Characterization of stratified L-topological spaces by convergence of stratified L-filters

Thesis submitted in fulfilment of the requirements for the degree of Master of Science of Rhodes University.

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June 30, 2010

Abstract

For the case where L is an ecl-premonoid, we explore various characterizations of SL-topological spaces, in particular characterization in terms of a convergence function $\lim\colon \mathcal{F}_L^S(X) \to L^X$. We find we have to introduce a new axiom , $\mathbf{L}\otimes$ on the lim function in order to completely describe SL-topological spaces, which is not required in the case where L is a frame. We generalize the classical Kowalski and Fischer axioms to the lattice context and examine their relationship to the convergence axioms. We define the category of stratified L-generalized convergence spaces, as a generalization of the classical convergence spaces and investigate conditions under which it contains the category of stratified L-topological spaces as a reflective subcategory. We investigate some subcategories of the category of stratified L-generalized convergence spaces obtained by generalizing various classical convergence axioms.

Contents

Acknowledgements

I would like to thank my supervisor Dr. Gunther Jäger for his encouragement and patience throughout the past two years, and for reading and commenting on huge documents at short notice. My thanks also to Dad and Mum, Shaun and Judy for love and support, especially the financial support. I couldn't have done it without you guys. Thanks to countless other friends for keeping me motivated.

Introduction

The purpose of the research conducted for this thesis was threefold. First, we sought to extend the category theoretic results for stratified L-topological spaces obtained by Jäger [21, 22, 23, 24] for the case where L is a frame to the more general case where L is an ecl-premonoid. Secondly we endeavoured to characterize stratified L-topological spaces in terms of a limit function, specifying filter convergence. Lastly we defined the category of stratified L-generalized convergence spaces in the ecl-premonoid context and studied the relationships between several of its subcategories.

In Chapter 1 we present the background set theory and category theory necessary for understanding the rest of the text. We cover abstract and concrete categories, cartesian closedness, reflective subcategories and what it means for a category $\underline{\mathbf{A}}$ to be topological over a category $\underline{\mathbf{B}}$. We also present some theorems used in proofs in later chapters.

Chapter 2 begins with the definition of the category of classical topological spaces. We continue with characterizations of topological spaces in terms of interior, neighbourhood filter and convergence function. As an aid to developing intuition for the general ecl-premonoid case we examine the relationship between alternative convergence axioms, in particular the Kowalski and Fischer axioms, which play an important role later. Lastly we define the category of convergence spaces, a cartesian closed supercategory of the category of topological spaces.

Chapter 3 contains the lattice theory background necessary for understanding the main research. In the first section we cover properties of the lattice L. Here we present an extension of a proof from Höhle ([18]), showing that the underlying lattice of any GL-monoid is a frame. Next we look at properties of L-sets, in particular, how their properties are influenced by the properties of L, and properties of images and inverse images of L-sets. In the final section we cover stratified L-filter theory, developing concepts of the infimum of a family of stratified L-filters and the supremum of two stratified L-filters, properties of image and inverse images of stratified L-filters and the product of two stratified L-filters.

In Chapter 4 we begin by defining the category of stratified L-topological spaces for the ecl-premonoid case (cf. [20]). We then obtain characterizations for stratified L-topological spaces in terms of interior operators, neighbourhood spaces and convergence functions, generalized to the lattice context. In generalizing the characterization by convergence from the frame case to the ecl-premonoid case we obtain what seems to be a new convergence axiom, **L**$\otimes$, related to whether the neighbourhood filter is actually a stratified L filter or merely a function $\mathcal{U}^x \colon L^X \to L$. **L**$\otimes$ is always satisfied in the frame case. In the classical case one of the convergence axioms states that the convergence function lim factors through infima of sets of stratified L-filters. In the general ecl-premonoid case (indeed even in the frame case [23]) this axiom is not quite sufficient in our scheme for characterization of stratified L-topological spaces. We try to investigate why. We try to replace the condition guaranteeing idempotency of the interior operator with generalizations of the Kowalski and Fischer iterated limit axioms. Here we discover that the form of the axioms changes slightly from the frame case to take into account that we are using the implication defined by the GL-monoid operation.

In Chapter 5 we define the category of stratified L-generalized convergence spaces and prove some of its categorical properties. We present a new lemma showing that the requirement that all objects in the category satisfy the **L**$\otimes$ axiom is equivalent a monotonicity condition **M** on the lattice L is satisfied. This criterion sharpens a previously know result where it was known that **M** implies that all stratified L-generalized convergence spaces satisfy **L**$\otimes$. The monotonicity condition is sufficiently general to cover the important special cases of frames and GL-monoids with square roots. We prove that the **L**$\otimes$ axiom is independent of several of the more basic convergence axioms and is not satisfied by all stratified L-generalized convergence spaces in the general ecl-premonoid case. In the final section of the chapter we generalize the definitions of various subcategories of the category of stratified L-generalized convergence spaces and investigate their properties. We enter into a short discussion of the form of the axiom which defines stratified L-limit spaces, as there are several possibilities for generalizing the axiom from the frame to the ecl-premonoid case. We present a new proof that the category of stratified L-principal convergence spaces is topological in the ecl-premonoid case.

Lastly we summarize our progress and outline some open problems and interesting directions for possible future research in Chapter 6.

Chapter 1

Category theory

Properties of topological spaces (and many other branches of mathematics) can be placed conveniently within the framework of category theory [34]. Hence before the investigation of the spaces we are interested in, we here present a summary of the category theoretic definitions and results needed in the main text. A short discussion of the necessary set theory and notation conventions precedes the category theory. The main reference for this chapter is Adámek et al. [1].

1.1 Set theory

A full discussion of the necessary set theory which provides a foundation for category theory is beyond the scope of this text, however in order to facilitate proofs later on a summary of some theory is appropriate. For further details see e.g. [1, 17]. Briefly, a *class* is a collection of *objects* determined by a logical condition. A class which is a *member* (i.e. an object) of another class is called a *set*. We shall rarely if ever use the definition just given in proving that some class is a set, instead we shall usually rely on the following axioms given in Herrlich and Strecker [17].

1. For each set X and each property P, we can form the set $\{ x \in X \mid P(x) \}$ of members of X which possess the property P. Note that this means that any subclass of a set is also a set.

2. For each set X we can form the set $\mathcal{P}(X)$ of all subsets of X.

3. For all sets X, Y we can form the usual $\{X, Y\}$ (pair), (X, Y) (ordered pair), $X \cup Y$ (union), $X \cap Y$ (intersection), $X \backslash Y$ (complement), $X \times Y$ (cartesian product), and Y^X (set of all functions from Y to X), and all of these constructions are sets.

4. For any set I and any family $(X_i)_{i \in I}$ of sets, the following constructions

are also sets: $\{ X_i \mid i \in I \}$ (the image set of the indexing function), $\bigcup_{i \in I} X_i$, $\bigcap_{i \in I} X_i$ (if $I \neq \emptyset$), $\prod_{i \in I} X_i$.

5. The classes of the natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$ and real numbers $\mathbb{R}$ are sets.

The usual conventions (see e.g. [16]) have been observed for set theoretic notation throughout this text. In some examples properties of $\mathbb{R}$ and $\mathbb{N}$ are used without explicit mention. Further details may be found in e.g. Suppes [38]. The abbreviation 'iff' for 'if and only if' has been used throughout.

Relations

Because we will be dealing with lattices, we will need some definitions from the theory of relations. These definitions are basic but they are included for completeness.

Definition 1.1.1 [16]: Let X be a set. Then a *relation* on X is a subset $\mathrm{R} \subseteq X \times X$. $(x, y) \in \mathrm{R}$ is usually denoted $x \,\mathrm{R}\, y$.

Definition 1.1.2 [16]: A relation R on a set X is said to be

1. reflexive iff $\forall\, x \in X, \quad x \,\mathrm{R}\, x$.
2. transitive iff $\forall\, x, y, z \in X, \quad x \,\mathrm{R}\, y$ and $y \,\mathrm{R}\, z \Rightarrow x \,\mathrm{R}\, z$.
3. symmetric iff $\forall\, x, y \in X, \quad x \,\mathrm{R}\, y \Rightarrow y \,\mathrm{R}\, x$.
4. anti-symmetric iff $\forall\, x, y \in X, \quad x \,\mathrm{R}\, y$ and $y \,\mathrm{R}\, x \Rightarrow x = y$.

Definition 1.1.3 [16, 38]: A relation R on a set X is said to be

1. a quasi-ordering iff it is reflexive and transitive.
2. a partial ordering iff it is reflexive, anti-symmetric and transitive.
3. an equivalence relation iff it is reflexive, symmetric and transitive.

Example 1.1.4 : The real numbers $\mathbb{R}$ with the usual order relation $\leq$ form a partially ordered set. A further example is given by the power set $\mathcal{P}(X)$ of X, ordered by inclusion, i.e. for $A, B \in \mathcal{P}(X)$, $A \leq B \Leftrightarrow A \subseteq B$. Equality,'=', is an equivalence relation on $\mathbb{R}$.

1.2 Abstract Categories

We begin with an informal definition in order to familiarize ourselves with the concepts, and later sharpen this to a formal definition (Definition 1.2.1). Loosely speaking, a *category* consists of:

1. A class of *objects*, in our case usually structured sets e.g. vector spaces.
2. A class of *morphisms* between objects obeying the rules
   (a) For every object there is an *identity* morphism.
   (b) *Composition* of morphisms is associative.

   In our case morphisms will usually be functions between the structured sets which satisfy some axioms. For example for the category **VEC** (the category of vector spaces and morphisms between them) the morphisms are linear functions between vector spaces i.e. functions f which satisfy the axiom

   $$f(ax + by) = af(x) + bf(y)$$

   for all vectors x, y and all scalars a, b. Note that for a vector space $(X, +, \cdot)$, the identity function id_X is a morphism (i.e. the identity function is linear). Composition in the category **VEC** is the usual composition of functions and is of course associative.

For the rest of this chapter, examples will usually relate to the following categories:

SET The category whose objects are sets and morphisms the functions between them. Composition is the usual function composition and identity on a set X is the usual identity function.

VEC The category of vector spaces, already introduced. Objects are vector spaces and morphisms are linear functions between vector spaces.

TOP The category of topological spaces. Objects are topological spaces i.e. structures (X, τ) where X is a non-empty set and τ is a *topology*. Morphisms are *continuous functions* and composition is the usual function composition.

Functors are functions between categories. An example is the *forgetful functor* from **TOP** to **SET**, which maps topological spaces (X, τ) to their underlying sets X and continuous functions $\phi\colon (X, \tau_X) \to (Y, \tau_Y)$ to the corresponding set functions $\phi\colon X \to Y$.

Although the categories described so far all consist of *structured sets* and morphisms which are in some sense *structure preserving functions*, this need not be the case. In fact these categories are all examples of *concrete categories* which are a special case of the general definition of an *abstract* category. Objects in a category need not be structured sets, morphisms need not be functions and furthermore, even if the morphisms are functions, composition need not be the usual composition of functions. However, throughout this text we will only really be dealing with "nice" concrete categories whose objects *are* structured sets and in which morphisms *are* functions and composition *is* function composition. We begin our discussion with abstract categories and their properties.

Abstract Categories

Definition 1.2.1 is the formal expression of the rather vague definition of a category given previously.

Definition 1.2.1 [1]: A *category* $\underline{\mathbf{A}} = (\mathcal{O}, \text{hom}, \text{id}, \circ)$ consists of

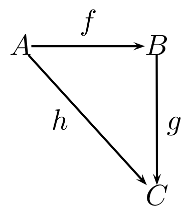
1. A class $\mathcal{O}$ of $\underline{\mathbf{A}}$-objects.
2. A function hom which associates with each pair (A, B) of $\underline{\mathbf{A}}$-objects a *set* $\text{hom}(A, B)$. The members of the set $\text{hom}(A, B)$ are called $\underline{\mathbf{A}}$-morphisms from A to B. The sets $\text{hom}(A, B)$ are further required to be pairwise disjoint.
3. A function id which associates with each $\underline{\mathbf{A}}$-object A a morphism $\text{id}_A \in \text{hom}(A, A)$.
4. A *morphism composition* $\circ$ which satisfies the following conditions
   (a) $g \circ f \in \text{hom}(A, C)$ is defined wherever $f \in \text{hom}(A, B)$ and $g \in \text{hom}(B, C)$.
   (b) Wherever defined, $h \circ (g \circ f) = (h \circ g) \circ f$. (Associativity)
   (c) If $f \in \text{hom}(A, B)$ then $f \circ \text{id}_A = f$ and $\text{id}_B \circ f = f$.

Remark 1.2.2 : We often refer to the class $\mathcal{O}$ of $\underline{\mathbf{A}}$-objects as $\text{Ob}\,(\underline{\mathbf{A}})$. Similarly we define $\text{Mor}\,(\underline{\mathbf{A}}) = \cup\{\, \text{hom}(A, B) \mid A, B \in \text{Ob}\,(\underline{\mathbf{A}})\,\}$, the class of $\underline{\mathbf{A}}$-morphisms. For $f \in \text{hom}(A, B)$ we usually write $A \xrightarrow{f} B$ or $f\colon A \to B$. A is referred to as the *domain* of f, B is the *codomain*. The sets $\text{hom}(A, B)$ are required to be pairwise disjoint so that each morphism $f \in \text{Mor}\,(\underline{\mathbf{A}})$ has a *unique* domain and codomain. We define the functions

$$\text{dom}\colon \text{Mor}\,(\underline{\mathbf{A}}) \to \text{Ob}\,(\underline{\mathbf{A}}) \quad (A \xrightarrow{f} B) \mapsto A$$

$$\text{cod}\colon \text{Mor}\,(\underline{\mathbf{A}}) \to \text{Ob}\,(\underline{\mathbf{A}}) \quad (A \xrightarrow{f} B) \mapsto B$$

For $h\colon A \to C$, $f\colon A \to B$ and $g\colon B \to C$ we may express the statement $h = g \circ f$ by saying that the triangle



commutes.

Example 1.2.3 : All of the categories mentioned so far (**SET**, **VEC** and **TOP**) satisfy the requirements of Definition 1.2.1. We expand further on **TOP** (see e.g. [1]), since it will be one of the most important examples in the following discussion.

Objects Objects are *topological spaces* i.e. structures (X, τ) satisfying:

$\emptyset \neq X$, X is a set, $\tau \subseteq \mathcal{P}(X)$

τ is called a *topology* on X.Members of τ are called *open sets* and must satisfy the following axioms:

O1 $\emptyset, X \in \tau$.

O2 $A, B \in \tau \;\Rightarrow\; A \cap B \in \tau$ (Finite intersections of open sets are open).

03 $\mathcal{A} \subseteq \tau \Rightarrow \cup\mathcal{A} \in \tau$ (Arbitrary unions of open sets are open).

Morphisms Morphisms between topological spaces (X, τ_X) and (Y, τ_Y) are *continuous* functions $\phi\colon X \to Y$ satisfying the axiom
$\forall\, V \in \tau_Y, \quad \phi^{\leftarrow}(V) \in \tau_X$ (Inverse images of open sets are open).

Identity The identity function $\mathrm{id}_X\colon X \to X$ is continuous between the topological space (X, τ) and itself.

Composition Composition is the usual function composition.

Remark 1.2.4 : Consider the topological spaces (X, τ_1) and (X, τ_2) defined by $\tau_1 = \mathcal{P}(X)$ and $\tau_2 = \{\emptyset, X\}$ (these are both easily verified to be topologies on X). Then the identity function id_X is continuous between (X, τ_1) and (X, τ_2), as well as being continuous between (X, τ_1) and (X, τ_1). Since in general $\tau_1 \neq \tau_2$, it seems that we are violating the condition that hom-sets must be pairwise disjoint. To get around this problem, we define a morphism in **TOP** as a triple $((X, \tau_X), \phi\colon X \to Y, (Y, \tau_Y))$. In other words, we regard $\mathrm{id}_X\colon (X, \tau_1) \to (X, \tau_2)$ as a *different* morphism to $\mathrm{id}_X\colon (X, \tau_1) \to (X, \tau_1)$.

Subcategories

We often have situations where objects and morphisms of one category $\underline{\mathbf{A}}$ may be regarded as naturally belonging to a larger category $\underline{\mathbf{B}}$. The formal definition of a subcategory (Definition 1.2.5) ensures that morphisms and identities of the subcategory behave as expected (i.e. in the same way in both categories $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$).

Definition 1.2.5 [1]: Let $\underline{\mathbf{A}}, \underline{\mathbf{B}}$ be categories. $\underline{\mathbf{A}}$ is a *subcategory* of $\underline{\mathbf{B}} \Leftrightarrow$

1. $\mathrm{Ob}\,(\underline{\mathbf{A}}) \subseteq \mathrm{Ob}\,(\underline{\mathbf{B}})$.
2. $\forall\, A, A' \in \mathrm{Ob}\,(\underline{\mathbf{A}}), \quad \hom_{\underline{\mathbf{A}}}(A, A') \subseteq \hom_{\underline{\mathbf{B}}}(A, A')$.
3. $\forall\, A \in \mathrm{Ob}\,(\underline{\mathbf{A}})$, id_A in $\underline{\mathbf{A}}$ is the same as id_A in $\underline{\mathbf{B}}$, i.e. the identity function id in $\underline{\mathbf{A}}$ is the restriction of the identity in $\underline{\mathbf{B}}$ to $\mathrm{Ob}\,(\underline{\mathbf{A}})$.
4. $g \circ_{\underline{\mathbf{A}}} f = g \circ_{\underline{\mathbf{B}}} f$ wherever defined i.e. morphism composition in $\underline{\mathbf{A}}$ is the same as morphism composition in $\underline{\mathbf{B}}$.

$\underline{\mathbf{A}}$ is a *full* subcategory of $\underline{\mathbf{B}} \Leftrightarrow$

1. $\underline{\mathbf{A}}$ is a subcategory of $\underline{\mathbf{B}}$.
2. $\forall\, A, A' \in \mathrm{Ob}\,(\underline{\mathbf{A}}), \quad \hom_{\underline{\mathbf{A}}}(A, A') = \hom_{\underline{\mathbf{B}}}(A, A')$.

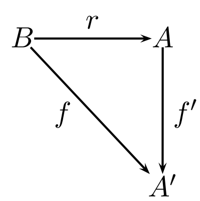
Example 1.2.6 : Let $\underline{\mathbf{SET_i}}$ be the category consisting of objects all sets and morphisms all injective functions between sets. Then $\underline{\mathbf{SET_i}}$ is easily seen to be a category and $\underline{\mathbf{SET_i}}$ is a non-full subcategory of $\underline{\mathbf{SET}}$.

The concept of a subcategory is useful because it allows us to apply concepts related to the supercategory without modification to the subcategory. If we further restrict the concept of a subcategory we obtain special subcategories which have "nice" relationships to the categorical properties of the supercategory. One such restricted definition which has proved useful is that of a reflective subcategory.

Definition 1.2.7 [1]: Let $\underline{\mathbf{A}}$ be a subcategory of $\underline{\mathbf{B}}$, $B \in \mathrm{Ob}\,(\underline{\mathbf{B}})$.
$A \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ is an $\underline{\mathbf{A}}$-*reflection* for $B \Leftrightarrow$

$$\exists\, r \in \hom_{\underline{\mathbf{B}}}(B, A)\ \forall\, A' \in \mathrm{Ob}\,(\underline{\mathbf{A}})\ \forall\, f \in \hom_{\underline{\mathbf{B}}}(B, A') \\ \exists!\, f' \in \hom_{\underline{\mathbf{A}}}(A, A'), \quad f = f' \circ r.$$

In other words, A is an $\underline{\mathbf{A}}$-reflection for B if there is an $r : B \to A$ such that for every f from B to an $\underline{\mathbf{A}}$-object A' there is a unique $\underline{\mathbf{A}}$-morphism f' from A to A' such that the triangle



commutes.

r may be referred to as an $\underline{\mathbf{A}}$-*reflection arrow for* B.

Definition 1.2.8 [1]: $\underline{\mathbf{A}}$ is a *reflective subcategory* of $\underline{\mathbf{B}}$ $\Leftrightarrow$

1. $\underline{\mathbf{A}}$ is a subcategory of $\underline{\mathbf{B}}$.

2. $\forall\, B \in \mathrm{Ob}\,(\underline{\mathbf{B}})\ \ \exists\, A_B \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ such that A_B is an $\underline{\mathbf{A}}$-reflection for B.

Remark 1.2.9 : Examples of reflective subcategories will be given later in the chapters on classical topological spaces and L-topological spaces.

Lemma 1.2.10 [1]: Let $\underline{\mathbf{A}}$ be a reflective subcategory of $\underline{\mathbf{B}}$. Then

$\underline{\mathbf{A}}$ is a full subcategory of $\underline{\mathbf{B}}$ $\Leftrightarrow$

$$\forall\, A \in \mathrm{Ob}\,(\underline{\mathbf{A}}), \quad A \xrightarrow{\mathrm{id}_A} A \text{ is an } \underline{\mathbf{A}}\text{-reflection arrow.}$$

Proof:
Assume $\underline{\mathbf{A}}$ is a full subcategory of $\underline{\mathbf{B}}$. Let $A, A' \in \mathrm{Ob}\,(\underline{\mathbf{A}})$, $f \in \hom_{\underline{\mathbf{B}}}(A, A')$. Then $f \in \hom_{\underline{\mathbf{A}}}(A, A')$ and $f = f \circ \mathrm{id}_A$. Now let $f' \in \hom_{\underline{\mathbf{A}}}(A, A')$, $f = f' \circ \mathrm{id}_A$. Then $f' = f$, so $\exists!\, f' \in \hom_{\underline{\mathbf{A}}}(A, A'), \quad f = f' \circ \mathrm{id}_A$. Thus $A \xrightarrow{\mathrm{id}_A} A$ is an $\underline{\mathbf{A}}$-reflection arrow.

Now assume $\forall\, A \in \mathrm{Ob}\,(\underline{\mathbf{A}}), \quad A \xrightarrow{\mathrm{id}_A} A$ is an $\underline{\mathbf{A}}$-reflection arrow. Let $f \in \hom_{\underline{\mathbf{B}}}(A, A')$. Then $\exists!\, f' \in \hom_{\underline{\mathbf{A}}}(A, A') \quad f = f' \circ \mathrm{id}_A = f'$. Thus $f \in \hom_{\underline{\mathbf{A}}}(A, A')$. Therefore $\underline{\mathbf{A}}$ is a full subcategory of $\underline{\mathbf{B}}$. ■

Functors

A functor F from a category $\underline{\mathbf{A}}$ to a category $\underline{\mathbf{B}}$ is simply a function mapping objects of $\underline{\mathbf{A}}$ to objects of $\underline{\mathbf{B}}$ and morphisms of $\underline{\mathbf{A}}$ to morphisms of $\underline{\mathbf{B}}$. In order to be useful, however, such a function must preserve some of the structure of the category $\underline{\mathbf{A}}$. Hence we define a functor as a function between $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ preserving composition and identities. Definition 1.2.11 is the formal definition.

Definition 1.2.11 [1]: Let $\underline{\mathbf{A}}$, $\underline{\mathbf{B}}$ be categories. $F\colon \underline{\mathbf{A}} \to \underline{\mathbf{B}}$ is a *functor* $\Leftrightarrow$

1. F assigns to each $A \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ an object $F(A) \in \mathrm{Ob}\,(\underline{\mathbf{B}})$.

2. F assigns to each $f \in \hom_{\underline{\mathbf{A}}}(A, A')$ a morphism $F(f) \in \hom_{\underline{\mathbf{B}}}(F(A), F(A'))$ in such a way that:

   (a) $F(g \circ f) = F(g) \circ F(f)$ wherever $g \circ f$ is defined.

   (b) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathrm{Ob}\,(\underline{\mathbf{A}})$.

The action of a functor on a morphism is usually represented as

$$F(A \xrightarrow{f} A') = F(A) \xrightarrow{F(f)} F(A')$$

We often leave out the brackets and write $F(A \xrightarrow{f} A') = FA \xrightarrow{Ff} FA'$.

Example 1.2.12 : Let $\underline{\mathbf{A}}$ be a category. The identity functor $\mathrm{id}_{\underline{\mathbf{A}}}$ defined by

$$\mathrm{id}_{\underline{\mathbf{A}}}(A \xrightarrow{f} A') = A \xrightarrow{f} A'$$

is a functor.

Lemma 1.2.13 [1]: Let $F\colon \underline{\mathbf{A}} \to \underline{\mathbf{B}}, G\colon \underline{\mathbf{B}} \to \underline{\mathbf{C}}$ be functors. Define the composite functor $G \circ F\colon \underline{\mathbf{A}} \to \underline{\mathbf{C}}$ by

$$(G \circ F)(A \xrightarrow{f} A') = G(FA) \xrightarrow{G(Ff)} G(FA')$$

Then $G \circ F$ is a functor.

Definition 1.2.14 [1]: Let $F\colon \underline{\mathbf{A}} \to \underline{\mathbf{B}}$ be a functor. Then

1. F is an *embedding* $\Leftrightarrow$ F is injective on morphisms.

2. F is *faithful* $\Leftrightarrow$ all hom-set restrictions

$$F\colon \hom_{\underline{\mathbf{A}}}(A, B) \to \hom_{\underline{\mathbf{B}}}(FA, FB)$$

   are injective.

3. F is *full* $\Leftrightarrow$ all hom-set restrictions are surjective.

4. F is an *isomorphism* $\Leftrightarrow$ $\exists G\colon \underline{\mathbf{B}} \to \underline{\mathbf{A}} \quad G \circ F = \mathrm{id}_{\underline{\mathbf{A}}}$ and $F \circ G = \mathrm{id}_{\underline{\mathbf{B}}}$.

$\underline{\mathbf{A}}$ is *isomorphic* to $\underline{\mathbf{B}}$ $\Leftrightarrow$ there exists an isomorphism $F\colon \underline{\mathbf{A}} \to \underline{\mathbf{B}}$.

Example 1.2.15 : The identity functor is an isomorphism $\mathrm{id}_{\underline{\mathbf{A}}}\colon \underline{\mathbf{A}} \to \underline{\mathbf{A}}$. Let $U\colon \underline{\mathbf{TOP}} \to \underline{\mathbf{SET}}$ be the forgetful functor which maps $(X, \tau_X) \xrightarrow{\phi} (Y, \tau_Y)$ to $X \xrightarrow{\phi} Y$. Then U is faithful and full but is not an embedding since, with reference to Remark 1.2.4, we can define two morphisms $\phi, \psi \in \mathrm{Mor}\,(\underline{\mathbf{TOP}})$ such that $\mathrm{cod}\,\phi = (X, \tau_1) \neq (X, \tau_2) = \mathrm{cod}\,\psi$ and

$$U((X, \tau_1) \xrightarrow{\phi} (X, \tau_1)) = X \xrightarrow{\mathrm{id}_X} X = U((X, \tau_1) \xrightarrow{\psi} (X, \tau_2))$$

so U is not injective on morphisms.

Lemma 1.2.16 [1]: Let $\underline{\mathbf{A}}$ be a subcategory of $\underline{\mathbf{B}}$. We define the *inclusion functor* $E\colon \underline{\mathbf{A}} \hookrightarrow \underline{\mathbf{B}}$ by $E(A \xrightarrow{f} A') = A \xrightarrow{f} A'$. Then

1. E is an embedding.
2. E is a full functor $\Leftrightarrow$ $\underline{\mathbf{A}}$ is a full subcategory of $\underline{\mathbf{B}}$.

Lemma 1.2.17 [1]:

1. A composite of embeddings is an embedding.
2. $F\colon \underline{\mathbf{A}} \to \underline{\mathbf{B}}$ is an embedding $\Leftrightarrow$ F is faithful and injective on objects.
3. $F\colon \underline{\mathbf{A}} \to \underline{\mathbf{B}}$ is an isomorphism $\Leftrightarrow$ F is faithful, full and bijective on objects.

Lemma 1.2.18 [1]: Let F be a functor between categories $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$. If F is a full functor, or is injective on objects, then the *image* of $\underline{\mathbf{A}}$ under F, $F(\underline{\mathbf{A}})$, is a subcategory of $\underline{\mathbf{B}}$.

1.3 Concrete categories

The definition of an abstract category ignores the fact that objects in a category may have a structure of their own. Concrete categories provide a way to apply category theory to categories where the structures of the objects in the category are of interest. *Constructs* are categories consisting of structured sets and structure preserving functions between those sets, and are the primary examples of concrete categories.

Concrete categories

Definition 1.3.1 [1]: A *concrete category* $(\underline{\mathbf{A}}, U)$ over a *base category* $\underline{\mathbf{X}}$ is a category $\underline{\mathbf{A}}$ together with a *faithful* functor $U\colon \underline{\mathbf{A}} \to \underline{\mathbf{X}}$.
A concrete category over $\underline{\mathbf{SET}}$ is called a *construct.*

Example 1.3.2 : Any category $\underline{\mathbf{A}}$ can be regarded as a concrete category $(\underline{\mathbf{A}}, \mathrm{id}_{\underline{\mathbf{A}}})$ over $\underline{\mathbf{A}}$. The category $\underline{\mathbf{TOP}}$ cat be regarded via the forgetful functor $U\colon \underline{\mathbf{TOP}} \to \underline{\mathbf{SET}}$ as a construct $(\underline{\mathbf{TOP}}, U)$.

Remark 1.3.3 [1]: For a concrete category $(\underline{\mathbf{A}}, U)$ over a category $\underline{\mathbf{X}}$, because U is faithful, we can regard $\hom_{\underline{\mathbf{A}}}(A, B)$ as a subset of $\hom_{\underline{\mathbf{X}}}(UA, UB)$ in the following way: we define the convention that

$$\phi\colon UA \to UB \in \hom_{\underline{\mathbf{A}}}(A, B) \Leftrightarrow \ \exists\, f \in \hom_{\underline{\mathbf{A}}}(A, B) \quad Uf = \phi$$

For $f \in \hom_{\underline{\mathbf{A}}}(A, B)$ we say that f is *identity carried* if

$$U(A \xrightarrow{f} B) = UA \xrightarrow{\mathrm{id}_{UA}} UB$$

Note that if f is identity carried then $UA = UB$ but A is not necessarily equal to B.

Example 1.3.4 : Consider the concrete category $\underline{\mathbf{TOP}}$. Let U be the forgetful functor from $\underline{\mathbf{TOP}}$ into $\underline{\mathbf{SET}}$. We know that when we say $\phi \colon X \to Y$ is a morphism from (X, τ_X) to (Y, τ_Y) we really mean that there is $f = ((X, \tau_X), \phi, (Y, \tau_Y))$ such that $f \in \hom_{\underline{\mathbf{TOP}}}((X, \tau_X), (Y, \tau_Y))$ and $Uf = \phi$. Since U is faithful this morphism f is necessarily unique and we may simply refer to it as ϕ as well, and we write $\phi \colon (X, \tau_X) \to (Y, \tau_Y)$.

Definition 1.3.5 [1]: Let $(\underline{\mathbf{A}}, U)$ be a concrete category over $\underline{\mathbf{X}}$. Let $X \in \mathrm{Ob}\,(\underline{\mathbf{X}})$. The $\underline{\mathbf{A}}$-*fibre* of X is the class defined by

$$\mathrm{Fibre}_{\underline{\mathbf{A}}}(X) = \{\, A \in \mathrm{Ob}\,(\underline{\mathbf{A}}) \mid U(A) = X \,\}.$$

Ordering on the elements of $\mathrm{Fibre}_{\underline{\mathbf{A}}}(X)$ is achieved by

$$A \leq B \Leftrightarrow \mathrm{id}_X \in \hom_{\underline{\mathbf{A}}}(A, B).$$

This turns $\mathrm{Fibre}_{\underline{\mathbf{A}}}(X)$ into a quasi-ordered class (i.e. $\leq$ is reflexive and transitive). $(\underline{\mathbf{A}}, U)$ is said to be *fibre-small* if each $\underline{\mathbf{A}}$-fibre is a *set* (as opposed to being a proper class). $(\underline{\mathbf{A}}, U)$ is said to be *amnestic* if each $\underline{\mathbf{A}}$-fibre is a *partially ordered* class.

Example 1.3.6 : Let X be a set. The $\underline{\mathbf{TOP}}$-fibre of X is the class

$$\mathrm{Fibre}_{\underline{\mathbf{TOP}}}(X) = \{\, (Y, \tau) \in \mathrm{Ob}\,(\underline{\mathbf{TOP}}) \mid U(Y, \tau) = X \,\}.$$

In other words, the class of all topological spaces over X. This is a set since $\mathcal{P}(\mathcal{P}(X))$ is a set and the class of all topologies over X can be considered as a subset of $\mathcal{P}(\mathcal{P}(X))$. Hence $\underline{\mathbf{TOP}}$ is fibre small. In this fibre,

$$\begin{aligned} (X, \tau_1) \leq (X, \tau_2) &\Leftrightarrow \mathrm{id}_X \in \hom_{\underline{\mathbf{TOP}}}((X, \tau_1), (X, \tau_2)) \\ &\Leftrightarrow \tau_2 \subseteq \tau_1 \end{aligned}$$

Now if $(X, \tau_1) \leq (X, \tau_2)$ and $(X, \tau_2) \leq (X, \tau_1)$ then $(X, \tau_1) = (X, \tau_2)$, so $\underline{\mathbf{TOP}}$ is amnestic.

Concrete subcategories

We now wish to have a mechanism which enables us to consider certain concrete categories as sub-structures of other concrete categories. In other words, subcategories of concrete categories which are *in themselves concrete* and can therefore be expected to have the same concrete categorical properties as their supercategories.

The concept we are looking for is that of a *concrete subcategory* (Definition 1.3.7).

Definition 1.3.7 [1]: Let $(\underline{\mathbf{A}}, U), (\underline{\mathbf{B}}, V)$ be concrete categories over $\underline{\mathbf{X}}$. Then $(\underline{\mathbf{A}}, U)$ is a concrete subcategory of $(\underline{\mathbf{B}}, V) \Leftrightarrow$

1. $\underline{\mathbf{A}}$ is a subcategory of $\underline{\mathbf{B}}$.
2. $U = V \circ E$, where $E \colon \underline{\mathbf{A}} \hookrightarrow \underline{\mathbf{B}}$ is the inclusion functor.

Remark 1.3.8 : The second condition of Definition 1.3.7 expresses the fact that if $(\underline{\mathbf{A}}, U)$ is a concrete subcategory of $(\underline{\mathbf{B}}, V)$ (over $\underline{\mathbf{X}}$), then $\underline{\mathbf{A}}$-objects must have the same underlying $\underline{\mathbf{X}}$-objects when regarded as $\underline{\mathbf{B}}$-objects.

Example 1.3.9 : A trivial example of a concrete subcategory is $\underline{\mathbf{SET_i}}$ (defined in Example 1.2.6) considered as a concrete subcategory of $\underline{\mathbf{SET}}$. $\underline{\mathbf{SET_i}}$ may be regarded as a construct via the inclusion functor E while $\underline{\mathbf{SET}}$ may be regarded as a construct via it's identity functor $\mathrm{id}_{\underline{\mathbf{SET}}}$. Since $\underline{\mathbf{SET_i}}$ is a subcategory of $\underline{\mathbf{SET}}$ and $E = \mathrm{id}_{\underline{\mathbf{SET}}} \circ E$, we may regard $(\underline{\mathbf{SET_i}}, E)$ as a concrete subcategory of $(\underline{\mathbf{SET}}, \mathrm{id}_{\underline{\mathbf{SET}}})$.

Another example is the category $\underline{\mathbf{HAUS}}$ (see e.g. [5]), the category of all Hausdorff spaces, whose objects are the Hausdorff topological spaces and whose morphisms are the continuous functions between them. $\underline{\mathbf{HAUS}}$ can naturally be considered as a subcategory of $\underline{\mathbf{TOP}}$, so there is an inclusion functor $E \colon \underline{\mathbf{HAUS}} \hookrightarrow \underline{\mathbf{TOP}}$. We know that $(\underline{\mathbf{TOP}}, U)$, where U is the forgetful functor into $\underline{\mathbf{SET}}$, is a construct. We can therefore regard $(\underline{\mathbf{HAUS}}, U \circ E)$ as a subconstruct of $(\underline{\mathbf{TOP}}, U)$.

Definition 1.3.10 [1]: Let $(\underline{\mathbf{A}}, U), (\underline{\mathbf{B}}, V)$ be concrete categories over $\underline{\mathbf{X}}$. Then $(\underline{\mathbf{A}}, U)$ is a *concretely reflective* subcategory of $(\underline{\mathbf{B}}, V) \Leftrightarrow$

1. $(\underline{\mathbf{A}}, U)$ is a concrete subcategory of $(\underline{\mathbf{B}}, V)$.
2. $\forall\, B \in \mathrm{Ob}\,(\underline{\mathbf{B}})\ \exists\, A_B \in \mathrm{Ob}\,(\underline{\mathbf{A}})\ \exists\, r \in \hom_{\underline{\mathbf{B}}}(B, A_B)$,
   (a) $Vr = \mathrm{id}_{VB}$.
   (b) r is an $\underline{\mathbf{A}}$-reflection arrow for B (see Definition 1.2.7).

In other words, $(\underline{\mathbf{A}}, U)$ is a concretely reflective subcategory of $(\underline{\mathbf{B}}, V)$ iff for every $\underline{\mathbf{B}}$-object B there is an identity-carried $\underline{\mathbf{A}}$-reflection arrow.

Remark 1.3.11 : Examples of concretely reflective subcategories will be given later in the chapters on classical topological spaces and L-topological spaces.

Concrete functors

Definition 1.3.12 [1]: Let $(\underline{\mathbf{A}}, U), (\underline{\mathbf{B}}, V)$ be concrete categories over $\underline{\mathbf{X}}$. F is a *concrete functor* between concrete categories $(\underline{\mathbf{A}}, U)$ and $(\underline{\mathbf{B}}, V) \Leftrightarrow$

1. $F\colon \underline{\mathbf{A}} \to \underline{\mathbf{B}}$ is a functor.

2. $U = V \circ F$.

Remark 1.3.13 : The second condition is easier to understand if one thinks in terms of a functor mapping a construct to another construct, e.g. mapping for example topological spaces to neighbourhood spaces. What the condition says is that the *underlying set* does not change.

Note that since both U and V are faithful, this implies that F must be faithful too. This means that *every concrete functor is faithful* [1].

Definition 1.3.14 [1]: Let $(\underline{\mathbf{A}}, U), (\underline{\mathbf{B}}, V)$ be concrete categories over $\underline{\mathbf{X}}$. Let $F\colon (\underline{\mathbf{A}}, U) \to (\underline{\mathbf{B}}, V)$, $G\colon (\underline{\mathbf{A}}, U) \to (\underline{\mathbf{B}}, V)$ be concrete functors. We define

$$F \leq G \Leftrightarrow \ \forall A \in \mathrm{Ob}\,(\underline{\mathbf{A}}) \quad F(A) \leq G(A).$$

Note that $U = V \circ F = V \circ G$ so that $V(F(A)) = V(G(A))$ meaning that $F(A)$ and $G(A)$ are always members of the same fibre of $\underline{\mathbf{X}}$.

Definition 1.3.15 [1]: Let $(\underline{\mathbf{A}}, U), (\underline{\mathbf{B}}, V)$ be concrete categories over $\underline{\mathbf{X}}$. $F\colon (\underline{\mathbf{A}}, U) \to (\underline{\mathbf{B}}, V)$ is a *concrete isomorphism* $\Leftrightarrow$

1. F is a concrete functor between $(\underline{\mathbf{A}}, U)$ and $(\underline{\mathbf{B}}, V)$.

2. $F\colon \underline{\mathbf{A}} \to \underline{\mathbf{B}}$ is an isomorphism (see Definition 1.2.14).

Example 1.3.16 : Let $(\underline{\mathbf{A}}, U)$ be a concrete category over $\underline{\mathbf{X}}$. Then $\mathrm{id}_{\underline{\mathbf{A}}}$ is a concrete isomorphism between $(\underline{\mathbf{A}}, U)$ and $(\underline{\mathbf{A}}, U)$.

1.4 Topological categories

Many of the useful properties of topological spaces and related constructions such as neighbourhood spaces may be regarded as properties of the corresponding categories. In this way notions such as topological spaces, limit spaces and uniform spaces may be drawn together and given a common framework through category theory [34]. Accordingly we define what it means for a category to be *topological.*

Definition 1.4.1 [1]: Let $\underline{\mathbf{A}}$ be a category. $\mathcal{S}$ is a *source* in $\underline{\mathbf{A}} \Leftrightarrow$

$$\mathcal{S} = (A, (f_i)_{i \in I})$$

where $A \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ and $(f_i)_{i\in I} \in \mathrm{Mor}\,(\underline{\mathbf{A}})^I$ is a class of morphisms indexed by the class I with the property that $\forall\, i \in I \quad \mathrm{dom}\, f_i = A$. Alternative notation for $\mathcal{S}$ is $(A \xrightarrow{f_i} A_i)_{i\in I}$. A is referred to as the *domain* of the source $\mathcal{S}$ while the A_i are referred to as the *codomain.*

Definition 1.4.2 [1]: Let $G : \underline{\mathbf{A}} \to \underline{\mathbf{B}}$ be a functor. $\mathcal{T}$ is a *G-structured source* in $\underline{\mathbf{B}} \Leftrightarrow$

1. $\mathcal{T}$ is a source in $\underline{\mathbf{B}}$.
2. the codomain of $\mathcal{T}$ is given by a family $(GA_i)_{i\in I}$ where $\forall\, i \in I \quad A_i \in \mathrm{Ob}\,(\underline{\mathbf{A}})$.

Definition 1.4.3 [1]: Let $\underline{\mathbf{A}}, U)$ be a concrete category over $\underline{\mathbf{X}}$. We recall the convention (see Remark 1.3.3) that

$$\psi \in \hom_{\underline{\mathbf{A}}}(A', A) \Leftrightarrow\ \exists\, f \in \hom_{\underline{\mathbf{A}}}(A', A), \quad Uf = \psi.$$

We define $\mathcal{S}$ is an *initial* source in $(\underline{\mathbf{A}}, U) \Leftrightarrow$

1. $\mathcal{S} = (A \xrightarrow{f_i} A_i)_{i\in I}$ is a source in $\underline{\mathbf{A}}$.
2. $\forall\, A' \in \mathrm{Ob}\,(\underline{\mathbf{A}})\ \forall\, \psi \colon UA' \to UA$

$$(\ \forall\, i \in I \quad (Uf_i) \circ \psi \in \hom_{\underline{\mathbf{A}}}(A', A_i)) \Leftrightarrow \psi \in \hom_{\underline{\mathbf{A}}}(A', A).$$

Definition 1.4.4 [1]: Let $(\underline{\mathbf{A}}, U)$ be a concrete category over $\underline{\mathbf{X}}$. $(\underline{\mathbf{A}}, U)$ is a *topological* category over $\underline{\mathbf{X}} \Leftrightarrow$ every U-structured source $(X \xrightarrow{\phi_i} UA_i)_I$ has a *unique* initial lift $(A \xrightarrow{f_i} A_i)_I$ i.e. there exists a unique initial source $(A \xrightarrow{f_i} A_i)_I$ such that

$$\forall\, i \in I \quad U(A \xrightarrow{f_i} A_i) = X \xrightarrow{\phi_i} UA_i.$$

Another way of saying this is that $(\underline{\mathbf{A}}, U)$ has *initial structures.*

Remark 1.4.5 : Throughout the following discussion we will be dealing with constructs (concrete categories over $\underline{\mathbf{SET}}$) which are topological over $\underline{\mathbf{SET}}$. Preuss ([34]) defines a construct $(\underline{\mathbf{A}}, U)$ to be *topological* if it satisfies the following three conditions:

1. $(\underline{\mathbf{A}}, U)$ is topological over $\underline{\mathbf{Set}}$ (in the sense already defined).

2. $(\underline{\mathbf{A}}, U)$ is fibre small.

3. $(\underline{\mathbf{A}}, U)$ satisfies the *terminal separator property* i.e. for any singleton set $\{x\}$ there exists exactly one $A \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ such that $UA = \{x\}$.

We will use the definition given by Adámek et al. ([1]), but wherever possible we will prove the extra properties needed for Preuss's stronger definition as well, in order that our results may apply more widely.

Adámek et al. define a construct $(\underline{\mathbf{A}}, U)$ to be *well-fibred* iff for all $X \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$ with $|X| \leq 1$ there exists a unique $A \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ such that $UA = X$. For us this is essentially the same as the terminal separator property since we consider constructs where one of the conditions on the objects is that the underlying sets are non-empty.

Example 1.4.6 : The category $\underline{\mathbf{TOP}}$ satisfies the terminal separator property. Consider a one point set $X = \{x\}$. Now both $\emptyset$ and X must be members of any topology τ on X, but these are the only possible subsets of X, so the only possible topology on X is $\mathcal{P}(X) = \{\emptyset, X\}$.

Lemma 1.4.7 [1]: Let $(\underline{\mathbf{A}}, U)$ be a concrete category which is amnestic over its base category $\underline{\mathbf{X}}$. Then for any U-structured source $(X \xrightarrow{\phi_i} UA_i)_{i\in I}$ in $\underline{\mathbf{X}}$, if that source has an initial lift $(A \xrightarrow{f_i} A_i)_{i\in I}$, that initial lift is *unique.*

Proof:
Let $(X \xrightarrow{\phi_i} UA_i)_{i\in I}$ be a U-structured source in $\underline{\mathbf{X}}$ which has an initial lift $(A \xrightarrow{f_i} A_i)_{i\in I}$ and another initial lift $(B \xrightarrow{g_i} A_i)_{i\in I}$. Now $A, B \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ and $\mathrm{id}_{UA} = \mathrm{id}_X \in \hom_{\underline{\mathbf{X}}}(UA, X)$. By definition

$$\forall\, i \in I \quad \phi_i \circ \mathrm{id}_X \in \hom_{\underline{\mathbf{A}}}(A, A_i).$$

Thus since both sources are initial, we have that $\mathrm{id}_X \in \hom_{\underline{\mathbf{A}}}(A, A)$, which tells us nothing new, but also $\mathrm{id}_X \in \hom_{\underline{\mathbf{A}}}(A, B)$. Thus we have that $A \leq B$. Similarly $B \leq A$. Since $(\underline{\mathbf{A}}, U)$ is amnestic, $B = A$.

For the morphisms, select $i \in I$. We have

$$U(A \xrightarrow{f_i} A_i) = U(B \xrightarrow{g_i} A_i) = U(A \xrightarrow{g_i} A_i).$$

Since U is faithful, $f_i = g_i$. So we have proved that

$$(A \xrightarrow{f_i} A_i)_{i \in I} = (B \xrightarrow{g_i} A_i)_{i \in I}.$$

■

Proving that a construct has initial structures

To prove that a construct $(\underline{\mathbf{A}}, U)$ has initial structures (i.e. is topological over $\underline{\mathbf{Set}}$) we need the following steps:

1. We define a U-structured source $(X \xrightarrow{\phi_i} X_i)_I$ in $\underline{\mathbf{SET}}$ where $(X_i, \xi_i) \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ and $\phi_i : X \to X_i$ are functions.

2. We now need to find a structure ξ_X on X such that $(X, \xi_X) \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ and $\forall\, i \in I \quad \phi_i \in \hom_{\underline{\mathbf{A}}}((X, \xi_X), (X_i, \xi_i))$.

