t is Abelian for all $t \in (0,\mu(e)]$. Let $a, b \in$ supp μ. Since supp $\mu = \cup\{\mu^t\colon 0 < t \leq \mu(e)\}$, $a \in \mu^{t_1}$ and $b \in \mu^{t_2}$ for some $t_1, t_2 \in (0,\mu(e)]$. Suppose $t_1 < t_2$. Then $\mu^{t_2} \subseteq \mu^{t_1}$. Hence $a, b \in \mu^{t_1}$. Therefore $ab = ba$.

The proof is complete.

DEFINITION : 1.1.14 [20]

A fuzzy set in $\mathcal{G}$ is called a *fuzzy point* if and only if it takes the value 0 for all $y \in \mathcal{G}$ except one, say, $x \in \mathcal{G}$. If its value at x is λ, $0 < \lambda \leq 1$, we denote this fuzzy point by x_λ, where the point x is called its support. The fuzzy point x_λ is said to be in the fuzzy set μ if $\lambda \leq \mu(x)$, and we write $x_\lambda \in \mu$.

Let μ be a fuzzy subgroup of $\mathcal{G}$.

In this thesis the fuzzy subgroup μ_e is defined by

$$\mu_e(x) = \begin{cases} \mu(e) & x = e \\ 0 & x \neq e. \end{cases}$$

DEFINITION : 1.1.15 [1]

Let μ be a fuzzy subgroup of $\mathcal{G}$ and ν a normal fuzzy subgroup of $\mathcal{G}$ such that $\nu \leq \mu$. The *quotient* μ/ν is a fuzzy subset of $\mathcal{F}_\nu = \{x\nu : x \in \mathcal{G}\}$ defined by $\mu/\nu\,(x\nu) = \mu(x)$ for all $x \in \mathcal{G}$.

It is trivial that μ/ν is a fuzzy subgroup of $\mathcal{F}_\nu$.

If $\nu \leq \mu$ are fuzzy subgroups of $\mathcal{G}$, we shall say that ν is a fuzzy subgroup of μ, or ν is contained in μ.

1.2 : ISOMORPHISM AND QUOTIENT FUZZY SUBGROUPS

In defining isomorphism of fuzzy subgroups of μ and ν in $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, we must ensure that μ and ν turn out to be essentially the same when we rename the elements of $\mathcal{G}_1$ and $\mathcal{G}_2$, i.e. as functions, they must behave in a similar manner. For example, if $\mu(x) > \mu(y)$ and $f : \text{supp}\ \mu \longrightarrow \text{supp}\ \nu$ is an isomorphism, then we must have that $\nu(f(x)) > \nu(f(y))$.

Further, the following properties are desirable :

Im μ must be equipotent to Im ν. $\mathcal{G}_1$ and $\mathcal{G}_2$ need not be isomorphic.

If $\mu = \alpha\nu$ for some fixed $\alpha \in \mathbb{R}^+$, we want μ and ν to be isomorphic.

If $f : \text{supp}\ \mu \longrightarrow \text{supp}\ \nu$ is an isomorphism, we want $f(\mu)$ to be equal to ν. Also there must be a one–to–one correspondence between the fuzzy subgroups of μ and the fuzzy subgroups of ν.

The definition below does have the above properties :

DEFINITION : 1.2.1

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be groups and μ and ν fuzzy subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. An isomorphism f : supp $\mu \longrightarrow$ supp ν is a *fuzzy isomorphism* of μ onto ν if $\exists$ a constant $k \in \mathbb{R}^+$ such that $\mu(x) = k\, \nu(f(x))$ for all $x \in$ supp $\mu \setminus \{e\}$.

We then say that μ is *isomorphic* to ν and write $\mu \simeq \nu$.

EXAMPLE : 1.2.2.

Let $S_3 = \{e, a, a^2, b, ab, a^2b\}$, $a^3 = e = b^2$, and $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under addition modulo 6.

Define $\mu : S_3 \longrightarrow [0,1]$ by $\mu(e) = 1$, $\mu(a) = {}^1/_2 = \mu(a^2)$; $\mu(b) = 0 = \mu(ab) = \mu(a^2b)$. Then μ is a fuzzy subgroup of S_3 and supp $\mu = A_3 = \{e, a, a^2\}$.

Define $\nu : \mathbb{Z}_6 \longrightarrow [0,1]$ by $\nu(\bar{0}) = 1$; $\nu(\bar{2}) = {}^1/_3 = \nu(\bar{4})$; $\nu(\bar{1}) = \nu(\bar{5}) = 0 = \nu(\bar{3})$.

Therefore ν is a fuzzy subgroup of $\mathbb{Z}_6$, supp $\nu = \{\bar{0}, \bar{2}, \bar{4}\} \simeq \mathbb{Z}_3$.

Define f : supp $\mu \longrightarrow$ supp ν by $f(e) = \bar{0}$; $f(a) = \bar{2}$; $f(a^2) = \bar{4}$.

$$\begin{aligned} \mu(e) &= \nu(\bar{0}) &&= \nu(f(e)). \\ \mu(a) &= {}^1/_2 &&= {}^3/_2\, \nu(\bar{2}) = {}^3/_2\, \nu(f(a)) \\ \text{and} \quad \mu(a^2) &= {}^3/_2\, \nu(f(a^2)). \\ \mu(x) &= 0 &&= {}^3/_2\, \nu(f(x)),\ x \neq e, a, a^2. \end{aligned}$$

Therefore $\mu \simeq \nu$.

Note that $\mathbb{Z}_6$ is not isomorphic to S_3.

Im $\mu \neq$ Im ν, but Im μ is equivalent to Im ν.

PROPOSITION : 1.2.3.

Let μ and ν be fuzzy subgroups of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Suppose $\mu \simeq \nu$. Then μ is fuzzy Abelian if and only if ν is fuzzy Abelian.

PROOF :

Trivial.

PROPOSITION : 1.2.4.

Let μ_1 and μ_2 be fuzzy subgroups of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, such that $\mu_1 \simeq \mu_2$. Then, given any $t \in (0, \mu_1(e)]$, there exists $s \in (0, \mu_2(e)]$ such that $\mu_1^t \simeq \mu_2^s$.

PROOF :

Let $f : \text{supp } \mu_1 \longrightarrow \text{supp } \mu_2$ be an isomorphism such that $\mu_1(x) = k\, \mu_2 (f(x))$ for all $x \in \text{supp } \mu_1 \setminus \{e\}$, k fixed. Define $g : \mu_1^t \longrightarrow \mu_2^{t/k}$ by $g = f|\, \mu_1^t$. Let $x \in \mu_1^t$. Then $\mu_1(x) \geq t$ and $k\, \mu_2 (f(x)) \geq t$. Hence $f(x) \in \mu_2^{t/k}$. So g is well–defined.

Clearly g is an injective homomorphism. Let $y \in \mu_2^{t/k}$. Then $\mu_2(y) \geq t/k$.
Now $y = f(x)$ for some $x \in \text{supp } \mu_1$. Therefore $k\, \mu_2 (f(x)) \geq t$. Hence $x \in \mu_1^t$.
Therefore g is onto.
The proof is complete.

PROPOSITION : 1.2.5.

Let μ and ν be fuzzy subgroups of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Suppose $\mu \simeq \nu$. If μ is fuzzy normal in supp μ, then ν is fuzzy normal in supp ν.

PROOF :

Straightforward.

Before proceeding with the notion of isomorphism, we have to give a weaker notion of a fuzzy quotient. The quotient given in Definition : 1.1.15 [1] is not good for the second Isomorphism Theorem, (see Chapter 3). Also we do not have the crisp analogue that $\mu/\mu \simeq \mu_e$.

MOTIVATION :

Let μ be a normal fuzzy subgroup of $\mathscr{G}$. Let f : $\mathscr{G} \longrightarrow \mathscr{G}/\mathscr{H}$ be the canonical homomorphism where $\mathscr{H}$ is a normal subgroup of $\mathscr{G}$ contained in supp μ. Then f(μ) is a fuzzy subgroup of $\mathscr{G}/\mathscr{H}$, (Proposition 1.1.10).

$$\text{Let } \nu(x) = \begin{cases} \mu(e) & x \in \mathscr{H} \\ 0 & x \notin \mathscr{H}. \end{cases}$$

Then ν is a normal fuzzy subgroup of $\mathscr{G}$ and supp $\nu = \mathscr{H} \subseteq$ supp μ.

Now $f(\mu)(x\mathscr{H}) = \sup\{\mu(a) : a \text{ supp } \nu = x \text{ supp } \nu\}$

This motivates the following definition :

DEFINITION : 1.2.6

Let μ and ν be fuzzy subgroups of $\mathscr{G}$, where ν is fuzzy normal and supp $\nu \subseteq$ supp μ. The *fuzzy quotient* group μ modulo ν, denoted by μ/ν, is the function
$\mu/\nu : \mathscr{G}/\text{supp } \nu \longrightarrow [0,1]$ defined by
$\mu/\nu \,(x \text{ supp } \nu) = \sup\{\mu(a) : a \text{ supp } \nu = x \text{ supp } \nu\}$. If f : $\mathscr{G} \longrightarrow \mathscr{G}/\text{supp } \nu$ is the canonical homomorphism, then $f(\mu) = \mu/\nu$.

To distinguish between the quotients in Definitions 1.2.6 and 1.1.15, we call the quotient in definition 1.1.15 the strong fuzzy quotient and denote it by $(\mu/\nu)_S$. So whenever we mention a quotient μ/ν, definition 1.2.6 is to be assumed.

THE RELATIONSHIP BETWEEN $(\mu/\nu)_S$ AND μ/ν :

Let $\mathscr{F}_\nu = \{x\nu : x \in \mathscr{G}\}$, $E_\nu = \{x \in \mathscr{G} : \nu(x) = \nu(e)\}$. $\mathscr{F}_\nu \simeq \mathscr{G}/E_\nu$. So we can write $(\mu/\nu)_S(x\nu) = (\mu/\nu)_S(xE_\nu) = \mu(x)$ for all $x \in \mathscr{G}$.

If supp $\nu = E_\nu$, then $\mathscr{G}/E_\nu = \mathscr{G}/\text{supp } \nu \simeq \mathscr{F}_\nu$ and hence $(\mu/\nu)_S = (\mu/\nu)$.

However, if $E_\nu \subsetneq \text{supp } \nu$, then

$(\mu/\nu)_s(x E_\nu) \leq \mu/\nu\,(x \text{ supp } \nu)$ for all $x \in \mathcal{G}$, and $\mathcal{G}/\text{supp } \nu \simeq \mathcal{F}_\nu / \mathcal{F}_\nu^{\text{supp } \nu}$, where $\mathcal{F}_\nu^{\text{supp } \nu} = \{x\nu : x \in \text{supp } \nu\}$. So we see that $(\mu/\nu)_s$ can be obtained from μ/ν by demanding that supp ν be replaced by $E_\nu \subseteq \text{supp } \nu$.

PROPOSITION : 1.2.7

Let μ be a fuzzy subgroup of $\mathcal{G}$. Then $\mu/\mu_e \simeq \mu$ and $\mu/\mu \simeq \mu_e$.

PROOF :

Define f : supp $\mu/\mu_e \longrightarrow$ supp μ by f(x supp μ_e) = x. Clearly f defines a fuzzy isomorphism between μ/μ_e and μ.

Define g : supp $\mu_e \longrightarrow$ supp μ/μ by g(e) = supp μ.

It is trivial that $\mu(e) = \mu/\mu(g(e))$. Therefore $\mu/\mu \simeq \mu_e$.

REMARK : 1.2.8

(a) Since $(\mu/\mu)_s\,(x\mu) = \mu(x)$ for all $x \in \mathcal{G}$, $(\mu/\mu)_s$ need not be isomorphic to μ_e.

(b) It is possible for μ_1/ν to be equal to μ_2/ν with $\mu_1 \neq \mu_2$, (contrary to the crisp case).

For example let $S_3 = \{e, a, a^2, b, ab, a^2b\}$, $a^3 = e = b^2$.

S_3 is the symmetric group on 3 symbols.

$$\text{Let } \mu_1(x) = \begin{cases} 1 & x = e \\ 1/2 & x = a, a^2 \\ 1/4 & x \notin A_3 \end{cases}$$

A_3 is the alternating group of degree 3, i.e. A_3 is the set of all even permutations in S_3.

$$\text{Let } \mu_2(x) = \begin{cases} 1 & x = e \\ 1/3 & x = a,\ a^2 \\ 1/4 & x \notin A_3 \end{cases}$$

$$\nu(x) = \begin{cases} 1 & x = e \\ 1/4 & x = a,\ a^2 \\ 0 & x \notin A_3 \end{cases}$$

ν is fuzzy normal.

μ_1, μ_2 are fuzzy subgroups of S_3, and $\mu_1 \neq \mu_2$.

It is routine to check that $\mu_1/\nu = \mu_2/\nu$.

1.3 : NORMALITY OF A FUZZY SUBGROUP IN ANOTHER FUZZY SUBGROUP AND PRODUCTS OF FUZZY SUBGROUPS.

The notion of fuzzy normality does not have the crisp analogue that a group $\mathcal{G}$ is normal in itself or that $\mathcal{H}$ is normal in $\mathcal{G}$. This notion is needed in solvable fuzzy subgroups. For example if we have a chain $\mu \geq \mu_1 \geq \cdots \geq \mu_n$ of fuzzy subgroups, we want to define what is meant by the chain being normal. It is too strong to require that each μ_i be fuzzy normal. We only require that μ_i be normal in μ_{i-1}.

MOTIVATION :

Let $\nu \leq \mu$ be fuzzy subgroups of $\mathcal{G}$ and a_λ a fuzzy point in μ, i.e. $\mu(a) \geq \lambda > 0$. The product $a_\lambda\nu$, defined by $a_\lambda\nu(x) = \lambda \wedge \nu(a^{-1}x)$ for all $x \in \mathcal{G}$, can be viewed as another version of a left fuzzy coset of ν since supp $a_\lambda\nu$ = a supp ν, a crisp left coset. Likewise for νa_λ defined as $\nu a_\lambda(x) = \lambda \wedge \nu(xa^{-1})$. If $a_\lambda\nu = \nu a_\lambda$ for all fuzzy points a_λ, then $a\nu = \nu a$ for all $a \in \mathcal{G}$, i.e. ν is fuzzy normal.

PROPOSITION : 1.3.1

Let ν be a fuzzy subgroup of $\mathcal{G}$ such that $a_\lambda\nu = \nu a_\lambda$ for all $a \in \mathcal{G}$ and λ fixed.

Then supp ν is normal in $\mathcal{G}$.

PROOF :

We have $\lambda \wedge \nu(a^{-1}x) = \lambda \wedge \nu(xa^{-1})$ for all $x \in \mathcal{G}$.
Therefore $\lambda \wedge \nu(a^{-1}xa) = \lambda \wedge \nu(xaa^{-1}) = \lambda \wedge \nu(x)$.
Let $x \in$ supp ν. Therefore $\lambda \wedge \nu(x) > 0$ implies that $\nu(a^{-1}xa) > 0$.
Hence $a^{-1}xa \in$ supp ν for all $a \in \mathcal{G}$.
The proof is complete.

The above discussion motivates the following definition :

DEFINITION : 1.3.2

Let ν and μ be fuzzy subgroups of $\mathcal{G}$ such that $\nu \leq \mu$. ν is a *normal fuzzy subgroup* of μ iff $a_\lambda \nu = \nu a_\lambda$ for all fuzzy points a_λ in μ, i.e. iff $\lambda \wedge \nu(a^{-1}x) = \lambda \wedge \nu(xa^{-1})$ for all $x \in \mathcal{G}$ and all fuzzy points a_λ in μ. We write $\nu \triangleleft \mu$.

Note that $\nu \triangleleft \mu$ iff $a_\lambda \; \nu a_\lambda^{-1} = \nu e_\lambda$ for all λ such that $a_\lambda \in \mu$.

PROPOSITION : 1.3.3

A fuzzy subgroup ν in $\mathcal{G}$ is normal in itself, i.e. $\nu \triangleleft \nu$.

PROOF :

Let $a_\lambda \in \nu$.

$$
\begin{aligned}
a_\lambda \nu(x) = \lambda \wedge \nu(a^{-1}x) \quad & \geq \lambda \wedge \nu(a) \wedge \nu(x) \\
& = \lambda \wedge \nu(x) \\
& = a_\lambda(a) \wedge \nu(aa^{-1}x) \\
& \geq \lambda \wedge \nu(a^{-1}x) \\
& = a_\lambda \nu(x).
\end{aligned}
$$

Therefore $a_\lambda \nu = \lambda \wedge \nu$.

Similarly, $\nu a_\lambda = \lambda \wedge \nu$.

Hence $a_\lambda \nu = \nu a_\lambda$ for all $a_\lambda \in \nu$.

The proof is complete.

PROPOSITION : 1.3.4

$\nu \lhd \mu$ if and only if $\nu^t \lhd \mu^t$ for each $t \in [0,1]$.

PROOF :

Suppose $\nu \lhd \mu$.
Let $a \in \mu^t$ and $x \in \nu^t$, i.e. $a_t \in \mu$ and $x_t \in \nu$.

Hence $a_t\nu = \nu a_t$, i.e. $a_t \nu a_t^{-1} = \nu e_t$.

Therefore $\nu(a^{-1}xa) \wedge t = \nu(x) \wedge t = t$.
Hence $\nu(a^{-1}xa) \geq t$; i.e. $a^{-1}xa \in \nu^t$.

Conversely, suppose $\nu^t \lhd \mu^t$ for all $t \in [0,1]$.
Let $a_\lambda \in \mu$, and $x \in \mathcal{G}$.

Case $x \notin$ supp ν :

If $a^{-1}xa \in$ supp ν, then $a^{-1}xa \in \nu^t$ for some $t \leq \lambda$. Hence $x_t \in \nu$ since $\nu^t \lhd \mu^t$.

Therefore $x \in$ supp ν. Contradiction.
Hence $\lambda \wedge \nu(x) = 0 = \lambda \wedge \nu(a^{-1}xa)$.

Case $x_\lambda \in \nu$:

Since $\nu^\lambda \lhd \mu^\lambda$, we have $(a^{-1}xa)_\lambda \in \nu$.

Hence $\lambda \wedge \nu(x) = \lambda = \lambda \wedge \nu(a^{-1}xa)$.

Case $x_\lambda \notin \nu$, $x \in$ supp ν :

$x \in \nu^t$ for some $t < \lambda$. Therefore $a_t \in \mu$.

Therefore $t \wedge \nu(x) = t = t \wedge \nu(a^{-1}xa)$.
Suppose $\lambda \wedge \nu(x) \gneq \lambda \wedge \nu(a^{-1}xa)$. (*)
If LHS $= \lambda$, then RHS $= \nu(a^{-1}xa)$.
i.e. $\nu(a^{-1}xa) < \lambda \leq \nu(x)$.
Hence $x_\lambda \in \nu$.

Contradiction !

Therefore LHS of (*) = $\nu(x) < \lambda$

i.e. $\nu(a^{-1}xa) < \nu(x) < \lambda$.
Let $t_1 = \nu(x)$.

Therefore $x_{t_1} \in \nu$ and $a_{t_1} \in \mu$.

Hence $t_1 \wedge \nu(x) = t_1 \wedge \nu(a^{-1}xa)$ since $\nu^{t_1} \triangleleft \mu^{t_1}$.

Therefore $\nu(x) = \nu(a^{-1}xa)$.
Hence $\lambda \wedge \nu(x) = \lambda \wedge \nu(a^{-1}xa)$.
Therefore $e_\lambda \nu = a_\lambda \, \nu \, a_\lambda^{-1}$.

Hence $a_\lambda \nu = \nu a_\lambda$.

The proof is complete.

REMARK :

If supp $\nu \triangleleft$ supp μ, it does not follow that $\nu \triangleleft \mu$.
Consider $\mathcal{G} = S_3 = \{e, a, a^2, b, ab, a^2b\}$, where $b^2 = e = a^3$.

Let $$\mu(x) = \begin{cases} 1 & x = e \\ 1/2 & x \neq e, \end{cases}$$ and

$$\nu(x) = \begin{cases} 1 & x = e \\ 1/2 & x = b \\ 1/3 & \text{otherwise} . \end{cases}$$

ν and μ are fuzzy subgroups of $\mathcal{G}$.
Now supp $\nu = S_3 =$ supp μ.

So supp $\nu \triangleleft$ supp μ.

But $\nu^{1/2} = \{x \in \mathcal{G} : \nu(x) \geq 1/2\} = \{e,b\}$

and $\mu^{1/2} = \{x \in \mathcal{G} : \mu(x) \geq 1/2\} = S_3$,

so $\nu^{1/2}$ is not normal in $\mu^{1/2}$.
Hence ν is not normal in μ.

DEFINITION : 1.3.5 [15]

Let μ and ν be fuzzy subsets of $\mathcal{G}$. The *product* $\mu\nu : \mathcal{G} \longrightarrow [0,1]$ is defined by

$$\mu\nu(x) = \sup_{x = x_1x_2} \mu(x_1) \wedge \nu(x_2).$$

PROPOSITION : 1.3.6

Let μ, ν, ξ be fuzzy subsets of $\mathcal{G}$.

Then $(\mu\nu)\xi(x) = \sup_{x = x_1x_2x_3} \min(\mu(x_1), \nu(x_2), \xi(x_3))$.

Hence $(\mu\nu)\xi = \mu(\nu\xi)$.

PROOF :

Let $0 \neq a = \sup_{x = x_1x_2x_3} \min(\mu(x_1), \nu(x_2), \xi(x_3))$.

Let $\epsilon \in (0,a)$. Therefore there exist x_1, x_2, x_3 such that

$$\begin{aligned} a - \epsilon &< \min(\mu(x_1), \nu(x_2), \xi(x_3)) \\ &= \min[\min(\mu(x_1), \nu(x_2)), \xi(x_3)] \end{aligned}$$

Therefore $a - \epsilon < \min(\mu(x_1), \nu(x_2))$. Let $y_1 = x_1x_2$.

Hence $a - \epsilon < \sup_{y_1 = a_1a_2} \min(\mu(a_1), \nu(a_2)) = \mu\nu(y_1)$.

Therefore $a - \epsilon < \mu\nu(y_1) \wedge \xi(x_3)$
$\leq (\mu\nu)\xi(x)$.

Hence $a \leq (\mu\nu)\xi(x)$.

Now let $\epsilon \in (0, (\mu\nu)\xi(x))$.

Therefore there exist x_1, x_2 such that $x = x_1x_2$ and

$$\begin{aligned} (\mu\nu)\xi(x) - \epsilon &< \min(\mu\nu(x_1), \xi(x_2)) \\ &= \left[\sup_{x_1 = y_1y_2} \mu(y_1) \wedge \nu(y_2)\right] \wedge \xi(x_2). \end{aligned}$$

There exist y_1,y_2 such that

$$\xi(x_2) \wedge \sup_{x_1 = y_1y_2} \mu(y_1) \wedge \nu(y_2) - \epsilon < \mu(y_1) \wedge \nu(y_2) \wedge \xi(x_2).$$

Therefore $(\mu\nu)\xi(x) - \epsilon < (\mu(y_1) \wedge \nu(y_2)) \wedge \xi(x_2) + \epsilon$.

Hence $(\mu\nu)\xi(x) - 2\epsilon < a$.
Therefore $(\mu\nu)\xi(x) \leq a$.
Hence $a = (\mu\nu)\xi(x)$.
Similarly, $a = \mu(\nu\xi)(x)$.
The case when $a = 0$ is easy.

Therefore the proof is complete.

PROPOSITION : 1.3.7
If μ is a fuzzy subgroup of $\mathcal{G}$, then $\mu^2 = \mu$, hence $\mu^n = \mu$ for all natural numbers n.

PROOF :

Trivial.

PROPOSITION : 1.3.8

Let μ be a fuzzy subset of $\mathcal{G}$. Then μ is a fuzzy subgroup of $\mathcal{G}$ if and only if $\mu^2 = \mu$ and $\mu(x) = \mu(x^{-1})$ for all $x \in \mathcal{G}$.

PROOF :

$\Longrightarrow$: Obvious

$\Longleftarrow$: Let $x, y \in \mathcal{G}$. Now $\mu(xy) = \mu^2(xy)$. Let $z = xy$.

Therefore $\mu(xy) = \mu^2(z) = \sup\limits_{z = z_1 z_2} \mu(z_1) \wedge \mu(z_2)$

$$\geq \mu(x) \wedge \mu(y).$$

The proof is complete.

PROPOSITION : 1.3.9

Let μ and ν be fuzzy subgroups of $\mathcal{G}$ such that $\mu\nu = \nu\mu$. Then $\mu\nu$ is a fuzzy subgroup of $\mathcal{G}$.

PROOF :

Clearly $\mu\nu(x) = \mu\nu(x^{-1})$ for all $x \in \mathcal{G}$.

$$\mu\nu = \mu^2\nu^2 = \mu[(\mu\nu)\nu] \text{ by Proposition 1.3.6}$$
$$= \mu[(\nu\mu)\nu]$$
$$= (\mu\nu)(\mu\nu) = (\mu\nu)^2.$$

Therefore by Proposition 1.3.8, $\mu\nu$ is a fuzzy subgroup of $\mathcal{G}$.

PROPOSITION : 1.3.10

Let μ_1, μ_2, μ be fuzzy subgroups of $\mathcal{G}$ such that

(i) $\mu_1, \mu_2 \leq \mu$ and

(ii) $\mu_1 \triangleleft \mu$.

Then $\mu_1\mu_2$ is a fuzzy subgroup of $\mathcal{G}$.

PROOF :

Let $a_\lambda \in \mu$ and $x \in \mathcal{G}$. Then

$$\lambda \wedge \mu_1(a^{-1}x) = \lambda \wedge \mu_1(xa^{-1}) \text{ ——— } \otimes$$

Let $x = x_1x_2$ and $x_2^{-1}x_1x_2 = a_1$.

Suppose $\mu_1\mu_2(x) \neq 0$.

Then $\mu_1\mu_2(x)$
$$= \sup_{x = x_1x_2} \mu_1(x_1) \wedge \mu_2(x_2)$$
$$= \sup_{x = x_1x_2} \mu_1(x_2\, a_1\, x_2^{-1}) \wedge \lambda_2, \text{ where } \lambda_2 = \mu_2(x_2) \neq 0,$$
$$= \sup_{x = x_2a_1} \lambda_2 \wedge \mu_1(a_1) \qquad \text{by } \otimes$$
$$= \sup_{x = x_2a_1} \mu_2(x_2) \wedge \mu_1(a_1)$$
$$= \mu_2\mu_1(x).$$

If $\mu_1\mu_2(x) = 0$, it is easily shown that $\mu_2\mu_1(x) = 0$. Hence, by Proposition 1.3.9, $\mu_1\mu_2$ is a fuzzy subgroup of $\mathcal{G}$.

Further results on products will be given in Chapter 3.

The following easy result will be useful later :

PROPOSITION : 1.3.11

Let μ be a fuzzy subgroup of $\mathscr{G}$ and $\mathscr{H}$ a normal subgroup of $\mathscr{G}$.

Define $\nu : \mathscr{G} \longrightarrow [0,1]$ by $\nu(x) = \begin{cases} \mu(x) & x \in \mathscr{H} \\ 0 & x \notin \mathscr{H} \end{cases}$

Then $\nu \lhd \mu$.

PROOF :

Similar to the proof of Proposition 1.3.4.

We end this chapter by proving that the fuzzy union of all the left fuzzy cosets of the form $a_\lambda \nu$, where $\nu \lhd \mu$, is a fuzzy subgroup of $\mathscr{G}$.

THEOREM : 1.3.12

Let $\nu \leq \mu$ be fuzzy subgroups of $\mathscr{G}$ such that $\nu \lhd \mu$. Define $w : \mathscr{G} \longrightarrow I$ by $w(x) = \sup_{a_\lambda \in \mu} a_\lambda \nu(x)$. Then w is a fuzzy subgroup of $\mathscr{G}$.

PROOF

$$\begin{aligned} w(x) &= \sup_{a_\lambda \in \mu} \lambda \wedge \nu(a^{-1}x) \\ &= \sup_{a_\lambda \in \mu} \lambda \wedge \nu(x\, a^{-1}) \quad \text{since } \nu \lhd \mu \\ &= \sup_{a_\lambda \in \mu} \lambda \wedge \nu(a\, x^{-1}) \qquad \text{since } \nu(x) = \nu(x^{-1}) \; \forall \; x \in \mathscr{G} \\ &= \sup_{a_\lambda^{-1} \in \mu} \lambda \wedge \nu(a\, x^{-1}) \\ &= \sup_{a_\lambda \in \mu} \lambda \wedge \nu(a^{-1}\, x^{-1}) \\ &= \sup_{a_\lambda \in \mu} a_\lambda \nu(x^{-1}) = w(x^{-1}). \end{aligned}$$

Next we show that $w(xy) \geq w(x) \wedge w(y)$, $x, y \in \mathcal{G}$.

$$w(x) = \sup_{a_\lambda \in \mu} a_\lambda \nu(x) \quad \text{and} \quad w(y) = \sup_{b_\beta \in \mu} b_\beta \nu(y).$$

Let $\lambda, \beta \in (0,1]$ such that $a_\lambda, b_\beta \in \mu$. Let $\alpha = \lambda \wedge \beta$. Then $(ab)_\alpha \in \mu$.

$$\begin{aligned} w(xy) &= \sup_{c_\lambda \in \mu} c_\lambda \nu(xy) \\ &\geq \sup_{a_\lambda,\, b_\beta \in \mu} (ab)_\alpha \nu(xy) \\ &= \sup_{a_\lambda,\, b_\beta \in \mu} \lambda \wedge \beta \wedge \nu(b^{-1}a^{-1}xy) \\ &= \sup_{a_\lambda,\, b_\beta \in \mu} \lambda \wedge \beta \wedge \nu(a^{-1}xy\, b^{-1}) \text{ since } \nu \triangleleft \mu, \\ &\geq \sup_{a_\lambda,\, b_\beta \in \mu} \lambda \wedge \nu(a^{-1}x) \wedge \beta \wedge \nu(y\, b^{-1}) \\ &= \sup_{a_\lambda,\, b_\beta \in \mu} [a_\lambda \nu(x) \wedge b_\beta \nu(y)] \\ &= \alpha_1, \text{ say.} \end{aligned}$$

Let $\alpha_3 = w(x)$, $\alpha_2 = w(y)$.

Let $\alpha_4 = \alpha_3 \wedge \alpha_2$.

We claim that $\alpha_4 = \alpha_1$:

Clearly $\alpha_1 \leq \alpha_4$.

$\alpha_4 = \sup_{a_\lambda \in \mu} a_\lambda \nu(x) \wedge \alpha_2$. Let $\epsilon \in (0, \alpha_2 \wedge \alpha_3)$.

Therefore $\alpha_2 - \epsilon < b_{\beta_0} \nu(y)$ for some $b_{\beta_0} \in \mu$.

Hence $a_\lambda \nu(x) \wedge (\alpha_2 - \epsilon) \leq a_\lambda \nu(x) \wedge b_{\beta_0} \nu(y)$
$\leq \alpha_1$ for all $a_\lambda \in \mu$.

Therefore $\sup_{a_\lambda \in \mu} [a_\lambda \nu(x) \wedge (\alpha_2 - \epsilon)] \leq \alpha_1$.

Hence $\alpha_3 \wedge (\alpha_2 - \epsilon) \leq \alpha_1$ for all $\epsilon \in (0, \alpha_2 \wedge \alpha_3)$.

Therefore $\alpha_3 \wedge \alpha_2 \leq \alpha_1$; i.e. $\alpha_4 \leq \alpha_1$.

Hence $w(xy) \geq w(x) \wedge w(y)$, as required.

REMARK : 1.3.13

The fuzzy subgroup w, defined above, can be thought of as another version of a fuzzy quotient μ/ν since in the crisp case the set

$$\{aN : a \in \mathcal{G}\} = \mathcal{G}/N \quad \text{for } N \triangleleft \mathcal{G}.$$

Unfortunately the collection $\{a_\lambda \ \nu : a_\lambda \in \mu\}$ does not form a group except if λ is fixed. So we do not pursue this apparent new notion of a fuzzy quotient.

CHAPTER 2

FUZZY CONGRUENCE RELATIONS AND NORMAL FUZZY SUBGROUPS

INTRODUCTION

Given a fuzzy subset ν of a group $\mathcal{G}$, is there a smallest fuzzy subgroup μ of $\mathcal{G}$ containing ν ? Of course the answer is in the affirmative. But we want to know the structure of μ. For example we know that given fuzzy subgroups μ and ν such that $\mu(e) = \nu(e)$, the smallest fuzzy subgroup containing μ and ν is $\mu\nu$ (see Chapter 3). In this chapter we answer the above question by studying fuzzy subgroups generated by fuzzy subsets. We also construct, from a given fuzzy subgroup μ, some fuzzy subgroups that are normal in μ. In [43] Mukherjee and Bhattacharya introduced a notion of fuzzy normalizer. This fuzzy normalizer is not a fuzzy subset, but a crisp set in which the fuzzy subgroup is fuzzy normal, which is basically a crisp normalizer. In this chapter we present a fuzzy normalizer $N(\mu)$ of a fuzzy subgroup μ in which μ is fuzzy normal. A notion of a fuzzy centre is also introduced, and it is easily shown that it behaves like the crisp centre.

