SOME GENERAL CONVERGENCE THEOREMS ON FIXED POINTS

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Abstract

In this thesis, we first obtain coincidence and common fixed point theorems for a pair of generalized nonexpansive type mappings in a normed space. Then we discuss two types of convergence theorems, namely, the convergence of Mann iteration procedures and the convergence and stability of fixed points. In addition, we discuss the viscosity approximations generated by (ψ, ϕ)-weakly contractive mappings and a sequence of nonexpansive mappings and then establish Browder and Halpern type convergence theorems on Banach spaces. With regard to iteration procedures, we obtain a result on the convergence of Mann iteration for generalized nonexpansive type mappings in a Banach space which satisfies Opial's condition. And, in the case of stability of fixed points, we obtain a number of stability results for the sequence of (ψ, ϕ)- weakly contractive mappings and the sequence of their corresponding fixed points in metric and 2-metric spaces. We also present a generalization of Fraser and Nadler type stability theorems in 2-metric spaces involving a sequence of metrics.

Keywords: Fixed points, coincidence points, common fixed points, generalized nonexpansive type mapping, Mann iteration, viscosity approximations, stability, (ψ, ϕ)-weakly contractive mappings, (G)-convergence, (H)-convergence, 2-metric spaces.

American Mathematical society subject classification(2010): 47H10, 54H25.

Declaration

Except for the references that have been accurately cited and discussed herein, the content of this thesis represents my own work. The entire thesis has neither been nor is concurrently being submitted to any other academic institution for the purpose of obtaining a degree.

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Symbols and Notations

$\in$	Belongs to
$\forall$	For all
$\exists$	There exists
$\subset$	Subset of
$\bigcup$	Union
$\Rightarrow$	Implies
$\Leftrightarrow$	Logical equivalence, if and only if
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of non-negative reals
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{N}$	The set of natural numbers
$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$	The set of naturals including infinity
(X, d)	A metric space with a metric d
$d(x, y)$	Distance from x to y
$(X, \|.\|$	A normed space
(X, ρ)	A 2-metric space with a 2-metric ρ
sup	Supremum
inf	Infimum
$\lim_n x_n = x$	$\lim_{n\to\infty} x_n = x$
lim sup	Limit supremum
lim inf	Limit infimum
$x_n \to x$	$\{x_n\}$ converges strongly to x.
max	Maximum
$\prod_{n\in\mathbb{N}} X_n$	A sequence in X such that $x_n \in X_n \ \forall n \in \mathbb{N}$
$cl(X)$	Closure of X

Dedication

This thesis is dedicated to my family
whose prayers and sacrifices made it possible.

Introduction

0.1 General Background

Many problems arising in different areas of mathematics, such as optimization, variational analysis and differential and integral equations, can be modeled by the equation

$$x = Tx,$$

where T is generally a nonlinear operator and x, an element of a topological space X. The solutions to this equation are called fixed points of T and the theorems concerning the existence and properties of fixed points are known as fixed point theorems. Such theorems have broad applications in proving existence and uniqueness of solutions of various functional equations.

If T is a contraction mapping defined on a complete metric space X, the Banach fixed point theorem (also known as the contraction mapping theorem or contraction mapping principle) establishes that T has a unique fixed point and for any $x \in X$, the sequence of Picard iterates $\{T^n x\}$ strongly converges to the fixed point of T (recall that T is a contraction if $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ and $0 \leq k < 1$). The Banach fixed point theorem first appeared in explicit form in Banach's [4] 1922 thesis. Because of its importance and usefulness for mathematical theory, it has become a very popular tool to show the existence of solutions of nonlinear Volterra integral equations, nonlinear integro-differential equations in Banach spaces, and to show the convergence of algorithms in computational mathematics.

Generalizations of the above principle have been extensively investigated by many authors (see [16], [30], [39], [71], [75]). An excellent reference in this context is due to Rhoades [75] which presents a survey of 125 contractive conditions, out of which 25 have been found to be independent. Among these 25 conditions, one of the most general conditions is due to Ćirić [16]. Subsequent developments in this direction could be found in Kincses and Totic [44], Jachymski [36] and Collaco and Carvalho E silva [17]. Further, from the application point of view, two notable generalizations of the Banach contraction principle are the contractive mappings (mappings T satisfying the condition $d(Tx, Ty) < d(x, y)$ for all $x, y \in X,\ x \neq y$) by Edelstein [21] and nonexpansive mappings (mappings T satisfying the condition $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$) by Browder [12]. These mappings have again found a wide range of applications in the theory of monotone operators and variational inequalities (see Deimling [18] and Zeidler [103, 104]). Other interesting generalizations include the conditions by Rakotch [70] and Boyd and Wong [11] where the constant k is replaced by a function $k(x, y)$ by suitably defining the family of functions $\{k(x, y)\}$ and an upper semicontinuous function ϕ respectively. The Boyd and Wong type mappings are known as nonlinear contractions or ϕ-contractions. The minimum common property for the above classes of mappings is that they are all continuous. For an excellent discussion on metric fixed point theory, we refer to Goebel and Kirk [26] and Khamsi and Kirk [40] among others.

In 1972, Krasnoselski$\tilde{i}$ et al. [45] introduced the notion of weakly contractive mappings which includes the classical Banach contraction as a special case and is closely related to the nonlinear contractions of Boyd and Wong [11]. Later on Alber and Guerre-Delabriere [1] obtained certain fixed point theorems in Hilbert spaces for weakly contractive mappings and acknowledged that their results were true at least for uniformly smooth and uniformly convex Banach spaces(cf. [26]). Subsequently, Rhoades [80] extended some of their results to complete metric spaces under less restrictive conditions and thus established that his results are still valid for arbitrary Banach spaces. On the other hand, in 1976 Delbosco [19] initiated the study of contractive conditions with the so called altering distance function ψ. In fact, Delbosco [19] considered only the case in which ψ is a power function. Subsequently, his result was extended by Skof [95] in 1977 and Khan et al. [41] in 1984. Recently, Dutta and Choudhury [20] introduced the notion of (ψ, ϕ)- weakly contractive mappings where ψ and ϕ are functions from positive reals into itself satisfying certain conditions. They obtained a fixed point theorem for the above class of mappings, which in turn generalizes the above results of Rhoades [80]. This class of mappings has been further improved by Bose and Roychowdhury [10].

Fixed points have been used as a study tool to investigate the relationship between the convergence of sequences of mappings on a metric (resp. Banach) space and the sequence of their fixed points. This area of research is called the stability of fixed points and has many useful applications (cf. Istratescu and Istratescu [35]). In this context, the first result was obtained by Bonsall [9] where he proved that the pointwise convergence of a sequence of contraction mappings $\{T_n\}$ on a metric space implies the convergence of their corresponding fixed points. In addition, the limit mapping also turns out to be a contraction mapping. He also used his result successfully to obtain a solution of Cauchy's initial value problem. For a related result, we refer to Sonnenschein [96]. Subsequent results by Nadler Jr. [59] (see also Fraser and Nadler [22]) and others address mainly the problem of replacing the completeness of the space by the existence of fixed points and various relaxations on the contraction constant. For related results in this direction using various contractive conditions on different settings, we refer to [48, 61, 82, 90, 93, 94]. Diverse aspects of the stability results appear in subjects such as data dependence of fixed points, approximation theory, iteration methods for operator equations and techniques of proof in fixed point theory among others (cf. Rus [82]).

Iterative construction of fixed points is an interesting area of nonlinear analysis. In linear spaces, various iteration schemes have been successfully applied to fixed point problems and also to obtain solutions of operator equations. In most of the cases the contractive condition is strong enough, not only to guarantee the existence of a unique fixed point, but also to obtain that fixed point by repeated iteration of the function. The most commonly used iteration procedure to approximate fixed points is the method of successive approximations (or Picard iteration), given by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \cdots \quad \text{and } x_0 \in X.$$

Since the Picard iteration may, in general, need not converge to a fixed point for certain kinds of mappings such as nonexpansive mappings, other iteration procedures were considered. In fact, a nonexpansive mapping need not have a fixed point (for example, a translation mapping is a fixed point free nonexpansive mapping). As an illustration, let us consider the following example of a nonexpansive mapping with a fixed point and whose iteration does not converge to the fixed point in question. Define $Tx = 1 - x$ for all $x \in [0, 1]$. Then T is a nonexpansive self mapping of $[0, 1]$ with a unique fixed point at $x = \dfrac{1}{2}$, but if we choose our starting point at $x = a$, $x \neq \frac{1}{2}$, then the repeated iteration yields $\{1 - a, a, 1 - a\}$, which clearly does not converge to $\dfrac{1}{2}$ (cf. Rhoades [77]). The sequence thus obtained does not converge to any point of $[0, 1]$.

Such considerations have compelled mathematicians to look for other types of iteration schemes, beyond the method of successive approximations, to construct and locate fixed points for linear and nonlinear mappings. We feel that the motivating factor behind the study of these methods were the summability methods(cf. Mann [49] and Ishikawa [34]).

There is another notion of stability in fixed point theory which is related to the stability of iteration procedures. Ostrowski [65] appears to be the first to discuss the stability of iterative procedures on metric spaces. However, a formal definition of the stability of general iterative procedures is due to Harder and Hicks [28, 29]. For an excellent discussion on this topic, one may refer to Berinde [8], Jachymski [37], Osilike [63, 64], Rhoades [78, 79] among others.

0.2 The Present Thesis

It is known that common fixed point (and coincidence point) theorems are generalizations of fixed point theorems. Over the past few decades, there have been a lot of activity in fixed point theory and a number of authors took interest in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems. Most of these results assume notions such as weak commutativity [83], R-weakly commutativity [68] or R-subweakly commutativity [84]. In *Chapter 1* of this thesis, under the assumption of R-subweakly commutativity, we prove coincidence and common fixed point theorems for a pair of generalized nonexpansive type mappings in a Normed space. We also establish the weak convergence of a sequence of Mann iterates of a generalized nonexpansive type mapping in a Banach space which satisfies Opial's condition [62].

Fixed point theory for nonexpansive mappings has its origin in the existence theorems of Browder(1965) and Halpern(1967) among others. These theorems have been extended in several directions by many authors. For example, in 2000, Moudafi generalized the above results of Browder and Halpern in the direction of the so called viscosity approximations generated by a contraction mapping and a nonexpansive mapping. The viscosity approximation method (VAM) for solving nonlinear operator equations has recently attracted much attention. The advantage of this method is that one can find a particular solution to the associated problem, and in most cases this particular solution solves some variational inequality. In *Chapter 2*, we discuss the viscosity approximations generated by a (ψ, ϕ)-weakly contractive mapping and a sequence of nonexpansive mappings and prove that under certain appropriate conditions, the iterative scheme defined by the VAM converges strongly to a unique fixed point which solves some variational inequality.

In respect of the stability of fixed points, uniform convergence and pointwise convergence play an important role. However, when the domain of definition of all mappings in question is neither the same space nor a unique nonempty subset of it, the above notions do not work. This difficulty has recently been overcome by Barbet and Nachi [5, 6] where some new notions of convergence called (G)-convergence and (H)-convergence have been introduced and utilized to obtain stability results in a metric space. These results have been further generalized by Mishra et al.[51, 52, 53, 55, 56] in different settings. The above results generalize the earlier results of Bonsall [9] and Nadler Jr. [59] among others. In *Chapter 3*, we consider a sequence $\{T_n\}$ of (ψ, ϕ)-weakly contractive mappings which are only defined on a subset X_n of the metric space (X, d) and obtain stability results using the notions of (G)-convergence and (H)-convergence.

In 1962, S. Gähler [23, 24, 25] introduced and studied the notion of 2-metric spaces in a series of papers. The study of the Banach contraction on a 2-metric space was initiated by Iseki et al.[33]. They [33] proved that a Banach contraction on a bounded complete 2-metric space possesses a unique fixed point. The requirement of boundedness of the space was dispensed with subsequently by Rhoades [76] and Lal and Singh [46] independently. Many contractive type principles on 2-metric spaces have been proved by Khan [42], Lal and Das [47], Sharma [86, 87, 88], Singh [89] and others. However, Hsiao [31] showed that all these contractivity conditions don't have a wide range of applications since they imply colinearity of the sequence of iterates starting with any point. To adress this infirmity, Aliouche and Simpson [2] recently considered a 2-metric space that satisfies an additional quadratic axiom and they assumed that the 2-metric defined on the space is globally bounded. With this hypothesis on 2-metric and under appropriate compactness conditions, they proved that a contractive mapping has either a fixed point or a fixed line. We would like to note here that the objection raised by Hsiao has however, no bearing on stability of fixed points if the existence of fixed point is assumed as suggested by Nadler Jr. [59]. A number of stability results under pointwise and uniform convergence have been studied in 2-metric spaces by many authors (see Rhoades [76], Singh [90] and Singh and Ram [91, 92] for details). In *Chapter 4*, we generalize the above results and prove stability of fixed points for (ψ, ϕ)-weakly contractive mappings under (G)-convergence and (H)-convergence introduced by Barbet and Nachi [6] in a 2-metric space setting.

In 1969, Fraser and Nadler [22] investigated stability of fixed points under pointwise convergence for a sequence of contractive maps $\{T_n\}$ in a metric space which also involve a sequence of metrics. In addition, they demonstrated that under pointwise convergence of the sequence of metrics, the sequence of fixed points does not converge to the fixed point of the limit mapping of $\{T_n\}$. However, the above conclusion may hold if the pointwise convergence of the sequence of metrics is replaced by uniform convergence. In *Chapter 5*, motivated by Fraser and Nadler [22], Nachi [58] and Mishra et al. [54], we obtain stability of fixed points for (ψ, ϕ)-weakly contractive mappings in 2-metric spaces.

Definitions, theorems, corollaries and remarks are numbered per chapter and sequentially per section, for example, Definition 1.3.5 means the fifth definition of the third section of Chapter 1.

To the best of our knowledge, the results stated below are our own major results in this thesis:

Theorem 1.3.2, Theorem 1.3.4, Theorem 1.4.2, Theorem 2.3.1, Theorem 2.3.2, Theorem 3.4.1, Theorem 3.4.2, Theorem 3.4.4, Theorem 3.5.1, Theorem 4.3.1, Theorem 4.3.2, Theorem 4.3.4, Theorem 4.4.1, Theorem 5.3.1, Theorem 5.3.2 and Theorem 5.4.1.

