RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS

A STUDY OF BARRED PREFERENTIAL ARRANGEMENTS WITH APPLICATIONS TO NUMERICAL APPROXIMATION IN ELECTRIC CIRCUITS

by

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Abstract

In 1854 Cayley proposed an interesting sequence 1,1,3,13,75,541,... in connection with analytical forms called trees. Since then there has been various combinatorial interpretations of the sequence. The sequence has been interpreted as the number of preferential arrangements of members of a set with n elements. Alternatively the sequence has been interpreted as the number of ordered partitions; the outcomes in races in which ties are allowed or geometrically the number of vertices, edges and faces of simplicial objects. An interesting application of the sequence is found in combination locks. The idea of a preferential arrangement has been extended to a wider combinatorial object called barred preferential arrangement with multiple bars. In this thesis we study barred preferential arrangements combinatorially with application to resistance of certain electrical circuits. In the process we derive some results on cyclic properties of the last digit of the number of barred preferential arrangements. An algorithm in python has been developed to find the number of barred preferential arrangements.

Contents

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Dedicated to my late Brother Malinda Nkonkobe.

Preface

Cayley in [9] proposed the sequence 1,1,3,13,75,541,... with its combinatorial interpretation, as the number of trees with a certain number of knots. Since then the sequence has been given various combinatorial interpretations and has been connected to a number of other combinatorial sequences (for instance in [25],[24],[40]). In [31] the sequence has been interpreted as the number of ordered partitions of elements of the set $X_n = \{1, 2, 3, \ldots, n\}$. In [27] Murali has connected the sequence to the number of preferential fuzzy sets of certain sets. Recently in [2] the authors generalised the idea of a preferential arrangement to barred preferential arrangements with multiple bars.

In this thesis we thoroughly investigate the properties of preferential arrangements and barred preferential arrangements. We then apply the results obtained here to determine the equivalence resistance in electrical circuits. Another application results considered in this study pertain to combination locks, this we discuss in chapter Two. We conclude the thesis by raising some useful questions of generalisation of the results obtained here. We determine the numerical values of combinatorial entities of varying parameters using a computer program in python. In the last few pages we have included the actual program and illustrative pages. Now we briefly discuss below the contents of each chapter.

In chapter One we gather some useful known concepts. We define factorial polynomials, finite differences and Stirling numbers of both kinds. We connect Stirling numbers to Leibniz Harmonic triangle. We also establish a relationship between the harmonic sequence of finite differences.

We introduce preferential arrangements in chapter Two. We study their connection to locks which use combinations. We then discuss a relationship between preferential arrangements and races in-which ties are allowed.

In chapter Three we study a relationship between preferential arrangements and the resistance of a hypercube. We also analyse combinatorial arguments used in obtaining equivalence resistance of certain electric circuits. We discuss barred preferential arrangements in chapter Four. We then study their relationship with equivalence resistance of certain resistive structures.

Throughout the thesis we have adequately acknowledge results to their respective sources. Some of the results presented in this study is no-where mentioned in the literature we consulted; the wok is our own original contribution. For instance: equation 7; equation 39; theorem 18; theorem 19; theorem 28 and the proof of theorem 23. The python program in appex 1.

1. **Chapter One**

1.1. **Finite differences.** .

In this section we state definitions on finite differences and theorems.

Definition 1. [8] *"For a function $t(z)$ (a discrete function) in which each z_i has an image h_i such that the $z_i's$ are equally spaced $\Delta t(z_i)$ is defined as $\Delta t(z_i) = t(z_{i+1}) - t(z_i)$".*

This is called the forward difference of $t(z)$.

Definition 2. [8] *"For a function $t(z)$ (a discrete function) in which each z_i has an image h_i such that the $z_i's$ are equally spaced $\nabla t(z_i)$ is defined as $\nabla t(z_i) = t(z_i) - t(z_{i-1})$".*

This is called the backward difference of $t(z)$.

Definition 3. [8] *"For a function $t(z)$ (a discrete function) in which each z_i has an image h_i such that the $z_i's$ are equally spaced $E^s(t(z_i))$ is defined as $E^s(t(z_i)) = t(z_{i+s})$ where $s \in \mathbb{Z}$".*

This is called the shift operator.

We now show relationships between the three stated difference operators.

For any $t(z)$ as defined above the following relationship holds (see [14]);

$$1 + \Delta = E \tag{1}$$

We illustrate the above relationship in what follows,
$(1 + \Delta)t(z_i) = t(z_i) + \Delta t(z_i) = t(z_i) + [t(z_{i+1}) - t(z_i)] = t(z_{i+1})$
$= E(t(z_i))$.
Also for any $t(z)$ as defined above we have (see[14]);

$$1 - \nabla = E^{-1} \tag{2}$$

We end this section with the following theorem,

Theorem 1. [14] *For* $m \geq 0$,

$$\Delta^m t(z_0) = \sum_{s=0}^{m} \binom{m}{s} (-1)^s t(z_{m-s})$$

We prove the theorem in the following way (see [14]): The LHS of the statement of the theorem is LHS=$\Delta^m t(z_0)$. Using equation 1 above we have LHS=$(E-1)^m t(z_0)$. Making use of the binomial theorem we have LHS=$\sum_{s=0}^{m} \binom{m}{s} (-1)^s E^{m-s} t(z_0)$ $\Rightarrow$LHS=$\sum_{s=0}^{m} \binom{m}{s} (-1)^s t(z_{m-s})$. From this the statement of the theorem follows.

1.2. **Factorial polynomials.** .

In this section we introduce and discuss some identities on factorial polynomials. The study of factorial polynomials has become one of the corner stones of numerical analysis throughout the world. Most modern book in numerical analysis includes a section on factorial polynomials. Factorial polynomials have been studied in various contexts in numerical analysis and related fields for many centuries (for instance in [8],[14],[19]).

Definition 4. [14] *“ We define a falling factorial polynomial as,*
$l_{(n)} = l(l-1)(l-2)\cdots(l-n+1)$, *where* $n \in \mathbb{Z}^+$ *”.*

We observe that $\frac{l(l-1)(l-2)\cdots(l-n+1)}{n!} = \binom{l}{n}$, when $l \in \mathbb{R}$, $n \in \mathbb{Z}^+$ (see [34]). This together with definition 4 above imply that $\binom{l}{n} = \frac{l_{(n)}}{n!}$ (see [14]). For instance,
$\binom{0.5}{3} = \frac{0.5(0.5-1)(0.5-2)}{3!} = \frac{1}{16}$. We end this section by stating without proof two theorems on factorial polynomials,

Theorem 2. [14] *For* $n \in \mathbb{Z}^+$

$$l_{(n)} = \tfrac{1}{l-n} l_{(n+1)}$$

Theorem 3. [14] *For* $n \in \mathbb{Z}^+$

$$\Delta\, l_{(n)} = n l_{(n-1)}$$

where Δ *is the forward difference operator.*

1.3. **Stirling numbers of the first kind.** .

We consider the factorial polynomial $l_n = l(l-1)(l-2)\cdots(l-n+1)$, we write the factorial polynomial as a sum of ordinary polynomials. For an example:

I. for $n = 2$ we have, $l_2 = l(l-1) = l^2 - l$

II. for $n = 3$ we have, $l_3 = l(l-1)(l-2) = l^3 - 3l^2 + 2l$

The coefficient of l^i when l_n is written as a sum of ordinary polynomials is a Stirling number of the first kind[14]. We will denote by $s(n,i)$ this Stirling number. For instance in the above two examples;

I. $s(2,2) = 1$ and $s(2,1) = -1$

II. $s(3,3) = 1$, $s(3,2) = -3$ and $s(3,1) = 2$.

For all $n \geq 0$ we define $s(n,0) = 0$. We can view $s(n,0) = 0$ as the coefficient of l^0 in the expansion of $l_{(n)}$ which is 0.

We will denote by $\left[{n \atop i}\right]$ the absolute value of $s(n,i)$ (see [6]). The numbers $\left[{n \atop i}\right]$ are known as unsigned Stirling numbers of the first kind[21]. The numbers $\left[{n \atop i}\right]$ have a combinatorial interpretation as the number of permutations of an n-element set having i distinct cycles [6]. So from examples I and II above; the number of permutations of a 3-element set having 2 cycles is 3; this is so since $s(3,2) = -3 \Rightarrow \left[{3 \atop 2}\right] = 3$.

We now illustrate in the form of a table, few values of $\left[{n \atop i}\right]$ for various n and i,

TABLE 1. A table for the unsigned Stirling numbers of the first kind[14].

$i\backslash n$	1	2	3	4	5	6	7
1	1	1	2	6	24	120	720
2		1	3	11	50	274	1764
3			1	6	35	225	1624
4				1	10	85	735
5					1	15	175
6						1	21
7							1

We end this section by stating two theorems without proof on the numbers $\left[{n \atop i}\right]$.

Theorem 4. [6] *For* $n, s \in \mathbb{Z}^+$

$$\left[{n \atop n-i}\right] = \sum_{1 \le s_1 < s_2 < s_3 \cdots < s_i < n} s_1 s_2 s_3 \cdots s_i$$

Theorem 5. [6] *For* $n, s \in \mathbb{Z}^+ : n > s$ *we have,*

$$\left[{n \atop s}\right] = \left[{n-1 \atop s-1}\right] + (n-1)\left[{n-1 \atop s}\right]$$

1.4. **Stirling numbers of the second kind.** .

When an ordinary polynomial l^n is written as a sum of factorial polynomials, for an example I. $l^2 = l_{(2)} + l_{(1)}$. This is so since
$l_{(2)} = l(l-1) = l^2 - l$, $l_{(1)} = l$, so $l_{(2)} + l_{(1)} = l^2 - l + l = l^2$
II. $l^3 = l_{(3)} + 3l_{(2)} + l_{(1)}$. The coefficients of the $l_{(i)}$'s when we write an ordinary polynomial l^n in terms of factorial polynomials are called

the Stirling numbers of the second kind[6]. We have illustrated in the above two examples how ordinary polynomials can be written in terms of factorial polynomials. We will denote by $\left\{ {n \atop i} \right\}$ the coefficient of $l_{(i)}$ when a polynomial l^n is written in terms of factorial polynomials.

For the above two examples;
for example I we have $\left\{ {2 \atop 1} \right\} = 1$ and $\left\{ {2 \atop 2} \right\} = 1$.
for example II we have $\left\{ {3 \atop 1} \right\} = 1$, $\left\{ {3 \atop 2} \right\} = 3$ and $\left\{ {3 \atop 3} \right\} = 1$.

The numbers $\left\{ {n \atop i} \right\}$ seem to first appear in Stirling's book [36]. For $i > n$ and $n = 0 = i$ we have $\left\{ {n \atop i} \right\} = 0$. There is a combinatorial interpretation of the numbers $\left\{ {n \atop i} \right\}$, for $n \geq 0$ the number $\left\{ {n \atop i} \right\}$ is the number of ways of partitioning an n-element set into i non-empty blocks/subsets [20]. We illustrate this idea using the set $X_3 = \{1, 2, 3\}$ and find $\left\{ {3 \atop 2} \right\}$. The possible blocks when we partition X_3 into 2 non-empty subsets are a)$\{3, 1\}\{2\}$ b) $\{3, 2\}\{1\}$ c) $\{1, 2\}\{3\}$, which is equal to 3, as was shown in example II above that $\left\{ {3 \atop 2} \right\} = 3$. We now illustrate with a table few values of $\left\{ {n \atop i} \right\}$ for various n and i.

TABLE 2. A table for the Stirling numbers of the second kind[14],[36],[39].

$i \backslash n$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2		1	3	7	15	31	63
3			1	6	25	90	301
4				1	10	65	350
5					1	15	140
6						1	21
7							1

In [39],[36] James Stirling proposed the following equation,

$$\frac{1}{(r-1)(r-2)\cdots(r-j)} = \sum_{t=j}^{\infty}(-1)^{t-j}\left\{ t \atop j \right\}\frac{1}{r^t} \tag{3}$$

Which upon rearrangement, letting $r = -n$ and multiplying by $j!$ we obtain the following factorial equation as observed in [31],

$$\frac{j!}{(n-1)(n-2)\cdots(n-j)} = \sum_{t=j}^{\infty} j!\left\{ t \atop j \right\}\frac{1}{n^t} \tag{4}$$

Equation 4 is a factorial form of equation 3. We will make reference to equation 4 in Chapter Three.

We end this section by stating without proof two theorems on the numbers $\left\{ n \atop i \right\}$ which are very much similar to the theorems on the numbers $\left[n \atop i \right]$ we stated in section 1.3 above.

Theorem 6. [6] *For $n, s_i \in \mathbb{Z}^+$: $s_i < n$ for $1 \leq i < n$ we have,*

$$\left\{ n+i \atop i \right\} = \sum_{1\leq s_1\leq s_2\leq s_3\cdots\leq s_i\leq n} s_1 s_2 s_3 \cdots s_i$$

Theorem 7. [6] *For $n, s \in \mathbb{N}$: $n > s$ we have,*

$$\left\{ n \atop s \right\} = \left\{ n-1 \atop s-1 \right\} + s\left\{ n-1 \atop s \right\}$$

1.5. **Pascal's triangle.** .

We now introduce Pascal's arithmetic triangle.

Pascal's arithmetic triangle is an infinite rectangular array of numbers, where the first row and column contain all ones (see [22], pp 3).

Note 1.

"Each other entry in the triangle is the sum of the entries immediately to its north and to its west"[22].

We illustrate few values of Pascal's triangle in the form of a table.

Table 3. Pascal's arithmetic triangle[22]

1	1	1	1	1	....
1	2	3	4	5	....
1	3	6	10	15	....
1	4	10	20	35	....
1	5	15	35	70	....
1	6	21	36	106	....
1	7	28	64	170	....
1	8	36	100	270	....
1	9	45	145	415	....
1	10	55	200	615	....
1	11	66	266	881	....
1	12	78	344	1225	....
⋮	⋮	⋮	⋮	⋮	....

We now state properties of the entries in the table.

Property 1. [35] *"Each entry x in the table, not in the first column is the difference of the entry y immediately to its south and the entry k immediately to the west of y".*

For an example, the entry 35 in the fourth row is equal to $70 - 35$.

Property 2. [35] *" Each entry x in the table, not in the first row or column, is the sum of the entry y to its north and all the entries to the left of y".*

For an example the entry 10 in the third row is equal to $4+3+2+1$. Rotating table 3 above at an angle of 45 degrees results to the below more familiar Pascal's triangle[22].

TABLE 4. Pascal's arithmetic triangle[22]

1	1	1	1	1	......
1	2	3	4	5	......
1	3	6	10	15	......
1	4	10	20	35	......
1	5	15	35	70	......
1	6	21	36	106	......
1	7	28	64	170	......
1	8	36	100	270	......
1	9	45	145	415	......
1	10	55	200	615	......
1	11	66	266	881	......
1	12	78	344	1225	......
⋰	⋰	⋰	⋰	⋰	......

We will denote the first row of table 4 as row zero and the first column on each row as column zero. From now onwards when referring to Pascal's triangle we will be referring to table 4 unless stated otherwise.