3. Next we need to prove that $((X, \xi_X) \xrightarrow{\phi_i} (X_i, \xi_i))_I$ is an initial source in $\underline{\mathbf{A}}$. To do this, let $(Y, \xi_Y) \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ and $\psi : Y \to X$ be a function. We need to prove that

$$\forall\, i \in I \quad \phi_i \circ \psi \in \hom_{\underline{\mathbf{A}}}((Y, \xi_Y), (X_i, \xi_i)) \\ \Rightarrow \psi \in \hom_{\underline{\mathbf{A}}}((Y, \xi_Y), (X, \xi_X)).$$

4. Lastly we need to prove that $((X, \xi_X) \xrightarrow{\phi_i} (X_i, \xi_i))_I$ is the *unique* lift of $(X \xrightarrow{\phi_i} X_i)_I$ which is initial in $(\underline{\mathbf{A}}, U)$. In this we are assisted by the fact that this is *automatic* if we are considering an amnestic construct (Lemma 1.4.7). All of the categories we will be considering are amnestic constructs.

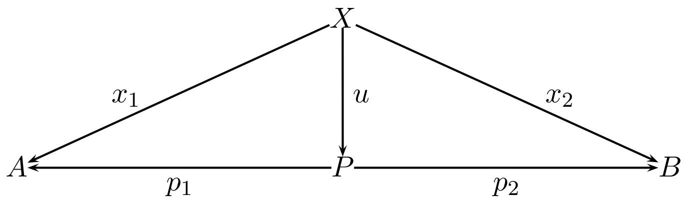
Examples will be given in later chapters and the procedure should become clear.

1.5 Cartesian closed categories

Definition 1.5.1 [3]: Let $\underline{\mathbf{A}}$ be a category. Then $T \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ is a *terminal object* iff

$$\forall\, A \in \mathrm{Ob}\,(\underline{\mathbf{A}}), \qquad \exists\,! f \in \hom_{\underline{\mathbf{A}}}(A, T).$$

Let $A, B \in \mathrm{Ob}\,(\underline{\mathbf{A}})$. Then a *product diagram* for A and B consists of an object $P \in \mathrm{Ob}\,(\underline{\mathbf{A}})$ and morphisms $p_1 \colon P \to A, p_2 \colon P \to B$ such that given any $X \in \mathrm{Ob}\,(\underline{\mathbf{A}}), x_1 \colon X \to A, x_2 \colon X \to B$ there exists a *unique* $u \colon X \to P$ such that the following diagram commutes:



i.e. such that $x_1 = p_1 \circ u$ and $x_2 = p_2 \circ u$.

$\underline{\mathbf{A}}$ is said to have *all binary products* iff a product diagram exists for every $A, B \in \mathrm{Ob}\,(\underline{\mathbf{A}})$. $\underline{\mathbf{A}}$ is said to have *all finite products* iff it has a terminal object and all binary products.

Remark 1.5.2 : For the object P in Definition 1.5.1, we normally write $P = A \times B$. For the morphism u we normally write $u = \langle x_1, x_2 \rangle$.

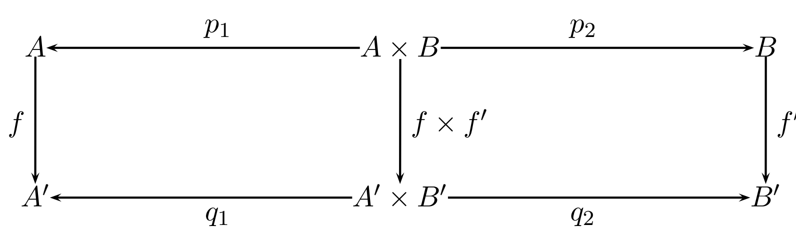
Example 1.5.3 : Any one element set $\{x\}$ together with $\mathcal{P}(\{x\}) = \{\emptyset, \{x\}\}$ is a terminal object in the category $\underline{\mathbf{TOP}}$.

Example 1.5.4 : Let $A, B \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$, i.e. A and B are sets. Let $A \times B = \{\, (a, b) \mid a \in A \text{ and } b \in B \,\}$. Let $\pi_A \colon A \times B \to A \quad (a, b) \mapsto a$ and $\pi_B \colon A \times B \to B \quad (a, b) \mapsto b$. Then $(A \times B, \pi_A, \pi_B)$ is a product diagram in $\underline{\mathbf{SET}}$ for A and B.

Any one-element set $\{x\}$ is a terminal object for the category $\underline{\mathbf{SET}}$, hence $\underline{\mathbf{SET}}$ has all finite products.

Lemma 1.5.5 [3]: Products are unique up to isomorphism.

Definition 1.5.6 [3]: Let $\underline{\mathbf{A}}$ be a category with binary products. Let $f \colon A \to B$ and $f' \colon A' \to B'$ be morphisms in $\underline{\mathbf{A}}$. We define the morphism $f \times f' \colon A \times A' \to B \times B'$ such that the following diagram commutes:



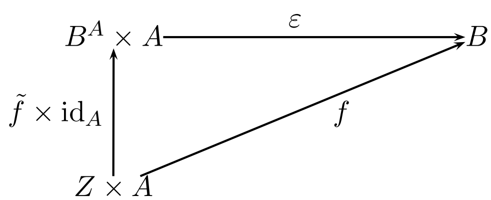
i.e. $f \times f' = \langle f \circ p_1, f' \circ p_2 \rangle$.

Example 1.5.7 : Let $\emptyset \neq X, Y, U, V \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$ and let $f \colon X \to U$, $g \colon Y \to V$ be functions. Then we know that the normal cartesian products $X \times Y, U \times V$ and projection functions $\pi_X, \pi_Y, \pi_U, \pi_V$ form product diagrams for X, Y and U, V, thus we can define $f \times g \colon X \times Y \to U \times V$. Now $\pi_U((f \times g)(x, y)) = (f \circ \pi_X)(x, y) = f(x)$ for $(x, y) \in X \times Y$ and similarly $\pi_V((f \times g)(x, y)) = g(y)$, thus $f \times g(x, y) = (f(x), g(y))$.

Definition 1.5.8 [3]: Let $\underline{\mathbf{A}}$ be a category with binary products and let $A, B \in \mathrm{Ob}(\underline{\mathbf{A}})$. Then an *exponential of* A *and* B consists of an object $B^A \in \mathrm{Ob}(\underline{\mathbf{A}})$ (usually called an *exponential object*) and a morphism $\varepsilon\colon B^A \times A \to B$ (usually called the *evaluation mapping*) such that

$$\forall\, Z \in \mathrm{Ob}(\underline{\mathbf{A}})\ \forall\, f\colon Z \times A \to B \quad \exists!\,\tilde{f}\colon Z \to B^A$$

such that $\varepsilon \circ (\tilde{f} \times \mathrm{id}_A) = f$, i.e. there exists a unique $\tilde{f}$ (the *transpose of* f) such that the diagram below commutes.



Example 1.5.9 : Let $A, B \in \mathrm{Ob}(\underline{\mathbf{SET}})$. Define

$$B^A = \{\, g \mid g \text{ is a function between } A \text{ and } B \,\}.$$

Define $\varepsilon\colon B^A \times A \to B \quad (g, a) \mapsto g(a)$. Let $Z \in \mathrm{Ob}(\underline{\mathbf{SET}})$ and let $f\colon Z \times A \to B$ be a function. For $z \in Z$, define $f_z\colon A \to B$ by $f_z(a) = f(z, a)$. Now $f_z \in B^A$ by the definition. Define $\tilde{f}\colon Z \to B^A \quad z \mapsto f_z$. Now let $(z, a) \in Z \times A$. Then $f(z, a) = f_z(a) = \varepsilon(f_z, a) = \varepsilon \circ (\tilde{f} \times \mathrm{id}_A)(z, a)$. Lastly assume $h\colon Z \to B^A$ has the property that $\varepsilon \circ (h \times \mathrm{id}_A) = f$. Let $z \in Z$. Then $\forall\, a \in A, \quad h(z)(a) = \tilde{f}(z)(a) = f(z, a)$, thus $h = \tilde{f}$. We have proved that (B^A, ε) is an exponential of A and B in the category $\underline{\mathbf{SET}}$.

Definition 1.5.10 [3]: A category $\underline{\mathbf{A}}$ is *cartesian closed* iff

1. $\underline{\mathbf{A}}$ has all finite products.
2. $\underline{\mathbf{A}}$ has all exponentials, i.e. for every pair of objects $A, B \in \mathrm{Ob}(\underline{\mathbf{A}})$ there exists an exponential of A and B.

Example 1.5.11 : By Examples 1.5.4 and 1.5.9, $\underline{\mathbf{SET}}$ is cartesian closed.

1.6 Categorical theorems

In this section we present some useful theorems which are used later in proving properties of different categories.

Theorem 1.6.1 [1]: Let $(\underline{\mathbf{A}}, U)$ be a concrete subcategory of a concrete category $(\underline{\mathbf{B}}, V)$ over $\underline{\mathbf{X}}$. Let $E : \underline{\mathbf{A}} \hookrightarrow \underline{\mathbf{B}}$ be the inclusion functor. Then

$(\underline{\mathbf{A}}, U)$ is a full, concretely reflective subcategory of $(\underline{\mathbf{B}}, V) \Leftrightarrow$
There exists a functor $G\colon (\underline{\mathbf{B}}, V) \to (\underline{\mathbf{A}}, U)$ such that

$$G \circ E = \mathrm{id}_{\underline{\mathbf{A}}} \quad \text{and} \quad E \circ G \geq \mathrm{id}_{\underline{\mathbf{B}}}.$$

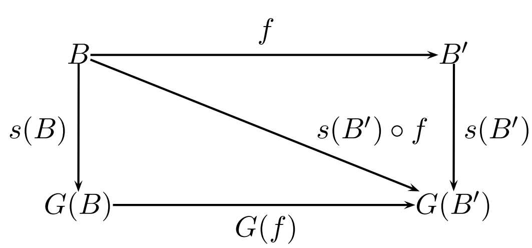
Proof:
Assume that $(\underline{\mathbf{A}}, U)$ is a full, concretely reflective subcategory of $(\underline{\mathbf{B}}, V)$. We define the functor G as follows: we know that

$$\forall\, B \in \mathrm{Ob}\,(\underline{\mathbf{B}})\ \exists \text{ an identity-carried } \underline{\mathbf{A}}\text{-reflection arrow, } r_B\colon B \to A_B.$$

We define a selection function $s\colon \mathrm{Ob}\,(\underline{\mathbf{B}}) \to \mathrm{Mor}\,(\underline{\mathbf{B}})$

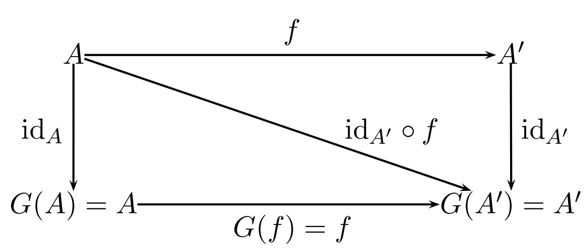
$$B \mapsto \begin{cases} \mathrm{id}_B & B \in \mathrm{Ob}\,(\underline{\mathbf{A}}) \\ \text{One of the } r_B\text{'s} & \text{otherwise} \end{cases}$$

This is possible by the axiom of choice. Now we define the functor G by the diagram below.



We can do this since for each $B \xrightarrow{f} B' \in \mathrm{Mor}\,(\underline{\mathbf{B}})$, the morphism $s(B') \circ f$ is uniquely defined. The morphism $G(f)$ is defined as the unique morphism such that $s(B') \circ f = G(f) \circ s(B)$, which is possible since $s(B)$ is an $\underline{\mathbf{A}}$-reflection arrow.

Now let $A \xrightarrow{f} A' \in \mathrm{Mor}\,(\underline{\mathbf{A}})$. Then $E(A \xrightarrow{f} A') = A \xrightarrow{f} A'$. We calculate $G \circ E(A \xrightarrow{f} A')$ using the diagram below.



So we have that $G \circ E = \mathrm{id}_{\underline{\mathbf{A}}}$.

Let $B \in \mathrm{Ob}\,(\underline{\mathbf{B}})$. We know that $s(B)\colon B \to G(B)$ is *identity carried.* Hence $B \leq G(B) = E \circ G(B)$ and we have that $E \circ G \geq \mathrm{id}_{\underline{\mathbf{B}}}$.

For the converse, assume that there exists a functor $G\colon (\underline{\mathbf{B}}, V) \to (\underline{\mathbf{A}}, U)$ such that

$$G \circ E = \mathrm{id}_{\underline{\mathbf{A}}} \qquad E \circ G \geq \mathrm{id}_{\underline{\mathbf{B}}}$$

Let $B \in \mathrm{Ob}\,(\underline{\mathbf{B}})$. We define $A_B = GB \in \mathrm{Ob}\,(\underline{\mathbf{A}})$. Because $E \circ G \geq \mathrm{id}_{\underline{\mathbf{B}}}$ we have

$$\exists\, r \in \hom_{\underline{\mathbf{B}}}(B, A_B) \quad Vr = \mathrm{id}_{VB}$$

so r is identity carried. For each $B \in \mathrm{Ob}\,(\underline{\mathbf{B}})$ we select one $r_B \in \hom_{\underline{\mathbf{B}}}(B, A_B)$ such that $Vr_B = \mathrm{id}_{VB}$. This is possible by the axiom of choice. We now prove that r_B is an $\underline{\mathbf{A}}$-reflection arrow.

Let $A \in \mathrm{Ob}\,(\underline{\mathbf{A}}), f \in \hom_{\underline{\mathbf{B}}}(B, A)$. Then $Gf \in \hom_{\underline{\mathbf{A}}}(A_B, A)$ since $GA = A$ from $G \circ E = \mathrm{id}_{\underline{\mathbf{A}}}$. So $(Gf) \circ r_B \in \hom_{\underline{\mathbf{B}}}(B, A)$. Now G is a concrete functor, thus $U \circ G = V$. Also $(\underline{\mathbf{A}}, U)$ is a concrete subcategory of $(\underline{\mathbf{B}}, V)$, so $U = V \circ E$. So we have $V \circ E \circ G = V$.

$$\begin{aligned} V((Gf) \circ r_B) = V(Gf) \circ V(r_B) = V(Gf) \circ \mathrm{id}_{VB} \\ = V(Gf) = V \circ E \circ G(f) = V(f). \end{aligned}$$

Since $(Gf) \circ r_B, f \in \hom_{\underline{\mathbf{B}}}(B, A)$ and V is faithful we have $(Gf) \circ r_B = f$. We need to prove that $f' \circ r_B = f \Rightarrow f' = Gf$. Assume $f' \circ r_B = f$. Then $V(f' \circ r_B) = V((Gf) \circ r_B)$ and $V(f') = V(Gf)$, so $f' = Gf$ since V is faithful.

Lastly we need to prove that $(\underline{\mathbf{A}}, U)$ is a full subcategory of $(\underline{\mathbf{B}}, V)$. Let $A, A' \in \mathrm{Ob}\,(\underline{\mathbf{A}}), f \in \hom_{\underline{\mathbf{B}}}(A, A')$. Then $Gf \in \hom_{\underline{\mathbf{A}}}(A, A')$ and

$$V(A \xrightarrow{Gf} A') = V \circ E(A \xrightarrow{Gf} A') = VA \xrightarrow{V \circ E \circ G(f)} VA' = VA \xrightarrow{Vf} VA'.$$

Because V is faithful we have $f = Gf$, thus $f \in \hom_{\underline{\mathbf{A}}}(A, A')$. ■

Theorem 1.6.2 [1]: Let $(\underline{\mathbf{A}}, U), (\underline{\mathbf{B}}, V)$ be concrete categories over a category $\underline{\mathbf{X}}$. Then
$(\underline{\mathbf{A}}, U)$ is isomorphic to a full, reflective subcategory of $(\underline{\mathbf{B}}, V) \Leftrightarrow$ There exist functors $F\colon (\underline{\mathbf{A}}, U) \to (\underline{\mathbf{B}}, V)$, $G\colon (\underline{\mathbf{B}}, V) \to (\underline{\mathbf{A}}, U)$ such that the following conditions are satisfied:

1. F is an embedding.
2. $G \circ F = \mathrm{id}_{\underline{\mathbf{A}}}$ and $F \circ G \geq \mathrm{id}_{\underline{\mathbf{B}}}$.

Proof:
Assume that $(\underline{\mathbf{A}}, U)$ is isomorphic to a full, reflective subcategory $(\underline{\mathbf{C}}, W)$ of

$(\underline{\mathbf{B}}, V)$. Let E be the inclusion functor $E\colon (\underline{\mathbf{C}}, W) \hookrightarrow (\underline{\mathbf{B}}, V)$. By Theorem 1.6.1, there exists a functor $K\colon (\underline{\mathbf{B}}, V) \to (\underline{\mathbf{C}}, W)$ such that

$$K \circ E = \mathrm{id}_{\underline{\mathbf{C}}} \quad \text{and} \quad E \circ K \geq \mathrm{id}_{\underline{\mathbf{B}}}.$$

Since $(\underline{\mathbf{A}}, U)$ is isomorphic to $(\underline{\mathbf{C}}, W)$, there exists an isomorphism functor $H\colon (\underline{\mathbf{A}}, U) \to (\underline{\mathbf{C}}, W)$ with inverse H^{-1}. We define the functors F and G by

$$F = E \circ H \quad \text{and} \quad G = H^{-1} \circ K.$$

Then F is an embedding since H and E are both embeddings and a composite of embeddings is an embedding (see Lemma 1.2.17). We calculate

$$G \circ F = H^{-1} \circ K \circ E \circ H = H^{-1} \circ \mathrm{id}_{\underline{\mathbf{C}}} \circ H = \mathrm{id}_{\underline{\mathbf{A}}},$$
$$F \circ G = E \circ H \circ H^{-1} \circ K = E \circ K \geq \mathrm{id}_{\underline{\mathbf{B}}}.$$

For the converse, assume that we have functors $F\colon (\underline{\mathbf{A}}, U) \to (\underline{\mathbf{B}}, V)$ and $G\colon (\underline{\mathbf{B}}, V) \to (\underline{\mathbf{A}}, U)$ such that F is an embedding and

$$G \circ F = \mathrm{id}_{\underline{\mathbf{A}}} \quad \text{and} \quad F \circ G \geq \mathrm{id}_{\underline{\mathbf{B}}}.$$

Since F is an embedding, it is injective on objects (see Lemma 1.2.17) and hence $F(\underline{\mathbf{A}})$, the image of $\underline{\mathbf{A}}$ under F, is a subcategory of $\underline{\mathbf{B}}$ (see Lemma 1.2.18). Let E be the inclusion functor $E\colon F(\underline{\mathbf{A}}) \hookrightarrow \underline{\mathbf{B}}$. Then with $W = V \circ E$, we can regard $(F(\underline{\mathbf{A}}), W)$ as a concrete subcategory of $(\underline{\mathbf{B}}, V)$.

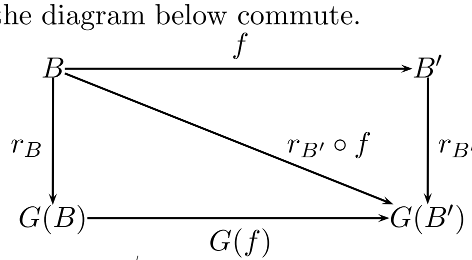
Since F is an embedding, $(\underline{\mathbf{A}}, U)$ is *isomorphic* to $(F(\underline{\mathbf{A}}), W)$. The isomorphism functor is simply $F\colon (\underline{\mathbf{A}}, U) \to (F(\underline{\mathbf{A}}), W)$. We define $K\colon (\underline{\mathbf{B}}, V) \to (F(\underline{\mathbf{A}}), W)$ by $K = F \circ G$. Now $E \circ K = F \circ G \geq \mathrm{id}_{\underline{\mathbf{B}}}$ and $K \circ E = K \circ E \circ F \circ F^{-1} = F \circ G \circ F \circ F^{-1} = F \circ \mathrm{id}_{\underline{\mathbf{A}}} \circ F^{-1} = \mathrm{id}_{F(\underline{\mathbf{A}})}$. Thus by Theorem 1.6.1, $(F(\underline{\mathbf{A}}), W)$ is a full, reflective subcategory of $(\underline{\mathbf{B}}, V)$, which is isomorphic to $(\underline{\mathbf{A}}, U)$. ■

Theorem 1.6.3 [1]: Let $(\underline{\mathbf{A}}, U)$ be a full concretely reflective subcategory of an amnestic category $(\underline{\mathbf{B}}, V)$, with base category $\underline{\mathbf{X}}$. Then

$$(\underline{\mathbf{B}}, V) \text{ is topological over } \underline{\mathbf{X}} \Rightarrow (\underline{\mathbf{A}}, U) \text{ is topological over } \underline{\mathbf{X}}.$$

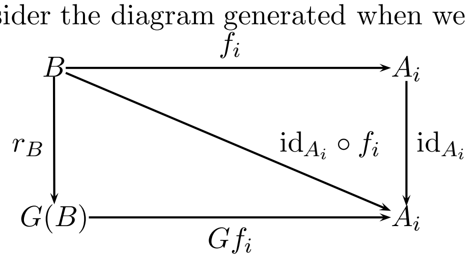
Proof:
Since $(\underline{\mathbf{A}}, U)$ is a full, concretely reflective subcategory of $(\underline{\mathbf{B}}, V)$, by the axiom of choice we can associate with each $B \in \mathrm{Ob}\,(\underline{\mathbf{B}})$ an identity carried $\underline{\mathbf{A}}$-reflection arrow r_B. Furthermore for those $B \in \mathrm{Ob}\,(\underline{\mathbf{A}})$, we can specify (by Lemma 1.2.10), that this reflection arrow shall be the identity on B,

namely id_B. We define a functor $G\colon (\underline{\mathbf{A}}, U) \to (\underline{\mathbf{B}}, V)$ in such a way as to make the diagram below commute.



Now let $(X \xrightarrow{\phi_i} UA_i)_{i\in I}$ be a U-structured source in $\underline{\mathbf{X}}$. Then $(X \xrightarrow{\phi_i} VA_i)_{i\in I}$ is a V-structured source which has a unique initial lift $(B \xrightarrow{f_i} A_i)_{i\in I}$ in $\underline{\mathbf{B}}$, since $\underline{\mathbf{B}}$ is topological over $\underline{\mathbf{X}}$. We prove that $(GB \xrightarrow{Gf_i} A_i)_{i\in I}$ is an initial lift of $(X \xrightarrow{\phi_i} UA_i)_{i\in I}$ in $\underline{\mathbf{A}}$.

Consider the diagram generated when we attempt to find Gf_i:

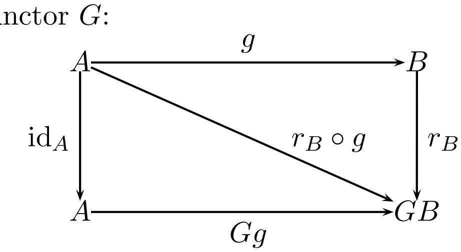


From the diagram we have $(Gf_i) \circ r_B = f_i$ so

$$V(Gf_i) = V(Gf_i) \circ \mathrm{id}_{VB} = V(Gf_i) \circ V(r_B) = V(Gf_i \circ r_B) = Vf_i = \phi_i$$

Thus $(GB \xrightarrow{Gf_i} A_i)_{i\in I}$ is a lift of $(X \xrightarrow{\phi_i} UA_i)_{i\in I}$ in $\underline{\mathbf{A}}$. We now prove that it is initial.

Let $A \in \mathrm{Ob}\,(\underline{\mathbf{A}}), \psi \in \hom_{\underline{\mathbf{X}}}(UA, X),\ \forall\, i \in I \quad \phi_i \circ \psi \in \hom_{\underline{\mathbf{A}}}(A, A_i)$. We need to prove that $\psi \in \hom_{\underline{\mathbf{A}}}(A, GB)$. We know that $\psi \in \hom_{\underline{\mathbf{X}}}(VA, X)$ and also $\forall\, i \in I \quad \phi_i \circ \psi \in \hom_{\underline{\mathbf{B}}}(A, A_i)$. Since $(B \xrightarrow{f_i} A_i)_{i\in I}$ is an initial lift in $\underline{\mathbf{B}}$, we have that $\psi \in \hom_{\underline{\mathbf{B}}}(A, B)$, i.e. $\exists\, g \in \hom_{\underline{\mathbf{B}}}(A, B) \quad Vg = \psi$. Let's look at the diagram of what happens when we map g to $\mathrm{Mor}\,(\underline{\mathbf{A}})$ using the functor G:



So we have $Gg = r_B \circ g$ and $U(Gg) = V \circ E(Gg) = V(r_B \circ g) = \mathrm{id}_{VB} \circ Vg = \psi$ thus $\psi \in \hom_{\underline{\mathbf{A}}}(A, GB)$.

Since $(\underline{\mathbf{B}}, V)$ is amnestic, $(\underline{\mathbf{A}}, U)$ is also amnestic and thus $(GB \xrightarrow{Gf_i} A_i)_{i\in I}$ is automatically a unique initial lift of $(X \xrightarrow{\phi_i} UA_i)_{i\in I}$. Thus $(\underline{\mathbf{A}}, U)$ is topological over $\underline{\mathbf{X}}$. ■

Chapter 2

Classical Topological Spaces

In this chapter we introduce the classical topological spaces and the corresponding categories which we will later generalize to the lattice case. The purpose is to provide us with some concrete examples which we may keep in mind as we generalize in Chapters 4 and 5. In the first section we collect some preliminary results which we will need later. Next, the category **TOP** is defined and some of its categorical properties are stated. In the following section we define the category **TCS** of topological convergence spaces and show that it is isomorphic to **TOP**. We pay particular attention to the classical Kowalski and Fischer axioms since they will play an important role in later chapters. Lastly we weaken the **TCS** axioms to obtain a category **CONV**, which is topological over **SET**, cartesian closed, and contains **TCS** as a reflective subcategory.

2.1 Filter Theory

One of the original strands in the study of topological spaces arose from the study of open sets in metric spaces [39]. One very useful property of such open sets is that they can be characterized by convergence of *sequences* i.e. functions whose domain is $\mathbb{N}$[36]. It was hoped initially that topological spaces could be characterized by convergence of sequences as well, but this turns out not to be true [39]. However, topological spaces can be characterized by the convergence of *filters* (see e.g. [29]), which are a generalized form of sequence. In this section we will develop some filter theory to the point where we can use it in Section 2.3 to formulate axioms which characterize topological spaces. Although many proofs in Section 2.3 are not given, the purpose of presenting the necessary filter theory in the classical case is to make comparison with the general L-filter case easier and more intuitive.

Definition 2.1.1 [5]: Let X be a non-empty set. Then $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *filter* on $X \Leftrightarrow$

F1 $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$.

F2 $\forall A, B \subseteq X, \quad A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$.

F3 $\forall A, B \subseteq X, \quad A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$.

The set of all filters on X is denoted $\mathcal{F}(X)$. We define an order on $\mathcal{F}(X)$ by

$$\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}(X), \quad \mathcal{F} \leq \mathcal{G} \Leftrightarrow \mathcal{F} \subseteq \mathcal{G}.$$

Example 2.1.2 : For $x \in X$ we define $[x] = \{ A \subseteq X \mid x \in A \}$. $[x] \in \mathcal{F}(X)$ is the *point filter at* x. The set $\{X\}$ is a filter on X.

Remark 2.1.3 : For all $\mathcal{F} \in \mathcal{F}(X), \quad X \in \mathcal{F}$, since $\mathcal{F} \neq \emptyset$, thus $\exists A \subseteq X$ in $\mathcal{F}$ and by **F2**, $X \in \mathcal{F}$.

Lemma 2.1.4 [5]: Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$. Let $(\mathcal{F}_i)_{i\in I} \in \mathcal{F}(X)^I$ be a collection of filters on X indexed by the class I. Then $\bigcap_{i\in I} \mathcal{F}_i \in \mathcal{F}(X)$ and furthermore $\bigcap_{i\in I} \mathcal{F}_i$ is the largest filter $\mathcal{G}$ on X such that $\forall i \in I, \quad \mathcal{G} \subseteq \mathcal{F}_i$.

A mapping $\phi\colon X \to Y$ induces mappings $\phi\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ and $\phi^{\leftarrow}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ in the following way. We define, for $A \subseteq X$ and $B \subseteq Y$

$$\phi(A) = \{ y \in Y \mid \exists x \in A \quad \phi(x) = y \},$$
$$\phi^{\leftarrow}(B) = \{ x \in X \mid \phi(x) \in B \}.$$

Lemma 2.1.5 [39]: Let $\emptyset \neq X, Y \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \mathcal{F} \in \mathcal{F}(X)$. Let $\phi\colon X \to Y$ be a function. We define

$$\phi(\mathcal{F}) = \{ B \subseteq Y \mid \phi^{\leftarrow}(B) \in \mathcal{F} \}.$$

Then $\phi(\mathcal{F}) \in \mathcal{F}(Y)$. We call $\phi(\mathcal{F})$ the image of $\mathcal{F}$ under ϕ.

Remark 2.1.6 : Note that the definition of $\phi(\mathcal{F})$ differs from the usual definition of the image of a collection of subsets $\mathcal{A}$ of X under ϕ. The definition given in Lemma 2.1.5 is equivalent to

$$\phi(\mathcal{F}) = \{ B \subseteq Y \mid \exists F \in \mathcal{F}, \quad \phi(F) \subseteq B \}.$$

The latter is given by $\phi(\mathcal{A}) = \{ \phi(A) \mid A \in \mathcal{A} \}$. This should cause no confusion provided that we are aware of it.

Lemma 2.1.7 [5]: Let $\emptyset \neq X, Y \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \mathcal{F} \in \mathcal{F}(Y)$. Let $\phi\colon X \to Y$ be a function. We define

$$\phi^{\leftarrow}(\mathcal{F}) = \{\, \phi^{\leftarrow}(A) \mid A \in \mathcal{F} \,\}.$$

Then $(\ \forall A \in \mathcal{F}, \quad \phi^{\leftarrow}(A) \neq \emptyset) \Leftrightarrow \phi^{\leftarrow}(\mathcal{F}) \in \mathcal{F}(X)$. We call $\phi^{\leftarrow}(\mathcal{F})$ the inverse image of $\mathcal{F}$ under ϕ.

Lemma 2.1.8 [5]: Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \mathcal{F}, \mathcal{G} \in \mathcal{F}(X)$. We define $\mathcal{F} \vee \mathcal{G} \subseteq \mathcal{P}(X)$ by

$$A \in \mathcal{F} \vee \mathcal{G} \Leftrightarrow \ \exists A_1 \in \mathcal{F}, A_2 \in G, \quad A_1 \cap A_2 \subseteq A$$

Then $\mathcal{F} \vee \mathcal{G} \in \mathcal{F}(X) \Leftrightarrow \ \forall F \in \mathcal{F}\ \forall G \in \mathcal{G}, \quad F \cap G \neq \emptyset$. If this condition is satisfied then $\mathcal{F} \vee \mathcal{G}$ is the least upper bound of $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{F}(X)$.

Lemma 2.1.9 [5]: Let $\emptyset \neq X, Y \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$ and let π_X, π_Y be the usual projection functions from $X \times Y$ to X, Y. Let $\mathcal{F} \in \mathcal{F}(X), \mathcal{G} \in \mathcal{F}(Y)$. Then $\pi_X^{\leftarrow}(\mathcal{F}), \pi_Y^{\leftarrow}(\mathcal{G}) \in \mathcal{F}(X \times Y)$, and furthermore $\pi_X^{\leftarrow}(\mathcal{F}), \pi_Y^{\leftarrow}(\mathcal{G})$ satisfy the condition of Lemma 2.1.8. Thus we can define

$$\mathcal{F} \times \mathcal{G} = \pi_X^{\leftarrow}(\mathcal{F}) \vee \pi_Y^{\leftarrow}(\mathcal{G}).$$

Lemma 2.1.10 [7, 28]: Let $\emptyset \neq J, X \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \mathcal{G} \in \mathcal{F}(J)$. Let $(\mathcal{F}_j)_{j \in J} \in \mathcal{F}(X)^J$ be a collection of filters on X indexed by J. We define the compression operator κ by

$$\kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J}) = \bigcup_{G \in \mathcal{G}} \bigcap_{j \in G} \mathcal{F}_j$$

Then $\kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J}) \in \mathcal{F}(X)$.

Proof:

F1 By Remark 2.1.3 and Lemma 2.1.4, $J \in \mathcal{G}$ and $\bigcap_{j \in J} \mathcal{F}_j \in \mathcal{F}(X)$ so $\emptyset \neq \kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J})$. Similarly since $\forall G \in \mathcal{G}, \quad \bigcap_{j \in G} \mathcal{F}_j \in \mathcal{F}(X)$, by the definition $\emptyset \notin \kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J})$.

F2 Let $A, B \subseteq X, A \in \kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J}), A \subseteq B$. Then by the definition of $\kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J})$, $\exists G \in \mathcal{G}\ \forall j \in G, \quad A \in \mathcal{F}_j$. By **F2**, $\exists G \in \mathcal{G}\ \forall j \in G, \quad B \in \mathcal{F}_j$. Thus $B \in \kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J})$.

F3 Let $A, B \in \kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J})$. Then by the definition of $\kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J})$, $\exists G, H \in \mathcal{G}, \quad A \in \bigcap_{j \in G} \mathcal{F}_j$ and $B \in \bigcap_{k \in H} \mathcal{F}_k$. Now $\emptyset \neq G \cap H \in \mathcal{G}$ by **F1** and **F3**, we have $\forall j \in G, \quad A \in \mathcal{F}_j$ and $\forall k \in H, \quad B \in \mathcal{F}_k$. Thus $\forall i \in G \cap H, \quad A, B \in \mathcal{F}_i$. By **F3**, $\forall i \in G \cap H, \quad A \cap B \in \mathcal{F}_i$. So we have $A \cap B \in \bigcap_{i \in G \cap H} \mathcal{F}_i$ and $A \cap B \in \kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J})$. ∎

2.2 Topological Spaces

Topological spaces and the category **TOP** have already been defined in Chapter 1 (Example 1.2.3). We restate the definition formally.

Definition 2.2.1 [5, 39]: Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \tau \subseteq \mathcal{P}(X)$. Then (X, τ) is a topological space $\Leftrightarrow$

O1 $\emptyset, X \in \tau$

O2 $A, B \in \tau \Rightarrow A \cap B \in \tau$

03 $\mathcal{A} \subseteq \tau \Rightarrow \cup\mathcal{A} \in \tau$

τ is referred to as a *topology* on X.

Definition 2.2.2 [1]: We define the category **TOP** by

Objects Topological spaces (X, τ)

Morphisms Functions $\phi\colon X \to Y$ between topological spaces (X, τ_X) and (Y, τ_Y) which satisfy $\forall V \in \tau_Y, \quad \phi^{\leftarrow}(V) \in \tau_X$. These are called continuous functions.

Identity The identity morphism for a topological space (X, τ) is the usual $\mathrm{id}_X\colon (X, \tau) \to (X, \tau)$.

Composition Morphism composition is the usual function composition.

Example 2.2.3 : In the **TOP**-fibre of X, i.e. the class of all topologies on the fixed set X, topologies are ordered by $\tau_1 \leq \tau_2 \Leftrightarrow \tau_2 \subseteq \tau_1$, when the order is defined as in definition 1.3.5. In this fibre, the smallest element is $(X, \mathcal{P}(X))$ and the largest element is (X, τ_0) where $\tau_0 = \{\emptyset, X\}$.

Remark 2.2.4 : There is the obvious forgetful functor $U\colon$ **TOP** $\to$ **SET**. Thus (**TOP**$, U$) can be regarded as a concrete category over **SET**, i.e. a construct. Through abuse of notation we also refer to this category as **TOP**.

Lemma 2.2.5 [1]: The category **TOP** is fibre-small, amnestic and has the terminal separator property.

Remark 2.2.6 : The fact that **TOP** is amnestic, fibre small and has the terminal separator property was proved in Examples 1.3.6 and 1.4.6.

Theorem 2.2.7 (see e.g. [1]): The category **TOP** is topological over **SET**.

Theorem 2.2.8 [1, 2, 11]: The category **TOP** is not cartesian closed.

2.3 Characterizations of Topological Spaces

It is well known (see e.g. [5, 39]) that classical topological spaces may be viewed in several different but equivalent ways. The formulation is usually chosen depending on the problem the topological machinery is being applied to. In this section we explore the various standard axiom schemes which describe topological spaces. It will be shown in Chapter 4 that these axiom schemes can be translated into the more general setting of L-sets, where they similarly suffice to characterize L-topological spaces.

Definition 2.3.1 [5, 39]: Let (X, τ) be a topological space. We define the interior operator int by

I0 $$\text{int}\colon \mathcal{P}(X) \to \tau \qquad A \mapsto \text{int}A = \bigcup \{ V \in \tau \mid V \subseteq A \}.$$

In cases where no ambiguity can arise we will use the shortened notation A° for $\text{int}A$.

Lemma 2.3.2 [5, 39]: Let (X, τ) be a topological space. Then the interior operator int has the following properties for all $A, B \subseteq X$

I1 $A \in \tau \Leftrightarrow A = A^\circ$.

I2 $A^\circ \subseteq A$.

I3 $X^\circ = X$.

I4 $A^\circ \cap B^\circ = (A \cap B)^\circ$.

I5 $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$.

I6 $(A^\circ)^\circ = A^\circ$.

Remark 2.3.3 : Note that property **I5** follows from **I4**. We state it here separately for comparison with the general ecl-premonoid case in chapter 4.

Definition 2.3.4 : Let X be a non-empty set and let $\text{int}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ be an operator satisfying the properties **I2** to **I6** of Lemma 2.3.2. Then (X, int) is an *interior space*.

Lemma 2.3.5 [5, 39]: The properties of Lemma 2.3.2 *characterize* topological spaces, i.e. if (X, int) is an interior space then it can be mapped uniquely to a topological space (X, τ) via **I1**, and the topological space so defined can be mapped back to the *same* interior space (X, int) via **I0**. In this way the topological axioms **O1**–**O3** describe essentially the same object as the interior axioms **I2**–**I6**.

Remark 2.3.6 : Lemmas 2.3.2 and 2.3.5 are proved in the general ecl-premonoid case as Lemmas 4.2.2 and 4.2.4 respectively.

Definition 2.3.7 [5, 39]: Let (X, int) be an interior space. We define the neighbourhood filter $\mathcal{U}^x$ at $x \in X$ by

N0 $$\mathcal{U}^x = \{ A \subseteq X \mid x \in A^\circ \}.$$

Lemma 2.3.8 [5, 39]: The neighbourhood filter at $x \in X$ has the following properties:

N1 $\forall A \subseteq X, \quad A \in \mathcal{U}^x \Leftrightarrow x \in A^\circ.$

N2 $\forall A \in \mathcal{U}^x \quad x \in A.$

N3 $X \in \mathcal{U}^x.$

N4 $\forall A, B \subseteq X, \quad A, B \in \mathcal{U}^x \Rightarrow A \cap B \in \mathcal{U}^x.$

N5 $\forall A, B \subseteq X, \quad A \subseteq B$ and $A \in \mathcal{U}^x \Rightarrow B \in \mathcal{U}^x.$

N6 $\forall A \subseteq X, \quad A \in \mathcal{U}^x \Rightarrow \ \exists B \in \mathcal{U}^x \ \forall y \in B, \quad A \in \mathcal{U}^y.$

Remark 2.3.9 : By the axioms **N2**, **N3**, **N4** and **N5**, $\mathcal{U}^x \in \mathcal{F}(X)$.

Definition 2.3.10 [5, 39]: Let X be a non-empty set and let $(\mathcal{U}^x)_{x \in X}$ be a collection of filters on X indexed by X satisfying axioms **N2**–**N6** of Lemma 2.3.8. Then $(X, (\mathcal{U}^x)_{x \in X})$ is a *neighbourhood space*.

Lemma 2.3.11 [5, 39]: The properties of Lemma 2.3.8 *characterize* interior spaces, i.e. if $(X, (\mathcal{U}^x)_{x \in X})$ is a neighbourhood space then it can be mapped uniquely to an interior space (X, int) via **N1**, and the interior space so defined can be mapped back to the *same* neighbourhood space $(X, (\mathcal{U}^x)_{x \in X})$ via **N1** again. In this way the interior axioms **I2**–**I6** describe essentially the same object as the neighbourhood axioms **N2**–**N6**.

Remark 2.3.12 : Lemmas 2.3.8 and 2.3.11 are proved in the general ecl-premonoid case as Lemmas 4.2.6 and 4.2.9 respectively.

A convergence structure on a set X is defined by most authors (see e.g. [9]) as a function $\tau X \to \mathcal{P}(\mathcal{F}(X))$. From this point of view, for $x \in A$, $\tau(x)$ is interpreted as the set of all filters on X which converge to x. We can equivalently specify a function $\lim \colon \mathcal{F}(X) \to \mathcal{P}(X)$ which specifies the set of points $\lim \mathcal{F}$ to which each filter $\mathcal{F}$ converges. This is the viewpoint we shall adopt as standard, since it is closely related to the concepts in later chapters.

Definition 2.3.13 [9]: Let $(X, (\mathcal{U}^x)_{x \in X})$ be a neighbourhood space. We define the limit function

Lp $\quad \lim\colon \mathcal{F}(X) \to \mathcal{P}(X), \qquad \lim \mathcal{F} = \{\, x \in X \mid \mathcal{U}^x \subseteq \mathcal{F} \,\}.$

Lemma 2.3.14 (see [21] for the frame case): The lim function satisfies the following properties:

L0 $\forall\, x \in X, \quad \mathcal{U}^x = \bigcap \{\, \mathcal{F} \in \mathcal{F}(X) \mid x \in \lim \mathcal{F} \,\}.$

L1 $\forall\, x \in X, \quad x \in \lim[x].$

Lp $\forall\, \mathcal{F} \in \mathcal{F}(X), \quad \lim \mathcal{F} = \{\, x \in X \mid \mathcal{U}^x \subseteq \mathcal{F} \,\}.$

Lt $\forall\, x \in X, \quad A \in \mathcal{U}^x \Rightarrow\ \exists\, B \in \mathcal{U}^x\ \forall\, y \in B, \quad A \in \mathcal{U}^y.$

Remark 2.3.15 : The axiom scheme given above is a mixture of the standard axioms for neighbourhood spaces and the standard axioms for convergence relations (see e.g. [7, 26]) translated into axioms in terms of the lim function. The reason this approach has been adopted is to make more clear the comparison between the classical case and the more general ecl-premonoid case.

Definition 2.3.16 (see [21] for the frame case): Let X be a non-empty set and let $\lim\colon \mathcal{F}(X) \to \mathcal{P}(X)$ be a function satisfying axioms **L1**, **Lp** and **Lt** of Lemma 2.3.14, with $\mathcal{U}^x$ defined by **L0**. Then $(X, \lim)$ is a *topological convergence space.*

Lemma 2.3.17 (see [21] for the frame case): The properties of Lemma 2.3.14 *characterize* neighbourhood spaces, i.e. if $(X, \lim)$ is a topological convergence space then it can be mapped uniquely to a neighbourhood space $(X, (\mathcal{U}^x)_{x \in X})$ via **L0**, and the neighbourhood space so obtained can be mapped back to the *same* topological convergence space $(X, \lim)$ via **Lp**. In this way the neighbourhood axioms **N2**–**N6** describe essentially the same object as the topological convergence axioms.

Remark 2.3.18 : Lemmas 2.3.14 and 2.3.17 are proved in the general ecl-premonoid case as Lemmas 4.2.11 and 4.2.14 respectively.

Remark 2.3.19 : From Lemmas 2.3.5, 2.3.11 and 2.3.17, we have that topological spaces have at least four equivalent characterizations, hence in the future when a topological space is defined it will be possible to use either it's 'canonical' axioms, or the interior operator, or the neighbourhood filter, or the lim function in proofs.

Lemma 2.3.20 (see [21] for the frame case): Let $(X,\tau_X),(Y,\tau_Y)$ be topological spaces. Let $\phi\colon X\to Y$ be a function. Then

$$\begin{aligned}\phi \text{ is a continuous function} &\Leftrightarrow\ \forall\, V\in\tau_Y,\quad \phi^{\leftarrow}(V)\in\tau_X\\ &\Leftrightarrow\ \forall\, x\in X\ \forall\, B\subseteq Y,\quad B\in\mathcal{U}_Y^{\phi(x)}\Rightarrow\phi^{\leftarrow}(B)\in\mathcal{U}_X^x\\ &\Leftrightarrow\ \forall\,\mathcal{F}\in\mathcal{F}(X),\quad \phi(\lim_X\mathcal{F})\subseteq\lim_Y\phi(\mathcal{F}).\end{aligned}$$

Remark 2.3.21 : Lemma 2.3.20 is proved in the general ecl-premonoid case as Lemma 4.3.2.

Definition 2.3.22 (see [21] for the frame case): We define the category **TCS** of *topological convergence spaces* by

Objects Spaces $(X,\lim)$ which satisfy axioms **L1**, **Lp** and **Lt**, with $\mathcal{U}^x$ defined by **L0**.

Morphisms Functions $\phi\colon(X,\lim_X)\to(Y,\lim_Y)$ which satisfy

$$\forall\,\mathcal{F}\in\mathcal{F}(X),\quad \phi(\lim_X\mathcal{F})\subseteq\lim_Y\phi(\mathcal{F}).$$

Identity The identity morphism for a topological convergence space $(X,\lim)$ is the usual $\mathrm{id}_X\colon(X,\lim)\to(X,\lim)$.

Composition Morphism composition is the usual function composition.

Theorem 2.3.23 : The category **TOP** is isomorphic to the category **TCS**.

Proof:
We define functors

$$F\colon \underline{\mathbf{TOP}}\to\underline{\mathbf{TCS}}$$
$$(X,\tau_X)\xrightarrow{\phi}(Y,\tau_Y)\mapsto(X,\lim_{\tau_X})\xrightarrow{\phi}(Y,\lim_{\tau_Y}).$$

$$G\colon \underline{\mathbf{TCS}}\to\underline{\mathbf{TOP}}$$
$$(X,\lim_X)\xrightarrow{\phi}(Y,\lim_Y)\mapsto(X,\tau_{\lim_X})\xrightarrow{\phi}(Y,\tau_{\lim_Y}).$$

Then by Lemmas 2.3.5, 2.3.11, 2.3.17 and 2.3.20 we have

$$F \circ G = \mathrm{id}_{\underline{\mathbf{TOP}}} \quad \text{and} \quad G \circ F = \mathrm{id}_{\underline{\mathbf{TCS}}}.$$

So **<u>TOP</u>** is isomorphic to **<u>TCS</u>**. ■

2.4 Alternatives to the Lp and Lt axioms

It is desirable, in Lemma 2.3.14, that for topological convergence spaces we should express the **Lp** and **Lt** axioms entirely in terms of some simple (or at least, simple-looking) axiom on the function lim, rather than as it is in the Lemma, where we have to first define the neighbourhood filter $\mathcal{U}^x$ and only then check if the filter satisfies the conditions. In fact, this is possible and we can replace the **Lp** axiom with an equivalent condition on lim. We have two different axioms which achieve the desired result for the **Lt** axiom, both equivalent to each other, at least in the presence of the axioms **L1** and **Lp**. The first axiom **K** is due to Kowalski [28], the second, **F**, is due to Fischer [7].