Chapter 1

Some existence and convergence theorems for nonexpansive type mappings

1.1 Introduction

In this chapter some existence and convergence theorems for a class of nonexpansive type mappings are obtained in a normed space. Specifically, in Section 1.3, we obtain coincidence and common fixed point theorems while in Section 1.4, the weak convergence of Mann iterations (cf. Mann [49]) to a common fixed point for the above class of mappings is discussed. The results obtained herein generalize certain results of Kim et al. [43], Rhoades and Temir [81] and Shahzad [85] among others.

The results of this chapter appear in *International Journal of Analysis (2013)*, Art. ID 539723

1.2 Preliminaries

In this section we review the basic definitions and well known properties on normed spaces.

Definition 1.2.1. Let C be a nonempty subset of a normed space X. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.2.1}$$

for all $x, y \in C$. Suppose $S : C \to C$ is another mapping on C. Then the mapping T is said to be S-nonexpansive if

$$\|Tx - Ty\| \leq \|Sx - Sy\| \tag{1.2.2}$$

for all $x, y \in C$.

The class of S-nonexpansive mappings is more general than nonexpansive mappings (see for reference [43], [66] and [81]).

Now we extend the above notion of S-nonexpansive mappings to a more general class of nonexpansive mappings.

Definition 1.2.2. Let X be a normed space, C a nonempty subset of X and $S, T : C \to C$. We say that T is a *generalized S-nonexpansive type mapping* if

$$\|Tx - Ty\| \leq M(x, y) \tag{1.2.3}$$

for all $x, y \in C$, where

$$M(x, y) = \max\left\{\|Sx - Sy\|, \frac{\|Sx - Tx\| + \|Sy - Ty\|}{2}, \frac{\|Sx - Ty\| + \|Sy - Tx\|}{2}\right\}.$$

Further, T will be called a generalized nonexpansive type mappings if

$$\|Tx - Ty\| \leq m(x, y) \tag{1.2.4}$$

for all $x, y \in C$, where

$$m(x, y) = \max\left\{\|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2}, \frac{\|x - Ty\| + \|y - Tx\|}{2}\right\}.$$

Let C be a nonempty subset of a normed space X and $S, T : C \to X$. A point $z \in C$ is called a coincidence point of S and T if $Sz = Tz$ and a common fixed point if $Sz = Tz = z$.

Throughout this chapter, $F(S)$ and $F(T)$ will denote the set of fixed points of S and T respectively.

Definition 1.2.3. Let X be a normed space, C a nonempty subset of X and $S, T : C \to C$. The pair of mappings (S, T) is called:

(i) commuting if $TSx = STx$ for all $x \in C$.

(ii) weakly commuting (see [83]) if for all $x \in C$,

$$\|STx - TSx\| \leq \|Sx - Tx\|.$$

(iii) R-weakly commuting (see [68]) if for all $x \in C$, there exists $R > 0$ such that

$$\|STx - TSx\| \leq R\|Sx - Tx\|.$$

The following example illustrates that weakly commuting mappings are R-weakly commuting but the converse is not true in general.

Example 1.2.1. [67] Let $X = C = [1, \infty)$ be endowed with the usual norm $\|x\| = |x|$. Let $S, T : C \to C$ be mappings defined by

$$Sx = x^2 \text{ and } Tx = 2x - 1 \text{ for all } x \in X.$$

Then

$$\|STx - TSx\| = \|2x^2 - 4x + 2\| = 2|x^2 - 2x + 1|$$

and

$$\|Sx - Tx\| = |x^2 - 2x + 1|.$$

Therefore $\|STx - TSx\| = 2|x^2 - 2x + 1| = 2\|Sx - Tx\|$ and the pair (S, T) is R-weakly commuting with $R = 2$ but not weakly commuting.

In general, Commuting $\Rightarrow$ Weakly commuting $\Rightarrow$ R-weakly commuting.

Definition 1.2.4. (cf. [43]). Let X be a normed space and C a nonempty subset of X. The set C is called q-starshaped with $q \in C$, if for all $x \in C$, the segment $[q, x] = \{(1 - t)q + tx\}$ joining q to x is contained in C, where $0 \leq t \leq 1$.

Further, if C is a nonempty q-starshaped subset of a normed space X, then the mapping $S : C \to C$ is said to be q-affine if

$$S(tx + (1 - t)q) = tSx + (1 - t)q$$

for all $x \in C$ and $0 \leq t \leq 1$.

Definition 1.2.5. (cf. [43]). Let X be a normed space, C a nonempty subset of X and $S, T : C \to C$ such that $F(S) \neq \emptyset$. Suppose $q \in F(S)$ and C is q-starshaped. Then the pair of mappings (S, T) is called R-subweakly commuting on C if for all $x \in C$, there exists a real number $R > 0$ such that

$$\|STx - TSx\| \leq R \, dist(Sx, [q, Tx]),$$

where $dist(Sx, [q, Tx]) = \inf\{\|Sx - y\| : y \in [q, Tx]\}$.

We note that R-subweakly commuting mappings are R-weakly commuting but the converse is not true in general.

Example 1.2.2. Let $X = \mathbb{R}$ with the usual norm $\|x\| = |x|$ and $C = [0, 10]$. Define $S, T : C \to C$ by

$$Tx = \frac{x+1}{2}, \qquad Sx = \frac{x}{2}.$$

Then

$$\|TSx - STx\| = \left|\frac{x+2}{4} - \frac{x+1}{4}\right| = \frac{1}{4}$$

and

$$\|Tx - Sx\| = \left|\frac{x+1}{2} - \frac{x}{2}\right| = \frac{1}{2} \text{ for all } x \in C.$$

Therefore

$$\|TSx - STx\| = \frac{1}{4} = \frac{1}{2}\left(\frac{1}{2}\right) = R\,\|Tx - Sx\|$$

holds for $R = \dfrac{1}{2}$ and T and S are R-weakly commuting on C.

On the other hand, $q = 0 \in F(S)$ and for all $x \in C$,

$$\|Sx - [Tx, q]\| = \left|\frac{x}{2} - \left[\frac{x+1}{2}, 0\right]\right| = 0.$$

So, there does not exist any $R > 0$ such that for all $x \in C$,

$$\|TSx - STx\| = \frac{1}{4} \leq R\,\|Sx - [Tx, q]\| = 0$$

holds. Thus T and S are not R-subweakly commuting on C.

Definition 1.2.6. Let C be a nonempty subset of a normed space X and $T : C \to C$. Let $\{x_n\}$ be a sequence in X. We denote the weak and strong convergence of $\{x_n\}$ to x by $x_n \rightharpoonup x$ and $x_n \to x$ respectively. The mapping T is said to be *demicontinuous* if $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $Tx_n \rightharpoonup Tx$.

Definition 1.2.7. A Banach space X is said to satisfy the Opial's condition (see [62]), if whenever a sequence $\{x_n\}$ in X converges weakly to x ($x_n \rightharpoonup x$), then

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$$

for all $y \in X$, $y \neq x$.

We note that the L^p spaces , $p \neq 2$ do not satisfy Opial's condition while all l^p spaces $(1 < p < \infty)$ do (see for details Goebel and Kirk [26]).

1.3 Coincidence and Common fixed point Theorems

The following common fixed point theorem is due to Shahzad [85, Theorem 2.1]. For related results we refer to [3],[66], [73] and [84].

Theorem 1.3.1. *Let (X, d) be a metric space and C a nonempty subset of X. Let $S, T : C \to C$ be a pair of mappings such that*

(i) $T(C) \subseteq S(C)$;

(ii) $d(Tx, Ty) \leq k \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \dfrac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}$, $k \in (0, 1)$;

(iii) *the pair (S, T) is R-weakly commuting on C.*

If $cl(T(C))$ is complete and T is continuous, then $F(S) \cap F(T) \cap C$ is a singleton.

Now we obtain a more general version of the above theorem, where the continuity condition on T has been dispensed with and the completeness of $cl(T(C))$ (the closure of $T(C)$), has been replaced by the completeness of $T(C)$.

Theorem 1.3.2. *Let (X, d) be a metric space and C a nonempty subset of X. Let $S, T : C \to C$ be a pair of mappings such that*

(i) $T(C) \subseteq S(C)$;

(ii) $d(Tx, Ty) \leq k \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \dfrac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}$, $k \in (0, 1)$;

(iii) *the pair (S, T) is R-weakly commuting on C.*

Then we have the following:

(a) *$F(S) \cap F(T) \cap T(C)$ is a singleton if $T(C)$ is complete.*

(b) *$F(S) \cap F(T) \cap S(C)$ is a singleton if $S(C)$ is complete.*

Proof. Pick $x_0 \in C$. Since $T(C) \subseteq S(C)$, we can construct a sequence $\{x_n\}$ in C such that $Sx_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. By (ii), we have

$$\begin{aligned}
d(Sx_n, Sx_{n+1}) &= d(Tx_n, Tx_{n-1}) \\
&\leq k \max\Big\{d(Sx_n, Sx_{n-1}), d(Sx_n, Tx_n), d(Sx_{n-1}, Tx_{n-1}), \\
&\qquad \frac{d(Sx_n, Tx_{n-1}) + d(Sx_{n-1}, Tx_n)}{2}\Big\} \\
&= k \max\Big\{d(Sx_n, Sx_{n-1}), d(Sx_n, Sx_{n+1}), d(Sx_{n-1}, Sx_n), \\
&\qquad \frac{d(Sx_n, Sx_n) + d(Sx_{n-1}, Sx_{n+1})}{2}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= k \max \Bigg\{ d(Sx_n, Sx_{n-1}), d(Sx_n, Sx_{n+1}), d(Sx_{n-1}, Sx_n), \\
&\quad \frac{d(Sx_{n-1}, Sx_{n+1})}{2} \Bigg\} \\
&\leq k \max \Bigg\{ d(Sx_n, Sx_{n-1}), d(Sx_n, Sx_{n+1}), d(Sx_{n-1}, Sx_n), \\
&\quad d(Sx_n, Sx_{n+1}) \Bigg\} \\
&= k \max \{ d(Sx_n, Sx_{n-1}), d(Sx_n, Sx_{n+1}) \}
\end{aligned}
$$

If

$$\max \{ d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1}) \} = d(Sx_n, Sx_{n+1})$$

then

$$d(Sx_n, Sx_{n+1}) \leq kd(Sx_n, Sx_{n+1})$$

a contradiction. Therefore we have

$$d(Sx_n, Sx_{n+1}) \leq kd(Sx_{n-1}, Sx_n,).$$

Since $k < 1$, $\{Sx_n\}$ is a Cauchy sequence in C (see[43] and [85]).

(a) Suppose that $T(C)$ is complete. Then there exists a point $z \in T(C)$ such that $Tx_n \to z \in T(C)$. Thus, $Sx_n \to z$. Since $z \in T(C) \subseteq S(C)$, there exists $u \in C$ such that $z = Su$. Again by (ii), we have

$$d(Tu, Tx_n) \leq k \max \left\{ d(Su, Sx_n), d(Su, Tu), d(Sx_n, Tx_n), \frac{d(Su, Tx_n) + d(Sx_n, Tu)}{2} \right\}.$$

Making $n \to \infty$, yields

$$d(Tu, Su) \quad \leq \quad k \max \left\{ 0, d(Su, Tu), \frac{d(Su, Tu)}{2} \right\} = kd(Su, Tu) < d(Su, Tu),$$

a contradiction. Therefore $d(Tu, Su) = 0$ and $Su = Tu = z$.

Since the pair (S, T) is R-weakly commuting on C, it follows that

$$d(STu, TSu) \leq Rd(Su, Tu) = Rd(z, z) = 0.$$

Therefore $d(STu, TSu) = 0$ and $Sz = Tz$.

Again by (ii), we have

$$d(Tz, Tx_n) \leq k \max\left\{d(Sz, Sx_n), d(Sz, Tz), d(Sx_n, Tx_n), \frac{d(Sz, Tx_n) + d(Sx_n, Tz)}{2}\right\}$$

Making $n \to \infty$, yields

$$\begin{aligned} d(Tz, z) &\leq k \max\left\{d(Sz, z), d(Sz, Tz), d(z, z), \frac{d(Sz, z) + d(z, Tz)}{2}\right\} \\ &= k \max\{d(Sz, z), 0, d(Sz, z)\} \\ &= kd(Sz, z) = kd(Tz, z), \end{aligned}$$

which implies that $z = Tz = Sz$.

To prove the uniqueness of z, let us assume that $\tilde{z}$ is another common fixed point of T and S.

ie., $\tilde{z} = T\tilde{z} = S\tilde{z}$.

Hence

$$\begin{aligned} d(z, \tilde{z}) = d(Tz, T\tilde{z}) &\leq k \max\left\{d(Sz, S\tilde{z}), d(Sz, Tz), d(S\tilde{z}, T\tilde{z}), \right. \\ &\qquad \left. \frac{d(Sz, T\tilde{z}) + d(S\tilde{z}, Tz)}{2}\right\} \\ &\leq k \max\left\{d(z, \tilde{z}), d(z, z), d(\tilde{z}, \tilde{z}), \frac{d(z, \tilde{z}) + d(\tilde{z}, z)}{2}\right\} \\ &\leq k \max\left\{d(z, \tilde{z}), 0, 0, d(z, \tilde{z})\right\} \\ &= kd(z, \tilde{z}). \end{aligned}$$

Thus $(1-k)d(z,\tilde{z}) \leq 0$.

Since $k<1$, $d(z,\tilde{z})=0$, which implies $z=\tilde{z}$. Hence the unicity is proved.

Since $z \in T(C)$, we conclude that $F(S) \cap F(T) \cap T(C) = \{z\}$.

(b) Suppose $S(C)$ is complete. Then $Sx_n \to z$ for some $z \in S(C)$ and there exist $u \in C$ such that $z = Su$. As in part (a), we can show that $Sz = Tz = z$. Thus $F(S) \cap F(T) \cap S(C) = \{z\}$. □

Recently Kim et al. [43] obtained the following result for S-nonexpansive type mappings in a normed space.

Theorem 1.3.3. *Let C be a nonempty q-star shaped subset of a normed space X and $S, T : C \to C$ two mappings satisfying the following conditions:*

(i) *the mapping T is S-nonexpansive and S is q-affine with $q \in F(S)$;*

(ii) $T(C) \subseteq S(C)$

(iii) *the pair (S,T) is R-subweakly commuting;*

Suppose $S(C)$ is compact. Then we have the following:

(a) *There exists $y \in S(C)$ such that $Sy = Ty$.*

(b) *If S or T is demicontinuous, then $y \in F(S) \cap F(T)$.*

We extend the above theorem for generalized S-nonexpansive type mappings. In the sequel we will need the following Corollary 1.3.1 and Proposition 1.3.1.