Pascal proposed the following way of calculating the value of the entry in the i^{th} column of the j^{th} row in Pascal's triangle (see [22]);

$$(5)\quad C(j,i) = C(j-1,i-1) + C(j-1,i) \quad where \quad C(j,i) = \begin{cases} 1, & \text{if } i = j \\ 0, & i > j \end{cases}$$

This is in-line with the definition we gave in note 1 on how the entry in the i^{th} column of row j of Pascal's triangle is generated (referring to table 3: see page 13 above). Equation 5 is known in [22] as pascal's identity (since it was proposed by Blaise Pascal). The numbers $C(j, i)$ have a set theoretic meaning; they are the number of subsets of size i on set having j elements[22].

In [22] Leavitt observed that; the left most entry in any given row of Pascal's triangle is equal to one and any other entry in the row is a fraction of the immediate preceding entry. Hence in [22] the author proposed the following recurrence relation for calculating the value of the entry in the i^{th} column of the j^{th} row (an alternative to Pascal's method: in equation 5 above),

$$(6)\ \ U_i^j = U_{i-1}^j \frac{j+1-i}{i} \qquad where \quad U_0^j = 1 \quad for\ all\ j = 0, 1, 2, 3......$$

Equation 6 is an alternative way of generating the entries of a given row in Pascal's triangle independent of the preceding rows of the triangle, as opposed to the traditional way of generating each row; where each successive row is generated from the preceding row (see([35], prop 1).

We illustrate in the form of a table Leavitt's method by calculating U_4^4.

U_4^4	$= U_3^4(\frac{4+1-4}{4})$	$= U_3^4$	$= 4(\frac{1}{4})$	=1
U_3^4	$= U_2^3(\frac{4+1-3}{3})$	$= U_2^4(\frac{2}{3})$	$= 6(\frac{2}{3})$	=4
U_2^4	$= U_1^4(\frac{4+1-2}{2})$	$= U_1^4(\frac{3}{2})$	$= \frac{4}{1}\frac{3}{2}$	=6
U_1^4	$= U_0^4(\frac{4+1-1}{1})$	$= U_0^4(\frac{4}{1})$	$= (\frac{4}{1})$	=4
U_0^4	=1			=1

TABLE 5. Table for calculation of U_4^4 using equation 6

The last column of table 5 is row 5 of pascal's triangle.

We observe that the right most entry in each row of Pascal's triangle is always equal to one and that each other entry in that row is a fraction of the immediate successive entry. Hence we propose the following recurrence relation, which is a dual of equation 6 above.

$$U_i^j = U_{i+1}^j \frac{i+1}{j-i} \qquad where \quad U_j^j = 1 \quad for\ all\ j = 0, 1, 2, 3...... \tag{7}$$

Equation 7 suggests that, the entry in column i of row j can be calculated using the entries in successive columns of row j.

We illustrate the method described in equation 7 in the form of a table by calculating U_0^3.

U_0^3	$= U_1^3(\frac{1}{3})$	$= U_1^3(\frac{1}{3})$	$= 3(\frac{1}{3})$	$=1$
U_1^3	$= U_2^3(\frac{2}{2})$	$= U_2^3(\frac{2}{2})$	$= \frac{3}{1}\frac{2}{2}$	$=3$
U_2^3	$= U_3^3(\frac{3}{1})$	$= U_3^3(\frac{3}{1})$	$= (\frac{3}{1})$	$=3$
U_3^3	$=1$			$=1$

TABLE 6. Table for calculation of U_0^3 using equation 7

The last column of table 6 is the fourth row of Pascal's triangle (see page 14). From equation 7 we deduce that when enumerating the entries of a given row of Pascal's triangle we can start from the right most entry and work backwards to obtain the other entries of that row.

Remark. Equation 7 is no-where mentioned in the literature we consulted, it is our own original contribution.

1.6. **The Harmonic triangle.** .

We now introduce another triangle of numbers known as Leibniz triangle (The harmonic triangle). The harmonic triangle is formed by repeated differences of the harmonic sequence (see [4]) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7} \ldots$ which we illustrate in the form of a table as follows.

TABLE 7. Leibniz harmonic triangle[31]

1/1	1/2	1/3	1/4	1/5	....
1/2	1/6	1/12	1/20	1/30	....
1/3	1/12	1/30	1/60	1/105	....
1/4	1/20	1/60	1/140	1/280	....
1/5	1/30	1/105	1/280	1/630	....
1/6	1/42	1/168	1/504		....
1/7	1/56	1/252	1/840		....
1/8	1/72	1/360			....
1/9	1/90				....
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....

We now state some properties of the harmonic triangle (referring to table 7). Some of these properties are analogous to those of Pascal's triangle considered in table 3 (see page 13 above).

Property* 1. [35] *"Each entry x not in the first row is the difference of the entry y to the north of x and the entry k to the east of y".*

For an example the entry 1/12 in the second row is equal to
1/3 -1/4. This property is analogous to property 1 of table 3 (see page 13 above).

Property* 2. [35] *"Each entry x is the sum of the entry y to it's south and all the entries to the east of y".*

For an example the entry 1/2 in the first row is equal to $1/6 + 1/12 + 1/20 + \cdots$ This property is analogous to property 2 of Pascal's triangle in table 3 (see page 13 above).

Upon justifying table 7 to the right we obtain the following table;

TABLE 8. Leibniz harmonic triangle[4]

1	1/2	1/3	1/4	1/5	1/6	....
	1/2	1/6	1/12	1/20	1/30	....
		1/3	1/12	1/30	1/60	....
			1/4	1/20	1/60	....
				1/5	1/30	....
					1/6	....

We now show that each entry in the harmonic triangle (referring to table 8) is given by (see [4])

$$\beta(\zeta,\epsilon) \;=\; \int_0^1 (1-r)r^{\zeta-1}r^{\epsilon-1}\,dr \quad for\; some \quad \zeta \geq 1\, and\, \epsilon \geq 1 \tag{8}$$

In table 8 we number both rows and columns starting from zero onwards. We denote the entry in column i of row j by Z_j^i. Which in the notation of finite differences in chapter One is $\Delta^j t(z_{i-j})$ i.e $Z_j^i = \Delta^j t(z_{i-j})$ (where each $t(z_k)$ is a term in the harmonic sequence)[4]. Which is a j^{th} forward difference in the harmonic triangle. Adjusting theorem 1 of chapter One (see page 8 above) we have (see [4])

$$\Delta^j t(z_{i-j}) = \sum_{s=0}^{j} (-1)^s \binom{j}{s} t(z_{i-j+s}) \tag{9}$$

So we have,

$\Delta^j t(z_{i-j}) = t(z_{i-j}) - \binom{j}{1} t(z_{i-j+1}) + \binom{j}{2} t(z_{i-j+2}) + \cdots + (-1)^j t(z_i)$

Where the $t(z_k)$'s are the elements of the harmonic sequence we considered above. We define $t(z_k) = \frac{1}{k+1}$ for each k (see [4]). So (see [4]);

$$\Delta^j t(z_{i-j}) = \frac{1}{i-j+1} - \binom{j}{1}\frac{1}{i-j+2} + \binom{j}{2}\frac{1}{i-j+3} + \cdots + \frac{(-1)^j}{i+1}$$

Also $\int_0^1 (1-r)^j r^{i-j}\,dr = \int_0^1 [r^{i-j} - \binom{j}{1} r^{i-j+1} + \binom{j}{2} r^{i-j+2} + \cdots + (-1)^r r^i]\,dr$
$= \frac{1}{i-j+1} - \binom{j}{1}\frac{1}{i-j+2} + \binom{j}{2}\frac{1}{i-j+3} + \cdots + \frac{(-1)^j}{i+1}$ (see [4]).

Hence $Z_j^i = \int_0^1 (1-r)^j r^{i-j}\,dr = \beta(j+1, i-j+1)$ where $i \geq j \geq 0$ (see [4]).

We observe that (in table 8: see page 19 above) the sum of each row is the first entry of the preceding row[4]. For an example the sum of the third row is; $1/3 + 1/12 + 1/30 + 1/60 + \cdots = 1/2$. This is so since $\sum_{i=j}^{\infty} Z_j^i = \sum_{i=j}^{\infty} \int_0^1 (1-r)^j r^{i-j}\,dr = \int_0^1 (1-r)^{j-1}\,dr = Z_{j-1}^{j-1}$ (see [4]).

We also observe that there is symmetry within each column of table 8 (see [4]). This is because of the following equality (see [4])

$$\beta(i-j+1, j+1) = \beta(j+1, i-j+1)$$

We recognise the function $\beta(\zeta, \epsilon)$ defined in equation 8 above as a Beta integral (see [33] pp 210). Hence each entry in the harmonic sequence can be represented as a Beta integral[4].

We recall Pascal's triangle;

TABLE 9. The arithmetic triangle (right justified)[22]

1	1	1	1	1	1	1	....
	1	2	3	4	5	6	....
		1	3	6	10	15	....
			1	4	10	20	....
				1	5	15	....
					1	6	....
						1	....

We now show a relationship between the harmonic and the arithmetic triangle (see [35]).

Note 2. [35] *"We observe that each entry in column i of Pascal's triangle (in table 9 above) is the reciprocal of multiplying $i+1$ by the corresponding entry in the harmonic triangle (in table 8: see page 19 above)".*

For example the entry 6 in column 4 of table 9 is given by
$6 = [5 \times 1/30]^{-1}$. This is because (referring to table 8)
$Z_j^i = \beta(j+1, i-j+1) = \frac{j!(i-j)!}{(i+1)!} = \frac{1}{(i+1)\binom{i}{j}}$ (see [4]).
From note 2 above we deduce that the sum in each column of table 8 (see page 19 above) is given by (see [4])

$$\sum_{j=0}^{i} Z_j^i = \sum_{j=0}^{i} \frac{1}{(i+1)\binom{i}{j}} \tag{10}$$

We will refer to equation 10 in chapter Three.

Note 3. *The column sums of table 8 (see page 19 above) are given by*[23][31]*;*
$1/1, 1, 5/6, 2/3, 8/15, \cdots$

Remark. A convolution of a generating function $L(x) = \sum_{s=0}^{\infty} h_s x^s$ and a generating function $F(x) = \sum_{z=0}^{\infty} u_z x^z$ is the generating function $S(x) = \sum_{c=0}^{\infty} q_c x^c$ where $q_c = \sum_{k=o}^{c} h_k u_{c-k}$ (see [42]).

We observe that (see [23]) the sequence in note 3 is formed by the convolution of the sequence $1, 1/2, 1/2^2.1/2^3 \cdots$ and the sequence $1, 1/(1+1), 1/(1+2), \cdots, 1/(n+1), \cdots$ This is because the sum of column n of Pascal's triangle in table 9 (see page 21 above) is equal to 2^n. Also to get column n of the Harmonic triangle we need to multiply column n of Pascal's triangle by $n+1$ and invert the terms (referring to note 2 above)[23]. Hence from this we compute as follows the generating function for the column sums of table 8 (see [23]): The generating function of the sequence $1, 1/2, 1/2^2.1/2^3 \cdots$ is $M(x) = \frac{2}{2-x}$ and the generating function for the sequence $1/1, 1/(1+1), 1/(1+2), \cdots, 1/(n+1), \cdots$ is $J(x) = log(\frac{1}{1-x})$ (see [31]). Which both can be obtained by manipulating the Macleurin serieses of $R(x) = \frac{1}{1-x}$ and $W(x) = log(1-x)$ respectively. Hence the generating function of the sequence $1, 1, 5/6, 2/3, 8/15. \cdots$ is the product of $M(x)$ and $J(x)$[23]: which is;

$$Y(x) = \frac{log(1/(1-x)}{1-x} \tag{11}$$

So $Y(x)$ in equation 11 when viewed as a convolution of generating functions is $Y(x) = \sum_{n=0}^{\infty} T_n x^n$(see [31]); where (see [31])

$$T_n = \sum_{r=1}^{n} (\frac{1}{r})(\frac{1}{2})^{n-r} = \frac{1}{2^n} \sum_{r=1}^{n} \frac{2^r}{r} \tag{12}$$

We will make reference to equation 12 in chapter Three.

We end this section by stating two theorems without proof on harmonic numbers in connection with Stirling numbers of the first kind.

We define the i^{th} order harmonic number D_i^n as $D_i^n = \sum\limits_{s=1}^{n} \frac{1}{s^i}$ (see[1]). When $i = 1$ the number D_1^n is the sum of the first n entries of either the first row or the first column of table 7 above (see page 18 above).

Lemma 1. [1] *For $n \in \mathbb{N}$ we have,*

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! D_1^{n-1}$$

Lemma 2. [1] *For $n \in \mathbb{N}$ we have,*

$$\begin{bmatrix} n \\ 3 \end{bmatrix} = \frac{(n-1)!}{2!} \left[D_1^{n-1} - D_2^{n-1} \right]$$

2. **Chapter Two**

2.1. **Preferential arrangements.** .

Let's consider the set $X_3 = \{1, 2, 3\}$; unordered set partitions of X_3 are as follows :

I. {1,2,3}

II. {1,2} {3}

III. {1,3} {2}

IV. {2,3} {1}

V. {1} {2} {3}

There is 1 way of ordering the partition in I. There are 2!=2 ways of ordering the partitions in II,III,IV. There are 3!=6 ways of ordering the partition in V. Hence the number of ordered partitions of X_3 is 1+ 2+2+2+6=13. We recall that $\left\{ {n \atop s} \right\}$ is the number of ways of partitioning an n-element set into s non-empty blocks (see page 11 above). As a result we have the following; there are $\left\{ {3 \atop s} \right\}$ ways of partitioning X_3 into s non-empty blocks. There are s factorial ways of ordering those blocks. Taking the product and summing over s (where $0 \leq s \leq 3$) we have $0!\left\{ {3 \atop 0} \right\} + 1!\left\{ {3 \atop 1} \right\} + 2!\left\{ {3 \atop 2} \right\} + 3!\left\{ {3 \atop 3} \right\} = 0 + 1 \times 1 + 2 \times 3 + 6 \times 1 = \sum_{s=0}^{3} s!\left\{ {3 \atop s} \right\} = 13$ (we have made use of the table on page 11 for the numbers $\left\{ {n \atop i} \right\}$)[31]. In general for $X_n = \{1, 2 \ldots, n\}$ the number of ordered partitions of X_n is given by $\sum_{s=0}^{n} s!\left\{ {n \atop s} \right\}$ (see [31],[38]). We denote the number of ordered partitions of X_n by q_n and the set of all the ordered partitions/preferential arrangements of X_n by Q_n i.e $|Q_n| = q_n$.

A table for few values of q_n,

TABLE 10. A table for values of q_n

n	q_n
0	1
1	1
2	3
3	13
4	75
5	541
6	4683
7	47293
8	545855
9	7087261
10	102247563
11	1622632573
$\vdots$	$\vdots$

The sequence q_n for $n \geq 0$ seems to first appear in [9], a 1859 paper by Arthur Cayle in connection with analytical forms called trees. Since then the sequence has been studied by mathematicians in various contexts over many centuries until to date (for instance in [25],[24],[40]); more recently the 2013 paper [2] by Pippenger et al. The sequence q_n has a connection with the well known Fubini's Theorem in calculus (see [10],[34]). The number q_n for $n \geq 0$ is the number of ways of exchanging the order of integrals when integrating a continuous function $J(x_1, \ldots, x_n)$ of n variables [10],[34]; In two dimensions the number of possible ways of integrating $J(x_1, x_2)$ over the rectangle

$B = a \leq x_1 \leq b$: $C = c \leq x_2 \leq d$: are

I $\int\limits_{B\times C} J(x_1, x_2)\, dx_1 x_2$

$II. \int\limits_{B}\int\limits_{C} J(x_1, x_2)\, dx_1\, dx_2$

$III. \int\limits_{C}\int\limits_{B} J(x_1, x_2)\, dx_2\, dx_1.$

The number is 3 which is equal to q_2. As a result of the above connection of the numbers q_n to Fubini's theorem the numbers q_n are sometimes called Fubini numbers. Louis Comtet seems to be the first to call these numbers Fubini numbers in [10]. In this study we will call the numbers q_n preferential arrangements. The word preferential arrangements for the numbers q_n seem to be due to Gross in [19].