Lemma 2.4.1 : Let X be a non-empty set and let $\lim\colon \mathcal{F}(X) \to \mathcal{P}(X)$ be a function satisfying the axiom **L1** of Lemma 2.3.14. Then axiom **Lp** of Lemma 2.3.14 (with $\mathcal{U}^x$ defined by **L0**) is equivalent to:

L2 $\qquad \forall\, \mathcal{F}, \mathcal{G} \in \mathcal{F}(X), \quad \mathcal{F} \subseteq \mathcal{G} \Rightarrow \lim \mathcal{F} \subseteq \lim \mathcal{G}$

and

LpW2

$$\forall\, \emptyset \neq I \in \mathrm{Ob}\,(\underline{\mathbf{SET}}),\ \forall\, (\mathcal{F}_i)_{i\in I} \in \mathcal{F}(X)^I, \quad \lim(\bigcap_{i\in I} \mathcal{F}_i) = \bigcap_{i\in I} \lim \mathcal{F}_i.$$

i.e. **Lp** $\Leftrightarrow$ **L2** and **LpW2**, provided that **L1** is satisfied.

Proof:
Assume that axiom **Lp** is satisfied, i.e.

Lp $\qquad \forall\, \mathcal{F} \in \mathcal{F}(X), \quad \lim \mathcal{F} = \{\, x \in X \mid \mathcal{U}^x \subseteq \mathcal{F} \,\}.$

Define $\mathcal{U}^x$ by

L0 $\qquad \mathcal{U}^x = \bigcap \{\, \mathcal{F} \in \mathcal{F}(X) \mid x \in \lim \mathcal{F} \,\}.$

We need **L1** so that this intersection is non-empty. We prove that **Lp** $\Rightarrow$ **L2**. Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}(X), \mathcal{F} \subseteq \mathcal{G}$. Then

$$x \in \lim \mathcal{F} \Leftrightarrow \mathcal{U}^x \subseteq \mathcal{F} \Rightarrow \mathcal{U}^x \subseteq \mathcal{G} \Leftrightarrow x \in \lim \mathcal{G}.$$

Next we prove that **Lp** $\Rightarrow$ **LpW2**. Let I be a non-empty set indexing a family of filters $(\mathcal{F}_i)_{i\in I}$. Then

$$\begin{aligned} x \in \bigcap_{i\in I} \lim \mathcal{F}_i &\Leftrightarrow \ \forall\, i \in I, \quad x \in \lim \mathcal{F}_i \\ &\Leftrightarrow \ \forall\, i \in I, \quad \mathcal{U}^x \subseteq \mathcal{F}_i \\ &\Leftrightarrow \mathcal{U}^x \subseteq \bigcap_{i\in I} \mathcal{F}_i \Leftrightarrow x \in \lim \bigcap_{i\in I} \mathcal{F}_i. \end{aligned}$$

Now assume that axioms **L2** and **LpW2** are satisfied. We prove that (**L2** and **LpW2**) $\Rightarrow$ **Lp**. Let $x \in X$. We define $\mathcal{U}^x$ via **L0**. Then it is clear that $x \in \lim \mathcal{F} \Rightarrow \mathcal{U}^x \subseteq \mathcal{F}$, for $\mathcal{F} \in \mathcal{F}(X)$. For the converse, assume that $\mathcal{U}^x \subseteq \mathcal{F}$. Then by **L2**, $\lim \mathcal{U}^x \subseteq \lim \mathcal{F}$. Now by **LpW2**

$$\lim \mathcal{U}^x = \lim\Big(\bigcap_{x\in\lim\mathcal{G}} \mathcal{G}\Big) = \bigcap_{x\in\lim\mathcal{G}} (\lim \mathcal{G}).$$

So $x \in \lim \mathcal{U}^x$. Thus $x \in \lim \mathcal{F}$. We have proved that $x \in \lim \mathcal{F} \Leftrightarrow \mathcal{U}^x \subseteq \mathcal{F}$, which is the axiom **Lp**. ■

Definition 2.4.2 : Let X be a non-empty set and let $\lim\colon \mathcal{F}(X) \to \mathcal{P}(X)$ be a function satisfying the axioms **L1** and **Lp** of Lemma 2.3.14. Then $(X, \lim)$ is a *principal convergence space*.

Remark 2.4.3 : Kent and Richardson ([26]) use a weaker axiom scheme where a *convergence space* (their terminology) $(X, \lim)$ is defined as one which satisfies the axioms **L1**, **L2** and **Kent**:

Kent $\quad \forall\, \mathcal{F} \in \mathcal{F}(X)\ \forall\, x \in X, \quad x \in \lim \mathcal{F} \Rightarrow x \in \lim(\mathcal{F} \cap [x])$

Under these conditions they are able to prove that the axiom **F** is equivalent to $(X, \tau_{\lim})$ being a topological space. Previously Kowalsky ([28]) had proved that a *pretopological space* (again, Kent and Richardson's terminology) $(X, \lim)$ (i.e. one which satisfies **L1**, **L2** and $\forall\, x \in X, \quad x \in \lim \mathcal{U}^x$) is equivalent to a topological space $(X, \tau_{\lim})$ iff it satisfies the axiom **K**. Thus all the results in this section are implied by the results previously obtained by Kent and Richardson and Kowalsky. The reason that they are stated here in this formulation is for comparison with results in Chapters 4 and 5, where we follow Jäger's development of convergence theory for the frame case ([21, 23, 24, 22]).

Lemma 2.4.4 : In a principal convergence space $(X, \lim)$, $\forall\, x \in X, \quad \mathcal{U}^x \in \mathcal{F}(X)$. Furthermore $\forall\, x \in X, \quad x \in \lim \mathcal{U}^x$.

Proof:
Let $x \in X$. Define $\mathcal{U}^x$ by **L0**. Then by **L1**, the intersection exists, hence by Lemma 2.1.4, $\mathcal{U}^x \in \mathcal{F}(X)$. Now by **Lp**, $\lim \mathcal{U}^x = \{ z \in X \mid \mathcal{U}^z \subseteq \mathcal{U}^x \}$. But $\mathcal{U}^x \subseteq \mathcal{U}^x$, so $x \in \lim \mathcal{U}^x$. ∎

Lemma 2.4.5 : Let $(X, \lim)$ be a principal convergence space. Then the axiom **Lt** of Lemma 2.3.14 is equivalent to:

K $\quad \forall \mathcal{G} \in \mathcal{F}(X)\ \forall (\mathcal{F}_y)_{y \in X} \in \mathcal{F}(X)^X\ \forall x \in X,$

$$x \in \lim \mathcal{G} \text{ and } \forall y \in X, \quad y \in \lim \mathcal{F}_y \Rightarrow x \in \lim \kappa(\mathcal{G}, (\mathcal{F}_y)_{y \in X})$$

where $\kappa(\mathcal{G}, (\mathcal{F}_y)_{y \in X}) = \bigcup_{G \in \mathcal{G}} \bigcap_{z \in G} \mathcal{F}_z$ is the compression operator defined in Lemma 2.1.10.

Proof:
We prove that $\mathbf{K} \Rightarrow \mathbf{Lt}$. Assume the axiom **K**. Let $x \in X$. Now by Lemma 2.4.4, $x \in \lim \mathcal{U}^x$ and $\forall y \in X, \quad y \in \lim \mathcal{U}^y$. Taking $\mathcal{G} = \mathcal{U}^x$ and $\mathcal{F}_y = \mathcal{U}^y$ in the statement of **K**, we deduce that $x \in \lim \kappa(\mathcal{U}^x, (\mathcal{U}^y)_{y \in X})$. Therefore by **Lp**, $\mathcal{U}^x \subseteq \kappa(\mathcal{U}^x, (\mathcal{U}^y)_{y \in X})$. Let $U \in \mathcal{U}^x$. Then by the definition of $\kappa(\mathcal{G}, (\mathcal{F}_y)_{y \in X})$,
$\exists V \in \mathcal{U}^x\ \forall y \in V, \quad U \in \mathcal{U}^y$.

Now we prove that $\mathbf{Lt} \Rightarrow \mathbf{K}$. Let $\mathcal{G} \in \mathcal{F}(X)$, $\forall y \in X, \quad \mathcal{F}_y \in \mathcal{F}(X)$. Let $x \in X, x \in \lim \mathcal{G}$ and $\forall y \in X, \quad y \in \lim \mathcal{F}_y$. Let $U \in \mathcal{U}^x$. Then by **Lt**, $\exists V \in \mathcal{U}^x\ \forall y \in V, \quad U \in \mathcal{U}^y$. Now $\mathcal{U}^x \subseteq \mathcal{G}$ and $\forall y \in V, \quad \mathcal{U}^y \subseteq \mathcal{F}_y$ by **Lp**, so $U \in \bigcap_{y \in V} \mathcal{U}^y$ and thus $U \in \bigcup_{V \in \mathcal{G}} \bigcap_{y \in V} \mathcal{U}^y \subseteq \kappa(\mathcal{G}, (\mathcal{F}_y)_{y \in X})$. ∎

Lemma 2.4.6 : Let $(X, \lim)$ be a principal convergence space. Then the axiom **K** of Lemma 2.4.5 is equivalent to:

F $\quad \forall \emptyset \neq J \in \mathrm{Ob}\,(\underline{\mathbf{SET}})\ \forall \phi \colon J \to X,\ \forall \mathcal{G} \in \mathcal{F}(J)$

$$\forall (\mathcal{F}_j)_{j \in J} \in \mathcal{F}(X)^J\ \forall x \in X,$$

$$x \in \lim \phi(\mathcal{G}) \text{ and } \forall j \in J, \quad \phi(j) \in \lim \mathcal{F}_j \Rightarrow x \in \lim \kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J})$$

where $\kappa(\mathcal{G}, (\mathcal{F}_j)_{j \in J}) = \bigcup_{G \in \mathcal{G}} \bigcap_{z \in G} \mathcal{F}_z$ is the compression operator defined in Lemma 2.1.10.

Proof:
It is obvious, by taking $J = X$ and $\phi = \mathrm{id}_X$, that **F** implies **K**. We prove the converse. Let $\emptyset \neq J \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$, $\mathcal{G} \in \mathcal{F}(J)$, $(\mathcal{F}_j)_{j\in J} \in \mathcal{F}(X)^J$ and $x \in X$ such that $x \in \lim \phi(\mathcal{G})$ and $\forall\, j \in J, \quad \phi(j) \in \lim \mathcal{F}_j$. Now by **K** and **Lp**, and by Lemma 2.4.4, we have that $\mathcal{U}^x \subseteq \kappa(\mathcal{U}^x, (\mathcal{U}^y)_{y\in X})$. Let $A \in \kappa(\mathcal{U}^x, (\mathcal{U}^y)_{y\in X})$. We define the set A° by $z \in A^\circ \Leftrightarrow A \in \mathcal{U}^z$. Now, $A \in \kappa(\mathcal{U}^x, (\mathcal{U}^y)_{y\in X})$ implies that $A \in \bigcup_{B\in\phi(\mathcal{G})} \bigcap_{y\in B} \mathcal{U}^y$, since $\mathcal{U}^x \subseteq \phi(\mathcal{G})$ by **Lp**. Thus there exists $B \in \phi(\mathcal{G})$ such that $\forall\, y \in B, \quad A \in \mathcal{U}^y$. By our definition of A°, this implies that $\exists\, B \in \phi(\mathcal{G})$ such that $B \subseteq A^\circ$. $\phi(\mathcal{G})$ is a filter, thus $A^\circ \in \phi(\mathcal{G})$, so by the definition of $\phi(\mathcal{G})$ (2.1.5), $\phi^\leftarrow(A^\circ) \in \mathcal{G}$. Now $j \in \phi^\leftarrow(A^\circ) \Rightarrow \phi(j) \in A^\circ \Rightarrow A \in \mathcal{U}^{\phi(j)} \subseteq \mathcal{F}_j$, thus $A \in \bigcap_{j\in\phi^\leftarrow(A^\circ)} \mathcal{F}_j$. Thus $A \in \bigcup_{G\in\mathcal{G}} \bigcap_{j\in G} \mathcal{F}_j = \kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J})$. We have proved that

$$\mathcal{U}^x \subseteq \kappa(\mathcal{U}^x, (\mathcal{U}^y)_{y\in X}) \subseteq \kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J}).$$

By Lemma 2.4.4, $x \in \lim \mathcal{U}^x$, thus by **L2**, which is implied by **Lp**, we have that $x \in \lim \kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J})$. ■

2.5 Convergence Spaces

We now define a new category, the category of *convergence spaces*, $\underline{\mathbf{CONV}}$. A major reason (see e.g. [35]) for the definition of $\underline{\mathbf{CONV}}$ was that the category $\underline{\mathbf{TOP}}$ is not cartesian closed [2, 11, 31]. However $\underline{\mathbf{CONV}}$ *is* cartesian closed [33, 35] and contains $\underline{\mathbf{TCS}}$ (isomorphic to $\underline{\mathbf{TOP}}$) as a reflective subcategory [33].

Definition 2.5.1 [9, 28]: Let X be a non-empty set and let $\lim\colon \mathcal{F}(X) \to \mathcal{P}(X)$ be a function satisfying the axioms **L1** (see Lemma 2.3.14) and **L2** (see Lemma 2.4.1). Then $(X, \lim)$ is a *convergence space*.

Definition 2.5.2 (See e.g. [33, 35]): We define the category $\underline{\mathbf{CONV}}$ of *convergence spaces* by

Objects Convergence spaces $(X, \lim)$.

Morphisms Functions $\phi\colon (X, \lim_X) \to (Y, \lim_Y)$ which satisfy

$$\forall\, \mathcal{F} \in \mathcal{F}(X), \quad \phi(\lim_X \mathcal{F}) \subseteq \lim_Y \phi(\mathcal{F}).$$

Identity The identity morphism for a convergence space $(X, \lim)$ is the usual $\mathrm{id}_X\colon (X, \lim) \to (X, \lim)$.

Composition Morphism composition is the usual function composition.

Theorem 2.5.3 [9]: The category **CONV** is topological over **SET**. Furthermore it is amnestic, fibre small and has the terminal separator property.

Example 2.5.4 : Let $\emptyset \neq X \in \mathrm{Ob}\,(\mathbf{SET})$ and let $(X_i, \lim_i)_{i\in I}$ be a family of convergence spaces indexed by the set I. Let ϕ_i be a function from X to X_i for each $i \in I$. Define

$$\lim\colon \mathcal{F}(X) \to \mathcal{P}(X) \quad \mathcal{F} \mapsto \bigcap_{i\in I} \phi_i^{\leftarrow}(\lim_i \phi(\mathcal{F}))$$

Then $(X, \lim)$ is an initial structure in **TCS**.

We now show the basic structures involved in the proof that **CONV** is cartesian closed, i.e. we show how finite products and exponentials are formed in **CONV**.

Lemma 2.5.5 [33]: Let $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\mathbf{CONV})$. Define

$$\lim_X \times \lim_Y \colon \mathcal{F}(X\times Y) \to \mathcal{P}(X \times Y)$$
$$\mathcal{F} \mapsto \pi_X^{\leftarrow}(\lim_X \pi_X(\mathcal{F})) \wedge \pi_Y^{\leftarrow}(\lim_Y \pi_Y(\mathcal{F}))$$

Then $((X\times Y, \lim_X \times \lim_Y), \pi_X, \pi_Y)$ are a product diagram for the objects $(X, \lim_X)$ and $(Y, \lim_Y)$ in **CONV**. For the one element set $\{x\}$, we define $\lim\colon \mathcal{F}(\{x\}) \to \mathcal{P}(\{x\}) \quad \mathcal{F} \mapsto \{x\}$. Then the object $(\{x\}, \lim)$ is a terminal object in **CONV**. Thus **CONV** has all finite products.

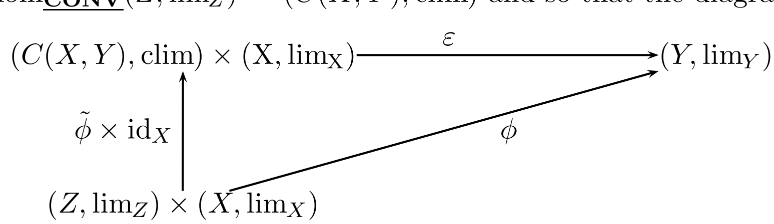
Lemma 2.5.6 [6, 33]: Let $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\mathbf{CONV})$. We define $C(X, Y) = \hom_{\mathbf{CONV}}((X, \lim_X), (Y, \lim_Y))$. We also define

$$\varepsilon\colon C(X,Y) \times X \to Y \quad (g, x) \mapsto g(x)$$

Now define $\mathrm{clim}\colon \mathcal{F}(C(X, Y)) \to \mathcal{P}(C(X, Y))$ by

$$f \in \mathrm{clim}\,\mathcal{F} \Leftrightarrow \ \forall x \in X \ \forall \mathcal{G} \in \mathcal{F}(X)$$
$$x \in \lim_X \mathcal{G} \Rightarrow f(x) \in \lim_Y \varepsilon(\mathcal{F}\times\mathcal{G})$$

Then $(C(X,Y), \mathrm{clim}) \in \mathrm{Ob}\,(\mathbf{CONV})$ and
$\varepsilon \in \hom_{\mathbf{CONV}}((C(X,Y), \mathrm{clim}), (\mathrm{Y}, \lim_{\mathrm{Y}}))$. Further, $((C(X,Y), \mathrm{clim}), \varepsilon)$ is an exponential in **CONV** for the objects $(X, \lim_X)$ and $(Y, \lim_Y)$, in the sense that if ϕ is a morphism between $(Z \times X, \lim_Z \times \lim_X)$ and $(Y, \lim_Y)$ then we can define a unique $\tilde{\phi}$ given by $\tilde{\phi}(z) = \phi(z, -)$ such that $\tilde{\phi} \in \hom_{\mathbf{CONV}}(Z, \lim_Z) \to (C(X,Y), \mathrm{clim})$ and so that the diagram



commutes.

Theorem 2.5.7 [33, 35]: The category **CONV** is cartesian closed.

Theorem 2.5.8 [33]: The category **TCS** is a full, reflective subcategory of the category **CONV**.

Chapter 3

Lattices, L-sets and L-filters

$\mathcal{P}(X)$, the *power set* of a set X, is usually defined as the set of all subsets of X. We could also view it as the set of all *characteristic functions* over X, since there is a bijective relationship between $\mathcal{P}(X)$ and $\{0,1\}^X$ given by $A \mapsto \chi_A$ where $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in X \backslash A \end{cases}$ is the characteristic function of $A \subseteq X$.

If we have $A, B \in \mathcal{P}(X)$, then we can obtain $\chi_{A\cap B}$ and $\chi_{A\cup B}$ from

$$\chi_{A\cap B}(x) = \min\{\chi_A(x), \chi_B(x)\}$$
$$\chi_{A\cup B}(x) = \max\{\chi_A(x), \chi_B(x)\}$$

L-sets over a base set X are simply the extension of the idea of characteristic functions to a set L other than $\{0,1\}$. A function $a\colon X \to L \quad x \mapsto a(x)$ is called an L-set. The value $a(x)$ is interpreted as the *degree of membership of x in a.* By analogy with the relationships between $A \cap B, A \cup B$ and their characteristic functions, we seek to define an analogue of union and intersection for our L-sets. In order to do this we require that L satisfies some properties. Most importantly, L must be a *lattice.*

The first section of this chapter details the lattice structure of the set L which is used as the basis for L-sets throughout the rest of the text. The following section explores properties of L-sets using the structure previously defined, and the final section contains some definitions and results concerning L-filters, the generalization of classical filters to the L-set case.

3.1 Lattice theory

The properties of the set L determine the *logic* that we have available to us. The set $\{0,1\}$ with $0 \leq 1$ has a number of nice properties which mean that when we translate statements back and forth between characteristic functions and the subsets of X that they represent, the statements are made in terms of *boolean* logic. When we change $\{0, 1\}$ to L, we are also switching from boolean logic statements to *many-valued* logic.

We want to be able to define operations on L-sets which are generalizations of union and intersection. In order to do this we require that L is a *complete lattice,* so that we have sup ($\bigvee$) and inf ($\bigwedge$) operations (the analogues of boolean *or* and *and* respectively) available to us. We will need to have some form of *implication operator* on L (analogous to the boolean implication) available to us. This leads us to define a *frame*. It turns out that we can relax the axioms further and define an *ecl-premonoid*. This is the framework for L which is used throughout the rest of this text. A lattice is a partially ordered set which satisfys some additional axioms. We begin by restating the definition of a partially ordered set from Chapter 1 (Definition 1.1.3).

Definition 3.1.1 [4]: $(L, \leq)$ is a *partially ordered set* (a *poset*) $\Leftrightarrow$

L is a non-empty set and

$\leq$ is a relation on L (i.e. a subset of $L \times L$) which satisfies

PO1 $\forall\, \alpha \in L, \quad \alpha \leq \alpha.$ (reflexive)

PO2 $\forall\, \alpha, \beta \in L, \quad \alpha \leq \beta$ and $\beta \leq \gamma \Rightarrow \alpha \leq \gamma.$ (transitive)

PO3 $\forall\, \alpha, \beta, \gamma \in L, \quad \alpha \leq \beta$ and $\beta \leq \alpha \Rightarrow \alpha = \beta.$ (anti-symmetric)

If $(L, \leq)$ satisfies the additional condition

POL $\forall\, \alpha, \beta \in L, \quad \alpha \leq \beta$ or $\beta \leq \alpha.$ (linearity)

then $(L, \leq)$ is called a *linearly ordered set* or *chain* and $\leq$ is called a *linear order* on L.

Example 3.1.2 : Let X be a non-empty set. Then the subsethood relation on $\mathcal{P}(X)$ is reflexive, anti-symmetric and transitive. It is not linearly ordered. Thus $(\mathcal{P}(X), \subseteq)$ is a poset which is not a chain. The real numbers $\mathbb{R}$ and the natural numbers $\mathbb{N}$ are both posets under the usual ordering $\leq$.

Definition 3.1.3 [4]: Let $(L, \leq)$ be a poset and let $\beta, \gamma \in L, A \subseteq L$. Then

1. β is a *lower bound* of $A \Leftrightarrow \ \forall\, \alpha \in A, \quad \beta \leq \alpha.$
2. γ is an *upper bound* of $A \Leftrightarrow \ \forall\, \alpha \in A, \quad \alpha \leq \gamma.$
3. β is the *greatest lower bound* of $A \Leftrightarrow \beta$ is a lower bound of A and $\forall\, \delta \in L, \quad \delta$ is a lower bound of $A \Rightarrow \delta \leq \beta.$
4. γ is the *least upper bound* of $A \Leftrightarrow \gamma$ is an upper bound of A and $\forall\, \delta \in L, \quad \delta$ is an upper bound of $A \Rightarrow \gamma \leq \delta.$

Remark 3.1.4 : For a subset A of L, the greatest lower bound and the least upper bound are unique if they exist [4]. The set of lower bounds for A is denoted, in this text, by $\mathrm{lb}(A)$, similarly the set of upper bounds for A is denoted $\mathrm{ub}(A)$. We denote the greatest lower bound of A by $\bigwedge A$ and the least upper bound of A by $\bigvee A$. We refer to $\bigwedge A$ as the *meet* of A and $\bigvee A$ as the *join* of A. For a two element subset $\{\alpha, \beta\} \subseteq L$ we write $\alpha \wedge \beta$ for $\bigwedge\{\alpha, \beta\}$ and $\alpha \vee \beta$ for $\bigvee\{\alpha, \beta\}$.

Definition 3.1.5 [4]: $(L, \leq)$ is a *lattice* $\Leftrightarrow$

1. $(L, \leq)$ is a poset.
2. $\forall \{\alpha, \beta\} \subseteq L \; \exists \lambda, \mu \in L, \quad \lambda = \alpha \wedge \beta, \quad \mu = \alpha \vee \beta$. i.e. every two element subset of L has a meet and a join.

$(L, \leq)$ is a *complete lattice* $\Leftrightarrow$

1. $(L, \leq)$ is a poset.
2. $\forall A \subseteq L \; \exists \lambda, \mu \in L, \quad \lambda = \bigwedge A, \quad \mu = \bigvee A$. i.e. every subset of L has a meet and a join.

In a complete lattice, $\bigwedge L$ is denoted $\perp$, the *smallest element*, and $\bigvee L$ by $\top$, the *largest element*. A complete lattice $(L, \leq)$ which satisfies the *frame law*

FL $$\forall \alpha \in L \; \forall B \subseteq L, \quad \alpha \wedge (\bigvee B) = \bigvee_{\beta \in B} (\alpha \wedge \beta)$$

is called a *frame* or a *complete Heyting algebra* [25].

Remark 3.1.6 : For a complete lattice $(L, \leq)$, we define $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \perp$. This is consistent with the given definition of $\bigwedge A$ and $\bigvee A$ for a given $A \subseteq L$, since $\mathrm{lb}(\emptyset) = L$ and $\mathrm{ub}(\emptyset) = L$.

Example 3.1.7 : Let X be a non-empty set, then we already know that $(\mathcal{P}(X), \subseteq)$ is a poset. Any collection $\mathcal{A}$ of subsets of X has a meet $(\cap\mathcal{A})$ and a join $(\cup\mathcal{A})$. Thus $(\mathcal{P}(X), \subseteq)$ is a complete lattice. In addition the subsethood relation satisfies the frame law, thus $(\mathcal{P}(X), \subseteq)$ is a frame. The smallest element is $\emptyset$ and the largest element is X.

The interval $[0, 1]$ together with the usual $\leq$ relation on $\mathbb{R}$ is a frame [36].

Definition 3.1.8 [20]: Let $(L, \leq)$ be a frame. We may define an *implication operator* $\rightarrow$ on L by

$$\rightarrow : L \times L \rightarrow L \quad \alpha \rightarrow \beta = \bigvee \{ \lambda \in L \mid \alpha \wedge \lambda \leq \beta \}.$$

Obviously we can define an implication operator for any complete lattice. However, the frame law gives the implication operator some of the desirable properties of the usual boolean implication. These will be explored further after we have defined *GL-monoids*, which are a generalization of frames.

Lemma 3.1.9 : The $\bigwedge$ and $\bigvee$ operations have the following properties:

1. Let $A \subseteq L, \gamma \in L$. Then
$$(\forall \alpha \in A, \quad \gamma \leq \alpha) \Leftrightarrow \gamma \leq \bigwedge A$$
$$\text{and } (\forall \alpha \in A, \quad \alpha \leq \gamma) \Leftrightarrow \bigvee A \leq \gamma.$$

2. Let $\alpha, \beta, \gamma \in L$. Then
$$\alpha \leq \beta \Rightarrow \alpha \wedge \gamma \leq \beta \wedge \gamma \quad \text{and} \quad \alpha \leq \beta \Rightarrow \alpha \vee \gamma \leq \beta \vee \gamma.$$

3. Let $A, B \subseteq L$. Then
$$A \subseteq B \Rightarrow \bigwedge B \leq \bigwedge A \quad \text{and} \quad A \subseteq B \Rightarrow \bigvee A \leq \bigvee B.$$

4. Let $A = \{ \alpha_{ij} \mid i \in I, j \in J \}$ be a subset of L indexed by sets I and J. Then
$$\bigvee A = \bigvee_{i \in I, j \in J} \alpha_{ij} = \bigvee_{j \in J} (\bigvee_{i \in I} \alpha_{ij}) = \bigvee_{i \in I} (\bigvee_{j \in J} \alpha_{ij})$$
$$\text{and } \bigwedge A = \bigwedge_{i \in I, j \in J} \alpha_{ij} = \bigwedge_{j \in J} (\bigwedge_{i \in I} \alpha_{ij}) = \bigwedge_{i \in I} (\bigwedge_{j \in J} \alpha_{ij}).$$

Proof:

1. Let $A \subseteq L, \gamma \in L$. Then $\forall \alpha \in A, \quad \gamma \leq \alpha$ implies that γ is a lower bound for A, thus by the definition of $\bigwedge A$ (Definition 3.1.3), $\gamma \leq \bigwedge A$. The converse follows from the transitivity of $\leq$. The proof for $\bigvee A$ is similar.

2. Let $\alpha, \beta, \gamma \in L, \alpha \leq \beta$. Then $\alpha \wedge \gamma \leq \alpha$, but $\alpha \leq \beta$, thus by transitivity $\alpha \wedge \gamma \leq \beta$. Also $\alpha \wedge \gamma \leq \gamma$. Thus $\alpha \wedge \gamma$ is a lower bound of $\{\beta, \gamma\}$ and so $\alpha \wedge \beta \leq \beta \wedge \gamma$.

3. Let $A, B \subseteq L, A \subseteq B$. Then $\forall \beta \in B, \quad \bigwedge B \leq \beta$. Since $A \subseteq B$, we have $\forall \alpha \in A, \quad \bigwedge B \leq \alpha$. Thus $\bigwedge B$ is a lower bound for A, therefore $\bigwedge B \leq \bigwedge A$. Also we have $\forall \beta \in B, \quad \bigvee B \geq \beta$. Therefore $\forall \alpha \in A, \quad \bigvee B \geq \alpha$. Thus $\bigvee B$ is an upper bound for A and $\bigvee A \leq \bigvee B$.

4. Let $A = \{ \alpha_{ij} \mid i \in I, j \in J \}$ be a subset of L indexed by sets I and J. Let $m \in I, n \in J$. Then $\alpha_{mn} \leq \bigvee_{i \in I, j \in J} \alpha_{ij}$. Thus $\bigvee_{n \in J} \alpha_{mn} \leq \bigvee_{i \in I, j \in J} \alpha_{ij}$. Finally $\bigvee_{m \in I}(\bigvee_{n \in J} \alpha_{mn}) \leq \bigvee_{i \in I, j \in J} \alpha_{ij}$. Now let $i \in I, j \in J$. Then $\alpha_{ij} \leq \bigvee_{m \in I} \alpha_{mj} \leq \bigvee_{m \in I}(\bigvee_{n \in J} \alpha_{mn})$. Therefore $\bigvee_{i \in I, j \in J} \alpha_{ij} \leq \bigvee_{m \in I}(\bigvee_{n \in J} \alpha_{mn})$. The proof for $\bigwedge$ is similar.

■

As mentioned before, we seek to relax the conditions on the set L in order that our topological theory may apply as widely as possible. One of the first definitions which relaxes the requirements of a frame is that of a *GL-monoid*. The 'GL' is an abbreviation of 'generalized logic'.

Definition 3.1.10 [18]: $(L, \leq, *)$ is a *GL-monoid* $\Leftrightarrow$

1. $(L, \leq)$ is a complete lattice,
2. $*\colon L \times L \to L$ is an operator satisfying:

   GL1 $\forall \alpha \in L, \quad \alpha * \top = \alpha$. (identity)

   GL2 $\forall \alpha, \beta \in L, \quad \alpha * \beta = \beta * \alpha$. (commutativity)

   GL3 $\forall \alpha, \beta, \gamma \in L, \quad \alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$. (associativity)

   GL4 $\forall \alpha, \beta \in L, \quad \alpha \leq \beta \Rightarrow \exists \delta \in L, \quad \alpha = \beta * \delta$. (divisibility)

   GL5 $\forall \beta \in L \; \forall A \subseteq L, \quad \beta * (\bigvee A) = \bigvee_{\alpha \in A}(\beta * \alpha)$. ($*$ distributes over $\bigvee$)

For a GL-monoid, we can define an implication operator analogous to the frame implication, given by

$$\to\colon L \times L \to L \quad \alpha \to \beta = \bigvee \{ \lambda \in L \mid \alpha * \lambda \leq \beta \}.$$

A GL-monoid is called a *complete MV-algebra* if the condition

MV $$\forall \alpha \in L, \quad (\alpha \to \bot) \to \bot = \alpha$$

holds [19].

Example 3.1.11 : Any frame $(L, \leq)$ is an example of a GL-monoid, with $* = \wedge$. By Lemma 3.1.16, any continuous T-norm is an example of a GL-Monoid operation on the set $[0, 1]$. The Lukasiewicz T-norm T_L (see Example 3.1.13 below) is an example of an MV-algebra operation on $[0, 1]$ where the $*$ operation is not $\wedge$ [20].

Definition 3.1.12 [37]: $T\colon [0, 1] \times [0, 1] \to [0, 1]$ is a *T-norm* $\Leftrightarrow$

T1 $\forall \alpha, \beta \in [0, 1], \quad T(\alpha, \beta) = T(\beta, \alpha)$. (commutativity)

T2 $\forall \alpha, \beta, \gamma \in [0, 1], \quad T(T(\alpha, \beta), \gamma) = T(\alpha, T(\beta, \gamma))$. (associativity)

T3 $\forall\, \alpha, \beta, \gamma \in [0, 1], \quad \beta \leq \gamma \Rightarrow T(\alpha, \beta) \leq T(\alpha, \gamma)$. (monotonicity)

T4 $\forall\, \alpha \in [0, 1], \quad T(\alpha, 1) = \alpha$. (identity)

A T-norm is *continuous* iff it is continuous with respect to the usual topologies on $[0, 1] \times [0, 1]$ and $[0, 1]$.

Example 3.1.13 : The most important examples of T-norms are the following:

1. $T_m(\alpha, \beta) = \alpha \wedge \beta$, the minimum T-norm.
2. $T_p(\alpha, \beta) = \alpha \cdot \beta$, the product T-norm.
3. $T_L(\alpha, \beta) = (\alpha + \beta - 1) \vee 0$, the Lukasiewicz T-norm.
4. $T_D(\alpha, \beta) = \begin{cases} \min \alpha, \beta & \alpha = 1 \text{ or } \beta = 1 \\ 0 & \text{otherwise} \end{cases}$, the drastic product.

Of these, T_m, T_p and T_L are continuous, while T_D is not [27].

We now prove that any continuous T-norm is a GL-monoid operation on $[0, 1]$. In order to do this we require two auxiliary lemmas before the main proof.

Lemma 3.1.14 [27]: Let T be a T-norm. Then

$$\forall\, \alpha \in [0, 1], \quad T(0, \alpha) = T(\alpha, 0) = 0.$$

Proof:
By axiom **T4**, we have that $T(0, 1) = 0$. Let $\alpha \in [0, 1]$. Then $\alpha \leq 1$. Hence $T(0, \alpha) \leq T(0, 1) = 0$ by **T3**. Thus by **T1**, $T(0, \alpha) = T(\alpha, 0) = 0$. ■

Lemma 3.1.15 [36]: Let $\emptyset \neq A \subseteq \mathbb{R}$. Then if $\bigvee A$ exists, $M = \bigvee A \Leftrightarrow$

1. M is an upper bound of A.
2. $\forall\, \varepsilon \in \mathbb{R}\; \varepsilon > 0\; \exists\, \alpha \in A, \quad \alpha \in (M - \varepsilon, M] \subseteq \mathbb{R}$.

Proof:
Let $M = \bigvee A, \varepsilon > 0$. Then by definition, $M \in \text{ub}(A)$. Assume $\nexists\, \alpha \in A$ such that $\alpha \in (M - \varepsilon, M]$. Then $M - \varepsilon < M$ and $M - \varepsilon \in \text{ub}(A)$, which is a contradiction.

Conversely, assume that $M \in \mathrm{ub}(A)$ and $\forall \varepsilon > 0, \quad (M-\varepsilon, M] \ni \alpha \in A$. Assume that $M \neq \bigvee A$. Then $\exists N < M, \quad N = \bigvee A$ since $M \in \mathrm{ub}(A)$. Let $\varepsilon = M - N$. But then by assumption $\exists \alpha \in A$ such that $\alpha \in (M-\varepsilon, M] = (N, M]$, which is a contradiction. ■

Lemma 3.1.16 [27]: Any continuous T-norm T is GL-monoid operation on $[0,1]$ i.e. $([0,1], \leq, T)$ is a GL-monoid.

Proof:
The properties **GL1**,**GL2** and **GL3** follow from the conditions **T4**, **T1** and **T2** respectively. For **GL4** we note that for $\beta \in [0,1]$, the function $T(\cdot, \beta)$ defined by $T(\cdot,\beta)(\alpha) = T(\alpha,\beta)$ is continuous. Now $T(\cdot,\beta)(0) = 0$ and $T(\cdot,\beta)(1) = \beta$ by Lemma 3.1.14 and by axiom **T4**. Let $\alpha \in [0,1], \alpha \leq \beta$. Then by the intermediate value Theorem [36] $\exists \gamma \in [0,1], \quad \alpha = T(\cdot,\beta)(\gamma) = T(\gamma,\beta)$.
We first prove **GL5** for the empty set. In $[0,1]$, $\bigvee \emptyset = 0$ as for any complete lattice. Let $\beta \in [0,1]$. Then

$$T(\beta, \bigvee \emptyset) = T(\beta, 0) = 0 = \bigvee \{\, T(\beta,\alpha) \mid \alpha \in \emptyset \,\}$$

Now let $\emptyset \neq A \subseteq [0,1], \beta \in [0,1]$.
If $\forall \varepsilon > 0 \ \exists \alpha \in A, \quad \alpha \in (T(\beta, \bigvee A) - \varepsilon, T(\beta, \bigvee A)]$, then by Lemma 3.1.15, $T(\beta, \bigvee A) = \bigvee \{\, T(\beta,\alpha) \mid \alpha \in A \,\}$. Let $\sigma = \bigvee A, \varepsilon > 0$. T is continuous, so

$$\exists \delta > 0 \ \forall \gamma \in [0,1], \quad |\sigma - \gamma| < \delta \Rightarrow |T(\beta,\sigma) - T(\beta,\gamma)| < \varepsilon$$

Let $\gamma \in (\sigma - \frac{\delta}{2}, \sigma] \cap A$. This set is non-empty by Lemma 3.1.15. Then $\gamma \leq \sigma$, so by **T3**, $T(\beta,\gamma) \leq T(\beta,\sigma)$. Thus $T(\beta,\gamma) \in (T(\beta,\sigma) - \varepsilon, T(\beta,\sigma)]$. ■

Lemma 3.1.17 : Let $(L, \leq, *)$ be a GL-monoid. Then

1. $\forall \alpha, \beta, \gamma \in L, \quad \alpha \leq \beta \Rightarrow \alpha * \gamma \leq \beta * \gamma$.
2. $\forall \alpha, \beta \in L, \quad \alpha * \beta \leq \alpha \wedge \beta$.

Proof:
Let $\alpha, \beta \in L, \alpha \leq \beta$. Then $\alpha \vee \beta = \beta$. Thus $\gamma * \beta = \gamma * (\alpha \vee \beta) = (\gamma * \alpha) \vee (\gamma * \beta)$ by **GL5**. So $\gamma * \alpha \leq \gamma * \beta$.

Now let α, β be arbitrary members of L. $\beta \leq \top$ so $\alpha * \beta \leq \alpha * \top = \alpha$ by what we have just proved and by **GL1**. Also $\alpha \leq \top$ so $\alpha * \beta \leq \top * \beta = \beta$. Therefore $\alpha * \beta \in \mathrm{lb}(\{\alpha, \beta\})$ and $\alpha * \beta \leq \alpha \wedge \beta$. ■

Lemma 3.1.18 [18]: Let $(L, \leq, *)$ be a GL-monoid. Then the implication operator defined by

$$\rightarrow : L \times L \rightarrow L \quad \alpha \rightarrow \beta = \bigvee \{ \lambda \in L \mid \alpha * \lambda \leq \beta \}$$

has the following properties:

1. $\forall \alpha, \beta, \delta \in L, \quad \delta \leq \alpha \rightarrow \beta \Leftrightarrow \alpha * \delta \leq \beta.$
2. $\forall \alpha, \beta \in L, \quad \alpha * (\alpha \rightarrow \beta) \leq \beta.$
3. $\forall \alpha, \beta \in L, \quad \alpha \leq (\alpha \rightarrow \beta) \rightarrow \beta.$
4. $\forall \alpha, \beta, \gamma \in L, \quad \alpha * (\beta \rightarrow \gamma) \leq \beta \rightarrow (\alpha * \gamma).$
5. $\forall \alpha, \beta, \gamma \in L, \quad \alpha \leq \beta \Rightarrow \gamma \rightarrow \alpha \leq \gamma \rightarrow \beta.$
6. $\forall \alpha, \beta . \gamma \in L, \quad \alpha \leq \beta \Rightarrow \alpha \rightarrow \gamma \geq \beta \rightarrow \gamma.$
7. $\forall \alpha \in L, B \subseteq L, \quad \alpha \rightarrow (\bigwedge B) = \bigwedge_{\beta \in B} (\alpha \rightarrow \beta).$
8. $\forall \alpha \in L, B \subseteq L, \quad (\bigvee B) \rightarrow \alpha = \bigwedge_{\beta \in B} (\beta \rightarrow \alpha).$
9. $\forall \alpha \in L, \quad \alpha \rightarrow \top = \top, \quad \top \rightarrow \alpha = \alpha, \quad \bot \rightarrow \alpha = \top.$
10. $\forall \alpha, \beta \in L, \quad \alpha \leq \beta \Leftrightarrow \alpha \rightarrow \beta = \top.$

Proof:

1. Let $\alpha, \beta, \delta \in L, \delta \leq \alpha \rightarrow \beta$. Then by the definition, $\delta \leq \bigvee \{ \lambda \mid \alpha * \lambda \leq \beta \}$. Therefore $\alpha * \delta \leq \alpha * \bigvee \{ \lambda \mid \alpha * \lambda \leq \beta \}$ by Lemma 3.1.17. Thus $\alpha * \delta \leq \bigvee \{ \alpha * \lambda \mid \alpha * \lambda \leq \beta \} \leq \beta$ by **GL5**.

   Now let $\alpha * \delta \leq \beta$. Then $\delta \in \{ \lambda \mid \alpha * \lambda \leq \beta \}$. Thus $\delta \leq \bigvee \{ \lambda \mid \alpha * \lambda \leq \beta \} = \alpha \rightarrow \beta$.

2. Let $\alpha, \beta \in L$. We have $\alpha \rightarrow \beta \leq \alpha \rightarrow \beta$. Then by what we have just proved in 1 above, $\alpha * (\alpha \rightarrow \beta) \leq \beta$.

3. Let $\alpha, \beta \in L$. We have $\alpha * (\alpha \rightarrow \beta) \leq \beta$. Then by 1 above, $\alpha \leq (\alpha \rightarrow \beta) \rightarrow \beta$.

4. Let $\alpha, \beta, \gamma \in L$. Then $\beta * \beta \rightarrow \gamma \leq \gamma$. Thus $\alpha * \beta * (\beta \rightarrow \gamma) \leq \alpha * \gamma$. Then by 1 above, $\alpha * (\beta \rightarrow \gamma) \leq \beta \rightarrow (\alpha * \gamma)$.

5. Let $\alpha \leq \beta, \gamma \in L$. Then $\gamma * (\gamma \rightarrow \alpha) \leq \alpha$, so $\gamma * (\gamma \rightarrow \alpha) \leq \beta$. Thus by 1 above, $\gamma \rightarrow \alpha \leq \gamma \rightarrow \beta$.

6. Let $\alpha \leq \beta, \gamma \in L$. Then $\beta * (\beta \to \gamma) \leq \gamma$. Now $\alpha * (\beta \to \gamma) \leq \beta * (\beta \to \gamma)$ by Lemma 3.1.17, thus $\alpha * (\beta \to \gamma) \leq \gamma$. Finally $\beta \to \gamma \leq \alpha \to \gamma$.

7. Let $\alpha, \delta \in L, B \subseteq L$. Then

$$\begin{aligned}
\delta \leq \alpha \to (\bigwedge B) &\Leftrightarrow \delta * \alpha \leq \bigwedge B \\
&\Leftrightarrow \ \forall \beta \in B, \quad \delta * \alpha \leq \beta \\
&\Leftrightarrow \ \forall \beta \in B, \quad \delta \leq \alpha \to \beta \\
&\Leftrightarrow \delta \leq \bigwedge_{\beta \in B} (\alpha \to \beta).
\end{aligned}$$

8. Let $\alpha, \delta \in L, B \subseteq L$. Then

$$\begin{aligned}
\delta \leq (\bigvee B) \to \alpha &\Leftrightarrow \delta * (\bigvee B) \leq \alpha \\
&\Leftrightarrow \bigvee_{\beta \in B} (\delta * \beta) \leq \alpha \\
&\Leftrightarrow \ \forall \beta \in B \quad \delta * \beta \leq \alpha \\
&\Leftrightarrow \ \forall \beta \in B \quad \delta \leq \beta \to \alpha \\
&\Leftrightarrow \delta \leq \bigwedge_{\beta \in B} (\beta \to \alpha).
\end{aligned}$$

9. Let $\alpha \in L$. Then $\alpha \to \top = \bigvee\{\, \lambda \mid \alpha * \lambda \leq \top \,\}$. But $\alpha * \top \leq \top$, thus $\top \in \{\, \lambda \mid \alpha * \lambda \leq \top \,\}$. Finally $\alpha \to \top \geq \top$ and by anti-symmetry $\alpha \to \top = \top$.

   Next we have $\top \to \alpha = \bigvee\{\, \lambda \mid \top * \lambda \leq \alpha \,\} = \bigvee\{\, \lambda \mid \lambda \leq \alpha \,\} = \alpha$.

   Finally $\bot \to \alpha = \bigvee\{\, \lambda \mid \bot * \lambda = \bot \leq \alpha \,\} = \top$.

10. Let $\alpha, \beta \in L$. Assume $\alpha \leq \beta$. Then $\top * \alpha \leq \beta$ and by what we proved earlier, $\top \leq \alpha \to \beta$, i.e. $\alpha \to \beta = \top$. Now assume $\alpha \to \beta = \top$. Then $\top \leq \alpha \to \beta$ and again $\top * \alpha \leq \beta$, thus $\alpha \leq \beta$.

■

For any GL-monoid we can show that the underlying lattice is a frame. We generalize a proof in Höhle ([18]).

Lemma 3.1.19 [18]: Let $(L, \leq, *)$ be a GL-monoid. Then

$$\forall \alpha, \beta \in L, \quad \alpha * (\alpha \to \beta) = \alpha \wedge \beta.$$

Proof:
Let $\alpha, \beta \in L$. From Lemma 3.1.18, we have $\alpha * (\alpha \to \beta) \leq \beta$. We also have $\alpha \to \beta \leq \top = \alpha \to \alpha$. Thus by Lemma 3.1.18 again, $\alpha * (\alpha \to \beta) \leq \alpha$.

Now $\alpha \wedge \beta \leq \alpha$. By divisibility, $\exists\, \gamma \in L, \quad \alpha * \gamma = \alpha \wedge \beta$. Then $\gamma \leq \alpha \to (\alpha \wedge \beta) \leq \alpha \to \beta$. Thus $\alpha \wedge \beta = \alpha \wedge \gamma \leq \alpha * (\alpha \to \beta)$. ∎

Lemma 3.1.20 (see [18] for the case of B finite): Let $(L, \leq, *)$ be a GL-monoid. Then

$$\forall\, \alpha \in L\ \forall\, B \subseteq L, \quad \alpha \wedge \bigvee B = \bigvee_{\beta \in B} (\alpha \wedge \beta).$$

Proof:
Let $\alpha \in L, B \subseteq L$. Then by Lemmas 3.1.18 and 3.1.19, we have

$$\begin{aligned} \alpha \wedge \bigvee B = \bigvee B \wedge \alpha &= (\bigvee B) * ((\bigvee B) \to \alpha) \\ &= \bigvee_{\beta \in B} (\beta * ((\bigvee B) \to \alpha)) \leq \bigvee_{\beta \in B} (\beta * (\beta \to \alpha)) \\ &= \bigvee_{\beta \in B} (\beta \wedge \alpha). \end{aligned}$$

On the other hand, we have $\forall\, \beta \in B, \quad \alpha \wedge \bigvee B \geq \alpha \wedge \beta$. Thus $\alpha \wedge \bigvee B \geq \bigvee_{\beta \in B} (\alpha \wedge \beta)$, and the result follows. ∎

Definition 3.1.21 [20]: Let $(L, \leq, *)$ be a GL-monoid. Then $(L, \leq, *)$ has *square roots* iff there exists an operator $S \colon L \to L$ satisfying the conditions

Sq1 $\forall\, \alpha \in L, \quad \alpha = S(\alpha) * S(\alpha)$.