Corollary 1.3.1. *Let (X,d) be a metric space and C a nonempty subset of X. Let $S, T : C \to C$ be a pair of mappings such that*

(i) *$T(C) \subseteq S(C)$;*

(ii) $d(Tx,Ty) \leq k \max\left\{d(Sx,Sy), \frac{d(Sx,Tx)+d(Sy,Ty)}{2}, \frac{d(Sx,Ty)+d(Sy,Tx)}{2}\right\}$, $k \in (0,1)$;

(iii) *the pair (S,T) is R-weakly commuting on C.*

Then we have the following:

(a) *$F(S) \cap F(T) \cap T(C)$ is a singleton if $T(C)$ is complete.*

(b) *$F(S) \cap F(T) \cap S(C)$ is a singleton if $S(C)$ is complete.*

Proof. The proof can be obtained by replacing the condition

$$d(Tx,Ty) \leq k \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{d(Sx,Ty)+d(Sy,Tx)}{2}\right\}$$

by

$$d(Tx,Ty) \leq k \max\left\{d(Sx,Sy), \frac{d(Sx,Tx)+d(Sy,Ty)}{2}, \frac{d(Sx,Ty)+d(Sy,Tx)}{2}\right\}$$

in the proof of the Theorem 1.3.2. □

Proposition 1.3.1. *Let C be a nonempty q-star shaped subset of a normed space X and $S,T : C \to C$ two mappings such that*

(i) *T is a generalized S-nonexpansive type mapping and S is q-affine with $q \in F(S)$;*

(ii) *$T(C) \subseteq S(C)$*

(iii) *the pair (S,T) is R-subweakly commuting;*

(iv) *$S(C)$ is complete.*

Then there exist exactly one point x_λ such that

$$x_\lambda = Sx_\lambda = (1-\lambda)q + \lambda Tx_\lambda$$

for all $\lambda \in (0,1)$.

Proof. Define $T_\lambda : C \to C$ by $T_\lambda x = (1-\lambda)q + \lambda Tx$ for all $x \in C$ and for each $\lambda \in (0,1)$.

Since (S,T) is R-subweakly commuting and S is q-affine, we have

$$\begin{aligned} \|ST_\lambda x - T_\lambda Sx\| &= \|[(1-\lambda)q + \lambda STx] - [(1-\lambda)q + \lambda TSx]\| \\ &= \lambda\|TSx - STx\| \\ &\leq \lambda R\|Sx - T_\lambda x\| \end{aligned}$$

for all $x \in C$. Thus, the pair (S, T_λ) is R- weakly commuting on C.

Also

$$\begin{aligned} \|T_\lambda x - T_\lambda y\| &= \lambda\|Tx - Ty\| \\ &\leq \lambda \max\left\{\|Sx - Sy\|, \frac{\|Sx - Tx\| + \|Sy - Ty\|}{2}, \frac{\|Sx - Ty\| + \|Sy - Tx\|}{2}\right\} \end{aligned}$$

for all $x, y \in C$. For $x \in C$, we have $Tx \in T(C) \subseteq S(C)$, ie, there exists a point $y \in C$ such that $Tx = Sy \in S(C)$.

Observe that

$$T_\lambda x = (1-\lambda)q + \lambda Tx = (1-\lambda)q + \lambda Sy \in S(C).$$

It follows that $T_\lambda(C) \subseteq S(C)$ for all $\lambda \in (0,1)$. Now for each $\lambda \in (0,1)$, we conclude that

(i)* $T_\lambda(C) \subseteq S(C)$,

(ii)* $\|T_\lambda x - T_\lambda y\| \leq \lambda \max\left\{\|Sx - Sy\|, \frac{\|Sx - Tx\| + \|Sy - Ty\|}{2}, \frac{\|Sx - Ty\| + \|Sy - Tx\|}{2}\right\}$

(iii)* $S(C)$ is complete.

(iv)* (S, T_λ) is R-weakly commuting on C.

Therefore by Corollary 1.3.1, there exists exactly one point $x_\lambda \in S(C)$ such that

$$x_\lambda = Sx_\lambda = T_\lambda x_\lambda,$$

which implies that $x_\lambda = Sx_\lambda = (1 - \lambda)q + \lambda T x_\lambda$. □

Now we obtain a common fixed point theorem for generalized S -nonexpansive type mappings.

Theorem 1.3.4. *Let C be a nonempty subset of a normed space X. Let $S, T : C \to C$ be two mappings satisfying conditions (i) – (iii) of Proposition 1.3.1. Suppose $S(C)$ is compact. Then we have the following:*

(a) *S and T have a coincidence point $y \in S(C)$.*

(b) *If S or T is demicontinuous, then $y \in F(S) \cap F(T)$.*

Proof. Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ such that $\lambda_n \to 1$. By Proposition 1.3.1, there exists exactly one point $x_{\lambda_n} \in S(C)$ such that

$$x_{\lambda_n} = Sx_{\lambda_n} = (1 - \lambda_n)q + \lambda_n T x_{\lambda_n}$$

for all $n \in \mathbb{N}$.

Set $x_{\lambda_n} := y_n$. Since $S(C)$ is compact, there exist a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{j\to\infty} Sy_{n_j} = y \in S(C).$$

Thus $y = Su$ for some $u \in C$.

The assumption (ii) implies that $\{Ty_{n_j}\}$ is bounded. It follows that

$$\begin{aligned} \|y_{n_j} - Ty_{n_j}\| &= \|(1-\lambda_{n_j})q + \lambda_{n_j}Ty_{n_j} - Ty_{n_j}\| \\ &= (1-\lambda_{n_j})\|q - Ty_{n_j}\| \to 0 \qquad \text{as } \ j \to \infty \end{aligned}$$

Thus $\lim_{j\to\infty} Ty_{n_j} = \lim_{j\to\infty} y_{n_j} = y$. By the condition (1.2.3), we have

$$\begin{aligned} \|Ty_{n_j} - Tu\| &\le \max\left\{\|Sy_{n_j} - Su\|, \frac{\|Sy_{n_j} - Ty_{n_j}\| + \|Su - Tu\|}{2}, \right. \\ &\qquad \left. \frac{\|Sy_{n_j} - Tu\| + \|Su - Ty_{n_j}\|}{2}\right\} \\ &= \max\left\{\|Sy_{n_j} - y\|, \frac{\|Sy_{n_j} - Ty_{n_j}\| + \|y - Tu\|}{2}, \right. \\ &\qquad \left. \frac{\|Sy_{n_j} - Tu\| + \|y - Ty_{n_j}\|}{2}\right\}. \end{aligned}$$

Making $j \to \infty$, we get

$$\|y - Tu\| \ \le \ \max\left\{0, \frac{1}{2}\|y - Tu\|, \frac{1}{2}\|y - Tu\|\right\} = \frac{1}{2}\|y - Tu\|,$$

a contradiction. Therefore $\|y - Tu\| = 0$ and $Tu = y$.

(a) Since the pair (S, T) is R-subweakly commuting, we have

$$\begin{aligned} \|STu - TSu\| &\le R \ dist(Su, [Tu, q]) \\ &\le R\|Su - [(1-\lambda_n)q + \lambda_n Tu\| \end{aligned}$$

which on taking the limit as $n \to \infty$ gives $\|STu - TSu\| \le 0$. Thus $Sy = Ty$.

(b) Suppose S is demicontinuous. Since $\lim_{m\to\infty} x_m = \lim_{m\to\infty} Sx_m = y$, it follows from the demicontinuity of S that $Sy = y$.

But $Sy = Ty$. Thus we conclude that $y \in F(S) \cap F(T)$.

Similarly we can prove that $y \in F(S) \cap F(T)$ when T is demicontinuous. $\square$

The following example shows the generality of Theorem 1.3.4 over Theorem 1.3.3.

Example 1.3.1. Let $X = \mathbb{R}$ (set of reals) with norm $\|x\| = |x|$ and $C = [0, 4]$. Define $S, T : C \to C$ by

$$Tx = \begin{cases} 1 & \text{if } x \in \{1,3,4\}, \\ 2 & \text{if } x = 2, \\ \frac{3}{2} & \text{otherwise;} \end{cases} \quad \text{and } Sx = \begin{cases} 1 & \text{if } x = 1, \\ 4 & \text{if } x \in \{2,3\}, \\ 2 & \text{if } x = 4, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

For $x = 2$ and $y = 3$, we have

$$\|Tx - Ty\| = 1 > 0 = \|Sx - Sy\|,$$

and the condition (i) of Theorem 1.3.3 is not satisfied. Further, it can be easily verified that S and T satisfy all the hypotheses of Theorem 1.3.4 and $S1 = T1 = 1$, is a common fixed point of S and T.

1.4 Convergence of Mann iteration for a pair of mappings

Recently, Rhoades and Temir [81] obtained the following theorem.

Theorem 1.4.1. *Let X be a Banach space and C a closed convex subset of X which satisfies the Opial's condition. Let $S, T : C \to C$ be mappings such that*

(i) *T is S-nonexpansive;*

(ii) *S is nonexpansive.*

Suppose $\{k_n\}$ *is a real sequence in* $(0,1)$. *Then the sequence of Mann iterates defined for an arbitrary* $x_0 \in C$ *by*

$$x_{n+1} = (1-k_n)x_n + k_n T x_n, \qquad n \in \{0\} \cup \mathbb{N}$$

converges weakly to a common fixed point of S *and* T.

The following theorem extends Theorem 1.4.1 to generalized S -nonexpansive type mappings.

Theorem 1.4.2. *Let* X *be a Banach space and* C *a closed convex subset of* X *which satisfies Opial's condition. Let* $S, T : C \to C$ *be such that*

(i) T *is generalized* S*-nonexpansive type;*

(ii) S *is nonexpansive;*

(iii) $F(S) \cap F(T) \neq \phi$.

Suppose $\{k_n\}$ *is a real sequence in* $(0,1)$. *Then the sequence of Mann iterates defined for an arbitrary* $x_0 \in C$ *by*

$$x_{n+1} = (1-k_n)x_n + k_n T x_n, \qquad n \in \{0\} \cup \mathbb{N}$$

converges weakly to a common fixed point of S *and* T.

Proof. If $F(S) \cap F(T)$ is singleton, then the proof is complete. Assume that $F(S) \cap F(T)$ is not a singleton. Let $z \in F(S) \cap F(T)$. Then

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1-k_n)x_n + k_n T x_n - z\| \\ &= \|(1-k_n)(x_n - z) + k_n (T x_n - z)\|. \end{aligned} \tag{1.4.1}$$

Since T is generalized S-nonexpansive type, we have

$$\|x_{n+1} - z\| \leq (1 - k_n)\|x_n - z\| + k_n M(x_n, z). \tag{1.4.2}$$

Now the following cases arise.

Case 1. $M(x_n, z) = \|Sx_n - Sz\|$. Then

$$\|Tx_n - z\| \leq \|Sx_n - Sz\|.$$

Since S is nonexpansive on C, the above inequality reduces to

$$\|Tx_n - z\| \leq \|x_n - z\|.$$

Case 2. $M(x_n, z) = \dfrac{\|Sx_n - Tx_n\| + \|Sz - Tz\|}{2}$. Then

$$\begin{aligned}
\|Tx_n - z\| &\leq \frac{\|Sx_n - Tx_n\| + \|Sz - Tz\|}{2} \\
&= \frac{\|Sx_n - Tx_n\|}{2} \\
&\leq \frac{\|Sx_n - z\| + \|z - Tx_n\|}{2} \\
&= \frac{\|Sx_n - Sz\| + \|z - Tx_n\|}{2},
\end{aligned}$$

which implies that

$$\|Tx_n - z\| \leq \|Sx_n - Sz\|.$$

Nonexpansiveness of S on C implies

$$\|Tx_n - z\| \leq \|x_n - z\|.$$

Case 3. $M(x_n, z) = \dfrac{\|Sx_n - Tz\| + \|Sz - Tx_n\|}{2}$. Then

$$\begin{aligned}\|Tx_n - z\| &\le \frac{\|Sx_n - Tz\| + \|Sz - Tx_n\|}{2} \\ &= \frac{\|Sx_n - Tz\| + \|Tx_n - z\|}{2} \\ &= \frac{\|Sx_n - Sz\| + \|Tx_n - z\|}{2},\end{aligned}$$

which implies that

$$\|Tx_n - z\| \le \|Sx_n - Sz\|.$$

Again, since S is nonexpansive on C, it follows that

$$\|Tx_n - z\| \le \|x_n - z\|.$$

Therefore in all the cases, we get

$$\|Tx_n - z\| \le \|x_n - z\|. \tag{1.4.3}$$

By (1.4.2) and (1.4.3), we get

$$\begin{aligned}\|x_{n+1} - z\| &\le (1 - k_n)\|x_n - z\| + k_n\|x_n - z\| \\ &= \|x_n - z\|\end{aligned}$$

Thus, for $k_n \neq 0$, $\{\|x_n - z\|\}$ is a nonincreasing sequence. Hence, $\lim_{n\to\infty} \|x_n - z\|$ exists. Now we show that $\{x_n\}$ converges weakly to a common fixed point of S and T. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be two subsequences of $\{x_n\}$ which converge weakly to z and $\tilde{z}$ in $F(S) \cap F(T)$ respectively. We will show that $z = \tilde{z}$. Suppose the contrary. Since X

satisfies Opial's condition and $\lim_{n\to\infty}\|x_n - z\|$ exists, we get

$$\begin{aligned}\lim_{n\to\infty}\|x_n - z\| &= \lim_{k\to\infty}\|x_{n_k} - z\| < \lim_{k\to\infty}\|x_{n_k} - \tilde{z}\| \\ &= \lim_{n\to\infty}\|x_n - \tilde{z}\| = \lim_{j\to\infty}\|x_{m_j} - \tilde{z}\| \\ &< \lim_{j\to\infty}\|x_{m_j} - z\| = \lim_{n\to\infty}\|x_n - z\|,\end{aligned}$$

a contradiction. Hence $z = \tilde{z}$. □

Corollary 1.4.1. *Theorem* 1.4.1

Proof. It comes form Theorem 1.4.2, when $M(x,y) = \|Sx - Sy\|$. □

Corollary 1.4.2. *Let X be a Banach space and C a closed convex subset of X which satisfies Opial's condition. Let $T : C \to C$ be generalized nonexpansive type mapping.*

Suppose $\{k_n\}$ is a real sequence in $(0,1)$. Then the sequence of Mann iterates defined for an arbitrary $x_0 \in C$ by

$$x_{n+1} = (1 - k_n)x_n + k_n T x_n, \qquad n \in \{0\} \cup \mathbb{N}$$

converges weakly to the fixed point of T.