We now state a theorem concerning the numbers q_n.

Theorem 8. [24] *For* $n \geq 0$

$$q_n = \delta_n + \sum_{s=1}^{n} \binom{n}{s} q_{n-S}$$

Where δ_n is Kronecker's delta, which is equal to 1 if $n = 0$ and 0 otherwise. The recurrence relation in theorem 8 seems to first appear in [24] and was half a century later independently proposed by Pippenger in [31].

In [31] the recurrence relation is proved combinatorially in the following way: In a preferential arrangement of fixed $n \geq 1$ elements, we first consider elements which are tied in the top block. We assume there are s of them, where $1 \leq s \leq n$. There are $\binom{n}{s}$ ways of selecting the s elements. Since the s elements are to be in the same block, there is 1 way of arranging them. The other $n - s$ elements can be preferentially arranged in q_{n-s} ways into other blocks. So we have a total of $\binom{n}{s}(1)q_{n-s}$ ways of preferentially arranging the n-elements when there are s elements in the top block. Out of the n elements, there can be a minimum of 1 element which is in the top block and a maximum of n elements in the top block. So s ranges from 1 to n. Hence we obtain the following,

$$\boxed{q_n = \sum_{s=1}^{n} \binom{n}{s} q_{n-s}}$$

Incorporating the case $q_0 = 1$ by introducing Kronecker's delta to the equation we obtain the result of the theorem.

It is well known (see [9] [25] [31] [19]) that the sequence q_n for $n \geq 0$ has the exponential generating function

$$q(m) = \frac{1}{2 - e^m} = \sum_{n=0}^{\infty} \frac{q_n \times m^n}{n!} \tag{13}$$

The generating function in (13) seems to first appear in [9]. It is obtained in [31] using the following argument: From theorem 8 we have, $q_n = \delta_n + \sum_{s=1}^{n} \binom{n}{s} q_s$. Adding q_n on both sides then multiplying the equation by $\frac{m^n}{n!}$ and then summing over $n \geq 0$ we obtain $\sum_{n=0}^{\infty} \frac{2q_n \times m^n}{n!} = \sum_{n=0}^{\infty} \frac{m^n}{n!} \delta_n + \sum_{n=0}^{\infty} \frac{m^n}{n!} \left(\sum_{s=0}^{n} \binom{n}{s} q_s \right)$ (see [31]). Since $\delta_n = 0$ for all $n > 0$ we have

$$\sum_{n=0}^{\infty} \frac{2q_n \times m^n}{n!} = 1 + \sum_{n=0}^{\infty} \frac{m^n}{n!} \left(\sum_{s=0}^{n} \binom{n}{s} q_s \right) \text{ (see [31])}$$

So,

$$2 \sum_{n=0}^{\infty} \frac{m^n}{n!} q_n = 2 \frac{1}{2 - e^m} = 1 + \sum_{n=0}^{\infty} \frac{m^n}{n!} \left(\sum_{s=0}^{n} \binom{n}{s} q_s \right) \text{ (see [31])}$$

That is,

$$2 \frac{1}{2 - e^m} = 1 + \sum_{n=0}^{\infty} \frac{m^n}{n!} \left(\sum_{s=0}^{n} \binom{n}{s} q_s \right) \text{ (see [31])} \tag{14}$$

We let $q(m) = \frac{1}{2 - e^m}$.

" A convolution of two generating functions $L(x) = \sum\limits_{s=0}^{\infty} \frac{h_s \times x^s}{s!}$ and $F(x) = \sum\limits_{z=0}^{\infty} \frac{u_z \times x^z}{z!}$, is the generating function $S(x) = \sum\limits_{c=0}^{\infty} \frac{q_c \times x^c}{c!}$; where $q_c = \sum\limits_{k=o}^{c} \binom{c}{k} h_k u_{c-k}$ (see [31]). " From this we deduce that the summation $\sum\limits_{n=0}^{\infty} \frac{m^n}{n!} (\sum\limits_{s=0}^{n} \binom{n}{s} q_s)$ on (14) is a convolution of the generating functions $\sum\limits_{n=0}^{\infty} \frac{m^n}{n!}$ and $\sum\limits_{n=0}^{\infty} \frac{q_n \times m^n}{n!}$. Hence

$$2q(m) = 1 + \sum_{n=0}^{\infty} \frac{m^n}{n!} \times \sum_{n=0}^{\infty} \frac{q_n \times m^n}{n!} \text{ (see [31])}$$

But we know that $\sum\limits_{n=0}^{\infty} \frac{m^n}{n!} = e^m$ (see [31]); this together with $\sum\limits_{n=0}^{\infty} \frac{q_n \times m^n}{n!} = q(m)$ imply that

$$2q(m) = 1 + q(m) \times e^m \text{ (see [31])}$$

Making $q(m)$ the subject we obtain (13) above as required[31].

In [31] the author proposed the following identity for $t \geq 0$,

$$q_n = \frac{1}{2} \sum_{t=0}^{\infty} \frac{t^n}{2^t}, \quad \textit{where } n \geq 0 \tag{15}$$

We prove the identity in the following way (see[31]); since the generating function of the sequence q_n is given in (13) as,

$q(m) = \frac{1}{2-e^m} = \sum\limits_{n=0}^{\infty} \frac{q^t m^n}{n!}$ this implies that $q(m) = \frac{1}{2} \frac{1}{1-\frac{1}{2}e^m} = \frac{1}{2} \sum\limits_{t=0}^{\infty} \frac{e^{tm}}{2^t}$ $= \frac{1}{2} \sum\limits_{t=0}^{\infty} \frac{1}{2^t} \left(\sum\limits_{n=0}^{\infty} \frac{(tm)^n}{n!} \right) = \sum\limits_{n=0}^{\infty} \left(\frac{1}{2} \sum\limits_{t=0}^{\infty} \frac{t^n}{2^t} \right) \frac{m^n}{n!}$

Hence we obtain,

$$\sum_{n=0}^{\infty} \frac{q_n m^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{1}{2} \sum_{t=0}^{\infty} \frac{t^n}{2^t} \right) \frac{m^n}{n!}$$

The coefficients of $\frac{m^n}{n!}$ should be the same on both sides of the equation (see[31]). Hence $q_n = \frac{1}{2}\sum_{t=0}^{\infty} \frac{t^n}{2^t}$ as required [31]. Equation 15 above is a way of writing q_n as an infinite series.

The generating function of the number of preferential arrangements given by $q(m) = \frac{1}{2-e^m}$ satisfies the following differential equation (see[31]);

$$2[q(m)]^2 = \frac{d}{dm}(q(m)) + q(m) \tag{16}$$

Which is a Bernoulli differential equation[31].

Theorem 9. [19] *For all* $n \geq 1$

$$q_n = \sum_{s=1}^{n} \binom{n}{s} q_{n-s}$$

The statement of theorem 9 is the same as the statement in theorem 8 but the proof presented here is different. We prove the theorem in the following way (see [19]):

From equation 15 above we have,

$$q_n = \frac{1}{2}\sum_{n=0}^{\infty} \frac{t^n}{2^t} \quad \text{where } t \geq 0 \tag{17}$$

This implies that
$q_{n-s} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{t^{n-s}}{2^t} \Rightarrow \sum_{s=0}^{n-1} \binom{n}{s} q_{n-s} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\sum_{s=0}^{n-1} \binom{n}{s} t^{n-s} 2^{-t}\right)$
$= \frac{1}{2}\sum_{n=0}^{\infty} \Big((t+1)^n - 1\Big) 2^{-t} = \sum_{n=0}^{\infty} (t+1)^n 2^{-(t+1)} - \frac{1}{2}\sum_{n=0}^{\infty} 2^{-t}$
$= \frac{1}{2}\sum_{n=0}^{\infty} (t+1)^n 2^{-t} - \frac{1}{2}\sum_{n=0}^{\infty} 2^{-t}$ So,

$$\sum_{s=0}^{n-1} \binom{n}{s} q_{n-s} = \frac{1}{2}\sum_{n=0}^{\infty} (t+1)^n 2^{-t} - \frac{1}{2}\sum_{n=0}^{\infty} 2^{-t} \tag{18}$$

We first evaluate the term $\frac{1}{2}\sum_{n=0}^{\infty} (t+1)^n 2^{-t}$ say $y1 = \frac{1}{2}\sum_{n=0}^{\infty} (t+1)^n 2^{-t}$. Let $t+1 = j$ then $y1 = \sum_{n=1}^{\infty} j^n 2^{-j} = \sum_{n=0}^{\infty} j^n 2^{-j}$. Applying equation 17 above we have, $y1 = 2q_n$

Let the second term of equation 18 be $y2$ i.e $y2 = \frac{1}{2}\sum\limits_{n=0}^{\infty} 2^{-t}$. We recognise the summation on $y2$ as a geometric series with common ratio $\frac{1}{2}$; hence $y2 = 1$. Combining $y1$ and $y2$ we have, $\sum\limits_{s=0}^{n-1} \binom{n}{s} q_{n-s} = 2q_n - 1 \Rightarrow q_n = \sum\limits_{s=1}^{n-1} \binom{n}{s} q_{n-s} + 1$. This completes the proof.

Theorem 10. [19] *For all* $n \geq 1$

$$q_n = \sum_{s=0}^{\infty} \nabla^s s^n$$

We prove the theorem as follows (see[19]); from (13) above we have $q(m) = \frac{1}{2-e^m}$. This implies that $q(m) = \frac{1}{1-(e^m-1)} = \sum\limits_{s=0}^{\infty}(e^m - 1)^s$. So $\frac{d^n}{dm^n}\Big(q(m)\Big) = \frac{d^n}{dm^n}\left(\sum\limits_{s=0}^{\infty}(e^m - 1)^s\right) = \sum\limits_{s=0}^{\infty}\frac{d^n}{dm^n}(e^m - 1)^s$. This implies that $\frac{d^n}{dm^n}\Big(q(m)\Big) = \sum\limits_{s=0}^{\infty}\frac{d^n}{dm^n}\sum\limits_{u=0}^{s}\binom{s}{u}(-1)^s e^{(s-u)x}$. So, $\frac{d^n}{dm^n}\Big(q(m)\Big)|_{s=0} = \sum\limits_{s=0}^{\infty}\sum\limits_{u=0}^{s}\binom{s}{u}(-1)^s(s-u)^n$. By definition of $q(m)$ the n^{th} derivative of $q(m)$ evaluated at zero is equal to q_n (this is so since $q(m)$ is the exponential generating function of q_n) hence $q_n = \sum\limits_{s=0}^{\infty}\sum\limits_{u=0}^{s}\binom{s}{u}(-1)^s(s-u)^n$. A property of the backward difference operator is that $\nabla^s = [1 - E^{-1}]^s = \sum\limits_{u=0}^{s}\binom{s}{u}(-1)^u E^{-u}$ (where E is the shift operator)(see[14], page 52). So, $\sum\limits_{u=0}^{s}\binom{s}{u}(-1)^u(s-u)^n = \nabla^s s^n$. Hence $q_n = \sum\limits_{s=o}^{\infty}\nabla^s s^n$ as required.

2.2. **Locks which use combinations.** .

We now discuss how preferential arrangements can be interpreted as valid combinations on a combination lock having n buttons (see [40]). A combination lock is a lock which uses a sequence of digits to lock and unlock. We denote by $W(n)$ the set of valid combinations on a combination lock having n buttons (where by valid we mean pushes on the buttons without repetition, where all the buttons are pushed: not necessarily all at the same time). The possible sequences of a combination lock having n buttons correspond to preferential arrangements of an n element set i.e each sequence on a combination lock having n buttons can be interpreted as a unique preferential arrangement of an n element set [40]. For illustration we consider a combination lock with 2 buttons $W(2)$. The set $W(2)$ is given by $W(2) = \left\{ \begin{array}{c} \{1\},\{2\}; \\ \{2\},\{1\}; \\ \{1,2\} \end{array} \right\}$. We interpret the first row of $W(2)$ as the button $\{1\}$ is to be pushed first and the button $\{2\}$ is to be pushed second. The second row of $W(2)$ means the button $\{2\}$ is to be pushed first and the button $\{1\}$ is to be pushed second. The third row of $W(2)$ means that both buttons are to be pushed at the same time. The order $|W(2)| = 3 = q_2 = |Q_2|$ (where Q_2 is the set of all preferential arrangements of a 2 element set). Clearly from $W(2)$ we get a natural way of interpreting each combination as a unique preferential arrangement of Q_2 (see[40]). In general each element of $W(n)$ can be interpreted as a unique element of Q_n and vice (where Q_n is the set of all preferential arrangements of an n-element set). Hence $|W(n)| = q_n = |Q_n|$ (see [40]). So in this section by q_n we mean the number of valid combinations on a combination lock having n buttons.

Theorem 11. [40] *For all* $n \geq 0$ *we have,*

$$\frac{1}{2(\ln 2)^n} \leq u_n \leq \frac{1}{(\ln 2)^n}$$

We prove the theorem as follows (see[40]): We recall from theorem 8 on page 26 above that $q_n = \sum_{s=0}^{n} \binom{n}{s} \times q_{n-s}$. Now $q_n = \binom{n}{1} q_{n-1} + \binom{n}{2} q_{n-2} + \cdots + \binom{n}{n} q_0 \Rightarrow \frac{q_n}{n!} = \left(\frac{q_{n-1}}{(n-1)! \times 1!} + \cdots + \frac{q_0}{n! \times 0!} \right)$(see [40]).

By letting $u_n = \frac{q_n}{n!}$ we obtain the equation(see [40]);

$$(19) \qquad u_n = \left(\frac{u_{n-1}}{1!} + \cdots + \frac{u_o}{n!}\right) \quad where\ n > 0.$$

We prove the theorem by applying induction on n (see [40]). We assume the statement is true for all $n* \in \mathbb{Z}^+$ such that $n* \leq n$. We prove the theorem by applying induction on each term of equation 19. We first show that $u_n \leq \frac{1}{(\ln 2)^n}$. By equation 19 we have,
$u_n = \left(\frac{u_{n-1}}{1!} + \cdots + \frac{u_o}{n!}\right) \Rightarrow u_n \leq \left(\frac{1}{(\ln 2)^{n-1}\times 1!} + \cdots + \frac{1}{(\ln 2)^n \times n!}\right)$
$= \frac{1}{(\ln 2)^n}\left(\frac{(\ln 2)^1}{1!} + \cdots + \frac{(\ln 2)^n}{n!}\right) = \frac{1}{(\ln 2)^n}\left(e^{\ln 2} - 1\right) = \frac{1}{(\ln 2)^n}$.
So $u_n \leq \frac{1}{(\ln 2)^n}$. We now show that $\frac{1}{2(\ln 2)^n} \leq u_n$. Also applying induction on each term of equation 19 we obtain,
$u_n \geq \left(\frac{1}{2(\ln 2)^{n-1}\times 1!} + \cdots + \frac{1}{2(\ln 2)^n \times n!}\right) = \frac{1}{2(\ln 2)^n}\left(\frac{(\ln 2)^1}{1!} + \cdots + \frac{(\ln 2)^n}{n!}\right)$
$= \frac{1}{2(\ln 2)^n}\left(e^{\ln 2} - 1\right) = \frac{1}{2(\ln 2)^n}$. So $u_n \geq \frac{1}{2(\ln 2)^n}$. Thus the theorem.