In [9] Murali studied fuzzy congruence relations on algebras. In Section 2.2 we present the work done by us in [10]. Here we unite the two notions of fuzzy normality and congruence in a fuzzy subgroup setting. In particular we prove that every fuzzy congruence relation determines a normal fuzzy subgroup. Conversely, given a normal fuzzy subgroup, we can associate a fuzzy congruence relation. The association between normal fuzzy subgroups and fuzzy congruence relations is bijective and unique.

Subsequent to the work in [10], Sidky and Ghanim studied fuzzy congruence relations on semigroups in [52]. In the most recent publication, Kuroki [58] has also studied fuzzy congruence relations and normal fuzzy subgroups. The difference between the notions of fuzzy equivalence relation given in [10] and [52] is that in [52] the sup–min transitivity of Definition 3.1 in [10] is replaced by the sup – T transitivity, where T is a t–norm. The substitution property in the definition of a fuzzy congruence relation is also replaced by T–compatible and T–equivalence. In this thesis we do not use a t–norm.

2.1 : FUZZY SUBGROUPS GENERATED BY FUZZY SUBSETS :

DEFINITION : 2.1.1

Let $\mathcal{G}$ be a group and ν a fuzzy subset of $\mathcal{G}$, $\nu \neq 0$. The smallest fuzzy subgroup of $\mathcal{G}$ containing ν, denoted by $< \nu >$, is called the *fuzzy subgroup of $\mathcal{G}$ generated* by ν.

NOTE :

1. If $\{\nu_\lambda : \lambda \in \Delta\}$ is a set of fuzzy subgroups, then $\bigwedge_{\lambda \in \Delta} \nu_\lambda = \inf_{\lambda \in \Delta} \nu_\lambda$ is also a fuzzy subgroup of $\mathcal{G}$. Therefore definition 2.1.1 makes sense.

2. By $< \{\nu_\lambda : \lambda \in \Delta\} >$ we mean $< \bigvee_{\alpha \in \Delta} \nu_\alpha >$, where $\bigvee_{\alpha \in \Delta} \nu_\alpha = \sup_{\alpha \in \Delta} \nu_\alpha$.

3. The notion of fuzzy generation coincides with the notion of crisp generation when $[0,1]$ is $\{0,1\}$.

PROPOSITION : 2.1.2

Let ν be a fuzzy subset of $\mathcal{G}$ and $\mu = < \nu >$. Then supp $\mu = <$ supp $\nu >$.

PROOF :
Straightforward.

DEFINITION : 2.1.3

Let ν be the supremum of a finite number of fuzzy points. If $\mu = < \nu >$, then μ is said to be *finitely generated.* If $\mu = < a_\lambda >$ for some fuzzy point a_λ, then μ is said to be *cyclic.*

It is clear that a level subgroup of a finitely generated fuzzy subgroup is finitely generated.

PROPOSITION : 2.1.4

Let ν be a fuzzy subset of $\mathcal{G}$. Let $x \in \mathcal{G}$ such that $x = a_1^{k_1}a_2^{k_2} \cdots a_n^{k_n}$, where each k_i is an integer for each $i \in \{1,2,\cdots,n\}$. Choose $\lambda_i \in (0,1]$ such that $\lambda_i \leq \nu(a_i)$, and let $\lambda = \inf\{\lambda_1,\cdots,\lambda_n\}$. Define $\omega : \mathcal{G} \longrightarrow [0,1]$ by $\omega(x) = \sup \lambda$, where the supremum is taken over all the n–tuples $\{\lambda_1,\cdots,\lambda_n\}$, $n \in \mathbb{N}$. Then

(i) ω is a fuzzy subgroup of $\mathcal{G}$,

(ii) $\omega = < \nu >$.

PROOF :

(i) Suppose $\lambda \leq \nu(a)$. Now $a = aa^{-1}a$ implies that $\omega(a) \geq \lambda$. Hence $\omega \geq \nu$.
$x = a_1^{k_1}a_2^{k_2} \cdots a_n^{k_n} \Leftrightarrow x^{-1} = a_n^{-k_n} \cdots a_2^{-k_2}a_1^{-k_1}$.
So it is easy to see that $\omega(x) = \omega(x^{-1})$.
Let $\omega(y) = \text{Sup } \beta$, where $\beta = \inf\{\beta_1,\cdots,\beta_s\}$, $y = b_1^{m_1}b_2^{m_2} \cdots b_s^{m_s}$ such that $\beta_i \leq \nu(b_i)$, $i = 1,\cdots,s$, and $\omega(x) = \text{Sup } \lambda$, where $\lambda = \inf\{\lambda_1,\cdots,\lambda_n\}$, $x = a_1^{k_1}a_2^{k_2} \cdots a_n^{k_n}$ such that $\lambda_i \leq \nu(a_i)$ for each $i = 1,2,\cdots,n$.
$\lambda \wedge \beta = \inf\{\lambda_1,\cdots,\lambda_n,\beta_1,\cdots,\beta_s\}$ and $xy = a_1^{k_1} \cdots a_n^{k_n}b_1^{m_1} \cdots b_s^{m_s}$.
So it is easy to see that $\omega(xy) \geq \sup \lambda \wedge \beta = \sup \lambda \wedge \sup \beta = \omega(x) \wedge \omega(y)$.
Hence ω is a fuzzy subgroup of $\mathcal{G}$.

(ii) Let $\mu = < \nu >$. Since $\omega \geq \nu$, it is obvious that $\omega \geq \mu$.
Let $\lambda \leq \omega(x)$, $x \in \mathcal{G}$. So $x \in \text{supp } \omega$.
$\omega(x) = \sup \beta$, where $\beta = \inf(\beta_1,\cdots,\beta_n\}$, $x = b_1^{k_1}b_2^{k_2} \cdots b_n^{k_n}$ such that $\beta_i \leq \nu(b_i)$ for each $i = 1,2,\cdots,n$. Hence

$$\begin{aligned} \mu(x) &\geq \inf\{\mu(b_1), \mu(b_2),\cdots, \mu(b_n)\} \\ &\geq \inf\{\nu(b_1), \nu(b_2),\cdots, \nu(b_n)\} \\ &\geq \inf\{\beta_1,\cdots, \beta_n\} = \beta. \end{aligned}$$

Therefore $\mu(x) \geq \omega(x)$. Hence $\mu = \omega$.

DEFINITION : 2.1.5

Let μ be a fuzzy subgroup of $\mathcal{G}$. The *commutator fuzzy subgroup* of μ, denoted by μ', is the smallest normal fuzzy subgroup of μ such that μ/μ' is fuzzy Abelian.

The above definition makes sense since $\mu \lhd \mu$ and $\mu/\mu \simeq \mu_e$, hence μ/μ is fuzzy Abelian.

THEOREM : 2.1.6

Let μ be a fuzzy subgroup of $\mathcal{G}$. Let $w = < \{a_\lambda\, b_\beta\, a_\lambda^{-1}\, b_\beta^{-1} : a_\lambda, b_\beta \in \mu\} >$. Then $w \lhd \mu$; $w \geq \mu'$ and $^{\mu}/w$ is fuzzy Abelian.

PROOF :

Clearly $\omega \leq \mu$. We claim that $\omega \lhd \mu$: Let $\beta \leq \mu(a)$ and $x \in \mathcal{G}$.
$a_\beta^{-1}\omega a_\beta(x) = \beta \wedge \omega(axa^{-1})$ and $e_\beta\omega(x) = \beta \wedge \omega(x)$.
Suppose $\omega(x) = \sup \alpha$, $\alpha = \inf\{\lambda_1,\cdots,\lambda_n,\beta_1,\cdots,\beta_n\}$ where
$x = (a_1b_1a_1^{-1}b_1^{-1})^{k_1} \cdots (a_nb_na_n^{-1}b_n^{-1})^{k_n}$ such that
$\lambda_i \leq \mu(a_i)$ and $\beta_i \leq \mu(b_i)$, $i = 1,2,\cdots,n$.
$axa^{-1} = (aa_1a^{-1})(ab_1a^{-1})(aa_1^{-1}a^{-1})(ab_1^{-1}a^{-1}) \cdots (aa_na^{-1})(ab_na^{-1})(aa_n^{-1}a^{-1})(ab_n^{-1}a^{-1})$
$\Leftrightarrow x = a_1b_1a_1^{-1}b_1^{-1} \cdots a_nb_na_n^{-1}b_n^{-1}$.
Also $\mu(aa_ia^{-1}) \geq \mu(a) \wedge \mu(a_i) \geq \beta \wedge \lambda_i$, $i = 1,2,\cdots,n \Leftrightarrow \mu(a_i) \geq \beta \wedge \lambda_i$, $i = 1,2,\cdots,n$.
Therefore $\beta \wedge \omega(x) = \beta \wedge \omega(axa^{-1})$. This is also true for the case $\omega(x) = 0$.
Hence $a_\beta\omega = \omega a_\beta$ for all $a_\beta \in \mu$.
It is easy to show that μ/ω is fuzzy Abelian and $\omega \geq \mu'$.
This completes the proof.

CONSTRUCTION OF FUZZY SUBGROUPS FROM A GIVEN FUZZY SUBGROUP.

Let μ be a fuzzy subgroup of $\mathcal{G}$. Define $\mu_2 : \mathcal{G} \longrightarrow [0,1]$ by $\mu_2(x) = \mu(x)\mu(x)$, $x \neq e$, and $\mu_2(e) = \mu(e)$.

In general, $\mu_n(x) = [\mu(x)]^n$, $x \neq e$, and $\mu_n(e) = \mu(e)$.

PROPOSITION : 2.1.7

Let μ be a fuzzy subgroup of $\mathcal{G}$. Then μ_2 is a fuzzy subgroup of $\mathcal{G}$.
Also μ_n is a fuzzy subgroup of $\mathcal{G}$ for each natural number n.

PROOF :

Clearly $\mu_2(x) = \mu_2(x^{-1})$,

$\mu_2(xy) \quad = \mu(xy)\mu(xy)$, and

$\mu(xy) \quad \geq \mu(x) \wedge \mu(y)$.

Suppose $\quad \mu(x) \leq \mu(y)$. Then $\mu(xy) \geq \mu(x)$.

Hence $\quad \mu(xy)\ \mu(xy) \geq \mu(x)\ \mu(x) = \mu_2(x)$.

i.e. $\quad \mu_2(xy) \geq \mu_2(x) \geq \mu_2(x) \wedge \mu_2(y)$.

So μ_2 is a fuzzy subgroup of $\mathcal{G}$.

Similarly μ_n is a fuzzy subgroup of $\mathcal{G}$.

It is easy to see that μ is fuzzy normal if and only if each μ_n is fuzzy normal.

The following fuzzy groups will be used in the study of nilpotent groups :

DEFINITION : 2.1.8

Let μ and ν be fuzzy subgroups of $\mathcal{G}$. If $h_\lambda \in \mu$ and $k_\beta \in \nu$, then $[h_\lambda, k_\beta] = h_\lambda^{-1}\, k_\beta^{-1}\, h_\lambda\, k_\beta$, and $[\mu,\nu] = < \{[h_\lambda, k_\beta] : h_\lambda \in \mu, k_\beta \in \nu\} >$.
Let $x = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n} \in \mathcal{G}$, where $k_i \in \mathbb{Z}$, $i = 1,2,\cdots,n$. Let $\lambda = \inf\{\lambda_1,\cdots,\lambda_n\}$, $a_i = [h_i,k_i]$ such that $\xi_i \leq \mu(h_i)$ and $\beta_i \leq \nu(k_i)$ and $\lambda_i = \xi_i \wedge \beta_i$ for each $i = 1,2,\cdots,n$. It is easy to show that $[\mu,\nu](x) = \sup \lambda$, where the supremum is taken over all the n–tuples $\{\lambda_1,\cdots,\lambda_n\}$, $n \in \mathbb{N}$.

PROPOSITION : 2.1.9

$[\mu,\nu] = [\nu,\mu]$.

PROPOSITION : 2.1.10

$[\mu,\nu] = \mu_e$ if and only if $k_\lambda h_\beta = h_\beta k_\lambda$ for all $k_\lambda \in \nu$, $h_\beta \in \mu$.

PROOF :

$\Rightarrow$: Let $x = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$, $a_i = [h_i,k_i]$ and $\lambda_i = \xi_i \wedge \beta_i$ such that

$\xi_i \leq \mu(h_i)$ and $\beta_i \leq \nu(k_i)$ for each $i = 1,2,\cdots,n$. Let $\lambda = \inf\{\lambda_1,\cdots,\lambda_n\}$.

Then $\mu_e(x) = \sup \lambda$.

$$\mu_e(x) = \begin{cases} \mu(e) & x = e \\ 0 & x \neq e. \end{cases}$$

Now $\lambda_i \leq \mu_e(a_i)$ implies $\mu_e(x) \geq \mu_e(a_1) \wedge \cdots \wedge \mu_e(a_n) = \mu_e(a_j) \geq \lambda_j$ for some $j \in \{1,2,\cdots,n\}$

Suppose $\lambda_j \neq 0$. So $\mu_e(a_i) = \mu(e)$ for all $i = 1,2,\cdots,n$. Therefore $a_i = e$, and the result follows.

Suppose $\mu_e(a_i) = 0$ for all $i = 1,2,\cdots,n$.

$a_i = [h_i,k_i]$ implies $h_i \notin \operatorname{supp} \mu$ or $k_i \notin \operatorname{supp} \nu$. Either case yields the desired result.

The converse is obvious.

DEFINITION : 2.1.11

Define a sequence of fuzzy subgroups of μ as follows :

$$\gamma_1(\mu) = \mu,\ \gamma_2(\mu) = [\gamma_1(\mu),\ \mu],\ \gamma_3(\mu) = [\gamma_2(\mu),\ \mu],\ \cdots .$$

PROPOSITION : 2.1.12

Let μ be a fuzzy subgroup of $\mathcal{G}$. Then $\mu \geq \gamma_1(\mu) \geq \gamma_2(\mu) \geq \cdots$ and $\gamma_i(\mu) \triangleleft \mu$ for all $i \in \mathbb{N}$.

PROOF :

$\gamma_2(\mu) = [\gamma_1(\mu),\ \mu] = [\mu,\ \mu] \leq \mu$.

$\gamma_3(\mu) = [\gamma_2(\mu),\ \mu] \leq [\gamma_1(\mu),\ \mu] = \gamma_2(\mu)$.

Hence, by induction, $\gamma_{n+1}(\mu) \leq \gamma_n(\mu)$.

Let $a_\lambda \in \mu$. We will show that $a_\lambda\ \gamma_i(\mu) = \gamma_i(\mu) a_\lambda$.

Let $a^{-1}y = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$ such that for each $i = 1,\cdots,n$, $a_i = [h_i,k_i]$ such that $\xi_i \leq \gamma_{i-1}(\mu)(h_i)$ and $\beta_i \leq \mu(k_i)$. Let $\lambda_i = \xi_i \wedge \beta_i$ and $\beta = \inf\{\lambda_1,\cdots,\lambda_n\}$. In fact we can assume that $a^{-1}y = a_1 a_2 \cdots a_n$.

Then $\lambda \wedge \gamma_i(\mu)(a^{-1}y) \quad = \sup(\beta \wedge \lambda) = \lambda \wedge \sup \beta$

$$\gamma_i(\mu) = [\gamma_{i-1}(\mu),\mu] = [\mu, \gamma_{i-1}(\mu)].$$

$$(ya^{-1})_\beta = a_\beta(a_{1\lambda_1} \cdots a_{n\lambda_n})a_\beta^{-1}.$$

Assume, without loss of generality, that $a_{1\lambda_1} \cdots a_{n\lambda_n} = h_{i\xi_i}^{-1}k_{i\beta_i}^{-1}\, h_{i\xi_i}k_{i\beta_i}$. Therefore $(ya^{-1})_\beta = [a_\beta h_{i\xi_i}a_\beta^{-1}, a_\beta k_{i\beta_i}a_\beta^{-1}]$.

Hence $\lambda \wedge \gamma_i(\mu)(y\, a^{-1}) \geq \lambda \wedge \gamma_i(\mu)(a^{-1}y)$.

By symmetry, $\lambda \wedge \gamma_i(\mu)(a^{-1}y) \geq \lambda \wedge \gamma_i(\mu)(y\, a^{-1})$.

Hence $a_\lambda\, \gamma_i(\mu) = \gamma_i(\mu)a_\lambda$ for all $i \in \mathbb{N}$.

THE NORMALIZER OF A FUZZY SUBGROUP :

DEFINITION : 2.1.13

Let μ be a fuzzy subgroup of $\mathcal{G}$. The fuzzy subset $N(\mu)$, of $\mathcal{G}$, defined by

$$N(\mu)(x) = \begin{cases} \sup\{\lambda : x_\lambda\mu = \mu\, x_\lambda,\ x_\lambda \text{ a fuzzy point in } \mathcal{G}\} \\ 0 \quad \text{otherwise} \end{cases}$$

is called the *normalizer* of μ in $\mathcal{G}$.

THEOREM : 2.1.14

Let μ be a fuzzy subgroup of $\mathcal{G}$. Then

(i) $N(\mu)$ is a fuzzy subgroup of $\mathcal{G}$;

(ii) $\mu \lhd N(\mu)$;

(iii) $N(\mu)$ is the largest fuzzy subgroup of $\mathcal{G}$ such that $\mu \lhd N(\mu)$.

PROOF :

(i) $x_\lambda\mu = \mu x_\lambda$ if and only if $x_\lambda^{-1}\mu = \mu x_\lambda^{-1}$.

Hence $N(\mu)(x) = 0$ if and only if $N(\mu)(x^{-1}) = 0$.

Suppose $N(\mu)(x) = \sup\{\lambda : x_\lambda\mu = \mu x_\lambda\}$

Therefore $N(\mu)(x) = \sup\{\lambda : x_\lambda^{-1}\mu = \mu\, x_\lambda^{-1}\} = N(\mu)(x^{-1})$.

We now claim that $N(\mu)(xy) \geq N(\mu)(x) \wedge N(\mu)(y)$. If $N(\mu)(x) = 0$, there is nothing to prove.
Suppose $N(\mu)(x) > 0$.

Let $N(\mu)(y) = \quad \sup\{\beta_j : y_{\beta_j}\mu = \mu\, y_{\beta_j}\} > 0$.

Take any λ_i and any β_j, and let $\alpha_{ij} = \lambda_i \wedge \beta_j$.

$$\begin{aligned}(xy)_{\alpha_{ij}}\mu(a) &= \alpha_{ij} \wedge \mu(y^{-1}\, x^{-1}a) \\ &= \alpha_{ij} \wedge \mu(x^{-1}\, a\, y^{-1}) \text{ since } y_{\alpha_{ij}}\, \mu = \mu\, y_{\alpha_{ij}} \\ &= \alpha_{ij} \wedge \mu(a\, y^{-1}x^{-1}) \\ &= (\mu)(xy)_{\alpha_{ij}}(a).\end{aligned}$$

$N(\mu)(xy) = \sup\{\lambda : (xy)_\lambda\mu = \mu(xy)_\lambda\} \geq \sup\{\alpha_{ij} : x_{\lambda_i}\mu = \mu\, x_{\lambda_i}$ and $y_{\beta_j}\mu = \mu\, y_{\beta_j}\}$. Therefore

$$\begin{aligned}N(\mu)(xy) \quad &\geq \sup\{\lambda_i \wedge \beta_j : x_{\lambda_i}\mu = \mu\, x_{\lambda_i} \text{ and } y_{\beta_j}\mu = \mu\, y_{\beta_j}\} \\ &= \sup\{\lambda_i : x_{\lambda_i}\mu = \mu\, x_{\lambda_i}\} \wedge \sup\{\beta_j : y_{\beta_j}\mu = \mu\, y_{\beta_j}\} \\ &= N(\mu)(x) \wedge N(\mu)(y).\end{aligned}$$

(ii) Clearly $\mu \leq N(\mu)$.
Let $a_\lambda \in N(\mu)$. We want to show that $a_\lambda\mu = \mu\, a_\lambda$.

Case $N(\mu)(a) = \lambda$:

So $\lambda = \sup\{\lambda_i : a_{\lambda_i}\mu = \mu\, a_{\lambda_i}\}$. Now $\lambda_i \wedge \mu(a^{-1}x) = \lambda_i \wedge \mu(x\, a^{-1})$ for all $x \in \mathscr{G}$ and all a_{λ_i} satisfying $a_{\lambda_i}\, \mu = \mu\, a_{\lambda_i}$.

Hence $\sup_i[\lambda_i \wedge \mu(a^{-1}x)] = \sup_i[\lambda_i \wedge \mu(x\, a^{-1})]$.

Therefore $(\sup_i \lambda_i) \wedge \mu(a^{-1}x) = (\sup_i \lambda_i) \wedge \mu(x\, a^{-1})$

i.e. $\quad \lambda \wedge \mu(a^{-1}x) = \lambda \wedge \mu(x\, a^{-1})$

i.e. $\quad a_\lambda\mu = \mu\, a_\lambda$.

Case N(μ)(a) $\not\geq \lambda$:

$\exists\ \lambda_i$, $a_{\lambda_i}\mu = \mu\, a_{\lambda_i}$, such that $\lambda_i \not\geq \lambda$.

Now $\lambda_i \wedge \mu(a^{-1}x) = \lambda_i \wedge \mu(x\, a^{-1})$ for all $x \in \mathcal{G}$.

Hence $\lambda_i \wedge \lambda \wedge \mu(a^{-1}x) = \lambda_i \wedge \lambda \wedge \mu(x\, a^{-1})$

i.e. $\lambda \wedge \mu(a^{-1}x) = \lambda \wedge \mu(x\, a^{-1})$

i.e. $a_\lambda\, \mu = \mu\, a_\lambda$.

This proves (ii).

(iii) Let $\mu \lhd w$, for some fuzzy subgroup w in $\mathcal{G}$. We want to show that $w \leq N(\mu)$. Let $a_\lambda \in w$. Then $a_\lambda\, \mu = \mu\, a_\lambda$.

Hence $N(\mu)(a) = \sup\{\lambda_i : a_{\lambda_i}\mu = \mu\, a_{\lambda_i}\} \geq \lambda$.

Hence $N(\mu)(a) \geq w(a)$.

This completes the proof.

PROPOSITION : 2.1.15

Let μ be a fuzzy subgroup of $\mathcal{G}$ such that $\mu(\text{supp}\ \mu) \subset [\alpha, \mu(e)]$ for some $\alpha > 0$. Then supp $N(\mu) = N(\text{supp}\ \mu)$.

PROOF :

Let $a \in N(\text{supp}\ \mu)$. So supp $\mu = a(\text{supp}\ \mu)a^{-1}$.
If $a \in$ supp μ, there exists $\lambda > 0$ such that $a_\lambda \in \mu$. So $a_\lambda\mu = \mu a_\lambda$.

Hence $a \in$ supp $N(\mu)$.

Suppose $a \notin$ supp μ. Let $x \in \mathcal{G}$.
Suppose $a^{-1}x \in$ supp μ. Then $xa^{-1} = a(a^{-1}x)a^{-1} \in$ supp μ and conversely.

Now $\mu(a^{-1}x) \geq \alpha$ and $\mu(xa^{-1}) \geq \alpha$.
So $\alpha \wedge \mu(a^{-1}x) = \alpha = \alpha \wedge \mu(xa^{-1})$.

Suppose $a^{-1}x \notin$ supp μ. So $xa^{-1} \notin$ supp μ.
Therefore $\alpha \wedge \mu(a^{-1}x) = 0 = \alpha \wedge \mu(xa^{-1})$.
Hence $a_\alpha\mu = \mu a_\alpha$.

Therefore $a \in$ supp $N(\mu)$.

$$\begin{aligned}\text{Supp } N(\mu) &= \{x \in \mathscr{G} : x_\lambda \mu = \mu\, x_\lambda \text{ for some } \lambda > 0\} \\ &\subset \{x \in \mathscr{G} : x(\text{supp } \mu) = (\text{supp } \mu)\, x\} \\ &= N(\text{supp } \mu).\end{aligned}$$

Hence $N(\text{supp } \mu) = \text{supp } N(\mu)$.

DEFINITION : 2.1.16

Let μ be a fuzzy subgroup of $\mathscr{G}$. The *centre* of μ, denoted by $Z(\mu)$, is a fuzzy subset of $\mathscr{G}$ defined by

$$Z(\mu)(x) = \begin{cases} \sup\{\lambda,\ x_\lambda \in \mu \text{ such that } x_\lambda b_\beta = b_\beta x_\lambda \text{ for all } b_\beta \in \mu\} \\ 0 \qquad\qquad \text{if } x \notin Z(\text{supp } \mu) \end{cases}.$$

Clearly $Z(\mu) \neq 0$ since $\mu(e) > 0$.
Let $a \in \mathscr{G}$. Then $\mu(a) \geq Z(\mu)(a)$.

PROPOSITION : 2.1.17

$Z(\mu) \triangleleft \mu$.

PROOF :

First we prove that $Z(\mu)$ is a fuzzy subgroup of $\mathscr{G}$. Let $x \in \mathscr{G}$ such that $Z(\mu)(x) > 0$. So $Z(\mu)(x) = \sup \lambda,\ x_\lambda \in \mu$ such that $x_\lambda b_\beta = b_\beta x_\lambda$ for all $b_\beta \in \mu$.

We claim that $x_\lambda b_\beta = b_\beta x_\lambda$ if and only if $x_\lambda^{-1} b_\beta = b_\beta\, x_\lambda^{-1}$.

($\Longrightarrow$): Let $\alpha = \lambda \wedge \beta$.

Therefore $(xb)_\alpha = (bx)_\alpha$, hence $\alpha \wedge b_\alpha(x^{-1}y) = \alpha \wedge b_\alpha(y\, x^{-1})$ for all $y \in \mathscr{G}$.

If $y = xb$, then LHS $= \alpha$. Hence RHS $= \alpha$. So $xb = bx$.

Now $x_\lambda^{-1} b_\beta(y) = \lambda \wedge b_\beta(xy)$ (1)

and $b_\beta x_\lambda^{-1}(y) = \lambda \wedge b_\beta(y\, x)$ (2)

If (1) $= 0$ and (2) $\neq 0$, then $y = bx^{-1} = x^{-1}b$, contradicting the fact that (1) $= 0$. So (1) and (2) are either both zero or both nonzero.
If (1) and (2) are > 0, then $y = x^{-1}b = bx^{-1}$, hence (1) $=$ (2).
Therefore $Z(\mu)(x) = Z(\mu)(x^{-1}) > 0$.

Clearly $Z(\mu)(x) = 0$ if and only if $Z(\mu)(x^{-1}) = 0$.

Next we show that $Z(\mu)(xy) \geq Z(\mu)(x) \wedge Z(\mu)(y)$.

Consider $Z(\mu)(x) = \sup\{\lambda_i : x_{\lambda_i} b_\beta = b_\beta x_{\lambda_i}\}$ and

$Z(\mu)(y) = \sup\{\beta_j : y_{\beta_j} b_\beta = b_\beta y_{\beta_j}\}$ as required in Definition 2.1.16.

Let $\alpha_{ij} = \lambda_i \wedge \beta_j$.

Therefore $x_{\alpha_{ij}} b_\beta = b_\beta x_{\alpha_{ij}}$ and $y_{\alpha_{ij}} b_\beta = b_\beta y_{\alpha_{ij}}$.

$$\begin{aligned} \text{Now } (xy)_{\alpha_{ij}} b_\beta &= x_{\alpha_{ij}}(y_{\alpha_{ij}} b_\beta) &&= x_{\alpha_{ij}}(b_\beta y_{\alpha_{ij}}) \\ &= (x_{\alpha_{ij}} b_\beta) y_{\alpha_{ij}} &&= b_\beta(x_{\alpha_{ij}} y_{\alpha_{ij}}) = b_\beta(xy)_{\alpha_{ij}}. \end{aligned}$$

Therefore

$$\begin{aligned} Z(\mu)(xy) &= \sup\{\lambda : (xy)_\lambda b_\beta = b_\beta(xy)_\lambda\} \\ &\geq \sup\{\lambda_i \wedge \beta_j : x_{\lambda_i} b_\beta = b_\beta x_{\lambda_i} \text{ and } y_{\beta_j} b_\beta = b_\beta y_{\beta_j}\} \\ &= \sup\{\lambda_i : x_{\lambda_i} b_\beta = b_\beta x_{\lambda_i}\} \wedge \sup\{\beta_j : y_{\beta_j} b_\beta = b_\beta y_{\beta_i}\} \\ &= Z(\mu)(x) \wedge Z(\mu)(y). \end{aligned}$$

If $Z(\mu)(x) = 0$, the result is trivial. Hence $Z(\mu)$ is a fuzzy subgroup of $\mathcal{G}$.

We now show that $Z(\mu) \triangleleft \mu$:

Let $a_\lambda \in \mu$. We will show that $a_\lambda Z(\mu) = Z(\mu) a_\lambda$.

$a_\lambda Z(\mu)(x) = \lambda \wedge Z(\mu)(a^{-1}x) = \lambda \wedge \sup \lambda_i$ and $(a^{-1}x)_{\lambda_i} b_\beta = b_\beta(a^{-1}x)_{\lambda_i}$,

$(a^{-1}x)_{\lambda_i} \in \mu$ and $Z(\mu)\, a_\lambda(x) = \lambda \wedge Z(\mu)(x\, a^{-1}) = \lambda \wedge \sup \lambda_i$,

$(x\, a^{-1})_{\lambda_i} b_\beta = b_\beta(x\, a^{-1})_{\lambda_i}$. $(a^{-1}x)_{\lambda_i} b_\beta = b_\beta(a^{-1}x)_{\lambda_i}$ for all $b_\beta \in \mu$ if and only if $(x\, a^{-1})_{\lambda_i} b_\beta = b_\beta(x\, a^{-1})_{\lambda_i}$ for all $b_\beta \in \mu$. Hence $a_\lambda Z(\mu) = Z(\mu) a_\lambda$ for all $a_\lambda \in \mu$.

The proof is complete.

PROPOSITION : 2.1.18

$Z(\text{supp } \mu) = \text{supp } Z(\mu)$. Hence $Z(\mu)$ is fuzzy Abelian.

PROOF :

obvious.

The notions of fuzzy centre and normalizer are good extensions of the crisp analogues. More precisely

PROPOSITION : 2.1.19

(i) $\chi_{Z(\mathcal{G})} = Z(\chi_{\mathcal{G}})$;

(ii) If $H \leq \mathcal{G}$, then $\chi_{N(H)} = N(\chi_H)$.

PROOF :

(i) Easy.

(ii) $\chi_H(\text{supp}\ \chi_H) = \{1\} \subset [\alpha,1]$ for some $\alpha > 0$.

So supp $N(\chi_H) = N(\text{supp}\ \chi_H) = N(H)$ by Proposition 2.1.15.

$N(H) = \{x \in \mathcal{G} : x H x^{-1} = H\}$.

Supp $N(\chi_H) = \{x \in \mathcal{G} : \lambda \wedge \chi_H(x^{-1}a) = \lambda \wedge \chi_H(a x^{-1})$ for some $\lambda > 0\}$.

Now let $x \in N(H)$. So $x H x^{-1} = H$.

Suppose $\chi_H(x^{-1}a) \neq 0$. Then $x^{-1}a \in H$. So $x^{-1}a = h$ for some $h \in H$.

Hence $a x^{-1} = x h x^{-1} \in H$. So $\chi_H(x^{-1}a) \neq 0 \Leftrightarrow \chi_H(a x^{-1}) \neq 0$.

Hence $\lambda \wedge \chi_H(x^{-1}a) = \chi_H(a x^{-1}) \wedge \lambda$ for all $\lambda \in (0,1]$.

i.e. $x_1 \chi_H = \chi_H x_1$ for all $x \in N(H)$.

$$\text{So } N(\chi_H)(x) = \begin{cases} \sup\{\lambda : x_\lambda \mu = \mu x_\lambda, \mu = \chi_H\} \\ 0 \quad \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & x \in N(H) \\ 0 & x \notin N(H) \end{cases} = \chi_{N(H)}(x).$$

The proof is complete.

For the sake of completeness we define the notion of fuzzy centralizer :

DEFINITION : 2.1.20

The *centralizer of a fuzzy point* $a_\beta \in \mu$, denoted by $C_\mu(a_\beta)$, is defined by

$$C_\mu(a_\beta)(x) = \begin{cases} \sup\{\lambda,\ x_\lambda \in \mu \text{ such that } x_\lambda a_\beta = a_\beta x_\lambda\} \\ 0 \quad \text{otherwise.} \end{cases}$$

It is clear that $C_\mu(a_\beta)$ is a fuzzy subgroup of $\mathcal{G}$ and that $C_\mu(a_\beta) \leq \mu$.