Proof. It comes from Theorem 1.4.2, when S is an identity mapping on X. □

Corollary 1.4.3. *Let X be a Banach space and C a closed convex subset of X which satisfies Opial's condition. Let $T : C \to C$ be a nonexpansive mapping.*

Suppose $\{k_n\}$ is a real sequence in $(0,1)$. Then the sequence of Mann iterates defined for arbitrary $x_0 \in C$ defined

$$x_{n+1} = (1 - k_n)x_n + k_n T x_n, \qquad n \in \{0\} \cup \mathbb{N}$$

converges weakly to the fixed point of T.

Proof. It comes from Corollary 1.4.2 when $m(x,y) = \|x - y\|$. $\square$

Chapter 2

Viscosity approximations with weakly contractive mappings

2.1 Introduction

In this chapter, we study viscosity approximations with (ψ, ϕ)- weakly contractive mappings. We show that Browder and Halpern type convergence theorems imply Moudafi's viscosity approximations. Our results generalize a number of convergence theorems including a strong convergence theorem of Song and Liu [97].

2.1.1 Some generalizations of contraction mappings

Definition 2.1.1. Let (X, d) be a metric space and $T : X \to X$. Then T is called a contraction mapping if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \tag{2.1.1}$$

for all x, $y \in X$.

$T : X \to X$ is called nonlinear contraction [11, 14], if

$$d(Tx, Ty) \leq \alpha\,(d(x, y)) \tag{2.1.2}$$

for all $x, y \in X$. We note that $\alpha(0) = 0$.

In 1968, Browder [14] proved that if α is right continuous and nondecreasing, then T has a unique fixed point. Subsequently, this result was extended in 1969 by Boyd and Wong [11], who observed that it sufficed to assume only the right- upper semicontinuity of α. $T : X \to X$ is called weakly contractive [1, 80], if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \tag{2.1.3}$$

for all x,$y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

The above concept was initially introduced by Krasnoselski$\tilde{i}$ et al. [45] where ϕ was assumed as a continuous function and $\phi(0) = 0$. Later on, condition(2.1.3) was rediscovered in a Hilbert space by Alber and Guerre-Delabriere [1], who assumed additionally that ϕ is nondecreasing. They proved that weakly contractive mappings possess a unique fixed point in a Hilbert space. In 2001, Rhoades [80] extended the above result of [1] to complete metric spaces. Clearly, this is a special form of the Boyd-Wong [11] condition with $\alpha(t) = t - \phi(t)$.

In this thesis, we shall use the following class of mappings satisfying the so called (ψ, ϕ) condition (see for details [20, 10, 15]).

$T : X \to X$ is called (ψ, ϕ)-weakly contractive if

$$\psi\left(d(Tx, Ty)\right) \leq \psi\left(d(x, y)\right) - \phi\left(d(x, y)\right) \tag{2.1.4}$$

for all $x, y \in X$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, ϕ is non-increasing and ψ is increasing(strictly).

Remark 2.1.1. Recently Jachymski [38] observed that the theorem proved by Dutta and Choudhury [20, Theorem 2.1] is equivalent to Browder's [14] theorem, which means that the two theorems deal with the same class of mappings. In particular, if T satisfies (2.1.4), then there exists a continuous and nondecreasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is a nonlinear contraction, ie; (2.1.2) holds.

Remark 2.1.2. It is interesting to note that if one takes $\phi(t) = (1-k)t$, where $0 < k < 1$ and $\psi(t) = t$, then (2.1.4) reduces to (2.1.1). The condition (2.1.3) can be recovered easily by taking $\psi(t) = t$ in (2.1.4). In fact, the weakly contractive mappings are also closely related to nonlinear contraction. If $\phi(t) = t - \alpha(t)$, then (2.1.3) turns into (2.1.2). Again if $\alpha(t) = kt$, then (2.1.2) reduces to (2.1.1). Therefore

$$(2.1.1) \Rightarrow (2.1.2) \Rightarrow (2.1.3) \Rightarrow (2.1.4).$$

This shows the generality of (ψ, ϕ)- weakly contractive mappings over its counter parts.

Now we provide an example of a (ψ, ϕ)-weakly contractive mapping on a metric space which shows that the above implication is not reversible.

Example 2.1.1. [20] Let $X = [0,1] \cup \{2, 3, 4, \cdots\}$ and

$$d(x,y) = \begin{cases} |x-y|, & \text{if } \ x, y \in [0,1]; x \neq y \\ x+y, & \text{if } \ \text{at least one of } x \text{ or } y \notin [0,1] \text{ and } x \neq y \\ 0, & \text{if } \ x = y. \end{cases}$$

Then (X, d) is a complete metric space[11].

Let $\psi : [0, \infty) \to [0, \infty)$ be defined as

$$\psi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1 \\ t^2, & \text{if } t > 1. \end{cases}$$

and $\phi : [0, \infty) \to [0, \infty)$ be defined as

$$\phi(t) = \begin{cases} \frac{1}{2}t^2, & \text{if } 0 \leq t \leq 1 \\ \frac{1}{2}, & \text{if } t > 1. \end{cases}$$

Let $T : X \to X$ be defined as

$$T(x) = \begin{cases} x - \frac{1}{2}x^2, & \text{if } 0 \leq x \leq 1 \\ x - 1, & \text{if } x \in \{2, 3, \cdots\}. \end{cases}$$

It is seen that the condition(2.1.4) remains valid for ψ,ϕ and T constructed as above.

2.2 Preliminaries

In this section, we recall some basic definitions and strong convergence theorems for nonexpansive mappings which will be used in the remaining section of the chapter. Throughout this chapter, we assume that E is a Banach space over the real scalar field.

Theorem 2.2.1. [20, Theorem 2.1]. *Let X be a complete metric space and $T : X \to X$ be a (ψ, ϕ)-weakly contractive mapping. Then T has a unique fixed point.*

Definition 2.2.1. Let K be a nonempty subset of a Banach space E. In Chapter 1 we defined that $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K$$

Let E be a real Banach space with its dual space E^* and K a nonempty closed convex subset of E. Let $\langle x, x^* \rangle$ be the dual pairing between $x \in E$ and $x^* \in E^*$, and $J : E \to 2^{E^*}$ be the normalized duality mapping on E defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \ \forall x \in E.$$

E is said to be *smooth* or to have a *Gâteaux differentiable norm* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, \ y \in E$ with $\|x\| = \|y\| = 1$.

Definition 2.2.2. [7]. Let K be a nonempty closed convex subset of a Banach space E and C a nonempty subset of K. A retraction from K to C is a mapping $Q : K \to C$ such that $Qx = x$ for $x \in C$. A retraction Q from K to C is called sunny if Q satisfies the property: $Q(Qx + t(x - Qx)) = Qx$ for $x \in K$ and $t > 0$ whenever $Qx + t(x - Qx) \in K$. A retraction Q from K to C is sunny nonexpansive if Q is both sunny and nonexpansive.

The well known way to find the fixed point of a nonexpansive mapping T is to use a contraction to approximate it (Browder [13]). More precisely, take $t \in (0, 1)$ and for $x \in K$ define a contraction $T_t : K \to K$ by $T_t x = tu + (1 - t)Tx$, where $u \in K$ is fixed. Then by the Banach contraction principle, T_t has a unique fixed point x_t in K, that is,

$$x_t = tu + (1 - t)Tx_t. \tag{2.2.1}$$

Halpern [27], introduced the following explicit iteration scheme for a sequence $\{\alpha_n\}$ of real numbers in $(0, 1)$ and an arbitrary $u \in K$:

$$x_0 \in K, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \tag{2.2.2}$$

where $T : K \to K$ is a nonexpansive mapping.

In the case of T having a fixed point, Browder [13] (respectively, Halpern [27]) proved that $\{x_t\}$ (respectively, $\{x_n\}$) converges strongly to the fixed point of T that is nearest to u in a Hilbert space. These theorems have been extended in several directions by many authors (cf. Reich [74] and Xu [100, 101]).

Theorem 2.2.2. [74]. *Let K be a bounded closed convex subset of a uniformly smooth Banach space E and $T : K \to K$ a nonexpansive mapping. Fix $u \in K$ and define a sequence $\{y_k\}$ in K by $y_k = (1-k)Ty_k + ku$ for $k \in (0,1)$. Then $\{y_k\}$ converges strongly to Qu as k tends to $+0$, where Q is the unique sunny nonexpansive retraction from K onto $F(T)$, where $F(T) = \{x \in K : T(x) = x\}$.*

Theorem 2.2.3. [100] . *Let E, K, T, Q and u be as in Theorem 2.2.2. Define a sequence $\{y_n\}$ in K by $y_1 \in K$ and $y_{n+1} = (1-\alpha_n)Ty_n + \alpha_n u$ for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying*

(C_1) $\lim\limits_{n\to\infty} \alpha_n = 0$

(C_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(C_3) $\lim\limits_{n\to\infty} \dfrac{\alpha_n}{\alpha_{n+1}} = 1.$

Then $\{y_n\}$ converges strongly to Qu.

In 2000, Moudafi [57] introduced the viscosity approximation method and proved that in a real Hilbert space H, for a given $x_0 \in K \subset H$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \alpha_n S(x_n) + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \tag{2.2.3}$$

where $S : K \to K$ is a contraction mapping and $\{\alpha_n\} \subseteq (0, 1)$ satisfying certain conditions, converges strongly to the unique solution $\tilde{x} \in F(T)$ of the following variational inequality:

$$\langle (I - S)\tilde{x}, \tilde{x} - x\rangle \geq 0, \quad \forall x \in F(T).$$

Moudafi in [57] generalized the results of Browder and Halpern for viscosity approximations. Subsequently, Xu [102] extended Moudafi's results to the framework of uniformly smooth Banach spaces. In 2007, Suzuki [98] replaced the contraction mapping S in condition (2.2.3) by Meir-Keeler type contractions (cf.[50]) to find a fixed point of a nonexpansive mapping T. Recently, Song and Liu [97] considered the following viscosity approximations:

$$y_n = \alpha_n S y_n + (1 - \alpha_n) T_n y_n, \quad n \in \mathbb{N} \tag{2.2.4}$$

$$x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) T_n x_n, \quad n \in \mathbb{N}, \tag{2.2.5}$$

where S is a weakly contractive mapping and $\{T_n\}$ a sequence of nonexpansive mappings.

In the next section, we extend the above viscosity approximations by Song and Liu for a more general class of (ψ, ϕ)-weakly contractive mappings and establish strong convergence theorems for a sequence of nonexpansive mappings.

2.3 Browder and Halpern type convergence results

Our main results are prefaced by the following lemmas.

Lemma 2.3.1. [72, 99]. *Let K be a nonempty convex subset of a smooth Banach space E and C a nonempty subset of K. Let J be the duality mapping from E into*

E^*, and $Q : K \to C$ a retraction. Then Q is both sunny and nonexpansive if and only if the following holds:

$$\langle x - Qx, J(y - Qx)\rangle \leq 0, \tag{2.3.1}$$

for all $x \in K$ and $y \in C$.

Lemma 2.3.2. [69, Page 302]. *Let $\{a_n\}_{n=1}^{\infty}$ satisfy the condition*

$$a_{n+1} \leq \omega a_n + \sigma_n, \quad n \in \mathbb{N}$$

where $a_n \geq 0$, $\sigma_n \geq 0$, $\lim\limits_{n\to\infty} \sigma_n = 0$ and $0 \leq \omega < 1$. Then $\lim\limits_{n\to\infty} a_n = 0$.

Let $\{T_n\}$ be a sequence of nonexpansive mappings with $F = \bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$ on a closed convex subset K of a Banach space E and $\{\alpha_n\}$ a sequence in $(0, 1]$ with $\lim\limits_{n\to\infty} \alpha_n = 0$. $(E, K, \{T_n\}, \{\alpha_n\})$ is said to have ***Browder's property*** if for each $u \in K$, a sequence $\{y_n\}$ in K defined by

$$y_n = (1 - \alpha_n)T_n y_n + \alpha_n u, \tag{2.3.2}$$

for $n \in \mathbb{N}$, converges strongly.

Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then $(E, K, \{T_n\}, \{\alpha_n\})$ is said to have ***Halpern's property*** if for each $u \in K$, a sequence $\{y_n\}$ in K defined by

$$y_{n+1} = (1 - \alpha_n)T_n y_n + \alpha_n u \tag{2.3.3}$$

for $n \in \mathbb{N}$, converges strongly.

Lemma 2.3.3. [98, Proposition 4]. *Let $(E, K, \{T_n\}, \{\alpha_n\})$ have Browder's property. For each $u \in K$, put $Qu = \lim\limits_{n\to\infty} y_n$, where $\{y_n\}$ is a sequence in K defined by (2.3.2). Then Q is a nonexpansive mapping on K.*

Lemma 2.3.4. [98, Proposition 5]. *Let $(E, K, \{T_n\}, \{\alpha_n\})$ have Halpern's property. For each $u \in K$, put $Qu = \lim_{n\to\infty} y_n$, where $\{y_n\}$ is a sequence in K defined by* (2.3.3). *Then the following hold: (i) Qu does not depend on the initial point y_1. (ii) Q is a nonexpansive mapping on K.*

First we prove the following:

Proposition 2.3.1. *Let K be a convex subset of a smooth Banach space E. Let C be a subset of K and Q a unique sunny nonexpansive retraction from K onto C. Suppose $S : K \to K$ is a (ψ, ϕ)-weakly contractive mapping where ψ is strictly increasing and convex and $T : K \to K$ a nonexpansive mapping. Then*

(i) the composite mapping TS is a (ψ, ϕ)-weakly contractive on K;

(ii) for each $t \in (0, 1)$, the mapping $T_t = (1-t)T + tS$ is a (ψ, ϕ)-weakly contractive mapping on K and x_t is a unique solution of the fixed point equation:

$$x_t = tSx_t + (1-t)Tx_t; \tag{2.3.4}$$

(iii) $z = Q(Sz)$ if and only if $z \in K$ is a unique solution of the variational inequality:

$$\langle Sz - z, J(y-z)\rangle \leq 0, \quad \forall y \in C. \tag{2.3.5}$$

Proof. **(i)** For any $x, y \in K$, we have

$$\|TSx - TSy\| \leq \|Sx - Sy\|.$$

Since ψ is strictly increasing and S is a (ψ, ϕ)-weakly contractive mapping, the above inequality reduces to

$$\begin{aligned}\psi(\|TSx - TSy\|) &\leq \psi(\|Sx - Sy\|)\\ &\leq \psi(\|x-y\|) - \phi(\|x-y\|).\end{aligned}$$

So, TS is a (ψ, ϕ)-weakly contractive mapping.