Theorem 12. [40] *For all $n \geq 0$, $q_n = \frac{d^n}{dx^n}\left(\frac{1}{2-e^x}\right)\Big|_{x=0}$*

We prove the theorem in the following way (see[40]): We consider u_n as defined in (19) of theorem 11 above. The ordinary generating function $q(x)$ of u_n is;

$$(20) \qquad q(x) = \sum_{n\geq 0} u_n x^n$$

From equation 20 we have,
$q(x) = \sum_{n=0}^{\infty} u_n x^n = 1 + \sum_{n=1}^{\infty} u_n x^n = 1 + \sum_{n=1}^{\infty}\left(\sum_{s=1}^{n}\frac{u_{n-s}}{s!}\right)x^n$.

This implies that $q(x) = 1 + \sum_{s=1}^{\infty}\sum_{n=s}^{\infty}\frac{u_{n-s}}{s!}x^n = 1 + \sum_{s=1}^{\infty}\frac{x^s}{s!}\sum_{n=s}^{\infty}u_{n-s}x^{n-s}$
$= 1 + \left(\sum_{s=1}^{\infty}\frac{x^s}{s!}\right)\left(\sum_{n=0}^{\infty}u_{n-s}x^{n-s}\right) = 1 + (e^x - 1)(q(x))$.
So $q(x) = 1 + (e^x - 1)q(x) \Rightarrow q(x) = \frac{1}{2-e^x}$. The number u_n is the coefficient of x^n in the Maclaurin series of $q(x)$. So $u_n = \frac{q^n(0)}{n!}$. We recall that $q_n = n! \times u_n$ this implies that, $q_n = q^n(0) = \left(\frac{1}{2-e^x}\right)\Big|_{x=0}$.
This completes the proof.

Theorem 13. [40] *For all* $n \geq 0$ *we have* $q_n = \frac{1}{2}\sum_{s=0}^{\infty} \frac{s^n}{2^s}$

We prove the theorem as follows (see[40]): The theorem follows when we rewrite the generating function $q(x) = \frac{1}{2-e^x}$ in the following way[40]; $q(x) = \frac{1}{2-e^x} = \frac{1/2}{1-\frac{e^x}{2}} = \frac{1}{2}\sum_{s=0}^{\infty}\left(\frac{e^x}{2}\right)^s$(We recognize a geometric series). Now by theorem 12 above we have; $q_n = \frac{d^n}{dx^n}\frac{1}{2}\sum_{s=0}^{\infty}\left(\frac{e^x}{2}\right)^s\Big|_{x=0} = \frac{1}{2}\sum_{s=0}^{\infty}\frac{d^n}{dx^n}\left(\frac{e^x}{2}\right)^s\Big|_{x=0}$. Thus the theorem.

Corollary 1. [40] *For all* $n \geq 0$, $q_n \approx \frac{n!}{2(\ln 2)^{n+1}}$

The corollary is a corollary of theorem 13 above. We prove the corollary in the following way (see [40]); If we replace the sum on the statement of theorem 13 by an integral we have $\frac{1}{2}\int_{s=0}^{\infty} \frac{s^n}{2^s}\, dx$. We let $l = s\ln 2$. Then we have $\frac{1}{2\times(\ln 2)^{n+1}}\int_{l=0}^{\infty} e^{-l}l^n\, dl$ (we recognize a Gamma function)[40]. This implies that $\frac{1}{2\times(\ln 2)^{n+1}}\int_{l=0}^{\infty} e^{-l}l^n\, dl = \frac{n!}{2\times(\ln 2)^{n+1}}$ (see [40]). Hence $q_n \approx \frac{n!}{2(\ln 2)^{n+1}}$ (see [40]). This completes the proof.

The error bounds of the approximation in corollary 1 are as follows;

Corollary 2. [40] *For all* $n \geq 0$ *we have,* $\left|q_n - \frac{n!}{2(\ln 2)^{n+1}}\right| < \frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n$

We end this section by stating the following corollary of theorem 13;

Corollary 3. [40] *For all* $n \geq 0$ $\lim_{n\to\infty} \frac{q_n}{n!/2(\ln 2)^{n+1}} = 1$

We prove the corollary in the following way (see [40]). Using corollary 2 above we have
$-\frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n < q_n - \frac{n!}{2(\ln 2)^{n+1}} < \frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n$. This implies that
$\frac{n!}{2(\ln 2)^{n+1}} - \frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n < q_n < \frac{n!}{2(\ln 2)^{n+1}} + \left(\frac{n}{e\ln 2}\right)^n$. From this we obtain

$$\left|\frac{q_n}{n!/2(\ln 2)^{n+1}} - 1\right| < \frac{n^n(\ln 2)^{n+1}}{(e\ln 2)^n \times n!} \tag{21}$$

In proving the corollary we will make use of Stirling's approximation of $n!$ which is given by $\lim_{n\to\infty} \frac{\sqrt{2\pi n} n^n}{e^n} = n!$ (see[29],[30])[40]). To show that the corollary holds we need to show that the right hand side of (21) approaches 0 as n approaches infinity[40]. Taking the limit of the right hand side of (21) we have

$\lim_{n\to\infty} \frac{n^n \ln 2}{e^n n!} = \lim_{n\to\infty} \frac{\sqrt{2\pi n} n^n \ln 2}{\sqrt{2\pi n} e^n n!} = \lim_{n\to\infty} \frac{\sqrt{2\pi n} n^n \ln 2}{\sqrt{2\pi n} e^n n!}$

$= \lim_{n\to\infty} \left(\frac{(n/e)^n \sqrt{2\pi n}}{n!}\right)\left(\frac{\ln 2}{\sqrt{2\pi n} n!}\right)$

$= \lim_{n\to\infty} \left(\frac{(n/e)^n \sqrt{2\pi n}}{n!}\right) \lim_{n\to\infty} \left(\frac{\ln 2}{\sqrt{2\pi n} n!}\right)$ (see [40]). Applying Stirling's approximation of $n!$ to the first limit and evaluating the second limit we have $\lim_{n\to\infty} \left(\frac{(n/e)^n \sqrt{2\pi n}}{n!}\right) \lim_{n\to\infty} \left(\frac{\ln 2}{\sqrt{2\pi n} n!}\right) = 1 \times 0$ (see [40]). This completes the proof.

2.3. **Races.** .

Preferential arrangements can be interpreted as outcomes of races in which ties are allowed[24]. If n runners run a races: runners who finish at the same time are interpreted as a single block of a preferential arrangement of n-elements, where we interpret an outcome of a race as a preferential arrangement of the n-elements[24]. By choice we interpret runners who finish first in the race as the first block of the preferential arrangement of the n-elements. Each position in-which runners finish in, is interpreted as a single block of the preferential arrangement[24]. A race having n runners has a maximum of n possible positions (in that case each runner finishes alone) and there is a minimum of one position in-which the runners can finish in (in that case all the runners finish at the same time). Clearly there is a 1-1 correspondence between the total number of preferential arrangements of an n-element set and number of possible outcomes of a race of n runners. In this section we interpret q_n as the number of outcomes of a race in which ties are allowed.

Theorem 14. [24] *For all* $n \geq 0$,

$$q_{n+1} = \sum_{s=0}^{n} \binom{n+1}{s} q_s \quad \text{where } q_0 = 1,$$

We prove the theorem in the following way (see [24]): Let s be the number of runners who do not finish in the last block in a race of $n+1$ runners[24]. There are $\binom{n+1}{s}$ ways of selecting those s runners. There are q_s ways in which those runners can finish on other positions other than the last position. Hence the product $\binom{n+1}{s} \times q_s$ is the number of possible outcomes of a race of $n+1$ runners in which s of them do not finish on the last position. There can be a minimum of 0 runners who do not finish in the last position (in that case all the $n+1$ runners finish at the same time) and there can be a maximum of n runners who do not finish in the last position. This is so since there has to be some one who finishes last. Hence s runs from 0 to n. Thus the theorem.

Theorem 15. [24] *For all* $n \geq 1$, $q_n = \sum_{s=1}^{n} \sum_{t=1}^{s} (-1)^{s-1} \binom{s}{t} t^n$

The theorem gives a way of interpreting possible outcomes of a race as onto mappings between two sets (see [24]). We prove the theorem in the following way (see [24]): We consider a set $\xi_n = \{1, 2, 3 \ldots, n\}$ and a set $\psi_s = \{1, 2, \ldots, s\}$. The number of possible functions $f : \xi_n \to \psi_s$ is s^n. This is so since for each element of ξ_n there are s choices to map it to an element of ψ_s, hence the total number is s^n. From the number s^n we want the number of those functions which are onto[24]. If a fixed element of ψ_s is to be omitted in all mappings of the n elements of ξ_n, then the number of possible mappings is given by $(s-1)^n$. If two fixed elements of ψ_s are to be omitted in mapping elements of ξ_n to ψ_s, the

the number of possible functions is given by $(s-2)^n$. Hence if t fixed elements of ξ_n are omitted then the total number of possible mappings is given by $(s-t)^n$ where $1 \leq t \leq s-1$ (see [24]). Also there are $\binom{s}{t}$ ways of selecting the t elements. Using the inclusion/exclusion principle the number of functions which are not onto is given by (see [24]) $\chi_n = \binom{s}{1}(s-1)^n + \binom{s}{2}(s-2)^n - \binom{s}{3}(s-3)^n + \cdots + (-1)^s\binom{s}{s+1}(s-s)^n = \sum_{i=1}^{s-1}(-1)^{i+1}\binom{s}{i}(s-i)^n$. Making the change of variable $t = s-i$ we obtain $\chi_n = \sum_{t=1}^{s-1}(-1)^{s+1-t}\binom{s}{t}t^n$. The number of onto functions from ξ_n to ψ_s is given by (see [24]): $s^n - \chi_n = s^n - \sum_{t=1}^{s-1}(-1)^{s+1-t}\binom{s}{t}t^n = s^n + \sum_{t=1}^{s-1}(-1)^{s-t}\binom{s}{t}t^n = \sum_{t=1}^{s}(-1)^{s-t}\binom{s}{t}t^n$; s runs from 1 to n since χ_n has n elements. So we have; $\sum_{s=1}^{n}\sum_{t=1}^{s}(-1)^{s-t}\binom{s}{t}t^n$. Each mapping is interpreted as an outcome of a race. Where elements having the same image are interpreted as runners who finish at the same time[24]. The mappings need to be onto because a mapping which is not onto would not make much sense as an outcome of a race. Hence $q_n = \sum_{s=1}^{n}\sum_{t=1}^{s}(-1)^{s-1}\binom{s}{t}t^n$ as required.

We conclude this section by stating an identity on q_n.

Theorem 16. [24] *For all $n \geq 1$ and $1 \leq s \leq n$,*

$$q_n = \sum_{s=1}^{n} s(q_{n-1,s-1} + q_{n-1,s})$$

We prove the theorem in the following way (see [24]); We consider a race having n runners; we mark one fixed runner[24]. We assume the n runners finish in s positions (where $1 \leq s \leq n$). We denote by $q_{n,k}$ the outcomes of a race having n runners where the n runners finish in k positions. In all possible outcomes of the race the marked runner finishes alone or with a group of other runners[24]. In the case the marked runner finishes alone: The $n-1$ other runners finish in $q_{n-1,s-1}$ ways (since the s^{th} position will be occupied by the marked runner)[24]. There are s positions in-between, before and after the positions in which the $n-1$ runners finish in. We insert the marked runner in any one of them[24]. Hence the number of possible outcomes when the marked runner finishes alone is given by $s \times q_{n-1,s-1}$ (see [24]). We now consider the case the marked runner finishes with a group of other runners[24]. Firstly the $n-1$ other runners finish in the s positions in $q_{n-1,s}$ ways[24]. We put the marked runner in any one of the s positions [24]. So the number of outcomes when the marked runner finishes with a group of other runners is $s \times q_{n-1,s}$ (see [24]). Combining the two types of outcomes we obtain the total number of outcomes of a race of the n runners being given by $s \times q_{n-1,s-1}+ s \times q_{n-1,s}$ and this is equal to q_n (by definition of q_n) [24]. Summing over s we obtain the result.

Theorem 17. [19] *For all* $n \geq 1$

$$q_n = \sum_{s=0}^{\infty} \nabla^s s^n$$

We prove the theorem as follows (see [19]); By equation 13 (see 27 above) we have $q(m) = \frac{1}{2-e^m}$. This implies that $q(m) = \frac{1}{1-(e^m-1)} = \sum_{s=0}^{\infty}(e^m-1)^s$. So $\frac{d^n}{dm^n}\Big(q(m)\Big) = \frac{d^n}{dm^n}\left(\sum_{s=0}^{\infty}(e^m-1)^s\right) = \sum_{s=0}^{\infty}\frac{d^n}{dm^n}(e^m-1)^s$. This implies that $\frac{d^n}{dm^n}\Big(q(m)\Big) = \sum_{s=0}^{\infty}\frac{d^n}{dm^n}\sum_{u=0}^{s}\binom{s}{u}(-1)^s e^{(s-u)x}$. So $\frac{d^n}{dm^n}\Big(q(m)\Big)|_{s=0} = \sum_{s=0}^{\infty}\sum_{u=0}^{s}\binom{s}{u}(-1)^s(s-u)^n$. By definition of $q(m)$ the n^{th} derivative of $q(m)$ evaluated at zero is equal to q_n (this is so since $q(m)$ is the exponential generating function of q_n). Hence $q_n = \sum_{s=0}^{\infty}\sum_{u=0}^{s}\binom{s}{u}(-1)^s(s-u)^n$. A property of the backward difference operator is that $\nabla^s = [1 - E^{-1}]^s = \sum_{u=0}^{s} \binom{s}{u}(-1)^u E^{-u}$ (where E is the shift operator)(see[14], page 52). So, $\sum_{u=0}^{s} \binom{s}{u}(-1)^u (s-u)^n = \nabla^s s^n$. Hence $q_n = \sum_{s=o}^{\infty} \nabla^s s^n$ as required.

3. Chapter Three

3.1. A look at a three dimensional cube problem using physics principles. .

In a recent paper[31], preferential arrangements appear in connection with electric circuits as part of a solution to an electric circuit problem. In this section we state the electric circuit problem. We will later show how preferential arrangements discussed in chapter Two are connected to electric circuits. We also discuss how some of the identities discussed in chapter Two play a role in equations concerning resistive structures.