Also $\chi_{C_{\mathcal{G}}(a)} = C_{\chi_{\mathcal{G}}}(\chi_a)$, $a \in \mathcal{G}$.

DEFINITION : 2.1.21

The *centralizer of a fuzzy subgroup* μ is a fuzzy subset of $\mathcal{G}$ defined by

$$C(\mu)(a) = \begin{cases} \sup\{\lambda \wedge \beta,\ a_\lambda \in \chi_{\mathcal{G}} \text{ such that } a_\lambda b_\beta = b_\beta a_\lambda \text{ for all } b_\beta \in \mu\} \\ 0 \quad \text{otherwise.} \end{cases}$$

It is not hard to prove that $C(\mu)$ is a fuzzy subgroup of $\mathcal{G}$, and that
$\text{supp } C(\mu) = C_{\mathcal{G}}(\text{supp } \mu)$.
In case $\mu = \chi_{\mathcal{G}}$, then $Z(\mu) = C(\mu)$.

2.2 : FUZZY CONGRUENCE RELATIONS INDUCED BY NORMAL FUZZY SUBGROUPS

DEFINITION : 2.2.1 [8]

A *fuzzy relation* ν on $\mathcal{G}$ is a mapping $\nu : \mathcal{G} \times \mathcal{G} \longrightarrow [0,1]$.
Denote the set of all fuzzy relations on $\mathcal{G}$ by $\Pi(\mathcal{G})$.
For $\mu \in \Pi(\mathcal{G})$, let $t_0 = \sup\{\mu(x,y) : (x,y) \in \mathcal{G} \times \mathcal{G}\}$

If $t_0 = 0$, then we have the empty relation

$$\mu(x,y) = 0 \text{ for all } x, y \in \mathcal{G}.$$

In this section assume that $0 < t_0 \leq 1$. Two operations are defined on $\Pi(\mathcal{G}) \times \Pi(\mathcal{G})$, one, called the composition and denoted by $\mu \circ \nu$, is defined as

$$\mu \circ \nu(x,y) = \sup_{z \in \mathcal{G}} [\mu(x,z) \wedge \nu(z,y)]$$

for $\mu, \nu \in \Pi(\mathcal{G})$.

The other, called the multiplication and denoted by $\mu.\nu$ or simply $\mu\nu$, is defined as $\mu\nu(x,y) = \sup\{[\mu(x_1,y_1) \wedge \nu(x_2,y_2)] : x = x_1x_2 \text{ and } y = y_1y_2\}$ for $\mu, \nu \in \Pi(\mathcal{G})$. (See [9]).

DEFINITION : 2.2.2 [8]

A fuzzy relation μ on $\mathcal{G}$ is said to be a *fuzzy equivalence relation* on $\mathcal{G}$ if

(i) $\mu(x,x) = t_0$ for all $x \in \mathcal{G}$, (Reflexive).

(ii) $\mu(x,y) = \mu(y,x)$ for all $x, y \in \mathcal{G}$, (Symmetric).

(iii) $\mu \circ \mu \leq \mu$, where $\circ$ denotes the composition, (Transitive).

It is readily checked that if μ is a fuzzy equivalence relation, then μ is idempotent for $\circ$, i.e. $\mu \circ \mu = \mu$. Furthermore, for each $t \in [0, t_0]$, the t–cut, μ^t, is a crisp equivalence relation, where μ^t is the relation $x\ \mu^t y \Leftrightarrow \mu(x,y) \geq t$. In particular, the t_0–cut μ^{t_0} is a crisp equivalence relation and as such yields a partition of $\mathcal{G}$ in the crisp sense. The t_0–cut classes of $\mathcal{G}$ under this partition are denoted by $\bar{x}$, $\bar{y}$, $\bar{e}$, etc., containing representative elements x, y, e respectively.

For each t_0–cut class $\bar{x}$, for $x \in \mathcal{G}$, a fuzzy subset $\mu_{\bar{x}} : \mathcal{G} \longrightarrow [0,1]$ is defined as $\mu_{\bar{x}}(a) = \mu(x,a)$ for all $x \in \mathcal{G}$.

Now for each $t \in [0,t_0]$, the collection $\{ C_t^{\bar{x}} : x \in \mathcal{G}\}$ of t–cuts is a crisp partion of $\mathcal{G}$.

DEFINITION : 2.2.3

A fuzzy equivalence relation μ on $\mathcal{G}$ is called a fuzzy congruence relation of $\mathcal{G}$ if $\mu\mu \leq \mu$.

The relation $\mu\mu \leq \mu$ can be thought of as a substitution property as is well known in crisp congruence relation on a group or a general algebra. Moreover, one can interpret in the crisp case a congruence relation as an equivalence relation E which is at the same time a subgroup of $\mathcal{G} \times \mathcal{G}$. Analogously, a fuzzy equivalence relation which is at the same time a fuzzy subgroup of $\mathcal{G} \times \mathcal{G}$ is called a fuzzy congruence relation. It is easily checked that for each $t \in [0,t_0]$, μ^t is a congruence relation if and only if μ is a fuzzy congruence relation on $\mathcal{G}$.

CONGRUENCE AND NORMAL SUBGROUPS :

We now turn our attention to the relationship between fuzzy congruence relations on $\mathcal{G}$ on the one hand, and normal fuzzy subgroups on the other. Firstly we have

THEOREM : 2.2.4

Let μ be a normal fuzzy subgroup of $\mathcal{G}$. Define $\nu : \mathcal{G} \times \mathcal{G} \longrightarrow [0,1]$ by $\nu(x,y) = \mu(xy^{-1})$. Then ν is a fuzzy congruence relation on $\mathcal{G}$.

PROOF :

$\nu(x,y) = \mu(xy^{-1}) \leq \mu(e) = t_0$. Also
$\nu(x,x) = \mu(e) = t_0$ for all $x \in \mathcal{G}$.
Hence ν is reflexive.

$$\nu(x,y) = \mu(xy^{-1}) = \mu[(xy^{-1})^{-1}] = \mu(yx^{-1}) = \nu(y,x) \text{ for all } x, y \in \mathcal{G}.$$

$$\begin{aligned} \nu \circ \nu(x,y) &= \sup_{z \in \mathcal{G}} [\nu(x,z) \wedge \nu(z,y)] \\ &= \sup_{z \in \mathcal{G}} [\mu(xz^{-1}) \wedge \mu(zy^{-1})] \\ &\leq \mu(xz^{-1}zy^{-1}) \\ &= \mu(xy^{-1}) \\ &= \nu(x,y). \end{aligned}$$

Finally we show that $\nu\nu \leq \nu$.

$$\nu\nu(x,y) = \sup\{[\nu(x_1,y_1) \wedge \nu(x_2,y_2)] : x = x_1x_2 \text{ and } y = y_1y_2\}$$

whereas $\nu(x,y) = \mu(x\,y^{-1}) = \mu(x_1x_2y_2^{-1}y_1^{-1})$ for every representation $x = x_1x_2$, $y = y_1y_2$.

But $\mu(x_1x_2y_2^{-1}y_1^{-1})$ $= \mu(x_1y_1^{-1}y_1x_2y_2^{-1}y_1^{-1})$

$\geq \mu(x_1y_1^{-1}) \wedge \mu(y_1x_2y_2^{-1}y_1^{-1})$

$= \mu(x_1y_1^{-1}) \wedge \mu(x_2y_2^{-1})$

$= \nu(x_1,y_1) \wedge \nu(x_2,y_2)$ since μ is fuzzy normal.

Hence $\nu\nu(x,y) \leq \nu(x,y)$.

This completes the proof.

The following is a sort of converse to the above Theorem. Every fuzzy congruence relation determines a normal fuzzy subgroup.

THEOREM : 2.2.5

Let μ be a fuzzy congruence relation on $\mathcal{G}$. Then there is a normal fuzzy subgroup ν of $\mathcal{G}$ such that $\mu(x,y) = \nu(xy^{-1})$.

PROOF :

Clearly $\mu(x,x) = t_0$ for all $x \in \mathcal{G}$. μ^{t_0} is a crisp congruence relation on $\mathcal{G}$. Let $[e]_{\mu^{t_0}}$ be the class containing the identity e in the partition of $\mathcal{G}$ yielded by μ^{t_0}.
Define $\nu : \mathcal{G} \longrightarrow [0,1]$ by $\nu(x) = \mu(x,e)$ for all $x \in \mathcal{G}$.

(i) ν is well defined.

(ii) $\mu(x,e) = \mu(x^{-1},e)$ for all $x \in \mathcal{G}$:

Suppose $x \in [e]_{\mu^{t_0}}$. Then $x\ \mu^{t_0} e$ and $x^{-1}\ \mu^{t_0}\ x^{-1}$. Hence $e\ \mu^{t_0}\ x^{-1}$, and this implies that $\mu(x^{-1},e) = t_0 = \mu(x,e)$.

Suppose $x \notin [e]_{\mu^{t_0}}$. Then $\mu(x,e) < t_0$, and also $\mu(x^{-1},e) < t_0$ since $[e]_{\mu^{t_0}} \leq \mathcal{G}$. Let $t_1 = \mu(x,e)$ and $t_2 = \mu(x^{-1},e)$. If $t_1 < t_2$, then $x \in [e]_{\mu^{t_1}}$ and $x \notin [e]_{\mu^{t_2}}$. Also $x^{-1} \in [e]_{\mu^{t_2}}$ implies that $x \in [e]_{\mu^{t_2}}$.

This is a contradiction.

A similar contradiction arises if we assume that $t_2 < t_1$. Therefore $t_1 = t_2$, i.e. $\nu(x) = \nu(x^{-1})$ for all $x \in \mathcal{G}$.

(iii) $\nu(xy) = \mu(xy,e)$
$\geq \mu(x,e) \wedge \mu(y,e)$ since μ is a fuzzy congruence relation,
$= \nu(x) \wedge \nu(y)$.

(iv) $\mu(x,e) = \mu(xy,y)$ for all x, y $\in \mathcal{G}$:

Let $t_1 = \mu(xy,y)$ and $t_2 = \mu(x,e)$. If $t_1 > t_2$, then $[e]_{\mu^{t_1}} \subseteq [e]_{\mu^{t_2}}$.

$xy\ \mu^{t_1} y$ and $y^{-1}\mu^{t_1}y^{-1}$ implies that $x\ \mu^{t_1} e$.

Hence $\mu(x,e) \geq t_1$, and this implies that $t_2 \geq t_1$, a contradiction !

A similar contradiction results if $t_1 < t_2$.

Therefore $\mu(x,e) = \mu(xy,y)$.
So $\mu(xy^{-1},e) = \mu(x,y)$. Hence $\nu(xy^{-1}) = \mu(x,y)$.

(v) ν is fuzzy normal in $\mathcal{G}$.

$\nu(a^{-1}xa) = \mu(a^{-1}x\,a,e)$
$=\mu(xa,a)$
$= \mu(x,e)$ for all x, a $\in \mathcal{G}$.

So $\nu(a^{-1}xa) = \nu(x)$.

This completes the proof.

THEOREM 2.2.6

Let μ be a fuzzy congruence relation on $\mathcal{G}$, and $t_0 = \sup_{x,\, y \in \mathcal{G}} \mu(x,y)$. The collection $\{\mu_{\bar{x}} : x \in \mathcal{G}\}$ is a fuzzy partition of $\mathcal{G}$ in the sense that $\sup_{x \in \mathcal{G}} \mu_{\bar{x}} = \Pi_{\mathcal{G}}$, where $\Pi_{\mathcal{G}}(x) = t_0$ for all $x \in \mathcal{G}$, and $\mu_{\bar{x}} \wedge \mu_{\bar{y}}(a) < t_0$ for $\bar{x} \neq \bar{y}$, for all $a \in \mathcal{G}$. Furthermore $\{\mu_{\bar{x}} : x \in \mathcal{G}\}$ is a group under a suitably defined binary operation. The fuzzy subset $\mu_{\bar{x}}$ of $\mathcal{G}$ is precisely the left fuzzy coset $x\mu$ of $\mu_{\bar{e}}$ associated with $x \in \mathcal{G}$, where $\bar{e} = [e]_{\mu^{t_0}}$.

PROOF :

It is easy to check that $\sup_{x \in \mathcal{G}} \mu_{\bar{x}} = \Pi_{\mathcal{G}}$.

We now show that $\mu_{\bar{x}} \wedge \mu_{\bar{y}}(a) < t_0 \; \forall \; a \in \mathcal{G}, \bar{x} \neq \bar{y}$.

Suppose $\mu_{\bar{x}} \wedge \mu_{\bar{y}}(a) = t_o$ for some $a \in \mathcal{G}$. Then $\mu_{\bar{x}}(a) = t_o = \mu_{\bar{y}}(a)$.

Hence $\mu(x,a) = \mu(y,a)$ for all $x \in \bar{x}$, $y \in \bar{y}$. Therefore $x \, \mu^{t_o} a$ and $y \, \mu^{t_o} a$ implies that $x \, \mu^{t_o} y$. So $y \in \bar{x} \cap \bar{y} = \emptyset$, a contradiction.

Now define a binary operation as follows :

$\mu_{\bar{x}} \mu_{\bar{y}} = \mu_{\overline{xy}}$, where $\overline{xy}$ is the class containing xy for $x \in \bar{x}$, $y \in \bar{y}$.

The multiplication is well defined :

Let $x, x_1 \in \bar{x}$ and $y, y_1 \in \bar{y}$.

We must show that $\mu(xy,a) = \mu(x_1y_1,a)$ for all $a \in \mathcal{G}$.

Now $x \, \mu^{t_o} \, x_1$ and $y \, \mu^{t_o} \, y_1$ implies that $xy \, \mu^{t_o} \, x_1y_1$. So $x_1y_1 \in \overline{xy}$.

Hence $\mu(xy,x_1y_1) = t_o$. If $a \in \overline{xy}$, then $\mu(xy,a) = t_0 = \mu(x_1y_1,a)$.

If $a \notin \overline{xy}$, then $\mu(xy,a) < t_o$ and $\mu(x_1y_1,a) < t_o$.

Therefore $\mu(xy,a) \geq \mu(xy,x_1y_1) \wedge \mu(x_1y_1,a)$
$= \mu(x_1y_1,a)$.

Similarly $\mu(x_1y_1,a) \geq \mu(xy,a)$.

Hence $\mu(xy,a) = \mu(x_1y_1,a)$.

So $\mu_{\overline{xy}}(a) = \mu(x_1y_1,a)$.

We next show that $\{\mu_{\bar{x}} : x \in \mathcal{G}\}$ is a group under the binary operation defined above.

$\mu_{\bar{e}}\mu_{\bar{x}} = \mu_{\bar{x}}$ since $\mu_{\overline{ex}}(a) = \mu(ex,a) = \mu(x,a) = \mu_{\bar{x}}(a)$ for all $a \in \mathcal{G}$.

Similarly $\mu_{\bar{x}}\mu_{\bar{e}} = \mu_{\bar{x}}$.

Define $\bar{x}^{-1} = [x^{-1}]_{\mu^{t_o}}$ for $x \in \mathcal{G}$.

Then $\mu_{\bar{x}} \mu_{\bar{x}^{-1}} = \mu_{\bar{e}}$. So $(\mu_{\bar{x}})^{-1} = \mu_{\bar{x}^{-1}}$.

The associativity $(\mu_{\bar{x}} \mu_{\bar{y}})\mu_{\bar{z}} = \mu_{\bar{x}}(\mu_{\bar{y}} \mu_{\bar{z}})$ follows from the same property in $\mathcal{G}$.

Finally we recall that the left fuzzy coset of μ associated with $x \in \mathcal{G}$ is $x\mu$ defined by

$$(x\mu)(y) = \mu(x^{-1}y) \text{ for all } y \in \mathcal{G}.$$

By Theorem 2.2.5, $\mu_{\bar{e}}$ is a fuzzy normal subgroup of $\mathscr{G}$. We claim that $x\mu_{\bar{e}} = \mu_{\bar{x}}$ for all $x \in \mathscr{G}$:

$$\begin{aligned} x\mu_{\bar{e}}(y) &= \mu_{\bar{e}}(x^{-1}y) = \mu(x^{-1}y,e) = \mu(xx^{-1}y,x) = \mu(y,x) \\ &= \mu(x,y) = \mu_{\bar{x}}(y) \ \text{ for all } y \in \mathscr{G}. \end{aligned}$$

This completes the proof.

CHAPTER 3

DIRECT PRODUCTS OF FUZZY SUBGROUPS AND THE FUZZY ISOMORPHISM THEOREMS.

INTRODUCTION

The product of two fuzzy subgroups was introduced in Chapter 1. In Proposition 1.3.10 we proved that if μ_1 and μ_2 are fuzzy subgroups of μ such that μ_1 or μ_2 is normal in μ, then $\mu_1\mu_2$ is a fuzzy subgroup of $\mathscr{G}$. It is easy to show that normality of μ_1 or μ_2 in μ is essential for $\mu_1\mu_2$ to be a fuzzy subgroup. It is also easily shown that if $\mu\nu$ is a fuzzy subgroup of $\mathscr{G}$, then it is the smallest fuzzy subgroup of $\mathscr{G}$ containing both μ and ν on condition $\mu(e) = \nu(e)$. Analogues of the Dedekind and Modular laws are presented and proved in this chapter. In [14] Sherwood introduced the notion of an external direct product of fuzzy subgroups. In [11] we introduced the notion of an internal direct product and then proved that the internal and the external direct products of normal fuzzy subgroups are isomorphic. When $\mathscr{G}$ is a finite group, we show that $\mu\nu$ is an internal direct product if and only if every nonzero level subgroup of $\mu\nu$ is an internal direct product of the corresponding level subgroups of μ and ν.

In Section 3.2 we introduce a notion of a fuzzy kernel of a homomorphism. It turns out that the fuzzy kernel is the homomorphic pre–image of the trivial fuzzy subgroup whose support is the identity element of the underlying group. This fuzzy kernel is a generalization of the fuzzy kernel given in [11]. If $f : \mathscr{G} \longrightarrow \mathscr{G}'$ is a group homomorphism, and μ is a fuzzy subgroup of $\mathscr{G}$, it is easily checked that the fuzzy kernel of f associated with μ is fuzzy normal in μ.

Finally we state and prove analogues of the three well–known isomorphism theorems in group theory. This is an improvement of part of the work done in [11]. We also show, by means of an example, that the second isomorphism theorem fails if we use the quotient of Mukherjee and Bhattacharya introduced in [1].

Subsequent to the work that we have done on direct products, it is interesting to note that Malik, Mordeson and Nair in [59] defined an internal direct product of fuzzy subgroups as in [11]. Finally they proved that if ν is a compatible and divisible fuzzy subgroup in a fuzzy subgroup μ, then ν is a direct factor of μ. In this thesis we have not studied divisible and pure fuzzy subgroups.

3.1 : PRODUCTS AND DIRECT PRODUCTS OF FUZZY SUBGROUPS.

By proposition 1.3.10, if μ is fuzzy normal in $\mathcal{G}$, and ν is a fuzzy subgroup of $\mathcal{G}$, then $\mu\nu$ is a fuzzy subgroup of $\mathcal{G}$. We show, by an example, that if μ and ν are both not fuzzy normal in some larger fuzzy group, then $\mu\nu$ need not be a fuzzy subgroup of $\mathcal{G}$.

EXAMPLE : 3.1.1

Let $\mathcal{G} = S_3 = \{e, a, a^2, b, ab, a^2b\}$, $b^2 = e = a^3$.

Let $H = \{e,b\}$ and $K = \{e, ab\}$ and $\mu = \chi_H$, $\nu = \chi_K$.

Then μ and ν are not fuzzy normal. Now $\mu\nu(a^2b) = 0$, while $\mu\nu(a^2) = 1 = \mu\nu(b)$.
i.e. $\mu\nu(a^2b) \not\geq \mu\nu(a^2) \wedge \mu\nu(b)$.

THEOREM : 3.1.2

Let μ, ν be fuzzy subgroups of $\mathcal{G}$ where μ is fuzzy normal. Then $\mu\nu$ is the smallest fuzzy subgroup of $\mathcal{G}$ containing both μ and ν.

PROOF :

For any $x \in \mathcal{G}$

$\mu\nu(x) \geq \min(\mu(x), \nu(e))$

$= \mu(x)$ since $\nu(e) = \mu(e)$.

So $\mu\nu \geq \mu$. Similarly $\mu\nu \geq \nu$.

Let ξ be a fuzzy subgroup of $\mathcal{G}$ such that $\mu \leq \xi$ and $\nu \leq \xi$.

Let $a = \mu\nu(x) > 0$, $x \in \mathcal{G}$. For any $0 < \epsilon < a$, there exist $x_1,x_2 \in \mathcal{G}$, such that

$x = x_1,x_2$, with $a-\epsilon < \mu(x_1) \wedge \nu(x_2)$.

Therefore $\xi(x_1) \wedge \xi(x_2) > a-\epsilon$ and $\xi^2(x) > a-\epsilon$.

Hence $\xi(x) > a-\epsilon$ since $\xi^2 = \xi$.

So $\xi(x) \geq a = \mu\nu(x)$. If $\mu\nu(x) = 0$, then $\mu\nu(x) \leq \xi(x)$. Therefore $\xi \geq \mu\nu$.

The proof is complete.

REMARK : 3.1.3

The above theorem is not necessarily true if $\mu(e) \neq \nu(e)$.

PROPOSITION : 3.1.4

Let μ, μ_1, μ_2 be fuzzy subgroups of $\mathcal{G}$ such that $\mu = \mu_1\mu_2$. Then supp μ = (supp μ_1)(supp μ_2).

PROOF :

Straightforward.

We now state and prove the Dekekind and the Modular laws :

PROPOSITION : 3.1.5 (DEDEKIND LAW)

Let μ, ν, ξ be normal fuzzy subgroups of $\mathcal{G}$ with $\mu < \xi$. Then $\mu\nu \wedge \xi = \mu(\nu \wedge \xi)$.

PROOF :

$$(\mu\nu \wedge \xi)(x) = \sup_{x = x_1x_2} (\min(\mu(x_1), \nu(x_2)) \wedge \xi(x))$$

$$= \sup_{x = x_1x_2} (\mu(x_1) \wedge \nu(x_2) \wedge \xi(x))$$

$$\geq \sup_{x = x_1x_2} (\mu(x_1) \wedge \nu(x_2) \wedge \xi(x_1) \wedge \xi(x_2))$$

$$= \sup_{x = x_1x_2} (\mu(x_1) \wedge \nu(x_2) \wedge \xi(x_2)), \text{ since } \mu < \xi.$$

So $$(\mu\nu \wedge \xi)(x) \geq \sup_{x = x_1x_2} \min(\mu(x_1), (\nu \wedge \xi)(x_2))$$

$$= \mu(\nu \wedge \xi)(x),$$

i.e. $$\mu\nu \wedge \xi \geq \mu(\nu \wedge \xi).$$

Conversely

$$\mu(\nu \wedge \xi)(x) \geq \min_{x = x_1x_2} [\mu(x_1), \nu \wedge \xi(x_2)]$$

$$= \min[\mu(x_1), \nu(x_2), \xi(x_2)]$$

$$\geq \min[\mu(x_1), \nu(x_2), \xi(x_1) \wedge \xi(x_2)] \geq \min[\mu(x_1), \nu(x_2), \xi(x_1) \wedge \xi(x)]$$

$$= \min[\mu(x_1), \nu(x_2), \xi(x)] \quad \text{since } \mu < \xi.$$

Therefore $\mu(\nu \wedge \xi)(x) \geq \min(\mu(x_1), \nu(x_2)) \wedge \xi(x)$.

Hence $\mu(\nu \wedge \xi)(x) \geq \mu\nu(x) \wedge \xi(x)$ and therefore $\mu(\nu \wedge \xi) = \mu\nu \wedge \xi$.

PROPOSITION : 3.1.6 (MODULAR LAW)

Let μ, ν and ξ be normal fuzzy subgroups of $\mathcal{G}$ with $\mu \leq \nu$. Suppose also that $\mu \wedge \xi = \nu \wedge \xi$ and $\mu\xi = \nu\xi$. Then $\mu = \nu$.

PROOF :

$$\begin{aligned}
\mu(x) = \mu^2(x) &= \sup_{x = x_1x_2} \mu(x_1) \wedge \mu(x_2) \\
&\geq \sup_{x = x_1x_2} \mu(x_1) \wedge (\mu \wedge \xi)(x_2) \\
&= \mu(\mu \wedge \xi)(x) \\
&= \mu(\nu \wedge \xi)(x) \\
&= \mu\xi \wedge \nu(x) \quad \text{by the Dedekind Law,} \\
&= (\nu\xi \wedge \nu)(x) \quad \text{since } \mu\xi = \nu\xi \\
&= \nu(x) \quad \text{since } \nu \leq \nu\xi
\end{aligned}$$

i.e. $\mu \geq \nu$. Therefore $\mu = \nu$.

The following obvious result will be useful later.

PROPOSITION : 3.1.7

Let μ, ν, ξ, α be fuzzy subgroups of $\mathcal{G}$. If $\mu \leq \nu$ and $\xi \leq \alpha$, then $\mu\xi \leq \nu\alpha$.

PROOF :

Straightforward.

PROPOSITION 3.1.8

Let μ, μ_1, μ_2 be fuzzy subgroups of $\mathcal{G}$ with $\mu_1, \mu_2 \triangleleft \mu$. Suppose also that $\mu_1 \wedge \mu_2 = \mu_e$, and $\mu = \mu_1\mu_2$. Then supp μ = supp $\mu_1 \otimes$ supp μ_2, where $\otimes$ denotes the internal direct product.

PROOF :

Straightforward.

This motivates the following definition :

DEFINITION : 3.1.9

Let μ, μ_1, μ_2 be fuzzy subgroups of $\mathcal{G}$. Then μ is the *fuzzy internal direct product* of μ_1 and μ_2, and we write $\mu = \mu_1 \otimes \mu_2$, if

(i) $\mu_1, \mu_2 \triangleleft \mu$,

(ii) $\mu = \mu_1\mu_2$

(iii) $\mu_1 \wedge \mu_2 = \mu_e$.

PROPOSITION : 3.1.10

Let $\mathcal{G}$ be a finite group with μ_1,μ_2 fuzzy subgroups of $\mathcal{G}$ and μ_1 fuzzy normal. Then $(\mu_1\mu_2)^t = \mu_1^t\mu_2^t$, where $\mu_i^t = \{x \in \mathcal{G} : \mu_i(x) \geq t\}$, i = 1,2.

PROOF :

Let $x \in \mu_1^t\mu_2^t$. Then there exist $x_1 \in \mu_1^t$ and $x_2 \in \mu_2^t$, $x = x_1x_2$, such that

$$\min(\mu_1(x_1), \mu_2(x_2)) \geq t.$$

Thus $\mu_1\mu_2(x) \geq t$, and hence $x \in (\mu_1\mu_2)^t$.

Conversely, let $x \in (\mu_1\mu_2)^t$; i.e. $\mu_1\mu_2(x) \geq t$.

Since $\mathcal{G}$ is finite, there exist $y_1,y_2 \in \mathcal{G}$, $x = y_1y_2$, such that $\mu_1\mu_2(x) = \mu_1(y_1) \wedge \mu_2(y_2)$.

So $\mu_1(y_1) \geq t$ and $\mu_2(y_2) \geq t$. Hence $x \in \mu_1^t\mu_2^t$.

This completes the proof.

PROPOSITION : 3.1.10′

Let $\mathcal{G}$ be a group with μ_1, μ_2 fuzzy subgroups of $\mathcal{G}$ and μ_1 fuzzy normal. If $\mu_1\mu_2$ has the sup property, then $(\mu_1\mu_2)^t = \mu_1^t\mu_2^t$, where $\mu_i^t = \{x \in \mathcal{G} : \mu_i(x) \geq t\}$, i = 1,2.

PROOF :

Similar to Proposition 3.1.10.

THEOREM : 3.1.11

Let $\mathscr{G}$ be a finite group and μ_1,μ_2 be fuzzy subgroups of $\mathscr{G}$, where μ_1 is fuzzy normal. Then $\mu_1\mu_2$ is a fuzzy internal direct product if and only if for each $t \in (0, \mu_1\mu_2(e)]$, $(\mu_1\mu_2)^t$ is the internal direct product of μ_1^t and μ_2^t.

PROOF :

($\Longrightarrow$): If $\mu_1\mu_2$ is a fuzzy internal direct product, then $\mu_1,\mu_2 \triangleleft \mu_1\mu_2$ and $\mu_1 \wedge \mu_2 = \mu_e$.
By proposition 3.1.10, $(\mu_1\mu_2)^t = \mu_1^t\mu_2^t$. Clearly $\mu_1^t \triangleleft \mathscr{G}$. By Proposition 1.3.4,
$\mu_2^t \triangleleft (\mu_1\mu_2)^t$.
Clearly $\mu_1^t \cap \mu_2^t = \{e\}$. So $(\mu_1\mu_2)^t = \mu_1^t \otimes \mu_2^t$.

($\Longleftarrow$): Conversely, suppose $(\mu_1\mu_2)^t = \mu_1^t \otimes \mu_2^t$ for all $t \in (0, \mu_1\mu_2(e)]$.
μ_1^t and $\mu_2^t \triangleleft \mu_1^t\mu_2^t$.

By proposition 1.3.4, $\mu_1,\mu_2 \triangleleft \mu_1\mu_2$. We now argue that $\mu_1 \wedge \mu_2 = \mu_e$.
Let $x \in \text{supp } \mu_1 \setminus \{e\}$. Thus $x \in \mu_1^t$ for some $t \in (0, \mu_1(e)]$ and $x \notin \mu_2^t$ since
$\mu_1^t \cap \mu_2^t = \{e\}$. Hence $0 \leq \mu_2(x) < t$.
We claim that $\mu_2(x) = 0$:
Suppose $0 < \mu_2(x) = t_1 < t$. Then $\mu_1^t \subsetneq \mu_1^{t_1}$. So $x \in \mu_1^{t_1}$ and $x \in \mu_2^{t_1}$.
Therefore $x = e$, a contradiction. Hence $\mu_2(x) = 0$.
If $x \notin \text{supp } \mu_1$, then $\mu_1(x) = 0$. Hence $\mu_1\mu_2 = \mu_1 \otimes \mu_2$.

THEOREM : 3.1.11′

Let $\mathscr{G}$ be a group and μ_1,μ_2 be fuzzy subgroups of $\mathscr{G}$ where μ_1 is fuzzy normal. If $\mu_1\mu_2$ has the sup property, then $\mu_1\mu_2$ is a fuzzy internal direct product if and only if for each $t \in (0, \mu(e)]$, $(\mu_1\mu_2)^t$ is the internal direct product of μ_1^t and μ_2^t.

PROOF :

Similar to Theorem 3.1.11.

REMARK : 3.1.12

For the converse of Theorem 3.1.11, $\mathcal{G}$ need not be finite.

DEFINITION : 3.1.13 [14]

Let μ and ν be fuzzy subgroups of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. The fuzzy external direct product of μ and ν is the mapping $\mu \times \nu : \mathcal{G}_1 \times \mathcal{G}_2 \longrightarrow [0,1]$ defined by $\mu \times \nu(x_1,x_2) = \mu(x_1) \wedge \nu(x_2)$, $x_i \in \mathcal{G}_i$, $i = 1,2$.

PROPOSITION : 3.1.14

(i) $\mu \times \nu$ is a fuzzy subgroup of $\mathcal{G}_1 \times \mathcal{G}_2$.

(ii) supp $(\mu \times \nu) =$ supp $\mu \times$ supp ν.

(iii) $(\mu \times \nu)^t = \mu^t \times \nu^t \; \forall \; t \in (0, \mu(e) \wedge \nu(e)]$.

PROOF :

Straightforward.

THEOREM : 3.1.15

Let μ and ν be fuzzy subgroups of $\mathcal{G}$. If $\mu\nu$ is a fuzzy internal direct product, then $\mu\nu = \mu \otimes \nu \simeq \mu \times \nu$.

PROOF :

By proposition 3.1.14, supp $\mu \times \nu =$ supp $\mu \times$ supp ν. By Proposition 3.1.8, supp $\mu \otimes \nu =$ supp $\mu \otimes$ supp ν. Define f : supp $\mu \times \nu \longrightarrow$ supp $\mu \otimes \nu$ by $f(a,b) = ab$ for $a \in$ supp μ, $b \in$ supp ν.

f is a crisp isomorphism.

We will show that $\mu \otimes \nu\,(f(a,b)) = \mu \times \nu((a,b))$:

$\mu \otimes \nu(f(a,b)) = \mu \otimes \nu(ab) = \sup_{ab\,=\,xy} \mu(x) \wedge \nu(y) > 0$ since $a \in \text{supp}\,\mu$ and $b \in \text{supp}\,\nu$.