(ii) For each fixed $t \in (0,1)$ and $\varphi(s) = t\phi(s)$, we have

$$\begin{aligned} \|T_t x - T_t y\| &= \|(tSx + (1-t)Tx) - (tSy + (1-t)Ty)\| \\ &\leq (1-t)\|Tx - Ty\| + t\|Sx - Sy\| \\ &\leq (1-t)\|x - y\| + t\|Sx - Sy\|. \end{aligned}$$

Since ψ is strictly increasing, the above inequality reduces to

$$\psi(\|T_t x - T_t y\|) \leq \psi((1-t)\|x - y\| + t\|Sx - Sy\|).$$

Further, since ψ is convex, we have

$$\begin{aligned} \psi(\|T_t x - T_t y\|) &\leq (1-t)\psi(\|x - y\|) + t\psi(\|Sx - Sy\|) \\ &\leq (1-t)\psi(\|x - y\|) + t[\psi(\|x - y\|) - \phi(\|x - y\|)] \\ &= \psi(\|x - y\|) - t\phi(\|x - y\|) \\ &= \psi(\|x - y\|) - \varphi(\|x - y\|). \end{aligned}$$

Thus, T_t is a (ψ, ϕ)-weakly contractive mapping. By Theorem 2.2.1, it can be seen that T_t has a unique fixed point x_t in K.

(iii) By Theorem 2.2.1, there exists a unique element $z \in K$ such that $z = Q(Sz)$. By Lemma 2.3.1, such a $z \in C$ satisfies (2.3.5). Next we show that the variational inequality (2.3.5) has a unique solution z. Assume $p \in C$ is another solution of (2.3.5). That is,

$$\langle Sp - p, J(z - p) \rangle \leq 0 \tag{2.3.6}$$

and

$$\langle Sz - z, J(p - z) \rangle \leq 0 \tag{2.3.7}$$

Adding (2.3.6) and (2.3.7), we get

$$\begin{aligned} 0 &\geq \langle p - z - (Sp - Sz), J(p - z)\rangle \\ &= \|p - z - (Sp - Sz)\|\|p - z\| \\ &\geq \|p - z\|^2 - \|Sp - Sz\|\|p - z\| \\ &= \|p - z\|[\|p - z\| - \|Sp - Sz\|], \end{aligned}$$

which implies that

$$\|p - z\| - \|Sp - Sz\| \leq 0 \quad \text{or} \quad \|p - z\| \leq \|Sp - Sz\|.$$

Since ψ is strictly increasing and S is (ψ, ϕ)-weakly contractive, we have

$$\begin{aligned} \psi(\|p - z\|) &\leq \psi(\|Sp - Sz\|) \\ &\leq \psi(\|p - z\|) - \phi(\|p - z\|). \end{aligned}$$

Therefore

$$\phi(\|p - z\|) \leq 0,$$

which implies that $p = z$. □

First we discuss the Browder type convergence.

Theorem 2.3.1. *Let $(E, K, \{T_n\}, \{\alpha_n\})$ have Browder's property. For each $u \in K$, let $Qu = \lim_{n\to\infty} y_n$, where $\{y_n\}$ is a sequence in K defined by (2.2.1). Let $S : K \to K$ be a (ψ, ϕ)-weakly contractive mapping where ψ is strictly increasing and convex and ϕ is nonincreasing. Define $\{x_n\}$ in K by $x_1 \in K$ and*

$$x_n = \alpha_n S x_n + (1 - \alpha_n) T_n x_n, \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to the unique point $z \in K$ satisfying $Q(Sz) = z$.

Proof. We note that Proposition 2.3.1(ii) assures the existence and uniqueness of $\{x_n\}$. It follows from Proposition 2.3.1(i) and Lemma 2.3.3 that QS is a (ψ, ϕ)-weakly contractive mapping on K. Then by Theorem 2.2.1, there exists the unique element $z \in K$ such that $Q(Sz) = z$. Define a sequence $\{y_n\}$ in K by

$$y_n = \alpha_n Sz + (1 - \alpha_n)T_n y_n, \quad n \in \mathbb{N}.$$

Then by the assumption, $\{y_n\}$ converges strongly to $Q(Sz)$.

Now for every $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_n - y_n\| &\le (1 - \alpha_n)\|T_n x_n - T_n y_n\| + \alpha_n \|Sx_n - Sz\| \\ &\le (1 - \alpha_n)\|x_n - y_n\| + \alpha_n \|Sx_n - Sz\|, \end{aligned}$$

and

$$\|x_n - y_n\| \le \|Sx_n - Sz\|.$$

Since ψ is strictly increasing, we have

$$\begin{aligned} \psi(\|x_n - y_n\| &\le \psi(\|Sx_n - Sz\|) \\ &\le \psi(\|x_n - z\|) - \phi(\|x_n - z\|) \\ &\le \psi(\|x_n - y_n\| + \|y_n - z\|) - \phi(\|x_n - y_n\| + \|y_n - z\|). \end{aligned}$$

Making $n \to \infty$, we obtain

$$\lim_{n\to\infty} \psi(\|x_n - y_n\|) \le \lim_{n\to\infty} \psi(\|x_n - y_n\| + \|y_n - z\|) - \lim_{n\to\infty} \phi(\|x_n - y_n\| + \|y_n - z\|).$$

Since $\{y_n\}$ converges strongly to z, we have

$$\lim_{n\to\infty} \psi(\|x_n - y_n\|) \le \lim_{n\to\infty} \psi(\|x_n - y_n\|) - \lim_{n\to\infty} \phi(\|x_n - y_n\|).$$

Therefore

$$\lim_{n\to\infty} \phi(\|x_n - y_n\|) \leq 0.$$

This implies

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Hence

$$\lim_{n\to\infty} \|x_n - z\| \leq \lim_{n\to\infty} (\|x_n - y_n\| + \|y_n - z\|) = 0.$$

Consequently, $\{x_n\}$ converges strongly to z and the conclusion holds. $\square$

Now we have the following result by Song and Liu [97] as a special case of Theorem 2.3.1.

Corollary 2.3.1. [97, Theorem 3.1]. *Let $(E, K, \{T_n\}, \{\alpha_n\})$ have Browder's property. For each $u \in K$, let $Qu = \lim_{n\to\infty} y_n$, where $\{y_n\}$ is a sequence in K defined by* (2.2.1)*. Let $S : K \to K$ be a weakly contractive mapping. Define $\{x_n\}$ in K by*

$$x_n = \alpha_n S x_n + (1 - \alpha_n) T_n x_n, \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to the unique point $z \in K$ satisfying $Q(Sz) = z$.

Proof. This comes from Theorem 2.3.1 when $\psi(t) = t$. $\square$

We now discuss Halpern type convergence.

Theorem 2.3.2. *Let $(E, K, \{T_n\}, \{\alpha_n\})$ have Halpern's property. For each $u \in K$, let $Qu = \lim_{n\to\infty} y_n$, where $\{y_n\}$ is a sequence in K defined by* (2.2.2)*. Let $S : K \to K$ be a (ψ, ϕ)-weakly contractive mapping where ψ is strictly increasing and convex and ϕ is nonincreasing. Define $\{x_n\}$ in K by $x_1 \in K$ and*

$$x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) T_n x_n, \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to the unique point $z \in K$ satisfying $Q(Sz) = z$.

Proof. It follows from Proposition 2.3.1(i) and Lemma 2.3.4 that QS is a (ψ, ϕ)-weakly contractive mapping on K. Then by Theorem 2.2.1, there exists a unique element $z \in K$ such that $z = Q(Sz)$. Thus we may define a sequence $\{y_n\}$ in K by

$$y_{n+1} = \alpha_n Sz + (1 - \alpha_n) T_n y_n, \quad n = 0, 1, 2, \cdots$$

Then by the assumption, $y_n \to Q(Sz)$ as $n \to \infty$. For every n, we have

$$\|x_{n+1} - y_{n+1}\| \leq \|\alpha_n (Sx_n - Sz) + (1 - \alpha_n)(T_n x_n - T_n y_n)\|.$$

Since ψ is strictly increasing, the above inequality reduces to

$$\begin{aligned} \psi(\|x_{n+1} - y_{n+1}\|) &\leq \alpha_n \psi(\|Sx_n - Sz\|) + (1 - \alpha_n)\psi(\|T_n x_n - T_n y_n\|) \\ &\leq (1 - \alpha_n)\psi(\|x_n - y_n\|) + \alpha_n[\psi(\|x_n - z\|) - \phi(\|x_n - z\|)] \\ &\leq (1 - \alpha_n)\psi(\|x_n - y_n\|) + \alpha_n \psi(\|x_n - z\|). \end{aligned}$$

Thus by Lemma 2.3.2 we get

$$\lim_{n\to\infty} \psi(\|x_n - y_n\|) = 0.$$

Since ψ is continuous,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Hence

$$\lim_{n\to\infty} \|x_n - z\| \leq \lim_{n\to\infty} \|x_n - y_n\| + \lim_{n\to\infty} \|y_n - z\| = 0$$

Consequently, we obtain the strong convergence of $\{x_n\}$ to $z = Q(Sz)$. □

The following result by Song and Liu [97] can be obtained as a special case of Theorem 2.3.2 when $\psi(t) = t$.

Corollary 2.3.2. [97]. *Let* $(E, K, \{T_n\}, \{\alpha_n\})$ *have Halpern's property. For each* $u \in K$, *let* $Qu = \lim_{n\to\infty} y_n$, *where* $\{y_n\}$ *is a sequence in* K *defined by* (2.2.2). *Let* $S : K \to K$ *be a weakly contractive mapping. Define* $\{x_n\}$ *in* K *by* $x_1 \in K$ *and*

$$x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) T_n x_n, \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to the unique point* $z \in K$ *satisfying* $Q(Sz) = z$.

Chapter 3

Stability of fixed points in metric spaces

3.1 Introduction

In this chapter, stability results for the class of (ψ, ϕ)-weakly contractive mappings using some general notions of convergence called (G)-convergence and (H)-convergence are proved in a metric space. We first present some preliminary notions and results that are needed in the sequel.

3.2 Preliminaries

We begin this section by stating some definitions on mappings which are generalizations of Banach contraction mapping principle and recalling certain notions of general convergence due to Barbet and Nachi [6] in metric spaces. We then present

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the above mentioned notions of convergence for (ψ, ϕ)-weakly contractive mappings in the setting of metric spaces.

3.2.1 Some general notions of convergence of type (G) and (H)

Definition 3.2.1. [6] Let (X, d) be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings. Then:

(i) T_∞ is called a (G)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G), where

(G) $Gr(T_\infty) \subset \liminf Gr(T_n)$: $\forall x \in X_\infty$, $\exists\{x_n\}_{n\in\mathbb{N}}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_n d(x_n, x) = 0 \text{ and } \lim_n d(T_n x_n, T_\infty x) = 0,$$

and $Gr(T)$ stands for the graph of T.

The following notion of (G^-) convergence is weaker than (G)-convergence.

(ii) T_∞ is called a (G^-)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G^-), where

(G^-) $Gr(T_\infty) \subset \limsup Gr(T_n)$: $\forall x \in X_\infty$, $\exists\{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$, which has a subsequence $\{x_{n_j}\}$ such that

$$\lim_n d(x_{n_j}, x) = 0 \text{ and } \lim_n d(T_{n_j} x_{n_j}, T_\infty x) = 0.$$

(iii) T_∞ is called an (H)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H), where

(H) If $\forall \{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$, $\exists \{y_n\}_{n\in\mathbb{N}} \subset X_\infty$ such that

$$\lim_n d(x_n, y_n) = 0 \text{ and } \lim_n d(T_n x_n, T_\infty y_n) = 0.$$

Remark 3.2.1. We note the following essential features of the above limits.

(i) pointwise convergence $\Rightarrow$ (G) - convergence. However, the above implication is not reversible unless $\{T_n\}_{n\in\mathbb{N}}$ is equicontinuous on a common domain of definition.

Example 3.2.1. [6] Consider the family $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ defined by $T_n x = \dfrac{nx}{1+nx}$ and $T_\infty(x) = 1$ for all $x \in \mathbb{R}_+$. Then the map T_∞ is a (G)-limit of $\{T_n\}$ but pointwise convergence is not satisfied.

(ii) a (G)-limit need not be unique (see Example 3.2.2). However if T_n is a (ψ, ϕ)-weakly contractive mapping for all $n \in \mathbb{N}$, then it is unique (see Proposition 3.4.1).

Example 3.2.2. [6] Consider $X_n = \mathbb{R}(n \in \overline{\mathbb{N}})$ and the sequence $\{T_n : \mathbb{R} \to \mathbb{R}\}_{n\in\overline{\mathbb{N}}}$ of mappings defined by $T_n x = \dfrac{nx}{1+nx}$ for all $x \in \mathbb{R}$. Then $T_\infty(x) = 1$ for all $x \in \mathbb{R}_+$, $T_\infty(0) = 0$. Clearly T_∞ is a (G)-limit of $\{T_n\}$. Let $T'_\infty : \mathbb{R} \to \mathbb{R}$ be defined by $T'_\infty(x) = T_\infty(x)$ if $x \neq 0$ and $T'_\infty(0) = \dfrac{1}{2}$. Then T'_∞ is also a (G)-limit of $\{T_n\}$, indeed the point $x = 0$ is the limit of the sequence $\{x_n = \dfrac{1}{n}\}_{n\in\mathbb{N}}$ such that $\{T_n x_n\}$ converges to $T'_\infty(0)$.

(iii) an (H)-limit need not be unique.