We begin this section by stating definitions which will later be helpful in the discussion;

Definition 5. *Ohm's current law*

"The current C in any portion of an electric circuit is directly proportional to the potential difference E between it's extremities, and inversely proportional to its resistance R i.e $C = \frac{E}{R}$"[7].

Definition 6. *Electromotive force (EMF)*

" An electromotive force is the potential difference between terminals of any current generator when no current is allowed to flow"[7].

Definition 7. *Kirchoff's law of current*

"In any branching network of wires the algebraic sum of all the current meeting at a point is zero"[7].

Definition 8. *Kirchoff's law of potential difference*

"In any closed path in a network the algebraic sum of the $C \times R$ products is equal to the EMF acting in that path"[7].

Definition 9. *Parallel connection*

Resistors R_1 and R_2 are in parallel connection if they are connected as in the figure below[7],

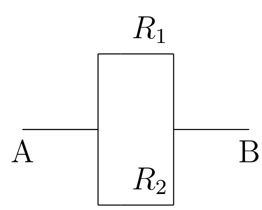

FIGURE 1

Where a current flows into point A and extracted at point B; the equivalence resistance R_e of the circuit is $\frac{1}{R_e} = \frac{1}{R_1} + \frac{1}{R_2}$ (see[7]). This can be generalised to any number of resistors, which are connected in parallel, in the same way as for the two resistors considered above[7].

Definition 10. *Series Connection*

Resistors R_1 and resistor R_2 between end point A and B are said to be connected in series if they are connected in the following way (see [7]);

A R_1 R_2 B

FIGURE 2

The equivalence resistance R_e of the circuit is given by $R_e = R_1 + R_2$ (see [7]). This can also be generalised to any number of resistor in the same way as for two resistors[7].

The electric problem we are concerned with is;

"Consider a 3-dimensional cubical network in which all the edges are replaced by 1 ohm resistors. What is the equivalence resistance of the circuit, taken along opposite diagonal corners, if a current of 1 ampere flows into one of the chosen opposite diagonal corners?"[7]

Many forms of the problem have been a subject of study for over a century now (for instance in [16], [31], [7], [13]). The problem seems to first appear in [7]. In [13] Zave and Mullin generalised the problem to n-dimensions. They proposed the following question:

"What would be the equivalence resistance of an n-dimensional cube [a hypercube] (taken along any chosen most remotive vertices) if the edges of the cube are all replaced by R ohm resistors and a current of C amperes flows into one of the chosen vertices? [13]*"*

A further question that arises as a result of the above problem is the following (see [13]): What would be the equivalence resistance of the other four platonic solids and their generalised forms (an n-dimensional regular polyhedra) when subject to the same conditions as the hypercube being considered above (hypercube: a cube in n dimensions); where the resistance is taken along any two chosen most remotive vertices? (see [13]) In his book on mathematical puzzles Martin Gardner discusses the solutions of the electric problem for the other four platonic solids in three dimensions (see [16]) .

We now consider an example of the electrical problem in the case of a cube in 3 dimensions. We show how the cube problem could be solved using physics principles.

Example 1.

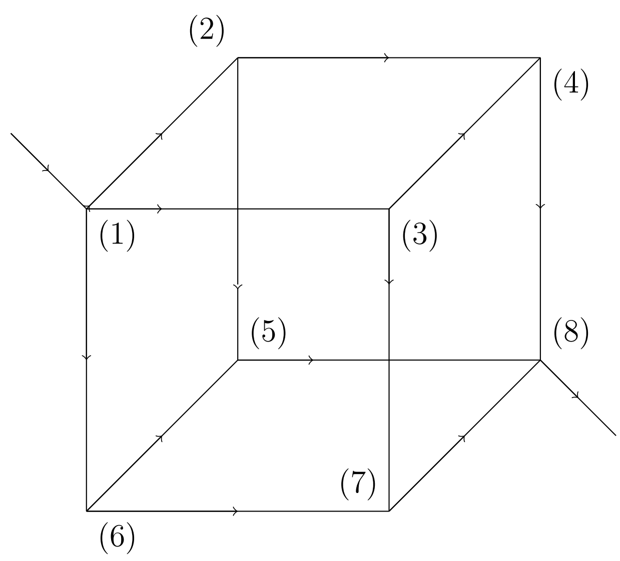

FIGURE 3. Electric circuit

Consider the 3-dimensional cube in example 1. The question is "What is the equivalence resistance of the cube (taken along the vertices (1) and (8)) if all the edges are replaced by 1 ohm resistors and a current of 1 ampere flows into (1) (we denote the current by A_3)" (see [7]). The argument we use in answering the question is that of [7]. We answer the question using physics principles in the following way (see [7],[5]);

To answer the question we need to know what is the potential difference (PD) between the vertices (1) and (8) (see [7]). To calculate this potential difference we make use of the definitions 5,6,7 and 8 defined above.

We argue as follows: Since the cube is symmetric then along the edges (1)–(2), (1)–(6) and (1)–(3) there will be the same amount of current flowing in them [7]. By definition 7 on each of the 3 mentioned edges there will be a current of $\frac{1}{3}$ amperes flowing in it[7]. The current flowing on the edge (1)–(3) splits at the vertex (3) resulting in a current of $\frac{1}{6}$ amperes flowing along each of the edges (3)–(4) and (3)–(7) (see [7]). The current flowing on the edge (2)–(4) and the current flowing on the edge (3)–(4) meet at the vertex (4), by definition 7 the current flowing along the edge (4)–(8) is $\frac{1}{3}$ amperes. A similar argument can be used to determine the current flowing along the other edges.

To determine the potential difference (PD) between the vertices (1) and (8) we make use of the path (1)–(3)–(4)–(8). (By definition 8) the potential difference between the vertices (1) and (8) is the sum of the potential differences along the edges (1)–(3), (3)–(4) and (4)–(8) (see [7])

i.e PD (between (1) and (8)) is equal to the current on edge (1)–(3) × 1 ohm + current on edge (3) – (4) × 1 ohm + current on edge (4)–(8) × 1 ohm $= \frac{1}{3}\, amperes \times 1\, ohm + \frac{1}{6}\, amperes \times 1\, ohm + \frac{1}{3}\, amperes \times 1\, ohm = \frac{5}{6}\, volts.$ By Ohm's law $R_e = \frac{PD}{A_3} \Rightarrow R_e = \frac{5/6\, volts}{1\, ampere} = \frac{5}{6} ohms$ (see [7]). Hence the equivalence resistance of the 3-dimensional cube is $R_e = \frac{5}{6}\, ohms$.

Remark. The path that we have used from (1) to (8) is a matter of choice, other paths could be used.

The method we have used above to solve the problem in example 1 is known in [5] as "The RF cafe method" by Blattenberger Kirt, although the method pre-dates him. This method of solving the problem goes back as far as [7] a 1914 book on electromagnetism by Brook and Poyser.

3.2. **A combinatorial solution to the cube problem.** .

The method used above to solve the problem in example 1 of section 3.2 can be very much tedious in dimensions higher than 3 hence more efficient ways of calculating the equivalence resistance are needed. We now show how the problem being considered in example 1 (see page 42) can be solved using combinatorial arguments as proposed in [31].

We consider a hypercube (n-dimensional cube) Ψ . Let us denote the number of m-dimensional faces bounding an n-dimensional cube by L_m where $0 \leq m \leq n$. Where faces of dimension zero are vertices on the hypercube. One dimensional faces are straight lines on the hypercube. Two dimensional faces are squares on the hypercube and the n-dimensional face being the hypercube itself. In [11]:pp 122 Coxeter proposed the following formula for determining the number of m faces of an n-dimensional cube, $L_m = 2^{n-m}\binom{n}{m}$. Hence Ψ has 2^n vertices which are denoted by L_0 (see [41]).

Each vertex of Ψ has n coordinates of which each coordinate is an entry from the set $\{0,1\}^n$ (see [11],[41]). Each coordinate is connected to n other coordinates[41]. We fix a vertex μ of a long diagonal of Ψ. Vertices at distance s from μ are those vertices of Ψ with s co-ordinates (from the combinations of zeros and ones) being different from those of μ, when we compare corresponding co-ordinates[41].

We now give a combinatorial argument of an identity as proposed in [31] using the same argument as in [31].

Let's consider μ as defined in the above paragraph. For a vertex λ at distance h from μ, there are h co-ordinates of λ which are different from those of μ. From the n co-ordinates of μ there are $\binom{n}{h}$ ways of choosing the h co-ordinates. Hence there are $\binom{n}{h}$ vertices at distance h from μ. Each of those $\binom{n}{h}$ vertices has $n-h$ vertices which are the same as those of μ. For each of the $n-k$ co-ordinates taking the compliment of one of the $n-h$ co-ordinates, the result are the co-ordinates of a vertex of Ψ which is at distance $h+1$ from μ, at distance 1 from λ. There are $\binom{n-h}{1} = n-h$ ways of doing this. i.e each of the $\binom{n}{h}$ vertices at distance h from μ is connected to $n-h$ vertices at distance $h+1$ from μ. Hence there are $(n-h)\times\binom{n}{h}$ parallel connections between vertices at distance h and distance $h+1$ from μ [31].

The maximum distance between vertices of Ψ is n, Hence h runs from 0 to $n-1$. Let T_n denote the equivalence resistance of an n-dimensional cube. By Ohm's current law (see page 39 above) we have (see [31])

(22)
$$T_n = \sum_{h=0}^{n-1} \frac{1}{(n-h)\times\binom{n}{h}}$$

Example 1 on page 42 is a special case of equation 22 when $n = 3$.

$$T_3 = \frac{1}{3\times\binom{3}{0}} + \frac{1}{2\times\binom{3}{1}} + \frac{1}{1\times\binom{3}{2}} = \frac{5}{6}\ ohms$$

Which confirms the result we obtained in example 1 using the physics principles.

Since $(n-h)\times\binom{n}{h} = n\times\binom{n-1}{h}$ then

(23)
$$T_n = \sum_{h=0}^{n-1} \frac{1}{(n-h)\times\binom{n}{h}} = \frac{1}{n}\sum_{h=0}^{n-1}\frac{1}{\binom{n-1}{h}}$$

On equation 23 letting $n-1=i$, we have

$$T_n = \sum_{h=0}^{i} \frac{1}{(i+1)\binom{i}{h}} \tag{24}$$

We recognise equation 24 as equation 10 of chapter One (see page 21 above). What this means is that the equivalence resistance T_n of an n-dimensional cube can be interpreted as the sum of the entries in the n^{th} column of table 8 (differences of Harmonic sequence)[31]. For example the sum of the entries in the third column of table 8 (see 19) is $1/3 + 1/6 + 1/3 = 5/6 = T_3$. This agrees with the answer we got in example 1 about the equivalence resistance of a 3-dimensional cube (see page 42 above).

Now substituting $y = h+1$ in equation 23 we obtain;

$$T_n = \sum_{y=1}^{n} \frac{1}{y\binom{n}{y}} \tag{25}$$

We have obtained equation 25 from equation 23 algebraically. Equation 25 was proposed in [26]. The two equations are also combinatorial equivalent. For a fixed vertex r of a hypercube in finding the equivalence resistance the former uses vertices at distance k and those vertices at distance $k+1$ from r, hence the limits are 0 and $n-1$. The latter uses vertices which are at distance k and those at distance $k-1$ from r, hence the limits of equation 25 are from 1 to n (see [26]). Equation 25 can be independently obtained using the same kind of argument as the one we used to obtain equation 23 above, basing the argument on vertices which are at distance k and those at $k-1$ from a fixed vertex r (see [26]). The arguments used above in obtaining (22), (23),(24) are that of [31].

The appearance of inverses of binomial coefficients in equation 23 leads to to the study of the numbers $Q_n = \sum_{h=0}^{n} \frac{1}{\binom{n}{h}}$ (see [31]). The relationship between T_n and Q_n is given by (see [31]);

$$T_n = \frac{1}{n} Q_{n-1} \tag{26}$$

In [31] the author proposed the following equation as an alternative formula for T_n.

$$T_n = \sum_{r=1}^{n} (\frac{1}{r})(\frac{1}{2})^{n-r} = \frac{1}{2^n} \sum_{r=1}^{n} \frac{2^r}{r} \tag{27}$$

We see that equation 27 above is exactly the same as equation 12 (see page 23 above). This is further evidence that the resistance of an n-dimensional cube can be viewed as sums of rows of the harmonic triangle[31].

3.3. **Numerical approximations in electric circuits.** .

We now discuss concept from analysis which will help us in numerically approximating the equivalence resistance T_n of an n-dimensional cube and other resistive structures.

We define the O-order notation as; "given a function $L(x) : \mathbb{R} \longrightarrow \mathbb{R}$ and $u(x) : \mathbb{R} \longrightarrow \mathbb{R}$ then $L(x) = O\left(u(x)\right)$ as $x \to x_0$ in $\mathbb{R}$; if both $L(x)$ and u(x) possess limits in $\mathbb{R}$ as $x \to x_0$ and if $\exists$ a constant $\epsilon \in \mathbb{R}$ such that $|L(x)| \leq \epsilon\, |u(x)|$" [28]. What this means is that $\frac{|L(x)|}{|u(x)|}$ is bounded above by ϵ (see [28]).

For any functions $u(x)$ and $p(x)$ from $\mathbb{R}$ to $\mathbb{R}$, the following algebraic properties hold for the O-order notation,

1. $O\left(u(x)\right) + O\left(u(x)\right) = O\left(u(x)\right)$ (see [3])
2. $O\left(u(x)\right) O\left(p(x)\right) = O\left(f(x)p(x)\right)$ (see [3])

Definition 11. *"A sequence of functions $\{W_q(x)\}$ for q=1.2.3,$\cdots$ is said to be an asymptotic sequence if for all q, $W_{q+1}(x) = W_q(x)$ as $x \to x_0$"*[28].

What definition 11 means is that $\lim\limits_{x\to x_0} \frac{W_{q+1}(x)}{W_q(x)} = 0$ (see [28]).

Definition 12. *"An asymptotic sequence* $\{W_q(x)\}$ *of functions for* $x \to x_0$ *is said to be an asymptotic expansion/approximation of a function* $u(x)$ *if* $u(x) = \sum_{q=1}^{k-1} e_q W_q(x) + O\left(W_k(x)\right)$*"*[28].

What definition 12 implies is that the error in the approximation is of the same magnitude as the first terms omitted[28]. An asymptotic expansion of a given function may or not be divergent. An asymptotic expansion would be more useful if it where divergent than convergent[28].

The analytic forms for T_n and Q_n given in section 3.2 give little intuitiveness on how the sequences of T_n and Q_n behave for very large values of n [31]. In this section we state numerical approximations of T_n and Q_n which give much more incite on the behaviours of T_n and Q_n for large values of n.