So we can assume that $x \in \text{supp}\,\mu$ and $y \in \text{supp}\,\nu$. Hence, if $ab = xy$, then $a = x$ and $b = y$. So $\mu \otimes \nu(f(a,b)) = \mu(a) \wedge \nu(b) = \mu \times \nu(a,b)$.

This completes the proof.

We end this section by defining a direct product of more than two fuzzy subgroups.

DEFINITION : 3.1.16

Let $\{\mu_i : i = 1,2,\cdots,n\}$ be a collection of fuzzy subgroups of $\mathcal{G}$. Let μ be a fuzzy subgroup of $\mathcal{G}$. Then μ is the *internal direct product* of the μ_i, $i = 1,\cdots,n$, and we write $\mu = \mu_1 \otimes \cdots \otimes \mu_n$, in case

(i) $\mu_i \triangleleft \mu$ for all $i = 1,\cdots,n$,

(ii) $\mu_i \wedge < \bigvee_{i \neq j} \mu_j > = \mu_e$, $i = 1,\cdots,n$, and

(iii) $\mu = < \bigvee_{i=1}^{n} \mu_i >$

Further results on direct products will be given in Chapter 4.

We first need the isomorphism theorems.

3.2 THE ISOMORPHISM THEOREMS

DEFINITION : 3.2.1

Let $f : \mathcal{G} \longrightarrow \mathcal{G}^1$ be a homomorphism of a group $\mathcal{G}$ into a group $\mathcal{G}^1$. Let μ be a fuzzy subgroup of $\mathcal{G}$. The *fuzzy kernel* of f corresponding to μ is the fuzzy subgroup μ_E of $\mathcal{G}$ defined by

$$\mu_E(x) = \begin{cases} \mu(x) & x \in \ker f \\ 0 & x \notin \ker f \end{cases},$$

where ker f denotes the usual crisp kernel of f.

PROPOSITION : 3.2.2

$\mu_E = f^{-1}(f(\mu)_e) \wedge \mu$, where $f(\mu)_e$ is the fuzzy point of $\mathcal{G}^1$ having support f(e).

PROOF :

$[f^{-1}(f(\mu)_e) \wedge \mu](x) = \mu(x) \wedge f(\mu)_e(f(x))$.

For $x \in \ker f$, $[f^{-1}(f(\mu)_e \wedge \mu](x) = \mu(x) \wedge f(\mu)(f(e))$
$= \mu(x) \wedge \mu(e)$
$= \mu(x)$,

otherwise $x \notin \ker f$ and $[f^{-1}(f(\mu)_e \wedge \mu](x) = \mu(x) \wedge 0 = 0$.

Therefore $[f^{-1}(f(\mu)_e \wedge \mu](x) = \mu_E(x)$.

PROPOSITION : 3.2.3

The fuzzy kernel $\mu_E \triangleleft \mu$.

DEFINITION : 3.2.4

Let μ and ν be fuzzy subgroups of $\mathcal{G}$ and $\mathcal{G}^1$ respectively. A homomorphism $f : \text{supp } \mu \longrightarrow \text{supp } \nu$ is a *fuzzy epimorphism* of μ onto ν if $f(\mu) = \nu$. (We usually write $f : \mu \longrightarrow \nu$).

THEOREM : 3.2.5 (THE FIRST ISOMORPHISM THEOREM)

Let $\mathcal{G}$ and $\mathcal{H}$ be groups. Let μ and ν be fuzzy subgroups of $\mathcal{G}$ and $\mathcal{H}$ respectively. Let $f : \mu \longrightarrow \nu$ be a fuzzy epimorphism. Let μ_E be the fuzzy kernel of f corresponding to μ. Then $^{\mu}/\mu_E \simeq \nu$.

PROOF :

$\text{supp } \mu_E = \ker f$.

$\text{supp } \mu/\mu_E = \text{supp } \mu/\text{supp } \mu_E$ and μ/μ_E is indeed a fuzzy subgroup of $\text{supp } \mu/\mu_E$.

Define $\psi : \text{supp } \mu/\mu_E \longrightarrow \text{supp } \nu$ by

$$\psi(x \text{ supp } \mu_E) = f(x) \quad \text{for all } x \in \text{supp } \mu.$$

If x supp $\mu_E \in$ supp μ/μ_E, then μ/μ_E (x supp $\mu_E) > 0$.

Hence $\sup\{\mu(a)$: a supp $\mu_E =$ x supp $\mu_E\} > 0$. Therefore $f(\mu)(f(x)) > 0$,

i.e. $f(x) \in$ supp ν. Therefore ψ is well–defined.

Clearly ψ is a bijective homomorphism, i.e. supp $\mu/\mu_E \simeq$ supp ν.

$$\mu/\mu_E \text{ (x supp } \mu_E) = \sup_{f(a) = f(x)} \mu(a) = f(\mu)(f(x)) = \nu(\psi(\text{x supp } \mu_E)).$$

i.e. $\mu/\mu_E \simeq \nu$.

REMARK : 3.2.6

Theorem 3.2.5 still holds in terms of the strong quotient given in definition 1.1.15 [1].

THEOREM : 3.2.7 (THE SECOND ISOMORPHISM THEOREM)

Let μ and ν be fuzzy subgroups of $\mathcal{G}$, with ν fuzzy normal. Then

(a) $(\mu \wedge \nu) \triangleleft \mu$, and

(b) $\mu\nu/\nu \simeq \mu/(\mu \wedge \nu)$.

PROOF :

(a) Clear.

(b) Define f : supp $\mu\nu \longrightarrow$ supp $\mu/(\mu \wedge \nu)$ by $f(x_1x_2) = x_1$ supp$(\mu \wedge \nu)$, where $x_1 \in$ supp μ, $x_2 \in$ supp ν. f is a well–defined crisp isomorphism. We claim that $f(\mu\nu)(f(x)) = \mu/(\mu \wedge \nu)(f(x))$, $x \in$ supp $\mu\nu$:

$f(x) = f(x_1x_2) = x_1$ supp$(\mu \wedge \nu)$.

Let $\alpha_1 = f(\mu\nu)(f(x))$ and $\alpha_2 = \mu/(\mu \wedge \nu)(f(x))$.

$\alpha_1 = \sup\{\mu\nu(a)\colon f(a) = f(x)\} = \sup\{\mu\nu(a)\colon a_1$ supp $(\mu \wedge \nu) = x_1$ supp $(\mu \wedge \nu)$,

$a = a_1a_2$, $a_1 \in$ supp μ, $a_2 \in$ supp $\nu\}$.

Let $\epsilon \in (0, \alpha_1 \wedge \alpha_2)$. So there exists y, $y = y_1y_2$ and

y_1 supp $\mu \wedge \nu = x_1$

supp $\mu \wedge \nu$, such that $\alpha_1 - \epsilon < \mu\nu(y) = \sup_{y = z_1z_2} \mu(z_1) \wedge \nu(z_2)$.

Let $\beta_1 = \alpha_1 - \mu\nu(y)$. So $\alpha_1 = \beta_1 + \mu\nu(y)$. We can assume that ϵ is small enough so that $\mu\nu(y) > \epsilon$. So there exist z_1, z_2, $y = z_1 z_2$, such that

$\mu\nu(y) - \epsilon < \mu(z_1) \wedge \nu(z_2)$.

$f(y) = z_1 \operatorname{supp} \mu \wedge \nu = x_1 \operatorname{supp} \mu \wedge \nu$.

Let $\beta_2 = \mu\nu(y) - \mu(z_1) \wedge \nu(z_2) < \epsilon$. Therefore $\mu\nu(y) = \beta_2 + \mu(z_1) \wedge \nu(z_2)$.

$\alpha_2 = \sup\limits_{f(a) = f(x)} \mu(a)$ implies that there exists a_0, $f(a_0) = f(x)$, such that $\alpha_2 - \epsilon < \mu(a_0)$. Let $a_0 = b_1 b_2$, $b_1 \operatorname{supp} \mu \wedge \nu = x_1 \operatorname{supp} \mu \wedge \nu$.

Let $\beta_3 = \alpha_2 - \mu(a_0) < \epsilon$.

$$
\begin{aligned}
\text{Therefore } \alpha_2 &= \mu(a_0) + \beta_3 \\
&\geq \mu(z_1) + \beta_3 \text{ (otherwise use } z_1 \text{ in the place of } a_0\text{)}, \\
&\geq \mu(z_1) \wedge \nu(z_2) + \beta_3 \\
&> \mu\nu(y) - \epsilon + \beta_3 \\
&= \alpha_1 - \beta_1 - \epsilon + \beta_3.
\end{aligned}
$$

As $\epsilon \longrightarrow 0$, $\beta_i \longrightarrow 0$, $i = 1,3$. Hence $\alpha_2 \geq \alpha_1$. (1)

$$
\begin{aligned}
\alpha_1 = f(\mu\nu)(f(x)) &= \sup\{\mu\nu\ (a_1 a_2) : a_1 \operatorname{supp} \mu \wedge \nu = x_1 \operatorname{supp} \mu \wedge \nu\} \\
&\geq \sup\{\mu\ (a_1 a_2) : a_1 \operatorname{supp} \mu \wedge \nu = x_1 \operatorname{supp} \mu \wedge \nu\} \\
&= \mu/(\mu \wedge \nu)(f(x)) = \alpha_2.
\end{aligned}
$$

Therefore $\alpha_1 \geq \alpha_2$ (2)

(1) and (2) imply that $\alpha_1 = \alpha_2$.

Now $\ker f = \operatorname{supp} \nu$.

$$
\text{Let } (\mu\nu)_E(x) = \begin{cases} \mu\nu(x) & x \in \ker f \\ 0 & x \notin \ker f \end{cases}
$$

By the first isomorphism Theorem (Theorem 3.2.5)

$$(\mu\nu)/(\mu\nu)_E \simeq \mu/\mu \wedge \nu.$$

Furthermore $\mu\nu/(\mu\nu)_E \simeq \mu\nu/\nu$:

Let $\alpha : \operatorname{supp} \mu\nu/(\mu\nu)_E \longrightarrow \operatorname{supp} \mu\nu/\nu$ be given by

$\alpha(x_1 x_2 \operatorname{supp} (\mu\nu)_E) = x_1 x_2 \operatorname{supp} \nu = x_1 x_2 \operatorname{supp} (\mu\nu)_E$. Therefore α is a crisp isomorphism.

$\mu\nu/\nu\ (\alpha(x_1x_2\ \text{supp}\ (\mu\nu)_E)) = \mu\nu/\nu(x_1x_2\ \text{supp}\ \nu)$

$= \sup\{\mu\nu(a): a\ \text{supp}\ \nu = x_1x_2\ \text{supp}\ \nu\} = \sup\{\mu\nu(a): a\ \text{supp}\ (\mu\nu)_E$

$= x_1x_2\ \text{supp}\ (\mu\nu)_E\} = (\mu\nu)/(\mu\nu)_E\ (x_1x_2\ \text{supp}\ (\mu\nu)_E)$.

So $\mu\nu/(\mu\nu)_E \simeq \mu\nu/\nu$ and it follows that $\mu\nu/\nu \simeq \mu/\mu \wedge \nu$.

REMARK : 3.2.8

The assumption made throughout this thesis that $\mu(e) = \nu(e)$ for fuzzy subgroups of the same group $\mathscr{G}$ that are mentioned in a theorem has played a role in the proof of Theorem 3.2.7.

The following example shows that if we use the quotient given in Definition 1.1.15[1], the second isomorphism theorem fails :

EXAMPLE : 3.2.9

Let $\mathscr{G} = S_3 = \{e, a, a^2, b, ab, a^2b\}$, $a^3 = e = b^2$.

Define $\mu : S_3 \longrightarrow [0,1]$ by $\mu = \chi_{A_3}$. Then μ is a fuzzy normal subgroup of $\mathscr{G}$.

Define $\nu : S_3 \longrightarrow [0,1]$ by $\nu(e) = 1$, $\nu(a) = 1/2 = \nu(a^2)$, $\nu(b) = 1/4 = \nu(ab) = \nu(a^2b)$.

Then ν is a fuzzy normal subgroup of S_3.

Let $E\nu = \{x \in S_3 : \nu(x) = \nu(e)\}$. Supp $\mu = A_3 = \{e, a, a^2\}$. Supp $\nu = S_3$.

$$\begin{aligned}
\text{supp}\ \mu\nu/\nu &= \{x_1x_2\ \nu : x_1 \in \text{supp}\ \mu,\ x_2 \in \text{supp}\ \nu\} \\
&\simeq \{x_1x_2\ E\nu : x_1 \in \text{supp}\ \nu,\ x_2 \in \text{supp}\ \nu\} \\
&= (\text{supp}\ \mu\ \text{supp}\ \nu)/E\nu \\
&= \text{supp}\ \mu/E\nu \cdot \text{supp}\ \nu/E\ \nu \\
&= A_3/E\nu \cdot S_3/E\nu = S_3/E\nu \simeq S_3.
\end{aligned}$$

$$\begin{aligned}
\text{But supp}\ \mu/\mu \wedge \nu &= \{x(\mu \wedge \nu) : x \in \text{supp}\ \mu\} \\
&\simeq \{x\ E_{\mu \wedge \nu} : x \in \text{supp}\ \mu\} \\
&= A_3/E_{\mu \wedge \nu} \simeq A_3.
\end{aligned}$$

Therefore supp $\mu\nu/\nu$ is not isomorphic to supp $\mu/\mu \wedge \nu$.

Hence $\mu\nu/\nu$ is not isomorphic to $\mu/\mu \wedge \nu$.

THEOREM : 3.2.10 (THE THIRD ISOMORPHISM THEOREM)

Let $\nu \leq \mu \leq \xi$ be fuzzy subgroups of $\mathcal{G}$ such that μ and ν are normal in ξ. Then

(i) $\mu/\nu \triangleleft \xi/\nu$ and

(ii) $\xi/\mu \simeq (\xi/\nu)/(\mu/\nu)$.

PROOF :

(i) Let $(a \text{ supp } \nu)_\lambda \in \xi/\nu$. We can assume that $a_\lambda \in \xi$.

$(a \text{ supp } \nu)_\lambda \; \mu/\nu(x \text{ supp } \nu)$

$= \lambda \wedge \mu/\nu(a^{-1}x \text{ supp } \nu) = \lambda \wedge \sup\{\mu(y): y \text{ supp } \nu = a^{-1}x \text{ supp } \nu\}$

$= \sup\{\lambda \wedge \mu(a^{-1}xy): y \in \text{supp } \nu\} = \sup\{a_\lambda \, \mu(xy): y \in \text{supp } \nu\}$

$= \sup\{\mu a_\lambda(xy): y \in \text{supp } \nu\}$ since $\mu \triangleleft \xi$

$= \sup\{\mu a_\lambda(z): z \text{ supp } \nu = x \text{ supp } \nu\}$

$= \lambda \wedge \sup\{\mu(za^{-1}): z \text{ supp } \nu = x \text{ supp } \nu\}$

$= \lambda \wedge \sup\{\mu(y): ya \text{ supp } \nu = x \text{ supp } \nu\}$

$= \mu/\nu \, (a \text{ supp } \nu)_\lambda \, (x \text{ supp } \nu)$.

(ii) Define $f : \text{supp } \xi/\nu \longrightarrow \text{supp } \xi/\mu$ by $f(x \text{ supp } \nu) = x \text{ supp } \mu$. f is an epimorphism.

We claim that $f(\xi/\nu) = \xi/\mu$:

$$f(\xi/\nu)(x \text{ supp } \mu) = \sup\{\xi/\nu(a \text{ supp } \nu): a \text{ supp } \mu = x \text{ supp } \mu\}$$

$$= \sup_{a \text{ supp } \mu \,=\, x \text{ supp } \mu} \sup\{\xi(a'): a' \text{ supp } \nu = x \text{ supp } \nu\}$$

$$= \sup\{\xi(b): b \text{ supp } \mu = x \text{ supp } \mu\} = \xi/\mu \; (x \text{ supp } \mu).$$

$\ker f = \{x \text{ supp } \nu : x \in \text{supp } \mu\} = \text{supp } \mu/\nu$.

$$\text{Define } (\xi/\nu)_E \, (x \text{ supp } \nu) = \begin{cases} (\xi/\nu)(x \text{ supp } \nu) & x \in \text{supp } \mu \\ 0 & \text{otherwise} \end{cases}.$$

By the First Isomorphism Theorem (Theorem 3.2.5), $(\xi/\nu)/(\xi/\nu)_E \simeq \xi/\mu$.

We claim that $(\xi/\nu)/(\xi/\nu)_E \simeq (\xi/\nu)/(\mu/\nu)$:

Define $\alpha : \text{supp } (\xi/\nu)/(\xi/\nu)_E \longrightarrow \text{supp } (\xi/\nu)/(\mu/\nu)$ by

$$\alpha(x \text{ supp } \nu \text{ supp } (\xi/\nu)_E) = x \text{ supp } \nu \text{ supp } \mu/\nu.$$

α is a bijective homomorphism since supp $(\xi/\nu)_E$ = supp μ/ν.

$(\xi/\nu)/(\xi/\nu)_E$ (x supp ν supp $(\xi/\nu)_E$)

$= \sup\{\xi/\nu$(a supp ν): a supp ν supp $(\xi/\nu)_E$ = x supp ν supp $(\xi/\nu)_E\}$

$= (\xi/\nu)/(\mu/\nu)$ (α(x supp ν supp $(\xi/\nu)_E$).

Hence $(\xi/\nu)/(\mu/\nu) \simeq (\xi/\nu)$.

REMARK : 3.2.11

The above Theorem is still valid even if the quotients are replaced by the strong quotients given in Definition 1.1.15 [1].

CHAPTER 4

CYCLIC FUZZY SUBGROUPS AND THE BASIS THEOREM

INTRODUCTION :

In this chapter we first discuss cyclic fuzzy subgroups. In [50] Sidky and Mishref defined a fuzzy subgroup μ to be cyclic iff each level subgroup of μ is cyclic. Our notion of fuzzy cyclic is such that if μ is fuzzy cyclic, then all the nonzero level subgroups of μ are cyclic, but the zero level subgroup of μ need not be cyclic. Furthermore, our definition of fuzzy cyclic uses the notion of a fuzzy subgroup generated by a fuzzy subset. We also attempt to give characterizations of cyclic fuzzy subgroups and finitely generated fuzzy subgroups. Examples of cyclic fuzzy subgroups and fuzzy direct products are presented. We also define p–fuzzy subgroups and then prove that every finite Abelian fuzzy subgroup is a direct sum of p–fuzzy subgroups. The major result in this chapter is the Basis Theorem which asserts that every finite Abelian fuzzy subgroup is a direct sum of cyclic p–fuzzy subgroups. We end the chapter with a notion of dimension of a fuzzy subgroup that is also a fuzzy vector space over the field $\mathbb{Z}_p$, where p is a prime number.

4.1 : CYCLIC FUZZY SUBGROUPS

Let $\mathcal{G}$ be a group and x_λ a fuzzy point in $\mathcal{G}$. Recall that $< x_\lambda >$ is the smallest fuzzy subgroup of $\mathcal{G}$ containing x_λ. A fuzzy subgroup μ is cyclic in $\mathcal{G}$ if there exists a fuzzy point x_λ such that $\mu = < x_\lambda >$.

PROPOSITION : 4.1.1

Let $\mu = < x_\lambda >$ and

$$\nu(a) = \begin{cases} \lambda & a \in < x > \\ 0 & a \notin < x > \end{cases} \quad \forall\, a \in \mathcal{G}.$$

Then $\mu = \nu$.

PROOF :

Let $a_\beta \in \nu$. If $a = e$, then $e \in < x >$. Hence $\nu(e) = \lambda \geq \beta$. Now $\mu(x) \geq \lambda$ and $\mu(e) \geq \mu(x) \geq \lambda \geq \beta$. So $a_\beta = e_\beta \in \mu$.

Suppose $a \neq e$.
$\nu(a) \geq \beta > 0$. Hence $\nu(a) = \lambda \geq \beta$ and $a \in < x >$. So $a = x^m$ for some $m \in \mathbb{Z}$.

Therefore $\mu(a) \geq \mu(x) \geq \lambda \geq \beta$. So $a_\beta \in \mu$. Therefore $\nu \leq \mu$.

Now $x_\lambda \in \nu$ since $x \in < x >$. By definition of μ, $\mu \leq \nu$.

Hence $\mu = \nu$.

DEFINITION : 4.1.2

Let μ and ν be fuzzy subgroups of $\mathcal{G}$ such that $\nu \leq \mu$. ν *is cyclic in* μ in case there exists a fuzzy point $x_\lambda \in \mu$ such that $\nu = < x_\lambda >$.

We can apply proposition 4.1.1 to show that

$$\nu(a) = \begin{cases} \lambda & a \in < x >, a \neq e \\ \mu(e) & a = e \\ 0 & \text{otherwise} \end{cases},$$

where ν is the fuzzy subgroup given in Definition 4.1.2.

NOTE : (i) The condition $\nu(e) = \mu(e)$ is important in products of fuzzy subgroups.

(ii) By $A \backslash B$ we mean $\{x \in A : x \notin B\}$.

PROPOSITION : 4.1.3

Let $x_\lambda, y_\beta \in [0,1]^{\mathcal{G}}$, $x \neq y$.

Assume that $\mathcal{G}$ is Abelian and finite.

Let
$$\nu(a) = \begin{cases} \lambda & a \in < x > \backslash < y > \\ \beta & a \in < y > \backslash < x > \\ \lambda \vee \beta & a \in < x > \cap < y > \\ \lambda \wedge \beta & a \in < x >< y > \backslash < x > \cup < y > \\ 0 & \text{otherwise} \end{cases}.$$

Then $\nu = < x_\lambda, y_\beta >$ in $\mathcal{G}$.

PROOF :

Let $\mu = < x_\lambda, y_\beta >$.

We first note that $\nu(e) = \lambda \vee \beta$. Secondly $\exists$ $n \in \mathbb{Z}^+$ such that $y^n = e$.
So $e = y \cdots y$, hence $\mu(e) = < x_\lambda, y_\beta >(e) \geq \lambda \vee \beta = \nu(e)$.

By definition of $< x_\lambda, y_\beta >$, $\mu(a) \leq \lambda \vee \beta \;\; \forall \;\; a \in \mathcal{G}$.

Hence $\mu(e) = \lambda \vee \beta = \nu(e)$.
Let a, b $\in \mathcal{G}$. We want to show that ν is a fuzzy subgroup of $\mathcal{G}$.

Case a, b $\in < x > \setminus < y >$:

$ab \in < x >$: so $\nu(ab) = \lambda$ or $\lambda \vee \beta$.
$\lambda = \nu(a)$ implies that $\nu(ab) \geq \nu(a) \wedge \nu(b)$.
Similarly if a, b $\in < y > \setminus < x >$.

Case a, b $\in < x > \cap < y >$:

$ab \in < x > \cap < y >$.
Therefore $\nu(ab) = \lambda \vee \beta \geq \nu(a) \wedge \nu(b)$.

Case a, b $\in < x >< y > \setminus < x > \cup < y >$.

Therefore $ab \in < x >< y >$ since $\mathcal{G}$ is Abelian.
Now $\nu(a) \wedge \nu(b) = \lambda \wedge \beta \leq \nu(ab)$.

Case a $\in < x > \setminus < y >$, b $\in < y > \setminus < x >$:

Therefore $ab \in < x >< y >$. So this case is similar to the previous case.

Case a $\in < x > \setminus < y >$ and b $\in < x >< y > \setminus < x > \cup < y >$:

Similar to the previous case.

Case a $\in < x > \setminus < y >$ and b $\in < x > \cap < y >$:

Therefore $ab \in < x >$. So $\nu(ab) = \lambda$ or $\lambda \vee \beta$. Either case implies
$\nu(ab) \geq \nu(a) \wedge \nu(b) = \lambda$.

Case a $\notin < x >< y >$ and b $\in < x >< y >$:

Then $\nu(a) = 0 \leq \nu(ab)$. It is easy to show that $\nu(a) = \nu(a^{-1}) \;\; \forall \;\; a \in \mathcal{G}$.
So ν is indeed a fuzzy subgroup of $\mathcal{G}$. ν contains x_λ and y_β. Hence $\nu \geq \mu$.

Let $a_\xi \in \nu$. So $\nu(a) \geq \xi$.

Case (i) : $a \in \langle x \rangle \setminus \langle y \rangle$.
Therefore $a = x^m$ for some $m \in \mathbb{Z}$.
$\mu(a) = \sup\{\rho: a_\rho = a_{1\lambda_1} \cdots a_{n\lambda_1}, a_{i\lambda_i} \in x_\lambda \vee y_\beta\}$.
So $\lambda_i \leq \lambda \vee \beta$, $i = 1,\cdots,n$.
$a_\xi = x_\xi \cdots x_\xi$ and $\nu(a) = \lambda \geq \xi$.
Since $x_\lambda \in \mu$, then $x_\xi \in \mu$. Hence $a_\xi \in \mu$. Similarly if $a \in \langle y \rangle \setminus \langle x \rangle$.

Case (ii) : $a \in \langle x \rangle \cap \langle y \rangle$.
Similar to Case (i).

Case (iii) : $a \in \langle x \rangle\langle y \rangle \setminus \langle x \rangle \cup \langle y \rangle$.
$\nu(a) = \lambda \wedge \beta \geq \xi$ since $a_\xi \in \nu$.
$a = x^m y^n = (xy)^m y^{n-m}$, say, since $\mathcal{G}$ is Abelian.
Therefore $a_\xi = (xy)_\xi \cdots (xy)_\xi y_\xi \cdots y_\xi$.
$(xy)_{\lambda \wedge \beta} \in \mu$ since $x_\lambda, y_\beta \in \mu$.
Hence $(xy)_\xi, y_\xi \in \mu$. So $a_\xi \in \mu$. Therefore $\nu \leq \mu$.
Hence $\mu = \nu$.

The proof is complete.

DEFINITION : 4.1.4

A fuzzy subgroup μ of $\mathcal{G}$ is a *p-fuzzy subgroup* iff each nonzero level subgroup of μ is a p–group, where p is a fixed prime.

Clearly μ is a p–fuzzy subgroup iff supp μ is a p–group.

We aim to show that every fuzzy subgroup of a finite Abelian group is a direct sum of cyclic p–fuzzy subgroups. We begin by examining certain specific examples.

EXAMPLE : 4.1.5

Let $\mathcal{G}$ be the Klein 4–group. So $\mathcal{G} = \{0, x_1, x_2, x_3\}$, where $2x_i = 0$ for all $i = 1,2,3$, and $x_i + x_j = x_k$, i, j, k distinct.

Define $\mu : \mathcal{G} \longrightarrow [0,1]$ by

$$\mu(x) = \begin{cases} \alpha_0 & x = 0 \\ \alpha_1 & x = x_1, x_2 \\ \alpha_3 & x = x_3 \end{cases}$$

where $\alpha_0 > \alpha_3 > \alpha_1$.

Then μ is a fuzzy subgroup of $\mathcal{G}$.
Define $\mu_i : \mathcal{G} \longrightarrow [0,1]$ by

$$\mu_i = \begin{cases} \mu(x) & x \in \langle x_i \rangle \\ 0 & x \notin \langle x_i \rangle, \end{cases} \quad i = 1,2,3.$$

So

$$\mu_1(x) = \begin{cases} \alpha_0 & x = 0 \\ \alpha_1 & x \in \langle x_1 \rangle,\ x \neq 0 \\ 0 & x \notin \langle x_1 \rangle \end{cases}$$

Therefore μ_1 is cyclic in μ. Similarly, μ_2 and μ_3 are cyclic in μ.

$\mu_i \triangleleft \mu$ since $\mathcal{G}$ is Abelian.

It is not hard to see that $\mu_1 + \mu_2 + \mu_3 = \mu$.

NOTE : The + sign has replaced the product sign as is customary when dealing with Abelian groups.

The sum $\mu_1 + \mu_2 + \mu_3$ is not direct since for example $x_{3\lambda} \in (\mu_1 + \mu_2) \wedge \mu_3$ for some $\lambda \in (0,1]$.

We claim that $\mu = \mu_2 + \mu_3$ but $\mu \neq \mu_1 + \mu_2$:

$\mu(x_3) = \alpha_3$ and $(\mu_1 + \mu_2)(x_3) = (\mu_1 + \mu_2)(x_1 + x_2) = \mu_1(x_1) \wedge \mu_2(x_2) = \alpha_1 \neq \alpha_3$.

So $\mu \neq \mu_1 + \mu_2$.

Now $(\mu_2 + \mu_3)(x_1) = (\mu_2 + \mu_3)(x_2 + x_3) = \mu_2(x_2) \wedge \mu_3(x_3)$
$= \mu_2(x_2) = \mu(x_2) = \mu(x_1)$.

Clearly $(\mu_2 + \mu_3)(x_i) = \mu(x_i)$, i = 2,3.

Hence $\mu = \mu_2 + \mu_3$. In fact $\mu = \mu_2 \oplus \mu_3$.

Each μ_i is a 2–fuzzy subgroup.

EXAMPLE 4.1.6

Let $\mathcal{G} = \mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$. To avoid clumsiness we will omit the bars above the elements of $\mathbb{Z}_6$.

Let $H_1 = \{0,3\}$ and $H_2 = \{0,2,4\}$.

Define $\mu : \mathcal{G} \longrightarrow [0,1]$ by

$$\mu(x) = \begin{cases} \alpha_1 & x = 1,2,4,5 \\ \alpha_3 & x = 3 \\ 1 & x = 0 \end{cases}$$

where $1 > \alpha_3 > \alpha_1 > 0$. Then μ is a fuzzy subgroup of $\mathcal{G}$.

Let
$$\mu_i(x) = \begin{cases} \mu(x) & x \in H_i \\ 0 & x \notin H_i \end{cases}$$

μ_1 is a cyclic 2–fuzzy subgroup of μ, and μ_2 is a cyclic 3–fuzzy subgroup of μ.

We now show that $\mu = \mu_1 + \mu_2$.

$(\mu_1 + \mu_2)(5) = (\mu_1 + \mu_2)(3+2) = \mu_1(3) \wedge \mu_2(2) = \alpha_1 = \mu(5)$

$(\mu_1 + \mu_2)(4) = \mu_1(0) \wedge \mu_2(4) = \mu_2(4) = \mu(4)$.

Similarly $(\mu_1 + \mu_2)(x) = \mu(x)$ for all $x \in \mathcal{G}$.

Hence $\mu = \mu_1 \oplus \mu_2$.

4.2 : THE BASIS THEOREM

DEFINITION : 4.2.1

μ is finite in case supp μ is finite.

DEFINITION : 4.2.2

Let $m \in \mathbb{Z}^+$ and μ a fuzzy subgroup of $\mathcal{G}$. Then $m\mu$ is defined by

$$(m\mu)(x) = \begin{cases} \sup\{\beta : y_\beta \in \mu \text{ and } x = my\} \\ 0 \quad \text{otherwise} \end{cases}$$

It is then easy to see that

$$(m\mu)(x) = \begin{cases} \sup\{\mu(y) : x = my,\ y \in \operatorname{supp} \mu\} \\ 0 \qquad \text{otherwise} \end{cases}$$

Also $mx_\beta = (mx)_\beta$.

$(m\mu)(x) = \sup_{x = my} \mu(y) \leq \sup_{x = my} \mu(my) = \mu(x)$.

So $m\mu \leq \mu$.

PROPOSITION : 4.2.3

Let μ be a fuzzy subgroup of an Abelian group $\mathcal{G}$, and let $m \in \mathbb{Z}^+$. Then $m\mu$ is a fuzzy subgroup of $\mathcal{G}$.

PROOF :

Obvious.

THEOREM : 4.2.4

Every finite Abelian fuzzy subgroup μ is a direct sum of p–fuzzy subgroups.

PROOF :

Let $A_p = \{x_\lambda \in \mu : o(x) = p^s \text{ for some } s \in \mathbb{Z}^+\}$.

Let p be a prime number and define $\mu_p : \mathcal{G} \longrightarrow [0,1]$ by $\mu_p(x) = \sup\{\lambda : x_\lambda \in A_p\}$, otherwise $\mu_p(x) = 0$.

We will show that μ_p is a fuzzy subgroup of $\mathcal{G}$. Now $\mu_p(x+y) = \bigvee_{(x+y)_\lambda \in A_p} \lambda$,

$\mu_p(x) = \bigvee_{x_{\lambda_1} \in A_p} \lambda_1$ and $\mu_p(y) = \bigvee_{y_{\lambda_2} \in A_p} \lambda_2$.

Case (i) : $x, y \in \text{supp}\ \mu$.

$o(x+y) \mid \ell cm(o(x), o(y)) = p^s$ for some $s \in \mathbb{Z}^+$, and $\mu(x+y) \geq \mu(x) \wedge \mu(y) \geq \lambda_1 \wedge \lambda_2$.