(iv) when T_∞ is continuous and the condition $X_\infty \subset \liminf X_n$ is satisfied, then the following implications hold [6, Proposition 9]:

$$(H) \Rightarrow (G) \Rightarrow (G^-).$$

However, without the two restrictions above, we have the relationship.

$$(G) \Rightarrow (G^-), (H) \Rightarrow (G^-).$$

Further, a (G)-limit is not necessarily an (H)-limit.

Example 3.2.3. [6] Let $\{T_n : \mathbb{R}_+ \to \mathbb{R}\}_{n\in\overline{\mathbb{N}}}$ be defined by $T_n x = \dfrac{nx}{1+nx}$ and $T_\infty x = 1$ for all $x \in \mathbb{R}_+$. Then T_∞ is a (G)-limit of $\{T_n\}$. But the property (H) is not satisfied, since for the null sequence $\{x_n\}$ we get $|T_n 0 - T_\infty y_n| = 1$ for any sequence $\{y_n\}$ converging to 0.

(v) the interrelationship between the (H) convergence and uniform convergence is captured in [6, Proposition 10].

3.3 Convergence and Stability of Fixed Points in Metric Spaces

In this section, we first recall some fundamental results in stability of fixed points by Bonsall [9] and Nadler Jr. [59] followed by their generalizations by Barbet and Nachi [6] for sequences of mappings in variable domains.

Theorem 3.3.1. *[9] Let (X, d) be a complete metric space and T and $T_n (n = 1, 2, ...)$ be contraction mappings of X into itself with the same Lipschitz constant $0 < k < 1$, and with fixed points u and $u_n (n = 1, 2, ...)$, respectively. Suppose that $\lim_n T_n x = Tx$ for every $x \in X$. Then, $\lim_n u_n = u$.*

We have the following remarks with respect to Theorem 3.3.1:

(a) The condition that all the contraction mappings $T_n(n = 1, 2, ...)$ have the same Lipschitz constant k is too restrictive as one can easily see by the remarks and examples given in Nadler Jr.[59].

(b) The assumption that T is a contraction mapping is superfluous as this follows from the fact that $T_n(n = 1, 2, ...)$ is a contraction and d is continuous.

(c) the completeness condition may be replaced by the assumption of the existence of fixed points for the mappings T and $T_n(n = 1, 2, ...)$. Because there exist contraction mappings on spaces which are not complete and have a fixed point.

Under uniform convergence of the sequence $\{T_n\}$ to T and retaining the essence of (a), (b) and (c), the following stability result was obtained by Nadler Jr.[59].

Theorem 3.3.2. *Let (X, d) be a metric space and $T_n : X \to X$ be a mapping with at least one fixed point u_n, for each $n = 1, 2, ...$ and let $T : X \to X$ be a contraction mapping with fixed point u. If the sequence $\{T_n\}$ converges uniformly to T, then the sequence $\{u_n\}$ converges to u.*

The above theorems were generalized by Barbet and Nachi [6] using (G) and (H)-convergence where a number of supporting results were also obtained to arrive at the desired conclusions.

The following are the main stability results of Barbet and Nachi [6].

Theorem 3.3.3. *Let (X, d) be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for all $n \in \mathbb{N}$, S_n is a k-contraction from (X_n, d) into (X, d). If, for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of S_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Theorem 3.3.4. *Let (X, d) be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (H) and such that S_∞ is a k_∞-contraction. If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of S_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

3.4 Stability results under (G)- convergence

In this section, we present some stability results for a sequence of (ψ, ϕ)- weakly contractive mappings satisfying the property (G).

First we prove the following Proposition which ensures a unique G-limit for the sequence of mappings $\{T_n\}$.

Proposition 3.4.1. *Let X be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of (ψ, ϕ)-weakly contractive mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.*

Proof. Let $T_\infty, T^*_\infty : X_\infty \to X$ be two (G)- limits of the sequence $\{T_n\}$. Then for any point $x \in X_\infty$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ converging to x such that $\{T_n x_n\}$ and $\{T_n y_n\}$ converge to $T_\infty x$ and $T^*_\infty x$ respectively. Thus

$$\lim_{n\to\infty} d(T_n x_n, T_\infty x) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(T_n y_n, T^*_\infty x) = 0. \tag{3.4.1}$$

Since $\{x_n\}$ and $\{y_n\}$ converge to x, we have

$$d(x_n, y_n) \le d(x_n, x) + d(y_n, x) \to 0 \text{ as } n \to \infty. \tag{3.4.2}$$

Further,

$$d(T_\infty x, T^*_\infty x) \le d(T_\infty x, T_n x_n) + d(T_n x_n, T_n y_n) + d(T_n y_n, T^*_\infty x). \tag{3.4.3}$$

Since T_n is (ψ, ϕ)-weakly contractive for each $n \in \mathbb{N}$,

$$\psi\left(d(T_n x_n, T_n y_n)\right) \le \psi\left(d(x_n, y_n)\right) - \phi\left(d(x_n, y_n)\right),$$

which implies that

$$\psi\left(d(T_n x_n, T_n y_n)\right) \le \psi\left(d(x_n, y_n)\right).$$

As ψ is increasing, from the above inequality we have

$$d(T_n x_n, T_n y_n) \le d(x_n, y_n). \tag{3.4.4}$$

From (3.4.3) and (3.4.4) we get,

$$d(T_\infty x, T_\infty^* x)) \le d(T_\infty x, T_n x_n) + d(x_n, y_n) + d(T_n y_n, T_\infty^* x).$$

Letting $n \to \infty$ and using (3.4.1) and (3.4.2), the above expression tends to zero and we deduce that

$$T_\infty x = T_\infty^* x.$$

□

The following result in [6, Proposition 1] follows directly from Proposition 3.4.1.

Corollary 3.4.1. *Let X be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of k-contraction mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.*

We now prove the following theorem which is our first stability result.

Theorem 3.4.1. *Let X be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) such that* *for all* $n \in \mathbb{N}$*,* T_n *is a* (ψ, ϕ)*-weakly contractive mapping. If for all* $n \in \overline{\mathbb{N}}$*,* x_n *is a fixed point of* T_n*, then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x_∞*.*

Proof. Let x_n be a fixed point of T_n for each $n \in \overline{\mathbb{N}}$. Then, since the property (G) holds and $x_\infty \in X_\infty$, there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that $y_n \to x_\infty$ and $T_n y_n \to T_\infty x_\infty$. Therefore

$$\begin{aligned}\psi\left(d(x_n, x_\infty)\right) &= \psi\left(d(T_n x_n, T_\infty x_\infty)\right)\\ &\leq \psi\left(d(T_n x_n, T_n y_n) + d(T_n y_n, T_\infty x_\infty)\right).\end{aligned}$$

Taking the limit as $n \to \infty$ and using the continuity of ψ and ϕ, we obtain

$$\begin{aligned}\lim_{n\to\infty} \psi\left(d(x_n, x_\infty)\right) &\leq \lim_{n\to\infty} \psi\left(d(T_n x_n, T_n y_n)\right)\\ &\leq \lim_{n\to\infty} \left[\psi\left(d(x_n, y_n)\right) - \phi\left(d(x_n, y_n)\right)\right] \quad \text{(by condition (2.1.4))}\\ &\leq \lim_{n\to\infty} \left[\psi\left(d(x_n, x_\infty) + d(y_n, x_\infty)\right)\right] - \lim_{n\to\infty} \left[\phi\left(d(x_n, x_\infty) + d(y_n, x_\infty)\right)\right]\\ &= \lim_{n\to\infty} \psi\left(d(x_n, x_\infty)\right) - \lim_{n\to\infty} \phi\left(d(x_n, x_\infty)\right).\end{aligned}$$

Thus

$$\lim_{n\to\infty} \phi\left(d(x_n, x_\infty)\right) \leq 0.$$

By the property of ϕ, we get $\lim_{n\to\infty} d(x_n, x_\infty) = 0$. Hence the conclusion follows. □

The following result in [6, Theorem 2] follows from the above theorem in view of Remark 2.1.2.

Corollary 3.4.2. *Let* X *be a metric space,* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *a family of nonempty subsets of* X *and* $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ *a family of mappings satisfying the property* (G) *such that for all* $n \in \mathbb{N}$*,* T_n *is a* k*-contraction from* X_n *into* X*. If for all* $n \in \overline{\mathbb{N}}$*,* x_n *is a fixed point of* T_n*, then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x_∞*.*

When $X_n = X$ for all $n \in \overline{\mathbb{N}}$, X is complete and $\psi(t) = t$, $\phi(t) = (1-k)t$, for all $t > 0$ and $k \in (0,1)$, then we get the following result of Bonsall [9, Theorem 2] as a consequence of Theorem 3.4.1.

Corollary 3.4.3. *Let X be a complete metric space, and $\{T_n : X \to X\}$ a family of k-contraction mappings with Lipschitz constant $k < 1$ and such that the sequence $\{T_n\}_{n\in\mathbb{N}}$ converges pointwise to T_∞. Then for all $n \in \overline{\mathbb{N}}$, T_n has a unique fixed point x_n and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

The following theorem proves the existence of a fixed point for a (G)-limit of a sequence of (ψ, ϕ)-weakly contractive mappings.

Theorem 3.4.2. *Let X be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) such that for all $n \in \mathbb{N}$, T_n is a (ψ, ϕ)-weakly contractive mapping. Assume that for all $n \in \mathbb{N}$, x_n is a fixed point of T_n. Then*

$$\begin{aligned} T_\infty \text{ admits a fixed point } &\Leftrightarrow \{x_n\} \text{ converges and } \lim x_n \in X_\infty \\ &\Leftrightarrow \{x_n\} \text{ admits a subsequence converging to a point of } X_\infty. \end{aligned}$$

Proof. The necessary part follows from Theorem 3.4.1. To prove the sufficiency, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\lim_j x_{n_j} = x_\infty \in X_\infty$. By the property (G), there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that $y_n \to x_\infty$ and $T_n y_n \to T_\infty x_\infty$ as $n \to \infty$. For any $j \in \mathbb{N}$, we have

$$d(x_\infty, T_\infty x_\infty) \le d(x_\infty, x_{n_j}) + d(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}) + d(T_{n_j} y_{n_j}, T_\infty x_\infty). \tag{3.4.5}$$

By condition (2.1.4),

$$\begin{aligned} \psi\left(d(T_{n_j} x_{n_j}, T_{n_j} y_{n_j})\right) &\le \psi\left(d(x_{n_j}, y_{n_j})\right) - \phi\left(d(x_{n_j}, y_{n_j})\right) \\ &\le \psi\left(d(x_{n_j}, y_{n_j})\right) \end{aligned}$$

which implies that

$$d(T_{n_j}x_{n_j}, T_{n_j}y_{n_j}) \leq d(x_{n_j}, y_{n_j}).$$

Then, from (3.4.5),

$$\begin{aligned} d(x_\infty, T_\infty x_\infty) &\leq d(x_\infty, x_{n_j}) + d(x_{n_j}, y_{n_j}) + d(T_{n_j}y_{n_j}, T_\infty x_\infty) \\ &\leq d(x_\infty, x_{n_j}) + d(x_{n_j}, x_\infty) + d(y_{n_j}, x_\infty) + d(T_{n_j}y_{n_j}, T_\infty x_\infty). \end{aligned}$$

Now passing over to the limit as $j \to \infty$, we deduce that $T_\infty x_\infty = x_\infty$. □

Remark 3.4.1. Under the assumptions of Theorem3.4.2, and if,

1. $\liminf X_n \subset X_\infty$ (ie; the limit of any convergent sequence $\{z_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞), then T_∞ admits a fixed point $\Leftrightarrow$ $\{x_n\}$ converges.

2. $\limsup X_n \subset X_\infty$ (ie; the cluster point of any sequence $\{z_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞), then T_∞ admits a fixed point $\Leftrightarrow$ $\{x_n\}$ admits a convergent subsequence.

The following Proposition proves that the (G)- limit mapping $T_\infty : X_\infty \to X$ is a (ψ, ϕ) weakly contraction if each mapping $T_n : X_n \to X$ is a (ψ, ϕ) weakly contraction.

Proposition 3.4.2. *Let X be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for all $n \in \mathbb{N}$, T_n is a (ψ, ϕ)- weakly contractive mapping. Then T_∞ is (ψ, ϕ)-weakly contractive.*

Proof. Given two points x and y in X_∞. By the property (G), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ converging respectively to x and y such that the sequences $\{T_n x_n\}$ and $\{T_n y_n\}$ converge respectively to $T_\infty x$ and $T_\infty y$. For all $n \in \mathbb{N}$, by triangle inequality, we have

$$d(T_\infty x, T_\infty y) \leq d(T_\infty x, T_n x_n) + d(T_n x_n, T_n y_n) + d(T_n y_n, T_\infty y).$$

Since ψ is increasing,

$$\psi\left(d(T_\infty x, T_\infty y)\right) \leq \psi\left(d(T_\infty x, T_n x_n) + d(T_n x_n, T_n y_n) + d(T_n y_n, T_\infty y)\right).$$

Letting $n \to \infty$, and using the continuity of both ψ and ϕ we have,

$$\begin{aligned}\psi\left(d(T_\infty x, T_\infty y)\right) &\leq \lim_{n\to\infty} \psi\left(d(T_n x_n, T_n y_n)\right) \\ &\leq \lim_{n\to\infty} \left[\psi\left(d(x_n, y_n)\right) - \phi\left(d(x_n, y_n)\right)\right] \\ &= \psi\left(d(x, y)\right) - \phi\left(d(x, y)\right),\end{aligned}$$

and the conclusion holds. □

The following result in [6, Proposition 4] follows from Proposition 3.4.2.

Corollary 3.4.4. *Let X be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying property (G) such that, for all $n \in \mathbb{N}$, T_n is a k_n-contraction. Then T_∞ is a k-contraction where $\{k_n\}$ is a bounded (resp.convergent) sequence with $k =: \sup_n k_n$ (resp.$\lim_n k_n$).*

Under a compactness assumption, the existence of a fixed point of the (G)- limit mapping can be obtained from the existence of fixed points of the (ψ, ϕ)-weakly contractive mappings T_n.

Theorem 3.4.3. *Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for all $n \in \mathbb{N}$, T_n is a (ψ, ϕ)-weakly contractive mapping. Assume that $\limsup X_n \subset X_\infty$ and $\bigcup_{n\in\mathbb{N}} X_n$ is relatively compact. If for any $n \in \mathbb{N}$, T_n admits a fixed point x_n, then the (G)-limit mapping T_∞ admits a fixed point x_∞ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. Let x_n be the fixed point of T_n for each $n \in \mathbb{N}$. From the compactness condition, there exists a convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$. Now, by Remark 3.4.1, T_∞ admits a fixed point x_∞ and by Theorem 3.4.2, the sequence $\{x_n\}$ converges to x_∞. □

As a consequence of Theorem 3.4.3 and Remark 3.4.1, we have the following result in [6, Theorem7].