Sq2 $\forall\, \alpha, \beta \in L, \quad \beta * \beta \leq \alpha \Rightarrow \beta \leq S(\alpha)$.

If $(L, \leq, *)$ has square roots then we may define the *monoidal mean operator* $\circledast$ by

$$\circledast \colon L \times L \to L \qquad \alpha \circledast \beta = S(\alpha) * S(\beta).$$

An additional condition satisfied by several useful GL-monoids with square roots is

Sq3 $\qquad \forall\, \alpha, \beta \in L, \quad S(\alpha * \beta) = (S(\alpha) * S(\beta)) \vee S(\bot)$.

Example 3.1.22 : The T-norms T_m, T_p and T_L all have square roots [20]. For T_m, $S(\alpha) = \alpha$, for T_p, $S(\alpha) = \sqrt{\alpha}$ and for T_L, $S(\alpha) = \frac{\alpha+1}{2}$. The associated monoidal mean operators are given by, for T_m, $\alpha \circledast \beta = \alpha \wedge \beta$, for T_p, $\alpha \circledast \beta = \sqrt{\alpha \cdot \beta}$ and for T_L, $\alpha \circledast \beta = \frac{\alpha+\beta}{2}$.

Remark 3.1.23 : If a GL-monoid has square roots, then the square root operator S is unique [20].

Another type of operation on L, which is a generalization of the concept of GL-monoids, is a *completely lattice ordered premonoid* operation. Usually this is simply abbreviated to *cl-premonoid.*

Definition 3.1.24 [20]: $(L, \leq, \otimes)$ is a *cl-premonoid* $\Leftrightarrow$

1. $(L, \leq)$ is a complete lattice.
2. $\otimes \colon L \times L \to L$ is an operator satisfying:

   CLP1 $\forall \alpha \in L, \quad \alpha \leq \alpha \otimes \top$ and $\alpha \leq \top \otimes \alpha$.

   CLP2 $\forall \alpha, \beta, \gamma, \delta \in L, \quad \alpha \leq \beta$ and $\gamma \leq \delta \Rightarrow \alpha \otimes \gamma \leq \beta \otimes \delta$.

   CLP3 $\forall \beta \in L \; \forall \emptyset \neq A \subseteq L$,

$$\beta \otimes (\bigvee A) = \bigvee_{\alpha \in A} (\beta \otimes \alpha) \quad \text{and} \quad (\bigvee A) \otimes \beta = \bigvee_{\alpha \in A} (\alpha \otimes \beta).$$

Example 3.1.25 : Any GL-monoid $(L, \leq, *)$ is a cl-premonoid, with $\otimes = *$. The monoidal mean operator for a GL-monoid with square roots which satisfies condition **Sq3** is a cl-premonoid [20]. The T-norms T_m, T_p and T_L all satisfy this condition [20].

Remark 3.1.26 : Note that in condition **CLP3** of Definition 3.1.24, the $\otimes$ operation is only required to distribute over *non-empty* joins. Consider the monoidal mean operator associated with T_L, where we have $\alpha \circledast \beta = \frac{\alpha+\beta}{2}$. Taking $\beta = 1, A = \emptyset$, we have $\bigvee A = 0$. Then $\beta \circledast (\bigvee A) = \frac{1}{2}$, while $\bigvee_{\alpha \in A} (\beta \circledast \alpha) = \bigvee \emptyset = 0$. The reason that **CLP3** only refers to non-empty sets is so that we can include examples such as this.

If L is a GL-monoid under an operation $*$, and a cl-premonoid under an operation $\otimes$, then provided that the operations satisfy the domination condition of Definition 3.1.27, the resultant structure is called an *enriched cl-premonoid*, usually abbreviated *ecl-premonoid.*

Definition 3.1.27 [20]: $(L, \leq, *, \otimes)$ is an *enriched cl-premonoid* $\Leftrightarrow$

1. $(L, \leq, *)$ is a GL-monoid.

2. $(L, \leq, \otimes)$ is a cl-premonoid.

3. L satisfies the domination axiom. $\forall\, \alpha_1, \alpha_2, \beta_1, \beta_2 \in L$

$$(\alpha_1 \otimes \alpha_2) * (\beta_1 \otimes \beta_2) \leq (\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2).$$

An ecl-premonoid $(L, \leq, \otimes, *)$ has the *pseudo-bisymmetry property* iff $\forall\, \alpha_1, \alpha_2, \beta_1, \beta_2 \in L$

$$(\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2) = [(\alpha_1 \otimes \beta_1) * (\alpha_2 \otimes \beta_2)] \vee \\ [(\alpha_1 \otimes \bot) * (\alpha_2 \otimes \top)] \vee [(\bot \otimes \beta_1) * (\top \otimes \beta_2)].$$

Example 3.1.28 : A frame is an example of an enriched cl-premonoid which satisfies the pseudobisymmetry property, with $* = \otimes = \wedge$. Further examples of pseudobisymmetric ecl-premonoid structures are structures $(L, \leq, *, \circledast)$, where $(L, \leq, *)$ is a GL-monoid with square roots which satisfies condition **Sq3** and $\circledast$ is the associated monoidal mean operator [20].

Remark 3.1.29 : The pseudobisymmetry condition is mainly used in situations where we need a least upper bound for two L-filters. We will explicitly point out where the condition is used when it is necessary.

Lemma 3.1.30 : Let $(L, \leq, *, \otimes)$ be an ecl-premonoid. Then

1. $\forall\, \alpha, \beta \in L, \quad \alpha * \beta \leq \alpha \otimes \beta.$

2. $\forall\, \alpha, \beta, \gamma, \delta \in L, \quad (\alpha \to \beta) \otimes (\gamma \to \delta) \leq (\alpha \otimes \gamma) \to (\beta \otimes \delta).$

Proof:
Let $\alpha, \beta \in L$. Then by the domination axiom, **CLP1** and **CLP2**,

$$\alpha * \beta \leq (\alpha \otimes \top) * (\top \otimes \beta) \leq (\alpha * \top) \otimes (\top * \beta) = \alpha \otimes \beta.$$

Now let $\gamma, \delta \in L$. We have $\alpha * (\alpha \to \beta) \leq \beta$ and $\gamma * (\gamma \to \delta) \leq \delta$. Therefore by **CLP2**,

$$[\alpha * (\alpha \to \beta)] \otimes [\gamma * (\gamma \to \delta)] \leq \beta \otimes \delta.$$

But by domination

$$[\alpha * (\alpha \to \beta)] \otimes [\gamma * (\gamma \to \delta)] \geq (\alpha \otimes \gamma) * [(\alpha \to \beta) \otimes (\gamma \to \delta)].$$

Thus

$$(\alpha \otimes \gamma) * [(\alpha \to \beta) \otimes (\gamma \to \delta)] \leq \beta \otimes \delta.$$

Then by Lemma 3.1.18, we have

$$(\alpha \to \beta) \otimes (\gamma \to \delta) \leq (\alpha \otimes \gamma) \to (\beta \otimes \delta).$$

■

Throughout the rest of the text, L will be assumed to have the full ecl-premonoid structure. References to the set L are hence references to the ecl-premonoid $(L, \leq, *, \otimes)$. Also we will assume X to be a non-empty set, unless explicitly mentioned otherwise.

3.2 L-sets

L-sets are a generalization of characteristic functions to a lattice L. In our case we require the lattice to be an ecl-premonoid. $[0,1]$-sets, or fuzzy sets, were first defined and studied by Zadeh [40]. Later these were generalized to lattices by Goguen [15]. Based on what has been said in the introduction to this chapter, we define an L-set over X as a function from X to L. We denote by L^X the set of all such functions. We can extend the relation $\leq$ and the operations $\bigwedge, \bigvee, *, \otimes, \to$ to L^X as shown in this section.

Definition 3.2.1 : Let $a, b \in L^X, \Gamma \subseteq L^X$. Then

$$a \leq b \Leftrightarrow \ \forall x \in X \quad a(x) \leq b(x).$$

We may define the L-sets $\bigwedge \Gamma$ and $\bigvee \Gamma$ by

$$(\bigwedge \Gamma)(x) = \bigwedge_{g \in \Gamma} g(x) \quad \text{and} \quad (\bigvee \Gamma)(x) = \bigvee_{g \in \Gamma} g(x).$$

We may also define the L-sets $a * b, a \otimes b$ and $a \to b$ by

$$\begin{aligned} \forall x \in X, &\quad (a * b)(x) = a(x) * b(x), \\ \forall x \in X, &\quad (a \otimes b)(x) = a(x) \otimes b(x), \\ \forall x \in X, &\quad (a \to b)(x) = a(x) \to b(x). \end{aligned}$$

Let $\alpha \in L, A \subseteq X$. We define the L-set α_A by

$$\alpha_A(x) = \begin{cases} \alpha & x \in A \\ \perp & x \in X \backslash A \end{cases}.$$

Example 3.2.2 : Two special L-sets are $\top_X$ and $\bot_X$ which have the property

$$\forall\, a \in L^X \quad \bot_X \leq a \leq \top_X$$

Remark 3.2.3 : With Definition 3.2.1, we have that $(L^X, \leq, *, \otimes)$ forms an ecl-premonoid, with smallest element $\bot_X$ and largest element $\top_X$. The only possible problem is that the implication on L^X induced by the $*$ operation on L might be different from the implication induced by the $*$ operation on L^X. However we have by Lemma 3.2.4 (below) that they are the same, so that all the results of Section 3.1 apply to L^X. Note however though, that only the ecl-premonoid properties of L are passed on to L^X. In particular, if L is a linearly ordered set then L^X *need not* be linearly ordered. (For example $\{0, 1\}$ is linearly ordered but $\{0, 1\}^X$ is not, in general.)

Lemma 3.2.4 : Let $a, b \in L^X$. Define $a \to b$ and $a \rightsquigarrow b$ by

$$\forall\, x \in X, \quad (a \to b)(x) = a(x) \to b(x) = \bigvee \{\, \lambda \in L \mid a(x) * \lambda \leq b(x) \,\}$$
$$a \rightsquigarrow b = \bigvee \{\, l \in L^X \mid a * l \leq b \,\}$$

Then $a \to b = a \rightsquigarrow b$.

Proof:
Let $x \in X$. Then

$$\begin{aligned}(a \rightsquigarrow b)(x) &= \bigvee \{\, l(x) \mid a * l \leq b \,\} \\ &= \bigvee \{\, l(x) \mid \ \forall\, y \in X \quad a(y) * l(y) \leq b(y) \,\} \\ &\leq \bigvee \{\, l(x) \mid a(x) * l(x) \leq b(x) \,\} \\ &\leq \bigvee \{\, \lambda \mid a(x) * \lambda \leq b(x) \,\} = (a \to b)(x).\end{aligned}$$

To prove that $a \to b \leq a \rightsquigarrow b$ consider $\lambda \in L$ such that $a(x) * \lambda \leq b(x)$. Let $l = \lambda_{\{x\}}$. Then $\forall\, y \in X \quad a(y) * l(y) \leq b(y)$. Thus we have that $\lambda = l(x) \in \{\, l(x) \mid a * l \leq b \,\}$. Therefore

$$\{\, \lambda \in L \mid a(x) * \lambda \leq b(x) \,\} \subseteq \{\, l(x) \mid a * l \leq b \,\}$$
$$\text{thus} \quad \bigvee \{\, \lambda \in L \mid a(x) * \lambda \leq b(x) \,\} \leq \bigvee \{\, l(x) \mid a * l \leq b \,\}$$
$$\text{so finally} \quad (a \to b)(x) \leq (a \rightsquigarrow b)(x).$$

■

As we saw in Chapter 2, a mapping $\phi\colon X \to Y$ induces mappings $\phi\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ and
$\phi^{\leftarrow}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$

$$\forall\, A \subseteq X, \quad \phi(A) = \{\, y \in Y \mid \exists\, x \in A \quad \phi(x) = y \,\}.$$
$$\forall\, B \subseteq Y, \quad \phi^{\leftarrow}(B) = \{\, x \in X \mid \phi(x) \in B \,\}.$$

We now wish to generalize this to the L-set case in such a way that when we take $L = \{0, 1\}$ we regain our previous definition.

Definition 3.2.5 [40]: Let ϕ be a function between non-empty sets X and Y. Let $a \in L^X, b \in L^Y, x \in X, y \in Y$. We define

$$\phi(a)(y) = \bigvee \{\, a(x) \mid \phi(x) = y \,\},$$
$$\phi^{\leftarrow}(b)(x) = b(\phi(x)) = (b \circ \phi)(x).$$

We refer to $\phi(a)$ as the *image* of the L-set a under ϕ. $\phi^{\leftarrow}(b)$ is referred to as the *inverse image* of the L-set b under ϕ.

The properties stated below of Lemmas 3.2.6 and 3.2.9 are not difficult to prove. Some of them are described in [30]. We prove them anyway, to make sure.

Lemma 3.2.6 : Let $\phi\colon X \to Y, \psi\colon Y \to Z$ be functions. Let $a, a' \in L^X, b, b' \in L^Y, c \in L^Z$. Then

1. $\phi^{\leftarrow}(\phi(a)) \geq a$. If ϕ is injective then $\phi^{\leftarrow}(\phi(a)) = a$.
2. $\phi(\phi^{\leftarrow}(b)) \leq b$. If ϕ is surjective then $\phi(\phi^{\leftarrow}(b)) = b$.
3. $\psi(\phi(a)) = (\psi \circ \phi)(a)$.
4. $(\psi \circ \phi)^{\leftarrow}(c) = \phi^{\leftarrow}(\psi^{\leftarrow}(c))$.
5. $\phi(\top_X) = \top_{\phi(X)} \qquad \phi(\bot_X) = \bot_Y$.
6. $\phi^{\leftarrow}(\top_Y) = \top_X \qquad \phi^{\leftarrow}(\bot_Y) = \bot_X$.
7. $a \leq a' \Rightarrow \phi(a) \leq \phi(a')$.
8. $b \leq b' \Rightarrow \phi^{\leftarrow}(b) \leq \phi^{\leftarrow}(b')$.

Proof:

1. Let $x \in X$. Then

$$\phi^{\leftarrow}(\phi(a))(x) = \phi(a)(\phi(x)) = \bigvee_{\phi(z)=\phi(x)} a(z) \geq a(x).$$

Assume ϕ is injective. Then $\{ z \in X \mid \phi(z) = \phi(x) \} = \{x\}$, so $\phi^{\leftarrow}(\phi(a))(x) = a(x)$.

2. Let $y \in Y$. Then

$$\phi(\phi^{\leftarrow}(b))(y) = \bigvee_{\phi(x)=y} \phi^{\leftarrow}(b)(x) = \left\{ \begin{matrix} b(y) & y \in \phi(X) \\ \bot & y \notin \phi(X) \end{matrix} \right\} \leq b(y).$$

If ϕ is surjective then $\forall\, y \in Y \quad y \in \phi(X)$, so $\phi(\phi^{\leftarrow}(b))(y) = b(y)$.

3. Let $z \in Z$. Then

$$\begin{aligned} \psi(\phi(a))(z) = \bigvee_{\psi(y)=z} \phi(a)(y) &= \bigvee_{\psi(y)=z} \bigvee_{\phi(x)=y} a(x) \\ &= \bigvee_{(\psi\circ\phi)(x)=z} a(x) = (\psi \circ \phi)(a)(z). \end{aligned}$$

4. $(\psi \circ \phi)^{\leftarrow}(c) = c \circ \psi \circ \phi = \phi^{\leftarrow}(c \circ \psi) = \phi^{\leftarrow}(\psi^{\leftarrow}(c))$.

5. Let $y \in Y$. Then

$$\phi(\top_X)(y) = \bigvee_{\phi(x)=y} \top_X(x) = \left\{ \begin{matrix} \top & y \in \phi(X) \\ \bot & y \notin \phi(X) \end{matrix} \right\} = \top_{\phi(X)}(y)$$

$$\text{and} \quad \phi(\bot_X)(y) = \bigvee_{\phi(x)=y} \bot_X(y) = \bot = \bot_Y(y).$$

6. Let $x \in X$. Then

$$\begin{aligned} \phi^{\leftarrow}(\top_Y)(x) &= \top_Y(\phi(x)) = \top = \top_X(x) \\ \text{and}\phi^{\leftarrow}(\bot_Y)(x) &= \bot_Y(\phi(x)) = \bot = \bot_X(x). \end{aligned}$$

7. Let $a \leq a', y \in Y$. Then

$$\phi(a)(y) = \bigvee_{\phi(x)=y} a(x) \leq \bigvee_{\phi(x)=y} a'(x) = \phi(a')(y).$$

8. Let $b \leq b', x \in X$. Then

$$\phi^{\leftarrow}(b)(x) = b(\phi(x)) \leq b'(\phi(x)) = \phi^{\leftarrow}(b')(x).$$

■

Lemma 3.2.7 : Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), a \in L^X$. Then

$$\mathrm{id}_X(a) = a \quad \text{and} \quad \mathrm{id}_X^{\leftarrow}(a) = a.$$

Proof:
Let $x \in X$. Then $\mathrm{id}_X(a)(x) = \bigvee_{\mathrm{id}_X(z)=x} a(z) = a(x)$ and $\mathrm{id}_X^{\leftarrow}(a)(x) = a(\mathrm{id}_X(x)) = a(x)$. ■

Definition 3.2.8 : Let $\phi\colon X \to Y$ be a function between non-empty sets X and Y. Let $\Gamma \subseteq L^X$ and $\Delta \subseteq L^Y$. We define

$$\phi(\Gamma) = \{\, \phi(a) \mid a \in \Gamma \,\} \subseteq L^Y,$$
$$\phi^{\leftarrow}(\Delta) = \{\, \phi^{\leftarrow}(b) \mid b \in \Delta \,\} \subseteq L^X.$$

Lemma 3.2.9 : Let $\phi\colon X \to Y$ be a function between non-empty sets X and Y. Let $\Gamma \subseteq L^X, a, b \in L^X, \Delta \subseteq L^Y, c, d \in L^Y$. Then

1. $\phi^{\leftarrow}(\bigwedge \Delta) = \bigwedge \phi^{\leftarrow}(\Delta)$.
2. $\phi^{\leftarrow}(\bigvee \Delta) = \bigvee \phi^{\leftarrow}(\Delta)$.
3. $\phi^{\leftarrow}(c * d) = \phi^{\leftarrow}(c) * \phi^{\leftarrow}(d)$.
4. $\phi^{\leftarrow}(c \otimes d) = \phi^{\leftarrow}(c) \otimes \phi^{\leftarrow}(d)$.
5. $\phi^{\leftarrow}(c \to d) = \phi^{\leftarrow}(c) \to \phi^{\leftarrow}(d)$.
6. $\phi(\bigwedge \Gamma) \leq \bigwedge \phi(\Gamma)$.
7. $\phi(\bigvee \Gamma) = \bigvee \phi(\Gamma)$.
8. $\phi(a * b) \leq \phi(a) * \phi(b)$.
9. $\phi(a \otimes b) \leq \phi(a) \otimes \phi(b)$.

Proof:

1. Let $x \in X$. Then

$$\phi^{\leftarrow}(\bigwedge \Delta)(x) = (\bigwedge \Delta)(\phi(x)) = \bigwedge_{b \in \Delta} b(\phi(x)) = \bigwedge_{b \in \Delta} \phi^{\leftarrow}(b)(x) = (\bigwedge \phi^{\leftarrow}(\Delta))(x).$$

2. Let $x \in X$. Then

$$\begin{aligned}\phi^{\leftarrow}(\bigvee \Delta)(x) = (\bigvee \Delta)(\phi(x)) = \bigvee_{b\in\Delta} b(\phi(x)) \\ = \bigvee_{b\in\Delta} \phi^{\leftarrow}(b)(x) = (\bigvee \phi^{\leftarrow}(\Delta))(x).\end{aligned}$$

3. Let $x \in X$. Then

$$\phi^{\leftarrow}(c * d)(x) = (c * d)(\phi(x)) = c(\phi(x)) * d(\phi(x)) = (\phi^{\leftarrow}(c) * \phi^{\leftarrow}(d))(x).$$

4. Let $x \in X$. Then

$$\phi^{\leftarrow}(c \otimes d)(x) = (c \otimes d)(\phi(x)) = c(\phi(x)) \otimes d(\phi(x)) = (\phi^{\leftarrow}(c) \otimes \phi^{\leftarrow}(d))(x).$$

5. Let $x \in X$. Then

$$\begin{aligned}\phi^{\leftarrow}(c \to d)(x) = (c \to d)(\phi(x)) \\ = c(\phi(x)) \to d(\phi(x)) = (\phi^{\leftarrow}(c) \to \phi^{\leftarrow}(d))(x).\end{aligned}$$

6. Let $y \in Y$. Then

$$\phi(\bigwedge \Gamma)(y) = \bigvee_{\phi(x)=y} (\bigwedge \Gamma)(x) \leq \bigvee_{\phi(x)=y} a(x) = \phi(a)(y) \qquad \forall\, a \in \Gamma.$$

$$\text{Therefore } \phi(\bigwedge \Gamma)(y) \leq \bigwedge_{a\in\Gamma} \phi(a)(y) = (\bigwedge \phi(\Gamma))(y).$$

7. Let $y \in Y$. Then

$$\begin{aligned}\phi(\bigvee \Gamma)(y) = \bigvee_{\phi(x)=y} (\bigvee \Gamma)(x) = \bigvee_{\phi(x)=y} \bigvee_{a\in\Gamma} a(x) \\ = \bigvee_{a\in\Gamma} \bigvee_{\phi(x)=y} a(x) = \bigvee_{a\in\Gamma} \phi(a)(y) = (\bigvee \phi(\Gamma))(y).\end{aligned}$$

8. Let $y \in Y$. Then

$$\begin{aligned}\phi(a * b)(y) = \bigvee_{\phi(x)=y} a(x) * b(x) \\ \leq \bigvee_{\phi(x)=y} a(x) * \bigvee_{\phi(z)=y} b(z) = (\phi(a) * \phi(b))(y).\end{aligned}$$

9. Let $y \in Y$. Then

$$\begin{aligned}\phi(a \otimes b)(y) = &\bigvee_{\phi(x)=y} a(x) \otimes b(x) \\ &\leq \bigvee_{\phi(x)=y} a(x) \otimes \bigvee_{\phi(z)=y} b(z) = (\phi(a) \otimes \phi(b))(y).\end{aligned}$$

■

Lemma 3.2.10 (see e.g. [21] for the frame case): Let $\emptyset \neq X, Y, U, V \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$. Let $f: X \to U$ and $g: Y \to V$ be functions. Let π_1, π_2 denote the usual projections from $X \times Y$ to X, Y and let σ_1, σ_2 denote the usual projections from $U \times V$ to U, V respectively. We have the function

$$f \times g: X \times Y \to U \times V \quad (x, y) \mapsto (f(x), g(y)).$$

Let $a \in L^X, b \in L^Y$. We define $a \times b \in L^{X \times Y}$ by $a \times b = \pi_1^{\leftarrow}(a) * \pi_2^{\leftarrow}(b)$. Then

$$(f \times g)(a \times b) = \sigma_1^{\leftarrow}(f(a)) * \sigma_2^{\leftarrow}(g(b)) = f(a) \times g(b).$$

Proof:
Let $(u, v) \in U \times V$. Then

$$\begin{aligned}(f \times g)(a \times b)(u, v) &= \bigvee_{f \times g(x,y)=(u,v)} a \times b(x, y) = \bigvee_{\substack{f(x)=u \\ g(y)=v}} a(x) * b(y) \\ &= \bigvee_{\substack{f(x)=u \\ g(y)=v}} a(x) * \bigvee_{\substack{f(x)=u \\ g(y)=v}} b(y) = \bigvee_{f(x)=u} a(x) * \bigvee_{g(y)=v} b(y) \\ &= f(a)(u) * g(b)(v) = (\sigma_1^{\leftarrow}(f(a)) * \sigma_2^{\leftarrow}(g(b)))(u, v) \\ &= (f(a) \times g(b))(u, v).\end{aligned}$$

■

3.3 L-filters

In classical topology, a topology over X may be described by convergence of filters [39], which are special collections of subsets of X. It turns out that stratified L-topological spaces can be similarly described by convergence of stratified L-filters, which are a generalization of classical filters. This section collects the stratified L-filter results used in later chapters.

Definition 3.3.1 [20]: Let $\mathcal{F}: L^X \to L$. Then $\mathcal{F}$ is a *stratified L-filter* $\Leftrightarrow$

F1 $\mathcal{F}(\top_X) = \top, \mathcal{F}(\bot_X) = \bot$.

F2 $a \leq b \Rightarrow \mathcal{F}(a) \leq \mathcal{F}(b)$.

F3 $\mathcal{F}(a) \otimes \mathcal{F}(b) \leq \mathcal{F}(a \otimes b)$.

Fs $\alpha * \mathcal{F}(a) \leq \mathcal{F}(\alpha_X * a)$.

We define

$$\mathcal{F}_L^S(X) = \{\, \mathcal{F} \in L^{(L^X)} \mid \mathcal{F} \text{ is a stratified } L\text{-filter} \,\}.$$

Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$. We define an order on $\mathcal{F}_L^S(X)$ by

$$\mathcal{F} \leq \mathcal{G} \Leftrightarrow \; \forall\, a \in L^X \quad \mathcal{F}(a) \leq \mathcal{G}(a).$$

Example 3.3.2 : For $x \in X$ we define the *point filter at* x by

$$\forall\, a \in L^X, \quad [x](a) = a(x)$$

Then $[x]$ is a stratified L-filter.

Lemma 3.3.3 [20]: Let $(\mathcal{F}_i)_{i \in I}$ be a collection of stratified L-filters over X, indexed by a non-empty set I. We define a function

$$(\bigwedge_{i \in I} \mathcal{F}_i) \colon \text{Ł}^X \to L, \qquad (\bigwedge_{i \in I} \mathcal{F}_i)(a) = \bigwedge_{i \in I} \mathcal{F}_i(a).$$

Then $\bigwedge_{i \in I} \mathcal{F}_i \in \mathcal{F}_L^S(X)$ and furthermore $\bigwedge_{i \in I} \mathcal{F}_i$ is greatest lower bound of $\{\, \mathcal{F}_i \mid i \in I \,\}$.

Proof:

F1 We have

$$(\bigwedge_{i \in I} \mathcal{F}_i)(\bot_X) = \bigwedge_{i \in I} \mathcal{F}_i(\bot_X) = \bigwedge_{i \in I} \bot = \bot$$

$$\text{and } (\bigwedge_{i \in I} \mathcal{F}_i)(\top_X) = \bigwedge_{i \in I} \mathcal{F}_i(\top_X) = \bigwedge_{i \in I} \top = \top.$$

F2 Let $a, b \in L^X, a \leq b$. Then

$$(\bigwedge_{i \in I} \mathcal{F}_i)(a) = \bigwedge_{i \in I} \mathcal{F}_i(a) \leq \mathcal{F}_j(a) \leq \mathcal{F}_j(b) \text{ for all } j \in I.$$

$$\text{Therefore } (\bigwedge_{i \in I} \mathcal{F}_i)(a) \leq \bigwedge_{i \in I} \mathcal{F}_i(b) = (\bigwedge_{i \in I} \mathcal{F}_i)(b).$$

F3 Let $a, b \in L^X$. Then

$$(\bigwedge_{i\in I} \mathcal{F}_i)(a) \otimes (\bigwedge_{i\in I} \mathcal{F}_i)(b) \leq \mathcal{F}_j(a) \otimes \mathcal{F}_j(b) \leq \mathcal{F}_j(a \otimes b) \quad \forall\, j \in I.$$

$$\text{Therefore } (\bigwedge_{i\in I} \mathcal{F}_i)(a) \otimes (\bigwedge_{i\in I} \mathcal{F}_i)(b) \leq \bigwedge_{i\in I} \mathcal{F}_i(a \otimes b) = (\bigwedge_{i\in I} \mathcal{F}_i)(a \otimes b).$$

Fs Let $\alpha \in L, a \in L^X$. Then

$$\alpha * (\bigwedge_{i\in I} \mathcal{F}_i)(a) \leq \alpha * \mathcal{F}_j(a) \leq \mathcal{F}_j(\alpha_X * a) \quad \forall\, j \in I.$$

$$\text{hence} \quad \alpha * (\bigwedge_{i\in I} \mathcal{F}_i)(a) \leq \bigwedge_{i\in I} \mathcal{F}_i(a \otimes b) = (\bigwedge_{i\in I} \mathcal{F}_i)(\alpha_X * a).$$

Let $\mathcal{G} \in \mathcal{F}_L^S(X), \quad \forall\, i \in I, \quad \mathcal{G} \leq \mathcal{F}_i$. Let $a \in L^X$. Then $\forall\, i \in I, \quad \mathcal{G}(a) \leq \mathcal{F}_i(a)$. Thus $\mathcal{G}(a) \leq \bigwedge_{i\in I} \mathcal{F}_i(a) = (\bigwedge_{i\in I} \mathcal{F}_i)(a)$, so $\mathcal{G} \leq \bigwedge_{i\in I} \mathcal{F}_i$. Since $\forall\, j \in I, \quad \bigwedge_{i\in I} \mathcal{F}_i \leq \mathcal{F}_j$, we have that $\bigwedge_{i\in I} \mathcal{F}_i$ is the greatest lower bound of the set $\{ \mathcal{F}_i \mid i \in I \}$. ■

Example 3.3.4 : Let $A \subseteq X$. We define $[A] = \bigwedge_{x\in A}[x]$. The smallest possible filter on X is given by $\mathcal{F}_0 = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} \mathcal{F}$. It is not difficult to show (see [20]) that $\mathcal{F}_0(a) = \bigwedge_{x\in X} a(x) = \bigwedge_{\mathcal{F}\neq\mathcal{F}_0} \mathcal{F}$.

Lemma 3.3.5 [13, 14]: Let $(L, \leq, *, \otimes)$ be an ecl-premonoid which further satisfies the pseudo-bisymmetry condition. Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$. Define

$$\mathcal{H}\colon L^X \to L \quad a \mapsto \bigvee \{ \mathcal{F}(a_1) * \mathcal{G}(a_2) \mid a_1 * a_2 \leq a \}.$$

Then

$\mathcal{H}$ is an upper bound in $\mathcal{F}_L^S(X)$ for $\mathcal{F}$ and $\mathcal{G} \Leftrightarrow$

$$\forall\, a_1, a_2 \in L^X, \quad a_1 * a_2 = \perp_X \Rightarrow \mathcal{F}(a_1) * \mathcal{G}(a_2) = \perp.$$

If an upper bound for $\mathcal{F}$ and $\mathcal{G}$ exists in $\mathcal{F}_L^S(X)$, we define

$$\mathcal{F} \vee \mathcal{G} = \bigwedge \{ \mathcal{K} \in \mathcal{F}_L^S(X) \mid \mathcal{F} \leq \mathcal{K} \text{ and } \mathcal{G} \leq \mathcal{K} \}.$$

Remark 3.3.6 : By Lemma 3.3.3, we have that $\mathcal{F} \vee \mathcal{G} \in \mathcal{F}_L^S(X)$.

Lemma 3.3.7 [20]: Let $\emptyset \neq X, Y \in \text{Ob}\,(\underline{\textbf{SET}}), \phi\colon X \to Y, \mathcal{F} \in \mathcal{F}_L^S(X)$ We define $\phi(\mathcal{F})$ by

$$\phi(\mathcal{F})(a) = \mathcal{F}(\phi^{\leftarrow}(a)) \text{ for } a \in L^Y.$$

Then $\phi(\mathcal{F}) \in \mathcal{F}_L^S(Y)$.

Proof:

F1 We have

$$\phi(\mathcal{F})(\bot_Y) = \phi(\phi^{\leftarrow}(\bot_Y)) = \mathcal{F}(\bot_X) = \bot$$
$$\text{and } \phi(\mathcal{F})(\top_Y) = \phi(\phi^{\leftarrow}(\top_Y)) = \mathcal{F}(\top_X) = \top.$$

F2 Let $a, b \in L^Y, a \leq b$. Then $\phi^{\leftarrow}(a) \leq \phi^{\leftarrow}(b)$, hence

$$\phi(\mathcal{F})(a) = \mathcal{F}(\phi^{\leftarrow}(a)) \leq \mathcal{F}(\phi^{\leftarrow}(b)) = \phi(\mathcal{F})(b).$$

F3 Let $a, b \in L^Y$. Then

$$\begin{aligned}\phi(\mathcal{F})(a) \otimes \phi(\mathcal{F})(b) = \mathcal{F}(\phi^{\leftarrow}(a)) \otimes \mathcal{F}(\phi^{\leftarrow}(b)) &\leq \mathcal{F}(\phi^{\leftarrow}(a) \otimes \phi^{\leftarrow}(b)) \\ &= \mathcal{F}(\phi^{\leftarrow}(a \otimes b)) = \phi(\mathcal{F})(a \otimes b).\end{aligned}$$

Fs Let $\alpha \in L, a \in L^Y$. Then

$$\begin{aligned}\alpha * \phi(\mathcal{F})(a) &= \alpha * \mathcal{F}(\phi^{\leftarrow}(a)) \leq \mathcal{F}(\alpha_X * \phi^{\leftarrow}(a)) \\ &= \mathcal{F}(\phi^{\leftarrow}(\alpha_Y) * \phi^{\leftarrow}(a)) = \mathcal{F}(\phi^{\leftarrow}(\alpha_Y * a)) = \phi(\mathcal{F})(\alpha_Y * a).\end{aligned}$$

■

Lemma 3.3.8 : Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \mathcal{F} \leq \mathcal{G}$. Let $\phi\colon X \to Y$, $\psi\colon Y \to Z$ be functions between non-empty sets X, Y, and Z. Then

1. $\phi(\mathcal{F}) \leq \phi(\mathcal{G})$
2. $\psi(\phi(\mathcal{F})) = (\psi \circ \phi)(\mathcal{F})$

Proof:

1. Let $a \in L^Y$. Then $\phi(\mathcal{F})(a) = \mathcal{F}(\phi^{\leftarrow}(a)) \leq \mathcal{G}(\phi^{\leftarrow}(a)) = \phi(\mathcal{G})(a)$.
2. Let $a \in L^Z$. Then

$$\begin{aligned}\psi(\phi(\mathcal{F}))(a) = \phi(\mathcal{F})(\psi^{\leftarrow}(a)) &= \mathcal{F}(\phi^{\leftarrow}(\psi^{\leftarrow}(a))) \\ &= \mathcal{F}((\psi \circ \phi)^{\leftarrow}(a)) = (\psi \circ \phi)(\mathcal{F})(a).\end{aligned}$$

■

Lemma 3.3.9 [13]: Let $\emptyset \neq X, Y \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \phi\colon X \to Y, \mathcal{F} \in \mathcal{F}_L^S(Y)$. We define $\phi^{\leftarrow}(\mathcal{F})\colon L^X \to L$ by

$$\phi^{\leftarrow}(\mathcal{F})(a) = \bigvee\{\, \mathcal{F}(b) \mid \phi^{\leftarrow}(b) \leq a \,\} \text{ for } a \in L^X.$$

Then

$$\phi^{\leftarrow}(\mathcal{F}) \in \mathcal{F}_L^S(X) \Leftrightarrow \ \forall\, b \in L^Y, \quad \phi^{\leftarrow}(b) = \bot_X \Rightarrow \mathcal{F}(b) = \bot.$$

Proof:
Assume that $\phi^{\leftarrow}(\mathcal{F}) \in \mathcal{F}_L^S(X)$. Let $b \in L^Y$ such that $\phi^{\leftarrow}(b) = \bot_X$. Then $\bot = \phi^{\leftarrow}(\mathcal{F})(\bot_X) \geq \mathcal{F}(b)$. Thus $\mathcal{F}(b) = \bot$.

For the converse, assume that $\forall\, b \in L^Y, \quad \phi^{\leftarrow}(b) = \bot_X \Rightarrow \mathcal{F}(b) = \bot$. We prove the properties **F1**–**F3** and **Fs**:

F1 By our assumption, $\phi^{\leftarrow}(\mathcal{F})(\bot_X) = \bot$. Now $\phi^{\leftarrow}(\top_Y) = \top_X$, so $\phi^{\leftarrow}(\mathcal{F})(\top_X) \geq \mathcal{F}(\top_Y) = \top$.

F2 Let $a, b \in L^X, a \leq b$. Then

$$\phi^{\leftarrow}(\mathcal{F})(a) = \bigvee_{\phi^{\leftarrow}(c) \leq a} \mathcal{F}(c) \leq \bigvee_{\phi^{\leftarrow}(c) \leq b} \mathcal{F}(c) = \phi^{\leftarrow}(\mathcal{F})(b).$$

F3 Let $a, b \in L^X$. We calculate

$$\begin{aligned}\phi^{\leftarrow}(\mathcal{F})(a) \otimes \phi^{\leftarrow}(\mathcal{F})(b) &= \bigvee\{\, \mathcal{F}(a') \otimes \mathcal{F}(b') \mid \phi^{\leftarrow}(a') \leq a \text{ and } \phi^{\leftarrow}(b') \leq b \,\} \\ &\overset{\mathbf{F3}}{\leq} \bigvee\{\, \mathcal{F}(a' \otimes b') \mid \phi^{\leftarrow}(a' \otimes b') \leq a \otimes b \,\} \\ &\leq \bigvee\{\, \mathcal{F}(c') \mid \phi^{\leftarrow}(c') \leq a \otimes b \,\} = \phi^{\leftarrow}(\mathcal{F})(a \otimes b).\end{aligned}$$

Fs Let $\alpha \in L, a \in L^X$. Then

$$\begin{aligned}\alpha * \phi^{\leftarrow}(\mathcal{F})(a) &= \bigvee\{\, \alpha * \mathcal{F}(b) \mid \phi^{\leftarrow}(b) \leq a \,\} \leq \bigvee\{\, \mathcal{F}(\alpha_Y * b) \mid \phi^{\leftarrow}(b) \leq a \,\} \\ &\leq \bigvee\{\, \mathcal{F}(\alpha_Y * b) \mid \phi^{\leftarrow}(\alpha_Y * b) \leq \alpha_X * a \,\} \\ &\leq \bigvee\{\, \mathcal{F}(c) \mid \phi^{\leftarrow}(c) \leq \alpha_X * a \,\} = \phi^{\leftarrow}(\mathcal{F})(\alpha_X * a).\end{aligned}$$

Thus $\phi^{\leftarrow}(\mathcal{F}) \in \mathcal{F}_L^S(X)$. ■

Example 3.3.10 : Let $\emptyset \neq X, Y \in \mathrm{Ob}(\underline{\mathbf{SET}})$. Define $\pi_X \colon X \times Y \to X \quad (x,y) \mapsto x$. Let $b \in L^X$ such that $\pi_X^{\leftarrow}(b) = \bot_{X\times Y}$. Let $x \in X$. Then $\exists\, y \in Y, \quad (x,y) \in \pi_X^{\leftarrow}(\{x\})$. Thus $b(x) = b \circ \pi_X(x,y) = \bot$. So $b = \bot_X$. Thus by Lemma 3.3.9, $\forall\, \mathcal{F} \in \mathcal{F}_L^S(X), \quad \pi_X^{\leftarrow}(\mathcal{F}) \in \mathcal{F}_L^S(X \times Y)$.

Lemma 3.3.11 : Let $\emptyset \neq X, Y \in \mathrm{Ob}(\underline{\mathbf{SET}})$. Define $\pi_1 \colon X \times Y \to X \quad (x,y) \mapsto x$. Define $\pi_2 \colon X \times Y \to Y \quad (x,y) \mapsto y$. Let $\mathcal{F} \in \mathcal{F}_L^S(X), \mathcal{G} \in \mathcal{F}_L^S(Y)$. Then

$$\forall\, a_1, a_2 \in L^{X\times Y}, \quad a_1 * a_2 = \bot_{X\times Y} \Rightarrow \pi_1^{\leftarrow}(\mathcal{F})(a_1) * \pi_2^{\leftarrow}(\mathcal{G})(a_2) = \bot.$$

Thus by Lemma 3.3.5, if L is pseudo-bisymmetric, we may define

$$\mathcal{F} \times \mathcal{G} = \pi_1^{\leftarrow}(\mathcal{F}) \vee \pi_2^{\leftarrow}(\mathcal{G}).$$

Proof:
Let $a_1, a_2 \in L^{X\times Y}, a_1 * a_2 = \bot_{X\times Y}$. Then

$$\begin{aligned}
\pi_1^{\leftarrow}(\mathcal{F})(a_1) * \pi_2^{\leftarrow}(\mathcal{G})(a_2) &= \bigvee_{\pi_1^{\leftarrow}(b_1)\le a_1} \mathcal{F}(b_1) * \bigvee_{\pi_2^{\leftarrow}(b_2)\le a_2} \mathcal{G}(b_2) \\
&= \bigvee_{\pi_1^{\leftarrow}(b_1)\le a_1, \pi_2^{\leftarrow}(b_2)\le a_2} \mathcal{F}(b_1) * \mathcal{G}(b_2) \\
&\le \bigvee_{\pi_1^{\leftarrow}(b_1) * \pi_2^{\leftarrow}(b_2) = \bot_{X\times Y}} \mathcal{F}(b_1) * \mathcal{G}(b_2).
\end{aligned}$$

Now let $b_1 \in L^X, b_2 \in L^Y$ such that $\pi_1^{\leftarrow}(b_1) * \pi_2^{\leftarrow}(b_2) = \bot_{X\times Y}$. Let $\lambda = \mathcal{G}(b_2)$. Then $\mathcal{F}(b_1) * \mathcal{G}(b_2) = \mathcal{F}(b_1) * \lambda \overset{\mathbf{Fs}}{\le} \mathcal{F}(b_1 * \lambda_X)$. Let $x \in X$ and let $b_1(x) = \mu$. Then $b_1 * \lambda_X(x) = b_1(x) * \mathcal{G}(b_2) = \mu * \mathcal{G}(b_2) \overset{\mathbf{Fs}}{\le} \mathcal{G}(\mu_Y * b_2)$. Now let $y \in Y$. Then $\mu_Y * b_2(y) = b_1(x) * b_2(y) = (\pi_1^{\leftarrow}(b_1) * \pi_2^{\leftarrow}(b_2))(x,y) = \bot$. Thus $\mu_Y * b_2 = \bot_Y$, so $\mathcal{G}(\mu_Y * b_2) = \bot$. Then $b_1 * \lambda_X = \bot_X$, so $\mathcal{F}(b_1 * \lambda_X) = \bot$. Finally we have that $\mathcal{F}(b_1) * \mathcal{G}(b_2) = \bot$.

So we have proved that $\pi_1^{\leftarrow}(\mathcal{F})(a_1) * \pi_2^{\leftarrow}(\mathcal{G})(a_2) = \bot$. By Example 3.3.10, $\pi_1^{\leftarrow}(\mathcal{F}), \pi_2^{\leftarrow}(\mathcal{G}) \in \mathcal{F}_L^S(X \times Y)$. Then by Lemma 3.3.5, there exists $\mathcal{H} \in \mathcal{F}_L^S(X \times Y)$ such that $\pi_1^{\leftarrow}(\mathcal{F}) \le \mathcal{H}$ and $\pi_2^{\leftarrow}(\mathcal{G}) \le \mathcal{H}$. Thus we may define $\mathcal{F} \times \mathcal{G}$ as the least upper bound of $\pi_1^{\leftarrow}(\mathcal{F})$ and $\pi_2^{\leftarrow}(\mathcal{G})$. ■

Remark 3.3.12 [21]: If L is a frame then for $\mathcal{F} \in \mathcal{F}_L^S(X), \mathcal{G} \in \mathcal{F}_L^S(Y), a \in L^{X\times Y}$ we have

$$\mathcal{F} \times \mathcal{G}(a) = \bigvee \{\, \mathcal{F}(a_1) \wedge \mathcal{G}(a_2) \mid a_1 \times a_2 \le a \,\}.$$

Also note that in the proof of Lemma 3.3.11, the stratification condition was used to prove that $\mathcal{F} \times \mathcal{G}$ exists. Hence we are not sure if the product filter $\mathcal{F} \times \mathcal{G}$ exists for L-filters which are not stratified.

Lemma 3.3.13 (see [21] for the frame case): Let X, Y, π_1, π_2 be defined as in the statement of Lemma 3.3.11. Then

1. $\forall \mathcal{F} \in \mathcal{F}_L^S(X \times Y), \quad \pi_1(\mathcal{F}) \times \pi_2(\mathcal{F}) \leq \mathcal{F}$.
2. $\forall \mathcal{G}_1 \in \mathcal{F}_L^S(X)\, \forall \mathcal{G}_2 \in \mathcal{F}_L^S(Y), \quad \pi_i(\mathcal{G}_1 \times \mathcal{G}_2) \geq \mathcal{G}_i \quad , i \in \{1, 2\}$.
3. $\forall \mathcal{F} \in \mathcal{F}_L^S(X \times Y), \quad \pi_i(\pi_1(\mathcal{F}) \times \pi_2(\mathcal{F})) = \pi_i(\mathcal{F}) \quad , i \in \{1, 2\}$.

Proof:

1. Let $\mathcal{F} \in \mathcal{F}_L^S(X \times Y), a \in L^{X \times Y}$. We calculate
$$\pi_1^{\leftarrow}(\pi_1(\mathcal{F}))(a) = \bigvee_{\pi_1^{\leftarrow}(b) \leq a} \pi_1(\mathcal{F})(b) = \bigvee_{\pi_1^{\leftarrow}(b) \leq a} \mathcal{F}(\pi_1^{\leftarrow}(b)) \leq \mathcal{F}(a).$$
Thus $\pi_1^{\leftarrow}(\pi_1(\mathcal{F})) \leq \mathcal{F}$ and similarly $\pi_2^{\leftarrow}(\pi_2(\mathcal{F})) \leq \mathcal{F}$. Thus
$$\pi_1^{\leftarrow}(\pi_1(\mathcal{F})) \vee \pi_2^{\leftarrow}(\pi_2(\mathcal{F})) = \pi_1(\mathcal{F}) \times \pi_2(\mathcal{F}) \leq \mathcal{F}.$$
2. Let $\mathcal{G}_1 \in \mathcal{F}_L^S(X), \mathcal{G}_2 \in \mathcal{F}_L^S(Y), a \in L^X$. We calculate
$$\begin{aligned}\pi_1(\mathcal{G}_1 \times \mathcal{G}_2)(a) &= (\mathcal{G}_1 \times \mathcal{G}_2)(\pi_1^{\leftarrow}(a)) \\ &\geq \pi_1^{\leftarrow}(\mathcal{G}_1)(\pi_1(a)) = \bigvee_{\pi_1^{\leftarrow}(b) \leq \pi_1^{\leftarrow}(a)} \mathcal{G}_1(b) \geq \mathcal{G}_1(a).\end{aligned}$$
Thus $\pi_1(\mathcal{G}_1 \times \mathcal{G}_2) \geq \mathcal{G}_1$. Similarly $\pi_2(\mathcal{G}_1 \times \mathcal{G}_2) \geq \mathcal{G}_2$.
3. This is an immediate consequence of what we have just proved. ∎

The diagonal filter $\mathcal{G}(\mathcal{F}_{(\cdot)})$ of Definition 3.3.14 was first defined in connection with convergence by Gähler ([12]). We follow Jäger's notation and definition.