Hence $(x+y)_{\lambda_1 \wedge \lambda_2} \in A_p$.

$$\text{Therefore } \mu_p(x) \wedge \mu_p(y) = \bigvee_{x_{\lambda_1} \in A_p} \bigvee_{y_{\lambda_2} \in A_p} (\lambda_1 \wedge \lambda_2) \leq \bigvee_{(x+y)_\lambda \in A_p} \lambda.$$

Case (ii) : $x \in \text{supp}\ \mu$, $y \notin \text{supp}\ \mu$.

So $\mu_p(y) = 0$. Hence $\mu_p(x+y) \geq \mu_p(x) \wedge \mu_p(y)$.

If $x, y \notin \text{supp}\ \mu$, then $\mu_p(x+y) \geq \mu_p(x) \wedge \mu_p(y)$. So μ_p is a fuzzy subgroup of $\mathcal{G}$ and $\mu_p \leq \mu$.

We now show that $\mu = \sum_{p \mid o(\text{supp}\ \mu)} \mu_p$:

Let $x_\lambda \in \mu$, $x \neq 0$ and $o(x) = n$.

Therefore $n = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, where the p_i's are distinct primes and $s_i \in \mathbb{Z}^+$ for all $i = 1, \cdots, k$. Set $n_i = \dfrac{n}{p_i^{s_i}}$. So $n_i p_i^{s_i} = n$.

Also $(n_1, n_2, \cdots, n_k) = 1$ implies there exist $m_i \in \mathbb{Z}$, $i = 1, \cdots, k$, such that $\sum_{i=1}^{k} m_i n_i = 1$.

$$\text{Hence } \sum_{i=1}^{k} m_i n_i\, x = x \qquad (*)$$

Now $p_i^{s_i} m_i n_i\, x = n\, m_i\, x = m_i(nx) = 0$.

So $o(m_i n_i\, x) \mid p_i^{s_i}$. Therefore $o(m_i n_i\, x) = p_i^{t_i}$ for some $t_i \in \mathbb{Z}^+$.

Clearly $(m_i n_i\, x)_\lambda \in \mu$ since $x_\lambda \in \mu$.

Let us first show that

$$\mu_p(x) = \begin{cases} \mu(x) & x \in \text{supp } \mu \text{ such that } o(x) = p^s \text{ for some } s \in \mathbb{Z}^+. \\ 0 & \text{otherwise} \end{cases}$$

Now let $a_\beta \in \mu$, $o(a) = p^s$. $\mu_p(a) = \bigvee_{a_\xi \in A_p,\ A_p = \{x_\lambda \in \mu \,:\, o(x) = p^s\}} \xi$.

So $\mu_p(a) \geq \beta$, i.e. $a_\beta \in \mu_p$.

$$\begin{aligned}
\text{Now} \quad \mu(x) &\geq \mu_{p_1} + \cdots + \mu_{p_k}(m_1n_1\,x + \cdots + m_kn_k\,x), \text{ see } (*), \\
&\geq \mu_{p_1}(m_1n_1\,x) \wedge \cdots \wedge \mu_{p_k}(m_kn_k\,x) \\
&= \mu(m_1n_1\,x) \wedge \cdots \wedge \mu(m_kn_k\,x) \\
&\geq \mu(x). \\
\text{Hence } \mu(x) &= \mu_{p_1}(m_1n_1\,x) \wedge \cdots \wedge \mu_{p_k}(m_kn_k\,x) \\
&= \left(\sum_{i=1}^{k} \mu_{p_i}\right)(x).
\end{aligned}$$

The fact that $< \bigvee_{i \neq j} \mu_i > \wedge\ \mu_j = \mu_e$ follows straight from Group Theory since supp μ = supp $\mu_1 + \cdots +$ supp μ_k.

This completes the proof.

RECALL :

Let V be a vector space over a field F. μ is a fuzzy vector space of V in case μ is a fuzzy subgroup of V under addition such that $\mu(\alpha\, x) \geq \mu(x)$ for all $\alpha \in F$ and $x \in V$.

REMARK : 4.2.5

(i) We are assuming that $\mathbb{Z}_p = \{0,1,2,\cdots,p-1\}$ under addition and multiplication modulo p.

(ii) If supp $\mu = H_1 \oplus H_2$ and we define fuzzy subgroups μ_i by $\mu_i(x) = \mu(x)$ if $x \in H_i$ and $\mu_i(x) = 0$ for $x \notin H_i$, it does not follow that $\mu = \mu_1 + \mu_2$. See example 4.1.5.

However we have

LEMMA : 4.2.6

Let μ be a finite fuzzy vector space of the vector space $\mathcal{G}$ over the field $\mathbb{Z}_p$, where p is a prime number.

Suppose supp $\mu = \{0, x_1, x_2, \cdots, x_k, y_{k+1}, \cdots, y_n\} = < x_1 > \oplus \cdots \oplus < x_k >$.

Then there exist cyclic fuzzy subgroups $w_1, w_2, \cdots, w_k$ such that $\mu = w_1 \oplus \cdots \oplus w_k$.

PROOF :

The elements $x_1, \cdots, x_k$ form a basis for supp μ as a vector space over $\mathbb{Z}_p$.

Let k = 2. So supp $\mu = < x_1 > \oplus < x_2 >$.

Define $\mu_i : \mathcal{G} \longrightarrow [0,1]$ by

$$\mu_i(x) = \begin{cases} \mu(x) & x \in < x_i > \\ 0 & x \notin < x_i > \end{cases}.$$

We show that μ_i is cyclic :

Let $x \in < x_i >$, $x \neq 0$ (the additive identity element of $\mathcal{G}$). So $x = m_i x_i$, $m_i \in \mathbb{Z}_p$, $m_i \neq 0$.

Now $\mu_i(m_i x_i) \geq \mu_i(x_i) = \mu(x_i) = \mu(m_i^{-1} m_i x_i) \geq \mu(m_i x_i) = \mu_i(m_i x_i)$.

So $\mu_i(m_i x_i) = \mu_i(x_i)$.

So if $x \in < x_i >$, $x \neq 0$, then $\mu(x) = \mu(x_i)$. Hence μ_i is a cyclic fuzzy subgroup of μ.

$o(x_i) = p$ for all $i = 1, \cdots, k$. So each μ_i is a p–fuzzy subgroup for a fixed p.

Let $\mu(x_i) = \alpha_i$, $\mu(y_j) = \alpha_j$ with $\alpha_1 \leq \alpha_i$ for all $i = 1, \cdots, n$ and $\alpha_n \geq \alpha_i$ for all $i = 1, \cdots, n$.

The sum $\mu_1 + \mu_2$ may not be equal to μ by Remark 4.2.5.

Suppose $\mu(y_i) \neq (\mu_1 + \cdots + \mu_k)(y_i)$.

Replace x_1 or x_2 by y_n. So $\{x_1, y_n\}$ or $\{x_2, y_n\}$ spans supp μ.

Suppose $\{x_2, y_n\}$ is a basis for supp μ. So supp $\mu = < x_2 > \oplus < y_n >$.

Let $\nu_n(x) = \begin{cases} \mu(x) & x \in < y_n > \\ 0 & x \notin < y_n > \end{cases}.$

ν_n is a cyclic p–fuzzy subgroup of μ.

We now show that $\mu = \mu_2 + \nu_n$:

Let $x_1 = m_2x_2 + m_ny_n$.

Then $(\mu_2 + \nu_n)(x_1) \geq \mu_2(x_2) \wedge \nu_n(y_n)$

$$= \mu(x_2) \wedge \mu(y_n)$$

$$= \mu(x_2) \geq \mu(x_1) \geq (\mu_2 + \nu_n)(x_1).$$

Therefore $(\mu_2 + \nu_n)(x_1) = \mu(x_1)$.

Clearly $\mu(x_2) = (\mu_2 + \nu_n)(x_2)$:

Now let $y_i = m_2x_2 + m_ny_n$, $m_i \in \mathbb{Z}_p$.

$$\begin{aligned}(\mu_2+\nu_n)(y_i) &\geq \mu_2(x_2) \wedge \nu_n(y_n) \\ &= \mu_2(x_2) \\ &= \mu(y_i - m_ny_n) \\ &\geq \mu(y_i) \wedge \mu(m_ny_n) \\ &= \mu(y_i) \\ &\geq (\mu_2 + \nu_n)(y_i)\end{aligned}$$

Therefore $\mu(y_i) = (\mu_2 + \nu_n)(y_i)$.

The fact that $\mu_2 \wedge \nu_n = \mu_e$ is obvious.

Hence $\mu = \mu_2 \oplus \nu_n$.

If $\{x_1,y_n\}$ is a basis, it can be shown that $\mu = \mu_1 \oplus \nu_n$.

Now assume that if supp $\mu = < x_1 > \oplus \cdots \oplus < x_k >$, then there exist cyclic p–fuzzy subgroups $w_1,w_2,\cdots,w_k$, such that $\mu = w_1 \oplus \cdots \oplus w_k$.

Let supp $\mu = < x_1 > \oplus \cdots \oplus < x_{k+1} >$.

Note that $y_i = m_1x_1 + \cdots + m_{k+1}\, x_{k+1}$ implies that $\mu(y_i) \geq \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k \wedge \alpha_{k+1}$, where $\alpha_i = \mu_i(x_i)$, μ_i as defined at the beginning of this proof.

Also $\mu(y_n) \geq \alpha_i$ for all $i = 1,\cdots,k+1, \cdots,n$. We can assume without loss of generality that $\{y_n,x_1,\cdots,x_k\}$ is a basis for supp μ.

Let ν_n be as defined above.

Now supp $\mu = < y_n > \oplus < x_1 > \oplus \cdots \oplus < x_k >$. We will find cyclic p–fuzzy subgroups $w_1, \cdots w_k$, such that $\mu = w_1 \oplus \cdots \oplus w_k \oplus \nu_n$.

Let $H = < x_1 > \oplus \cdots \oplus < x_k >$.

Define $w : \mathcal{G} \longrightarrow [0,1]$ by $w(x) = \begin{cases} \mu(x) & x \in H \\ 0 & x \notin H \end{cases}$.

So supp $\omega = H$. By induction there exist cyclic p–fuzzy subgroups $w_1,\cdots,w_k$ such that $w = w_1 \oplus \cdots \oplus w_k$.

So if $x \in H$, then $\mu(x) = (w_1 \oplus \cdots \oplus w_k)(x)$.

Also $w_i = w$ on some cyclic group $< a_i >$ in H. So $w_i = \mu$ on $< a_i >$ and 0 elsewhere.

Let $y_s = m_1a_1 + \cdots + m_ka_k + m_ny_n \in$ supp μ.

Therefore

$$\begin{aligned}
(w + \nu_n)(y_s) &= (w_1 + \cdots + w_k + \nu_n)(m_1a_1 + m_2a_2 + \cdots + m_ka_k + m_ny_n) \\
&\geq w_1(a_1) \wedge \cdots \wedge w_k(a_k) \wedge \nu_n(y_n) \\
&= w_1(a_1) \wedge \cdots \wedge w_k(a_k) \\
&= (w_1 + \cdots + w_k)(m_1a_1 + \cdots + m_ka_k) \\
&= \mu(m_1a_1 + \cdots + m_ka_k) \\
&= \mu(y_s - m_ny_n) \\
&\geq \mu(y_s) \\
&\geq (w_1 + \cdots + w_k + \nu_n)(y_s).
\end{aligned}$$

So $\mu(y_s) = (w_1 + \cdots + w_k + \nu_n)(y_s)$.

The sum $(w_1 + \cdots + w_k + \nu_n)$ is clearly direct.

This completes the proof.

LEMMA : 4.2.7

An Abelian fuzzy subgroup μ, with $p\mu = \mu_e$, is a fuzzy vector space over $\mathbb{Z}_p$ and μ is a direct sum of cyclic fuzzy subgroups $\mu_1,\cdots,\mu_n$ such that $|\text{supp } \mu_i| = p$, where μ_i is finite, $\forall$ i = 1,$\cdots$,n.

PROOF :

Assume supp μ is finite.
From Group Theory, supp μ is a vector space over $\mathbb{Z}_p$. Clearly μ is a fuzzy vector space over $\mathbb{Z}_p$. Suppose that $\{x_1,\cdots,x_n\}$ is a basis for supp μ. By Lemma 4.2.6, there exist cyclic fuzzy subgroups $w_1,\cdots,w_n$, $|\text{supp } w_i| = p$, such that $\mu = w_1 \oplus \cdots \oplus w_n$.

This completes the proof.

REMARK :
For the rest of this chapter, we will sometimes denote a fuzzy point x_λ by (x,λ) to avoid clumsiness.

LEMMA : 4.2.8

Let μ be a p–fuzzy subgroup of $\mathcal{G}$.
Let $< (y_1,\beta_1), (y_2,\beta_2) > = < (y_1,\beta_1) > \oplus < (y_2,\beta_2) >$, where $(y_i,\beta_i) \in \mu$ such that $(y_i,\beta_i) = (px_i,\lambda_i)$, i = 1,2.
Then $< (x_1,\lambda_1), (x_2,\lambda_2) > = < (x_1,\lambda_1) > \oplus < (x_2,\lambda_2) >$.

PROOF :

Let $\nu = < (x_1,\lambda_1), (x_2,\lambda_2) >$; $\omega_i = < (x_i,\lambda_i) >$, i = 1,2. Clearly $\omega_1 \wedge \omega_2 = \mu_e$. Also $\omega_1 \oplus \omega_2 \leq \nu$. Let $(a,\lambda) \in \nu$. So $\lambda \leq \sup\{\beta\colon a = a_1 + \cdots + a_n, \beta = \beta_1 \wedge \cdots \wedge \beta_n,$ $(a_i,\beta_i) \in (x_1,\lambda_1) \vee (x_2,\lambda_2)\}$.

Therefore $a_i = x_1$ or x_2.

Suppose $a = mx_1 \neq 0$.

Then $(a,\lambda) = (mx_1,\lambda) \in \langle (x_1,\lambda_1) \rangle$ since $\omega_1(mx_1) = \lambda_1 = \nu(a) \geq \lambda$. In case $a = 0$, then $\omega_1(0) \geq \lambda$. Similarly when $a = mx_2$.

Suppose $a = m_1x_1 + m_2x_2$, $m_ix_i \neq 0 \; \forall \; i = 1,2$.

So $(a,\lambda) = (m_1x_1,\lambda) + (m_2x_2,\lambda) \in \omega_1 + \omega_2$ since $\omega_i\,(m_ix_i) = \lambda_i \geq \lambda_1 \wedge \lambda_2 = \nu(a) \geq \lambda$.

Therefore $\nu = \omega_1 \oplus \omega_2$.

NOTE : Lemma 4.2.8 can be extended to any finite number of fuzzy points.

THEOREM : 4.2.9 (THE BASIS THEOREM)

Every finite Abelian fuzzy subgroup μ is a direct sum of cyclic p–fuzzy subgroups.

PROOF :

In view of Theorem 4.2.4, we may assume that μ is a p–fuzzy group.
Let $m \in \mathbb{Z}^+$ such that $p^m\mu = \mu_e$.

If $m = 1$, then the theorem is just lemma 4.2.7. Suppose the theorem holds when $p^m\mu = \mu_e$. Let $p^{m+1}\mu = \mu_e$. Let $\nu = p\mu$. Then $p^m\nu = \mu_e$.

Therefore by induction, $\nu = \mu_1 \oplus \cdots \oplus \mu_t$, where each μ_i is a cyclic p–fuzzy subgroup.

So there exist $(y_i,\lambda_i) \in \nu$, such that $\mu_i = \langle (y_i,\lambda_i) \rangle$ for $i = 1,2,\cdots,t$.

So $(y_i,\lambda_i) \in p\mu$. We claim that $(y_i,\lambda_i) = p(y,\beta)$ for some $(y,\beta) \in \mu$:

$$\begin{aligned} \lambda_i \leq p\mu(y_i) &= \sup\{\mu(x) : y_i = px,\ x \in \text{supp}\ \mu\} \\ &= \mu(y_i),\ y_i = py,\ \text{since}\ \mu\ \text{is finite.} \end{aligned}$$

$(y,\lambda_i) \in \mu$ and $(y_i,\lambda_i) = (py,\lambda_i) = p(y,\lambda_i)$, and the claim is proved.

So if $(y_i,\lambda_i) \in p\mu$, then there exists $(z_i,\beta_i) \in \mu$ such that $(y_i,\lambda_i) = p(z_i,\beta_i)$. (*)

Let $\omega = \langle (z_i,\beta_i) : i = 1,2,\cdots,t \rangle$. Therefore $\omega = \langle (z_1,\beta_1) \rangle \oplus \cdots \oplus \langle (z_t,\beta_t) \rangle$ by lemma 4.2.8.

We will show that ω is a direct summand of μ.
Let $\mu[p](x) = \sup\{\beta : (x,\beta) \in \mu \text{ and } (px,\beta) = (0,\beta)\}$.

If $x \notin$ supp μ or if $px \neq 0$, then define $\mu[p](x) = 0$. So $p(\mu[p]) = \mu_e$. By lemma 4.2.7, $\mu[p]$ is a fuzzy vector space over $\mathbb{Z}_p$ and is also a direct sum of cyclic p–fuzzy subgroups.

Let $o(y_i) = k_i$. So $p(k_iz_i) = k_iy_i = 0$. Hence $o(k_iz_i) = p$. Let $\eta_i = \mu[p](k_iz_i)$. So $(k_iz_i,\eta_i) \in \mu[p]$.

Now $\{k_1z_1,\cdots,k_tz_t\}$ is a linearly independent subset of supp $\mu[p]$. So there exist $x_1,x_2,\cdots,x_s$ such that $\{k_1z_1,\cdots,k_tz_t, x_1,\cdots,x_s\}$ is a basis for supp $\mu[p]$. Therefore

$$\text{supp } \mu[p] = < k_1z_1 > \oplus \cdots \oplus < k_tz_t > \oplus < x_1 > \oplus \cdots \oplus < x_s >.$$

Let $\omega_i(x) = \begin{cases} \mu[p](x) & x \in < x_i > \\ 0 & x \notin < x_i > \end{cases}$, and $q_i = \begin{cases} \mu[p](x) & x \in < k_iz_i > \\ 0 & x \notin < k_iz_i > \end{cases}$.

Let $\xi = \omega_1 \oplus \cdots \oplus \omega_s$, and note that $\omega_i(x) = \mu[p](x) = \mu[p](m_ix_i) = \mu[p](x_i)$, where $x \in < x_i >$, as in lemma 4.2.6.

So ω_i is a cyclic p–fuzzy subgroup of μ for each i. Because of lemma 4.2.6, we can assume that $\mu[p] = q_1 \oplus \cdots \oplus q_t \oplus \omega_1 \oplus \cdots \oplus \omega_s$.

We claim that $\mu = \xi \oplus \omega$:

$\xi \wedge \omega = \mu_e$ follows straight from Group Theory. So we only have to show that $\mu = \xi + \omega$. Clearly $\xi + \omega \leq \mu$.

Let $(x,\beta) \in \mu$. Therefore $(px,\beta) \in p\mu = \nu$.

So $px = (c_1y_1) + \cdots + (c_ty_t)$, $c_i \in \mathbb{Z}$ and $\beta = \lambda_1 \wedge \cdots \wedge \lambda_t$, where λ_i is the degree of membership of c_iy_i.

So $p(x,\beta) = (c_1y_1,\lambda_1) + \cdots + (c_ty_t,\lambda_t) = \sum_{i=1}^{t} (pc_iz_i, \beta_i)$ by (*), (see Proposition 5.1.1).

Hence $p(x,\beta) - \sum_i (p\, c_iz_i,\beta_i) = (0,\beta)$,

i.e. $p((x,\beta) - \sum_i (c_iz_i,\beta_i)) = (0,\beta)$

So $(x,\beta) - \sum_i (c_iz_i,\beta_i) \in \mu[p]$.

Therefore $(x,\beta) - \sum_i (c_iz_i,\beta_i) = \sum_i (b_ik_iz_i,\eta_i) + \sum_j (a_jx_j,\rho_j)$, where $(a_jx_j,\rho_j) \in \omega_i$.

So $(x,\beta) = \sum_i ((c_i + b_ik_i)z_i, \beta_i \wedge \eta_i) + \sum_j (a_jx_j,\rho_j) \in \omega + \xi$.

Hence $\mu = \omega \oplus \xi = < (z_1,\beta_1) > \oplus \cdots \oplus < (z_t,\beta_t) > \oplus \omega_1 \oplus \cdots \oplus \omega_s$.

The proof is complete.

In the above theorem the fact that $\mu[p]$ is a fuzzy subgroup in μ is obvious.

REMARK :

If μ can be decomposed into cyclic fuzzy subgroups whose supports are of prime orders, then the Remak–Krull–Schmidt theorem in Chapter 5 will show that such a decomposition is unique up to isomorphism.

DEFINITION : 4.2.10

Let μ be a finite Abelian fuzzy subgroup of $\mathcal{G}$ which is also a fuzzy vector space over $\mathbb{Z}_p$. A *basis* for μ is a set $\{x_{1\lambda_1},\cdots,x_{n\lambda_n}\}$ of fuzzy points such that

(i) $\mu = < x_{1\lambda_1},\cdots,x_{n\lambda_n} >$ and

(ii) the set $\{x_1,\cdots,x_n\}$ is linearly independent.

The following result follows immediately from Group Theory.

PROPOSITION : 4.2.11

Let μ be a finite Abelian fuzzy subgroup of $\mathcal{G}$ with $p\mu = \mu_e$ for some prime p. Then any two decompositions of μ into a direct sum of cyclic fuzzy subgroups have the same number of summands.

PROOF :

Follows from Group Theory.

We end this chapter by defining a dimension for μ :

DEFINITION : 4.2.12

Let μ be a finite fuzzy vector space over $\mathbb{Z}_p$. Let $\{x_{1\lambda_1},\cdots,x_{n\lambda_n}\}$ be a basis for μ.

Let s be the number of distinct λ_i in the basis for μ. Then the *dimension* of μ is defined to be (n,s).

NOTE :

If $\dim(\mu) = (n,1)$, then Proposition 4.1.3 suggests that μ is of form

$$\mu(a) = \begin{cases} \lambda & a \in \langle x_1,\cdots,x_n \rangle = \operatorname{supp}\mu \\ 0 & a \notin \langle x_1,\cdots,x_n \rangle \end{cases}.$$

So $|R(\mu)| \leq 2$, where $R(\mu)$ is the range of μ. In fact it can be shown that if $\mu_i = \mu$ on $< x_i >$ and 0 elsewhere, then $\mu = \mu_1 \oplus \cdots \oplus \mu_n$. By Remark 4.2.5 this result is not true in general.

PROPOSITION : 4.2.13

Any two finite Abelian fuzzy subgroups that are vector spaces over $\mathbb{Z}_p$ and having the same demension (n,1) are isomorphic in the sense of definition 1.2.1.

PROOF :

This follows immediately from the note immediately after Definition 4.2.12. The isomorphism of the supports follows from linear algebra.

Finally, all the groups we have used in this chapter are finite. So let us end this chapter by giving an example of a fuzzy subgroup μ with infinite support such that μ is a direct product of some fuzzy subgroups.

EXAMPLE : 4.2.14

Let $\mathcal{G} = \mathbb{R}\backslash\{0\}$ under multiplication. So $\mathcal{G}$ is infinite. Define $\mu : \mathcal{G} \longrightarrow [0,1]$ by

$$\mu(a) = \begin{cases} 1/2 & a \in \langle 2 \rangle \setminus \{1\} \\ 1/3 & a \in \langle 2 \rangle\langle 3 \rangle \setminus \langle 2 \rangle \\ 1/4 & a \in \langle 2 \rangle\langle 5 \rangle \setminus \langle 2 \rangle \\ 1/4 & a \in \langle 3 \rangle\langle 5 \rangle \setminus \langle 3 \rangle \\ 1/4 & a \in \langle 2 \rangle\langle 3 \rangle\langle 5 \rangle \setminus \langle 2 \rangle\langle 3 \rangle \cup \langle 3 \rangle\langle 5 \rangle \cup \langle 2 \rangle\langle 5 \rangle, \\ 1 & a = 1. \end{cases}$$

Otherwise $\mu(a) = 0$.

$\mu(2\times 3) = 1/3 = \mu(3) \geq \mu(2) \wedge \mu(3)$.
$\mu(2\times 5) = 1/4 = \mu(5) \geq \mu(2) \wedge \mu(5)$
$\mu(3\times 5) = 1/4 = \mu(5) \geq \mu(3) \wedge \mu(5)$.

So it is easy to show that μ is a fuzzy subgroup of $\mathcal{G}$.

Define $\mu_i : \mathcal{G} \longrightarrow [0,1]$, i = 1,2,3, by

$\mu_1 = \mu$ on $< 2 >$ and $\mu_1 = 0$ elsewhere,

$\mu_2 = \mu$ on $< 3 >$ and $\mu_2 = 0$ elsewhere,

$\mu_3 = \mu$ on $< 5 >$ and $\mu_3 = 0$ elsewhere.

We show that $\mu = \mu_1 \oplus \mu_2 \oplus \mu_3$:

Clearly $\mu_i\mu_j \wedge \mu_k = \mu_e$, i,j,k distinct. Let $y = 2^{s_1} 3^{s_2} 5^{s_3}$, $s_i \in \mathbb{Z}\backslash\{0\}$.

Therefore $\mu_1\mu_2\mu_3(y) = \mu(2) \wedge \mu(3) \wedge \mu(5) = 1/4 = \mu(y)$.

If $y \notin < 2 >< 3 >< 5 >$, then $\mu(y) = 0 = \mu_1\mu_2\mu_3(y)$. So it easy to show that $\mu_1\mu_2\mu_3(x) = \mu(x) \; \forall \; x \in \mathcal{G}$. Hence $\mu = \mu_1 \oplus \mu_2 \oplus \mu_3$.

Supp $\mu = < 2 >< 3 >< 5 >$ is infinite, but finitely generated.

CHAPTER 5

THE FUZZY REMAK–KRULL–SCHMIDT THEOREM AND THE FUZZY JORDAN–HÖLDER THEOREM

INTRODUCTION :

This chapter is an extension of Chapter 4. In Section 5.1 we discuss decomposable and indecomposable fuzzy subgroups. In particular we state and prove the Remak–Krull–Schmidt theorem for fuzzy subgroups. In the proof of this theorem we follow the lattice–theoretic approach, (see for example Cohn [61]). This requires the notion of finite length for the lattice of fuzzy subgroups of a group $\mathcal{G}$. Our definition of finite length ensures that if μ is a fuzzy subgroup of a finite group $\mathcal{G}$, then the lattice of fuzzy subgroups of μ is of finite length. This notion of finite length reduces to the crisp notions of ascending chain condition (ACC) and descending chain condition (DCC). We end Section 5.1 with the Kuroš–Ore theorem for fuzzy subgroups. This theorem is a weaker version of the Remak–Krull–Schmidt theorem.

Section 5.2 is aimed at proving the Jordan–Hölder theorem for fuzzy subgroups. In the definition of a normal series of fuzzy subgroups, Bhattacharya and Mukherjee [1] require that each fuzzy subgroup in the series be fuzzy normal in the underlying group. If $\mu = \mu_1 \geq \cdots \geq \mu_k$ is called a normal series, we feel that it is too strong to require every fuzzy subgroup μ_i to be fuzzy normal in $\mathcal{G}$. Our definition of a normal series requires only that each μ_{i+1} be normal in μ_i. We begin Section 5.2 by proving a fuzzy version of the Zassenhaus lemma, which is a generalization of the second isomorphism theorem. This is followed by the Schreier theorem, which is the backbone of the Jordan–Hölder theorem. In defining a maximal chain of fuzzy subgroups, we have ensured that if $\mathcal{G}$ is finite, then any chain of fuzzy subgroups of $\mathcal{G}$ can be refined to a maximal chain. Finally, the Jordan–Hölder theorem is stated and proved.

5.1 THE REMAK–KRULL–SCHMIDT THEOREM

DEFINITION : 5.1.1

Let μ be a fuzzy subgroup of $\mathcal{G}$. μ is said to be *indecomposable* iff μ is not a fuzzy point, and if $\mu \simeq \mu_1 \otimes \mu_2$, then μ_1 or μ_2 is a fuzzy point. If μ is not indecomposable, it is said to be *decomposable.*

PROPOSITION : 5.1.2

Let μ be an indecomposable fuzzy subgroup of $\mathcal{G}$. If $\mu \simeq \mu_1 \otimes \mu_2$, then $\mu \simeq \mu_1$ or $\mu \simeq \mu_2$.

PROOF :

If $\mu \simeq \mu_1 \otimes \mu_2$, then μ_1 or μ_2 is a fuzzy point. Suppose μ_1 is a fuzzy point with support e. Let $f : \mu \longrightarrow \mu_1 \otimes \mu_2$ be a fuzzy isomorphism. Define $g : \mu \longrightarrow \mu_2$ by $g(x) = f(x)$. $f(x) \in \text{supp } \mu_1\mu_2 = \{e\}\text{supp } \mu_2 = \text{supp } \mu_2$. So g is a crisp isomorphism. There exists $k \in \mathbb{R}^+$ such that

$$\mu(x) = k\ \mu_1\mu_2(f(x)) \text{ for all } x \in \text{supp } \mu \setminus \{e\}.$$

Therefore $k \sup\limits_{f(x) = ab} \mu_1(a) \wedge \mu_2(b) = \mu(x)$.

Hence $k[\mu_1(e) \wedge \mu_2(f(x))] = \mu(x)$.

Therefore $k\ \mu_2(f(x)) = \mu(x)$ since $\mu_1(e) = \mu_2(e)$.

Hence $\mu_2 \simeq \mu$.

This completes the proof.

PROPOSITION : 5.1.3

Let $\mu \simeq \mu_1 \otimes \mu_2$, where μ_1,μ_2,μ are fuzzy subgroups of $\mathcal{G}$. Then there exist $\omega_1,\omega_2 \leq \mu$, ω_1,ω_2 fuzzy subgroups of $\mathcal{G}$, such that $\mu = \omega_1 \otimes \omega_2$, $\mu_i \simeq \omega_i$, $i = 1,2$. If μ_i is indecomposable, then ω_i is indecomposable.

PROOF :

Let $f : \mu \longrightarrow \mu_1 \otimes \mu_2$ be a fuzzy isomorphism. So there exists $k \in \mathbb{R}^+$ such that $\mu(x) = k\ \mu_1\mu_2(f(x))$ for all $x \in \text{supp } \mu \setminus \{e\}$.

$$\text{Let } \omega_i(x) = \begin{cases} k\ \mu_i(f(x)) & x \in \text{supp } \mu \setminus \{e\} \\ \mu_i(f(e)) & x = e \\ 0 & x \notin \text{supp } \mu \end{cases}$$

Clearly $\omega_i \leq \mu$, $i = 1,2$.

It is also clear that each ω_i is a fuzzy subgroup of $\mathcal{G}$. Let $x \in \text{supp}\,\mu \setminus \{e\}$.

$$\omega_1\omega_2(x) = \sup_{x = x_1x_2} \omega_1(x_1) \wedge \omega_2(x_2)$$

$$= k \sup_{f(x) = f(x_1)f(x_2)} \mu_1(f(x_1)) \wedge \mu_2(f(x_2))$$

$$= k\, \mu_1\mu_2(f(x)) = \mu(x)$$

Clearly $\omega_1\omega_2(e) = \mu(e)$.

It is also obvious that $\omega_1 \wedge \omega_2 = \mu_e$.

It is easy to check that $\omega_i \triangleleft \mu$, $i = 1,2$.

Hence $\mu = \omega_1 \otimes \omega_2$.

Define $g : \omega_i \longrightarrow \mu_i$ by $g(x) = f(x)$.

If $x \in \text{supp}\,\omega_i = \text{supp}\,f^{-1}(\mu_i) = f^{-1}(\text{supp}\,\mu_i)$, then $f(x) \in \text{supp}\,\mu_i$. So g is well–defined.

g is a crisp isomorphism.

$k\,\mu_i(g(x)) = k\,\mu_i(f(x)) = \omega_i(x)$.

So g is a fuzzy isomorphism.

Suppose μ_1 is indecomposable. Let $\omega_1 = \nu_1 \otimes \nu_2$. $\omega_1 = f^{-1}(k\,\mu_1)$. Hence $k\,\mu_1 = f(\nu_1\nu_2)$.

So $k\,\mu_1(f(x)) = \sup_{f(y) = f(x)} \nu_1\nu_2(y) = \nu_1\nu_2(x)$ since f is one–to–one.

Let $\nu_i^1(f(x)) = {}^1/_k\, \nu_i(x)$, $x \neq e$, and $\nu_i^1(f(e)) = \nu_i(e)$.