Corollary 3.4.5. *Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) such that for all $n \in \mathbb{N}$, T_n is a k-contraction. Assume that $\limsup X_n \subset X_\infty$ and $\bigcup_{n\in\mathbb{N}} X_n$ is relatively compact. If for all $n \in \mathbb{N}$, T_n admits a fixed point x_n, then the (G)-limit mapping T_∞ admits a fixed point.*

Now we present a stability result for (G^-) convergence, which is weaker than (G)-convergence, for a sequence of mappings $\{T_n\}$.

Theorem 3.4.4. *Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of (ψ, ϕ)-weakly contractive mappings satisfying the property (G^-). If for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. By the property (G^-), there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ which has a subsequence $\{y_{n_j}\}$ such that $y_{n_j} \to x_\infty$ and $T_{n_j} y_{n_j} \to T_\infty x_\infty$ as $j \to \infty$. We have

$$d(x_{n_j}, x_\infty) \le d(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}) + d(T_{n_j} y_{n_j}, T_\infty x_\infty),$$

which implies that

$$\psi\left(d(x_{n_j}, x_\infty)\right) \le \psi\left(d(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}) + d(T_{n_j} y_{n_j}, T_\infty x_\infty)\right).$$

Taking the limit as $j \to \infty$,

$$\lim_{j\to\infty} \psi\left(d(x_{n_j}, x_\infty)\right) \leq \lim_{j\to\infty} \psi\left(d(T_{n_j}x_{n_j}, T_{n_j}y_{n_j})\right).$$

Since each mapping T_{n_j} is (ψ, ϕ)-weakly contractive and ψ is increasing, we have

$$\begin{aligned}
\lim_{j\to\infty} \psi\left(d(x_{n_j}, x_\infty)\right) &\leq \lim_{j\to\infty} \left[\psi\left(d(x_{n_j}, y_{n_j})\right) - \phi\left(d(x_{n_j}, y_{n_j})\right)\right] \\
&\leq \lim_{j\to\infty} \left[\psi\left(d(x_{n_j}, x_\infty) + d(y_{n_j}, x_\infty)\right) - \phi\left(d(x_{n_j}, x_\infty) + d(y_{n_j}, x_\infty)\right)\right] \\
&= \lim_{j\to\infty} \psi\left(d(x_{n_j}, x_\infty)\right) - \lim_{j\to\infty} \phi\left(d(x_{n_j}, x_\infty)\right).
\end{aligned}$$

Hence,

$$\lim_{j\to\infty} \phi\left(d(x_{n_j}, x_\infty)\right) = 0.$$

Thus $\{x_{n_j}\}$ converges to x_∞, the fixed point of T_∞. □

The following result in [6, Theorem 8] follows from Theorem 3.4.4.

Corollary 3.4.6. *Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of a metric space X and let $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of k-contraction mappings satisfying the property (G^-). If for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

3.5 Stability results under (H)- convergence

Now, we present another stability result using the (H)-convergence as follows which is a generalization of uniform convergence.

Theorem 3.5.1. *Let X be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of a metric space X and let $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the* *property (H) such that T_∞ is a (ψ,ϕ)-weakly contractive mapping. If for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. By property (H), there exists a sequence $\{y_n\}$ in X_∞ such that $d(x_n, y_n) \to 0$ and $d(T_n x_n, T_\infty y_n) \to 0$. We have

$$d(x_n, x_\infty) \le d(T_n x_n, T_\infty y_n) + d(T_\infty y_n, T_\infty x_\infty).$$

Since ψ is increasing,

$$\psi\left(d(x_n, x_\infty)\right) \le \psi\left(d(T_n x_n, T_\infty y_n) + d(T_\infty y_n, T_\infty x_\infty)\right).$$

Taking the limit as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \psi\left(d(x_n, x_\infty)\right) &\le \lim_{n\to\infty} \psi\left(d(T_\infty y_n, T_\infty x_\infty)\right) \\
&\le \lim_{n\to\infty} \left[\psi\left(d(y_n, x_\infty)\right) - \phi\left(d(y_n, x_\infty)\right)\right] \\
&\le \lim_{n\to\infty} \left[\psi(d(x_n, y_n) + d(x_n, x_\infty) - \phi(d(x_n, y_n) + d(x_n, x_\infty))\right] \\
&= \lim_{n\to\infty} \psi\left(d(x_n, x_\infty)\right) - \lim_{n\to\infty} \phi\left(d(x_n, x_\infty)\right).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \phi\left(d(x_n, x_\infty)\right) = 0,$$

and hence the conclusion follows. □

The following result in [6, Theorem 11] follows directly from the above theorem.

Corollary 3.5.1. *Let X be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of a metric space X and let $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings satisfying the property (H) such that T_∞ is a k- contraction. If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Chapter 4

Stability results in 2-metric spaces

4.1 Introduction

In Chapter 3 we proved stability of fixed points using (G)- convergence and (H)-convergence for (ψ, ϕ)-weakly contractive mappings in metric spaces . In this chapter (Sections 4.2.4 and 4.3), we extend these results to 2-metric spaces. We note that these results may be considered as significant in the sense that the 2-metric spaces differ topologically from metric spaces in many ways (see Remark 4.2.1).

4.2 Preliminaries

S.Gähler introduced in the 1960's the notion of 2-metric space[23, 24, 25]. Since then, several mathematicians have been developing and introducing analogues in the setting of 2- metric spaces. Regarding fixed point theorems, the first result in these spaces was obtained by Iséki [32]. In this section we present the notion of 2-metric spaces and some related properties of these spaces and extend the notion of (G)-convergence and (H)-convergence to 2-metric spaces.

4.2.1 2-Metric Spaces

The following notion of 2-metric spaces is due to Gähler [23].

Definition 4.2.1. Let X be a nonempty set. A real valued function ρ on $X \times X \times X$ is said to be a 2- metric on X if

(d1) for any two distinct elements $x, y \in X$ there exists an element $z \in X$ such that $\rho(x, y, z) \neq 0$,

(d2) $\rho(x, y, z) = 0$ when at least two of x, y, z are equal,

(d3) $\rho(x, y, z) = \rho(z, x, y) = \rho(y, z, x)$ for all x, y, z in X and

(d4) $\rho(x, y, z) \leq \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z)$ for all x, y, z, u in X (triangle area inequality or simply TA-inequality).

The pair (X, ρ) is called a 2-metric space. It is easily seen that ρ is non-negative and it abstracts the properties of the area function for euclidean triangles in the same manner as a metric abstracts the properties of the length function. Thus geometrically $\rho(x, y, z)$ represents the area of a triangle formed by the points x, y and z in X.

4.2.2 Basic notions on 2-metric Spaces

We start this section with the following well known definitions:

Definition 4.2.2. Let $\{x_n\}$ be a sequence in a 2-metric space (X, ρ). Then:

(i) $\{x_n\}$ is said to be convergent with limit $z \in X$ if

$$\lim_{n\to\infty} \rho(x_n, z, a) = 0 \text{ for all } a \in X.$$

Notice that if the sequence $\{x_n\}$ converges to z, then

$$\lim_{n\to\infty} \rho(x_n, a, b) = \rho(z, a, b) \text{ for all } a, b \in X.$$

(ii) $\{x_n\}$ is said to be Cauchy if

$$\lim_{m,n\to\infty} \rho(x_m, x_n, a) = 0 \text{ for all } a \in X.$$

(iii) (X, ρ) is said to be complete if every Cauchy sequence in X is convergent.

Definition 4.2.3. A 2-metric space (X, ρ) is said to be bounded if there is a constant K such that $\rho(a, b, c) \leq K$ for all $a, b, c \in X$.

Remark 4.2.1. The following remarks capture some distinct features of topological properties of 2-metric spaces which differ from those of metric spaces.

(i) Given any metric space which consists of more than two points, there always exists a 2-metric compatible with the topology of the space. But the converse is not always true as one can find a 2-metric space which does not have a countable basis associated with one of its arguments (see Gähler [23, page 123]).

(ii) It is known that a 2-metric ρ is continuous in any one of its arguments. Generally, we cannot however assert the continuity of ρ in all three arguments. But if it is continuous in any two arguments, then it is continuous in all the three arguments (see Gähler [23, Theorem 20 and example on page 145]).

(iii) In a complete 2-metric space a convergent sequence need not be Cauchy.

Example 4.2.1. [60] Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$. Define $\rho : X \times X \times X \to [0, \infty)$ as

$$\rho(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \text{ are distinct and} \{\frac{1}{n}, \frac{1}{n+1}\} \subset \{x, y, z\} \text{ for some positive integer } n \\ 0 & \text{otherwise.} \end{cases}$$

Then (X,ρ) is a complete 2-metric space. The sequence $\{\frac{1}{n}\}$ converges to 0, but $\{\frac{1}{n}\}$ is not Cauchy.

(iv) In a 2-metric space (X,ρ) every convergent sequence is Cauchy whenever ρ is continuous. However, the converse need not be true.

Example 4.2.2. [60] Let $X = \{a\} \cup \{a_n : n = 1, 2, ...\} \cup \{b\} \cup \{b_n : n = 1, 2, ...\}$, where $a = (1, 0), b = (0, 0), a_n = (1 + \frac{1}{n}, 0)$ and $b_n = (0, \frac{1}{n})$. Define $\rho : X \times X \times X \to [0, \infty)$ as

$$\rho(x,y,z) = \begin{cases} 1 & \text{if } \{x,y,z\} = \{a_n, b_n, a\} \text{ or } \{a_n, b_n, b\} \text{ for some } n \in \mathbb{N} \text{ or} \\ & \{a_n, b_n, a_m\} \text{ or } \{a_n, b_n, b_m\} \text{ for some } m, n \in \mathbb{N} \text{ with } m \neq n \\ \Delta\ x\ y\ z & \text{otherwise,} \end{cases}$$

where $\Delta\ x\ y\ z$ is the area of the triangle formed by the points x, y and z. Then (X,ρ) is a complete 2-metric space and every convergent sequence in it is Cauchy. But ρ is not continuous on X, for $\{a_n\}$ converges to a, $\{b_n\}$ converges to b and $\{\rho(a_n, b_n, a)\}$ does not converge to zero.

Definition 4.2.4. Let (X,ρ) be a 2-metric space. A mapping $T : X \to X$ is called a k-contraction (or simply contraction)(cf. [33], [46]) if there exists a $k \in (0,1)$ such that:

$$\rho(Tx, Ty, a) \leq k\rho(x, y, a) \text{ for all } x, y, a \in X. \tag{4.2.1}$$

4.2.3 Some weakly contractive mappings in 2-metric spaces

In this section, we extend the weakly contractive conditions to 2-metric spaces.

Let (X,ρ) be a 2-metric space and $T : X \to X$.

T is called a nonlinear contraction if

$$\rho(Tx, Ty, a) \leq \alpha(\rho(x, y, a)) \tag{4.2.2}$$

for all $x, y \in X$, where $\alpha : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right and $\alpha(t) < t$ for $t > 0$. We note that $\alpha(0) = 0$.

T is called weakly contractive on X if

$$\rho(Tx, Ty, a) \leq \rho(x, y, a) - \phi(\rho(x, y, a)) \tag{4.2.3}$$

for all $x, y, a \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that $\phi(t) = 0$ if and only if $t = 0$.

T is called (ψ, ϕ)- weakly contractive if

$$\psi(\rho(Tx, Ty, a)) \leq \psi(\rho(x, y, a)) - \phi(\rho(x, y, a)) \tag{4.2.4}$$

for all $x, y \in X$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are both continuous functions such that $\psi(t), \phi(t) > 0$ for $t \in (0, \infty)$ and $\psi(0) = 0 = \phi(0)$. In addition, ϕ is nonincreasing and ψ is strictly increasing.

Remark 4.2.2. Notice that if one takes $\phi(t) = (1 - k)t$, where $0 < k < 1$, then (4.2.3) reduces to (4.2.1). When $\psi(t) = t$, then condition (4.2.4) recovers condition (4.2.3). If $\phi(t) = t - \alpha(t)$, then (4.2.3) turns into (4.2.2). Therefore

$$(4.2.1) \Rightarrow (4.2.2) \Rightarrow (4.2.3) \Rightarrow (4.2.4).$$

4.2.4 Some general notions of convergence of type(G) and (H) in 2-metric spaces

We first recall the following notions of convergence from [52]. We note that these notions are the extensions of corresponding notions introduced by Barbet and Nachi

[6] in the setting of metric spaces.

Definition 4.2.5. Let (X, ρ) be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of non-empty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings. Then:

(i) T_∞ is called a (G)-limit of sequence $\{T_n\}_{n\in\mathbb{N}}$, or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G), if the following condition hold:

(G) $Gr(T_\infty) \subset \liminf Gr(T_n)$: for all $x \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that for all $a \in X$,

$$\lim_n \rho(x_n, x, a) = 0 \text{ and} \lim_n \rho(T_n x_n, T_\infty x, a) = 0,$$

and $Gr(T)$ stands for the graph of T.

(ii) T_∞ is called a (G^-)-limit of sequence $\{T_n\}_{n\in\mathbb{N}}$, or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G^-), where

(G^-) $Gr(T_\infty) \subset \limsup Gr(T_n)$: for all $z \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ and which has a subsequence $\{x_{n_j}\}$ such that

$$\lim_j \rho(x_{n_j}, z, a) = 0 \text{ and} \lim_j \rho(T_{n_j} x_{n_j}, T_\infty z, a) = 0, \text{ for all } a \in X.$$

(iii) T_∞ is called an (H)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H), where

(H) For all sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that for all $a \in X$

$$\lim_n \rho(x_n, y_n, a) = 0 \text{ and } \lim_n \rho(T_n x_n, T_n y_n, a) = 0.$$

4.3 Stability under (G)-convergence

In this section, we present stability results under (G)-convergence for sequence of (ψ, ϕ)-weakly contractive mappings in 2-metric spaces. These results are the extensions of their counter parts which were obtained in Chapter 3 (Theorems 3.4.1, 3.4.2, 3.4.3 and 3.4.4).

Throughout, unless stated otherwise, X will denote a 2-metric space (X, ρ) with ρ continuous.