Lemma 3.3.14 [24]: Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \emptyset \neq J \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \mathcal{G} \in \mathcal{F}_L^S(J)$,
$(\mathcal{F}_j)_{j \in J} \in \mathcal{F}_L^S(X)^X$. For $a \in L^X$ we define the L-set $\mathcal{F}_{(\cdot)}(a) \in \mathcal{F}_L^S(J)$ by
$$\mathcal{F}_{(\cdot)}(a)(j) = \mathcal{F}_j(a).$$
We define the *diagonal filter* $\mathcal{G}(\mathcal{F}_{(\cdot)})$ by
$$\forall\, a \in L^X, \quad \mathcal{G}(\mathcal{F}_{(\cdot)})(a) = \mathcal{G}(\mathcal{F}_{(\cdot)}(a)).$$
Then $\mathcal{G}(\mathcal{F}_{(\cdot)}) \in \mathcal{F}_L^S(X)$.

Proof:

F1 $\mathcal{F}_{(\cdot)}(\bot_X)(j) = \mathcal{F}_j(\bot_X) = \bot$ so $\mathcal{F}_{(\cdot)}(\bot_X) = \bot_J$. Thus $\mathcal{G}(\mathcal{F}_{(\cdot)})(\bot_X) = \bot$. Similarly $\mathcal{G}(\mathcal{F}_{(\cdot)})(\top_X) = \top$.

F2 Let $a, b \in L^X, a \leq b$. Then $\mathcal{F}_{(\cdot)}(a)(j) = \mathcal{F}_j(a) \leq \mathcal{F}_j(b) = \mathcal{F}_{(\cdot)}(b)(j)$. Thus $\mathcal{G}(\mathcal{F}_{(\cdot)})(a) \leq \mathcal{G}(\mathcal{F}_{(\cdot)})(b)$.

F3 Let $a, b \in L^X$. Then

$$\begin{aligned}(\mathcal{F}_{(\cdot)}(a) \otimes \mathcal{F}_{(\cdot)}(b))(j) &= \mathcal{F}_j(a) \otimes \mathcal{F}_j(b) \\ &\leq \mathcal{F}_j(a \otimes b) = \mathcal{F}_{(\cdot)}(a \otimes b)(j).\end{aligned}$$

$$\begin{aligned}\mathcal{G}(\mathcal{F}_{(\cdot)})(a) \otimes \mathcal{G}(\mathcal{F}_{(\cdot)})(b) &\leq \mathcal{G}(\mathcal{F}_{(\cdot)}(a) \otimes \mathcal{F}_{(\cdot)}(b)) \\ &\leq \mathcal{G}(\mathcal{F}_{(\cdot)}(a \otimes b)) = \mathcal{G}(\mathcal{F}_{(\cdot)})(a \otimes b).\end{aligned}$$

Fs Let $\alpha \in L, a \in L^X$. Then

$$\alpha_J * \mathcal{F}_{(\cdot)}(a)(j) = \alpha * \mathcal{F}_j(a) \leq \mathcal{F}_j(\alpha_X * a) = \mathcal{F}_{(\cdot)}(\alpha_X * a)(j).$$

So

$$\begin{aligned}\alpha * \mathcal{G}(\mathcal{F}_{(\cdot)})(a) = \alpha * \mathcal{G}(\mathcal{F}_{(\cdot)}(a)) &\leq \mathcal{G}(\alpha_J \otimes \mathcal{F}_{(\cdot)}(a)) \\ &\leq \mathcal{G}(\mathcal{F}_{(\cdot)}(\alpha_X * a)) = \mathcal{G}(\mathcal{F}_{(\cdot)})(\alpha_X * a).\end{aligned}$$

■

Chapter 4

Stratified L-Topological Spaces

In this chapter we define stratified L-topological spaces and explore some of their properties. We prove that the category $\underline{\mathbf{SL-TOP}}$ of stratified L-topological spaces is topological. Next we prove that stratified L-topological spaces can be characterized in terms of interior operators, neighbourhood filters or limit functions in much the same way as their classical counterparts. In characterizing them by neighbourhood filter or limit function, we find that we have to introduce a new axiom, the $\mathbf{L}\otimes$ axiom. This axiom is always satisfied in the classical case. Next we prove that $\underline{\mathbf{SL-TOP}}$ is isomorphic to $\underline{\mathbf{SL-TCS}}$, the category of *stratified L-topological convergence spaces*.

Initially in our characterization by the limit function we use **Lp** and **Lt** axioms based on the classical ones, which make use of the L-neighbourhood filter, and the next section deals with the attempt to translate these to something entirely in terms of the limit function, as in [23, 24] for the special case where L is a frame. In this we are successful with the Fischer and Kowalski axioms, however our attempt to translate the **Lp** axiom as was done in Chapter 2 fails. The best we are able to do is show how the **Lp** axiom splits into two axioms, **LpW1** and **LpW2**, in the general case. We can state **LpW2** entirely in terms of the limit function, but **LpW1** still requires the L-neighbourhood filter. In the classical case the **LpW1** axiom is always true, and hence the **Lp** axiom is equivalent to the **LpW2** axiom in this case.

4.1 Stratified L-topological spaces

Definition 4.1.1 [20]: Let $(L, \leq, \otimes, *)$ be an ecl-premonoid. Then (X, Δ) is a *stratified L-topological space* $\Leftrightarrow$

$\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \Delta \subseteq L^X$.

LO1 $\bot_X, \top_X \in \Delta$.

LO2 $a, b \in \Delta \Rightarrow a \otimes b \in \Delta$.

LO3 $\Gamma \subseteq \Delta \Rightarrow \bigvee \Gamma \in \Delta$.

LOs $\alpha \in L, a \in \Delta \Rightarrow \alpha_X * a \in \Delta$.

For $(X, \Delta_X), (Y, \Delta_Y)$ stratified L-topological spaces, $\phi\colon X \to Y$ is *continuous* between (X, Δ_X) and $(Y, \Delta_Y) \Leftrightarrow$

$$\phi^{\leftarrow}(\Delta_Y) \subseteq \Delta_X.$$

We will abbreviate 'stratified L-topological space' as 'SL-topological space'.

Example 4.1.2 : Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$. Then (X, L^X) is an SL-topological space.

Lemma 4.1.3 : Let $\phi\colon X \to Y$ be continuous between SL-topological spaces (X, Δ_X) and (Y, Δ_Y). Let $\psi\colon Y \to Z$ be continuous between (Y, Δ_Y) and (Z, Δ_Z). Then $\psi \circ \phi\colon X \to Z$ is continuous between (X, Δ_X) and (Z, Δ_Z). Further id_X is continuous between (X, Δ_X) and itself.

Proof:
Let $c \in \Delta_Z$. Then $\psi^{\leftarrow}(c) \in \Delta_Y$ since ψ is continuous. So $\phi^{\leftarrow}(\psi^{\leftarrow}(c)) \in \Delta_X$. But $\phi^{\leftarrow}(\psi^{\leftarrow}(c)) = (\psi \circ \phi)^{\leftarrow}(c)$ (see Lemma 3.2.6) so we have

$$\forall\, c \in \Delta_Z \quad (\psi \circ \phi)^{\leftarrow}(c) \in \Delta_X.$$

For the second part, we have that $\mathrm{id}_X^{\leftarrow}(a) = a$ for all $a \in L^X$ (see Lemma 3.2.7) thus id_X is continuous between (X, Δ_X) and itself. ■

Lemma 4.1.4 : The class of all SL-topological spaces and continuous functions between them forms a construct, the concrete category $\underline{\mathbf{SL - TOP}}$.

Proof:
Define a class $\mathcal{O}$ by

$$\mathcal{O} = \{\, (X, \Delta) \mid (X, \Delta) \text{ is an S}L\text{-topological space} \,\}$$

Let $(X, \Delta_X), (Y, \Delta_Y) \in \mathcal{O}$. Define

$$\hom((X, \Delta_X), (Y, \Delta_Y)) = \{\, ((X, \Delta_X), \phi, (Y, \Delta_Y)) \mid \phi^{\leftarrow}(\Delta_Y) \subseteq \Delta_X \,\}.$$

Define

$$\text{id}(X, \Delta_X) = ((X, \Delta_X), \text{id}_X, (X, \Delta_X)).$$

Finally define

$$((Y, \Delta_Y) \xrightarrow{\psi} (Z, \Delta_Z)) \circ ((X, \Delta_X) \xrightarrow{\phi} (Y, \Delta_Y)) = (X, \Delta_X) \xrightarrow{\psi \circ \phi} (Z, \Delta_Z).$$

Then $(\mathcal{O}, \text{hom}, \text{id}, \circ) = \underline{\mathbf{SL-TOP}}$ is an abstract category in the sense of Definition 1.2.1. Now we simply define a forgetful functor

$$U: \underline{\mathbf{SL-TOP}} \to \underline{\mathbf{SET}} \quad (X, \Delta_X) \xrightarrow{\phi} (Y, \Delta_Y) \mapsto X \xrightarrow{\phi} Y.$$

Then $(\underline{\mathbf{SL-TOP}}, U)$ is a concrete category over $\underline{\mathbf{SET}}$, i.e. a construct, which we refer to by abuse of notation as $\underline{\mathbf{SL-TOP}}$ as well, since it should be clear from the context as to whether we are referring to it as an abstract or as a concrete category. ■

We now prove that $\underline{\mathbf{SL-TOP}}$ is topological over $\underline{\mathbf{SET}}$. In order to do this we will need some preliminary groundwork.

Lemma 4.1.5 [20]: Let $(X, \Delta_i)_{i \in I}$ be a non-empty collection of SL-topological spaces with a common base set X indexed by the class I. Then $\Delta = \bigcap_{i \in I} \Delta_i$ is a stratified L-topology on X.

Proof:

LO1 $\forall\, i \in I \quad \bot_X, \top_X \in \Delta_i$ so $\bot_X, \top_X \in \Delta$.

LO2 Let $a, b \in \Delta$. Then $\forall\, i \in I \quad a, b \in \Delta_i$ so $\forall\, i \in I \quad a \otimes b \in \Delta_i$, thus $a \otimes b \in \Delta$.

LO3 Let $\Gamma \subseteq \Delta$. Then $\forall\, i \in I \quad \Gamma \subseteq \Delta_i$ so $\forall\, i \in I \quad \bigvee \Gamma \in \Delta_i$, thus $\bigvee \Gamma \in \Delta$.

LOs Let $\alpha \in L, a \in \Delta$. Then $\forall\, i \in I \quad \alpha_X * a \in \Delta_i$ so $\alpha_X * a \in \Delta$.

■

Definition 4.1.6 [20]: Let (X, Δ) be an SL-topological space, $\Gamma \subseteq L^X$. Then

Γ is a *sub-base* for $\Delta \Leftrightarrow$

$$\Delta = \bigcap \{\, \Lambda \mid (X, \Lambda) \in \text{Ob}\,(\underline{\mathbf{SL-TOP}}) \text{ and } \Gamma \subseteq \Lambda \,\}.$$

We will denote the fact that Γ is a sub-base for Δ by $\Delta = \langle \Gamma \rangle$.

Lemma 4.1.7 : Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$. Then

$$\forall\, \Gamma \subseteq L^X \;\exists\, \Delta \subseteq L^X \quad (X, \Delta) \in \mathrm{Ob}\,(\underline{\mathbf{SL-TOP}}) \text{ and } \Delta = \langle \Gamma \rangle.$$

Proof:
Let $\Gamma \subseteq L$. We know that $(X, L^X) \in \mathrm{Ob}\,(\underline{\mathbf{SL-TOP}})$ and that $\Gamma \subseteq L^X$, thus $\{\, \Lambda \mid (X, \Lambda) \in \mathrm{Ob}\,(\underline{\mathbf{SL-TOP}}) \text{ and } \Gamma \subseteq \Lambda \,\}$ is non-empty and by Lemma 4.1.5 we have that

$$\Delta = \bigcap \{\, \Lambda \mid (X, \Lambda) \in \mathrm{Ob}\,(\underline{\mathbf{SL-TOP}}) \text{ and } \Gamma \subseteq \Lambda \,\} = \langle \Gamma \rangle$$

is a stratified L-topology on X. ■

Lemma 4.1.8 [20]: Let $(X, \Delta_X), (Y, \Delta_Y)$ be SL-topological spaces. Let Γ_Y be a sub-base for Δ_Y and let ϕ be a function from X to Y. Then

$$\phi \text{ is continuous from } (X, \Delta_X) \text{ to } (Y, \Delta_Y) \Leftrightarrow \phi^{\leftarrow}(\Gamma_Y) \subseteq \Delta_X.$$

Proof:
Let ϕ be continuous from (X, Δ_X) to (Y, Δ_Y), i.e. $\phi^{\leftarrow}(\Delta_Y) \subseteq \Delta_X$. Then $\Gamma_Y \subseteq \Delta_Y$ so $\phi^{\leftarrow}(\Gamma_Y) \subseteq \Delta_X$.

Now assume $\phi^{\leftarrow}(\Gamma_Y) \subseteq \Delta_X$. We shall prove that $\Lambda_Y = \{\, b \in L^Y \mid \phi^{\leftarrow}(b) \in \Delta_X \,\}$ is an SL-topology on Y. Then $\Gamma_Y \subseteq \Lambda_Y$. But Δ_Y is the *smallest* stratified L-topological space on Y containing Γ_Y, thus $\Delta_Y \subseteq \Lambda_Y$ and we have that $\phi^{\leftarrow}(\Delta_Y) \subseteq \Delta_X$

It remains to prove that $\Lambda_Y = \{\, b \in L^Y \mid \phi^{\leftarrow}(b) \in \Delta_X \,\}$ is a stratified L-topology on Y. The proof makes use of Lemmas 3.2.6 and 3.2.9.

LO1 $\phi^{\leftarrow}(\bot_Y) = \bot_X \in \Delta_X, \phi^{\leftarrow}(\top_Y) = \top_X \in \Delta_X$ so $\bot_Y, \top_Y \in \Lambda_Y$.

LO2 Let $a, b \in \Lambda_Y$. Then $\Delta_X \ni \phi^{\leftarrow}(a) \otimes \phi^{\leftarrow}(b) = \phi^{\leftarrow}(a \otimes b)$ so $a \otimes b \in \Lambda_Y$.

LO3 Let $\Sigma \subseteq \Lambda_Y$. Then $\phi^{\leftarrow}(\Sigma) \subseteq \Delta_X$. Thus $\Delta_X \ni \bigvee \phi^{\leftarrow}(\Sigma) = \phi^{\leftarrow}(\bigvee \Sigma)$ so $\bigvee \Sigma \in \Lambda_Y$.

LOs Let $\alpha \in L, a \in \Lambda_Y$. Then $\Delta_X \ni \alpha_X * \phi^{\leftarrow}(a) = \phi^{\leftarrow}(\alpha_Y * a)$ so $\alpha_Y * a \in \Lambda_Y$.

■

Theorem 4.1.9 [20]: $\underline{\mathbf{SL-TOP}}$ is topological over $\underline{\mathbf{SET}}$. Furthermore $\underline{\mathbf{SL-TOP}}$ is amnestic, fibre-small and has the terminal separator property.

Proof:
First, we prove that $\underline{\mathbf{SL-TOP}}$ is amnestic. To this end, let $(X, \Delta_1), (X, \Delta_2)$ be members of the same $\underline{\mathbf{SL-TOP}}$-fibre of X, and let the conditions $(X, \Delta_1) \leq (X, \Delta_2)$ and $(X, \Delta_2) \leq (X, \Delta_1)$ hold. Then we have $\mathrm{id}_X^{\leftarrow}(\Delta_1) \subseteq \Delta_2$, thus $\Delta_1 \subseteq \Delta_2$. Similarly we have $\Delta_2 \subseteq \Delta_1$. Hence $\Delta_2 = \Delta_1$.

Now we prove that $\underline{\mathbf{SL-TOP}}$ is topological over $\underline{\mathbf{SET}}$, following the procedure outlined in Chapter 1. Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$. Let $(X_i, \Delta_i)_{i \in I} \in \mathrm{Ob}\,(\underline{\mathbf{SL-TOP}})^I$ be a non-empty family of SL-topological spaces indexed by the class I. Let $(\phi_i \colon X \to X_i)_{i \in I}$ be a corresponding family of set functions. Define

$$\Delta_X = \langle \bigcup_{i \in I} \phi_i^{\leftarrow}(\Delta_i) \rangle$$

Then (X, Δ_X) is an SL-topological space by Lemma 4.1.7 and $\forall\, i \in I, \quad \phi_i$ is continuous between (X, Δ_X) and (X_i, Δ_i). Let (Y, Δ_Y) be an SL-topological space, $\psi : Y \to X$ be a function. We seek to prove that

$$\begin{aligned}\forall\, i \in I \quad \phi_i \circ \psi \text{ is continuous between } (Y, \Delta_Y) \text{ and } (X_i, \Delta_i) \Rightarrow \\ \psi \text{ is continuous between } (Y, \Delta_Y) \text{ and } (X, \Delta_X).\end{aligned}$$

Assume that $\forall\, i \in I \quad (\phi_i \circ \psi)^{\leftarrow}(\Delta_i) \subseteq \Delta_Y$. Then, by Lemma 4.1.8

$$\begin{aligned}\psi^{\leftarrow}(\Delta_X) \subseteq \Delta_Y &\Leftrightarrow \psi^{\leftarrow}(\langle \bigcup_{i \in I} \phi_i^{\leftarrow}(\Delta_i) \rangle) \subseteq \Delta_Y \\ &\Leftrightarrow \psi^{\leftarrow}(\bigcup_{i \in I} \phi_i^{\leftarrow}(\Delta_i)) \subseteq \Delta_Y \\ &\Leftrightarrow \ \forall\, i \in I \quad \psi^{\leftarrow}(\phi_i^{\leftarrow}(\Delta_i)) \subseteq \Delta_Y.\end{aligned}$$

But $\psi^{\leftarrow}(\phi_i^{\leftarrow}(\Delta_i)) = (\phi_i \circ \psi)^{\leftarrow}(\Delta_i)$. So we have, by our assumption, that $\psi^{\leftarrow}(\Delta_X) \subseteq \Delta_Y$. Hence

$$\begin{aligned}\forall\, i \in I \quad \phi_i \circ \psi \text{ is continuous between } (Y, \Delta_Y) \text{ and } (X_i, \Delta_i) \Rightarrow \\ \psi \text{ is continuous between } (Y, \Delta_Y) \text{ and } (X, \Delta_X).\end{aligned}$$

Since we have that $\underline{\mathbf{SL-TOP}}$ is amnestic, we have proved that $\underline{\mathbf{SL-TOP}}$ is topological over $\underline{\mathbf{SET}}$.

Let X be a non-empty set. Let $\Gamma \subseteq L^X$. Then (X, Γ) either is or is not an SL-topological space. Thus

$$\mathrm{Fibre}_{\underline{\mathbf{SL-TOP}}}(X) \subseteq (\mathcal{P}(L^X))^{\{0,1\}} \in \mathrm{Ob}\,(\underline{\mathbf{SET}}).$$

so $\underline{\mathbf{SL-TOP}}$ is fibre small.

Lastly, let $X = \{x\}, (X, \Delta) \in \mathrm{Ob}\,(\underline{\mathbf{SL-TOP}})$. Then by **LO1** and **LOs**, we have $\{\, \alpha_X \mid \alpha \in L \,\} \subseteq \Delta$. But $L^X = \{\, \alpha_X \mid \alpha \in L \,\}$. Therefore $\Delta = L^X$ and there is only one stratified L-topology on the one point set $\{x\}$. ■

4.2 Characterization of stratified L-topological spaces

In this section we explore the characterization of SL-topological spaces by interior, neighbourhood filter and convergence function, analogous to the classical characterization of topological spaces detailed in Chapter 2.

Definition 4.2.1 [8, 20]: Let (X, Δ) be a SL-topological space. We define the *interior operator on* L^X by

I0 $$\mathrm{int}\colon L^X \to \Delta, \quad \mathrm{int}(a) = \bigvee \{\, g \in \Delta \mid g \le a \,\}.$$

We will usually denote the interior operator acting on an L-set a as $\underline{a} = \mathrm{int}(a)$, except where the expanded notation makes more sense.

Lemma 4.2.2 [20]: The interior operator has the following properties:

I1 $\forall\, a \in L^X, \quad a \in \Delta \Leftrightarrow a \le \underline{a}$.

I2 $\forall\, a \in Ł^X, \quad \underline{a} \le a$.

I3 $\underline{\top_X} = \top_X$.

I4 $\forall\, a, b \in L^X, \quad \underline{a} \otimes \underline{b} \le \underline{a \otimes b}$.

I5 $\forall\, a, b \in L^X, \quad a \le b \Rightarrow \underline{a} \le \underline{b}$.

I6 $\forall\, a \in L^X, \quad \underline{\underline{a}} = \underline{a}$.

I7 $\forall\, \alpha \in L\ \forall\, a \in L^X, \quad \alpha_X * \underline{a} \le \underline{\alpha_X * a}$.

Proof:
Let (X, Δ) be an SL-topological space with the interior operation defined as in Definition 4.2.1. From the definition and properties of the $\bigvee$ operation (Lemma 3.1.9), **I2** is true. For **I3** we note that by **O1**, $\top_X \in \Delta$. Thus from the definition, $\underline{T_X} \ge \top_X$. Note that $\forall\, a \in L^X, \quad \underline{a} \in \Delta$ from **O3**. Taking $a, b \in L^X$, from **O2** and **I2** we have that $\Delta \ni \underline{a} \otimes \underline{b} \le a \otimes b$, hence **I4** follows. **I5** is immediate from the definition and from properties of the $\bigvee$ operation. Now let $a \in \Delta$. Then from the definition of $\underline{a}$, $a \le \underline{a}$. For the converse let $a \in L^X, a \le \underline{a}$. Then from **I2**, we have that $a = \underline{a}$. But $\underline{a} \in \Delta$ by **O3**. So **I6** is true. Finally let $\alpha \in L, a \in L^X$. Then $\Delta \ni \alpha_X * \underline{a} \le \alpha_X * a$ by **I2** and **Os**. Thus by the definition $\underline{\alpha_X * a} \ge \alpha_X * \underline{a}$. ■

Definition 4.2.3 : Let X be a non-empty set and let $\mathrm{int}\colon L^X \to L^X$ be an operator satisfying the properties **I2** to **I7** of Lemma 4.2.2. Then (X, int) is a *stratified L-interior space*. We will abbreviate 'stratified L-interior space' as 'SL-interior space'.

Lemma 4.2.4 [20]: The properties of Lemma 4.2.2 *characterize* SL-topological spaces, i.e. if (X, int) is an SL-interior space then it can be mapped uniquely to a SL-topological space (X, Δ) via **I1**, and the SL-topological space so defined can be mapped back to the *same* SL-interior space (X, int) via **I0**. In this way the axioms **O1**–**O3** describe essentially the same object as the interior axioms **I2**–**I7**.

Proof:
Let (X, int) be an SL-interior space. Define $a \in \Delta \Leftrightarrow a \le \underline{a}$ (this is the **I1** axiom). Then by definition, $\bot_X \le \underline{\bot_X}$. From **I3**, $\top_X \in \Delta$, so the **O1** axiom is satisfied. For **O2**, let $a, b \in \Delta$. Then $a \otimes b \le \underline{a} \otimes \underline{b} \le \underline{a \otimes b}$ from **I4** and **I1**. Let $\Gamma \subseteq \Delta$. Then $\bigvee \Gamma \le \bigvee_{g \in \Gamma} \underline{g} \le \underline{\bigvee \Gamma}$ by **I1** and **I5**, so $\bigvee \Gamma \le \underline{\bigvee \Gamma}$ and **O3** is satisfied. Finally let $\alpha \in L, a \in \Delta$. Then $\alpha_X * a \le \alpha_X * \underline{a} \le \underline{\alpha_X * a}$, by **I7**, so **Os** is satisfied.

Let $\underline{a}_\Delta$ denote the interior operator defined by Δ via **I0**, i.e. $\underline{a}_\Delta = \bigvee\{\, b \in \Delta \mid b \le a \,\}$. We seek to prove that $\underline{a}_\Delta = \underline{a}$ for all a. Let $a \in L^X$. Then from **I6**, $\underline{a} \in \Delta$ and from **I2**, $\underline{a} \le a$, thus $\underline{a}_\Delta \ge \underline{a}$. By definition and by **I5**, $\underline{a}_\Delta \le \underline{a}$, thus $\underline{a} = \underline{a}_\Delta$. ■

Definition 4.2.5 [20]: Let (X, int) be an SL-interior space. For $x \in X$ we define the *L-neighbourhood filter at x*, $\mathcal{U}^x$, by

N1 $$\forall\, a \in L^X, \quad \mathcal{U}^x(a) = \underline{a}(x).$$

Lemma 4.2.6 [20]: The L-neighbourhood filter at $x \in X$ has the following properties:

N1 $\forall\, a \in L^X, \quad \underline{a}(x) = \mathcal{U}^x(a)$.

N2 $\forall\, a \in L^X, \quad \mathcal{U}^x(a) \le a(x)$.

N3 $\mathcal{U}^x(\top_X) = \top$.

N4 $\forall\, a, b \in L^X, \quad \mathcal{U}^x(a) \otimes \mathcal{U}^x(b) \le \mathcal{U}^x(a \otimes b)$.

N5 $\forall\, a, b \in L^X, \quad a \le b \Rightarrow \mathcal{U}^x(a) \le \mathcal{U}^x(b)$.

N6 $\forall\, a \in L^X, \quad \mathcal{U}^x(a) \le \bigvee\{\, \mathcal{U}^x(b) \mid \ \forall\, y \in X \quad b(y) \le \mathcal{U}^y(a) \,\}$.

N7 $\forall\, \alpha \in L\ \forall\, a \in L^X, \quad \alpha * \mathcal{U}^x(a) \le \mathcal{U}^x(\alpha_X * a)$.

Proof:
Let (X, int) be an SL-interior space. Let $x \in X, a, b \in L^X$. Then $\mathcal{U}^x(a) = \underline{a}(x) \leq a(x)$ from **I2** so **N2** is true. From **I3**, $\mathcal{U}^x(\top_X) = \underline{\top_X}(x) = \top_X(x) = \top$, so **N3** is true. **N4** and **N5** similarly follow directly from **I4** and **I5**. For **N6**, note that $\forall\, y \in X, \quad \underline{a}(y) = \mathcal{U}^y(a) \geq \underline{a}(y)$. Thus $\underline{a} \in \{\, b \in L^X \mid \forall\, y \in X, \quad b(y) \leq \mathcal{U}^y(a)\,\}$. From **I6**, $\mathcal{U}^x(a) \leq \mathcal{U}^x(\underline{a})$. So we have $\mathcal{U}^x(a) \leq \bigvee\{\, \mathcal{U}^x(b) \mid \forall\, y \in X, \quad b(y) \leq \mathcal{U}^y(a)\,\}$. Finally for **N7**, let $\alpha \in L$. Then $\alpha * \mathcal{U}^x(a) = (\alpha_X * \underline{a})(x) \leq \underline{\alpha_X * a}(x) = \mathcal{U}^x(\alpha_X * a)$. ∎

Remark 4.2.7 : By the axioms **N2**, **N3**, **N4**, **N5** and **N7**, $\mathcal{U}^x \in \mathcal{F}_L^S(X)$.

Definition 4.2.8 : Let X be a non-empty set and let $(\mathcal{U}^x)_{x \in X}$ be a collection of filters on X indexed by X satisfying axioms **N2**–**N7** of Lemma 4.2.6. Then $(X, (\mathcal{U}^x)_{x \in X})$ is a *stratified L-neighbourhood space*. We will abbreviate 'stratified L-neighbourhood space' as 'SL-neighbourhood space'.

Lemma 4.2.9 [20]: The properties of Lemma 4.2.6 *characterize* SL-interior spaces, i.e. if $(X, (\mathcal{U}^x)_{x \in X})$ is an SL-neighbourhood space then it can be mapped uniquely to an SL-interior space (X, int) via **N1**, and the SL-interior space so defined can be mapped back to the *same* SL-neighbourhood space $(X, (\mathcal{U}^x)_{x \in X})$ via **N1** again. In this way the interior axioms **I2**–**I7** describe essentially the same object as the neighbourhood axioms **N2**–**N7**.

Proof:
Let $(X, (\mathcal{U}^x)_{x \in X})$ be an SL-neighbourhood space. For $a \in L^X$ and for each $x \in X$, define $\mathrm{int}(a)(x) = \underline{a}(x) = \mathcal{U}^x(a)$. Now let $a \in L^X$. From **N2** we have $\forall\, x \in X \underline{a}(x) \leq a(x)$, hence $\underline{a} \leq a$ and **I2** is satisfied. Similarly **N3** implies **I3**. For **I4**, let $a, b \in L^X, x \in X$. Then $(\underline{a} \otimes \underline{b})(x) = \mathcal{U}^x(a) \otimes \mathcal{U}^x(b) \leq \mathcal{U}^x(a \otimes b) = \underline{a \otimes b}(x)$ by **N4**. If $a \leq b$ then $\forall\, x \in X, \quad \underline{a}(x) \leq \underline{b}(x)$ by **N5**, so **I5** is satisfied. For **I6**, note that $\{\, \mathcal{U}^x(b) \mid \forall\, y \in X \quad b(y) \leq \mathcal{U}^y(a)\,\} = \{\, \mathcal{U}^x(b) \mid b \leq \underline{a}\,\}$. If $b \leq \underline{a}$ then by **I5**, $\underline{b} \leq \underline{\underline{a}}$. Now $\underline{a} \leq \underline{a}$, thus $\mathcal{U}^x(\underline{a}) = \bigvee\{\, \mathcal{U}^x(b) \mid \forall\, y \in X \quad b(y) \leq \mathcal{U}^y(a)\,\}$. Thus by **N6**, $\underline{a}(x) \leq \underline{\underline{a}}(x)$ for all $x \in X$. Finally for **I7**, let $\alpha \in L, a \in L^X, x \in X$. Then $(\alpha_X * \underline{a})(x) = \alpha * \mathcal{U}^x(a) \leq \mathcal{U}^x(\alpha_X * a) = \underline{\alpha_X * a}(x)$, thus **I7** is satisfied.

If we now define $\mathcal{U}^x_{\mathrm{int}}(a) = \underline{a}(x)$ then from **N1**, $\mathcal{U}^x_{\mathrm{int}}(a) = \mathcal{U}^x(a)$. ∎

Definition 4.2.10 [20]: Let $(X, (\mathcal{U}^x)_{x \in X})$ be an SL-neighbourhood space. We define the limit function $\lim\colon \mathcal{F}_L^S(X) \to L^X$ by defining the value of $\lim \mathcal{F}$ at each $x \in X$:

$$\textbf{Lp} \qquad \forall\, x \in X \ \forall\, \mathcal{F} \in \mathcal{F}_L^S(X), \quad \lim \mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a)).$$

Lemma 4.2.11 (see [21] for the frame case): The lim function satisfies the following properties:

L0 $\forall\, x \in X, \quad \mathcal{U}^x(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}(\lim\mathcal{F}(x) \to \mathcal{F}(a)).$

L1 $\forall\, x \in X, \quad \lim[x](x) = \top.$

L⊗ $\forall\, x \in X\ \forall\, a, b \in L^X, \quad \mathcal{U}^x(a) \otimes \mathcal{U}^x(b) \le \mathcal{U}^x(a \otimes b).$

Lp $\forall\, x \in X\ \forall\, \mathcal{F} \in \mathcal{F}_L^S(X), \quad \lim\mathcal{F}(x) = \bigwedge_{a\in L^X}(\mathcal{U}^x(a) \to \mathcal{F}(a)).$

Lt $\forall\, x \in X\ \forall\, a \in L^X, \quad \mathcal{U}^x(a) \le \bigvee\{\,\mathcal{U}^x(b) \mid \ \forall\, y \in X \quad b(y) \le \mathcal{U}^y(a)\,\}.$

Proof:
We prove first two minor results. Let $x \in X$. Then
$\lim\mathcal{U}^x(x) = \bigwedge_{a\in L^X}(\mathcal{U}^x(a) \to \mathcal{U}^x(a)) = \top$. Secondly let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \mathcal{F} \le \mathcal{G}$. Then by **Lp** and the fact that the implication is an increasing function in the second argument (Lemma 3.1.18), $\lim\mathcal{F} \le \lim\mathcal{G}$. To prove **L0**, first note that

$$\bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim\mathcal{F}(x) \to \mathcal{F}(a)) \le \lim\mathcal{U}^x(x) \to \mathcal{U}^x(a) = \mathcal{U}^x(a).$$

Also

$$\begin{aligned}\bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim\mathcal{F}(x) \to \mathcal{F}(a)) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} \Big(\bigwedge_{b\in L^X} (\mathcal{U}^x(b) \to \mathcal{F}(b)) \to \mathcal{F}(a)\Big) \\ &\ge \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} ((\mathcal{U}^x(a) \to \mathcal{F}(a)) \to \mathcal{F}(a)) \ge \mathcal{U}^x(a).\end{aligned}$$

For **L1**, let $x \in X$. Then $\lim[x](x) = \bigwedge_{a\in L^X}(\mathcal{U}^x(a) \Rightarrow a(x)) = \top$, by **N2**. The **L⊗** axiom is simply the **N4** axiom, **Lp** is the definition, and finally **Lt** is simply the **N6** axiom. ∎

Definition 4.2.12 : Let X be a non-empty set and let $\lim\colon \mathcal{F}_L^S(X) \to L^X$ be a function satisfying axioms **L1**, **L⊗**, **Lp** and **Lt** of Lemma 4.2.11, with $\mathcal{U}^x$ defined by **L0**. Then $(X, \lim)$ is a *stratified L-topological convergence space*. We will abbreviate 'stratified L-topological convergence space' as 'SL-topological convergence space'.

Remark 4.2.13 : The labels given to the axioms for SL-topological convergence spaces are more or less traditional (see e.g. [21, 24, 22, 7]), hence they have been retained although in later sections the labelling conventions do tend to become slightly bewildering.

Lemma 4.2.14 (see [21] for the frame case): The properties of Lemma 4.2.11 *characterize* SL-neighbourhood spaces, i.e. if $(X, \lim)$ is an SL-topological convergence space then it can be mapped uniquely to an SL-neighbourhood space $(X, (\mathcal{U}^x)_{x\in X})$ via **L0**, and the SL-neighbourhood space so obtained can be mapped back to the *same* SL-topological convergence space $(X, \lim)$ via **Lp**. In this way the neighbourhood axioms **N2**–**N7** describe essentially the same object as the topological convergence axioms.

Proof:
Let $(X, \lim)$ be an SL-topological convergence space. Define $\mathcal{U}^x(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}(\lim\mathcal{F}(x) \to \mathcal{F}(a))$. This is axiom **L0** in Lemma 4.2.11. From the definition, $\mathcal{U}^x(a) \le \lim[x](x) \to [x](a) = a(x)$, thus by **L1**, **N2** is satisfied. For **N3**, we calculate $\mathcal{U}^x(\top_X) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}(\lim\mathcal{F}(x) \to \mathcal{F}(\top_X)) = \top$. **N4** is simply the **L**$\otimes$ axiom. For **N5**, let $a, b \in L^X, a \le b$. Then for all $\mathcal{F} \in \mathcal{F}_L^S(X)$, $\mathcal{F}(a) \le \mathcal{F}(b)$, and by Lemma 3.1.18,

$$\mathcal{U}^x(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim\mathcal{F}(x) \to \mathcal{F}(a)) \le \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim\mathcal{F}(x) \to \mathcal{F}(b)) = \mathcal{U}^x(b).$$

N6 is given by the **Lt** axiom. Finally let $\alpha \in L, a \in L^X$. Then by Lemma 3.1.18,

$$\begin{aligned}\alpha * \mathcal{U}^x(a) = \alpha * \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim\mathcal{F}(x) \to \mathcal{F}(a)) &\le \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim\mathcal{F}(x) \to (\alpha * \mathcal{F}(a))) \\ &\le \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim\mathcal{F}(x) \to \mathcal{F}(\alpha_X * a)) = \mathcal{U}^x(\alpha_X * a).\end{aligned}$$

If we now define $\lim_{\mathcal{U}}\mathcal{F}(x) = \bigwedge_{a\in L^X}(\mathcal{U}^x(a) \to \mathcal{F}(a)$ for $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X$, then by **Lp**, $\lim_{\mathcal{U}} = \lim$. ■

4.3 Stratified L-topological convergence spaces

We now wish to define a category of SL-topological convergence spaces *isomorphic* to $\underline{\mathbf{SL-TOP}}$. Jäger does this for the case where L is a frame in [21]. In order achieve our goal we will have to find the condition on a function $\phi\colon X \to Y$ between two such convergence spaces $(X, \lim_X)$ and $(Y, \lim_Y)$ which is equivalent to ϕ being continuous between the corresponding SL-topological spaces (X, Δ_X) and (Y, Δ_Y). There is some background to be filled in before we proceed with the main proof.

Lemma 4.3.1 : Let $(X, \lim_X), (Y, \lim_Y)$ be SL-convergence spaces. Let $\phi\colon X \to Y$ be a function. Then

$$\forall \mathcal{F} \in \mathcal{F}_L^S(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}) \Leftrightarrow$$
$$\forall \mathcal{F} \in \mathcal{F}_L^S(X)\ \forall x \in X, \quad \lim_X \mathcal{F}(x) \leq \lim_Y \phi(\mathcal{F})(\phi(x)).$$

Proof:
From Lemma 3.2.6 we have that $\forall a \in L^X,\ \forall b \in L^Y, \quad \phi(a) \leq b \Leftrightarrow a \leq \phi^{\leftarrow}(b)$. Let $\mathcal{F} \in \mathcal{F}_L^S(X), a = \lim_X \mathcal{F}, b = \lim_Y \phi(\mathcal{F})$. Then immediately

$$\phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}) \Leftrightarrow \lim_X \mathcal{F} \leq \phi^{\leftarrow}(\lim_Y \phi(\mathcal{F})$$
$$\Leftrightarrow\ \forall x \in X, \quad \lim_X \mathcal{F}(x) \leq \lim_Y \phi(\mathcal{F})(\phi(x)).$$

■

Lemma 4.3.2 (see [21] for the frame case): Let $(X, \Delta_X), (Y, \Delta_Y)$ be SL-topological spaces. Let $(X, \lim_X), (Y, \lim_Y)$ be the corresponding SL-topological convergence spaces. Let $\phi\colon X \to Y$ be a function. Then

$$\phi \text{ is continuous} \Leftrightarrow \phi^{\leftarrow}(\Delta_Y) \subseteq \Delta_X$$
$$\Leftrightarrow\ \forall x \in X\ \forall b \in L^Y \quad \mathcal{U}^x_{\Delta_X}(\phi^{\leftarrow}(b)) \geq \mathcal{U}^{\phi(x)}_{\Delta_Y}(b)$$
$$\Leftrightarrow\ \forall \mathcal{F} \in \mathcal{F}_L^S(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}).$$

Proof:
We prove that

$$\phi^{\leftarrow}(\Delta_Y) \subseteq \Delta_X \Leftrightarrow\ \forall x \in X\ \forall b \in L^Y \quad \mathcal{U}^x_{\Delta_X}(\phi^{\leftarrow}(b)) \geq \mathcal{U}^{\phi(x)}_{\Delta_Y}(b).$$

Assume $\phi^{\leftarrow}(\Delta_Y) \subseteq \Delta_X$. Let $x \in X, b \in L^Y$. Then

$$\mathcal{U}^{\phi(x)}_{\Delta_Y}(b) = \bigvee_{b' \in \Delta_Y, b' \leq b} b'(\phi(x)) = \bigvee_{b' \in \Delta_Y, b' \leq b} \phi^{\leftarrow}(b')(x)$$
$$\leq \bigvee_{a' \in \Delta_X, a' \leq \phi^{\leftarrow}(b)} a'(x) = \mathcal{U}^x_{\Delta_X}(\phi^{\leftarrow}(b)).$$

Now we assume $\forall x \in X\ \forall b \in L^Y \quad \mathcal{U}^x_{\Delta_X}(\phi^{\leftarrow}(b)) \geq \mathcal{U}^{\phi(x)}_{\Delta_Y}(b)$. Let $b \in \Delta_Y, x \in X$. Then $b \leq \underline{b}_Y$. So

$$\phi^{\leftarrow}(b)(x) = b(\phi(x)) \leq \underline{b}_Y(\phi(x)) = \mathcal{U}^{\phi(x)}_{\Delta_Y}(b) \leq \mathcal{U}^x_{\Delta_X}(\phi^{\leftarrow}(b)) = \underline{\phi^{\leftarrow}(b)}_X(x).$$

This is true for all $x \in X$, thus $\phi^{\leftarrow}(b) \in \Delta_X$ and $\phi^{\leftarrow}(\Delta_Y) \subseteq \Delta_X$.

Next we prove that

$$\forall x \in X \ \forall b \in L^Y \quad \mathcal{U}^x_{\Delta_X}(\phi^{\leftarrow}(b)) \geq \mathcal{U}^{\phi(x)}_{\Delta_Y}(b)$$
$$\Leftrightarrow \ \forall \mathcal{F} \in \mathcal{F}^S_L(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}).$$

Assume $\forall x \in X \ \forall b \in L^Y \quad \mathcal{U}^x_{\Delta_X}(\phi^{\leftarrow}(b)) \geq \mathcal{U}^{\phi(x)}_{\Delta_Y}(b)$. Let $x \in X, \mathcal{F} \in \mathcal{F}^S_L(X)$.

$$\begin{aligned}
\lim_Y \phi(\mathcal{F})(\phi(x)) &= \bigwedge_{b \in L^Y} (\mathcal{U}^{\phi(x)}_{\Delta_Y}(b) \to \phi(\mathcal{F})(b)) \\
&\geq \bigwedge_{b \in L^Y} (\mathcal{U}^x_{\Delta_X}(\phi^{\leftarrow}(b)) \to \mathcal{F}(\phi^{\leftarrow}(b))) \\
&\geq \bigwedge_{a \in L^X} (\mathcal{U}^x_{\Delta_X}(a) \to \mathcal{F}(a)) = \lim \mathcal{F}(x).
\end{aligned}$$

Thus by Lemma 4.3.1, $\forall \mathcal{F} \in \mathcal{F}^S_L(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F})$.

Now assume $\forall \mathcal{F} \in \mathcal{F}^S_L(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F})$. Let $x \in X, b \in L^Y$. Then

$$\begin{aligned}
\mathcal{U}^x_{\Delta_X}(\phi^{\leftarrow}(b)) &= \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\lim_X \mathcal{F}(x) \to \mathcal{F}(\phi^{\leftarrow}(b))) \\
&\geq \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\lim_Y \phi(\mathcal{F})(\phi(x)) \to \phi(\mathcal{F})(b)) \\
&\geq \bigwedge_{\mathcal{G} \in \mathcal{F}^S_L(Y)} (\lim_Y \mathcal{G}(\phi(x)) \to \mathcal{G}(b)) = \mathcal{U}^{\phi(x)}_{\Delta_Y}(b).
\end{aligned}$$

■

Theorem 4.3.3 (see [21] for the frame case): The category **SL − TCS**, which has objects $(X, \lim)$ which are stratified L-topological convergence spaces and morphisms all functions ϕ which satisfy $\phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F})$ for $\mathcal{F} \in \mathcal{F}^S_L(X)$ and $(X, \lim_X), (Y, \lim_Y)$ SL-topological convergence spaces, is isomorphic to the category **SL − TOP**. Hence it is topological over **SET**, amnestic, fibre small and has the terminal separator property.

Proof:
We define functors

$$F : \underline{\mathbf{SL-TOP}} \to \underline{\mathbf{SL-TCS}}$$
$$(X, \Delta_X) \xrightarrow{\phi} (Y, \Delta_Y) \mapsto (X, \lim_{\Delta_X}) \xrightarrow{\phi} (Y, \lim_{\Delta_Y})$$

and

$$G\colon \underline{\mathbf{SL-TCS}} \to \underline{\mathbf{SL-TOP}}$$

$$(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \mapsto (X, \Delta_{\lim_X}) \xrightarrow{\phi} (Y, \Delta_{\lim_Y}).$$

Then by 4.2.11, 4.2.14 and 4.3.2 we have

$$F \circ G = \mathrm{id}_{\underline{\mathbf{SL-TOP}}} \quad \text{and} \quad G \circ F = \mathrm{id}_{\underline{\mathbf{SL-TCS}}}.$$

So $\underline{\mathbf{SL-TOP}}$ is isomorphic to $\underline{\mathbf{SL-TCS}}$. ■

4.4 Alternatives to the Lp and Lt axioms

As we did in the classical case, we now attempt to translate our axioms for topological convergence spaces into simple axioms entirely in terms of the lim function. In this section we are basically generalizing the work of Jäger ([21, 22, 23, 24]) from the frame case to the ecl-premonoid case. We begin with **Lp**.

The Lp axiom

Definition 4.4.1 : Let X be a non-empty set and let $\lim\colon \mathcal{F}_L^S(X) \to L^X$ be a function. Then $(X, \lim)$ is a *stratified L-preconvergence space*. We will abbreviate 'stratified L-preconvergence space' as 'SL-preconvergence space'.

Lemma 4.4.2 : Let $(X \lim)$ be an SL-preconvergence space. We define the axiom

L2 $\qquad \forall\, \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \quad \mathcal{F} \leq \mathcal{G} \Rightarrow \lim \mathcal{F} \leq \lim \mathcal{G}.$

Then

lim satisfies **Lp** $\Rightarrow$ lim satisfies **L2**.

Proof:
Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \mathcal{F} \leq \mathcal{G}, x \in X$. Then

$$\lim \mathcal{F}(x) \overset{\mathbf{Lp}}{=} \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a)) \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{G}(a)) \overset{\mathbf{Lp}}{=} \lim \mathcal{G}(x).$$

■

Lemma 4.4.3 : Let $(X, \lim)$ be an SL-preconvergence space. Then the following conditions are necessary and sufficient so that $\forall x \in X, \quad \mathcal{U}^x \in \mathcal{F}_L^S(X)$, when $\mathcal{U}^x$ is defined by **L0**:

LB $\forall x \in X, \quad \mathcal{U}^x(\bot_X) = \bot$.

L⊗ $\forall x \in X, \ \forall a, b \in L^X, \quad \mathcal{U}^x(a) \otimes \mathcal{U}^x(b) \leq \mathcal{U}^x(a \otimes b)$.

Proof:
If $\forall x \in X, \quad \mathcal{U}^x \in \mathcal{F}_L^S(X)$, then **LB** and **L⊗** are automatically true. We prove that together they imply the result. Assume **LB** and **L⊗**. Let $x \in X$.

F1 $\mathcal{U}^x(\top_X) \overset{\mathbf{L0}}{=} \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(\top_X)) = \top$. $\mathcal{U}^x(\bot_X) \overset{\mathbf{LB}}{=} \bot$.

F2 Let $a, b \in L^X, a \leq b$. Then

$$\mathcal{U}^x(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \overset{\mathbf{F2}}{\leq} \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(b)) = \mathcal{U}^x(b).$$

F3 This is simply the **L⊗** axiom.

Fs Let $\alpha \in L, a \in L^X$. Then by Lemma 3.1.18,

$$\begin{aligned}
\alpha * \mathcal{U}^x(a) &= \alpha * \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \\
&\leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\alpha * (\lim \mathcal{F}(x) \to \mathcal{F}(a))) \\
&\leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to (\alpha * \mathcal{F}(a))) \\
&\leq \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(\alpha_X * a)) = \mathcal{U}^x(\alpha_X * a).
\end{aligned}$$

■

Remark 4.4.4 : Note that if an SL-preconvergence space $(X, \lim)$ satisfies **L1**, then it automatically satisfies **LB**, since for $x \in X$,
$\mathcal{U}^x(\bot_X) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(\bot_X)) \leq \lim[x](x) \to \bot = \top \to \bot = \bot$.