Then $\mu_1 = \nu_1^1 \otimes \nu_2^1$ since

$$\nu_1^1\nu_2^1(f(x)) = \sup_{f(x) = f(ab)} \nu_1^1(f(a)) \wedge \nu_2^1(f(b))$$

$$= {}^1/_k \sup_{x = ab} \nu_1(a) \wedge \nu_2(b)$$

$$= {}^1/_k\, \nu_1\nu_2(x) = \mu_1(f(x)).$$

If μ_1 is indecomposable, then ν_2^1, say, is a fuzzy point. So ν_2 is also a fuzzy point. Hence ω_1 is indecomposable.

This completes the proof.

The above proposition allows us to replace $\simeq$ with $=$ in the definition of an indecomposable fuzzy subgroup, (Definition : 5.1.1).

PROPOSITION : 5.1.4

Let $\mu = \mu_1 \otimes \mu_2$. Then $\mu/\mu_1 \simeq \mu_2$.

PROOF :

Straightforward.

Let μ be a fuzzy subgroup of $\mathcal{G}$. Let $\mathcal{P}(\mu)$ be the set of all fuzzy subgroups ν such that $\nu \leq \mu$, where $\leq$ is defined by $\nu(x) \leq \mu(x) \; \forall \; x \in \mathcal{G}$, and $\mu(e) = \nu(e)$. Then $(\mathcal{P}(\mu), \leq)$ is a complete lattice. The supremum of ν_1 and ν_2 in $\mathcal{P}(\mu)$ is the smallest fuzzy subgroup of μ containing ν_1 and ν_2. In case ν_1 and ν_2 are normal, the supremum of ν_1 and ν_2 is the product $\nu_1\nu_2$.

DEFINITION : 5.1.5 [61]

A lattice $\mathcal{L}$ is a modular lattice or a Dedekind lattice iff $(c \vee a) \wedge b = (c \wedge b) \vee a$ $\forall \; a,b,c \in \mathcal{L}$ such that $a \leq b$.

The lattice $\mathcal{P}(\mu)$ need not be a modular lattice. Let $\mathcal{P}_n(\mu)$ be the subset of $\mathcal{P}(\mu)$ consisting of normal fuzzy subgroups in μ, i.e. if $\nu \in \mathcal{P}_n(\mu)$, then $\nu \triangleleft \mu$. In Chapter 3 (Proposition 3.1.5), we proved the Dedekind law for fuzzy subgroups in $\mathcal{P}_n(\mu)$. Hence $\mathcal{P}_n(\mu)$ is a modular lattice. $\mathcal{P}_n(\mu)$ need not be a distributive lattice.

DEFINITION : 5.1.6 [61]

Let $\mathcal{L}$ be a lattice with 0 and 1. $x_1,x_2 \in \mathcal{L}$ are related if there exists $y \in \mathcal{L}$ such that $x_1 \wedge y = 0 = x_2 \wedge y$ and $x_1 \vee y = 1 = x_2 \vee y$.

In the lattice $\mathcal{P}(\mu)$, $0 = \mu_e$ and $1 = \mu$. So in $\mathcal{P}_n(\mu)$, ν_1 and ν_2 are related iff there exists $\omega \in \mathcal{P}_n(\mu)$ such that $\nu_1 \wedge \omega = \mu_e = \nu_2 \wedge \omega$ and $\nu_1\omega = \mu = \nu_2\omega$. If $\mathcal{P}_n(\mu)$ is a distributive lattice, then ν_1 and ν_2 are related iff $\nu_1 = \nu_2$.

Now we want to define the notion of length on $\mathcal{F}(\mu)$.

DEFINITION : 5.1.7 [61]

Let $\mathscr{L}$ be a lattice. The length of $\mathscr{L}$ is the supremum of the number of nontrivial intervals (i.e. intervals with distinct end–points) in any chain. In particular, a lattice is of finite length when there is a finite bound on the lengths of its chains. The length of a point $a \in \mathscr{L}$, $\ell(a)$, is the length of [0,a]. It is then easy to show that

$$\ell(a) + \ell(b) = \ell(a \wedge b) + \ell(a \vee b) \; \forall \; a,b \in \mathscr{L}.$$

The following example shows that when defining finite length for $\mathscr{P}(\mu)$, we cannot just mimic the above definition.

EXAMPLE : 5.1.8

Let $\mathscr{G} = S_3 = \{e,a,a^2,b,ab,a^2b\}$, $a^3 = e = b^2$. Let

$$\omega(x) = \begin{cases} 1 & x = e \\ 1/2 & x = a,a^2 \\ 1/3 & \text{otherwise,} \end{cases} \quad \text{and} \quad \nu(x) = \begin{cases} 1 & x = e \\ 1/3 & x = a,a^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\nu < \omega$. We can construct infinitely many fuzzy subgroups between ν and ω as follows :

Let $\alpha_1 = 3/4$ and $\alpha_2 = 9/10$.

Let $\alpha\omega$ be defined by $\alpha\omega(x) = \begin{cases} \omega(e) & x = e \\ \alpha\omega(e) & x \neq e \end{cases}$, $\alpha \in (0,1]$.

Now $\alpha_1\omega \leq \omega$ and $\nu < \alpha_1\omega$.

Similarly, $\nu < \alpha_2\,(\alpha_1\omega) < \alpha_1\omega < \omega$.

Therefore $\nu < \cdots < \alpha_2(\alpha_1\omega) < \alpha_1\omega < \omega$ is an infinite chain of fuzzy subgroups, although $\mathscr{G}$ is a finite group.

Now let us look again at the fuzzy subgroups $\alpha\mu$, $\alpha \in (0,1]$. $\alpha\mu$ is fuzzy normal in $\mathscr{G} \Leftrightarrow \mu$ is fuzzy normal in $\mathscr{G}$, $\alpha\mu$ is cyclic $\Leftrightarrow \mu$ is cyclic, supp $(\alpha\mu)$ = supp μ and $\mu(x) > \mu(y) \Leftrightarrow \alpha\mu(x) > \alpha\mu(y)$.

So there is no essential difference between μ and $\alpha\mu$ except that the degree of membership of x in $\alpha\mu$ is always less than its degree of membership in μ. We would like to call such fuzzy sugbroups equivalent. But there are other fuzzy subgroups of μ that behave like $\alpha\mu$. So we want to put all these fuzzy subgroups in one class.

EXAMPLE : 5.1.9

Let S_3 be as before.

Let $\mu(x) = \begin{cases} 1 & x = e \\ 1/2 & x = a, a^2 \\ 1/4 & \text{otherwise} \end{cases}$ and $\nu(x) = \begin{cases} 1 & x = e \\ 3/4 & x = a, a^2 \\ 1/5 & \text{otherwise.} \end{cases}$ Although $\nu \neq \alpha\mu$,

ν and μ behave similarly, as described above.

This motivates the following definition :

DEFINITION : 5.1.10

Let μ and ν be fuzzy subgroups of $\mathcal{G}$ such that

(i) $\nu(x) > \nu(y) \Leftrightarrow \mu(x) > \mu(y)$, and
(ii) $\nu(x) = 0 \Leftrightarrow \mu(x) = 0$.

Then μ and ν are said to be **equivalent**, and we write $\mu \equiv \nu$ or $\nu \equiv \mu$.
Obviously the relation $\equiv$ is an equivalence relation on $\mathcal{P}(\mu)$.

EXAMPLE : 5.1.11

Let S_3 be as before.

Let $\mu(x) = \begin{cases} 1 & x = e \\ 1/2 & x = a, a^2 \\ 1/3 & \text{otherwise.} \end{cases}$ and $\nu(x) = \begin{cases} 1 & x = e \\ 1/5 & \text{otherwise.} \end{cases}$

$\nu < \mu$. Let $\nu < \omega < \mu$, where ω is a fuzzy subgroup of S_3. It is easy to see that $\omega \equiv \nu$ or $\omega \equiv \mu$.

Now let $\mathcal{G}$ be a finite group, say $|\mathcal{G}| = m$, m an odd number. Let μ be a fuzzy subgroup of $\mathcal{G}$. Then $|R(\mu)| \leq \frac{m-1}{2} + 1$, where $R(\mu)$ is the range of μ.

Suppose $|R(\mu)| = \frac{m-1}{2} + 1$. It is easy to see that the number of equivalence classes of fuzzy subgroups of $\mathcal{G}$ whose supports equal supp μ is less than or equal to the number of permutations of $\frac{m-1}{2}$ objects taken all at a time, which is equal to $\left(\frac{m-1}{2}\right)!$, a finite number.

Since a finite group must have finitely many subgroups, we conclude that there are finitely many equivalence classes of fuzzy subgroups of $\mathcal{G}$ induced by $\equiv$. In particular, if $\nu \leq \mu$ in a finite group $\mathcal{G}$, then there are only finitely many fuzzy subgroups between μ and ν, up to equivalence.

Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be a chain of fuzzy subgroups of $\mathcal{G}$ such that no two quotients of the form μ_i/μ_{i-1} are equivalent, and that each μ_i/μ_{i-1} is nontrivial, i.e. μ_i/μ_{i-1} is not isomorphic to μ_e, then the length of the chain is n–1.

Let us call two fuzzy subgroups ν and μ *distinct* if μ and ν are not equivalent. So the *length of a chain* $\mu_1 \leq \cdots \leq \mu_n$ is the number of nontrivial distinct fuzzy quotients of the form μ_i/μ_{i-1}. Note that we are assuming that each $\mu_{i-1} \triangleleft \mu_i$ so that the quotients are fuzzy subgroups.

The *length of a chain* $\mu_1 \geq \mu_2 \geq \mu \geq \cdots$ is the number of nontrivial distinct quotients of the form $\mu_1/\mu_2, \mu_2/\mu_3, \cdots, \mu_n/\mu_{n+1}, \cdots$.

The *length of a quotient* μ/ν is the supremum of the number of distinct nontrivial quotients in any chain of fuzzy subgroups between μ and ν. We write $\ell(\mu/\nu)$ for the length of μ/ν. μ/ν is of *finite length* in case $\ell(\mu/\nu)$ is finite.

EXAMPLE : 5.1.12

Let $\mathcal{G} = S_3$, $\mu(x) = \begin{cases} 1 & x = e \\ 1/2 & x = a, a^2 \\ 1/3 & \text{otherwise} \end{cases}$, $\nu(x) = \begin{cases} 1 & x = e \\ 1/4 & x = a, a^2 \\ 0 & \text{otherwise.} \end{cases}$

The only fuzzy subgroup, up to equivalence, that lies strictly between ν and μ is

$\omega(x) = \begin{cases} 1 & x = e \\ \alpha & \text{otherwise,} \end{cases}$ $1/4 < \alpha < 1/3$. $\omega/\nu\,(x \operatorname{supp} \nu) = \begin{cases} 1 & x \in \operatorname{supp} \nu \\ \alpha & x \notin \operatorname{supp} \nu \end{cases}$.

$\mu/\omega \simeq \mu_e$ and $\omega/\nu \equiv \mu/\nu$. It is now obvious that $\ell(\mu/\nu) = 1$.

We define the *length of a fuzzy subgroup* μ to be $\ell(\mu) = \ell(\mu/\mu_e)$. Let μ be as given in the above example. Then $\ell(\mu) = 2$.

Now $\mu\nu/\nu \simeq \mu/\nu \wedge \nu$ whenever the quotients are defined, (See Chapter 3).

Let $\omega_1/\omega_2 \equiv \nu_1/\nu_2$, where $\nu \leq \omega_2 \leq \omega_1 \leq \mu\nu$ and $\nu \leq \nu_2 \leq \nu_1 \leq \mu\nu$.

Therefore supp $\omega_2 =$ supp ν_2, also $\mu\omega_1/\omega_2 = \mu\nu/\omega_2 = \mu\omega_2/\omega_2 \simeq \mu/\mu \wedge \omega_2 = \mu/\mu \wedge \nu_2$.

Let f be the fuzzy isomorphism from $\mu\omega_1/\omega_2$ onto $\mu/\mu \wedge \omega_2 = \mu/\mu \wedge \nu_2$. Therefore $f(\omega_1/\omega_2) \equiv f(\nu_1/\nu_2)$ and $\mu \wedge \nu_2 \leq f(\omega_1/\omega_2), f(\nu_1/\nu_2) \leq \mu$.

This shows that to each quotient ω_1/ω_2, there corresponds a quotient ω'_1/ω'_2, $\mu \wedge \nu \leq \omega'_2 \leq \omega'_1 \leq \mu$, which behaves similarly to ω_1/ω_2.

Hence $\ell(\mu\nu/\nu) = \ell(\mu/\mu \wedge \nu)$.

DEFINITION : 5.1.13

The length of the lattice $\mathcal{P}(\mu)$ is $\ell(\mathcal{P}(\mu)) = \ell(\mu)$. So $\mathcal{P}(\mu)$ is of *finite length* in case $\ell(\mu) < \infty$.

If $\ell(\mu) < \infty$, it is not hard to prove that $\ell(\mu) \geq \ell(\mu/\nu) + \ell(\nu)$.

Now let $\mathcal{P}(\mu)$ be of finite length, and $\nu_1 \leq \nu_2 \leq \cdots$ be a normal chain in $\mathcal{P}(\mu)$. This chain must "stop" in the sense that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, ν_{n+1}/ν_n is equivalent to one of the quotients that appeared before. Hence supp ν_n and supp ν_{n+1} are subgroups that have appeared before in the chain

$$\text{supp } \nu_1 \subseteq \text{supp } \nu_2 \subseteq \cdots .$$

This shows that supp μ has the ACC. Similarly supp μ has the DCC whenever $\mathcal{P}(\mu)$ is of finite length.

PROPOSITION : 5.1.14

If $\mu \equiv \mu_1 \otimes \mu_2$, then there exist ν_1, ν_2 such that $\mu = \nu_1 \otimes \nu_2$ and $\nu_i \equiv \mu_i$ for each $i = 1,2$.

PROOF :

Let $\nu_i = \begin{cases} \mu(x) & x \in \text{supp } \mu_i \\ 0 & x \notin \text{supp } \mu_i \end{cases}, \quad i = 1,2.$

Clearly $\nu_1\nu_2$ is a direct product. Suppose $\mu(x) > \nu_1\nu_2(x)$ for some x.

Now $\nu_1\nu_2(x) = \nu_1(x_1) \wedge \nu_2(x_2) = \nu_2(x_2)$, say, where $x_i \in$ supp $\mu_i =$ supp ν_i, $i = 1,2$. $\nu_2(x_2) = \mu(x_2)$. So $\mu(x) > \mu(x_2)$.

Since $\mu \equiv \mu_1\mu_2$, $\mu_1\mu_2\,(x) > \mu_1\mu_2\,(x_2)$.
Therefore $\mu_1(x_1) \wedge \mu_2(x_2) > \mu_2(x_2)$ since $x_i \in \text{supp}\ \mu_i$.
Therefore $\mu_2(x_2) > \mu_2(x_2)$, an absurdity. Hence $\mu = \nu_1\nu_2$.

PROPOSITION : 5.1.15

Let $\mu = \mu_1 \otimes \mu_2$ and $\mu_1 = \nu_1 \otimes \nu_2$. Then

(a) $\nu_i \triangleleft \mu$ and

(b) if $\ell(\mu) < \infty$, then $\ell(\mu_i) < \infty$, $i = 1,2$.

PROOF :

(b) Suppose $\ell(\mu) < \infty$. Let $\mu_i = \omega_1 \geq \omega_2 \geq \cdots$. Therefore $\mu \geq \omega_1 \geq \omega_2 \geq \cdots$. Hence there exists $n_0 \in \mathbb{N}$ such that $\forall\ n \geq n_0\ \omega_n/\omega_{n+1}$ is equivalent to one of the fuzzy subgroups that have already appeared. This shows that $\ell(\mu_i) < \infty$, $i = 1,2$.

(a) We must show that $a_\lambda \nu_i = \nu_i a_\lambda\ \forall\ a_\lambda \in \mu$. Let $x \in \mathcal{G}$.

Case $a^{-1}x \in$ supp ν_i :

$\nu_i(a^{-1}x) = \mu_1(a^{-1}x)$ since $\mu_1 = \nu_1 \otimes \nu_2$.
Also $xa^{-1} \in \text{supp}\ \nu_i$ implies that $\nu_i(xa^{-1}) = \mu_1(xa^{-1})$. Hence

$$a_\lambda \nu_i(x) = \lambda \wedge \mu_1(a^{-1}x) = \lambda \wedge \mu_1(xa^{-1}) \text{ since } \mu_1 \triangleleft \mu,$$
$$= \lambda \wedge \nu_i(xa^{-1}) = \nu_i a_\lambda(x).$$

Case $a^{-1}x \notin$ supp ν_i :

$xa^{-1} \notin \text{supp}\ \nu_i$. Hence $a_\lambda \nu_i(x) = 0 = \nu_i a_\lambda(x)$.

This completes the proof.

PROPOSITION : 5.1.16

Let μ be a fuzzy subgroup of $\mathcal{G}$, $\mu \neq \mu_e$. If $\ell(\mu) < \infty$, then μ is a direct product of a finite number of indecomposable fuzzy subgroups.

PROOF :

If μ is indecomposable, there is nothing to prove. So suppose μ is decomposable, say $\mu = \nu_1 \otimes \mu_2$, $\nu_1, \mu_2 \neq \mu_e$. If μ_2 is decomposable, we can write $\mu_2 = \nu_2 \otimes \mu_3$. If μ_3 is decomposable, then $\mu_3 = \nu_3 \otimes \mu_4$.
By Proposition 5.1.15, $\mu = \nu_1 \otimes \nu_2 \otimes \nu_3 \otimes \mu_4$. By induction, we have $\mu = \nu_1 \otimes \cdots \otimes \nu_{n-1} \otimes \mu_n$, where each ν_i is indecomposable. Now we have the descending chain $\mu \geq \mu_2 \geq \mu_3 \geq \cdots$, where each $\mu_i \triangleleft \mu$ (by Proposition 5.1.15). Since $\ell(\mu) < \infty$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then μ_n/μ_{n+1} is one of the quotients that have appeared previously. So if $\mu_n = \nu_n \otimes \mu_{n+1}$, then $\mu_n/\mu_{n+1} \simeq \nu_n$ implies ν_n is isomorphic to a fuzzy subgroup that has appeared previously. In fact since supp μ has the DCC, we can assume $\mu_{n+1} = \mu_e$, hence $\mu_n = \nu_n$ is indecomposable.

Hence $\mu = \nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n$, where each ν_i is indecomposable.

PROPOSITION : 5.1.17

(a) Let ν_1, ν_2 be related in $\mathscr{P}_n(\mu)$. Then $\ell(\nu_1) = \ell(\nu_2)$.

(b) If $\nu\mu'_1 = \mu_1\mu'_1$ where $\mu_1 \wedge \mu'_1 = \mu_e$ and ν is related to μ_1, then $\ell(\nu \wedge \mu'_1) = 0$, hence $\nu \wedge \mu'_1 = \mu_e$.

PROOF :

(a) Let $\nu_1\omega = \mu = \nu_2\omega$ and $\nu_1 \wedge \omega = \mu_e = \nu_2 \wedge \omega$.
Now $\nu_1\omega/\omega = \nu_2\omega/\omega$ implies $\ell(\nu_1\omega/\omega) = \ell(\nu_2\omega/\omega)$,
hence $\ell(\nu_1/\nu_1 \wedge \omega) = \ell(\nu_2/\nu_2 \wedge \omega)$ by isomorphism.
But $\nu_1 \wedge \omega = \mu_e = \nu_2 \wedge \omega$, hence $\ell(\nu_1) = \ell(\nu_2)$.

(b) $\ell(\nu\mu'_1/\mu'_1) = \ell(\mu_1\mu'_1/\mu'_1) = \ell(\mu_1/\mu'_1 \wedge \mu_1) = \ell(\mu_1)$ since $\mu_1 \wedge \mu'_1 = \mu_e$.

Therefore $\ell(\nu/\nu \wedge \mu'_1) = \ell(\mu_1)$.

$$\begin{aligned} \ell(\nu) &\geq \ell(\nu/\mu'_1 \wedge \nu) + \ell(\mu'_1 \wedge \nu) \\ &= \ell(\mu_1) + \ell(\mu'_1 \wedge \nu). \end{aligned}$$

Therefore $\ell(\nu) - \ell(\mu_1) \geq \ell(\mu'_1 \wedge \nu)$.

But $\ell(\nu) = \ell(\mu_1)$, hence $\ell(\mu'_1 \wedge \nu) \leq 0$, therefore $\ell(\mu'_1 \wedge \nu) = 0$.

Hence $\mu'_1 \wedge \nu = \mu_e$.

In the next two theorems we will often write $\mu = \mu_1 \otimes \mu_2$ when we actually mean $\mu \simeq \mu_1 \otimes \mu_2$. This is not a bad notation because of Proposition 5.1.3.
If $\mu \simeq \mu_1 \otimes \mu_2 \simeq \mu_1 \otimes \nu_2$, we will still say that μ_2 is μ_1 – related to ν_2.

THEOREM : 5.1.18 (REMAK–KRULL–SCHMIDT)

Let μ be a fuzzy subgroup of $\mathcal{G}$ such that $\ell\,(\mathcal{P}(\mu)) < \infty$. If (1) $\mu = \mu_1 \otimes \mu_2 \cdots \otimes \mu_m$ and (2) $\mu = \nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n$, where each μ_i and ν_j are indecomposable fuzzy subgroups, then

(i) each μ_i is related to some ν_j ;

(ii) m = n, and for each r ∈ [0,n], there is a re–indexing so that $\mu \simeq \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_r \otimes \nu_{r+1} \otimes \cdots \otimes \nu_n$;

(iii) each μ_i is isomorphic to some ν_j.

PROOF :

$\mathcal{P}_n(\mu)$ is a modular lattice.

(i) Let $\mu'_i = \mu_1 \otimes \cdots \otimes \mu_{i-1} \otimes \mu_{i+1} \otimes \cdots \otimes \mu_m$ and
$\nu'_i = \nu_1 \otimes \cdots \otimes \nu_{i-1} \otimes \nu_{i+1} \otimes \cdots \otimes \nu_n$.

We note that $\omega \leq \prod_{i=1}^{m} (\omega\mu'_i \wedge \mu_i)$ for any $\omega \in \mathcal{P}_n(\mu)$. **(3)**

We aim to prove that each μ_i is μ'_i – related to some ν_j using induction on $\ell\,(\mu)$. **(4)**

Let $\ell\,(\mu) = 1$. Hence any quotient μ/ν, where $\nu \triangleleft \mu$, is equivalent to μ/μ_e or μ/μ. So if $\mu = \mu_1 \otimes \mu_2 = \nu_1 \otimes \nu_2$, where the μ_i's and ν_i's are indecomposable, then μ_1 or μ_2 equals μ_e and ν_1 or ν_2 equals μ_e. Hence μ_i is trivially μ'_i – related to some ν_j. Suppose now that **(4)** is true for a fuzzy subgroup ω whose length is less than $\ell\,(\mu)$. We consider two cases :

(a) **Case $\mu_1\nu'_j \neq \mu$ for some j :**

We may assume that supp $\mu_1\nu'_j \neq$ supp μ, otherwise we can replace the above condition by supp $\mu_1\nu'_j \neq$ supp μ.

Let $\omega_j = \nu_j \wedge \mu_1\nu'_j$, and $\omega = \omega_1\omega_2 \cdots \omega_n$. $\omega_j \leq \nu_j$ for all j, hence $\omega_j \wedge \omega'_j \leq \nu_j \wedge \nu'_j = \mu_e$. Therefore $\omega = \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n$.

If $\omega_j = \nu_j \;\forall$ j, then $\nu_j = \nu_j \wedge \mu_1\nu'_j$, hence $\nu_j \leq \mu_1\nu'_j$.

So $\mu = \nu_j\nu'_j \leq \mu_1\nu_j$.

Therefore $\mu = \mu_1\nu'_j \;\forall$ j, a contradiction. Hence there exists a j such that $\omega_j \neq \nu_j$. It also follows that supp $\omega_j \neq$ supp ν_j for some j.

Hence $\ell(\nu_j) \neq \ell(\omega_j)$, and ν_j/ω_j is not isomorphic to μ_e.

If $\nu_j/\omega_j \equiv \omega'/\omega_j$ for some $\omega' \leq \omega$, then supp ν_j = supp ω', hence ω' is indecomposable. But $\omega' = \omega_j \otimes (\omega' \wedge \omega'_j)$ by lemma 4.10 [61]. Therefore $\omega_j = \omega'$, and this contradicts the fact that ν_j/ω_j is not isomorphic to μ_e. Hence we conclude that $\ell(\omega) < \ell(\mu)$. So we can apply the induction hypothesis on $\ell(\omega)$.

By **(3)** above, $\mu_1 \leq \omega$. By lemma 4.10 [61], $\omega = \mu_1 \otimes (\omega \wedge \mu'_1)$. **(5)**

Since $\ell(\mu) < \infty$, each ω_i can be decomposed into a direct product of a finite number of indecomposable fuzzy subgroups. Suppose $\omega = u_1 \otimes e_1 \otimes u_2 \otimes e_2 \otimes \cdots \otimes u_n \otimes e_n$, where each u_i and e_i are indecomposable, and $\omega_i = u_i \otimes e_i$.

By induction, μ_1 is $\omega \wedge \mu'_1$ – related to u_i or e_i for some i, by comparing **(5)** and the above decomposition of the ω_i's. Suppose μ_1 is $\omega \wedge \mu'_1$ – related to u_1.

Therefore $\omega = \mu_1 \otimes (\omega \wedge \mu'_1) = u_1 \otimes (\omega \wedge \mu'_1)$.

Now $u_1\mu'_1 = u_1(\omega \wedge \mu'_1)\mu'_1 = \mu_1(\omega \wedge \mu'_1)\mu'_1 = \mu$. Since μ_1 and u_1 are related, $\ell(\mu_1) = \ell(u_1)$. Now $u_1\mu'_1 = \mu = \mu_1\mu'_1$, hence, by Proposition 5.1.17, $\ell(u_1 \wedge \mu'_1) = 0$. Therefore $u_1 \wedge \mu'_1 = \mu_e$. So $\mu = u_1 \otimes \mu'_1$.

Now $u_1 \leq \omega_1 \leq \nu_1$, hence by lemma 4.10 [61], $\nu_1 = u_1 \otimes (\nu_1 \wedge u'_1)$. But ν_1 is indecomposable, hence $\nu_1 = u_1 = \omega_1$, and $\nu_1 \wedge u'_1 = \mu_e$. So ν_1 is $\omega \wedge \mu'_1$ – related to μ_1. Now $\nu_1 = \omega_1 = \mu_1\nu'_1 \wedge \nu_1$, therefore $\nu_1 \leq \mu_1\nu'_1$, and so $\mu \leq \mu_1\nu'_1$. Therefore $\mu = \mu_1\nu'_1$. **(6)**

(b) **Case $\mu_1\nu'_j = \mu$ for all j.**

Suppose $\nu_j\mu'_1 \neq \mu$ for all j. Then, by applying (a) with the μ_i's and ν_i's interchanged, we find that ν_j is related to some μ_j. So we may replace ν_1 by μ_1', ν_2 by μ_2', and so on until we reach j such that $j' = 1$, say $1' = 1$. **(6)** implies that $\mu = \nu_1\mu'_1$, and this contradicts our supposition above. Therefore there exists j such that $\nu_j\mu'_1 = \mu$, say for $j = 1$. So $\nu_1\mu'_1 = \mu = \mu_1\mu'_1$.

Now $\nu_1\mu'_1/\mu'_1 = \mu_1\mu'_1/\mu'_1$, and so $\nu_1/(\mu'_1 \wedge \nu_1) \simeq \mu_1$ by the second isomorphism theorem. Therefore $\ell(\mu_1) = \ell(\nu_1/\mu'_1 \wedge \nu_1)$.

$$\begin{aligned} \ell(\nu_1) &\geq \ell(\nu_1/\nu_1 \wedge \mu'_1) + \ell(\nu_1 \wedge \mu'_1) \\ &= \ell(\mu_1) + \ell(\nu_1 \wedge \mu'_1) \end{aligned} \qquad \textbf{(7)}$$

So $\ell(\nu_1) - \ell(\mu_1) \geq 0$. By (b), $\mu = \mu_1\nu'_1 = \nu_1\nu'_1$, hence a similar argument shows that $\ell(\mu_1) - \ell(\nu_1) \geq 0$. Therefore $\ell(\mu_1) = \ell(\nu_1)$, hence **(7)** implies $\ell(\nu_1 \wedge \mu'_1) = 0$. So $\nu_1 \wedge \mu'_1 = \mu_e$. Therefore $\mu = \nu_1 \otimes \mu'_1 = \mu_1 \otimes \mu'_1$, i.e. μ_1 is μ'_1 – related to ν_1.

(ii) Suppose ν_i is ν'_i – related to μ_i, $i = 1,2,\cdots,n$.

Therefore $\mu = \mu_1 \otimes \nu'_1 = \mu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n$ by (i).

Suppose $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_{r-1} \otimes \nu_r \otimes \cdots \otimes \nu_n$.

Since ν_r is ν'_r – related to μ_r, $\nu_r \otimes \nu'_r = \mu_r \otimes \nu'_r$.

Therefore $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_{r-1} \otimes \mu_r \otimes \nu_{r+1} \otimes \cdots \otimes \nu_n$.

Suppose $n > m$. Then $\mu = \mu_1 \otimes \cdots \otimes \mu_m \otimes \nu_{m+1} \otimes \cdots \otimes \nu_n$, hence $\nu_{m+1} = \mu_e = \cdots = \nu_n$. Contradiction ! Therefore $n \leq m$.

By symmetry, $m \leq n$. Therefore $m = n$.

(iii) Suppose μ_1 is μ'_1 – related to ν_1. So $\mu_1\mu'_1 = \mu = \nu_1\mu'_1$ and $\mu_1 \wedge \mu'_1 = \mu_e = \nu_1 \wedge \mu'_1$. Hence $\mu_1\mu'_1/\mu'_1 \simeq \mu_1$ and $\nu_1\mu'_1/\mu'_1 \simeq \nu_1$ by the second isomorphism theorem. So $\mu_1 \simeq \nu_1$ since $\mu_1\mu'_1/\mu'_1 = \nu_1\mu'_1/\mu'_1$.

This completes the proof of the theorem.

DEFINITION : 5.1.19 [61]

Let μ_i be a fuzzy subgroup of μ. μ_i is *irreducible* in μ iff $\mu_i \neq \nu_1\nu_2$, where $\nu_1,\nu_2 \neq \mu_i$, $\nu_1,\nu_2 \in \mathscr{P}(\mu)$. A decomposition $\mu = \mu_1\mu_2 \cdots \mu_n$ is *irredundant* iff no μ_i can be omitted in the decomposition.

THEOREM : 5.1.20 (KUROŠ–ORE)

Let μ be a fuzzy subgroup of $\mathscr{G}$ such that
(1) $\mu = \mu_1\mu_2 \cdots \mu_r$ and (2) $\mu = \nu_1\nu_2 \cdots \nu_s$ are irredundant decompositions of μ where each factor is irreducible in μ. Then $r = s$, and for each $m \in [1,r]$, there is a re–indexing so that $\mu \simeq \mu_1\mu_2 \cdots \mu_m \nu_{m+1} \cdots \nu_r$.

PROOF :
This follows from theorem 4.14 [61].

5.2 : THE JORDAN–HÖLDER THEOREM

LEMMA : 5.2.1

Let μ, ν, μ^* be fuzzy subgroups of $\mathscr{G}$ such that $\mu, \nu \triangleleft \mu^*$. Then $\mu\nu \triangleleft \mu^*$.

PROOF :
Straightforward.

THEOREM : 5.2.2 (ZASSENHAUS LEMMA)

Let μ, ν, μ^*, ν^* be fuzzy subgroups of $\mathcal{G}$. Suppose also that $\nu \lhd \nu^*$ and $\mu \lhd \mu^*$. Then

(a) $\mu(\mu^* \wedge \nu) \lhd \mu(\mu^* \wedge \nu^*)$,

(b) $\nu(\mu \wedge \nu^*) \lhd \nu(\mu^* \wedge \nu^*)$, and

(c) $\mu(\mu^* \wedge \nu^*)/\mu(\mu^* \wedge \nu) \simeq \nu(\mu^* \wedge \nu^*)/\nu(\mu \wedge \nu^*)$.

PROOF :

(a) $\mu^* \wedge \nu \lhd \mu^* \wedge \nu^*$ since $\nu \lhd \nu^*$. Hence $\mu(\mu^* \wedge \nu) \lhd \mu^*(\mu^* \wedge \nu^*)$ since $\mu \lhd \mu^*$.
Therefore $\mu(\mu^* \wedge \nu) \lhd \mu(\mu^* \wedge \nu^*)$.
Similarly (b) holds.

(c) supp $(\mu \wedge \nu^*)$ and supp $(\mu^* \wedge \nu)$ are normal in
supp $(\mu^* \wedge \nu^*)$ = supp $\mu^* \cap$ supp ν^*.
Therefore D = supp $(\mu \wedge \nu^*)(\mu^* \wedge \nu) \lhd$ supp $(\mu^* \wedge \nu^*)$.
Let x supp $\nu(\mu \wedge \nu^*) \in$ supp $\nu(\mu^* \wedge \nu^*)/\nu(\mu \wedge \nu^*)$.