We start with an approximation of T_n (the resistance of an n-dimensional cube) as proposed in [31]. We recall from equation 23 that (see page 45);

$$T_n = \frac{1}{n} \sum_{h=0}^{n-1} \frac{1}{\binom{n-1}{h}}$$

We argue as follows (see [31]); We observe that the binomial coefficients increase as h increases from 0 to $\lfloor (n-1)/2 \rfloor$ and decrease as h increases from $\lceil (n-1)/2 \rceil$ to $n-1$ (see [31]). Also using symmetry of the binomial coefficients, the largest two terms of the summation are obtained when $h = 0$ and $h = n-1$, in-which case we have the two terms $\frac{1}{\binom{n-1}{0}} = \frac{1}{\binom{n-1}{n-1}} = 1$ (see [31]). The second largest terms are $\frac{1}{\binom{n-1}{1}}$ and $\frac{1}{\binom{n-1}{n-2}}$, in that case we have $\frac{1}{\binom{n-1}{1}} = \frac{1}{\binom{n-1}{n-2}} = \frac{1}{n-1} = O(\frac{1}{n})$ (see [31]). There are $n-4$ other terms, the largest two being $\frac{1}{\binom{n-1}{2}} = \frac{1}{\binom{n-1}{n-3}} = O(\frac{1}{n^2})$, so the sum of the $n-4$ terms is approximately $O(\frac{1}{n})$ (see [31]). Hence $T_n = \frac{1}{n}[\frac{2 \times 1!}{1} + O(\frac{1}{n}) + O(\frac{1}{n})] = \frac{2}{n}[\frac{1!}{1} + O(\frac{1}{n})]$ (see definition 12 above)[31]. We now refine the estimate by including the second largest two terms which sum to $\frac{2\times 1!}{n-1}$ as observed above[31]. The third largest two terms are $\frac{1}{\binom{n-1}{2}} = \frac{1}{\binom{n-1}{n-3}} = O(\frac{1}{n^2})$. There are $n-6$ other terms. The largest two being $\frac{1}{\binom{n-1}{3}} = \frac{1}{\binom{n-1}{n-4}} = O(\frac{1}{n^3})$, so the sum of the $n-6$ terms is approximately $O(\frac{1}{n^2})$ (see [31]). So $T_n = \frac{2}{n}[\frac{1}{1} + \frac{1!}{n-1} + O(\frac{1}{n^2})]$. In general (see [31]),

$$T_n = \frac{2}{n}[1 + \frac{1!}{n-1} + \frac{2!}{(n-1)(n-2)} + \cdots + \frac{h!}{(n-1)(n-2)\cdots(n-h)} + O(\frac{1}{n^{h+1}})] \tag{28}$$

Equation 28 is an approximation of T_n with the error in the approximation given by $O(\frac{1}{n^{h+1}})$ (see [31]). The relationship between n and j in equation 28 should be $n \geq 2j$, otherwise it would not make sense[31]. Equation 28 is equivalent to the equation;

$$\frac{nT_n}{2} = [1 + \frac{1!}{n-1} + \frac{2!}{(n-1)(n-2)} + \cdots + \frac{h!}{(n-1)(n-2)\cdots(n-h)} + O(\frac{1}{n^{h+1}})] \tag{29}$$

The argument used above in obtaining (28) and (29) is that of [31].

Applying equation 4 of chapter One (see page 12 above) to each term on the right of equation 29 we obtain the equation [31];

$$\frac{nT_n}{2} = \sum_{t=0}^{\infty} \left(\sum_{s=0}^{t} s! \left\{ {t \atop s} \right\} \right) \frac{1}{n^t} \tag{30}$$

Where the terms $\left\{ {t \atop s} \right\}$ are Stirling numbers of the second kind.
We recognise the terms $\sum_{s=0}^{t} s! \left\{ {t \atop s} \right\}$ in equation 30 above as the number of preferential arrangements of a t-element set (denoted by q_t) (see section 2.1 above) [31]. So equation 30 is (see [31])

$$\frac{nT_n}{2} = q_0 + \frac{q_1}{n} + \frac{q_2}{n^2} + \frac{q_3}{n^3} \cdots + \frac{q_j}{n^j} + \cdots \tag{31}$$

Equation 31 does not just give us more insight into how the sequence T_n behaves for large values of n but also shows a way of how T_n is related to preferential arrangements[31]. Hence the equivalence resistance of a hypercube T_n is related to the number of preferential arrangements of a t-element set through equation 31.

We recall from section 3.2 (see page 47 above) that;

$$Q_n = \sum_{h=0}^{n} \frac{1}{\binom{n}{h}}$$

We seek an asymptotic expansion of Q_n similar to that of T_n given in equation 31 above. In finding the asymptotic expansion of Q_n, we argue as follows (see [31]): The largest two terms in the summation above are the first and the last, which are $\frac{1}{\binom{n}{0}} = \frac{1}{\binom{n}{n}}$. There are $n-1$ other terms. The next largest terms are $\frac{1}{\binom{n}{1}} = \frac{1}{\binom{n}{n-1}}$ of which they are equal to $O(\frac{1}{n})$ (see [31]). There are $n-3$ other terms. The next largest terms are $\frac{1}{\binom{n}{2}} = \frac{1}{\binom{n}{n-2}}$ which are equal to $O(\frac{1}{n^2})$ (see [31]). Hence the sum of the $n-3$ terms is also $O(\frac{1}{n})$ (see [31]). So we have $Q_n = 1 + [O(\frac{1}{n}) + O(\frac{1}{n})] = 1 + O(\frac{1}{n})$ (see [31]). We further refine the approximation by extracting the second and the second last terms in the summation, we obtain (see [31]) $Q_n = 2 + \frac{2}{n} + O(\frac{1}{n(n-1)})$. When we keep refining we obtain (see [31])

$$\frac{Q_n}{2} = 1 + \frac{1}{n} + \cdots + \frac{h!}{n(n-1)\cdots(n-h)} + O(\frac{1}{n^{h+1}}) \tag{32}$$

When applying equation 4 of chapter One (see page 12 above) to equation 32 above, we acknowledge the result being the following equation as proposed in [31]

$$Q_n = \ a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \qquad where \quad a_p = \sum_{s=0}^{p} \left\{ {p \atop s} \right\} \times (s+1)! \tag{33}$$

But through personal communication with Pippenger the author of [31] on April 2014 we agreed that Q_n should be presented as in (34) below instead of equation 33 above,

$$(34)\quad Q_n = 1 + \frac{a_0}{n} + \frac{a_1}{n^2} + \frac{a_2}{n^3} \cdots \qquad \textit{where} \quad a_p = \sum_{s=0}^{p} \left\{ {p \atop s} \right\} \times (s+1)!$$

We recognise a_p in (34) as the number of preferential arrangements of a p-element set[31]. We will refer to equation 34 in chapter Four. Note the terms $\left\{ {p \atop s} \right\}$ are Stirling numbers of the second kind.

Equation 34 was obtained in [31] in the following way; multiplying by $\frac{1}{n}$ on both sides of equation 4 (see page 12) and rearranging other terms, we obtain (see [31]);

$$(35)\qquad \frac{j \times (j-1)!}{n(n-1)(n-2)\cdots(n-(j-1)-1)} = \sum_{t=j}^{\infty} j! \left\{ {t \atop j} \right\} \frac{1}{n^{t+1}}$$

Rearranging equation 32 above we have ;

$$(36)\qquad \frac{Q_n}{2} = 1 + \frac{1 \times 0!}{n} + \cdots + \frac{h \times (h-1)!}{n(n-1)\cdots(n-h)} + O(\frac{1}{n^{h+1}})$$

Applying equation 35 to each term of equation 36 above we obtain the following (see [31])

$$(37)\qquad \frac{Q_n}{2} = 1 + \sum_{p=0}^{\infty} 1 \times 0! \left\{ {t \atop 0} \right\} \frac{1}{n^{p+1}} + \sum_{p=1}^{\infty} 2 \times 1! \left\{ {t \atop 1} \right\} \frac{1}{n^{p+1}} + \cdots$$

The first term of equation 37 corresponds to the first term on the right hand side of equation 32; this is because $\left\{ {0 \atop 0} \right\}$=1 and for all $p \neq 0$ we have $\left\{ {p \atop 0} \right\}$=0. Collecting terms corresponding to each $p \geq 0$ we have (see [31]);

$$\frac{Q_n}{2} = 1 + \sum_{p=0}^{\infty} \sum_{s=0}^{p} [(s+1)! \left\{ {p \atop s} \right\}] \frac{1}{n^{p+1}} \equiv 1 + \frac{a_0}{n} + \frac{a_1}{n^2} + \frac{a_2}{n^3} + \cdots$$

Thus equation 34 above.

3.4. **An n-dimensional tetrahedral resistive structure.** .

In this section we study tetrahedral resistive structures. We also study their connection to the resistance of hypercubes.

Question 1. *On an n-dimensional tetrahedron Y_n, in which all the edges are 1 ohm resistors; we consider two vertices 1 and 2 which are furthest apart, we ask; What is the equivalence resistance of Y_n if a current of 1 ampere flows into the vertex 1 and extracted at the vertex 2 ?* [18]

We recursively construct an n-dimensional tetrahedron using the argument in [11].

A 0-dimensional tetrahedron is a point [11] (in that case we say the resistance is 0)[18].
A 1-dimensional tetrahedron is a straight line[11]; as in the figure below.

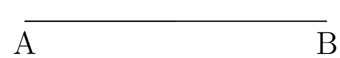

Figure 4

The vertices 1 and 2 in question 1 are the vertices A and B. The connection is a series connection so the resistance of the circuit is 1 *ohm* [18]. Hence $Y_1 = 1$ *ohm*. We note that the maximum number of vertices which are equidistant and non-collinear in a 1-dimensional plane is 2 (see [11]).

In constructing a 2-dimensional tetrahedron we connect using edges, a vertex C which is non-collinear to both A and B and equidistant to both A and B in figure 4 (see [11]). For the vertex C to be equidistant from A and B such that the three points A B C are non-collinear then C needs to be out of the 1-dimensional plane of the straight line[11]. Connecting C to A and B we obtain the 2-dimensional tetrahedron (see [11]);

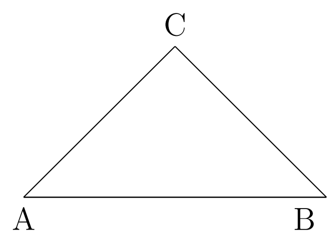

FIGURE 5. [18]

We note that the maximum number of vertices which are equidistant and non-collinear in a 2-dimensional plane is 3 (see [11]).

Each of the three edges is a 1 *ohm* resistor. All the Vertices in figure 5 are pairwise most remotive (furthest apart). The chosen two most remotive vertices can be any two of the three vertices 5 (see [18]). They all lead to the same answer. By choice we say the vertices A and B are the vertices 1 and 2 (see question 1 above) respectively. Measuring the resistance along vertices 1 and 2 we have two parallel connections. One is the edge A-B, which has resistance 1 *ohm* and the connection $A - C - B$, which has a resistance of 2 *ohms* (see [11]). Since the edges $A - C$ and $C - B$ are in series then the equivalence resistance Y_2 is given by; $\frac{1}{Y_2} = \frac{1}{1} + \frac{1}{2} \Rightarrow Y_2 = \frac{2}{3}$ *ohms* (see [18]).

Repeating the process we connect another vertex D which is equidistant and out of the plane of the three vertices in figure 5; we obtain a 3-dimensional tetrahedron, which in flattened form is [18];

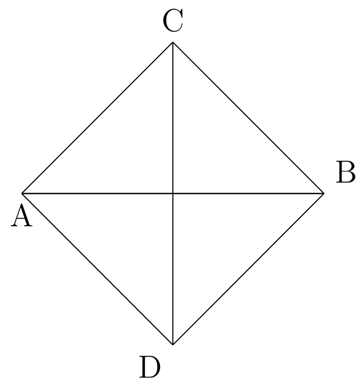

FIGURE 6. [18]

We note that the maximum number of vertices which are equidistant and non-collinear in a 3-dimensional plane is 4 (see [11]). Some of the vertices which are furthest apart are the vertices A and B. So the chosen vertices are by choice A and B. In calculating the equivalence resistance of the circuit we argue as follows (see [18]): For a current flowing into vertex A and extracted on vertex B; since the vertices C and D are equidistant from the vertex A, then the vertices C and D are at the same potential(see [18]). So there is no current flowing on the edge C-D (that means even if you were to touch the edge C-D, you would not be hit by an electric shock)[18]. So the edge $C - D$ can be removed (or thought of as not part of the circuit)(see [18]).

That results in the following figure (see [18]);

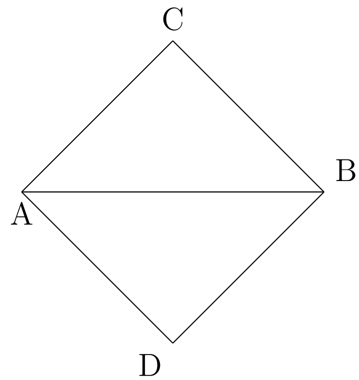

Figure 7. [18]

In figure 7 we have three parallel connections. We have the connection $A - B$ which has a resistance of 1 *ohm*. Secondly we have the connection $A - C - B$ which has resistance of 2 *ohms*. We also have the connection $A - D - B$, which is similar to the connection $A - C - B$, which also has a resistance of 2 *ohms*. Hence the equivalence resistance of the tetrahedron in figure 7; when a current of 1 ampere flows into the vertex A and extracted at the vertex B is; $\frac{1}{Y_3} = \frac{1}{1} + 2 \times \frac{1}{2} \Rightarrow Y_3 = \frac{1}{2}$ *ohms* (see [18]).

We generalise the tetrahedron to n dimensions in the following way (see [11]); In n dimensions the number of maximum points which are equidistant is $n + 1$ (see [11]). There are n parallel connections. One connection being a direct connection between two vertices A and B which are furthest apart. The other $n - 1$ connections are from A to B via another single vertex. Hence the equivalence resistance of an n-dimensional tetrahedron Y_n is given by (see [18])

$$\frac{1}{Y_n} = 1 + (n-1)\frac{1}{2} \Rightarrow Y_n = \frac{2}{n+1} \tag{38}$$

We now propose a recurrence relation satisfied by Y_n;

Theorem 18. *For* $n \geq 2$

$$Y_n = \frac{n}{n+1} \times Y_{n-1} \quad \textit{where } Y_1 = 1$$

In proving the recurrence relation, it is sufficient to show that the general term of the recurrence relation is; $Y_n = \frac{2}{n+1}$. We prove the recurrence relation using an inductive argument. For a fixed $n \in \mathbb{N}$, we assume the recurrence relation holds for all $k<n$. So the recur rence relation holds for $n-1$, hence $Y_{n-1} = \frac{n-1}{n} \times Y_{n-2} = \frac{2}{n}$. This is true by the induction hypothesis. By definition $Y_n = \frac{n}{n+1} \times Y_{n-1}$, this together with $Y_{n-1} = \frac{2}{n}$ imply that; $Y_n = \frac{n}{n+1} \times Y_{n-1} = \frac{n}{n+1} \times \frac{2}{n} = \frac{2}{n+1}$ as required.

Remark. Theorem 18 is no-where mentioned in the literature we consulted, it is our own original contribution.

We further ask; Could the equivalent resistance Y_n of an n-dimensional tetrahedron be related to the equivalence resistance T_n of an n-dimensional cube?