Lemma 4.4.5 : Let $(X, \lim)$ be an SL-preconvergence space satisfying **L1**, **L**$\otimes$ and **Lp**. Then

$$\forall\, x \in X, \quad \lim\mathcal{U}^x(x) = \top.$$

Proof:
Let $x \in X$. **LB** and **L**$\otimes$ are satisfied, thus by Lemma 4.4.3, $\mathcal{U}^x \in \mathcal{F}_L^S(X)$, so we can calculate $\lim\mathcal{U}^x$. Now

$$\lim\mathcal{U}^x(x) \overset{\mathbf{Lp}}{=} \bigwedge_{a\in L^X} (\mathcal{U}^x(a) \to \mathcal{U}^x(a)) = \top.$$

∎

Definition 4.4.6 [10, 23]: Let $(X, \lim)$ be an SL-preconvergence space. We define the *stratified α-level L-neighbourhood filter at $x \in X$, $\mathcal{U}_\alpha^x$*, by

Lα $$\mathcal{U}_\alpha^x = \bigwedge\{\, \mathcal{F} \in \mathcal{F}_L^S(X) \mid \alpha \le \lim\mathcal{F}(x)\,\}.$$

Lemma 4.4.7 : Let $(X, \lim)$ be an SL-preconvergence space. Then

$$\lim \text{ satisfies } \mathbf{Lp} \Rightarrow\ \forall\, \alpha \in L\ \forall\, x \in X, \quad \lim\mathcal{U}_\alpha^x(x) \ge \alpha.$$

Proof:
Assume **Lp**. Let $x \in X, \alpha \in L$. We have:

$$\begin{aligned}\lim\mathcal{U}_\alpha^x(x) &\overset{\mathbf{Lp}}{=} \bigwedge_{a\in L^X} (\mathcal{U}^x(a) \to \mathcal{U}_\alpha^x(a)) = \bigwedge_{a\in L^X} (\mathcal{U}^x(a) \to (\bigwedge_{\alpha\le\lim\mathcal{F}(x)} \mathcal{F}(a))) \\ &= \bigwedge_{a\in L^X} \bigwedge_{\alpha\le\lim\mathcal{F}(x)} (\mathcal{U}^x(a) \to \mathcal{F}(a)) = \bigwedge_{\alpha\le\lim\mathcal{F}(x)} \bigwedge_{a\in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a)) \\ &= \bigwedge_{\alpha\le\lim\mathcal{F}(x)} \lim\mathcal{F}(x) \ge \alpha.\end{aligned}$$

∎

Lemma 4.4.8 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Define $\mathcal{U}^x(a)$ via **L0** (see Lemma 4.2.11) and $\mathcal{U}_\alpha^x$ via **L**α. Then

$$\forall\, a \in L^X\ \forall\, x \in X, \quad \mathcal{U}^x(a) \ge \bigwedge_{\alpha\in L} (\alpha \to \mathcal{U}_\alpha^x(a)).$$

If $\lim$ satisfies **Lp** then equality holds.

Proof:
We begin by noting that $\alpha \leq \lim \mathcal{F}(x) \Rightarrow \mathcal{U}^x_\alpha \leq \mathcal{F}$, for $\alpha \in L, x \in X$, $\mathcal{F} \in \mathcal{F}^S_L(X)$. Let $\alpha_{\mathcal{F}} = \lim \mathcal{F}(x)$. Then $\mathcal{U}^x_{\alpha_{\mathcal{F}}} \leq \mathcal{F}$. Let $a \in L^X$. From **L0** and Lemma 3.1.18 we have:

$$\begin{aligned}\mathcal{U}^x(a) &\overset{\mathbf{L0}}{=} \bigwedge_{\mathcal{F}\in\mathcal{F}^S_L(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \\ &\geq \bigwedge_{\mathcal{F}\in\mathcal{F}^S_L(X)} (\alpha_{\mathcal{F}} \to \mathcal{U}^x_{\alpha_{\mathcal{F}}}(a)) \geq \bigwedge_{\alpha\in L} (\alpha \to \mathcal{U}^x_\alpha(a)).\end{aligned}$$

Now assume **Lp**. By Lemmas 3.1.18 and 4.4.7 we have

$$\begin{aligned}\mathcal{U}^x(a) &\overset{\mathbf{L0}}{\leq} \lim \mathcal{U}^x_\alpha(x) \to \mathcal{U}^x_\alpha(a), \quad \forall\, \alpha \in L \\ &\leq \alpha \to \mathcal{U}^x_\alpha(a), \quad \forall\, \alpha \in L.\end{aligned}$$

Thus $\mathcal{U}^x(a) \leq \bigwedge_{\alpha\in L}(\alpha \to \mathcal{U}^x_\alpha(a))$. ■

Lemma 4.4.9 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Then

$$\forall\, \mathcal{F} \in \mathcal{F}^S_L(X)\ \forall\, x \in X, \quad \lim \mathcal{F}(x) \leq \bigvee \{\, \alpha \in L \mid \mathcal{U}^x_\alpha \leq \mathcal{F}\,\}.$$

If lim satisfies **Lp** then equality holds.

Proof:
We have $\alpha \leq \lim \mathcal{F}(x) \Rightarrow \mathcal{U}^x_\alpha \leq \mathcal{F}$. Thus $\lim \mathcal{F}(x) \in \{\, \alpha \in L \mid \mathcal{U}^x_\alpha \leq \mathcal{F}\,\}$ and the result follows. Now assume **Lp**. Let $\mathcal{F} \in \mathcal{F}^S_L(X), \alpha \in L, \mathcal{U}^x_\alpha \leq \mathcal{F}$. Then by **L2** and Lemma 4.4.7, $\alpha \leq \lim \mathcal{U}^x_\alpha(x) \leq \lim \mathcal{F}(x)$. Thus $\bigvee \{\, \alpha \in L \mid \mathcal{U}^x_\alpha \leq \mathcal{F}\,\} \leq \lim \mathcal{F}(x)$. ■

Lemma 4.4.10 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. We define the axiom

LpW2 $\quad \forall\, \mathcal{F} \in \mathcal{F}^S_L(X)\ \forall\, x \in X, \quad \lim \mathcal{F}(x) = \bigvee \{\, \alpha \in L \mid \mathcal{U}^x_\alpha \leq \mathcal{F}\,\}.$

Then if lim satisfies **L2**, the following are equivalent:

1. lim satisfies **LpW2**.
2. $\forall\, \alpha \in L\ \forall\, \mathcal{F} \in \mathcal{F}^S_L(X)\ \forall\, x \in X, \quad \alpha \leq \lim \mathcal{F}(x) \Leftrightarrow \mathcal{U}^x_\alpha \leq \mathcal{F}$.
3. $\forall\, \alpha \in L\ \forall\, x \in X, \quad \alpha \leq \lim \mathcal{U}^x_\alpha(x)$.

4. $\forall\, (\mathcal{F}_i)_{i\in I} \in \mathcal{F}_L^S(X)^I, \quad \lim(\bigwedge_{i\in I} \mathcal{F}_i) = \bigwedge_{i\in I} \lim \mathcal{F}_i$.

Proof:

(1 ⇒ 2) It is always true that $\alpha \le \lim \mathcal{F}(x) \Rightarrow \mathcal{U}_\alpha^x \le \mathcal{F}$. Conversely assume $\mathcal{U}_\alpha^x \le \mathcal{F}$. Then $\alpha \in \{\, \beta \in L \mid \mathcal{U}_\beta^x \le \mathcal{F} \,\}$, thus by (1), $\alpha \le \lim \mathcal{F}(x)$.

(2 ⇒ 3) For $\alpha \in L, x \in X$, we have $\mathcal{U}_\alpha^x \le \mathcal{U}_\alpha^x$. Hence by (2), $\alpha \le \lim \mathcal{U}_\alpha^x(x)$.

(3 ⇒ 4) Let $x \in X, (\mathcal{F}_i)_{i\in I} \in \mathcal{F}_L^S(X)^I$. It is always true (by **L2**) that $\lim \bigwedge_{i\in I} \mathcal{F}_i(x) \le \bigwedge_{i\in I} \lim \mathcal{F}_i(x)$. Let $\alpha = \bigwedge_{i\in I} \lim \mathcal{F}_i(x)$. Then $\alpha \le \lim \mathcal{U}_\alpha^x(x) = \lim \bigwedge_{\alpha \le \lim \mathcal{G}(x)} \mathcal{G}(x) \le \lim \bigwedge_{i\in I} \mathcal{F}_i(x)$.

(4 ⇒ 1) It is always true (see Lemma 4.4.9), that $\lim \mathcal{F}(x) \le \bigvee\{\, \alpha \in L \mid \mathcal{U}_\alpha^x \le \mathcal{F} \,\}$. Let $\mathcal{U}_\alpha^x \le \mathcal{F}$. Then by **L2**,

$$\lim \mathcal{F}(x) \ge \lim \mathcal{U}_\alpha^x(x) = \lim(\bigwedge_{\alpha \le \lim \mathcal{G}(x)} \mathcal{G})(x) \overset{4}{=} \bigwedge_{\alpha \le \lim \mathcal{G}(x)} \lim \mathcal{G}(x) \ge \alpha.$$

Thus $\lim \mathcal{F}(x) \ge \bigvee\{\, \alpha \in L \mid \mathcal{U}_\alpha^x \le \mathcal{F} \,\}$. ■

Lemma 4.4.11 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. We define the axiom

LpW2' $\quad \forall\, (\mathcal{F}_i)_{i\in I} \in \mathcal{F}_L^S(X)^I, \quad \lim(\bigwedge_{i\in I} \mathcal{F}_i) = \bigwedge_{i\in I} \lim \mathcal{F}_i.$

Then

lim satisfies **LpW2** ⇔ lim satisfies **LpW2'**.

Proof:
We prove that **LpW2** ⇒ **L2**. Let $x \in X, \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \mathcal{F} \le \mathcal{G}$. Then $\lim \mathcal{F}(x) = \bigvee\{\, \alpha \in L \mid \mathcal{U}_\alpha^x \le \mathcal{F} \,\} \le \bigvee\{\, \alpha \in L \mid \mathcal{U}_\alpha^x \le \mathcal{G} \,\} = \lim \mathcal{G}(x)$. Hence by Lemma 4.4.10, **LpW2** ⇒ **LpW2'**. For the converse we simply need to prove that **LpW2'** ⇒ **L2**. Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$, $\mathcal{F} \le \mathcal{G}$. Then $\mathcal{F} \wedge \mathcal{G} = \mathcal{F}$ and $\lim \mathcal{F} = \lim \mathcal{F} \wedge \mathcal{G} \overset{\textbf{LpW2'}}{=} \lim \mathcal{F} \wedge \lim \mathcal{G} \le \lim \mathcal{G}$. Thus **LpW2'** ⇒ **L2**. Then by Lemma 4.4.10, **LpW2'** ⇒ **LpW2**. ■

We recognize the form of **LpW2'** in Lemma 4.4.11 as being the same as that of the classical **LpW2** axiom, which turned out to be equivalent to the classical **Lp** axiom. It would be quite nice if the generalized **LpW2** axiom also turned out to be equivalent to the generalized **Lp** axiom, since if so, we have achieved our goal of translating the **Lp** axiom into a simple condition on the lim function. Unfortunately this generalization does not hold (shown for the frame case in [23]), as the following chain of proofs shows.

Lemma 4.4.12 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Then

$$\lim \text{ satisfies } \mathbf{Lp} \Leftrightarrow \lim \text{ satisfies } \mathbf{Lp'},$$

where **Lp'** is given by:

Lp' $\forall\, \alpha \in L\; \forall\, x \in X$,

$$\{\, \mathcal{F} \in \mathcal{F}_L^S(X) \mid \alpha \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{F}(a)) \,\} = \{\, \mathcal{F} \in \mathcal{F}_L^S(X) \mid \alpha \leq \lim \mathcal{F}(x) \,\}.$$

Proof:
Lp' follows trivially from **Lp**. We prove the converse. Assume **Lp'**. Let $\alpha = \lim \mathcal{G}(x)$ for $\mathcal{G} \in \mathcal{F}_L^S(X), x \in X$. Then

$$\mathcal{G} \in \{\, \mathcal{F} \in \mathcal{F}_L^S(X) \mid \lim \mathcal{G}(x) \leq \lim \mathcal{F}(x) \,\} = \{\, \mathcal{F} \in \mathcal{F}_L^S(X) \mid \lim \mathcal{G}(x) \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{F}(a)) \,\}.$$

Thus $\lim \mathcal{G}(x) \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{G}(a))$. Now let $\alpha = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{G}(a))$. Similarly it follows that $\bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{G}(a)) \leq \lim \mathcal{G}(x)$. ■

If we weaken **Lp'**, we obtain the axiom **LpW1**.

Definition 4.4.13 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. We say that lim satisfies the axiom **LpW1** iff

LpW1 $\forall\, \alpha \in L\; \forall\, x \in X$,

$$\bigwedge \{\, \mathcal{F} \in \mathcal{F}_L^S(X) \mid \alpha \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{F}(a)) \,\} = \bigwedge \{\, \mathcal{F} \in \mathcal{F}_L^S(X) \mid \alpha \leq \lim \mathcal{F}(x) \,\}.$$

Remark 4.4.14 : Let $\alpha \in L, \mathcal{F} \in \mathcal{F}_L^S(X)$. Then

$$\alpha \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a)) \Leftrightarrow \forall\, a \in L^X, \quad \alpha \leq \mathcal{U}^x(a) \to \mathcal{F}(a)$$
$$\Leftrightarrow \forall\, a \in L^X, \quad \alpha * \mathcal{U}^x(a) \leq \mathcal{F}(a).$$

If we define $[\alpha * \mathcal{U}^x] = \bigwedge\{ \mathcal{F} \in \mathcal{F}_L^S(X) \mid \forall\, a \in L^X, \quad \alpha * \mathcal{U}^x(a) \leq \mathcal{F}(a) \}$, then we can state **LpW1** succinctly as

$$\forall\, \alpha \in L\ \forall\, x \in X, \quad [\alpha * \mathcal{U}^x] = \mathcal{U}^x_\alpha.$$

Lemma 4.4.15 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Then

lim satisfies **Lp** $\Leftrightarrow$ lim satisfies **LpW1** and **LpW2**.

Proof:
Lp obviously implies **LpW1**. We know from Lemma 4.4.9 that **Lp** implies **LpW2**. Assume now that lim satisfies **LpW1** and **LpW2**. Let $x \in X, \mathcal{F} \in \mathcal{F}_L^S(X)$. Then

$$\bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a)) = \bigwedge_{a \in L^X} \Big(\bigwedge_{\mathcal{G} \in \mathcal{F}_L^S(X)} (\lim \mathcal{G}(x) \to \mathcal{G}(a)) \to \mathcal{F}(a)\Big)$$
$$\geq \bigwedge_{a \in L^X} ((\lim \mathcal{F}(x) \to \mathcal{F}(a)) \to \mathcal{F}(a))$$
$$\geq \lim \mathcal{F}(x).$$

Now let $\alpha = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a))$. Then we have $\forall\, a \in L^X \quad \alpha \leq \mathcal{U}^x(a) \to \mathcal{F}(a)$, thus $\forall\, a \in L^X \quad \alpha * \mathcal{U}^x(a) \leq \mathcal{F}(a)$. Now $[\alpha * \mathcal{U}^x] \leq \mathcal{F}$ from the definition of $[\alpha * \mathcal{U}^x]$, and by **LpW1**, $[\alpha * \mathcal{U}^x] = \mathcal{U}^x_\alpha$, so $\mathcal{U}^x_\alpha \leq \mathcal{F}$. Since **LpW2** $\Rightarrow$ **L2**, we have

$$\lim \mathcal{F}(x) \overset{\mathbf{L2}}{\geq} \lim \mathcal{U}^x_\alpha(x) \overset{\mathbf{LpW2}}{\geq} \alpha = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a)).$$

Thus **Lp** is true. ∎

The following example shows that in an SL-preconvergence space $(X, \lim)$ satisfying axioms **L1**, **L2** and **L**$\otimes$, the **LpW1** axiom does not imply the **LpW2** axiom.

Example 4.4.16 [23]: Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), |X| > 1$. Define $\lim\colon \mathcal{F}_L^S(X) \to L^X$ for $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X$ by

$$\lim \mathcal{F}(x) = \begin{cases} \bot & \mathcal{F} = \mathcal{F}_0 \\ \top & \mathcal{F} \neq \mathcal{F}_0 \end{cases}.$$

where $\mathcal{F}_0 = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} \mathcal{F} = \bigwedge_{\mathcal{F} \neq \mathcal{F}_0} \mathcal{F}$ as in Example 3.3.4. Then it is easy to see that lim satisfies **L1** and **L2**. We show that lim does not satisfy **LpW2**:

$$\bigwedge_{\mathcal{F} \neq \mathcal{F}_0} \lim \mathcal{F}(x) = \top \neq \bot = \lim \bigwedge_{\mathcal{F} \neq \mathcal{F}_0} (x) = \lim \mathcal{F}_0(x).$$

Thus by Lemma 4.4.10, **LpW2** is not satisfied.

Now we show that lim does satisfy **LpW1**. Let $a \in L^X$. Then

$$\begin{aligned} \mathcal{U}^x(a) &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \\ &= (\bigwedge_{\mathcal{F} \neq \mathcal{F}_0} (\top \to \mathcal{F}(a))) \wedge (\bot \to \mathcal{F}_0(a)) \\ &= (\bigwedge_{\mathcal{F} \neq \mathcal{F}_0} \mathcal{F}(a)) \wedge \mathcal{F}_0(a) = \mathcal{F}_0(a). \end{aligned}$$

We also have for $\alpha \in L$ that

$$\mathcal{U}^x_\alpha = \left\{ \begin{matrix} \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} \mathcal{F} & , \alpha = \bot \\ \bigwedge_{\mathcal{F} \neq \mathcal{F}_0} \mathcal{F} & , \alpha > \bot \end{matrix} \right\} = \mathcal{F}_0.$$

Thus $\forall\, \alpha \in L\, \forall\, a \in L^X,\quad \alpha * \mathcal{U}^x(a) \leq \alpha \wedge \mathcal{U}^x(a) \leq \mathcal{U}^x(a) = \mathcal{U}^x_\alpha(a)$, so $[\alpha * \mathcal{U}^x] = \mathcal{U}^x_\alpha$. So lim satisfies **LpW1**.

The following example shows that in an SL-preconvergence space $(X, \lim)$ satisfying axioms **L1**, **L2** and **L⊗**, the **LpW2** axiom does not imply the **LpW1** axiom.

Example 4.4.17 [23]: Let $L = \{\bot, \alpha, \top\}$ with $\bot < \alpha < \top$ and let $X = \{x, y\}$. For $z \in X, \mathcal{F} \in \mathcal{F}_L^S(X)$ we define the *discrete stratified L-generalized convergence on X* by

$$\lim \mathcal{F}(z) = \begin{cases} \top & \mathcal{F} \geq [z] \\ \bot & \text{otherwise} \end{cases}.$$

Then lim satisfies **L1** and **L2**. We show that lim satisfies **LpW2**. Let $(\mathcal{F}_i)_{i \in I} \in \mathcal{F}_L^S(X)^I$ with $\bigwedge_{i \in I} \lim \mathcal{F}_i(z) = \top$. Then by definition, $\forall\, i \in I,\quad \mathcal{F}_i \geq [z]$. Thus $\bigwedge_{i \in I} \mathcal{F}_i \geq [z]$. Finally

$$\lim \bigwedge_{i \in I} \mathcal{F}_i(z) = \top = \bigwedge_{i \in I} \lim \mathcal{F}_i(z).$$

Now we show that lim does not satisfy **LpW1**. For $z \in X, a \in L^X$ we calculate

$$\begin{aligned} \mathcal{U}^z(a) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim \mathcal{F}(z) \to \mathcal{F}(a)) \\ &= (\bigwedge_{\mathcal{F}\geq[z]} (\top \to \mathcal{F}(a))) \wedge (\bigwedge_{\mathcal{F}\ngeq[z]} (\bot \to \mathcal{F}(a))) \\ &= (\bigwedge_{\mathcal{F}\geq[z]} \mathcal{F}(a)) \wedge \top = [z](a). \end{aligned}$$

Thus $\mathcal{U}^z = [z]$. We define

$$\mathcal{G}\colon L^X \to L \quad a \mapsto \begin{cases} \top & a = \top_X \\ \alpha & a(x) = \top \text{ and } a(y) \neq \top \\ \alpha & a(x) = \alpha \\ \bot & a(x) = \bot \end{cases}.$$

Then it is easy to show that $\mathcal{G} \in \mathcal{F}_L^S(X)$. Let $a \in L^X$ be defined by $a(x) = \top, a(y) = \alpha$. Then $[x](a) = \top > \alpha = \mathcal{G}(a)$. Thus $\mathcal{G} \ngeq [x]$, thus $\lim \mathcal{G}(x) = \bot$. But

$$\begin{aligned} \bigwedge_{b\in L^X} (\mathcal{U}^x(b) \to \mathcal{G}(b)) &= \bigwedge_{b\in L^X} (b(x) \to \mathcal{G}(b)) \\ &= (\top \to \top) \wedge (\top \to \alpha) \wedge (\alpha \to \alpha) \wedge (\bot \to \bot) = \alpha \neq \bot = \lim \mathcal{G}(x). \end{aligned}$$

Thus lim does not satisfy **Lp**. Hence by Lemma 4.4.15, lim does not satisfy **LpW1**.

Remark 4.4.18 : By Example 4.4.17, we have that even under the assumption that **L1**, **L2** and **L**$\otimes$ are satisfied, **Lp** $\not\Leftrightarrow$ **LpW2**. However we still have the hope that perhaps by assuming other axioms (e.g. **LK**) in addition to **L1** and **L2**, **Lp** will prove to be equivalent to **LpW2**. However (see Remark 4.4.29), this is not true for **LK** or **LF**. This is still an open question as to whether a formulation can be found to avoid the **LpW1** axiom.

Lemma 4.4.19 [23]: Let $(L, \leq, *, \otimes) = (\{0,1\}, \leq, \wedge, \wedge)$. Let $(X, \lim)$ be an SL-preconvergence space. Then the **LpW1** axiom is always satisfied.

Proof:
Let $x \in X$. We calculate

$$[0 \wedge \mathcal{U}^x] = \bigwedge \{ \mathcal{F} \in \mathcal{F}_L^S(X) \mid \ \forall a \in L^X, \quad 0 \wedge \mathcal{U}^x(a) \leq \mathcal{F}(a) \}$$

$$= \bigwedge\{\mathcal{F} \in \mathcal{F}_L^S(X)\} = \bigwedge_{0 \leq \lim \mathcal{F}(x)} \mathcal{F} = \mathcal{U}_0^x.$$

$$\begin{aligned}[1 \wedge \mathcal{U}^x] &= \bigwedge\{\, \mathcal{F} \in \mathcal{F}_L^S(X) \mid \ \forall\, a \in L^X, \quad 1 \wedge \mathcal{U}^x(a) \leq \mathcal{F}(a)\,\} \\ &= \bigwedge\{\, \mathcal{F} \in \mathcal{F}_L^S(X) \mid \ \forall\, a \in L^X, \quad \mathcal{U}^x(a) \leq \mathcal{F}(a)\,\} = \mathcal{U}^x.\end{aligned}$$

Let $a \in L^X$, then

$$\begin{aligned}\mathcal{U}^x(a) &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \\ &= \bigwedge_{\substack{\mathcal{F} \in \mathcal{F}_L^S(X) \\ \lim \mathcal{F}(x)=0}} (0 \to \mathcal{F}(a)) \wedge \bigwedge_{\substack{\mathcal{F} \in \mathcal{F}_L^S(X) \\ \lim \mathcal{F}(x)=1}} (1 \to \mathcal{F}(a)) \\ &= 1 \wedge \mathcal{U}_1^x(a) = \mathcal{U}_1^x(a).\end{aligned}$$

Therefore

$$[1 \wedge \mathcal{U}^x] = \mathcal{U}_1^x.$$

Thus in the case $L = \{0, 1\}$ it is true that $\forall\, \alpha \in L, \quad [\alpha * \mathcal{U}^x] = \mathcal{U}_\alpha^x$. ∎

Remark 4.4.20 : Lemma 4.4.19 shows that in the classical case the **Lp** axiom is equivalent to **LpW2**, as was shown in Chapter 2.

The Kowalski axiom

Definition 4.4.21 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Then lim satisfies the Kowalski axiom ⇔

LK $\quad \forall\, \mathcal{G} \in \mathcal{F}_L^S(X)\ \forall\, x \in X\ \forall\, (\mathcal{F}_y)_{y \in X} \in \mathcal{F}_L^S(X)^X$

$$\lim \mathcal{G}(x) * \bigwedge_{y \in X} \lim \mathcal{F}_y(y) \leq \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

The **LK** axiom generalizes the **K** axiom in the following way. Let $L = \{0, 1\}$ and let $(X, \lim)$ be an SL-preconvergence space. With each $A \subseteq \mathcal{P}(X)$ we associate $A_L \in L^X$ via

$$x \in A \Leftrightarrow A_L(x) = 1$$

With each $\mathcal{A} \subseteq \mathcal{P}(X)$ we associate $\mathcal{A}_L \colon L^X \to L$ via

$$A \in \mathcal{A} \Leftrightarrow \mathcal{A}_L(A_L) = 1.$$

In this way we identify the sets $\mathcal{F}_L^S(X)$ and $\mathcal{F}(X)$.

We define the limit function $\overline{\lim}\colon \mathcal{F}(X) \to \mathcal{P}(X)$ by

$$x \in \overline{\lim}\,\mathcal{F} \Leftrightarrow \lim \mathcal{F}_L(x) = 1.$$

Then

$$\lim \text{ satisfies the } \mathbf{LK} \text{ axiom} \Leftrightarrow \overline{\lim} \text{ satisfies the } \mathbf{K} \text{ axiom}.$$

Lemma 4.4.22 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Then

$$\lim \text{ satisfies } \mathbf{Lt} \Leftrightarrow \ \forall\, x \in X \quad \mathcal{U}^x \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)}).$$

Proof:
Assume **Lt**. Let $x \in X, a \in L^X$. Then by **Lt**, $\mathcal{U}^x(a) \leq \bigvee\{\, \mathcal{U}^x(b) \mid \ \forall\, y \in X, \quad b(y) \leq \mathcal{U}^y(a)\,\}$. Let $b \in L^X$. Now $\forall\, y \in X, \quad b(y) \leq \mathcal{U}^y(a) \Leftrightarrow b \leq \mathcal{U}^{(\cdot)}(a) \Rightarrow \mathcal{U}^x(b) \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)}(a))$. So by **Lt**, $\mathcal{U}^x(a) \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)})(a)$. For the converse, assume **Lt'**. Let $x \in X, a \in L^X$. Then $\mathcal{U}^{(\cdot)}(a) \in \{\, b \in L^X \mid b \leq \mathcal{U}^{(\cdot)}(a)\,\}$, thus $\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \in \{\, \mathcal{U}^x(b) \mid b \leq \mathcal{U}^{(\cdot)}(a)\,\}$. Finally $\mathcal{U}^x(a) \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)})(a) \leq \bigvee\{\, \mathcal{U}^x(b) \mid \ \forall\, y \in X, \quad b(y) \leq \mathcal{U}^y(a)\,\}$. ■

Lemma 4.4.23 (see [23] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space satisfying **L1**, **Lp** and **L⊗**. Then

$$\lim \text{ satisfies } \mathbf{Lt} \Leftrightarrow \lim \text{ satisfies } \mathbf{LK}.$$

Proof:
Assume **LK**. By Lemma 4.4.3, $\forall\, x \in X, \quad \mathcal{U}^x \in \mathcal{F}_L^S(X)$. Thus we can calculate $\lim \mathcal{U}^x(x) = \top$ for each $x \in X$. By **LK** we have

$$\top = \top * \top = \lim \mathcal{U}^x(x) * \bigwedge_{y \in X} \lim \mathcal{U}^y(y) \leq \lim(\mathcal{U}^x(\mathcal{U}^{(\cdot)}))(x).$$

By **Lp**, $\top = \lim \mathcal{U}^x(\mathcal{U}^{(\cdot)})(x) = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{U}^x(\mathcal{U}^{(\cdot)})(a))$, thus $\forall\, a \in L^X, \quad \mathcal{U}^x(a) \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)})(a)$ and we have proved that **LK** $\Rightarrow$ **Lt**.

Now assume **Lt**. Let $x \in X, \mathcal{G} \in \mathcal{F}_L^S(X), (\mathcal{F}_y)_{y \in X} \in \mathcal{F}_L^S(X)^X$. Let $\beta = \bigwedge_{y \in X} \lim \mathcal{F}_y(y)$. Then

$$\forall\, y \in X \quad \beta \leq \lim \mathcal{F}_y(y) \overset{\mathbf{Lp}}{=} \bigwedge_{a \in L^X} (\mathcal{U}^y(a) \to \mathcal{F}_y(a))$$

$$\begin{aligned}
&\text{so} \quad \forall\, y \in X\ \forall\, a \in L^X \quad \beta \le \mathcal{U}^y(a) \to \mathcal{F}_y(a)\\
&\text{thus} \quad \forall\, y \in X\ \forall\, a \in L^X \quad \beta * \mathcal{U}^y(a) \le \mathcal{F}_y(a)\\
&\text{therefore} \quad \forall\, a \in L^X \quad \beta_X * \mathcal{U}^{(\cdot)}(a) \le \mathcal{F}_{(\cdot)}(a)\\
&\text{finally} \quad \forall\, a \in L^X \quad \mathcal{G}(\mathcal{F}_{(\cdot)})(a) \ge \mathcal{G}(\beta_X * \mathcal{U}^{(\cdot)}(a)) \ge \beta * \mathcal{G}(\mathcal{U}^{(\cdot)})(a).
\end{aligned}$$

Hence

$$\begin{aligned}
\lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) &= \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{G}(\mathcal{F}_{(\cdot)})(a))\\
&\ge \bigwedge_{a \in L^X} (\mathcal{U}^x(\mathcal{U}^{(\cdot)})(a) \to (\beta * \mathcal{G}(\mathcal{U}^{(\cdot)})(a)))\\
&\ge \beta * \bigwedge_{a \in L^X} (\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \to \mathcal{G}(\mathcal{U}^{(\cdot)}(a)))\\
&\ge \beta * \bigwedge_{b \in L^X} (\mathcal{U}^x(b) \to \mathcal{G}(b))\\
&= \lim \mathcal{G}(x) * \bigwedge_{y \in X} \lim \mathcal{F}_y(y).
\end{aligned}$$

Thus we have proved that **Lt** $\Rightarrow$ **LK**. ■

We have proved with Lemma 4.4.23 that for SL-topological convergence spaces we may replace the **Lt** axiom with the **LK** axiom. However all of **L1**, **Lp** and **L**$\otimes$ are involved in the proof of Lemma 4.4.23, so we have no guarantee that we can leave any of them out and still replace the **Lt** axiom.

The Fischer axiom

Definition 4.4.24 (see [24] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Then lim satisfies the Fischer axiom $\Leftrightarrow$

LF $\quad \forall\, \emptyset \ne J \in \mathrm{Ob}\,(\underline{\mathbf{SET}})\ \forall\, \phi\colon J \to X\ \forall\, \mathcal{G} \in \mathcal{F}^S_L(J)$

$$\forall\, x \in X\ \forall\, (\mathcal{F}_j)_{j \in J} \in \mathcal{F}^S_L(X)^J$$

$$\lim \phi(\mathcal{G})(x) * \bigwedge_{j \in J} \lim \mathcal{F}_j(\phi(j)) \le \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

Lemma 4.4.25 (see [24] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Then

1. lim satisfies **LF** $\Rightarrow$lim satisfies **LK**.
2. lim satisfies **L1**, **L2** and **LF** $\Rightarrow$ lim satisfies **LpW2**.

Proof:
Let lim satisfy **LF**.

1. Let $J = X, \phi = \mathrm{id}_X$. Then **LK** follows immediately.
2. Assume **L1**. Let $x \in X$. Define $\phi\colon J \to X \quad j \mapsto x$. Let $\mathcal{G} = [J]$. Then for $a \in L^X$,

$$\phi(\mathcal{G})(a) = \mathcal{G}(\phi^{\leftarrow}(a)) = \bigwedge_{j \in J}([j]\phi^{\leftarrow}(a)) = a(x) = [x](a).$$

So $\phi(\mathcal{G}) = [x]$. Now for $b \in L^J$,

$$\mathcal{G}(\mathcal{F}_{(\cdot)})(b) = \bigwedge_{j \in J}[j](\mathcal{F}_{(\cdot)}(b)) = \bigwedge_{j \in J}\mathcal{F}_j(b).$$

So $\mathcal{G}(\mathcal{F}_{(\cdot)}) = \bigwedge_{j \in J}\mathcal{F}_j$. Finally we have

$$\lim[x](x) * \bigwedge_{j \in J}\lim\mathcal{F}_j(x) \overset{\mathbf{L1}}{=} \bigwedge_{j \in J}\lim\mathcal{F}_j(x) \leq \lim(\bigwedge_{j \in J}\mathcal{F}_j)(x).$$

From **L2**, $\lim(\bigwedge_{j \in J}\mathcal{F}_j)(x) \leq \bigwedge_{j \in J}\lim\mathcal{F}_j(x)$. Thus **LpW2** is satisfied.

■

The following example shows that in an SL-preconvergence space $(X, \lim)$ satisfying axioms **L1**, **L2** and **L**$\otimes$, the **LK** axiom does not imply the **LF** axiom.

Example 4.4.26 [24]: Let $X \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$ be an infinite set. We define, for $x \in X, \mathcal{F} \in \mathcal{F}_L^S(X)$,

$$\lim\mathcal{F}(x) = \begin{cases} \top & \exists \text{ finite } A \in \mathrm{Ob}\,(\underline{\mathbf{SET}}) \text{ such that } \mathcal{F} \geq [A] \\ \bot & \text{otherwise} \end{cases}.$$

Then **L1** and **L2** follow immediately. We now show that **LpW2** is not satisfied. Let A be an infinite subset of X. Assume $\exists\, B \subseteq A$ finite such that $[B] \leq [A]$. Then

$$[B](\top_B) = \bigwedge_{x \in B}\top_B(x) = \top > \bot = \bigwedge_{y \in A}\top_B(y) = [A](\top_B).$$

This contradicts our assumption. Thus there exists no finite $B \subseteq A$ such that $[B] \leq [A]$. Thus for $x \in X$,

$$\bigwedge_{y \in A}\lim[y](x) = \top \neq \bot = \lim[A](x) = \lim\bigwedge_{y \in A}[y](x)$$

and we have that **LpW2** is not satisfied. Therefore by Lemma 4.4.25, **LF** is not satisfied by lim.

Now we prove that lim satisfies **LK**. Let $x \in X, \mathcal{G} \in \mathcal{F}_L^S(X), (\mathcal{F}_y)_{y\in X} \in \mathcal{F}_L^S(X)^X$. Assume that

$$\lim \mathcal{G}(x) * \bigwedge_{y\in X} \lim \mathcal{F}_y(y) = \lim \mathcal{G}(x) \wedge \bigwedge_{y\in X} \lim \mathcal{F}_y(y) = \top.$$

Then $\exists\, A \subseteq X, (B_y)_{y\in X} \in \mathcal{P}(X)^X$ such that A is finite and $\forall\, y \in X, \quad B_y$ is finite and $\mathcal{G} \geq [A]$ and $\forall\, y \in X, \quad \mathcal{F}_y \geq [B_y]$. Hence we have $\mathcal{G}(\mathcal{F}_{(\cdot)}) \geq [A]([B_{(\cdot)}])$. For $a \in L^X$ we have

$$\begin{aligned}[A]([B_{(\cdot)}])(a) = [A]([B_{(\cdot)}](a)) &= \bigwedge_{y\in A} ([B_{(\cdot)}](a)(y)) \\ &= \bigwedge_{y\in A} ([B_y](a)) = \bigwedge_{y\in A} (\bigwedge_{z\in B_y} a(z)) \\ &= \bigwedge_{z\in\bigcup_{y\in A} B_y} a(z) = [\bigcup_{y\in A} B_y](a).\end{aligned}$$

Now A is a finite set and so are all the B_ys, thus $\bigcup_{y\in A} B_y$ is a finite set and therefore $\lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) = \top$. Thus **LK** is satisfied.

Lemma 4.4.27 (see [24] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space satisfying **L1** and **L2**. Then

lim satisfies **LK** and **LpW2** $\Leftrightarrow$ lim satisfies **LF**.

Proof:
We have that **LF** $\Rightarrow$ **LpW2** and **LK** from Lemma 4.4.25. We prove the converse. Let $\emptyset \neq J \in \mathrm{Ob}\,(\underline{\mathbf{SET}}), \phi\colon J \to X, \mathcal{G} \in \mathcal{F}_L^S(J), (\mathcal{F}_j)_{j\in J} \in \mathcal{F}_L^S(X)^J$. We define $(\mathcal{H}_y)_{y\in X}$ by

$$\mathcal{H}_y = \begin{cases} \bigwedge_{j\in\phi^{\leftarrow}(y)} \mathcal{F}_j & , \quad y \in \phi(J) \\ [y] & , \quad y \in X\backslash\phi(J) \end{cases}.$$

Then $\phi(\mathcal{G})(\mathcal{H}_{(\cdot)})(a) = \mathcal{G}(\phi^{\leftarrow}(\mathcal{H}_{(\cdot)}(a)))$. Let $j \in J$. We calculate

$$\phi^{\leftarrow}(\mathcal{H}_{(\cdot)}(a))(j) = \mathcal{H}_{(\cdot)}(a)(\phi(j)) = \mathcal{H}_{\phi(j)}(a) \leq \mathcal{F}_j(a) = \mathcal{F}_{(\cdot)}(a)(j).$$

Thus $\phi^{\leftarrow}(\mathcal{H}_{(\cdot)}(a)) \leq \mathcal{F}_{(\cdot)}(a)$, so $\phi(\mathcal{G})(\mathcal{H}_{(\cdot)}) \leq \mathcal{G}(\mathcal{F}_{(\cdot)})$.

Now we calculate

$$\bigwedge_{y\in X} \lim \mathcal{H}_y(y) = \big(\bigwedge_{y\in\phi(J)} \lim \mathcal{H}_y(y)\big) \wedge \big(\bigwedge_{y\in X\backslash\phi(J)} \lim \mathcal{H}_y(y)\big)$$

$$= \Big(\bigwedge_{y \in \phi(J)} \lim \bigwedge_{j \in \phi^{\leftarrow}(y)} \mathcal{F}_j(y) \Big) \wedge \Big(\bigwedge_{y \in X \setminus \phi(J)} \lim [y](y) \Big)$$

$$\overset{\mathbf{LpW2}}{\underset{\mathbf{L1}}{=}} \Big(\bigwedge_{y \in \phi(J)} \bigwedge_{j \in \phi^{\leftarrow}(y)} \lim \mathcal{F}_j(\phi(j)) \Big) \wedge \top$$

$$= \bigwedge_{j \in J} \lim \mathcal{F}_j(\phi(j)).$$

Let $x \in X$. From **LK** we have

$$\lim \phi(\mathcal{G})(x) * \bigwedge_{y \in X} \lim \mathcal{H}_y(y) \leq \lim \phi(\mathcal{G})(\mathcal{H}_{(\cdot)})(x).$$

Thus by what we have just proved,

$$\lim \phi(\mathcal{G})(x) * \bigwedge_{j \in J} \lim \mathcal{F}_j(\phi(j)) \leq \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

so the **LF** axiom is satisfied. ■

The following example shows that in an SL-preconvergence space $(X, \lim)$ satisfying axioms **L1**, **L2** and **L**$\otimes$, the **LF** axiom does not imply the **LpW1** axiom.

Example 4.4.28 [24]: Let $L = \{\bot, \alpha, \top\}$ with $\bot < \alpha < \top$ and let $X = \{x, y\}$. As in Example 4.4.17, for $z \in X, \mathcal{F} \in \mathcal{F}_L^S(X)$ we define the *discrete stratified L-generalized convergence on X* by

$$\lim \mathcal{F}(z) = \begin{cases} \top & \mathcal{F} \geq [z] \\ \bot & \text{otherwise} \end{cases}.$$

Then we know that $(X, \lim)$ satisfies **LpW2** and not **LpW1**. We show that lim satisfies **LK**. Let $x \in X, \mathcal{G} \in \mathcal{F}_L^S(X), (\mathcal{F}_y)_{y \in X} \in \mathcal{F}_L^S(X)^X$. Assume that

$$\lim \mathcal{G}(x) \wedge \bigwedge_{y \in X} \lim \mathcal{F}_y(y) = \top.$$

Then $\mathcal{G} \geq [x]$ and $\forall\, y \in X, \quad \mathcal{F}_y \geq [y]$. Thus $\mathcal{G}(\mathcal{F}_{(\cdot)}) \geq [x]([\cdot])$. Now for $a \in L^X$

$$[x]([\cdot])(a) = [x]([\cdot](a)) = [\cdot](a)(x) = [x](a).$$

thus $\mathcal{G}(\mathcal{F}_{(\cdot)}) \geq [x]$, so $\lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) = \top$. Thus **LK** is satisfied. By Lemma 4.4.27, **LF** is satisfied.

Remark 4.4.29 : By Example 4.4.28and by Lemma 4.4.27, there exists an SL-preconvergence space which satisfies all of **L1**, **L2**, **LpW2**, **LK** but does not satisfy **LpW1** and hence does not satisfy **Lp**. Thus we *must* include the **LpW1** axiom in order to obtain an SL-topological convergence space.

Lemma 4.4.30 (see [24] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space satisfying **L1**, **L2** and **L**$\otimes$. Then

$$\lim \text{ satisfies } \mathbf{LF} \text{ and } \mathbf{LpW1} \Leftrightarrow \lim \text{ satisfies } \mathbf{Lt} \text{ and } \mathbf{Lp}.$$

Proof:
If lim satisfies **LF** and **LpW1**, then by Lemma 4.4.27, lim also satisfies **LpW2** and **LK**. Thus by Lemmas 4.4.15 and 4.4.23, lim satisfies **Lp** and **Lt**. Conversely, assume lim satisfies **Lt** and **Lp**. Then by Lemmas 4.4.15 and 4.4.23, lim satisfies **LK**, **LpW1**, and **LpW2**. Therefore by Lemma 4.4.27, lim satisfies **LF** as well. ■

Chapter 5

Stratified L-convergence spaces

From Chapter 2 we know that $\underline{\textbf{TCS}}$, isomorphic to the category $\underline{\textbf{TOP}}$, is contained as a full, reflective subcategory in $\underline{\textbf{CONV}}$, the category of classical convergence spaces. $\underline{\textbf{CONV}}$ is both Cartesian closed and topological over $\underline{\textbf{SET}}$. In [21], Jäger proved that for the case where L is a frame, the generalization of $\underline{\textbf{CONV}}$, $\underline{\textbf{SL}-\textbf{GCS}}$, is topological, Cartesian closed and contains $\underline{\textbf{SL}-\textbf{TCS}}$ as a full, reflective subcategory. In this chapter we look at extending $\underline{\textbf{SL}-\textbf{GCS}}$ and various of its subcategories to the case where L is an ecl-premonoid, in the hope that a similar situation will apply.

5.1 Generalized convergence spaces

The objects of the category $\underline{\textbf{CONV}}$ are defined (see e.g. [28]) as those classical preconvergence spaces which satisfy the classical **L1** and **L2** axioms. Similarly, in [21], objects of the category $\underline{\textbf{SL}-\textbf{GCS}}$ are defined as those L-preconvergence spaces which satisfy **L1** and **L2**, in the case where L is a frame. Our initial generalization is simply to extend this definition to the ecl premonoid case.

Definition 5.1.1 (see [21, 22] for the frame case): We define the category $\underline{\textbf{SL}-\textbf{GCS}}$ by

Objects stratified L-generalized convergence spaces $(X, \lim)$.

Morphisms Functions $\phi : X \to Y$ between spaces $(X, \lim_X)$ and $(Y, \lim_Y)$ which satisfy

$$\forall \mathcal{F} \in \mathcal{F}_L^S(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}).$$

The identity function id_X is a morphism from $(X, \lim)$ to itself. Morphism composition is the usual function composition.

Definition 5.1.2 : Let $(X, \lim), (X, \overline{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. We define

$$\lim \leq \overline{\lim} \Leftrightarrow \ \forall\, \mathcal{F} \in \mathcal{F}_L^S(X) \quad \lim \mathcal{F} \leq \overline{\lim}\, \mathcal{F}.$$

Remark 5.1.3 : Both $(X, \lim)$ and $(X, \overline{\lim})$ are members of the X-fibre of $\underline{\mathbf{SL-GCS}}$. In terms of the ordering on the fibre defined in Definition 1.3.5, we have

$$\begin{aligned}(X, \lim) \leq & (X, \overline{\lim}) \\ & \Leftrightarrow \mathrm{id}_X \in \hom_{\underline{\mathbf{SL-GCS}}}((X, \lim), (X, \overline{\lim})) \\ & \Leftrightarrow \ \forall\, \mathcal{F} \in \mathcal{F}_L^S(X) \lim \mathcal{F} = \mathrm{id}_X(\lim \mathcal{F}) \leq \overline{\lim}\, \mathrm{id}_X(\mathcal{F}) = \overline{\lim}\, \mathcal{F} \\ & \Leftrightarrow \lim \leq \overline{\lim}\,.\end{aligned}$$

Theorem 5.1.4 (see [21] for the frame case): $\underline{\mathbf{SL-GCS}}$ is topological over $\underline{\mathbf{SET}}$. Furthermore it is amnestic, fibre-small and has the terminal separator property.

Proof:
Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$ and let $((X_i, \lim_i))_{i \in I}$ be a family of SL-generalized convergence spaces indexed by the class I. Let $(\phi_i \colon X \to X_i)_{i \in I}$ be a corresponding family of functions. For $\mathcal{F} \in \mathcal{F}_L^S(X)$ we define

$$\lim_X \mathcal{F} = \bigwedge_{i \in I} \phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{F}))$$

We prove that $\lim_X$ satisfies **L1** and **L2**. Let $x \in X$. Then

$$\lim_X [x](x) = \bigwedge_{i \in I} \phi_i^{\leftarrow}(\lim_i \phi_i([x]))(x) = \bigwedge_{i \in I} \lim_i [\phi_i(x)](\phi_i(x)) \overset{\mathbf{L1}}{=} \top$$

Thus $\lim_X$ satisfies **L1**. Now let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \mathcal{F} \leq \mathcal{G}$. Then

$$\forall\, i \in I, \quad \phi_i(\mathcal{F}) \leq \phi_i(\mathcal{G})$$
$$\text{hence by } \mathbf{L2} \ \forall\, i \in I, \quad \lim_i \phi_i(\mathcal{F}) \leq \lim_i \phi_i(\mathcal{G})$$
$$\text{by Lemma 3.2.6 } \ \forall\, i \in I, \quad \phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{F})) \leq \phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{G}))$$
$$\text{finally } \lim_X \mathcal{F} = \bigwedge_{i \in I} \phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{F})) \leq \bigwedge_{i \in I} \phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{G})) = \lim_X \mathcal{G}.$$

Thus $\lim_X$ satisfies **L2**. We have proved that $(X, \lim_X)$ is an SL-generalized convergence space, i.e. $(X, \lim_X) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. Now from Lemma 3.2.6, $\forall\, a \in L^X, \quad \phi_i(\phi_i^{\leftarrow}(a)) \leq a$, thus

$$\forall\, i \in I, \ \forall\, \mathcal{F} \in \mathcal{F}_L^S(X), \quad \phi_i(\lim_X \mathcal{F}) \leq \phi_i(\phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{F}))) \leq \lim_i \phi_i(\mathcal{F}).$$

Thus all of the functions ϕ_i are continuous between $(X, \lim_X)$ and $(X_i, \lim_i)$.