Therefore $\sup\{\nu(\mu^* \wedge \nu^*)(a) : a \text{ supp } \nu(\mu \wedge \nu^*) = x \text{ supp } \nu(\mu \wedge \nu^*)\} > 0$.

So there exists a_0 such that $\nu(\mu^* \wedge \nu^*)(a_0) > 0$

where a_0 supp $\nu(\mu \wedge \nu^*)$ = x supp $\nu(\mu \wedge \nu^*)$.

Hence there exist a_1, a_2, $a_0 = a_1a_2$, such that $\nu(a_1) \wedge (\mu^* \wedge \nu^*)(a_2) > 0$, where

a_1a_2 supp $\nu(\mu \wedge \nu^*)$ = x supp $\nu(\mu \wedge \nu^*)$.

So $a_1 \in$ supp ν and $a_2 \in$ supp $(\mu^* \wedge \nu^*)$.

Hence x supp $\nu(\mu \wedge \nu^*) = a_2$ supp $\nu(\mu \wedge \nu^*)$.

Define ψ : supp $\nu(\mu^* \wedge \nu^*)/\nu(\mu \wedge \nu^*) \longrightarrow$ supp $\mu^* \wedge \nu^*/\omega$, where
$\omega = (\mu \wedge \nu^*)(\mu^* \wedge \nu)$, by

ψ(x supp $\nu(\mu \wedge \nu^*)$) $= x_2$D, where D = supp ω,

x supp $\nu(\mu \wedge \nu^*) = x_1x_2$ supp $\nu(\mu \wedge \nu^*)$,

$x_1 \in$ supp ν and $x_2 \in$ supp$(\mu^* \wedge \nu^*)$.

It is routine to check that ψ is a well–defined crisp isomorphism.
We now argue that
$\nu(\mu^* \wedge \nu^*)/\nu(\mu \wedge \nu^*)$ (x supp $\nu(\mu \wedge \nu^*)$) = $\mu^* \wedge \nu^*/\omega$ (ψ(x supp $\nu(\mu \wedge \nu^*)$)) :
Let LHS = α_1 and RHS = α_2.

Therefore $\alpha_1 = \sup\{\nu(\mu^* \wedge \nu^*)(a) : a \text{ supp } \nu(\mu \wedge \nu^*) = x_2 \text{ supp } \nu(\mu \wedge \nu^*)\}$

x_2 as defined above.

$$\text{So} \qquad \alpha_1 = \sup_{a \in \text{supp } \nu(\mu \wedge \nu^*)} \nu(\mu^* \wedge \nu^*)(x_2 a)$$

$$\geq \sup_{a \in \text{supp } (\mu^* \wedge \nu)(\mu \wedge \nu^*)} \nu(\mu^* \wedge \nu^*)(x_2 a)$$

$$\geq \sup_{a \in D} \mu^* \wedge \nu^*(x_2 a)$$

$$= \sup_{yD = x_2 D} \mu^* \wedge \nu^*(y)$$

$$= \mu^* \wedge \nu^*/\omega\,(x_2 D).$$

$$= \mu^* \wedge \nu^*/\omega\,(\psi(x \text{ supp } \nu(\mu \wedge \nu^*))) = \alpha_2.$$

i.e. $\alpha_1 \geq \alpha_2$ **(1)**

Next we show that $\alpha_2 \geq \alpha_1$:

$\alpha_1 = \sup\{\nu(\mu^* \wedge \nu^*)(y) : y \text{ supp } \nu(\mu \wedge \nu^*) = x_2 \text{ supp } \nu(\mu \wedge \nu^*)\}$.

Let $\epsilon \in (0, \alpha_1 \wedge \alpha_2)$. There exists y_o, $y_o \text{ supp } \nu(\mu \wedge \nu^*) = x_2 \text{ supp } \nu(\mu \wedge \nu^*)$, such that $\alpha_1 \geq \nu(\mu^* \wedge \nu^*)(y_o) > \alpha_1 - \epsilon/2$. Let $\beta_1 = \alpha_1 - \nu(\mu^* \wedge \nu^*)(y_o)$. So $\alpha_1 = \nu(\mu^* \wedge \nu^*)(y_o) + \beta_1$, $0 \leq \beta_1 < \epsilon$.

$$\nu(\mu^* \wedge \nu^*)(y_o) = \sup_{y_o = y_1 y_2} \nu(y_1) \wedge (\mu^* \wedge \nu^*)(y_2).$$

So there exist $y_1, y_2, y_o = y_1 y_2$, $y_1 \in \text{supp } \nu$, $y_2 \in \text{supp } (\mu^* \wedge \nu^*)$, such that $\nu(y_1) \wedge (\mu^* \wedge \nu^*)(y_2) > \nu(\mu^* \wedge \nu^*)(y_o) - \epsilon/2 = \alpha_1 - \beta_1 - \epsilon/2$.

Therefore $\alpha_1 < \nu(y_1) \wedge (\mu^* \wedge \nu^*)(y_2) + \beta_1 + \epsilon/2$. (2)

Also, $x \text{ supp } \nu(\mu \wedge \nu^*) = y_o \text{ supp } \nu(\mu \wedge \nu^*)$.

So $x = y_o b'$, $b' \in \text{supp } \nu(\mu \wedge \nu^*)$, $y_o = y_1 y_2$.

Therefore $x \text{ supp } \nu(\mu \wedge \nu^*) = y_2 \text{ supp } \nu(\mu \wedge \nu^*)$.

$$\text{Hence } \alpha_2 = \mu^* \wedge \nu^*/\omega\,(\psi(x \text{ supp } \nu(\mu \wedge \nu^*)))$$

$$= \mu^* \wedge \nu^*/\omega\,(y_2 D) = \sup_{yD = y_2 D} \mu^* \wedge \nu^*(y).$$

There exists y_3, $y_3 D = y_2 D$, such that $\alpha_2 - \epsilon/2 < (\mu^* \wedge \nu^*)(y_3)$.

Let $\beta_2 = \alpha_2 - (\mu^* \wedge \nu^*)(y_3) < \epsilon$.

Therefore $\alpha_2 \quad = (\mu^* \wedge \nu^*)(y_3) + \beta_2$,

$\geq (\mu^* \wedge \nu^*)(y_2) + \beta_2$, otherwise use y_2 in the place of y_3.

So $\alpha_2 \geq (\mu^* \wedge \nu^*)(y_2) \wedge \nu(y_1) + \beta_2$.

So $\alpha_2 > \alpha_1 - \beta_1 - \epsilon/2 + \beta_2$ by (2).

As $\epsilon \longrightarrow 0$, $\beta_1, \beta_2 \longrightarrow 0$.

Hence $\alpha_2 \geq \alpha_1$ **(3)**

(3) and **(1)** imply that $\alpha_2 = \alpha_1$.

i.e. $\nu(\mu^* \wedge \nu^*)/\nu(\mu \wedge \nu^*) \simeq \mu^* \wedge \nu^*/\omega$.
By symmetry, $\mu(\mu^* \wedge \nu^*)/\mu(\mu^* \wedge \nu) \simeq \mu^* \wedge \nu^*/\omega$.
Therefore $\nu(\mu^* \wedge \nu^*)/\nu(\mu \wedge \nu^*) \simeq \mu(\mu^* \wedge \nu^*)/\mu(\mu^* \wedge \nu)$.

The proof is complete.

REMARK :

Zassenhaus Lemma is a generalization of the second isomorphism theorem (Theorem 3.2.7). (Set $\nu = \nu_e$ and $\mu^* = \chi_{\mathcal{G}}$).

DEFINITION : 5.2.3

Let μ be a fuzzy subgroup of $\mathcal{G}$ containing the fuzzy subgroups μ_i, $i = 1,2,\cdots,n$, satisfying

(i) $\quad \mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = \mu_e$ (*)

and (ii) $\quad \mu_i \triangleleft \mu_{i-1}$, $i = 2,\cdots,n$.

Then (*) is called a *normal series* or a *normal chain* of μ.

DEFINITION : 5.2.4

Let $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = \mu_e$ be a normal series of μ. A *refinement* of the series is a normal series obtained from the above series by inserting new fuzzy subgroups without removing any in the above series.

DEFINITION : 5.2.5

Two normal chains,
(1) $\nu = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_m = \mu$ and (2) $\nu = \nu_0 \leq \nu_1 \leq \cdots \leq \nu_n = \mu$, are said to be *isomorphic* if there is a one–to–one correspondence between the quotient fuzzy groups of (1) and (2) such that the corresponding quotient fuzzy groups are isomorphic.

PROPOSITION : 5.2.6

Let (1) and (2) be as in definition 5.2.5. If (1) and (2) are isomorphic, then their lengths are equal.

PROOF :

Let $\mu_k \leq \omega \leq \mu_{k+1}$.

Therefore $\omega/\mu_k \leq \mu_{k+1}/\mu_k \simeq \nu_{j+1}/\nu_j$, say. Hence there is a quotient $f(\omega/\mu_k) \leq \nu_{j+1}/\nu_j$, where $f : \mu_{k+1}/\mu_k \longrightarrow \nu_{j+1}/\nu_j$ is a fuzzy isomorphism. ω/μ_k and $f(\omega/\mu_k)$ behave similarly. Let $f(\omega/\mu_k) = \omega' \wedge \nu_{j+1}/\nu_j$. So $\nu_j \leq \omega' \wedge \nu_{j+1} \leq \nu_{j+1}$. The quotients μ_{k+1}/ω and $\nu_{j+1}/\omega' \wedge \nu_{j+1}$ behave similarly.

Hence (1) and (2) must have the same length.

DEFINITION 5.2.7

A chain $\nu = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_n = \mu$ of normal fuzzy subgroups is a *maximal chain* (or a *composition series*) iff whenever $\mu_{i-1} \leq \omega \leq \mu_i$, where $\omega \triangleleft \mu_i$, then $\omega/\mu_{i-1} \equiv \mu_i/\mu_{i-1}$ or $\omega/\mu_{i-1} \equiv \mu_{i-1}/\mu_{i-1}$. ν is a *maximal normal fuzzy subgroup* of μ iff $\nu \neq \mu$, and whenever $\nu \leq \omega \leq \mu$, where $\omega \triangleleft \mu$, $\omega/\nu \equiv \mu/\nu$ or $\omega/\nu \equiv \nu/\nu$.

EXAMPLE : 5.2.8

Let $\mathscr{G} = S_3$. Let $\mu(x) = \begin{cases} 1 & x = e \\ 3/4 & x = a, a^2 \\ 1/2 & \text{otherwise} \end{cases}$, and $\nu(x) = \begin{cases} 1 & x = e \\ 1/2 & \text{otherwise} \end{cases}$.

$\nu \neq \mu$ and $\nu \triangleleft \mu$.

It is easy to see that ν is a maximal normal fuzzy subgroup of μ.

Let $\nu_1(x) = \begin{cases} 1 & x = e \\ {}^1/_2 & x = a, a^2 \\ 0 & \text{otherwise} \end{cases}$. Let $\nu_1 \leq \omega \leq \mu$, where $\omega \triangleleft \mu$.

$\mu/\nu_1(x \text{ supp } \nu_1) = \begin{cases} 1 & x \in \text{supp } \nu_1 \\ {}^1/_2 & x \notin \text{supp } \nu_1 \end{cases}$, $\omega/\nu_1(x \text{ supp } \nu_1) = \begin{cases} 1 & x \in \text{supp } \nu_1 \\ \alpha & \text{otherwise} \end{cases}$,

for some $\alpha > 0$, therefore $\omega/\nu_1 \equiv \mu/\nu_1$ or $\omega/\nu_1 \equiv \nu_1/\nu_1$. So ν_1 is a maximal normal fuzzy subgroup of μ. Observe that $\nu_1 < \nu < \mu$, but both ν_1 and ν are maximal normal fuzzy subgroups of μ.

DEFINITION : 5.2.9

μ is *simple* in case μ_e is a maximal normal fuzzy subgroup of μ. So if $\mu_e \leq \omega \leq \mu$, where $\omega \triangleleft \mu$, then $\omega \equiv \mu$ or $\omega = \mu_e$.

THEOREM 5.2.10

μ/ν is simple if and only if ν is a maximal normal fuzzy subgroup of μ.

PROOF :

$\Longrightarrow$: Obvious

$\Longleftarrow$: Let ν be a maximal normal fuzzy subgroup of μ. Let $\omega/\nu \triangleleft \mu/\nu$. Let $f: \mu \longrightarrow \mu/\nu$ be the natural homomorphism. $\nu \leq \mu \wedge f^{-1}(\omega/\nu) = \xi \leq \mu$. By maximality of ν, $\xi/\nu \equiv \mu/\nu$ or $\xi/\nu \equiv \nu/\nu$. But $\xi/\nu = \omega/\nu$, hence $\omega/\nu \equiv \mu/\nu$ or $\omega/\nu \equiv \nu/\nu$. Thus μ/ν is simple.

PROPOSITION : 5.2.11

Let (1) $\nu = \mu_1 \leq \cdots \leq \mu_n = \mu$ be a maximal chain. Any refinement of this chain has the same length as the length of (1).

PROOF :

Let $\mu_k \leq \omega \leq \mu_{k+1}$, where $\omega \triangleleft \mu_{k+1}$. Since the chain is maximal, $\omega/\mu_k \equiv \mu_{k+1}/\mu_k$ or $\omega/\mu_k \equiv \mu_k/\mu_k$. Hence no new equivalence classes of quotients are formed when refining the chain. Hence the length of the chain remains unaltered by a refinement of the chain.

THEOREM : 5.2.12 (SCHREIER)

Any two normal chains of fuzzy subgroups between the same two fuzzy subgroups have isomorphic refinements.

PROOF :

Let (1) $\nu = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_m = \mu$ and (2) $\nu = \nu_0 \leq \nu_1 \leq \cdots \leq \nu_n = \mu$ be two normal chains. For $i = 1,2,\cdots,m$ and $j = 1,2,\cdots, n$, let $\mu_{ij} = (\mu_i \wedge \nu_j)\nu_{j-1}$, $\nu_{ji} = (\nu_j \wedge \mu_i)\mu_{i-1}$, $\mu_{0j} = (\mu_0 \wedge \nu_j)\nu_{j-1}$, $\nu_{0i} = (\nu_0 \wedge \mu_i)\mu_{i-1}$. By Zassenhaus lemma, $\mu_{ij}/\mu_{(i-1)j} \simeq \nu_{ji}/\nu_{(j-1)i}$, $i = 1,2,\cdots,m$; $j = 1,2,\cdots,n$.

The chains $\nu = \nu_{01} \leq \nu_{11} \leq \cdots \leq \nu_{n1} \leq \nu_{12} \leq \cdots \leq \nu_{n2} \leq \nu_{13} \leq \cdots \leq \nu_{nm} = \mu$ (3)

and $\nu = \mu_{01} \leq \mu_{11} \leq \cdots \leq \mu_{m1} \leq \mu_{12} \leq \cdots \leq \mu_{m2} \leq \mu_{13} \leq \cdots \leq \mu_{mn} = \mu$ (4)

refine (1) and (2) respectively. (3) and (4) are clearly isomorphic.

THEOREM : 5.2.13 (JORDAN—HÖLDER)

Let μ be a fuzzy subgroup of finite length. Any finite normal chain of fuzzy subgroups of μ can be refined to a maximal chain, and any two maximal chains between two given fuzzy subgroups have the same length.

PROOF :

Let (1) $\nu = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_n = \mu$ and

(2) $\nu = \nu_0 \leq \nu_1 \leq \cdots \leq \nu_m = \mu$ be two normal chains between ν and μ. Since $\ell(\mu) < \infty$, (1) can be refined to a maximal chain. Suppose now that (1) and (2) are maximal chains. By the Schreier theorem, (1) and (2) have isomorphic refinements (1)′ and (2)′ respectively. Since (1)′ and (2)′ are isomorphic, they have the same length. By Proposition 5.2.11, (1) and (2) have the same length.

This completes the proof.

CHAPTER 6

SOLVABILITY AND NILPOTENCY IN FUZZY SUBGROUPS.

6.1 : INTRODUCTION

DEFINITION : 6.1.1 [1]

Let μ be a fuzzy subgroup of $\mathcal{G}$. μ is *fuzzy solvable* if there is a chain of fuzzy normal subgroups $\mu = \nu_1 \geq \cdots \geq \nu_k$ with $\nu_k(x) = \nu_1(e)$ only when $x = e$, and $\nu_i(e) = \nu_1(e)$, $1 \leq i < k$, such that ν_i/ν_{i+1} is fuzzy Abelian.

Some comments about this definition :

(i) We deduce from the definition that a solvable fuzzy subgroup must be fuzzy normal. We feel this is a strong demand since in the crisp case a subgroup H of $\mathcal{G}$ can be solvable even if H is not normal in $\mathcal{G}$. For example let $\mathcal{G} = S_3 = \{e, a, a^2, b, ab, a^2b\}$, $a^3 = e = b^2$. Let $H = \{e,b\}$. H is not normal in $\mathcal{G}$, but H is solvable. Also, the demand that all the fuzzy subgroups in the solvable series for μ be fuzzy normal is strong.

(ii) By μ fuzzy Abelian, the authors mean that $E_\mu = \{x \in \mathcal{G} : \mu(x) = \mu(e)\}$ is Abelian. We remarked earlier that this definition of fuzzy Abelian is not acceptable to us since any fuzzy subgroup μ having $E_\mu = \{e\}$ is necessarily fuzzy Abelian although the other level subgroups need not be Abelian.

(iii) Let μ be fuzzy normal and $E_\mu = \{e\}$. In terms of Definition 6.1.1 it is straightforward to see that μ is fuzzy solvable.

(iv) If μ is fuzzy Abelian, μ need not be solvable, contrary to the crisp case.

EXAMPLE :

Consider $S_3 = \{e, a, a^2, b, ab, a^2b\}$, $a^3 = e = b^2$.

Define μ by $\mu(e) = 1$, $\mu(b) = 1/2$ and $\mu(x) = 0$, $x \neq e,b$.
μ is not fuzzy normal in S_3.

$E_\mu = \{e\}$ implies that μ is fuzzy Abelian. But μ is not solvable since μ is not fuzzy normal.

(v) If μ is fuzzy solvable, then supp μ need not be solvable :

EXAMPLE :

Let $\mathcal{G}$ be a non–solvable group. Define $\mu : \mathcal{G} \longrightarrow [0,1]$ by

$$\mu(x) = \begin{cases} 1 & x = e \\ 1/2 & x \neq e, \end{cases}$$ then μ is fuzzy normal in $\mathcal{G}$ and $E_\mu = \{e\}$.

Hence, by (iii) above, μ is fuzzy solvable. But supp $\mu = \mathcal{G}$ is not solvable.

(vi) The authors state only two analogues of the crisp case, *viz.* :

(a) A subgroup of a solvable group is solvable.
(b) A quotient of a solvable group is solvable.

We aim to give a more acceptable definition of a solvable fuzzy subgroup μ such that

(a) μ is not necessarily fuzzy normal,
(b) quotients used are those given in Definition 1.2.6.
(c) supp μ is solvable.

6.2 : SOLVABILITY IN FUZZY SUBGROUPS.

DEFINITION : 6.2.1

Let μ be a fuzzy subgroup of $\mathcal{G}$. Let $\mu = \mu_1 \geq \cdots \geq \mu_k = \mu_e$ be a normal series for μ. If μ_i/μ_{i+1} is fuzzy Abelian for all $i = 1,\cdots,k-1$, then the series is a *solvable series* for μ, and μ is said to be *fuzzy solvable* (or just solvable) in $\mathcal{G}$.

REMARK : 6.2.2

Definition 6.2.1 does not necessarily imply definition 6.1.1 since our notion of normality in the series is weaker than saying that each μ_i is fuzzy normal in $\mathcal{G}$.

On the other hand our definition is not necessarily weaker than Definition 6.1.1 since Definition 6.2.1 implies that supp μ is solvable, whereas Definition 6.1.1 does not necessarily imply that supp μ is solvable. It is now obvious that solvability in terms of Definition 6.1.1 does not imply solvability in terms of Definition 6.2.1.

From now on, whenever solvability is mentioned, we have Definition 6.2.1 in mind unless specified otherwise. The following two propositions are straightforward.

PROPOSITION : 6.2.3

Let μ be a solvable fuzzy subgroup of $\mathcal{G}$. Then supp μ is solvable in $\mathcal{G}$.

PROPOSITION : 6.2.4

If μ is fuzzy Abelian, then μ is solvable.

REMARK : 6.2.5

The notion of Abelian used above is the notion given in this thesis. If we use the notion of Abelian given in [1], Proposition 6.2.4 still holds. For example if E_μ is Abelian, then let

$$x\mu_e, y\mu_e \in E_{\mu/\mu_e} = \{x\mu_e : \mu/\mu_e\,(x\mu_e) = \mu/\mu_e\,(e\mu_e)\} = \{x\mu_e : \mu(x) = \mu(e)\}.$$

Then $x, y \in E_\mu$. So $xy = yx$. Hence $(x\mu_e)(y\mu_e) = xy\mu_e = yx\mu_e = (y\mu_e)(x\mu_e)$.

Hence μ/μ_e is fuzzy Abelian in terms of the definition in [1]. It now follows that μ is solvable in terms of Definition 6.2.1.

Note that this does not contradict (iv) in 6.1 since in our definition of solvability we do not require that μ be fuzzy normal.

In the above example the use of the strong quotient is not harmful since

μ/μ_e (x supp μ_e) $= \mu/\mu_e$ (e supp μ_e)

$\Leftrightarrow \mu(x) = \mu(e)$ also in terms of our quotient.

PROPOSITION : 6.2.6

A fuzzy subgroup of a solvable fuzzy subgroup is solvable.

PROOF :

Let $\nu \leq \mu$ be fuzzy subgroups of $\mathcal{G}$, where μ is fuzzy solvable. Let $\mu = \mu_1 \geq \mu_2 \cdots \geq \mu_n = \mu_e$ be a solvable series for μ. Consider $\nu = \mu_1 \wedge \nu \geq \mu_2 \wedge \nu \geq \cdots \geq \mu_n \wedge \nu = \mu_e$. Now $\nu \triangleleft \nu$ and $\mu_i \triangleleft \mu_{i-1}$, hence $\nu \wedge \mu_i \triangleleft \nu \wedge \mu_{i-1}$. It now follows that ν is solvable.

REMARK : 6.2.7

Using Definition 6.1.1 of solvability, it is not apparant how a non–fuzzy normal subgroup ν of a solvable fuzzy subgroup μ can be fuzzy solvable.

Suppose that in Proposition 6.2.6 we replace the quotients μ_i/μ_{i+1} by the strong quotients $(\mu_i/\mu_{i+1})_s$ given in Definition 1.1.15 [1], then it is not necessarily true that a fuzzy subgroup of a solvable fuzzy subgroup is solvable.

EXAMPLE : 6.2.8

Let $\mathcal{G} = S_4$. Let $\mu(x) = \begin{cases} c & {}^1/_3 < c \leq 1,\ x \in A_4 \\ 0 & x \notin A_4 \end{cases}$. Then μ is a fuzzy subgroup of $\mathcal{G}$.

We show that μ is solvable :

Let S be a normal subgroup of A_4 of order 4.

$$\text{Define } \mu_2(x) = \begin{cases} c_1 & 0 < c_1 < c,\ x \in S \\ 0 & x \notin S \end{cases}.$$

Let $\mu_3 = \mu_e$, $\mu_1 = \mu$. Each μ_i is fuzzy normal in $\mathcal{G}$ and $(\mu_i/\mu_{i+1})_s$ is fuzzy Abelian. Hence μ is solvable.

$$\text{Let } \nu(x) = \begin{cases} 1/3 & x \in A_4 \setminus \{e\} \\ c & 1/3 < c \leq 1,\ x = e\ . \\ 0 & x \notin A_4 \end{cases}$$

NOTE : We use the same c that is used in the definition of μ. ν is fuzzy normal in $\mathcal{G}$. Suppose that $\nu = \nu_1 \geq \cdots \geq \nu_n$ is a solvable series for ν. Then $(\nu/\nu_2)_s$ is fuzzy Abelian. Therefore for all $x,y \in \text{supp } \nu = A_4$, $xyx^{-1}y^{-1} \in E_{\nu_2} = \{e\}$. Hence A_4 is Abelian. Contradiction ! Therefore ν is not solvable.

It is clear that solvability in terms of the strong quotients implies solvability. We now show that a homomorphic image of a solvable group is solvable.

PROPOSITION : 6.2.9

Let μ be a fuzzy subgroup of $\mathcal{G}$. Let $f : \mathcal{G} \longrightarrow \mathcal{G}'$ be an epimorphism, where $\mathcal{G}'$ is a group. If μ is solvable in $\mathcal{G}$, then $f(\mu)$ is solvable in $\mathcal{G}'$.

PROOF :

Let $\mu = \mu_1 \geq \cdots \geq \mu_k = \mu_e$ be a solvable series for μ. Therefore $\mu_i \lhd \mu_{i-1}$.

We claim that $f(\mu_i) \lhd f(\mu_{i-1})$:

Let $f(a)_\lambda \in f(\mu_{i-1})$. We will show that $f(a)_\lambda\, f(\mu_i) = f(\mu_i)\, f(a)_\lambda$. We consider various cases :

(i) **Case $a_\lambda \in \mu_{i-1}$:**

Let $x \in \mathcal{G}$.

$f(e)_\lambda\, f(\mu_i)(f(x)) \qquad = \lambda \wedge f(\mu_i)(f(x))$

$= \sup\{\lambda \wedge \mu_i(y) : f(aya^{-1}) = f(axa^{-1})\}$

$= \sup\{\mu_i(a^{-1}(aya^{-1})a) \wedge \lambda : f(aya^{-1}) = f(axa^{-1})\}$

$= \sup\{\lambda \wedge \mu_i(aya^{-1}) : f(aya^{-1}) = f(axa^{-1})\}$ since $\mu_i \lhd \mu_{i-1}$

$= \sup\{\lambda \wedge \mu_i(z) : f(z) = f(axa^{-1})\}$

$= f(\mu_i)(f(axa^{-1})) \wedge \lambda$

Hence $(f(e)_\lambda f(\mu_i))(f(x)) = (f(a)_\lambda f(\mu_i) f(a)_\lambda^{-1})(f(x))$.

(ii) **Case $a_\lambda \notin \mu_{i-1}$**

Now $\lambda \leq \sup_{f(y) = f(a)} \mu_{i-1}(y)$.

Case $\lambda = \sup_{f(y) = f(a)} \mu_{i-1}(y)$:

Let $\epsilon \in (0,\lambda)$. There exists y_o, $f(y_o) = f(a)$, such that $\lambda - \epsilon < \mu_{i-1}(y_o)$, i.e. $(y_o)_{\lambda-\epsilon} \in \mu_{i-1}$.

Therefore by Case (i), $f(y_o)_{\lambda-\epsilon} f(\mu_i)(f(x)) = f(\mu_i) f(y_o)_{\lambda-\epsilon}(f(x))$.

Hence $f(a)_{\lambda-\epsilon} f(\mu_i)(f(x)) = f(\mu_i) f(a)_{\lambda-\epsilon}(f(x))$. ϵ is arbitrarily small.

Hence $f(a)_\lambda f(\mu_i)(f(x)) = f(\mu_i) f(a)_\lambda (f(x))$.

Case $\lambda < \sup_{f(a) = f(y)} \mu_{i-1}(y)$:

Therefore $\lambda < \mu_{i-1}(y)$ for some y satisfying $f(y) = f(a)$. Hence, as above, $f(a)_\lambda f(\mu_i)(f(x)) = f(\mu_i) f(a)_\lambda (f(x))$.

So the series $f(\mu) = f(\mu_i) \geq \cdots \geq f(\mu_k) = f(\mu)_{f(e)}$ is a normal series for $f(\mu)$. It is easy to show that each $f(\mu_i)/f(\mu_{i+1})$ is fuzzy Abelian. Hence $f(\mu)$ is solvable.

PROPOSITION : 6.2.10

Let $\nu \leq \mu$ be fuzzy subgroups of $\mathcal{G}$ such that $\nu \lhd \mu$. If μ is solvable, then μ/ν is solvable.

PROOF :

Let f : supp $\mu \longrightarrow$ supp μ/ν be the natural homomorphism. Therefore $f(\mu) = \mu/\nu$ is solvable by Proposition 6.2.9.

PROPOSITION : 6.2.11

Let μ and ν be fuzzy subgroups of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, such that $\mu(e) = \nu(e')$, where e′ is the identity in $\mathcal{G}_2$ and e is the identity in $\mathcal{G}_1$. Then $\mu \times \nu$ is solvable.

PROOF :

Straightforward.

PROPOSITION : 6.2.12

Let $\nu \leq \mu$ be fuzzy subgroups of $\mathcal{G}$ such that $\nu \triangleleft \mu$. Suppose μ/ν and ν are both solvable, then μ is solvable.

PROOF :

Let $\mu/\nu = \mu_1/\nu \geq \cdots \geq \mu_n/\nu = \nu/\nu$ and $\nu = \nu_1 \geq \cdots \geq \nu_k = \nu_e$ be solvable series for μ/ν and ν respectively.

Let $f : \mathcal{G} \longrightarrow \mathcal{G}/\text{supp } \nu$ be the natural homomorphism.
Let $\mu'_i = f^{-1}(\mu_i/\nu)$ and $\mu'_i{}' = \mu'_i \wedge \mu \geq \nu$.

CLAIM : $\mu'_i{}' /\nu = \mu'_i/\nu = \mu_i/\nu$.

It is clear that $\mu'_i/\nu = \mu_i/\nu$.

$\mu'_i{}' /\nu\, (x \text{ supp } \nu) = \mu'_i \wedge \mu/\nu\, (x \text{ supp } \nu)$

$= \sup\{f^{-1}(\mu_i/\nu)(a) \wedge \mu(a) : a \text{ supp } \nu = x \text{ supp } \nu\}$

$= \mu_i/\nu(x \text{ supp } \nu) \wedge \sup\{\mu(a) : a \text{ supp } \nu = x \text{ supp } \nu\}$

$= \mu_i/\nu \wedge \mu/\nu\, (x \text{ supp } \nu)$

$= \mu_i/\nu\, (x \text{ supp } \nu)$.

Hence $\mu/\nu = \mu'_1{}' /\nu \geq \cdots \geq \mu'_n{}' /\nu = \nu/\nu$ is a solvable series for μ/ν.

Therefore $\mu = \mu'_1{}' \geq \mu'_2{}' \geq \cdots \geq \mu'_n{}' \geq \nu \geq \nu_2 \geq \cdots \geq \nu_k = \nu_e$

is a normal series for μ. (*)

Note that $\mu_n'' = \mu_n' \wedge \mu = f^{-1}(\mu_n/\nu) \wedge \mu \geq \nu$ since if $x \in$ supp ν, then

$\mu_n''(x) = \mu_n/\nu\,(e \text{ supp } \nu) \wedge \mu(x) = \mu(e) \wedge \mu(x) = \mu(x) \geq \nu(x)$.

Let $x, y \in \text{supp } \mu_i'' = \text{supp } \mu_i' \wedge \mu = \text{supp } f^{-1}(\mu_i/\nu) \wedge \text{supp } \mu$.

Therefore $f(x), f(y) \in \text{supp } \mu_i/\nu$. Hence $f(xyx^{-1}y^{-1}) \in \text{supp } \mu_{i+1}/\nu$. This implies that $xyx^{-1}y^{-1} \in \text{supp } f^{-1}(\mu_{i+1}/\nu) = \text{supp } \mu_{i+1}'$.

Therefore $xyx^{-1}y^{-1} \in \text{supp } \mu_{i+1}''$, hence μ_i''/μ_{i+1}'' is fuzzy Abelian. Supp $\mu_n'' = \text{supp } \nu$. Hence μ_n''/ν is also fuzzy Abelian. Hence (*) is a solvable series for μ.

PROPOSITION : 6.2.13

Let μ and ν be fuzzy subgroups of $\mathcal{G}$ such that ν is fuzzy normal in $\mathcal{G}$. Suppose μ and ν are solvable. Then $\mu\nu$ is also solvable.

PROOF :

Define $f : \text{supp } \mu \longrightarrow \text{supp } \mu\nu/\nu$ by $f(a) = a \text{ supp } \nu$. f is a homomorphism.

We claim that $f(\mu) = \mu\nu/\nu$.

$$f(\mu)(x \text{ supp } \nu) = \sup\{\mu(a) : a \text{ supp } \nu = x \text{ supp } \nu\} \leq \mu\nu/\nu\,(x \text{ supp } \nu).$$

$$\begin{aligned}\mu\nu/\nu(x \text{ supp } \nu) &= \sup\{\mu\nu(a) : a \text{ supp } \nu = x \text{ supp } \nu\} \\ &= \sup\Big\{ \sup_{a = a_1a_2} \mu(a_1) \wedge \nu(a_2) : a \text{ supp } \nu = x \text{ supp } \nu \Big\} \\ &= \sup\{\mu(a_1) \wedge \nu(a_2) : a_1a_2 \text{ supp } \nu = x \text{ supp } \nu\},\end{aligned}$$

$a_2 \in \text{supp } \nu$.