We recall equation 28 of section 3.2 (see page 49)

$$T_n = \frac{2}{n}[1 + \frac{1!}{n-1} + \frac{2!}{(n-1)(n-2)} + \cdots + \frac{h!}{(n-1)(n-2)\cdots(n-h)} + O(\frac{1}{n^{h+1}})]$$

When we let $n+1$ play the role of n we obtain;

$$T_{n+1} = \frac{2}{n+1}[1 + \frac{1!}{n} + \frac{2!}{(n)(n-1)} + \cdots + \frac{h!}{(n)(n-1)\cdots(n-h+1)} + O(\frac{1}{n^{h+2}})]$$

We recognise the term $\frac{2}{n+1}$ as Y_n (the resistance of an n-dimensional tetrahedron we considered above). Hence

$$\frac{T_{n+1}}{Y_n} = [1 + \frac{1!}{n} + \frac{2!}{(n)(n-1)} + \cdots + \frac{h!}{(n)(n-1)\cdots(n-h+1)} + O(\frac{1}{n^{h+2}})] \tag{39}$$

3.5. **An n-dimensional octahedral resistive structure.** .

In this section, we study octahedral resistive structures.

Question 2. *"We ask what is the resistance of an n-dimensional octahedron F_n when measured along vertices furthest apart, when all the edges are 1 ohm resistors and a current of 1 ampere flows into one of the chosen furthest vertices and extracted at the other vertex ?"* [26]

We recursively construct an n-dimensional octahedron using the arguement in [17].
A 0-dimensional octahedron is a point; in this case the resistance is zero [17]. A 1-dimensional octahedron is a straight line [17], as shown in the figure below.

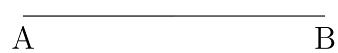

FIGURE 8. [17]

We let the distance between the vertices A and B of the 1-dimensional octahedron be $\sqrt{2}$ (for reasons which will be clear as we go on to higher dimensions)[17]. The connection between the vertices A and B is a series connection. The resistance between vertices A and B is 1 *ohm* hence the resistance of the circuit is $F_1 = 1\, ohm$.

To construct a 2-dimensional octahedron; we observe that the number of vertices of the 1-dimensional octahedron considered above is 2 times the number of dimensions[17].

We continue the process: we find on the perpendicular bisector (line going through the centre of the line $A - B$) two distinct points which are both at distance 1 from both the vertex A and the vertex B [17].

We connect via edges, the two new vertices C and D to both A and B; as shown in the figure below (see [17]);

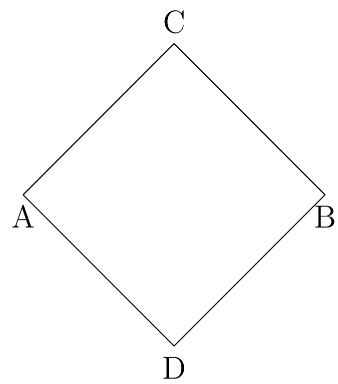

FIGURE 9. [17]

We calculate the equivalent resistance using a pair of vertices which are furthest apart. We obverse that the two new vertices C and D which we added in obtaining the 2-dimensional octahedron are among vertices which are furthest apart. Hence we use C and D in determining the equivalence resistance of the 2-dimensional octahedron. We assume that a flow of current of 1 ampere flows into the vertex C and is extracted from D. The equivalent resistance of the octahedron is (see [26]): $\frac{1}{F_2} = 2 \times \frac{1}{2} = 1 \Rightarrow F_2 = \frac{1}{1}$. We observe that the number of vertices of the 2-dimensional octahedron is 2 times the number of dimensions[17].

The idea of recursively constructing a higher dimensional octahedron can be generalised to n-dimensions in the following way (see [17]); An n-dimensional octahedron will have $2n$ vertices of which $2n-2$ of them are from those of the $n-1$ dimensional octahedron. The other two vertices A and B are the ones which lie on a perpendicular bisector of the $n-1$ dimensional octahedron(which are at a distance 1 unit from all the vertices of the $n-1$ dimensional octahedron)[17]. The vertices A and B are furthest apart. In the circuit there are $2n-2$ connections. Where each one is from A to B through one of the other $2n-2$ vertices. Hence the equivalence resistance F_n when a current of 1 ampere flows into A and extracted at B is $\frac{1}{F_n} = \frac{2n-2}{2} \Rightarrow F_n = \frac{1}{n-1}$ (see [26]).

We end this section with the theorem of the sequence F_n (where F_n is the equivalence resistance of an n-dimensional octahedron);

Theorem 19. *For $n \geq 2$ we have;*

$$F_n = F_{n-1} \times \frac{n-2}{n-1} \qquad \textit{where } F_1 = 1$$

In proving the recurrence relation in theorem 19, it is sufficient to show that the general term in the recurrence relation is given by $F_n = \frac{1}{n-1}$. We prove the theorem by induction. We argue as follows; we assume the recurrence relation holds for all $k<n$. We want to show that the general term of the recurrence relation is given by $F_n = \frac{1}{n-1}$. By induction hypothesis: $F_{n-1} = F_{n-2} \times \frac{n-3}{n-2}$. This implies that $F_{n-1} = \frac{1}{n-3} \times \frac{n-3}{n-2} = \frac{1}{n-2}$. The fact that $F_{n-1} = \frac{1}{n-2}$ together with the definition of F_n i.e $F_n = F_{n-1}\frac{n-2}{n-1}$ together imply that $F_n = \frac{1}{n-2}\frac{n-2}{n-1} = \frac{1}{n-1}$ as required.

Remark. Theorem 19 is no-where mentioned in the literature we consulted. It is our own original contribution.

4. **Chapter Four**

4.1. **Barred preferential arrangements.** .

In this section we study barred preferential arrangements. We discuss their connection to the resistance T_n of an n-dimensional cube. We also discuss a number of theorems on barred preferential arrangements.

A barred preferential arrangement is a preferential arrangement (see chapter Two) of elements of a set, which is ordered relative to a given number of bars[2].

Lets consider the set $X_2 = \{1, 2\}$. How many barred preferential arrangements are possible with two bars? We first enumerate all preferential arrangements of X_2 and then enumerate barred preferential arrangements of X_2 with two bars.

I. The preferential arrangements of X_2 are 1,2   2,1 and 12.

II. The barred preferential arrangements of X_2 with 2 bars are:

||1,2 |1,|2 |1,2| 1,||2   1,|2| 1,2 ||

||2,1 |2,|1 |2, 1| 2,||1   2,|1| 2,1 ||

||12 |12| 12||

Barred preferential arrangements with one bar seem to first appear in [31] a 2010 paper by Pippenger. Their generalisation to the case of multiple bars first appear in [2] a 2013 paper.

It does not matter whether the bar is before or after the comma it has the same interpretation as a barred preferential arrangement[2]; for instance 2,| 1| and 2|, 1| are interpreted the same way. Bars in a barred preferential arrangement are only allowed to go in-between blocks of a preferential arrangement; they are not allowed to go in-between elements of the same block [2]. A k number of bars separates a preferential arrangement into $k+1$ sections [2]. Where section 0 is the region before the left most bar and section k is the region after the right most bar [2]. For the barred preferential arrangement 1,||2; the two bars have separated the preferential arrangement 1,2 into three sections. The first section (section 0) has the number 1. The second section (section 1) is empty (it is the region between the two bars) and section 2 is the region having the number 2. For a fixed set $X_n = \{1, 2, 3, 4, \ldots, n\}$; we will denote the set of all barred preferential arrangements of X_n having k bars by Q_n^k and the number of them by q_n^k i.e $q_n^k = | Q_n^k |$.

We now illustrate with a table few values of q_n^k, for various values of n and k .

TABLE 11. Table of barred preferential arrangements

n/k	0	1	2	3	4
0	1	1	1	1	1
1	1	2	3	4	5
2	3	8	15	24	35
3	13	44	99	184	305
4	75	308	807	1704	3155
5	541	2612	7803	18424	37625
6	4683	25988	87135	227304	507035
7	47293	296564	1102419	3147064	7608305
8	545835	3816548	15575127	48278184	125687555
9	7087261	54667412	242943723	812387704	2265230825
10	102247563	862440068	4145495055	14872295784	44210200235
11	1622632573	14857100084	76797289539	294192418744	928594230305
12	28091567595	277474957988	1534762643847	6251984167464	20880079975955

We produced table 11 using the python code in appendix 1 (see page 75).

In the employment sector it is often of interest to grade candidates who have applied for a given vacancy. Grading those who qualify on their degree of being worthy for hiring and those who are not worthy of being hired on their degree of unworthiness[2]. A single bar can be used to distinguish those candidates who qualify from those who do not qualify for the vacancy[2]. A further single bar can be used to distinguish those who are over qualified for the vacancy[2].

Also when evaluating examination candidates, it is often of interest to grade the examination candidates with respect to a chosen passing mark. Two bars could be used to determine those candidates who qualify to write a supplementary examination from those who do not . Having the interpretation that all those below the first bar (from left) are below the sub-minimum to qualify for a supplementary and all those to the right of the second bar have reached the passing mark. A third bar could be used to determine those candidates who got first class.

We now discuss some identities on barred preferential arrangement. We denote the number of barred preferential arrangements of the set $X_n = \{1, 2, 3 \ldots, n\}$ having k bars is denoted by q_n^k.

We start with the equation (see [31]);

$$q_n^1 = \sum_{s=0}^{n} \left\{ {n \atop s} \right\} (s+1)! \tag{40}$$

We derive the equation in the following way; We consider the set X_n. The elements of X_n can be partitioned into s blocks in $\left\{ {n \atop s} \right\}$ (see chapter One on Stirling numbers) number of ways (where $0 \leq s \leq n$). The s blocks can then be ordered is $s!$ ways[31]. Hence there is a total of $\left\{ {n \atop s} \right\} \times s!$ preferential arrangements of X_n into s blocks [31]. Between the s blocks there are $s+1$ spaces[31]. As a result we have $s+1$ possible positions to insert a single bar[31]. Hence there is a total of $\left\{ {n \atop s} \right\} s! \times (s+1)$ barred preferential arrangements with s blocks, having one bar [31]. The set X_n can be partitioned into a minimum of 1 block and a maximum of n blocks. Hence s runs from 1 to n. So we have

$\sum_{s=1}^{n} \left\{ {n \atop s} \right\} (s+1)!$. Since $\left\{ {n \atop 0} \right\} = 0$, we may include the case $s = 0$. Hence we obtain equation 40 above[31]. Thus equation 40 enumerates the number of barred preferential arrangements of X_n having one bar[31].

We recall equation 34 of chapter three (see page 51 above); regarding the numbers Q_n in relation to the resistance of an n-dimensional cube;

$$(41) \quad Q_n = 1 + \frac{a_0}{n} + \frac{a_1}{n^2} + \frac{a_2}{n^3} \cdots \qquad where \quad a_p = \sum_{s=0}^{p} \left\{ {p \atop s} \right\} \times (s+1)!$$

We recognise the numbers a_p in equation 41 as the number of barred preferential arrangements of a p-element set having 1 bar [31]. Hence the resistance of a hypercube is related to barred preferential arrangements with one bar through equation 41 [31].

We now discuss barred preferential arrangements of X_n having k bars. We recall the number of barred preferential arrangements of X_n having k bars is denoted by q_n^k.

Theorem 20. [2]*For all $k \geq 0$ and $n \geq 1$ we have,*

$$q_n^k = \sum_{s=0}^{n} \left\{ {n \atop s} \right\} s! \left(\!\binom{k+1}{s}\!\right)$$

The symbol $\left(\!\binom{k+1}{s}\!\right) = \binom{k+s}{s}$ is the number of ways of choosing s elements with replacement from a $k+1$ element set having distinct elements[2]. We prove the theorem in the following way (see [2]); There are $\left\{ {n \atop s} \right\}$ ways of partitioning X_n into s unordered blocks[2]. There are $s!$ ways of ordering the blocks[2]. Hence we have $\left\{ {n \atop s} \right\} s!$ ordered partitions/preferential arrangements of X_n into s blocks[2]. There are $s+1$ distinct sections in between, before and after the s blocks. We select k of the $s+1$ sections with replacement[2]. That can be done in

$\left(\binom{s+1}{k}\right) = \binom{k+s}{k} = \binom{k+s}{s} = \left(\binom{k+1}{s}\right)$ ways[2]. Summing over s we obtain the result.

Theorem 21. [2] *For all $k \geq 0$,*

$$q^k(m) = \sum_{n=0}^{\infty} \frac{q_n^k m^n}{n!} = \frac{1}{(2-e^m)^{k+1}}$$

In the statement of the theorem $q^k(m)$ is the exponential generating function of the numbers q_n^k. We prove the theorem in the following way (see [2]); On the set X_n a barred preferential arrangement with k bars can be created in the following way. Choosing elements which are to the right of the right most bar[2]. Lets say there are s of them. There are $\binom{n}{s}$ ways of choosing the s elements from X_n. The s elements can be preferentially arranged among themselves in q_s^0 ways[2]. The remaining $n - s$ elements can be preferentially arranged in q_{n-s}^{k-1} ways with the remaining $k - 1$ bars[2]. Taking the product and summing over s we obtain (see [2])

$$q_n^k = \sum_{s=0} \binom{n}{s} q_s^0 q_{n-s}^{k-1}$$

From this we deduce that q_n^k is a convolution of two sequences, which are q_s^0 and q_{n-s}^{k-1}. The generating function of the sequence q_s^0 in equation 13 (see page 27 above) is $q^0(m) = \frac{1}{2-e^m} = \sum_{s=0}^{\infty} \frac{q_s \times m^s}{s!}$. Hence applying induction on k we obtain the result of the theorem. We conclude this section by stating two more theorems on the sequence of numbers q_n^k.

Theorem 22. [2] *For all $k \geq 1$*

$$q_n^k = \frac{1}{2^m m!} \sum_{s=0}^{n} \begin{bmatrix} k+1 \\ s+1 \end{bmatrix} q_{n+s}^0$$

Theorem 22 gives a way of writing the number q_n^k of barred preferential arrangements of X_n having k bars as a sum of the number of preferential arrangements of X_n with no bars, where $\begin{bmatrix} k+1 \\ s+1 \end{bmatrix}$ are Stirling numbers of the first kind.

Theorem 23. [2] *For all* $n, k \geq 1$*,*

$$q_n^k = \tfrac{1}{2k} q_{n+1}^{k-1} + \tfrac{1}{2} q_n^{k-1}$$

In proving the generating function, we give a proof by comparing coefficients of a specific generating function. We prove the theorem in the following way; by theorem 21 above we have;
$q^k(m) = \frac{1}{(2-e^m)^{k+1}} \Rightarrow 2k \times q^k(m) = \frac{2k}{(2-e^m)^{k+1}} = \frac{k}{(2-e^m)^k} + \frac{k \times e^m}{(2-e^m)^{k+1}}$.
We recognise $\frac{k \times e^m}{(2-e^m)^{k+1}}$ as $\frac{d}{dm}\left(\frac{1}{(2-e^m)^k}\right) = \sum\limits_{n=0}^{\infty} \frac{q_{n+1}^{k-1} \times m^n}{n!}$.
So $2k \times q^k(m) = k \sum\limits_{n=0}^{\infty} \frac{q_n^{k-1} \times m^n}{n!} + \sum\limits_{n=0}^{\infty} \frac{q_{n+1}^{k-1} \times m^n}{n!} \Rightarrow$
$\sum\limits_{n\geq 0} \frac{2k \times q_n^k \times m^n}{n!} = \sum\limits_{n\geq 0} \frac{k \times q_n^{k-1} \times m^n}{n!} + \sum\limits_{n\geq 0} \frac{q_{n+1}^{k-1} \times m^n}{n!}$. The coefficients of $\frac{m^n}{n!}$ must be equal on both sides hence $2k \times q_n^k = k \times q_n^{k-1} + q_{n+1}^{k-1}$.
Thus $q_n^k = \frac{1}{2} q_n^{k-1} + \frac{1}{2k} q_{n+1}^{k-1}$

Remark. The proof we have provided here of theorem 23 above, can be viewed as a generalisation of equation 16. The proof is an alternative algebraic proof to the one provided by the authors in [2] in proving theorem 23. The proof is no where mentioned in the literature we consulted, it is our own original contribution.