Next we prove that the source $\mathcal{S} = ((X, \lim_X) \xrightarrow{\phi_i} (X_i, \lim_i))_{i \in I}$ is initial. Let $(Y, \lim_Y)$ be an SL-generalized convergence space and let $\psi \colon Y \to X$ be a function. We seek to prove that

$$\forall\, i \in I, \quad \phi_i \circ \psi \text{ is continuous between } (Y, \lim_Y) \text{ and } (X_i, \lim_i)$$
$$\Rightarrow \psi \text{ is continuous between } (Y, \lim_Y) \text{ and } (X, \lim_X).$$

Assume that $\forall\, i \in I\ \forall\, \mathcal{F} \in \mathcal{F}_L^S(Y), \quad (\phi_i \circ \psi)(\lim_Y \mathcal{F}) \leq \lim_i (\phi_i \circ \psi)(\mathcal{F})$. Now $(\phi_i \circ \psi)(\lim_Y \mathcal{F}) = \phi_i(\psi(\lim_Y \mathcal{F}))$, and $(\phi_i \circ \psi)(\mathcal{F}) = \phi_i(\psi(\mathcal{F}))$. Thus

$$\forall\, i \in I, \quad \psi(\lim_Y \mathcal{F}) \leq \phi_i^{\leftarrow}((\phi_i \circ \psi)(\lim_Y \mathcal{F})) \leq \phi_i^{\leftarrow}(\lim_i \phi_i(\psi(\mathcal{F}))).$$

Thus $\psi(\lim_Y \mathcal{F}) \leq \bigwedge_{i \in I} \phi_i^{\leftarrow}(\lim_i \phi_i(\psi(\mathcal{F}))) = \lim_X \psi(\mathcal{F})$.

We now prove that $\underline{\mathbf{SL-GCS}}$ is amnestic. Let $(X, \lim), (X, \overline{\lim})$ be SL-generalized convergence spaces and let $(X, \lim) \leq (X, \overline{\lim}), \quad (X, \overline{\lim}) \leq (X, \lim)$. Then from Remark 5.1.3 and Definition 5.1.2, we have that $\forall\, \mathcal{F} \in \mathcal{F}_L^S(X), \quad \lim \mathcal{F} = \overline{\lim}\, \mathcal{F}$, thus $\lim = \overline{\lim}$ and $(X, \lim) = (X, \overline{\lim})$. Since $\mathcal{S}$ is initial in $\underline{\mathbf{SL-GCS}}$ and $\underline{\mathbf{SL-GCS}}$ is amnestic, we have proved that $\underline{\mathbf{SL-GCS}}$ is topological over $\underline{\mathbf{SET}}$.

Let $\lim_L^S(X) = \{ \lim \mid (X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}}) \}$. Then $\lim_L^S(X) \subseteq \mathcal{F}_L^S(X)^{(L^X)}$. Now $\mathcal{F}_L^S(X) \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$ since $\mathcal{F}_L^S(X) \subseteq L^{(L^X)}$, thus $\lim_L^S(X) \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$. Now $\mathrm{Fibre}_{\underline{\mathbf{SL-GCS}}}(X) = \{X\} \times \lim_L^S(X)$ and $\underline{\mathbf{SL-GCS}}$ is fibre small.

Lastly let $X = \{x\}$. Now $L^X = \{ \alpha_X \mid \alpha \in L \}$. Let $\mathcal{F} \in \mathcal{F}_L^S(X)$, $\alpha \in L$. Then $\mathcal{F}(\alpha_X) \geq \alpha * \mathcal{F}(\top_X) = \alpha = [x](\alpha_X)$, thus $\forall\, \mathcal{F} \in \mathcal{F}_L^S(X), \quad [x] \leq \mathcal{F}$. Then if $(X, \lim)$ is an SL-generalized convergence space, $\forall\, \mathcal{F} \in \mathcal{F}_L^S(X), \quad \lim \mathcal{F}(x) \overset{\mathbf{L2}}{\geq} \lim[x](x) \overset{\mathbf{L1}}{=} \top$. Thus $\lim \mathcal{F} = \top_X$ for $\mathcal{F} \in \mathcal{F}_L^S(X)$. Thus there is only one object in the $\underline{\mathbf{SL-GCS}}$-fibre of X. So $\underline{\mathbf{SL-GCS}}$ satisfies the terminal separator property. ∎

Cartesian closedness in $\underline{\mathbf{SL-GCS}}$

The material of this subsection is a little sketchy, as it was not a focus of the thesis. Only preliminary definitions have been made and no new results have been obtained. Nevertheless it seems logical to present the the definitions here, since some of the motivation for the definition of $\underline{\mathbf{SL-GCS}}$ was the hope that it would turn out to be cartesian closed. It is known that in the frame case (which covers the classical case as well), $\underline{\mathbf{SL-GCS}}$ is cartesian closed [21].

Lemma 5.1.5 (see [21] for the frame case): Let $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. Let π_X, π_Y denote the usual projections from $X \times Y$ to X, Y respectively. We define

$$\lim_X \times \lim_Y : \mathcal{F}_L^S(X \times Y) \to L^{X\times Y}$$
$$\mathcal{F} \mapsto \pi_X^{\leftarrow}(\lim_X \pi_X(\mathcal{F})) \wedge \pi_Y^{\leftarrow}(\lim_Y \pi_Y(\mathcal{F})).$$

Then $(X \times Y, \lim_X \times \lim_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$ and $((X \times Y, \lim_X \times \lim_Y), \pi_X, \pi_Y)$ forms a product (in the sense of Definition 1.5.1) for $(X, \lim_X), (Y, \lim_Y)$ in $\underline{\mathbf{SL-GCS}}$.

Proof:
This is a corollary of Theorem 5.1.4. ■

Corollary 5.1.6 : $\underline{\mathbf{SL-GCS}}$ has all finite products.

Lemma 5.1.7 [21]: Let L satisfy the pseudo-bisymmetry condition. Let $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. We define

$$C(X, Y) = \hom_{\underline{\mathbf{SL-GCS}}}((X, \lim_X), (Y, \lim_Y)).$$

We define
$$ev : C(X, Y) \times X \to Y \quad (g, x) \mapsto g(x).$$

For $\mathcal{F} \in \mathcal{F}_L^S(C(X, Y)), g \in C(X, Y)$, we define

$$\mathrm{clim}\,\mathcal{F}(g) = \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X)} \bigwedge_{x\in X} (\lim_X \mathcal{G}(x) \to \lim_Y ev(\mathcal{F} \times \mathcal{G})(g(x))).$$

Then if L is a frame, $(C(X, Y), \mathrm{clim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$ and ev is continuous between $(C(X, Y), \mathrm{clim})$ and $(Y, \lim_Y)$.

Lemma 5.1.8 [21]: Let L satisfy the pseudo-bisymmetry condition. Let $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. Let $\phi : (Z \times X, \lim_Z \times \lim_X) \to (Y, \lim_Y) \in \mathrm{Mor}\,(\underline{\mathbf{SL-GCS}})$. For $z \in Z$, we define $\phi_z : X \to Y$ by $\phi_z(x) = \phi(z, x)$. We define
$$\tilde{\phi} : Z \to Y^X \quad z \mapsto \phi_z.$$

Then if L is a frame, $\forall\, z \in Z, \quad \phi_z \in C(X, Y)$ and $\tilde{\phi} \in \hom_{\underline{\mathbf{SL-GCS}}}((Z, \lim_Z), (C(X, Y), \mathrm{clim}))$.

Lemma 5.1.9 [21]: Let L satisfy the pseudo-bisymmetry condition. Let $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. Then if L is a frame, the evaluation mapping ev together with the structure $(C(X, Y), \mathrm{clim})$ form an exponential in $\underline{\mathbf{SL-GCS}}$ for the objects $(X, \lim_X)$ and $(Y, \lim_Y)$. Thus if L is a frame the category $\underline{\mathbf{SL-GCS}}$ is cartesian closed.

$\underline{\mathbf{SL-TCS}}$ as a subcategory of $\underline{\mathbf{SL-GCS}}$

In the frame case (see [21]), we have that $\underline{\mathbf{SL-TCS}}$ is contained as a reflective subcategory in $\underline{\mathbf{SL-GCS}}$. We attempt to prove that this is true in the ecl-premonoid case.

Lemma 5.1.10 : $\underline{\mathbf{SL-TCS}}$ is a full subcategory of $\underline{\mathbf{SL-GCS}}$.

Proof:
We know that $\mathrm{Ob}\,(\underline{\mathbf{SL-TCS}}) \subseteq \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$, since all objects in $\underline{\mathbf{SL-TCS}}$ satisfy **L1** and **L2**. Let $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-TCS}})$. Then from the definition of $\underline{\mathbf{SL-TCS}}$ in Theorem 4.3.3 and from Definition 5.1.1,

$$\phi \in \hom_{\underline{\mathbf{SL-TCS}}}((X, \lim_X), (Y, \lim_Y)) \\ \Leftrightarrow \phi \in \hom_{\underline{\mathbf{SL-GCS}}}((X, \lim_X), (Y, \lim_Y)).$$

Thus $\underline{\mathbf{SL-TCS}}$ is a full subcategory of $\underline{\mathbf{SL-GCS}}$. ∎

Lemma 5.1.11 : Let $(X, \lim)$ be an SL-generalized convergence space. Define

L0 $\quad \forall\, x \in X\ \forall\, a \in L^X, \quad \mathcal{U}^x_{\lim}(a) = \displaystyle\bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)).$

N1 $\quad \forall\, x \in X\ \forall\, a \in L^X, \quad \underline{a}_{\lim}(x) = \mathcal{U}^x_{\lim}(a).$

I1 $\quad \Delta_{\lim} = \{\, a \in L^X \mid a \leq \underline{a}_{\lim}\,\}.$

Then

$$\lim \text{ satisfies } \mathbf{L}\otimes \Rightarrow (X, \Delta_{\lim}) \text{ is an S}L\text{-topological space.}$$

Proof:
We prove that $\Delta_{\lim}$ satisfies **LO1**–**LOs**. Note that $\mathcal{U}^x_{\lim} \in \mathcal{F}^S_L(X)$ for $x \in X$ by Lemma 4.4.3.

LO1 $\perp_X \leq \underline{\perp_X}_{\lim}$ by definition of $\perp_X$. Let $x \in X$. By **N1** and **L0**, $\underline{\top_X}_{\lim}(x) = \mathcal{U}^x_{\lim}(\top_X) = \top$. Thus $\top_X \in \Delta_{\lim}$.

L02 Let $a, b \in \Delta_{\lim}, x \in X$. Then

$$(a \otimes b)(x) = a(x) \otimes b(x) \overset{\mathbf{I1}}{\leq} \underline{a}_{\lim}(x) \otimes \underline{b}_{\lim}(x) \\ = \mathcal{U}^x_{\lim}(a) \otimes \mathcal{U}^x_{\lim}(b) \overset{\mathbf{L}\otimes}{\leq} \mathcal{U}^x_{\lim}(a \otimes b) = \underline{a \otimes b}_{\lim}(x).$$

Thus $a \otimes b \in \Delta_{\lim}$.

L03 We prove that $a \le b \Rightarrow \underline{a}_{\lim} \le \underline{b}_{\lim}$. Let $a, b \in L^X, a \le b$. Let $x \in X$. Then by **L0**, $\mathcal{U}^x_{\lim}(a) \le \mathcal{U}^x_{\lim}(b)$. Now let $\Gamma \subseteq \Delta_{\lim}$. Then

$$\bigvee \Gamma \le \bigvee_{g \in \Gamma} \underline{g}_{\lim} \le \underline{\bigvee \Gamma}_{\lim}.$$

Thus $\bigvee \Gamma \in \Delta_{\lim}$.

LOs Let $\alpha \in L, a \in \Delta_{\lim}, x \in X$. Then

$$\alpha_X * a(x) \le \alpha * \underline{a}_{\lim}(x) = \alpha * \mathcal{U}^x_{\lim}(a) \le \mathcal{U}^x_{\lim}(\alpha_X * a) = \underline{\alpha_X * a}_{\lim}(x).$$

Thus $\alpha_X * a \in \Delta_{\lim}$.

■

Lemma 5.1.12 was proved previously as Lemma 4.2.11.

Lemma 5.1.12 : Let (X, Δ) be an SL-topological space. Define $\lim_\Delta$ by

I0 $\quad \forall\, a \in L^X, \quad \underline{a}_\Delta = \bigvee \{\, b \in \Delta \mid b \le a \,\}.$

N1 $\quad \forall\, a \in L^X \; \forall\, x \in X, \quad \mathcal{U}^x_\Delta(a) = \underline{a}_\Delta(x).$

Lp $\quad \forall\, \mathcal{F} \in \mathcal{F}^S_L(X) \; \forall\, x \in X, \quad \lim_\Delta \mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}^x_\Delta(a) \to \mathcal{F}(a)).$

Then $(X, \lim_\Delta)$ is an SL-topological convergence space, and hence an SL-generalized convergence space.

Lemma 5.1.13 : Let $(X, \Delta) \in \mathrm{Ob}\,(\underline{\mathbf{SL-TOP}})$ and let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. Then

1. $\Delta_{(\lim_\Delta)} = \Delta$.
2. $\lim_{(\Delta_{\lim})} \ge \lim$.

Proof:

1. This is an obvious implication of Lemma 4.2.14.
2. Let $x \in X, a \in L^X$. Then

$$\begin{aligned} \mathcal{U}^x_{(\Delta_{\lim})}(a) &= \underline{a}_{\Delta_{\lim}}(x) = \bigvee \{\, b(x) \mid b \in \Delta_{\lim}, b \le a \,\} \\ &= \bigvee \{\, b(x) \mid b \le \underline{b}_{\lim}, b \le a \,\} \le \bigvee \{\, \underline{b}_{\lim}(x) \mid b \le a \,\} \\ &= \underline{a}_{\lim}(x) = \mathcal{U}^x_{\lim}(a). \end{aligned}$$

So we have that $\forall\, x \in X, \quad \mathcal{U}^x_{\Delta_{\lim}} \leq \mathcal{U}^x_{\lim}$.

Now let $x \in X, \mathcal{F} \in \mathcal{F}^S_L(X)$. Then

$$\begin{aligned}
\lim_{(\Delta_{\lim})} \mathcal{F}(x) &= \bigwedge_{a \in L^X} (\mathcal{U}^x_{\Delta_{\lim}}(a) \to \mathcal{F}(a) \geq \bigwedge_{a \in L^X} (\mathcal{U}^x_{\lim}(a) \to \mathcal{F}(a)) \\
&= \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\bigwedge_{\mathcal{G} \in \mathcal{F}^S_L(X)} (\lim \mathcal{G}(x) \to \mathcal{G}(a)) \to \mathcal{F}(a)) \\
&\geq \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} ((\lim \mathcal{F}(x) \to \mathcal{F}(a)) \to \mathcal{F}(a)) \geq \lim \mathcal{F}(x).
\end{aligned}$$

Thus we have proved that $\lim_{(\Delta_{\lim})} \geq \lim$. Note that this is true regardless of whether $\Delta_{\lim}$ is an SL-topology on X or not.

■

Lemma 5.1.14 : Let $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$ and let $\phi \in \hom_{\underline{\mathbf{SL-GCS}}}((X, \lim_X), (Y, \lim_Y))$. Then

$$(X, \Delta_{\lim_X}), (Y, \Delta_{\lim_Y}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-TOP}}) \Rightarrow \phi \in \hom_{\underline{\mathbf{SL-TOP}}}((X, \Delta_{\lim_X}), (Y, \Delta_{\lim_Y})).$$

Proof:

We know that $\phi \in \hom_{\underline{\mathbf{SL-GCS}}}((X, \lim_X), (Y, \lim_Y)) \Leftrightarrow$

$$\forall\, \mathcal{F} \in \mathcal{F}^S_L(X), \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}).$$

Let $b \in \Delta_{\lim_Y}$. Then $b \leq \underline{b}_{\lim}$ and $\forall\, y \in Y, \quad b(y) \leq \mathcal{U}^y_{\lim_Y}(b)$. Let $x \in X$. We calculate:

$$\begin{aligned}
\phi^{\leftarrow}(b)(x) = b(\phi(x)) &\leq \bigwedge_{\mathcal{G} \in \mathcal{F}^S_L(Y)} (\lim_Y \mathcal{G}(\phi(x)) \to \mathcal{G}(b)) \\
&\leq \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\lim_Y \phi(\mathcal{F})(\phi(x)) \to \phi(\mathcal{F})(b)) \\
&\leq \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\phi(\lim_X \mathcal{F})(\phi(x)) \to \mathcal{F}(\phi^{\leftarrow}(b))) \\
&\leq \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\lim_X \mathcal{F}(x) \to \mathcal{F}(\phi^{\leftarrow}(b))) = \mathcal{U}^x_{\lim_X}(\phi^{\leftarrow}(b)).
\end{aligned}$$

Thus $\phi^{\leftarrow}(b) \in \Delta_{\lim_X}$. So we have that $\phi^{\leftarrow}(\Delta_{\lim_Y}) \subseteq \Delta_{\lim_X}$. Thus if $(X, \Delta_{\lim_X}), (Y, \Delta_{\lim_Y})$ are SL-topological spaces, then $\phi \in \hom_{\underline{\mathbf{SL-TOP}}}((X, \Delta_{\lim_X}), (Y, \Delta_{\lim_Y}))$. ■

Theorem 5.1.15 (see [21] for the frame case): If the $\mathbf{L}\otimes$ axiom is satisfied by all $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$ then $\underline{\mathbf{SL-TCS}}$ is a reflective subcategory of $\underline{\mathbf{SL-GCS}}$.

Proof:
If the $\mathbf{L}\otimes$ axiom is satisfied by all $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$ then by Lemma 5.1.11, $\forall\,(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$, $\quad (X, \Delta_{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-TOP}})$. Thus by Lemmas 5.1.11, 5.1.12 and 5.1.14, we can define functors:

$$F\colon \underline{\mathbf{SL-GCS}} \to \underline{\mathbf{SL-TOP}}$$
$$F((X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y)) = (X, \Delta_{\lim_X}) \xrightarrow{\phi} (Y, \Delta_{\lim_Y}),$$
$$G\colon \underline{\mathbf{SL-TOP}} \to \underline{\mathbf{SL-GCS}}$$
$$G((X, \Delta_X) \xrightarrow{\phi} (Y, \Delta_Y)) = (X, \lim_{\Delta_X}) \xrightarrow{\phi} (Y, \lim_{\Delta_Y}).$$

Let $(X, \Delta_X), (Y, \Delta_Y)$ be SL-topological spaces. Then by Lemma 5.1.13,

$$\begin{aligned}(F \circ G)((X, \Delta_X) \xrightarrow{\phi} (Y, \Delta_Y)) &= F((X, \lim_{\Delta_X}) \xrightarrow{\phi} (Y, \lim_{\Delta_Y}))\\ &= (X, \Delta_{\lim_{\Delta_X}}) \xrightarrow{\phi} (Y, \Delta_{\lim_{\Delta_Y}})\\ &= (X, \Delta_X) \xrightarrow{\phi} (Y, \Delta_Y)\\ &= \mathrm{id}_{\underline{\mathbf{SL-TOP}}}((X, \Delta_X) \xrightarrow{\phi} (Y, \Delta_Y)).\end{aligned}$$

Thus $F \circ G = \mathrm{id}_{\underline{\mathbf{SL-TOP}}}$. Now let $(X, \lim)$ be an SL-generalized convergence space. Then by Lemma 5.1.13,

$$(G \circ F)(X, \lim) = G(X, \Delta_{\lim}) = (X, \lim_{\Delta_{\lim}}) \geq (X, \lim).$$

Thus $F \circ G \geq \mathrm{id}_{\underline{\mathbf{SL-GCS}}}$.

Finally by Theorem 4.3.3, we have that $\underline{\mathbf{SL-TOP}}$ is isomorphic to a reflective subcategory of $\underline{\mathbf{SL-GCS}}$. Since $\underline{\mathbf{SL-TOP}}$ is isomorphic to $\underline{\mathbf{SL-TCS}}$, we have that $\underline{\mathbf{SL-TCS}}$ is a reflective subcategory of $\underline{\mathbf{SL-GCS}}$. ■

Given that the $\mathbf{L}\otimes$ axiom is a requirement of Theorem 5.1.15, it would be quite convenient if every SL-generalized convergence space satisfied $\mathbf{L}\otimes$. Unfortunately this is not the case, as shown by the following counter-example, originated by Lu and Yao ([32]) in connection with proving that $\mathcal{U}^x_{\lim}$ is not always a stratified L-filter.

Example 5.1.16 [32]: Let $L = \{\bot, \alpha, \top\}$, with $\bot < \alpha < \top$. Define the $*$ operation by Table 5.1 (a). Then $(L, \leq, *)$ is a GL-monoid. We consider an ecl-premonoid $(L, \leq, *, *)$. The implication operator defined by $*$ is shown

$*$	$\bot$	α	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
α	$\bot$	$\bot$	α
$\top$	$\bot$	α	$\top$

(a)

$\to$	$\bot$	α	$\top$
$\bot$	$\top$	$\top$	$\top$
α	α	$\top$	$\top$
$\top$	$\bot$	α	$\top$

(b)

Table 5.1: $*$ and $\to$ operations on L

in Table 5.1 (b). Let $X = \{x, y\}$. For $\mu, \nu \in L$, define $(\mu, \nu) = \mu_{\{x\}} \vee \nu_{\{y\}}$. Now define the lim function by

$$\forall\, \mathcal{F} \in \mathcal{F}_L^S(X)\ \forall\, z \in X, \quad \lim \mathcal{F}(z) = \begin{cases} \mathcal{F}(\top, \alpha) & z = x \\ \mathcal{F}(\alpha, \top) & z = y \end{cases}.$$

lim obviously satisfies **L1** and **L2**. The lim function induces the neighbourhood filter at $z \in X$:

$$\forall\, a \in L^X, \quad \mathcal{U}_{\lim}^z(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(z) \to \mathcal{F}(a)).$$

Let a be the L-set $(\top, \alpha)$. Then $a * a = (\top, \bot)$. Now $\mathcal{U}_{\lim}^x(a) = \top$, trivially. We calculate

$$\begin{aligned} \mathcal{U}_{\lim}^x(a * a) &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \\ &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\mathcal{F}(\top, \alpha) \to \mathcal{F}(\top, \bot)) \\ &\le [y](\top, \alpha) \to [y](\top, \bot) = \alpha \to \bot = \alpha \\ &\ngeq \top * \top = \mathcal{U}_{\lim}^x(a) * \mathcal{U}_{\lim}^x(a). \end{aligned}$$

Thus lim does not satisfy the **L**$\otimes$ axiom.

For the case where L is a frame, we have that for all SL-generalized convergence spaces $(X, \lim)$, $\forall\, x \in X$, $\mathcal{U}_{\lim}^x \in \mathcal{F}_L^S(X)$ ([21]). In other words, the **L**$\otimes$ axiom is satisfied. When we examine the proof that $\mathcal{U}_{\lim}^x \in \mathcal{F}_L^S(X)$, it turns out to depend on the condition that $\forall\, \alpha \in L, \quad \alpha \le \alpha \wedge \alpha$, which is of course satisfied for L being a frame. The corresponding condition for L an ecl-premonoid is $\forall\, \alpha \in L, \quad \alpha \le \alpha \otimes \alpha$.

Definition 5.1.17 : Let $(L, \le, *, \otimes)$ be an ecl-premonoid. We say that L satisfies the *monotonicity condition* iff

M $\qquad \forall\, \alpha \in L, \quad \alpha \le \alpha \otimes \alpha.$

Lemma 5.1.18 : The ecl-premonoid L satisfies the monotonicity condition **M** iff

$$\forall\,(X,\lim)\in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}}),\quad \lim \text{ satisfies } \mathbf{L}\otimes.$$

Proof:
Let L satisfy the monotonicity condition **M** and $(X, \lim)$ be an SL-generalized convergence space. Let $x \in X, a, b \in L^X$. Then by Lemma 3.1.30 and **F3**,

$$\begin{aligned}\mathcal{U}^x(a)\otimes\mathcal{U}^x(b) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}(\lim\mathcal{F}(x)\to\mathcal{F}(a))\otimes\bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X)}(\lim\mathcal{G}(x)\to\mathcal{G}(b))\\ &\le \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}(\lim\mathcal{F}(x)\to\mathcal{F}(a))\otimes(\lim\mathcal{F}(x)\to\mathcal{F}(b))\\ &\le \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}((\lim\mathcal{F}(x)\otimes\lim\mathcal{F}(x))\to\mathcal{F}(a\otimes b))\\ &\overset{\mathbf{M}}{\le} \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}(\lim\mathcal{F}(x)\to\mathcal{F}(a\otimes b)=\mathcal{U}^x(a\otimes b)\end{aligned}$$

Thus lim satisfies $\mathbf{L}\otimes$.

Next, assume that L does not satisfy **M**. Then $\exists\,\alpha\in L,\quad \alpha\not\le\alpha\otimes\alpha$. We note that if $\alpha\to(\alpha\otimes\alpha)=\top$ then $\alpha\le\alpha\otimes\alpha$, so we know that $\alpha\to(\alpha\otimes\alpha)<\top$. We now construct an S$L$-generalized convergence space which does *not* satisfy $\mathbf{L}\otimes$. Let $X=\{x,y\}$. For $\mu,\nu\in L$ we define $(\mu,\nu)=\mu_{\{x\}}\vee\nu_{\{y\}}$. For $\mathcal{F}\in\mathcal{F}_L^S(X)$ we define

$$\lim\mathcal{F}(z)=\begin{cases}\mathcal{F}(\top,\alpha) & z=x\\ \mathcal{F}(\alpha,\top) & z=y\end{cases}.$$

Then $(X,\lim)\in\mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. Let $a=(\top,\alpha)$. Then $a\otimes a=(\top,\alpha\otimes\alpha)$ and $\mathcal{U}^x(a)=\top$. We calculate

$$\begin{aligned}\mathcal{U}^x(a\otimes a) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}(\lim\mathcal{F}(x)\to\mathcal{F}(a\otimes a))=\bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}(\mathcal{F}(a)\to\mathcal{F}(a\otimes a))\\ &\le [y](\top,\alpha)\to[y](\top,\alpha\otimes\alpha)=\alpha\to(\alpha\otimes\alpha)<\mathcal{U}^x(a)\otimes\mathcal{U}^x(a)\end{aligned}$$

Thus lim does not satisfy $\mathbf{L}\otimes$. ■

Remark 5.1.19 : Lemma 5.1.18 tells us that $\underline{\mathbf{SL-TCS}}$ is a reflective subcategory of $\underline{\mathbf{SL-GCS}}$ in the important special cases of the frame where $*=\otimes=\wedge$, and of GL-monoids $(L,\le,*)$ with monoidal mean operators where $\otimes=\circledast$. It does not apply to the general situation, e.g. where $\otimes=*$, since if $\alpha\le\alpha*\alpha$ for all α, then $*=\wedge$.

Difficulties with the L$\otimes$ axiom

In view of Theorem 5.1.15, it seems reasonable to change the definition of objects in $\underline{\mathbf{SL-GCS}}$ to those SL-preconvergence spaces which satisfy **L1**, **L2** and **L**$\otimes$. We experiment with this definition.

Definition 5.1.20 : We define the category $\underline{\mathbf{SL-GCS}\otimes}$ by

Objects SL-generalized convergence spaces $(X, \lim)$ additionally satisfying **L**$\otimes$.

Morphisms Functions $\phi : X \to Y$ between spaces $(X, \lim_X)$ and $(Y, \lim_Y)$ which satisfy

$$\forall\, \mathcal{F} \in \mathcal{F}_L^S(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}).$$

The identity function id_X is a morphism from $(X, \lim)$ to itself. Morphism composition is the usual function composition.

Theorem 5.1.21 : $\underline{\mathbf{SL-TCS}}$ is a reflective subcategory of $\underline{\mathbf{SL-GCS}\otimes}$.

Proof:
The proof is the same as the proof of Theorem 5.1.15, only now we have that every object in $\underline{\mathbf{SL-GCS}\otimes}$ satisfies **L**$\otimes$. ∎

Conjecture 5.1.22 : $\underline{\mathbf{SL-GCS}\otimes}$ is not topological over $\underline{\mathbf{SET}}$ in general.

Motivation:
Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$, $(X_i, \lim_i)_{i\in I} \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}\otimes})^I$ and let $(\phi_i \colon X \to X_i)_{i\in I}$ be a set of functions indexed by I. As previously we define $\lim_X$ by

$$\lim_X \mathcal{F} = \bigwedge_{i\in I} \phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{F})).$$

If $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}\otimes})$ then for $a, b \in L^X$, $x \in X$, we should have that

$$\mathcal{U}^x_{\lim_X}(a) \otimes \mathcal{U}^x_{\lim_X}(b) \leq \mathcal{U}^x_{\lim_X}(a \otimes b).$$

We attempt to prove this:

$$\mathcal{U}^x_{\lim_X}(a) \otimes \mathcal{U}^x_{\lim_X}(b) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim_X \mathcal{F}(x) \to \mathcal{F}(a)) \otimes \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X)} (\lim_X \mathcal{G}(x) \to \mathcal{G}(b))$$

$$\leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} ((\lim_X \mathcal{F}(x) \to \mathcal{F}(a)) \otimes (\lim_X \mathcal{F}(x) \to \mathcal{F}(b)))$$

$$\leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} ((\lim_X \mathcal{F}(x) \otimes \lim_X \mathcal{F}(x)) \to (\mathcal{F}(a) \otimes \mathcal{F}(b)))$$

$$\leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} ((\lim_X \mathcal{F}(x) \otimes \lim_X \mathcal{F}(x)) \to \mathcal{F}(a \otimes b)).$$

At this point, if we knew that $\lim_X \mathcal{F}(x) \otimes \lim_X \mathcal{F}(x) \geq \lim_X \mathcal{F}(x)$ (as would be implied for example by the monotonicity condition **M**), then we would be able to say that

$$\mathcal{U}^x_{\lim_X}(a) \otimes \mathcal{U}^x_{\lim_X}(b) \leq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\lim_X \mathcal{F}(x) \to \mathcal{F}(a \otimes b)) = \mathcal{U}^x_{\lim_X}(a \otimes b).$$

Possibly a different method of proof other than the naive approach used here might work but as it stands at the moment we are not *sure* that $\underline{\mathbf{SL-GCS}}\otimes$ is topological over $\underline{\mathbf{SET}}$. □

Remark 5.1.23 : We are not *sure* that $\underline{\mathbf{SL-GCS}}\otimes$ is topological over $\underline{\mathbf{SET}}$. Because the $\mathbf{L}\otimes$ axiom seems so difficult to work with, and we know that restricting the lattice via the monotonicity condition **M** can guarantee that all SL-generalized convergence spaces satisfy $\mathbf{L}\otimes$, we have preferred to instead work with the category $\underline{\mathbf{SL-GCS}}$ where spaces are only required to satisfy **L1** and **L2**. Together with **M** , this captures the important special cases of frames $(L, \leq, \wedge, \wedge)$ and GL-monoids with monoidal mean operators $(L, \leq, *, \circledast)$.

5.2 Subcategories of $\underline{\mathbf{SL-GCS}}$

Kent convergence spaces

Definition 5.2.1 (see [22] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Then lim satisfies the **L3w** axiom ⇔

L3w $\quad \forall\, \mathcal{F} \in \mathcal{F}_L^S(X)\ \forall\, x \in X, \quad \lim \mathcal{F}(x) \leq \lim(\mathcal{F} \wedge [x])(x).$

If lim additionally satisfies **L1** and **L2**, then $(X, \lim)$ is a *stratified* L*-Kent convergence space*. We abbreviate 'stratified L-Kent convergence space' as SL-Kent convergence space.

Definition 5.2.2 (see [22] for the frame case): We define the category $\underline{\mathbf{SL-KCS}}$ by

Objects SL-Kent convergence spaces $(X, \lim)$.

Morphisms Functions $\phi : X \to Y$ between spaces $(X, \lim_X)$ and $(Y, \lim_Y)$ which satisfy

$$\forall\, \mathcal{F} \in \mathcal{F}_L^S(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}).$$

The identity function id_X is a morphism from $(X, \lim)$ to itself. Morphism composition is the usual function composition.

Lemma 5.2.3 : $\underline{\mathbf{SL-KCS}}$ is a full subcategory of $\underline{\mathbf{SL-GCS}}$.

Proof:
It is obvious that $\mathrm{Ob}\,(\underline{\mathbf{SL-KCS}}) \subseteq \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$ and that for $(X, \lim_X), (Y, \lim_Y)$ in $\mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$,

$$\hom_{\underline{\mathbf{SL-KCS}}}((X, \lim_X), (Y, \lim_Y)) = \hom_{\underline{\mathbf{SL-GCS}}}((X, \lim_X), (Y, \lim_Y)).$$

Identity and morphism composition are the same in both categories. Thus $\underline{\mathbf{SL-KCS}}$ is a full subcategory of $\underline{\mathbf{SL-GCS}}$. ■

Lemma 5.2.4 : Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. We define

$$\overline{\lim}\,\mathcal{F}(x) = \bigvee_{\mathcal{G} \wedge [x] \leq \mathcal{F}} \lim \mathcal{G}(x).$$

Then $(X, \overline{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$ and $\overline{\lim} \geq \lim$.

Proof:
Let $x \in X$. Then

$$\overline{\lim}[x](x) = \bigvee_{\mathcal{G} \wedge [x] \leq [x]} \lim \mathcal{G}(x) \geq \lim[x](x) = \top.$$

Thus **L1** is satisfied. Now let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \mathcal{F} \leq \mathcal{G}$. Then

$$\overline{\lim}\,\mathcal{F}(x) = \bigvee_{\mathcal{H} \wedge [x] \leq \mathcal{F}} \lim \mathcal{H}(x) \leq \bigvee_{\mathcal{H} \wedge [x] \leq \mathcal{G}} \lim \mathcal{H}(x) = \overline{\lim}\,\mathcal{G}(x).$$

Thus **L2** is satisfied. Next we note that for $\mathcal{G} \in \mathcal{F}_L^S(X)$, $\mathcal{G} \wedge [x] \leq \mathcal{F} \Rightarrow \mathcal{G} \wedge [x] \leq \mathcal{F} \wedge [x]$, thus

$$\overline{\lim}\,\mathcal{F}(x) = \bigvee_{\mathcal{G} \wedge [x] \leq \mathcal{F}} \lim \mathcal{G}(x) \leq \bigvee_{\mathcal{G} \wedge [x] \leq \mathcal{F} \wedge [x]} \lim \mathcal{G}(x) = \overline{\lim}(\mathcal{F} \wedge [x])(x).$$

So **L3w** is satisfied. We have proved that $(X, \overline{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$. Finally, $\overline{\lim}\,\mathcal{F}(x) \geq \lim \mathcal{F}(x)$ for all $x \in X$, since $\mathcal{F} \wedge [x] \leq \mathcal{F}$. ■

Lemma 5.2.5 : Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL} - \mathbf{GCS}})$ and $(X, \widetilde{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL} - \mathbf{KCS}})$. Then

$$\lim \leq \widetilde{\lim} \Rightarrow \overline{\lim} \leq \widetilde{\lim}\,.$$

Proof:
Assume that $\lim \leq \widetilde{\lim}$. Let $x \in X, \mathcal{F} \in \mathcal{F}_L^S(X)$. Then

$$\begin{aligned}\overline{\lim}\,\mathcal{F}(x) = \bigvee_{\mathcal{G}\wedge[x]\leq\mathcal{F}} \lim\mathcal{G}(x) &\leq \bigvee_{\mathcal{G}\wedge[x]\leq\mathcal{F}} \widetilde{\lim}\,\mathcal{G}(x) \\ &\overset{\mathbf{L3w}}{\leq} \bigvee_{\mathcal{G}\wedge[x]\leq\mathcal{F}} \widetilde{\lim}(\mathcal{G}\wedge[x])(x) \leq \widetilde{\lim}\,\mathcal{F}(x).\end{aligned}$$

Thus $\overline{\lim} \leq \widetilde{\lim}$. ∎

Remark 5.2.6 : It is obvious that $\overline{\widetilde{\lim}} = \widetilde{\lim}$ since $\widetilde{\lim} \leq \widetilde{\lim}$.

Lemma 5.2.7 : Let $\phi\colon (X, \lim_X) \to (Y, \lim_Y) \in \mathrm{Mor}\,(\underline{\mathbf{SL} - \mathbf{GCS}})$. Then

$$\phi\colon (X, \overline{\lim}_X) \to (Y, \overline{\lim}_Y) \in \mathrm{Mor}\,(\underline{\mathbf{SL} - \mathbf{KCS}}).$$

Proof:
We have

$$\forall\,\mathcal{F} \in \mathcal{F}_L^S(X), \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}).$$

Let $x \in X, \mathcal{F} \in \mathcal{F}_L^S(X)$. We calculate

$$\begin{aligned}\overline{\lim}_Y\,\phi(\mathcal{F})(\phi(x)) &= \bigvee_{\mathcal{H}\wedge[\phi(x)]\leq\phi(\mathcal{F})} \lim_Y \mathcal{H}(\phi(x)) \\ &\geq \bigvee_{\phi(\mathcal{G})\wedge\phi([x])\leq\phi(\mathcal{F})} \lim_Y \phi(\mathcal{G})(\phi(x)) \\ &\geq \bigvee_{\mathcal{G}\wedge[x]\leq\mathcal{F}} \lim_X \mathcal{G}(x) = \overline{\lim}_X\,\mathcal{F}(x).\end{aligned}$$

Let $y \in Y$. Then

$$\phi(\overline{\lim}_X\,\mathcal{F})(y) = \bigvee_{\phi(x)=y} \overline{\lim}_X\,\mathcal{F}(x) \leq \bigvee_{\phi(x)=y} \overline{\lim}_Y\,\phi(\mathcal{F})(\phi(x)) \leq \overline{\lim}_Y\,\phi(\mathcal{F})(y).$$

Thus $\phi \in \hom_{\underline{\mathbf{SL-KCS}}}((X, \overline{\lim}_X), (Y, \overline{\lim}_Y))$. ■

Theorem 5.2.8 (see [22] for the frame case): $\underline{\mathbf{SL-KCS}}$ is a full reflective subcategory of $\underline{\mathbf{SL-GCS}}$.

Proof:
We know from Lemma 5.2.3 that $\underline{\mathbf{SL-KCS}}$ is a full subcategory of $\underline{\mathbf{SL-GCS}}$. We define a functor

$$F\colon \underline{\mathbf{SL-GCS}} \to \underline{\mathbf{SL-KCS}}$$
$$(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \mapsto (X, \overline{\lim}_X) \xrightarrow{\phi} (Y, \overline{\lim}_Y).$$

This is possible by Lemmas 5.2.4 and 5.2.7. We also have the inclusion functor

$$E\colon \underline{\mathbf{SL-KCS}} \hookrightarrow \underline{\mathbf{SL-GCS}}$$
$$(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \mapsto (X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y).$$

Let $(X, \lim_X) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. Then $E \circ F(X, \lim_X) = (X, \overline{\lim}_X) \geq (X, \lim_X)$ by Lemma 5.2.5. Now let $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$. Let $\phi \in \hom_{\underline{\mathbf{SL-KCS}}}((X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y))$. Then by Remark 5.2.6 and Lemma 5.2.3, $F \circ E((X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y)) = (X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y)$. Thus we have $F \circ E = \mathrm{id}_{\underline{\mathbf{SL-KCS}}}$ and $E \circ F \geq \mathrm{id}_{\underline{\mathbf{SL-GCS}}}$. Then by Theorem 1.6.2, $\underline{\mathbf{SL-KCS}}$ is a reflective subcategory of $\underline{\mathbf{SL-GCS}}$. ■

Limit spaces

Definition 5.2.9 (see [22] for the frame case):
Let $(X, \lim)$ be an SL-preconvergence space. Then lim satisfies the **L3** axiom $\Leftrightarrow$

L3 $\qquad \forall\, \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X) \quad \lim \mathcal{F} * \lim \mathcal{G} \leq \lim(\mathcal{F} \wedge \mathcal{G}).$

If lim additionally satisfies **L1** and **L2**, then $(X, \lim)$ is a *stratified L-limit space*. We abbreviate 'stratified L-limit space' as SL-limit space.

Definition 5.2.10 (see [22] for the frame case): We define the category $\underline{\mathbf{SL-LIM}}$ by

Objects SL-limit spaces $(X, \lim)$.

Morphisms Functions $\phi : X \to Y$ between spaces $(X, \lim_X)$ and $(Y, \lim_Y)$ which satisfy

$$\forall\, \mathcal{F} \in \mathcal{F}_L^S(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}).$$

The identity function id_X is a morphism from $(X, \lim)$ to itself. Morphism composition is the usual function composition.

Lemma 5.2.11 : $\underline{\mathbf{SL-LIM}}$ is a full subcategory of $\underline{\mathbf{SL-KCS}}$.

Proof:
We need to prove that $\mathrm{Ob}\,(\underline{\mathbf{SL-LIM}}) \subseteq \mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$. If this is the case then it is obvious that $\underline{\mathbf{SL-LIM}}$ is a full subcategory of $\underline{\mathbf{SL-KCS}}$. Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$. Let $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X$. Then

$$\lim \mathcal{F}(x) \overset{\mathbf{L1}}{=} \lim \mathcal{F}(x) * \lim[x](x) \overset{\mathbf{L3}}{\leq} \lim(\mathcal{F} \wedge [x])(x).$$

Thus **L3w** is satisfied and $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$. ■

Lemma 5.2.12 : Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$. We define

$$\overline{\lim}\, \mathcal{F} = \bigvee \{ \mathop{*}_{i=1}^{n} \lim \mathcal{F}_i \mid n \in \mathbb{N}, (\mathcal{F}_i)_{i \in I} \in \mathcal{F}_L^S(X)^{[n]}, \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F} \}.$$

Then $(X, \overline{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$. Furthermore $\overline{\lim} \geq \lim$.

Proof:
We prove that $\overline{\lim}$ satisfies **L1**, **L2** and **L3**. Let $x \in X$. Then $\lim[x] \in \{ *_{i=1}^{n} \lim \mathcal{F}_i \mid n \in \mathbb{N}, \bigwedge_{i=1}^{n} \mathcal{F}_i \leq [x] \}$. Thus $\overline{\lim}[x](x) \geq \lim[x](x) = \top$, so **L1** is satisfied. Now let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \mathcal{F} \leq \mathcal{G}$. Then it is obvious that $\overline{\lim}\, \mathcal{F} \leq \overline{\lim}\, \mathcal{G}$, so **L2** is satisfied.

For **L3**, let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$. Then

$$\begin{aligned}
\overline{\lim}\, \mathcal{F} * \overline{\lim}\, \mathcal{G} &= \bigvee \{ \mathop{*}_{i=1}^{m} \lim \mathcal{F}_i \mid \bigwedge_{i=1}^{m} \mathcal{F}_i \leq \mathcal{F} \} * \bigvee \{ \mathop{*}_{j=1}^{n} \lim \mathcal{G}_j \mid \bigwedge_{j=1}^{n} \mathcal{G}_j \leq \mathcal{G} \} \\
&\overset{!}{=} \bigvee \{ (\mathop{*}_{i=1}^{m} \lim \mathcal{F}_i) * (\mathop{*}_{j=1}^{n} \lim \mathcal{G}_j) \mid \bigwedge_{i=1}^{m} \mathcal{F}_i \leq \mathcal{F} \text{ and } \bigwedge_{j=1}^{n} \mathcal{G}_j \leq \mathcal{G} \} \\
&\leq \bigvee \{ \mathop{*}_{k=1}^{p} \lim \mathcal{H}_k \mid \bigwedge_{k=1}^{p} \mathcal{H}_k \leq \mathcal{F} \wedge \mathcal{G} \} = \overline{\lim}(\mathcal{F} \wedge \mathcal{G}).
\end{aligned}$$

Note in the step marked '!', that we have used the property of the $*$ operator, that it distributes over arbitrary joins. (Property **GL5** in Chapter 3). We have proved that $(X, \overline{\lim})$ is an SL-Limit space.

Finally, let $\mathcal{F} \in \mathcal{F}_L^S(X)$. Then $\lim \mathcal{F} \in \{ *_{i=1}^n \lim \mathcal{F}_i \mid n \in \mathbb{N}, \bigwedge_{i=1}^n \mathcal{F}_i \leq \mathcal{F} \}$. So $\overline{\lim} \geq \lim$. ∎

Lemma 5.2.13 : Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$, $(X, \widetilde{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$. Then

$$\lim \leq \widetilde{\lim} \Rightarrow \overline{\lim} \leq \widetilde{\lim}\,.$$

Proof:
Assume that $\lim \leq \widetilde{\lim}$. Let $\mathcal{F} \in \mathcal{F}_L^S(X)$. Then

$$\overline{\lim}\,\mathcal{F} = \bigvee \{ \mathop{*}_{i=1}^{n} \lim \mathcal{F}_i \mid \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F} \} \leq \bigvee \{ \mathop{*}_{i=1}^{n} \widetilde{\lim}\,\mathcal{F}_i \mid \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F} \}$$
$$\overset{\mathbf{L3}}{\leq} \bigvee \{ \widetilde{\lim} \bigwedge_{i=1}^{n} \mathcal{F}_i \mid \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F} \} \leq \widetilde{\lim}\,\mathcal{F}.$$

Thus $\overline{\lim} \leq \widetilde{\lim}$. ∎

Remark 5.2.14 : Obviously, $\overline{\widetilde{\lim}} = \widetilde{\lim}$ since $\widetilde{\lim} \leq \widetilde{\lim}$.

Lemma 5.2.15 : Let $(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \in \mathrm{Mor}\,(\underline{\mathbf{SL-KCS}})$. Then

$$(X, \overline{\lim}_X) \xrightarrow{\phi} (Y, \overline{\lim}_Y) \in \mathrm{Mor}\,(\underline{\mathbf{SL-LIM}}).$$

Proof:
Let $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X$. Then

$$\overline{\lim}_Y\,\phi(\mathcal{F})(\phi(x)) = \bigvee \{ \mathop{*}_{i=1}^{n} \lim_Y \mathcal{G}_i(\phi(x)) \mid \bigwedge_{i=1}^{n} \mathcal{G}_i \leq \phi(\mathcal{F}) \}$$
$$\geq \bigvee \{ \mathop{*}_{i=1}^{n} \lim_Y \phi(\mathcal{H}_i)(\phi(x)) \mid \bigwedge_{i=1}^{n} \phi(\mathcal{H}_i) \leq \phi(\mathcal{F}) \}$$
$$\geq \bigvee \{ \mathop{*}_{i=1}^{n} \lim_X \mathcal{H}_i(x) \mid \bigwedge_{i=1}^{n} \mathcal{H}_i \leq \mathcal{F} \} = \overline{\lim}_X\,\mathcal{F}(x).$$

Thus $\forall \mathcal{F} \in \mathcal{F}_L^S(X), \quad \phi(\overline{\lim}_X \mathcal{F}) \leq \overline{\lim}_Y \phi(\mathcal{F})$. Therefore $\phi \in \hom_{\underline{\mathbf{SL-LIM}}}((X, \overline{\lim}_X), (Y, \overline{\lim}_Y))$. ■

Theorem 5.2.16 (see [22] for the frame case): $\underline{\mathbf{SL-LIM}}$ is a full reflective subcategory of $\underline{\mathbf{SL-KCS}}$.