Hence

$$\begin{aligned}\mu\nu/\nu(x \text{ supp } \nu) &= \sup\{\mu(a_1) \wedge \nu(a_2) : a_1 \text{ supp } \nu = x \text{ supp } \nu\} \\ &\leq \sup\{\mu/\nu(a_1 \text{ supp } \nu) \wedge \nu/\nu(a_2 \text{ supp } \nu) : a_1 \text{ supp } \nu = x \text{ supp } \nu\} \\ &= \mu/\nu\,(x \text{ supp } \nu) \wedge \nu/\nu\,(e \text{ supp } \nu) \\ &= \mu/\nu\,(x \text{ supp } \nu).\end{aligned}$$

So $\mu\nu/\nu = \mu/\nu$, where $\mu/\nu\,(x \text{ supp } \nu) = \sup\{\mu(a) : a \text{ supp } \nu = x \text{ supp } \nu\}$.

So $f(\mu) = \mu\nu/\nu$.

Hence $\mu\nu/\nu$ is solvable.

So, by Proposition 6.2.12, $\mu\nu$ is solvable.

Let μ be a fuzzy subgroup of $\mathcal{G}$. Recall that the commutator fuzzy subgroup of μ, denoted by μ', is the smallest fuzzy subgroup of μ such that μ/μ' is fuzzy Abelian. By $\mu^{(2)}$ we mean $(\mu')'$, and by $\mu^{(n)}$ we mean $\mu^{(n-1)'}$.

PROPOSITION : 6.2.14

Let $\mu = \mu_1 \geq \cdots \geq \mu_n = \mu_e$ be a solvable series for μ ; then $\mu_i \geq \mu^{(i)}$ for all $i = 1,2,\cdots,n$.

PROOF :

Straightforward.

THEOREM : 6.2.15

Let μ be a fuzzy subgroup of $\mathcal{G}$. μ is solvable if and only if there exists $n \in \mathbb{N}$ such that $\mu^{(n)} = \mu_e$.

PROOF :

Let $\mu = \mu_1 \geq \cdots \geq \mu_n = \mu_e$ be a solvable series for μ. Therefore $\mu_n \geq \mu^{(n)}$. Hence $\mu^{(n)} = \mu_e$. Conversely, let $n \in \mathbb{N}$ such that $\mu^{(n)} = \mu_e$. Now $\mu \geq \mu' \geq \mu^{(2)} \geq \cdots \geq \mu^{(n)} = \mu_e$ is a normal series for μ. Clearly $\mu^{(i)}/\mu^{(i+1)}$ is fuzzy Abelian. Hence μ is solvable.

6.3 : NILPOTENCY IN FUZZY SUBGROUPS

Let μ and ν be fuzzy subgroups of $\mathcal{G}$.
Recall Definition 2.1.8 : $[\mu,\nu] = <\{[h_\lambda,k_\beta] : h_\lambda \in \mu, k_\beta \in \nu\}>$.

DEFINITION : 6.3.1

The descending central series of μ, (DCS), is the normal series $\mu = \gamma_1(\mu) \geq \gamma_2(\mu) \geq \cdots$ given in Definition 2.1.11.

DEFINITION : 6.3.2

A fuzzy subgroup μ is *nilpotent* if there exists $m \in \mathbb{Z}$ such that $\gamma_{m+1}(\mu) = \mu_e$.

PROPOSITION : 6.3.3

A fuzzy subgroup of a nilpotent fuzzy subgroup is nilpotent.

PROOF :

Let $\nu \leq \mu$ be fuzzy subgroups of $\mathcal{G}$ where μ is nilpotent. Therefore $\gamma_i(\nu) \leq \gamma_i(\mu)$, $i \in \mathbb{N}$. There exists $m \in \mathbb{Z}$ such that $\gamma_{m+1}(\mu) = \mu_e \geq \gamma_{m+1}(\nu)$.

i.e. $\gamma_{m+1}(\nu) = \mu_e$.

The proof is complete.

Let $f : \mu \longrightarrow \nu$. From now on assume $f^{-1}(\nu) = \mu \wedge f^{-1}(\nu)$, so that $f^{-1}(\nu) \leq \mu$.

THE ASCENDING CENTRAL SERIES (ACS) :

Let μ be a fuzzy subgroup of $\mathcal{G}$. Define a sequence of fuzzy subgroups of μ as follows :
Let $\mu^0 = \mu_e$, $\mu^1 = Z(\mu)$. Hence $\mu^1 \triangleleft \mu$. Let $\gamma_1 : \mu \longrightarrow \mu/\mu^1$ be the natural homomorphism, and μ^1_E the fuzzy kernel of γ_1 associated with μ^1. Then $\mu^1_E \triangleleft \mu$, and $\mu/\mu^1_E = \mu/\mu^1$. Let $\mu^0_E = \mu_e$. Let $\mu^2 = \gamma_1^{-1}(Z(\mu/\mu^1_E))$.

Then $\mu^2/\mu^1_E = Z(\mu/\mu^1_E) = \mu^2/\mu^1$.

$\mu^1 \leq \mu^2$: Let $x \in \text{supp}\,\mu^1$. Then $x\,\text{supp}\,\mu^1 = e\,\text{supp}\,\mu^1$.

So $\mu^2(x) = Z(\mu/\mu^1)(e\,\text{supp}\,\mu^1) = \mu/\mu^1(e\,\text{supp}\,\mu^1)$

$$= \mu(e) \geq \mu^1_E(x) \geq \mu^1(x)$$

We now claim that $\mu^2 \triangleleft \mu$:

Let $a_\lambda \in \mu$. We will show that $a_\lambda\,\mu^2 = \mu^2 a_\lambda$.

Let $x \in \mathcal{G}$. Therefore $a_\lambda\,\mu^2(x) = \lambda \wedge \mu^2(a^{-1}x) = \lambda \wedge \gamma_1^{-1}(Z(\mu/\mu^1))(a^{-1}x)$

$$= \lambda \wedge Z(\mu/\mu^1)(a^{-1}x\,\text{supp}\,\mu^1).$$

Now $Z(\mu/\mu^1) \triangleleft \mu/\mu^1$ and $(a \text{ supp } \mu^1)_\lambda \in \mu/\mu^1$
since $\sup\{\mu(x): x \text{ supp } \mu^1 = a \text{ supp } \mu^1\} \geq \lambda$.
Hence $a_\lambda\ \mu^2(x) = \lambda \wedge Z(\mu/\mu^1)(xa^{-1} \text{ supp } \mu^1)$
$= \mu^2 a_\lambda(x)$.

Let $\gamma_2 : \mu \longrightarrow \mu/\mu^2$ be the natural map, and μ^2_E the fuzzy kernel of γ_2 associated with μ^2. Therefore $\mu^2_E \triangleleft \mu$, and $\mu/\mu^2 = \mu/\mu^2_E$. Let $\mu^3 = \gamma_2^{-1}(Z(\mu/\mu^2_E))$. It can be shown that $\mu^2_E \leq \mu^3$, and $\mu^3/\mu^2_E = Z(\mu/\mu^2_E) = \mu^3/\mu^2$.

Clearly the quotients μ^3/μ^2 are fuzzy Abelian. By induction, we obtain a chain of normal fuzzy subgroups of μ :

$\mu^0 = \mu_e \leq \mu^1 \leq \mu^2 \leq \mu^3 \leq \cdots$ such that

$\mu^{i+1}/\mu^i_E = \mu^{i+1}/\mu^i$ is fuzzy Abelian. (In fact $\mu^{i+1}/\mu^i = Z(\mu/\mu^i)$).

The above series is called the ascending central series of μ.

We will prove later that (i) $\mu = \mu^m \Leftrightarrow \gamma_{m+1}(\mu) = \mu_e$, and (ii) $\gamma_{i+1}(\mu \leq \mu^{m-i}_E$. Before we do this, we need some lemmas :

LEMMA : 6.3.4

Let $\nu \triangleleft \mu$ and $\nu \leq \mu_1 \leq \mu$. Then $[\mu_1,\mu] \leq \nu$ implies that $\mu_1/\nu \leq Z(\mu/\nu)$.
If $\nu = \mu$ on supp ν, then the converse is also true.

PROOF :

($\Longrightarrow$): Let $(x \text{ supp } \nu)_\lambda \in \mu_1/\nu$, and $(y \text{ supp } \nu)_\beta \in \mu/\nu$.

Then $\sup\{\mu_1(a): a \text{ supp } \nu = x \text{ supp } \nu\} \geq \lambda > \lambda - \epsilon$ for $\epsilon \in (0, \lambda \wedge \beta)$.

So there exists a, $a \text{ supp } \nu = x \text{ supp } \nu$, such that $\mu_1(a) > \lambda - \epsilon$,

hence $a_{\lambda-\epsilon} \in \mu_1$.

Similarly there exists b, $b \text{ supp } \nu = y \text{ supp } \nu$, such that $\mu(b) > \beta - \epsilon$,
hence $b_{\beta-\epsilon} \in \mu$.

So $[\mu_1,\mu]\ (a^{-1}b^{-1}ab) \geq \lambda - \epsilon \wedge \beta - \epsilon$. (See the definition of $[\mu_1,\mu]$).

Hence $a^{-1}b^{-1}ab \in$ supp ν.

Therefore ab supp $\nu =$ ba supp ν. So $(xy \text{ supp } \nu)_{\lambda \wedge \beta} = (yx \text{ supp } \nu)_{\lambda \wedge \beta}$, i.e. $(x \text{ supp } \nu)_\lambda \in Z(\mu/\nu)$.

($\Longleftarrow$): $\mu_1/\nu \leq Z(\mu/\nu)$.

Let $c_\xi \in [\mu_1,\mu]$.

So $\xi \leq [\mu_1,\mu](c)$ $= \sup\{\xi' : c_{\xi'} = a_{1\lambda_1} \cdots a_{n\lambda_n}, a_{i\lambda_i} = [h_{i\xi_i}, k_{i\beta_i}], h_{i\xi_i} \in \mu_1, k_{i\beta_i} \in \mu\}$.

Now $(h_ik_i \text{ supp } \nu)_{\xi_i \wedge \beta_i} = (k_ih_i \text{ supp } \nu)_{\xi_i \wedge \beta_i}$

since $\mu_1/\nu \leq Z(\mu/\nu)$.

Therefore $[h_i,k_i] \in$ supp ν.

So $\nu([h_i,k_i]) = \mu([h_i,k_i]) \geq \xi_i \wedge \beta_i = \lambda_i$, hence $a_{i\lambda_i} \in \nu$.

Therefore $c_\xi \in \nu$ and $[\mu_1,\mu] \leq \nu$.

This completes the proof.

LEMMA : 6.3.4′

$\left[\, \mu_E^{m-i},\mu \,\right] \leq \mu_E^{m-i-1}$ if and only if $\mu^{m-i} / \mu^{m-i-1} \leq Z(\mu/\mu^{m-i-1})$.

PROOF :

This is similar to the proof of lemma 6.3.4.

LEMMA : 6.3.5

Let $f : \mu \longrightarrow \nu$ be a fuzzy epimorphism. Let μ_1 be a fuzzy subgroup of $\mathcal{G}$ such that $\mu_1 \leq Z(\mu)$. Then $f(\mu_1) \leq Z(\nu) = Z(f(\mu))$.

PROOF :

Let $b_\beta \in f(\mu_1)$, i.e. $\sup_{f(a) = b} \mu_1(a) \geq \beta$.

Let $\epsilon \in (0,\beta)$. There exists a_o, $f(a_o) = b$, such that $\mu_1(a_o) > f(\mu_1)(b) - \epsilon$.

So $\mu_1(a_o) > \beta - \epsilon$, i.e. $a_{o\beta-\epsilon} \in \mu_1$.

Hence $f(a_o)_{\beta-\epsilon} \in f(\mu_1)$.

Let $c_\lambda \in \nu = f(\mu)$. Assume $\epsilon \in (0,\beta \wedge \lambda)$. Therefore $\sup_{f(x) = c} \mu(x) \geq \lambda$.

There exists x_1, $f(x_1) = c$, such that $\mu(x_1) > \lambda-\epsilon$, i.e. $x_{1\lambda-\epsilon} \in \mu$.

By hypothesis, $(a_o x_1)_{\beta-\epsilon \wedge \lambda-\epsilon} = (x_1 a_o)_{\beta-\epsilon \wedge \lambda-\epsilon}$.

Therefore $f(a_o)f(x_1)_{\beta-\epsilon \wedge \lambda-\epsilon} = f(x_1)f(a_o)_{\beta-\epsilon \wedge \lambda-\epsilon}$.

i.e. $(bc)_{\beta-\epsilon \wedge \lambda-\epsilon} = (cb)_{\beta-\epsilon \wedge \lambda-\epsilon}$ for $\epsilon > 0$.

So $(bc)_{\beta \wedge \lambda} = (cb)_{\beta \wedge \lambda}$. Hence $b_\beta \in Z(\nu)$.

This completes the proof.

THEOREM : 6.3.6

For any fuzzy subgroup μ, $\mu = \mu^m$ if and only if $\gamma_{m+1}(\mu) = \mu_e$.

Moreover, $\gamma_{i+1}(\mu) \leq \mu_E^{m-i}$. (For the notation, see the construction of the DCS and ACS).

PROOF :

($\Longrightarrow$): $\mu^m = \mu$. $\gamma_1(\mu) = \mu$, $\gamma_2(\mu) = [\gamma_1(\mu),\mu]$, $\cdots$. We will prove, by induction on i, that $\gamma_{i+1}(\mu) \leq \mu_E^{m-i}$:

Let i = 0. LHS = μ = RHS.

Assume $\gamma_{i+1}(\mu) \leq \mu_E^{m-i}$.

Now $\mu^{m-i}/\mu^{m-i-1} = Z(\mu/\mu^{m-i-1})$.

Therefore by Lemma 6.3.4′, $[\mu_E^{m-i},\mu] \leq \mu_E^{m-i-1}$.

$$\text{Hence } \gamma_{i+2}(\mu) = [\gamma_{i+1}(\mu),\mu] \leq [\mu_E^{m-i},\mu] \leq \mu_E^{m-i-1} .$$

Hence $\gamma_{i+1}(\mu) \leq \mu_E^{m-i}$ for all $i \in \mathbb{N}$, $i \leq m$.

Let $i = m$. Therefore $\gamma_{m+1}(\mu) \leq \mu_E^o = \mu_e$.

($\Leftarrow$): $\gamma_{m+1}(\mu) = \mu_e$. We will prove, by induction on j, that $\gamma_{m+1-j}(\mu) \leq \mu^j$.

Let $j = 0$: Therefore LHS $= \gamma_{m+1}(\mu) = \mu_e =$ RHS.

Assume $\gamma_{m+1-j}(\mu) \leq \mu^j$.

Define $f : \mu/\gamma_{m+1-j}(\mu) \longrightarrow \mu/\mu^j$ by $f(x \text{ supp } \gamma_{m+1-j}(\mu)) = x \text{ supp } \mu^j$.

We claim that $f(\mu/\gamma_{m+1-j}(\mu)) = \mu/\mu^j$:

$$f(\mu/\gamma_{m+1-j}(\mu))\,(x \text{ supp } \mu^j) = \sup\{\mu/\gamma_{m+1-j}(\mu)(a \text{ supp } \gamma_{m+1-j}(\mu)) \colon a \text{ supp } \mu^j = x \text{ supp } \mu^j\}$$

$$= \sup\left\{\sup_{yD_1 = aD_1} \mu(y) \colon aD = xD\right\},$$ where $D_1 = \text{supp } \gamma_{m+1-j}(\mu)$, $D = \text{supp } \mu^j$,

$$\leq \sup\left\{\sup_{yD = aD} \mu(y) \colon aD = xD\right\}$$ since $D \geq D_1$.

$$= \sup_{aD = xD} \mu(a) = \mu/\mu^j\,(x \text{ supp } \mu^j).$$

Let $\epsilon \in (0, \mu/\mu^j\,(x \text{ supp } \mu^j))$.

There exists z_o, $z_oD = xD$, such that

$$\mu/\mu^j\,(x \text{ supp } \mu^j) - \epsilon < \mu(z_o) \leq f(\mu/\gamma_{m+1-j}(\mu))\,(x \text{ supp } \mu^j).$$

Since ϵ is arbitrarily small, the claim is proved.

Now $[\gamma_{m-j}(\mu), \mu] = \gamma_{m-j+1}(\mu)$.

Therefore, by Lemma 6.3.4, $\gamma_{m-j}(\mu)/\gamma_{m-j+1}(\mu) \leq Z(\mu/\gamma_{m+1-j}(\mu))$.

Therefore $f(\gamma_{m-j}(\mu)/\gamma_{m-j+1}(\mu)) \leq Z(f(\mu/\gamma_{m+1-j}(\mu)))$ by Lemma 6.3.5.

RHS $= Z(\mu/\mu^j) = \mu^{j+1}/\mu^j$.

$\gamma_{m-j}(\mu)/\mu^j \leq f(\gamma_{m-j}(\mu)/\gamma_{m+1-j}(\mu))$. Hence $\gamma_{m-j}(\mu)/\mu^j \leq \mu^{j+1}/\mu^j$. It is not hard to show that $\mu^{j+1} \geq \gamma_{m-j}(\mu)$.

Let $j = m-1$. So $\mu^m \geq \gamma_1(\mu) = \mu$.

But $\mu \geq \mu^m$. Hence $\mu = \mu^m$, and the proof is complete.

PROPOSITION : 6.3.7

If μ is nilpotent, then μ is solvable.

PROOF :

This follows from Theorem 6.3.6 and the construction of the ACS.

THEOREM : 6.3.8

Let μ be a fuzzy subgroup of $\mathcal{G}$. μ is nilpotent if and only if there exists a series $\mu = \mu_0 \geq \mu_1 \geq \cdots \geq \mu_n = \mu_e$ such that $\mu_i \triangleleft \mu$ and $\mu_i/\mu_{i+1} \leq Z(\mu/\mu_{i+1})$ for all $i = 0,1,2,\cdots,n-1$. (Such a series is called a central series of μ).

PROOF :

($\Longrightarrow$): The ascending central series of μ satisfies the above properties.

($\Longleftarrow$): Consider the ascending central series of μ :

$\mu^0 = \mu_e \leq \mu^1 = Z(\mu) \leq \mu^2 \leq \cdots \leq \mu^m \leq \cdots$

We will show that there exists $m \in \mathbb{Z}$ such that $\mu^m = \mu$.

Now $\mu_n = \mu_e$ implies that $\mu_{n-1}/\mu_n \simeq \mu_{n-1}$ and $Z(\mu/\mu_n) \simeq Z(\mu)$.

Therefore $\mu_{n-1} \leq Z(\mu) = \mu^1$.

Let $D = \text{supp } \mu^1$, $D_1 = \text{supp } \mu_{n-1}$.

$$\begin{aligned} \mu^2/\mu^1(xD) &= Z(\mu/\mu^1)(xD) \\ &\geq Z(\mu/\mu_{n-1})(xD_1) && \text{since } D \geq D_1 \\ &\geq \mu_{n-2}/\mu_{n-1}(xD_1) && \text{by hypothesis.} \end{aligned}$$

Therefore $\sup\limits_{aD = xD} \mu^2(a) \geq \sup\limits_{aD_1 = xD_1} \mu_{n-2}(a)$

$\mu^2(a) = Z(\mu/\mu^1)(aD) = Z(\mu/\mu^1)(xD) = \mu^2(x)$.

Therefore $\mu^2(x) \geq \sup\limits_{aD_1 = xD_1} \mu_{n-2}(a) \geq \mu_{n-2}(x)$.

So $\mu^2 \geq \mu_{n-2}$.

Assume $\mu^k \geq \mu_{n-k}$. We will show that $\mu^{k+1} \geq \mu_{n-k-1}$.

$$\begin{aligned} \mu^{k+1}/\mu^k(x \text{ supp } \mu^k) &= Z(\mu/\mu^k)(x \text{ supp } \mu^k) \\ &\geq Z(\mu/\mu_{n-k})(x \text{ supp } \mu_{n-k}) \\ &\geq \mu_{n-k-1}/\mu_{n-k}(x \text{ supp } \mu_{n-k}). \end{aligned}$$

Therefore
$\sup\{\mu^{k+1}(a)\text{: a supp } \mu^k = x \text{ supp } \mu^k\} \geq \sup\{\mu_{n-k-1}(a)\text{: a supp } \mu_{n-k} = x \text{ supp } \mu_{n-k}\}$.

We can show, as above, that $\mu^{k+1}(a) = \mu^{k+1}(x)$ for all a satisfying
a supp μ^k = x supp μ^k.

Therefore $\mu^{k+1}(x) \geq \mu_{n-k-1}(x)$.

Hence $\mu^k \geq \mu_{n-k}$ for all $k \leq n$.

Let k = n. Then $\mu^n \geq \mu_0 = \mu$. So $\mu = \mu^n$.

i.e. μ is nilpotent.

PROPOSITION : 6.3.9

A homomorphic image of a nilpotent fuzzy subgroup is nilpotent.

PROOF :

Let $f : \mu \longrightarrow \nu$ be a fuzzy epimorphism, where μ is a nilpotent fuzzy subgroup.
By Theorem 6.3.8, there is a normal series $\mu = \mu_0 \geq \mu_1 \geq \cdots \geq \mu_n = \mu_e$, $\mu_i \triangleleft \mu$ and $\mu_i/\mu_{i+1} \leq Z(\mu/\mu_{i+1})$, $i = 0,1,\cdots,n-1$.

Now $f(\mu) = f(\mu_0) \geq f(\mu_1) \geq \cdots \geq f(\mu_n) = f(\mu)_e$.

$f(\mu_i) \triangleleft f(\mu)$. We claim that $f(\mu_i)/f(\mu_{i+1}) \leq Z(f(\mu)/f(\mu_{i+1}))$:

Let $D = \text{supp } f(\mu_{i+1})$.

Let $(f(a)D)_\lambda \in f(\mu_i)/f(\mu_{i+1})$ and $(f(b)D)_\beta \in f(\mu)/f(\mu_{i+1})$.

Therefore $\sup\{f(\mu_i)(f(x))\text{: } f(x)D = f(a)D\} \geq \lambda$ and

$\sup\{f(\mu)(f(x))\text{: } f(x)D = f(b)D\} \geq \beta$.

Now

$\sup\left\{\sup_{f(c)=f(x)} \mu_i(c): f(x)D = f(a)D\right\} = \sup\{\mu_i(c): f(c)D = f(a)D\} \geq \lambda$ and

$\sup\left\{\sup_{f(c)=f(x)} \mu(c): f(x)D = f(b)D\right\} = \sup\{\mu(c)): f(c)D = f(b)D\} \geq \beta.$

Let $\epsilon \in (0, \lambda \wedge \beta)$. There exist c_o,c_1, $f(c_o)D = f(a)D$, $f(c_1)D = f(b)D$, such that $\mu_i(c_o) > \lambda-\epsilon$ and $\mu(c_1) > \beta-\epsilon$.

Therefore $c_{o\lambda-\epsilon} \in \mu_i$ and $c_{1\beta-\epsilon} \in \mu$.

By hypothesis, $(c_oc_1 \text{ supp } \mu_{i+1})_{\lambda-\epsilon \wedge \beta-\epsilon} = (c_1c_o \text{ supp } \mu_{i+1})_{\lambda-\epsilon \wedge \beta-\epsilon}$.

$c_oc_1 c_o^{-1}c_1^{-1} \in \text{supp } \mu_{i+1}$ implies that $f(c_o)\, f(c_1)\, f(c_o)^{-1}\, f(c_1)^{-1} \in \text{supp } f(\mu_{i+1})$ and

therefore $(f(c_o)\, f(c_1)D)_{\lambda-\epsilon \wedge \beta-\epsilon} = (f(c_1)\, f(c_0)D)_{\lambda-\epsilon \wedge \beta-\epsilon}$.

Hence $(f(a)\, f(b)D)_{\lambda-\epsilon \wedge \beta-\epsilon} = (f(b)\, f(a)D)_{\lambda-\epsilon \wedge \beta-\epsilon}$.

This is true for every ϵ arbitrarily small.

Hence $(f(a)\, f(b)D)_{\lambda \wedge \beta} = (f(b)\, f(a)D)_{\lambda \wedge \beta}$.

i.e. $(f(a)D)_\lambda \in Z(f(\mu)/f(\mu_{i+1}))$.

The proof is complete.

PROPOSITION : 6.3.10

A quotient of a nilpotent fuzzy subgroup is nilpotent.

PROOF :

Let $\nu \triangleleft \mu$, where μ is nilpotent.

$\gamma : \mu \longrightarrow \mu/\nu$ is a fuzzy epimorphism, where $\gamma(a) = a \text{ supp } \nu$.

Hence, by Proposition 6.3.9, μ/ν is nilpotent.

PROPOSITION : 6.3.11

A direct product of a finite number of nilpotent fuzzy groups is nilpotent.

PROOF :

Let $\mu = \mu_1 \times \mu_2$, where μ_i is nilpotent, i = 1,2.

Let $\mu_1 = \nu_o \geq \nu_1 \geq \cdots \geq \nu_n = (\mu_1)_e$ and

$\mu_2 = \omega_o \geq \omega_1 \geq \cdots \geq \omega_k = (\mu_2)_e$ be central series of μ_1 and μ_2 respectively.

We can assume, without loss of generality, that n = k.

Therefore $\mu_1 \times \mu_2 = \nu_0 \times \omega_0 \geq \cdots \geq \nu_n \times \omega_n = (\mu_1)_e \times (\mu_2)_e$ is a normal series of $\mu_1 \times \mu_2$.

It is straightforward to show that $\nu_i \times \omega_i/\nu_{i+1} \times \omega_{i+1} \leq Z(\mu_1 \times \mu_2/\nu_{i+1} \times \omega_{i+1})$.

This completes the proof.

PROPOSITION : 6.3.12

Let $\nu \leq Z(\mu)$, $\nu \triangleleft \mu$. If μ/ν is nilpotent, then μ is nilpotent.

PROOF :

Let $\mu/\nu = \xi_o \geq \xi_1 \geq \cdots \geq \xi_n = \nu/\nu$ be a central series of μ/ν. Let $\gamma : \mu \longrightarrow \mu/\nu$ be the natural map. Let $u_i = \gamma^{-1}(\xi_i)$ and $\omega_i = u_i \wedge \mu$.

Therefore $\omega_i \triangleleft \mu$. $\xi_i = \omega_i/\nu$.

Consider $\mu = \mu \wedge u_o \geq \omega_1 \geq \cdots \geq \omega_n \geq \nu \geq \nu_e$. (1)

(1) is a normal series.

We claim that (1) is a central series :

$$\omega_i/\omega_{i+1} \simeq (\omega_i/\nu)/(\omega_{i+1}/\nu) \leq Z((\mu/\nu)/(\omega_{i+1}/\nu))$$
$$\simeq Z(\mu/\omega_{i+1}).$$

Therefore $\omega_i/\omega_{i+1} \leq Z(\mu/\omega_{i+1})$.

$\omega_n/\nu = \nu/\nu \leq Z(\mu/\nu)$.

$Z(\mu) \geq \nu \simeq \nu/\nu_e$ and $Z(\mu/\nu_e) \simeq Z(\mu)$.

Therefore $\nu/\nu_e \leq Z(\mu/\nu_e)$. The claim is proved, and hence μ is nilpotent.

REMARK : 6.3.13

Many of the results in this section will not hold if the quotients are replaced by the strong quotients. This is because to show that μ/ν is Abelian, one takes x,y $\in$ supp μ and then show that $xyx^{-1}y^{-1} \in E_\nu$, where $E_\nu = \{x \in \mathcal{G} : \nu(x) = \nu(e)\}$. This is a strong condition.

PROPOSITION : 6.3.14

Let μ and ν be normal nilpotent fuzzy subgroups of ω, where ω is a fuzzy subgroup of $\mathcal{G}$. Then $\mu\nu$ is a normal nilpotent fuzzy subgroup of ω.

PROOF :

Clearly $\mu\nu \triangleleft \omega$.
Define f : $\mu \times \nu \longrightarrow \mu\nu/\mu \wedge \nu$ by f(a,b) = ab supp $\mu \wedge \nu$. It is easy to show that f is a homomorphism.

$\mu\nu/\mu \wedge \nu = f(\mu \times \nu)$. Hence $\mu\nu/\mu \wedge \nu$ is nilpotent since $\mu \times \nu$ is nilpotent.
We can then construct a central series for $\mu\nu$ using a central series for $\mu\nu/\mu \wedge \nu$.

PROPOSITION : 6.3.15

Let ν, $\omega \leq \mu$ be fuzzy subgroups of $\mathcal{G}$. Then $\nu \leq N_\mu(\omega)$ if and only if $[\omega,\nu] \leq \omega$.

PROOF :

($\Longrightarrow$): Let $a_\lambda \in [\omega,\nu]$.

So $\lambda \leq [\omega,\nu](a) = \sup\limits_{a_{\lambda_s} = a_{1\lambda_1} \cdots a_{n\lambda_n},\ a_{i\lambda_i} = [h_{i\xi_i}, k_{i\beta_i}],} \lambda_s$

$h_{i\xi_i} \in \omega$, $k_{i\beta_i} \in \nu \leq N_\mu(\omega)$.

By hypothesis $k_{i\beta_i}\omega = \omega\, k_{i\beta_i}$. (1)

$h^{-1}_{i\xi_i}\, k^{-1}_{i\beta_i}\, h_{i\xi_i}\, k_{i\beta_i}\, \omega(y) = \xi_i \wedge \beta_i \wedge \omega\,(k_i^{-1}\, h_i^{-1}\, k_i h_i y)$

$= \xi_i \wedge \beta_i \wedge \omega\,(h_i^{-1}(k_i h_i y\, k_i^{-1})) = \xi_i \wedge \beta_i \wedge \omega(k_i h_i\, y\, k_i^{-1})$

$= \xi_i \wedge \beta_i \wedge \omega\,(h_i\, y)$ by (1) above,

$= \xi_i \wedge \beta_i \wedge \omega(y)$ since $h_{i\xi_i} \in \omega$

$= e_{\xi_i \wedge \beta_i} \omega(y)$.

Therefore $[h_{i\xi_i}, k_{i\beta_i}]\omega = e_{\xi_i \wedge \beta_i} \omega \leq \omega$.

Hence $a_{\lambda_s} \in \omega$ and therefore $a_\lambda \in \omega$.

Hence $[\omega,\nu] \leq \omega$.

($\Leftarrow$): Let $a_\lambda \in \nu$. We will show that $a_\lambda b_\beta a_\lambda^{-1} \in \omega$ for all $b_\beta \in \omega$.

Now $a_\lambda b_\beta a_\lambda^{-1} b_\beta^{-1} \in \omega$ by hypothesis.

Therefore $b_\beta a_\lambda b_\beta^{-1} a_\lambda^{-1} \omega = e_{\lambda \wedge \beta} \omega$.

Therefore $a_\lambda b_\beta^{-1} a_\lambda^{-1} \omega = e_{\beta \wedge \lambda} \omega$ since $b_\beta \in \omega$.

So $(a\, b\, a^{-1})_{\lambda \wedge \beta} \in \omega$, as required.

PROPOSITION : 6.3.16

Let μ be nilpotent and $\nu \lneqq \mu$. Then $\nu \neq N_\mu(\nu)$.

PROOF :

We claim that there exists i such that $\nu \geq \gamma_{i+1}(\mu)$, but $\nu \not\geq \gamma_i(\mu)$ in the DCS of μ.

Let $\mu = \gamma_1(\mu) \geq \cdots \geq \gamma_{m+1}(\mu) = \mu_e$ be the DCS of μ.

$\nu \geq \gamma_{m+1}(\mu)$. If $\nu \not\geq \gamma_m(\mu)$, we are done.

Suppose $\nu \geq \gamma_m(\mu)$:

If $\nu \not\geq \gamma_{m-1}(\mu)$, we stop ; otherwise repeat the process. This process must come to an end since $\nu \not\geq \gamma_1(\mu)$. The claim is proved.

Now $[\gamma_i(\mu),\nu] \leq [\gamma_i(\mu),\mu] = \gamma_{i+1}(\mu)$.

Therefore $[\nu, \gamma_i(\mu)] \leq \nu$.

Hence, by Proposition 6.3.15, $\gamma_i(\mu) \leq N_\mu(\nu)$.

If $\nu = N_\mu(\nu)$, then $\gamma_i(\mu) \leq \nu$, which contradicts our claim above.

Hence $\nu \neq N_\mu(\nu)$ and the proof is complete.

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