4.2. **Barred preferential arrangements which are special.** .

In this section we study special barred preferential arrangements. We discuss a number of theorems on them. In the discussion of some theorems we use well known principles like the inclusion/exclusion principle. We end the chapter by establishing a cyclic property of the sequence q_n^k (the number of barred preferential arrangements of an n element set having k bars).

We consider a subset $E_n^k \subseteq Q_n^k$ (i.e a subset of the set of all barred preferential arrangements of X_n having k bars) of those barred preferential arrangements from Q_n^k in which non of the $k+1$ sections are empty (i.e those barred preferential arrangements from Q_n^k in which all the sections have at least one element). Barred preferential arrangements which are in E_n^k are termed in [2] as special barred preferential arrangements. We denote the cardinality of E_n^k by b_n^k i.e $b_n^k = |E_n^k|$.

We now discuss some identities on the numbers b_n^k.

Theorem 24. [2] *For all $k \geq 0$ and $n \geq 1$*

$$b_n^k = \sum_{s=0}^{n} \left\{ {n \atop s} \right\} s! \binom{s-1}{k}$$

We prove the theorem in the following way (see [2]);
There are $\left\{ {n \atop s} \right\}$ ways of partitioning of X_n into s unordered blocks[2]. There are $s!$ ways of ordering the blocks[2]. So we have $\left\{ {n \atop s} \right\} s!$ ordered partitions of X_n into s blocks[2]. There are $s-1$ distinct sections in between the s blocks[2]. We select k of them without replacement (without replacement because selection with replacement would result in having some empty sections)[2]. There are $\binom{s-1}{k}$ ways of doing the

selection[2]. Hence there are $\left\{ {n \atop s} \right\} s! \binom{s-1}{k}$ barred preferential arrangements of this type[2]. Summing over s we obtain the result of the theorem.

We denote the exponential generating function of the sequence b_n^k by $b^k(m)$. In constructing a generating function of the sequence b_n^k, we first consider the case $k = 0$.

Lemma 3. [2] *The exponential generating function of the sequence* b_n^o *is;*

$$b^0(m) = q^0(m) - 1 = \tfrac{e^m - 1}{2 - e^m}$$

We prove the lemma in the following way (see [2]); A generating function for preferential arrangements where there are no bars i.e $k = 0$ was derived in equation 13 (see page 27 above) as $q^0(m) = \frac{1}{2-e^m} = \sum\limits_{n \geq 0} \frac{q_n^0 \times m^n}{n!}$. In the case when $n = 0$ we have one section and the section is empty. So from $\frac{1}{2-e^m}$ we need to subtract the term $\frac{q_0^0 m^0}{0!} = 1$ (see [2]). Thus the result of the lemma.

Lemma 4. [2] *For all* $n, k \geq 1$

$$b_n^k = \sum_{s=0}^{n} b_s^{k-1} b_{n-s}^0$$

We prove the lemma in the following way (see [2]); From X_n we select s elements which are to the right of the left most bar[2]. There are $\binom{n}{s}$ ways of selecting them. These elements can be preferentially arranged to the right of this left most bar in b_s^{k-1} ways[2]. The remaining $n - s$ elements can be preferentially arranged to the left of the left most bar in b^0_{n-s} ways[2]. Hence we have $\binom{n}{s}b^{k-1}_s b^0_{n-s}$ number of outcomes. Summing over s we obtain the result of the lemma.

Theorem 25. [2]*For all $k \geq 0$ and $n \geq 1$,*

$$b^k(m) = (q^0(m) - 1)^{k+1} = \left(\frac{e^m-1}{2-e^m}\right)^{k+1}$$

We prove theorem in the following way (see [2]): The theorem follows from the identity in lemma 4 . Which shows that b^k_n is a convolution of the two sequences b^{k-1}_s and b^0_{n-s} [2]. Hence b^k_n has $b^{k-1}(m)b^0(m)$ as a generating function i.e
$b^k(m) = b^{k-1}(m)b^0(m)$ [2]. Since the generating function of b^0_n is given in lemma 3 as $b^0(m) = \frac{e^m-1}{2-e^m}$, then applying induction on k we obtain the result[2].

Theorem 26. [2] *For all $k \geq 0$ and $n \geq 1$,*

$$b^k_n = \sum_{s=0}^{k}(-1)^s\binom{k+1}{s}q^{k-s}_s$$

We prove theorem 26 using the inclusion/exclusion principle in the following way (see [2]); From the number q^k_n which is the number of all barred preferential arrangements of X_n having k bars, we subtract the number of those barred preferential arrangements which are not special i.e those barred preferential arrangements having some empty sections. We do that using the inclusion/exclusion principle in the following way[2]. A barred preferential arrangement of X_n with s empty sections is equivalent to preferentially arranging X_n with $k-s$ bars[2]. There are $\binom{k+1}{s}$ ways of choosing s empty sections from the $k+1$ [2]. That is equal to q^{k-s}_n, where $1 \leq s \leq k$ [2]. Subtracting the terms from q^k_n by using the inclusion/exclusion principle we have (see [2]),

$$b_n^k = \sum_{s=0}^{m} \binom{k+1}{s} (-1)^s q_n^{k-s},$$

which is the result of the theorem.

Theorem 27. [2] *For all $n, k \geq 0$*

$$b_n^k = (k+1)! \sum_{s=0}^{n} \binom{n}{s} \left\{ {s \atop k+1} \right\} q_{n-s}^k$$

We prove the theorem in following way (see [2]); We consider b_n^k: we suppose the sum of all the elements of X_n in the $k+1$ first blocks of the $k+1$ sections is s [2]. There are $\binom{n}{s}$ ways of choosing the s elements[2]. There are $\left\{ {s \atop k+1} \right\}$ ways of partitioning the s elements into $k+1$ blocks. There are $(k+1)!$ ways of distributing the s elements into $k+1$ sections[2]. The remaining $n-s$ elements of X_n can be preferentially arranged into the $k+1$ sections in q_{n-s}^k ways[2]. Hence we have, $(k+1)!\binom{n}{s}\left\{ {s \atop k+1} \right\} q_{n-s}^k$, number of outcomes[2]. Summing over s we obtain the result.

We end this chapter by proposing a theorem on barred preferential arrangements q_n^k.

Theorem 28. *For all $n \geq 1$ and a fixed $k \geq 0$ the last digit of q_n^k has a four cycle.*

In [19] the author has shown the property on theorem 28 holds for the case $k = 0$ (where $n \geq 1$). Here using the same argument we generalise the property from q_n^0 to q_n^k. The exponential generating function of q_n^k is given by (see theorem 21 above); $q^k(m) = \sum_{n \geq 0} \frac{q_n^k m^n}{n!} = \frac{1}{(2-e^m)^{k+1}}$ Writing $q^k(m)$ as a binomial series we have;

$q^k(m) = \frac{1}{2^{k+1}}\left(\frac{1}{1-\frac{e^x}{2}}\right)^{k+1} = \frac{1}{2^{k+1}}\sum_{s=0}^{\infty}\binom{-k-1}{s}\frac{e^{sx}}{2^s}$. From this we have the following equation;

$$q_n^k = \frac{d}{dm}\left(q^k(m)\right)|_{m=0} = \frac{1}{2^{k+1}}\sum_{s=0}^{\infty}\binom{-k-1}{s}\frac{s^n}{2^s} \tag{42}$$

On the statement of the theorem, to say that the last digit has a four cycle means that; q_n^k and q_{n+4}^k have the same last digit[19]. So if q_n^k and q_{n+4}^k are having the same last digit, that implies $q_{n+4}^k - q_n^k$ is divisible by 10 (this is what we need to show)[19]. By equation 42 above $q_{n+4}^k = \frac{1}{2^{k+1}}\sum_{s=0}^{\infty}\binom{-k-1}{s}\frac{s^{n+4}}{2^s} \Rightarrow q_{n+4}^k - q_n^k = \frac{1}{2^{k+1}}\sum_{s=0}^{\infty}\binom{-k-1}{s}\frac{1}{2^s}[s^{n+4} - s^n]$. We now show that $s^{n+4} - s^n$ is divisible by 10 (see [19]);

a)

I. If s is odd say $s = 2b+1$ (where $b \in \mathbb{Z}^+$) then $s^n = \underbrace{(2b+1)\cdots(2b+1)}_{n-factors}$ is odd. This is so since each product in the expansion of s^n is a multiple of two except the last one which is equal to 1. So $s^4 - 1$ is even. The two terms s^n and $s^4 - 1$ are of different parity (when one is even the other is odd)[19].

II. If s is even say $s = 2p$ (where $p \in \mathbb{Z}^+$) then $s^n = \underbrace{(2p)\cdots(2p)}_{n-factors}$ is even. This is so since each term in the expansion of s^n is a multiple of 2. Hence s^4-1 is odd. Also in this case s^n and s^4-1 are of different parity[19].

By I and II we deduce that $s^n(s^4 - 1)$ will always be divisible by 2. Hence $q_{n+4}^k - q_n^k$ is divisible by 2 (see [19]).

b)

From a) above we have $q_{n+4} - q_n = \frac{1}{2^{k+1}} \sum_{s=0}^{\infty} \binom{-k-1}{s} \frac{1}{2^s}(s^{n+4} - s^n) = \frac{1}{2^{k+1}} \sum_{s=0}^{\infty} \binom{-k-1}{s} \frac{1}{2^s}[s^n(s^4 - 1]$. This implies that;

$$q_{n+4} - q_n = \frac{1}{2^{k+1}} \sum_{s=0}^{\infty} \binom{-k-1}{s} \frac{1}{2^s}[s^{n-1}(s^5 - s)] \tag{43}$$

We recall Fermat's Theorem "If $w \in \mathbb{Z}$ then $w^m \equiv w \bmod m$, where m is a prime"(see [15]: A first course in Abstract algebra by John Fraleigh, page 184)[19]. Applying Fermat's theorem to equation 43, we deduce that $s^5 - s$ is divisible by 5 (see [19]). This implies that (Combining a) and b)) $q^k_{n+4} - q^k_n$ must be divisible by the lowest common multiple of 2 and 5 which is 10 (see[19]). Hence $q^k_{n+4} - q^k_n \equiv \; 0 \, mod \; 10$ (see [19]). From this the statement of the theorem follows.

To illustrate theorem 28 we consider the table below,

n/k	0	1	2	3	4
0	1	1	1	1	1
1	1	2	3	4	5
2	3	8	15	24	35
3	13	44	99	184	305
4	75	308	807	1704	3155
5	541	2612	7803	18424	37625
6	4683	25988	87135	227304	507035
7	47293	296564	1102419	3147064	7608305
8	545835	3816548	15575127	48278184	125687555
9	7087261	54667412	242943723	812387704	2265230825
10	102247563	862440068	4145495055	14872295784	44210200235
11	1622632573	14857100084	76797289539	294192418744	928594230305
12	28091567595	277474957988	1534762643847	6251984167464	20880079975955

TABLE 12. Table of barred preferential arrangements

We observe that on each column for $n \geq 1$ the last digit of q_n^k has a four cycle; for instance on the first column the cycle is 1,3,3,5; on the second column the four cycle is 2,8,4,8. This continues indefinitely.

Remark. Theorem 28 is no-where mentioned in the literature we consulted. It is our own original contribution.

Further Works

• Generalised Stirling numbers of the first kind $\left[{n \atop i}\right]_s$ are defined as: The number of permutations of the set $X_n = \{1, 2, \ldots, n\}$ having i cycles in which the first $1, 2 \ldots, s$ elements are in different cycles[6]. In our further works we will show how these Stirling numbers are related to q_n^k (i.e the number of barred preferential arrangement of an n-element set having k bars).

• Also the generalised Stirling numbers of the second kind denoted by $\left\{{n \atop i}\right\}_s$ are defined as : The number of partitions of the set $X_n = \{1, 2, 3 \ldots, n\}$ into i non-empty subsets in which the elements $1, 2 \ldots, s$ are in different subsets . In our further works we will also show how these Stirling numbers are related to q_n^k (i.e the number of barred preferential arrangement of an n-element set having k bars).

In proving the two above statements we will make use of the binomial series from calculus.

• Barred preferential arrangements considered in this study only allow bars to go in between blocks of a preferential arrangement; for further works what one could do is to study barred preferential which would allow bars to go in-between elements of the same block. That would lead to generalisations of some of the identities discussed above.

Appendix 1.

It turns out that the number of barred preferential arrangements increases exponentially even for small increments on n. Even for small values of n like 17 say, the number of bars is 6 (this is q_{17}^6). The number of barred preferential arrangements (q_{17}^6) turns out to be 3381768767046340670167; which would be very difficult to crank by hand if not impossible, hence we developed the below computer program for calculating the number of barred preferential arrangements q_n^k for us using python (the python version in [32]). At the end of this discussion we will show step by step how the number, $q_{17}^6 = 3381768767046340670167$ was obtained.

The program that we present below is based on theorem 5 of [2]. The program takes both as parameters the number n of elements in the underlying set and the number k of bars and return the total number of barred preferential arrangements.

Remark. The program is nowhere mentioned in the literature we consulted it is our own original contribution. It is what we used to generate entries of some of the tables in the thesis (for instance table 10, table 12).

The program starts here,

def stir(n,k):(The definition of stir(n,k) we present here can be found in [12])

s=n

y=k

```
if s<=0:
   return 1
elif y <=0:
   return 0
elif (s==0 and y==0)
   return -1
elif s==0 s==y
   return 1
elif s<y:
   return 0
else:
   a=stir(s-1,y)
   a=y*a
return (y*(stir(s-1,y)))+stir(s-1,y-1)
def pro(s):(The definition of pro(s) we present here can be found in[37]
   if s==0:
   return 1
   return s*pro(s-1)
def nck(n,k):( The definition of nck(n,k) we present here can be found
in [37])
   return pro(n)/(pro(k)*pro(n-k))
sum=0
i=0
n1=input('Enter the number of elements n of the underlying set')
n=int(n1)
k1=input('Enter the number k of bars')
k=int(k1)
while i<=n;
```

```
sum=sum+pro(i)*stir(n,i*nck(k+i,i))
i=i+1
if k<=0 or n<0:
print ('A negative value has been entered')
else: print(sum)
```

The code ends here.

In the next few pages we will show how the program works when you run it. We will demonstrate how the number q_{17}^{6} can be calculated using the program i.e barred preferential arrangements of a 17-element set having 6 bars.

Step One: The program asks you to enter the number of elements in the underlying set (if you enter a negative number then the program will return "a negative number has been inserted", in that case you will have to insert another number),

Step Two: We enter the number 17,

.

Step Three: The program asks you to enter the number of bars (which is represented by k in this instance),

.

Step four: We enter the number 6,

.

Step Five: The program returns the number of possible barred preferential arrangements.

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