Proof:
We know from Lemma 5.2.11 that $\underline{\mathbf{SL-LIM}}$ is a full subcategory of $\underline{\mathbf{SL-KCS}}$. We define a functor

$$F\colon \underline{\mathbf{SL-KCS}} \to \underline{\mathbf{SL-LIM}}$$
$$(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \mapsto (X, \overline{\lim}_X) \xrightarrow{\phi} (Y, \lim_Y).$$

This is possible by Lemmas 5.2.12 and 5.2.15. Since $\underline{\mathbf{SL-LIM}}$ is a subcategory of $\underline{\mathbf{SL-KCS}}$, we also have the inclusion functor

$$E\colon \underline{\mathbf{SL-LIM}} \hookrightarrow \underline{\mathbf{SL-KCS}}$$
$$(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \mapsto (X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y).$$

Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$. Then $E \circ F(X, \lim) = (X, \overline{\lim}) \geq (X, \lim)$. Now let $(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \in \mathrm{Mor}\,(\underline{\mathbf{SL-LIM}})$. Then $F \circ E((X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y)) = (X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y)$. Thus $F \circ E = \mathrm{id}_{\underline{\mathbf{SL-LIM}}}$ and $E \circ F \geq \mathrm{id}_{\underline{\mathbf{SL-KCS}}}$. By Theorem 1.6.2, $\underline{\mathbf{SL-LIM}}$ is a reflective subcategory of $\underline{\mathbf{SL-KCS}}$. ■

Discussion: The form of the L3 axiom

The classical **L3** axiom (see e.g. [26]) has the form

L3 $\qquad \forall \mathcal{F}, \mathcal{G} \in \mathcal{F}(X), \quad \lim \mathcal{F} \cap \lim \mathcal{G} \subseteq \lim(\mathcal{F} \cap \mathcal{G}).$

In [22], Jäger extends this to the case where L is a frame, resulting in the axiom

L3 $\qquad \forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \quad \lim \mathcal{F} \wedge \lim \mathcal{G} \leq \lim(\mathcal{F} \wedge \mathcal{G}).$

In extending to the ecl-premonoid case, we have available to us three possible generalizations of the $\wedge$ operation, namely the $\wedge$, $*$ and $\otimes$ operations, all of which coincide in the frame case. It is not immediately clear which of these should be used in the generalized **L3** axiom. We bear some principles in mind as we investigate. Firstly, we will try to choose our **L3** so that the category $\underline{\mathbf{SL-LIM}}$ (with objects those spaces $(X, \lim)$ satisfying **L1**, **L2** and **L3**) is a *reflective* subcategory of $\underline{\mathbf{SL-KCS}}$, as it is in the frame case ([22]). Ideally we should be able to extend Jäger's proofs almost without modification. Secondly, if possible in our proofs we should not require L to have properties beyond the usual ecl-premonoid properties. Lastly we should try to avoid strange constructions such as $\mathcal{F}*\mathcal{G}$ and $\mathcal{F}\otimes\mathcal{G}$ for $\mathcal{F},\mathcal{G}\in\mathcal{F}_L^S(X)$, since these will usually only be stratified L-filters if $(L,\leq,*,\otimes)$ satisfies extra conditions.

We thus have three possible candidates for our generalized **L3** axiom which we shall initially consider:

L3$\wedge$ $\quad \forall\,\mathcal{F},\mathcal{G}\in\mathcal{F}_L^S(X),\quad \lim\mathcal{F}\wedge\lim\mathcal{G}\leq\lim(\mathcal{F}\wedge\mathcal{G}).$

L3$*$ $\quad \forall\,\mathcal{F},\mathcal{G}\in\mathcal{F}_L^S(X),\quad \lim\mathcal{F}*\lim\mathcal{G}\leq\lim(\mathcal{F}\wedge\mathcal{G}).$

L3$\otimes$ $\quad \forall\,\mathcal{F},\mathcal{G}\in\mathcal{F}_L^S(X),\quad \lim\mathcal{F}\otimes\lim\mathcal{G}\leq\lim(\mathcal{F}\wedge\mathcal{G}).$

As we have already seen, **L3**$*$ results in $\underline{\mathbf{SL-LIM}}$ being a reflective subcategory of $\underline{\mathbf{SL-KCS}}$ and also does not require L to satisfy extra conditions when proving this. Thus **L3**$*$ satisfies our criteria. We now show why the other axioms were rejected.

Firstly, by induction we may restate the axioms as

L3$\wedge$ $\quad \forall\,n\in\mathbb{N},\ \forall\,(\mathcal{F}_i)_1^n\in\mathcal{F}_L^S(X)^{[n]},\quad \bigwedge_{i=1}^{n}\lim\mathcal{F}_i\leq\lim(\bigwedge_{i=1}^{n}\mathcal{F}_i)$

L3$*$ $\quad \forall\,n\in\mathbb{N},\ \forall\,(\mathcal{F}_i)_1^n\in\mathcal{F}_L^S(X)^{[n]},\quad \mathop{*}_{i=1}^{n}\lim\mathcal{F}_i\leq\lim(\bigwedge_{i=1}^{n}\mathcal{F}_i)$

L3$\otimes$ $\quad \forall\,n\in\mathbb{N},\ \forall\,(\mathcal{F}_i)_1^n\in\mathcal{F}_L^S(X)^{[n]},\quad \bigotimes_{i=1}^{n}\lim\mathcal{F}_i\leq\lim(\bigwedge_{i=1}^{n}\mathcal{F}_i)$

Where we define $\bigotimes_{1}^{n}a_i=a_n\otimes a_{n-1}\otimes\ldots\otimes a_1$.

Let $(X,\lim)\in\mathrm{Ob}\,(\underline{\mathbf{SL-KCS}})$. We need to define $\overline{\lim}$ such that $(X,\overline{\lim})\in\mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$ (i.e. satisfies **L1**, **L2** and **L3**) and $\overline{\lim}\geq\lim$. Furthermore, we need that if $(X,\widetilde{\lim})\in\mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$ and $\lim\leq\widetilde{\lim}$ then $\overline{\lim}\leq\widetilde{\lim}$.

If we are to use Jäger's proofs we need to define our $\overline{\lim}$ to be a generalization of the one used in his paper [22]. With this in mind we have three possibilities for $\overline{\lim}$ which are immediately obvious. Let $\mathcal{F}\in\mathcal{F}_L^S(X)$. We define:

$$\overline{\lim}_{\wedge}\,\mathcal{F}=\{\bigwedge_{1}^{n}\lim\mathcal{F}_i\mid n\in\mathbb{N},\bigwedge_{i=1}^{n}\mathcal{F}_i\leq\mathcal{F}\}.$$

$$\overline{\lim}_{*}\,\mathcal{F}=\{\mathop{*}_{1}^{n}\lim\mathcal{F}_i\mid n\in\mathbb{N},\bigwedge_{i=1}^{n}\mathcal{F}_i\leq\mathcal{F}\}.$$

$$\overline{\lim}_{\otimes}\mathcal{F} = \{\, \overset{n}{\underset{1}{\otimes}} \lim \mathcal{F}_i \mid n \in \mathbb{N}, \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F}\,\}.$$

If $\odot, \oplus \in \{\wedge, *, \otimes\}$ then we can restate the axiom and the definition of $\overline{\lim}\,\mathcal{F}$ as:

$$\textbf{L3}\odot \qquad \forall\, n \in \mathbb{N},\ \forall\, (\mathcal{F}_i)_1^n \in \mathcal{F}_L^S(X)^{[n]}, \quad \overset{n}{\underset{i=1}{\odot}} \lim \mathcal{F}_i \leq \lim(\bigwedge_{i=1}^{n} \mathcal{F}_i),$$

$$\overline{\lim}_{\odot}\mathcal{F} = \{\, \overset{n}{\underset{i=1}{\odot}} \lim \mathcal{F}_i \mid n \in \mathbb{N}, \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F}\,\}.$$

Now let $(X, \lim)$ satisfy **L1** and **L2**. Let $(X, \widetilde{\lim})$ satisfy **L1**, **L2** and **L3**$\oplus$ and let $\lim \leq \widetilde{\lim}$. We want $\overline{\lim}_{\odot} \leq \widetilde{\lim}$. We have

$$\begin{aligned}
\overline{\lim}_{\odot}\mathcal{F} &= \{\, \overset{n}{\underset{i=1}{\odot}} \lim \mathcal{F}_i \mid n \in \mathbb{N}, \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F}\,\} \\
&\leq \{\, \overset{n}{\underset{i=1}{\odot}} \widetilde{\lim} \mathcal{F}_i \mid n \in \mathbb{N}, \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F}\,\} \\
&\overset{?}{\leq} \{\, \widetilde{\lim} \bigwedge_{i=1}^{n} \mathcal{F}_i \mid n \in \mathbb{N}, \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F}\,\} \\
&\leq \widetilde{\lim}\,\mathcal{F}.
\end{aligned}$$

The step labelled '?' obviously follows if $\widetilde{\lim}$ satisfies the **L3**$\odot$ axiom, i.e. if **L3**$\oplus$ = **L3**$\odot$. Hence whichever of $\overline{\lim}_{\wedge}$, $\overline{\lim}_{*}$ or $\overline{\lim}_{\otimes}$ we use, we must use the corresponding axiom (**L3**$\wedge$, **L3**$*$ or **L3**$\otimes$ respectively). In particular, we cannot have a situation where our SL-limit spaces satisfy **L3**$*$ and $\overline{\lim}$ is defined as $\overline{\lim} = \overline{\lim}_{\wedge}$, since in this case the proof we have just been examining doesn't work. Remember, we are trying for a quick fix here by generalizing Jäger's frame proofs and hoping that they work in the ecl-premonoid case.

Now let's look at the proof that $\overline{\lim}_{\odot}$ satisfies the **L3**$\odot$ axiom. Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$. Then

$$\begin{aligned}
&\overline{\lim}_{\odot}\mathcal{F} \odot \overline{\lim}_{\odot}\mathcal{G} \\
&\qquad = \bigvee\{\, \overset{m}{\underset{i=1}{\odot}} \lim \mathcal{F}_i \mid \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F}\,\} \odot \bigvee\{\, \overset{n}{\underset{j=1}{\odot}} \lim \mathcal{G}_j \mid \bigwedge_{j=1}^{n} \mathcal{G}_j \leq \mathcal{G}\,\} \\
&\qquad \overset{?}{=} \bigvee\{\, (\overset{m}{\underset{i=1}{\odot}} \lim \mathcal{F}_i) \odot (\overset{n}{\underset{j=1}{\odot}} \lim \mathcal{G}_j) \mid \bigwedge_{i=1}^{n} \mathcal{F}_i \leq \mathcal{F} \text{ and } \bigwedge_{j=1}^{n} \mathcal{G}_j \leq \mathcal{G}\,\} \\
&\qquad \overset{!}{\leq} \bigvee\{\, \overset{p}{\underset{k=1}{\odot}} \lim \mathcal{H}_k \mid \bigwedge_{k=1}^{p} \mathcal{H}_k \leq \mathcal{F} \wedge \mathcal{G}\,\}.
\end{aligned}$$

For the step labelled '?', we have to assume that $\odot$ distributes over arbitrary joins. This is true in the *general* ecl-premonoid case for $\wedge, *$ but not for $\otimes$ ($\otimes$ does not distribute over empty joins in general). For the step labelled '!', we need to assume that $\odot$ is commutative, since otherwise we are unjustified in assuming that $(a_m \odot \ldots \odot a_1) \odot (b_n \odot \ldots \odot b_1) = a_m \odot \ldots a_1 \odot b_n \odot \ldots b_1$ for $a_i, b_j \in L^X$. This rules out the $\otimes$ operation since in the general case it is not commutative. If a space satisfies the **L3**$\wedge$ axiom, then it also satisfies the **L3**$*$ axiom by Lemma 3.1.17, thus the **L3**$\wedge$ axiom is stronger than the **L3**$*$ axiom and the category of limit spaces defined by **L3**$*$ includes those defined by **L3**$\wedge$. For the **L3**$*$ axiom, the category $\underline{\mathbf{SL-LIM}}$ so defined *is* a reflective subcategory of $\underline{\mathbf{SL-KCS}}$, as proved previously. This is the reason that we take **L3**$*$ as the generalization of **L3** to the ecl-premonoid case. We can define a category $\underline{\mathbf{SL-LIMS}}$ of *strong SL-limit spaces* which satisfy the axiom **L3**$\wedge$.

Principal convergence spaces

Definition 5.2.17 (see [21, 22] for the frame case): Let $(X, \lim)$ be an SL-preconvergence space. Then lim satisfies the **Lp** axiom $\Leftrightarrow$

$$\textbf{Lp} \qquad \forall\, \mathcal{F} \in \mathcal{F}_L^S(X)\ \forall\, x \in X, \quad \lim \mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}_{\lim}^x(a) \to \mathcal{F}(a)),$$

Where for $x \in X, a \in L^X$, $\mathcal{U}_{\lim}^x(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a))$. If lim additionally satisfies **L1** and **L2**, then $(X, \lim)$ is a *stratified L-principal convergence space*. We abbreviate 'stratified L-principal convergence space' as SL-principal convergence space.

Definition 5.2.18 (see [21, 22] for the frame case): We define the category $\underline{\mathbf{SL-PCS}}$ by

Objects SL-principal convergence spaces $(X, \lim)$.

Morphisms Functions $\phi : X \to Y$ between spaces $(X, \lim_X)$ and $(Y, \lim_Y)$ which satisfy

$$\forall\, \mathcal{F} \in \mathcal{F}_L^S(X) \quad \phi(\lim_X \mathcal{F}) \leq \lim_Y \phi(\mathcal{F}).$$

The identity function id_X is a morphism from $(X, \lim)$ to itself. Morphism composition is the usual function composition.

Lemma 5.2.19 : $\underline{\mathbf{SL-PCS}}$ is a full subcategory of $\underline{\mathbf{SL-LIM}}$.

Proof:

We prove that **Lp** implies **L3**. Let $(X, \lim)$ be an SL-principal convergence space. Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), x \in X$. Then

$$\begin{aligned}
\lim \mathcal{F}(x) * \lim \mathcal{G}(x) &\overset{\mathbf{Lp}}{=} \bigwedge_{a \in L^X} (\mathcal{U}^x_{\lim}(a) \to \mathcal{F}(a)) * \bigwedge_{b \in L^X} (\mathcal{U}^x_{\lim}(b) \to \mathcal{G}(b)) \\
&\leq \bigwedge_{a \in L^X} \big((\mathcal{U}^x_{\lim}(a) \to \mathcal{F}(a)) * (\mathcal{U}^x_{\lim}(a) \to \mathcal{G}(a)) \big) \\
&\leq \bigwedge_{a \in L^X} \big((\mathcal{U}^x_{\lim}(a) \to \mathcal{F}(a)) \wedge (\mathcal{U}^x_{\lim}(a) \to \mathcal{G}(a)) \big) \\
&= \bigwedge_{a \in L^X} (\mathcal{U}^x_{\lim}(a) \to (\mathcal{F} \wedge \mathcal{G})(a)) = \lim \mathcal{F} \wedge \mathcal{G}(x).
\end{aligned}$$

Thus $\mathrm{Ob}\,(\underline{\mathbf{SL-PCS}}) \subseteq \mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$.

Now for $(X, \lim_X), (Y, \lim_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-PCS}})$, obviously

$$\hom_{\underline{\mathbf{SL-PCS}}}((X, \lim_X), (Y, \lim_Y)) = \hom_{\underline{\mathbf{SL-LIM}}}((X, \lim_X), (Y, \lim_Y)).$$

Since composition and identity are defined to be the same for both categories, we have proved that $\underline{\mathbf{SL-PCS}}$ is a full subcategory of $\underline{\mathbf{SL-LIM}}$. ∎

Lemma 5.2.20 (see [21, 22] for the frame case): Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$. For $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X$ we define

$$\overline{\lim}\,\mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}^x_{\lim}(a) \to \mathcal{F}(a)).$$

Then $(X, \overline{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-PCS}})$ if $\lim$ satisfies the $\mathbf{L}\otimes$ axiom. Furthermore $\overline{\lim} \geq \lim$ (regardless of whether $\lim$ satisfies $\mathbf{L}\otimes$).

Proof:
We prove that $\overline{\lim}$ satisfies **L1** and **Lp**. We know from Lemma 4.4.2 that **Lp** implies **L2**. Let $x \in X, a \in L^X$. Then

$$\mathcal{U}^x_{\lim}(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \leq \lim[x](x) \to [x](a) \overset{\mathbf{L1}}{=} [x](a).$$

So $\overline{\lim}[x](x) = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to [x](a)) \geq \bigwedge_{a \in L^X} ([x](a) \to [x](a)) = \top$. Thus $\overline{\lim}$ satisfies **L1**.

Again, let $x \in X, a \in L^X$. Now

$$\mathcal{U}^x_{\overline{\lim}}(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\overline{\lim}\,\mathcal{F}(x) \to \mathcal{F}(a))$$

$$= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\bigwedge_{b\in L^X} (\mathcal{U}^x_{\lim}(b) \to \mathcal{F}(b)) \to \mathcal{F}(a))$$
$$\geq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} ((\mathcal{U}^x_{\lim}(a) \to \mathcal{F}(a)) \to \mathcal{F}(a)) \geq \mathcal{U}^x_{\lim}(a).$$

Since lim satisfies **L1**, we have that

$$\lim \text{ satisfies } \mathbf{L}\otimes \Leftrightarrow \ \forall\, x \in X, \quad \mathcal{U}^x_{\lim} \in \mathcal{F}_L^S(X).$$

(see Lemma 4.4.3). Thus, assuming that lim does satisfy $\mathbf{L}\otimes$, for $x \in X$ we can calculate

$$\overline{\lim}\,\mathcal{U}^x_{\lim}(x) = \bigwedge_{a\in L^X} (\mathcal{U}^x_{\lim}(a) \to \mathcal{U}^x_{\lim}(a)) = \top.$$

Then $\mathcal{U}^x_{\overline{\lim}}(a) \leq \overline{\lim}\,\mathcal{U}^x(x) \to \mathcal{U}^x(a) = \mathcal{U}^x(a)$. Thus if lim satisfies $\mathbf{L}\otimes$, $\mathcal{U}^x_{\lim} = \mathcal{U}^x_{\overline{\lim}}$ for all $x \in X$. Then $\overline{\lim}\,\mathcal{F}(x) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)}(\mathcal{U}^x_{\overline{\lim}}(a) \to \mathcal{F}(a)$ for all $x \in X, \mathcal{F} \in \mathcal{F}_L^S(X)$, so $\overline{\lim}$ satisfies the **Lp** axiom.

Finally, let $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X$. Then

$$\begin{aligned}\overline{\lim}\,\mathcal{F}(x) &= \bigwedge_{a\in L^X} (\mathcal{U}^x_{\lim}(a) \to \mathcal{F}(a)) \\ &= \bigwedge_{a\in L^X} (\bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X)} (\lim \mathcal{G}(x) \to \mathcal{G}(a)) \to \mathcal{F}(a)) \\ &\geq \bigwedge_{a\in L^X} ((\lim \mathcal{F}(x) \to \mathcal{F}(a)) \to \mathcal{F}(a)) \geq \lim \mathcal{F}(x).\end{aligned}$$

Thus $\overline{\lim} \geq \lim$. Note that this is true regardless of whether lim satisfies $\mathbf{L}\otimes$ or not. ■

Lemma 5.2.21 : Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL} - \mathbf{GCS}})$. Then

$$\lim \text{ satisfies } \mathbf{Lp} \not\Rightarrow \lim \text{ satisfies } \mathbf{L}\otimes.$$

Proof:
We prove that $(X, \lim)$ from our previous example (Example 5.1.16) satisfies the **Lp** axiom. Hence, since we have already proved that it does not satisfy the $\mathbf{L}\otimes$ axiom, we will have proved that $\mathbf{Lp} \not\Rightarrow \mathbf{L}\otimes$.

We know from Lemma 5.2.20 that $\overline{\lim} \geq \lim$ always. Now lim satisfies **Lp** iff $\overline{\lim} = \lim$. Assume that lim does not satisfy **Lp**. Then $\exists \mathcal{F} \in \mathcal{F}_L^S(X)\, \exists\, z \in X, \quad \overline{\lim}\,\mathcal{F}(z) > \lim \mathcal{F}(z)$. Now

$$\overline{\lim}\,\mathcal{F}(z) > \lim \mathcal{F}(z) \Leftrightarrow \bigwedge_{a\in L^X} (\mathcal{U}^z_{\lim}(a) \to \mathcal{F}(a)) > \lim \mathcal{F}(z)$$

$$\Rightarrow\ \forall\, a\in L^X,\quad \mathcal{U}^z_{\lim}(a)\to \mathcal{F}(a) > \lim \mathcal{F}(z).$$

Assume $z = x$. Then $\mathcal{U}^x_{\lim}(\top,\alpha) = \top$ and $\lim \mathcal{F}(x) = \mathcal{F}(a)$, thus $\exists\, a \in L^X,\ \ \mathcal{U}^x_{\lim}(a)\to\mathcal{F}(a) \not> \lim\mathcal{F}(x)$. But if we assume $z = y$ then similarly $\mathcal{U}^y_{\lim}(\alpha,\top) = \top$ and $\lim\mathcal{F}(y) = \mathcal{F}(\alpha,\top)$ and thus $\exists\, a \in L^X,\ \ \mathcal{U}^y_{\lim}(a) \to \mathcal{F}(a) \not> \lim\mathcal{F}(y)$. Thus we have that $\overline{\lim} = \lim$, and $\lim$ satisfies **Lp**. ■

Lemma 5.2.22 : Let $(X,\lim)\in \mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$, $(X,\widetilde{\lim})\in \mathrm{Ob}\,(\underline{\mathbf{SL-PCS}})$. Then

$$\lim \le \widetilde{\lim} \Rightarrow \overline{\lim} \le \widetilde{\lim}\,.$$

Proof:
Assume that $\lim \le \widetilde{\lim}$. Let $x\in X, a\in L^X$. Then

$$\mathcal{U}^x_{\lim}(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}^S_L(X)} (\lim\mathcal{F}(x)\to\mathcal{F}(a)) \\ \ge \bigwedge_{\mathcal{F}\in\mathcal{F}^S_L(X)} (\widetilde{\lim}\,\mathcal{F}(x)\to\mathcal{F}(a)) = \mathcal{U}^x_{\widetilde{\lim}}(a).$$

Now let $\mathcal{F}\in\mathcal{F}^S_L(X)$. Then

$$\overline{\lim}\,\mathcal{F}(x) = \bigwedge_{a\in L^X} (\mathcal{U}^x_{\lim}(a)\to\mathcal{F}(a)) \le \bigwedge_{a\in L^X} (\mathcal{U}^x_{\widetilde{\lim}}(a)\to\mathcal{F}(a)) \overset{\mathbf{Lp}}{=} \widetilde{\lim}\,\mathcal{F}(x).$$

Thus $\overline{\lim} \le \widetilde{\lim}$. ■

Remark 5.2.23 : Obviously, $\overline{\widetilde{\lim}} = \widetilde{\lim}$ since $\widetilde{\lim} \le \widetilde{\lim}$.

Lemma 5.2.24 : Let $(X,\lim_X)\xrightarrow{\phi}(Y,\lim_Y)\in \mathrm{Mor}\,(\underline{\mathbf{SL-LIM}})$. Then

$$(X,\overline{\lim}_X),(Y,\overline{\lim}_Y)\in \mathrm{Ob}\,(\underline{\mathbf{SL-PCS}}) \\ \Rightarrow (X,\overline{\lim}_X)\xrightarrow{\phi}(Y,\overline{\lim}_Y)\in \mathrm{Mor}\,(\underline{\mathbf{SL-PCS}}).$$

Proof:
Let $x\in X, b\in L^Y$. We calculate

$$\mathcal{U}^{\phi(x)}_{\lim_Y}(b) = \bigwedge_{\mathcal{H}\in\mathcal{F}^S_L(Y)} (\lim_Y \mathcal{H}(\phi(x))\to\mathcal{H}(b))$$

$$\leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X)} (\lim_Y \phi(\mathcal{G})(\phi(x)) \to \phi(\mathcal{G})(b))$$
$$\leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X)} (\lim_X \mathcal{G}(x) \to \mathcal{G}(\phi^{\leftarrow}(b))) = \mathcal{U}^x_{\lim_X}(\phi^{\leftarrow}(b)).$$

Now let $\mathcal{F} \in \mathcal{F}_L^S(X)$. Then

$$\begin{aligned}\overline{\lim}_Y \phi(\mathcal{F})(\phi(x)) &= \bigwedge_{b\in L^Y} (\mathcal{U}^{\phi(x)}_{\lim_Y}(b) \to \phi(\mathcal{F})(b)) \\ &\geq \bigwedge_{b\in L^Y} (\mathcal{U}^x_{\lim_X}(\phi^{\leftarrow}(b)) \to \mathcal{F}(\phi^{\leftarrow}(b))) \\ &\geq \bigwedge_{a\in L^X} (\mathcal{U}^x_{\lim_X}(a) \to \mathcal{F}(a)) = \overline{\lim}_X \mathcal{F}(x).\end{aligned}$$

Thus we have that $\phi \in \hom_{\underline{\mathbf{SL-LIM}}}((X, \overline{\lim}_X), (Y, \overline{\lim}_Y))$. Thus

$$(X, \overline{\lim}_X), (Y, \overline{\lim}_Y) \in \mathrm{Ob}\,(\underline{\mathbf{SL-PCS}}) \Rightarrow$$
$$\phi \in \hom_{\underline{\mathbf{SL-PCS}}}((X, \overline{\lim}_X), (Y, \overline{\lim}_Y)).$$

■

Theorem 5.2.25 (see [21, 22] for the frame case): If L satisfies condition **M**, then $\underline{\mathbf{SL-PCS}}$ is a full reflective subcategory of $\underline{\mathbf{SL-LIM}}$

Proof:
Assume that L satisfies **M**. By Lemma 5.1.18, this is equivalent to the condition that all $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$ satisfy $\mathbf{L}\otimes$. We know from Lemma 5.2.19 that $\underline{\mathbf{SL-PCS}}$ is a full subcategory of $\underline{\mathbf{SL-LIM}}$. We define a functor

$$F\colon \underline{\mathbf{SL-LIM}} \to \underline{\mathbf{SL-PCS}}$$
$$(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \mapsto (X, \overline{\lim}_X) \xrightarrow{\phi} (Y, \lim_Y).$$

This is possible by Lemmas 5.2.20 and 5.2.24. Note that $(X, \overline{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL-PCS}})$ is guaranteed by the fact that lim satisfies the $\mathbf{L}\otimes$ axiom. Since $\underline{\mathbf{SL-PCS}}$ is a subcategory of $\underline{\mathbf{SL-LIM}}$, we also have the inclusion functor

$$E\colon \underline{\mathbf{SL-PCS}} \hookrightarrow \underline{\mathbf{SL-LIM}}$$
$$(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \mapsto (X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y).$$

Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$. Then $E \circ F(X, \lim) = (X, \overline{\lim}) \geq (X, \lim)$. Now let $(X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y) \in \mathrm{Mor}\,(\underline{\mathbf{SL-PCS}})$. Then $F \circ E((X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y)) = (X, \lim_X) \xrightarrow{\phi} (Y, \lim_Y)$. Thus $F \circ E = \mathrm{id}_{\underline{\mathbf{SL-PCS}}}$ and $E \circ F \geq \mathrm{id}_{\underline{\mathbf{SL-LIM}}}$. By Theorem 1.6.2, $\underline{\mathbf{SL-PCS}}$ is a reflective subcategory of $\underline{\mathbf{SL-LIM}}$. ■

We know that if L satisfies **M**, then $\underline{\mathbf{SL-PCS}}$ is topological over $\underline{\mathbf{SET}}$, since it is a full reflective subcategory of $\underline{\mathbf{SL-GCS}}$. However it is interesting to note that we can prove that $\underline{\mathbf{SL-PCS}}$ is topological over $\underline{\mathbf{SET}}$ without invoking the $\mathbf{L}\otimes$ axiom or **M**. This raises the hope that $\underline{\mathbf{SL-PCS}}$ can be proved to be a reflective subcategory of $\underline{\mathbf{SL-GCS}}$ *in general.*

Theorem 5.2.26 : $\underline{\mathbf{SL-PCS}}$ is topological over $\underline{\mathbf{SET}}$. Furthermore it is amnestic, fibre small and has the terminal separator property.

Proof:
$\underline{\mathbf{SL-PCS}}$ is a full subcategory of $\underline{\mathbf{SL-GCS}}$ (Lemma 5.2.19), hence it is amnestic and fibre small. Since $\underline{\mathbf{SL-GCS}}$ has the terminal separator property, we need to prove that the unique generalized convergence structure on the one point set $\{x\}$ is also a principal convergence structure, in other words, satisfies the **Lp** axiom. Let $X = \{x\}$. From Theorem 5.1.4, the only convergence structure on X is given by $\lim \mathcal{F}(x) = \top$ for $\mathcal{F} \in \mathcal{F}_L^S(X)$. Now for $a \in L^X$, $\mathcal{U}_{\lim}^x(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^S(X)} \mathcal{F}(a)$. Thus

$$\overline{\lim}\,\mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}_{\lim}^x(a) \to \mathcal{F}(a)) = \top = \lim \mathcal{F}(x).$$

Thus lim satisfies **Lp**.

Let $\emptyset \neq X \in \mathrm{Ob}\,(\underline{\mathbf{SET}})$ and let $((X_i, \lim_i))_{i \in I}$ be a family of SL-principal convergence spaces indexed by the class I. Let $(\phi_i \colon X \to X_i)_{i \in I}$ be a corresponding family of functions. For $\mathcal{F} \in \mathcal{F}_L^S(X)$ we define

$$\lim_X \mathcal{F} = \bigwedge_{i \in I} \phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{F})).$$

Then we know from the proof of Theorem 5.1.4 that $\lim_X$ satisfies **L1** and **L2** and further that all of the functions ϕ_i are continuous (in the category $\underline{\mathbf{SL-GCS}}$) between $(X, \lim_X)$ and $(X_i, \lim_i)$. If we can prove that $\lim_X$ satisfies **Lp**, then $(X, \lim_X) \in \mathrm{Ob}\,(\underline{\mathbf{SL-PCS}})$ and all of the ϕ_is are continuous in $\underline{\mathbf{SL-PCS}}$.

Let $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X, i \in I$. Then

$$\overline{\lim}_X \mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}_{\lim_X}^x(a) \to \mathcal{F}(a)) \leq \bigwedge_{b \in L^{X_i}} (\mathcal{U}_{\lim_X}^x(\phi_i^{\leftarrow}(b)) \to \mathcal{F}(\phi_i^{\leftarrow}(b))).$$

Let $b \in L^{X_i}$. Then

$$\begin{aligned}
\mathcal{U}^x_{\lim_X}(\phi_i^{\leftarrow}(b)) &= \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X)} (\lim_X \mathcal{G}(x) \to \mathcal{G}(\phi_i^{\leftarrow}(b))) \\
&= \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X)} (\bigwedge_{j\in I} \phi_j^{\leftarrow}(\lim_j \phi_j(\mathcal{G}))(x) \to \phi_i(\mathcal{G})(b)) \\
&\geq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X)} (\lim_i \phi_i(\mathcal{G})(\phi_i(x)) \to \phi_i(\mathcal{G})(b)) \\
&\geq \bigwedge_{\mathcal{H}\in\mathcal{F}_L^S(X_i)} (\lim_i \mathcal{H}_i(\phi_i(x)) \to \mathcal{H}(b)) = \mathcal{U}^{\phi_i(x)}_{\lim_i}(b).
\end{aligned}$$

Thus

$$\begin{aligned}
\overline{\lim}_X \mathcal{F}(x) &\leq \bigwedge_{b\in L^{X_i}} (\mathcal{U}^x_{\lim_X}(\phi_i^{\leftarrow}(b)) \to \mathcal{F}(\phi_i^{\leftarrow}(b))) \\
&\leq \bigwedge_{b\in L^{X_i}} (\mathcal{U}^{\phi_i(x)}_{\lim_i}(b) \to \phi_i(\mathcal{F})(b)) \\
&= \overline{\lim}_i \phi_i(\mathcal{F})(\phi_i(x)) \overset{\mathbf{Lp}}{=} \lim_i \phi_i(\mathcal{F})(\phi_i(x)) \\
&= \phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{F}))(x).
\end{aligned}$$

Finally

$$\lim_X \mathcal{F}(x) \leq \bigwedge_{i\in I} \phi_i^{\leftarrow}(\lim_i \phi_i(\mathcal{F})(x) = \lim_X \mathcal{F}(x)$$

From Lemma 5.2.20, we know that $\overline{\lim}_X \geq \lim_X$, thus $\overline{\lim}_X = \lim_X$. Thus $\lim_X$ satisfies **Lp**.

We know from Theorem 5.1.4 that the source $\mathcal{S} = ((X, \lim_X) \xrightarrow{\phi_i} (X_i, \lim_i))_{i\in I}$ is initial. Since $\underline{\mathbf{SL-PCS}}$ is amnestic we have proved that $\underline{\mathbf{SL-PCS}}$ is topological over $\underline{\mathbf{SET}}$. ■

Discussion: Changing the definition of $\underline{\mathbf{SL-PCS}}$

Examining the proof of Theorem 5.2.26, we see that initial structures in $\underline{\mathbf{SL-PCS}}$ are formed in exactly the same way as in $\underline{\mathbf{SL-GCS}}$. This leads us to suspect that $\underline{\mathbf{SL-PCS}}$ *could* in fact be a reflective subcategory of $\underline{\mathbf{SL-LIM}}$, and possibly the proof of Theorem 5.2.25 can be improved to remove the dependence on the $\mathbf{L}\otimes$ axiom. In the final part of this section we look at an attempted approach which initially seems as if it might be successful, but ultimately fails.

We seek to find a functor F which maps objects and functors of $\underline{\mathbf{SL-LIM}}$ to $\underline{\mathbf{SL-PCS}}$ in such a way that $F(X, \lim) \geq (X, \lim)$. As a candidate for $F(X, \lim)$ we suggest $(X, \overline{\lim})$ where $\overline{\lim}$ is defined as in Lemma 5.2.27.

Lemma 5.2.27 : Let $(X, \lim)$ be an SL-generalized convergence space. As in Lemma 4.2.11, we define

L0 $\quad \forall\, x \in X \ \forall\, a \in L^X, \quad \mathcal{U}^x_{\lim}(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)).$

We further define

NF $\quad \forall\, x \in X, \quad \mathcal{N}^x_{\lim} = \bigwedge \{\, \mathcal{G} \in \mathcal{F}^S_L(X) \mid \mathcal{U}^x_{\lim} \leq \mathcal{G} \,\}.$

Then $\forall\, x \in X, \quad \mathcal{U}^x_{\lim} \leq [x]$ and $\mathcal{N}^x_{\lim} \in \mathcal{F}^S_L(X)$.

Proof:
Let $a \in L^X$. Then $\mathcal{U}^x_{\lim}(a) \leq \lim[x](x) \to [x](a) = \top \to [x](a) = [x](a)$. Thus $\mathcal{U}^x_{\lim} \leq [x]$. The set $\{\, \mathcal{G} \in \mathcal{F}^S_L(X) \mid \mathcal{U}^x_{\lim} \leq \mathcal{G} \,\}$ is non-empty, thus $\mathcal{N}^x_{\lim} \in \mathcal{F}^S_L(X)$. ■

Lemma 5.2.28 : Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. Define $\overline{\lim}$ as in Lemma 5.2.27. Then

$$\mathcal{U}^x_{\overline{\lim}} = \mathcal{N}^x_{\lim}.$$

Proof:
Let $a \in L^X$. Then

$$\begin{aligned}
\mathcal{U}^x_{\overline{\lim}}(a) &= \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\overline{\lim}\,\mathcal{F}(x) \to \mathcal{F}(a)) \\
&= \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} \Big(\bigwedge_{b \in L^X} (\mathcal{N}^x_{\lim}(b) \to \mathcal{F}(b)) \to \mathcal{F}(a) \Big) \\
&\geq \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} ((\mathcal{N}^x_{\lim}(a) \to \mathcal{F}(a)) \to \mathcal{F}(a)) \geq \mathcal{N}^x_{\lim}(a)
\end{aligned}$$

Thus $\mathcal{U}^x_{\overline{\lim}} \geq \mathcal{N}^x_{\lim}$. Also we have

$$\overline{\lim}\,\mathcal{N}^x_{\lim}(x) = \bigwedge_{a \in L^X} (\mathcal{N}^x_{\lim}(a) \to \mathcal{N}^x_{\lim}(a)) = \top$$

Thus $\mathcal{U}^x_{\overline{\lim}}(a) \leq \top \to \mathcal{N}^x_{\overline{\lim}}(a) = \mathcal{N}^x_{\overline{\lim}}(a)$. So $\mathcal{U}^x_{\overline{\lim}} = \mathcal{N}^x_{\lim}$. ■

Lemma 5.2.29 : Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL-GCS}})$. Define $\overline{\lim}$ as in Lemma 5.2.27. Then

$$\lim = \overline{\lim} \Leftrightarrow \lim \text{ satisfies } \mathbf{L}\otimes \text{ and } \mathbf{Lp}.$$

Proof:
Assume $\lim = \overline{\lim}$. Then $\mathcal{N}^x_{\lim} = \mathcal{U}^x_{\overline{\lim}} = \mathcal{U}^x_{\lim}$. Thus lim satisfies $\mathbf{L}\otimes$. Furthermore for $\mathcal{F} \in \mathcal{F}^S_L(X), x \in X$,

$$\lim \mathcal{F}(x) = \overline{\lim}\,\mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{N}^x_{\lim}(a) \to \mathcal{F}(a)) = \bigwedge_{a \in L^X} (\mathcal{U}^x_{\lim}(a) \to \mathcal{F}(a)).$$

Thus lim satisfies **Lp**.

For the converse, assume that lim satisfies $\mathbf{L}\otimes$ and **Lp**. Then $\mathcal{U}^x_{\lim} \in \mathcal{F}^S_L(X)$. Thus $\mathcal{U}^x_{\lim} = \mathcal{N}^x_{\lim}$. So for $\mathcal{F} \in \mathcal{F}^S_L(X), x \in X$,

$$\overline{\lim}\,\mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{N}^x_{\lim}(a) \to \mathcal{F}(a)) = \bigwedge_{a \in L^X} (\mathcal{U}^x_{\lim}(a) \to \mathcal{F}(a)) \overset{\mathbf{Lp}}{=} \lim \mathcal{F}(x).$$

Thus $\overline{\lim} = \lim$. ■

Lemma 5.2.30 : Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL} - \mathbf{GCS}})$. Define $\overline{\lim}$ as in Lemma 5.2.27. Then $(X, \overline{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL} - \mathbf{PCS}})$.

Proof:
By Lemma 5.2.28, $\mathcal{U}^x_{\overline{\lim}} = \mathcal{N}^x_{\lim} \in \mathcal{F}^S_L(X)$. Thus $\mathcal{N}^x_{\overline{\lim}} = \mathcal{U}^x_{\overline{\lim}} = \mathcal{N}^x_{\lim}$. Let $\mathcal{F} \in \mathcal{F}^S_L(X), x \in X$. Then

$$\overline{\overline{\lim}}\,\mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{N}^x_{\overline{\lim}}(a) \to \mathcal{F}(a)) = \bigwedge_{a \in L^X} (\mathcal{N}^x_{\lim}(a) \to \mathcal{F}(a)) = \overline{\lim}\,\mathcal{F}(x).$$

Thus by Lemma 5.2.29, $\overline{\lim}$ satisfies **Lp** and $\mathbf{L}\otimes$. It is obvious that $\overline{\lim}$ satisfies **L1** and **L2**, hence $(X, \overline{\lim}) \in \mathrm{Ob}\,(\underline{\mathbf{SL} - \mathbf{PCS}})$. ■

Lemma 5.2.31 : Let $(X, \lim) \in \mathrm{Ob}\,(\underline{\mathbf{SL} - \mathbf{GCS}})$. Define $\overline{\lim}$ as in Lemma 5.2.27. Then

$$\lim \geq \overline{\lim} \Rightarrow \lim \text{ satisfies } \mathbf{L}\otimes.$$

Proof:
By definition (see Lemma 5.2.27), $\forall\, x \in X, \quad \mathcal{U}^x_{\lim} \leq \mathcal{N}^x_{\lim}$. Assume $\overline{\lim} \geq \lim$. Let $x \in X, a \in L^X$ then by Lemma 5.2.28,

$$\mathcal{U}^x_{\lim}(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}^S_L(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a))$$

$$\geq \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X)} (\overline{\lim}\,\mathcal{F}(x) \to \mathcal{F}(a) = \mathcal{U}^x_{\overline{\lim}}(a) = \mathcal{N}^x_{\lim}(a).$$

Thus $\mathcal{U}^x_{\lim} = \mathcal{N}^x_{\lim} \in \mathcal{F}_L^S(X)$, thus lim satisfies $\mathbf{L}\otimes$. ■

Remark 5.2.32 : We know by Lemmas 5.2.19 and 5.2.21 that we have an example of an SL-limit space $(X, \lim)$ which does not satisfy $\mathbf{L}\otimes$. Thus for this limit space by Lemma 5.2.31, $\overline{\lim} \not\geq \lim$. Thus $(X, \overline{\lim}) \not\geq (X, \lim)$. Hence our proposed functor F fails to have the required properties and we cannot use this functor to prove that $\underline{\mathbf{SL-PCS}}$ is a reflective subcategory of $\underline{\mathbf{SL-LIM}}$.

Topological convergence spaces

Objects of $\underline{\mathbf{SL-TCS}}$ satisfy the axioms $\mathbf{L}\otimes$ and $\mathbf{Lt}$ in addition to the $\mathbf{L1}$, $\mathbf{L2}$ and $\mathbf{Lp}$ axioms. Hence $\mathrm{Ob}\,(\underline{\mathbf{SL-TCS}}) \subseteq \mathrm{Ob}\,(\underline{\mathbf{SL-PCS}})$. Again, we have that

$$\phi \in \hom_{\underline{\mathbf{SL-TCS}}}((X, \lim_X), (Y, \lim_Y)) \Leftrightarrow \phi \in \hom_{\underline{\mathbf{SL-PCS}}}((X, \lim_X), (Y, \lim_Y)).$$

Thus $\underline{\mathbf{SL-TCS}}$ is a full subcategory of $\underline{\mathbf{SL-PCS}}$.

We know that if we structure L so that the $\mathbf{L}\otimes$ axiom is satisfied for all SL-generalized convergence spaces over L, in other words, if L satisfies condition $\mathbf{M}$, then $\underline{\mathbf{SL-TCS}}$ is a full, reflective subcategory of $\underline{\mathbf{SL-GCS}}$ and there exist functors $F : \underline{\mathbf{SL-GCS}} \to \underline{\mathbf{SL-TCS}}$ and $G : \underline{\mathbf{SL-TCS}} \to \underline{\mathbf{SL-GCS}}$ such that G is the inclusion functor and

$$G \circ F \geq \mathrm{id}_{\underline{\mathbf{SL-GCS}}} \quad \text{and} \quad F \circ G = \mathrm{id}_{\underline{\mathbf{SL-TCS}}}.$$

If we now restrict F to the objects of $\underline{\mathbf{SL-PCS}}$, then we obtain

$$G \circ F \geq \mathrm{id}_{\underline{\mathbf{SL-PCS}}} \quad \text{and} \quad F \circ G = \mathrm{id}_{\underline{\mathbf{SL-TCS}}}.$$

so $\underline{\mathbf{SL-TCS}}$ is a full, reflective subcategory of $\underline{\mathbf{SL-PCS}}$ if L is structured so that the $\mathbf{L}\otimes$ axiom is satisfied by all L-convergence spaces.

Chapter 6

Conclusion

In summary, we have successfully characterized stratified L-topological spaces (X, Δ) by axioms on a convergence function $\lim\colon \mathcal{F}_L^S(X) \to L^X$. In particular, we found that such a function must satisfy the axioms **L1**, **Lp**, **Lt** and **L⊗**. The **L⊗** axiom is a new requirement which is always satisfied by lim in the previously known case where L is a frame. We showed that the **L⊗** axiom is *not* always satisfied and is not implied by **L1**, **L2** or **Lp**. We showed in the category **SL − GCS** whose objects satisfy **L1** and **L2**, that all such objects satisfy **L⊗** if and only if the monotonicity condition **M** is satisfied. The monotonicity condition includes the important special cases of frames and those ecl-premonoids defined by a GL-monoid and the associated monoidal mean operator.

We have formulated the generalizations to the ecl-premonoid case of the Kowalski and Fischer axioms, **LK** and **LF** and investigated their relationship to the generalized **Lp** and **Lt** axioms. Also we have generalized Jäger's previous results ([23, 24]) and shown that the **Lp** axiom is equivalent to the requirement that two simpler, independent axioms are satisfied, **LpW1** and **LpW2**.

We have defined the category **SL − GCS** of SL-convergence spaces, and shown that it is topological over **SET** and contains a reflective subcategory **SL − TCS**, isomorphic to **SL − TOP**, if the monotonicity condition is satisfied by L. Although strictly speaking outside the scope of the thesis, as a first step towards investigating cartesian closedness of **SL − GCS** we have made some definitions based on the structures used to prove cartesian closedness in the frame case. We have left as an open problem the question as to whether the category **SL − GCS⊗** whose objects satisfy **L1**, **L2** and **L⊗** is topological over **SET**.

We have investigated some subcategories of **SL − GCS**, defined by restricting the objects of **SL − GCS** with axioms which are generalizations of axioms which have proved important in the classical case (see e.g. [26]). In particular we have shown that **SL − KCS**, the category of SL-Kent convergence spaces, is a reflective subcategory of $\underline{\mathbf{SL-GCS}}$. Similarly $\underline{\mathbf{SL-LIM}}$, the category of S$L$-limit spaces, is a reflective subcategory of $\underline{\mathbf{SL-KCS}}$. The **L3** axiom which defines $\underline{\mathbf{SL-LIM}}$ required some care to generalize. We have shown that $\underline{\mathbf{SL-PCS}}$, the category of S$L$-principal convergence spaces, is a reflective subcategory of $\underline{\mathbf{SL-LIM}}$ if all spaces in $\mathrm{Ob}\,(\underline{\mathbf{SL-LIM}})$ satisfy $\mathbf{L}\otimes$. However we managed to prove that $\underline{\mathbf{SL-PCS}}$ is topological over $\underline{\mathbf{SET}}$, with initial stuctures formed in the same way as for $\underline{\mathbf{SL-LIM}}$. This leads us to speculate that perhaps $\underline{\mathbf{SL-PCS}}$ is a reflective subcategory of $\underline{\mathbf{SL-LIM}}$ in general, although we have left this as an open problem.

Probably the most useful issue which could be addressed by future research is a search for a simpler formulation of the $\mathbf{L}\otimes$ axiom. Although many of the important special cases are covered by restricting the lattice with the monotonicity condition, the fact remains that examples of SL-topological spaces exist even when L does not satisfy **M**. A simpler formulation of $\mathbf{L}\otimes$ would enable us for example to investigate conditions under which categories are topological over $\underline{\mathbf{SET}}$ when the objects are required to satisfy $\mathbf{L}\otimes$.

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