The following theorem gives a sufficient condition for the existence of a unique (G)-limit in a 2-metric space.

Proposition 4.3.1. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of (ψ, ϕ)-weakly contractive mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.*

Proof. Assume that $T_\infty : X_\infty \to X$ and $T^*_\infty : X_\infty \to X$ are (G)-limit mappings of the sequence $\{T_n\}$. Hence for any point $x \in X_\infty$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ converging to x such that $\{T_n x_n\}$ and $\{T_n y_n\}$ converge to T_∞ and T^*_∞ respectively. Therefore

$$\lim_{n\to\infty} \rho\left(T_n x_n, T_\infty x, a\right) = 0,\ \lim_{n\to\infty} \rho\left(\ T_n y_n, T^*_\infty x, a\right) = 0 \text{ for all } a \in X.$$

Since T_n is (ψ, ϕ)-weakly contractive for each $n \in \mathbb{N}$,

$$\psi(\rho(T_n x_n, T_n y_n, a)) \le \psi(\rho(x_n, y_n, a)) - \phi(\rho(x_n, y_n, a))$$

which implies that

$$\psi(\rho(T_n x_n, T_n y_n, a)) \le \psi(\rho(x_n, y_n, a)).$$

As ψ is increasing, from the above inequality we have

$$\rho(T_n x_n, T_n y_n, a) \leq \rho(x_n, y_n, a). \tag{4.3.1}$$

By the triangular area inequality and condition (4.3.1), for all $n \in \mathbb{N}$ and for any $a \in X$, we have

$$\begin{aligned}\rho(T_\infty x, T_\infty^* x, a) &\leq \rho(T_\infty x, T_\infty^* x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty^* x, a)\\ &\leq \rho(T_\infty x, T_\infty^* x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty^* x, T_n y_n)\\ &\quad +\rho(T_n x_n, T_n y_n, a) + \rho(T_n y_n, T_\infty^* x, a)\\ &\leq \rho(T_\infty x, T_\infty^* x, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty^* x, T_n y_n)\\ &\quad +\rho(x_n, y_n, a) + \rho(T_n y_n, T_\infty^* x, a) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence we deduce that $\lim_{n\to\infty} \rho(T_\infty x, T_\infty^* x, a) = 0$ and the unicity of the limit is established. □

When $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0,1)$ in the above proposition, we get the following result.

Corollary 4.3.1. *[53, Proposition 2.2] Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of k-contraction mappings. If $T_\infty : X_\infty \to X$ is a (G)-limit of $\{T_n\}$, then T_∞ is unique.*

When $\psi(t) = t$ and $\phi(t) = t - \alpha(t)$ in the above proposition, the following result is obtained.

Corollary 4.3.2. *Corollary* 4.3.1 *with k-contraction replaced by nonlinear contraction.*

The following theorem is our first stability result.

Theorem 4.3.1. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) such that for all $n \in \mathbb{N}$, T_n is a (ψ,ϕ)-weakly contractive mapping where ψ is increasing and ϕ is nonincreasing. If for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. Let x_n be a fixed point of T_n for each $n \in \overline{\mathbb{N}}$. Since the property (G) holds and $x_\infty \in X_\infty$, there exists a sequence $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ such that

$$\lim_{n\to\infty} \rho(y_n, x_\infty, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n y_n, T_\infty x_\infty, a) = 0 \text{ for all } a \in X.$$

We have

$$\begin{aligned}\psi\left(\rho(x_n, x_\infty, a)\right) &= \psi\left(\rho(T_n x_n, T_\infty x_\infty, a)\right)\\ &\le \psi\left(\rho(T_n x_n, T_\infty x_\infty, T_n y_n) + \rho(T_n x_n, T_n y_n, a) + \rho(T_n y_n, T_\infty x_\infty, a)\right).\end{aligned}$$

Making $n \to \infty$ in the above inequality and using the continuity of ψ and ϕ, we obtain

$$\begin{aligned}\lim_{n\to\infty}\psi\left(\rho(x_n, x_\infty, a)\right) &\le \lim_{n\to\infty}\psi\left(\rho(T_n x_n, T_\infty x_\infty, T_n y_n) + \rho(T_n x_n, T_n y_n, a) + \rho(T_n y_n, T_\infty x_\infty, a)\right)\\ &= \lim_{n\to\infty}\psi\left(\rho(T_n x_n, T_n y_n, a)\right)\\ &\le \lim_{n\to\infty}\left[\psi\left(\rho(x_n, y_n, a)\right) - \phi\left(\rho(x_n, y_n, a)\right)\right]\\ &\le \lim_{n\to\infty}[\psi\left(\rho(x_n, y_n, x_\infty) + \rho(x_n, x_\infty, a) + \rho(x_\infty, y_n, a)\right)\\ &\quad -\phi\left(\rho(x_n, y_n, x_\infty) + \rho(x_n, x_\infty, a) + \rho(x_\infty, y_n, a)\right)]\\ &= \lim_{n\to\infty}\psi\left(\rho(x_n, x_\infty, a)\right) - \lim_{n\to\infty}\phi\left(\rho(x_n, x_\infty, a)\right),\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \phi\left(\rho(x_n, x_\infty, a)\right) \le 0.$$

By the property of ϕ, we get $\lim_{n\to\infty} \rho(x_n, x_\infty, a) = 0$ and hence the conclusion. □

Corollary 4.3.3. *[53, Theorem 2.3] Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) such that, for all $n \in \mathbb{N}$, T_n is a k-contraction. If for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. It comes from Theorem 4.3.1 when $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0, 1)$. □

The existence of a fixed point for a (G)-limit mapping is characterized by the following result when it is a (ψ, ϕ)-weakly contractive mapping.

Theorem 4.3.2. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for any $n \in \mathbb{N}$, T_n is a (ψ, ϕ)-weakly contractive mapping. Assume that, for any $n \in \mathbb{N}$, x_n is a fixed point of T_n. Then*

$$\begin{aligned} T_\infty \text{ admits a fixed point} \quad &\Leftrightarrow \quad \{x_n\} \text{ converges and } \lim x_n \in X_\infty \\ &\Leftrightarrow \quad \{x_n\} \text{ admits a subsequence converging to a point of } X_\infty. \end{aligned}$$

Proof. The necessary part is already proved in Theorem 4.3.1. To prove the sufficiency, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = x_\infty \in X_\infty$. By the property (G), there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_{n\to\infty} \rho(y_n, x_\infty, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n y_n, T_\infty x_\infty, a) = 0 \text{ for all } a \in X.$$

Hence for any $a \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\rho(x_\infty, T_\infty x_\infty, a) &\leq \rho(x_\infty, x_{n_j}, a) + \rho\left(T_{n_j}x_{n_j}, T_\infty x_\infty, a\right) + \rho\left(x_\infty, T_\infty x_\infty, T_{n_j}x_{n_j}\right) \\ &\leq \rho(x_\infty, x_{n_j}, a) + \rho\left(T_{n_j}x_{n_j}, T_\infty x_\infty, T_{n_j}y_{n_j}\right) + \rho\left(T_{n_j}x_{n_j}, T_{n_j}y_{n_j}, a\right) \\ &\quad + \rho\left(T_{n_j}y_{n_j}, T_\infty x_\infty, a\right) + \rho\left(x_\infty, T_\infty x_\infty, T_{n_j}x_{n_j}\right) \\ &\leq \rho\left(x_\infty, x_{n_j}, a\right) + \rho\left(T_{n_j}x_{n_j}, T_\infty x_\infty, T_{n_j}y_{n_j}\right) + \rho\left(x_{n_j}, y_{n_j}, a\right) + \\ &\quad \rho\left(T_{n_j}y_{n_j}, T_\infty x_\infty, a\right) + \rho\left(x_\infty, T_\infty x_\infty, T_{n_j}x_{n_j}\right) \text{ by condition (4.3.1) .}\end{aligned}$$

The right hand side of the above expression tends to zero as $j \to \infty$ and hence $T_\infty x_\infty = x_\infty$, proving that x_∞ is a fixed point of T_∞. □

Corollary 4.3.4. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for any $n \in \mathbb{N}$, T_n is a nonlinear contraction. Assume that for any $n \in \mathbb{N}$, x_n is a fixed point of T_n. Then*

$$\begin{aligned}T_\infty \text{ admits a fixed point} &\Leftrightarrow \{x_n\} \text{ converges and } \lim x_n \in X_\infty \\ &\Leftrightarrow \{x_n\} \text{ admits a subsequence converging to a point of } X_\infty.\end{aligned}$$

Remark 4.3.1. Under the assumptions of Theorem 4.3.2, and if

(i) $\liminf X_n \subset X_\infty$ (i.e. the limit of any convergent sequence $\{z_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞), then

$$T_\infty \text{ admits a fixed point} \Leftrightarrow \{x_n\} \text{ converges.}$$

(ii) $\limsup X_n \subset X_\infty$ (i.e. the cluster point of any sequence $\{z_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞) then

$$T_\infty \text{ admits a fixed point} \Leftrightarrow \{x_n\} \text{ admits a convergent subsequence.}$$

The following proposition provides a sufficient condition under which a (G)-limit of a sequence of (ψ,ϕ)-weakly contractive mappings is again (ψ,ϕ)-weakly contractive.

Proposition 4.3.2. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{\ T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for any $n \in \mathbb{N}$, T_n is a (ψ,ϕ)-weakly contractive mapping. Then T_∞ is (ψ,ϕ)-weakly contractive.*

Proof. Given two points x and y in X_∞, by the property (G) there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ converging respectively to x and y such that the sequences $\{T_n x_n\}$ and $\{T_n y_n\}$ converge respectively to $T_\infty x$ and $T_\infty y$. For any $n \in \mathbb{N}$ and $a \in X$,

$$\begin{aligned}
\psi\left(\rho(T_\infty x, T_\infty y, a)\right) &\leq \psi\left(\rho(T_\infty x, T_\infty y, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty y, a)\right) \\
&\leq \psi\left(\rho(T_\infty x, T_\infty y, T_n x_n) + \rho(T_\infty x, T_n x_n, a) + \rho(T_n x_n, T_\infty y, T_n y_n)\right. \\
&\quad \left. + \rho\left(T_n x_n, T_n y_n, a\right) + \rho(T_n y_n, T_\infty y, a)\right).
\end{aligned}$$

Letting $n \to \infty$, and using the continuity of both ψ and ϕ we have

$$\begin{aligned}
\psi\left(\rho(T_\infty x, T_\infty y, a)\right) &\leq \lim_{n\to\infty} \psi\left(\rho(T_n x_n, T_n y_n, a)\right) \\
&\leq \lim_{n\to\infty} \left[\psi\left(\rho(x_n, y_n, a)\right) - \phi\left(\rho(x_n, y_n, a)\right)\right].
\end{aligned}$$

Hence we conclude that $\psi\left(\rho(T_\infty x, T_\infty y, a)\right) \leq \psi\left(\rho(x, y, a)\right) - \phi\left(\rho(x, y, a)\right).$ □

Corollary 4.3.5. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for any $n \in \mathbb{N}$, T_n is a k-contraction from X_n to X. Then T_∞ is a k-contraction.*

Proof. This comes from Proposition 4.3.2 when $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0,1)$. □

Under a compactness assumption, the existence of a fixed point of the (G)-limit mapping can be obtained from the existence of fixed points of the (ψ, ϕ)-weakly contractive mappings T_n.

Theorem 4.3.3. *Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a 2-metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for any $n \in \mathbb{N}$, T_n is a (ψ, ϕ)-weakly contractive mapping where ψ is increasing and ϕ is nonincreasing. Assume that $\limsup X_n \subset X_\infty$ and $\bigcup_{n\in\mathbb{N}} X_n$ is relatively compact. If for any $n \in \mathbb{N}$, T_n admits a fixed point x_n, then the (G)-limit mapping T_∞ admits a fixed point x_∞ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. Let x_n be the fixed point of T_n for $n \in \mathbb{N}$. From compactness condition, there exists a convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$. Now by Remark 4.3.1, T_∞ admits a fixed point x_∞ and by Theorem 4.3.1, the sequence $\{x_n\}$ converges to x_∞. □

Corollary 4.3.6. *[53, Theorem 2.10] Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a 2-metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for any $n \in \mathbb{N}$, T_n is a k-contraction. Assume that $\limsup X_n \subset X_\infty$ and $\bigcup_{n\in\mathbb{N}} X_n$ is relatively compact. If for any $n \in \mathbb{N}$, T_n admits a fixed point x_n, then the (G)-limit mapping T_∞ admits a fixed point x_∞ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. This comes from Theorem 4.3.3, when $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0,1)$. □

The following result establishes that a fixed point of a (G$^-$)-limit mapping is a cluster point of the sequence of fixed points associated with $\{T_n\}$.

Theorem 4.3.4. *Let $\{X_n\}$ be a family of nonempty subsets of a 2-metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of (ψ, ϕ)-weakly contractive mappings satisfying the property (G$^-$), where ψ is increasing and ϕ is nonincreasing. If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. By the property (G$^-$), there exists a sequence $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ which has a subsequence $\{y_{n_j}\}$ such that

$$\lim_{j\to\infty} \rho(y_{n_j}, x_\infty, a) = 0 \text{ and } \lim_{j\to\infty} \rho\left(T_{n_j} y_{n_j}, T_\infty x_\infty, a\right) = 0 \text{ for all } a \in X.$$

By the triangular area inequality, we have

$$\rho(x_{n_j}, x_\infty, a) \le \rho(T_{n_j} x_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j}) + \rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a) + \rho(T_{n_j} y_{n_j}, T_\infty x_\infty, a).$$

Since ψ is increasing,

$$\psi\left(\rho(x_{n_j}, x_\infty, a)\right) \le \psi\left(\rho(T_{n_j} x_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j}) + \rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a) + \rho(T_{n_j} y_{n_j}, T_\infty x_\infty, a)\right).$$

Since ψ is continuous, taking the limit as $j \to \infty$, we get

$$\begin{aligned}
\lim_{j\to\infty} \psi\left(\rho(x_{n_j}, x_\infty, a)\right) &\le \lim_{j\to\infty} \psi\left(\rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a)\right) \\
&\le \lim_{j\to\infty} \left[\psi\left(\rho(x_{n_j}, y_{n_j}, a)\right) - \phi\left(\rho(x_{n_j}, y_{n_j}, a)\right)\right] \\
&\le \lim_{j\to\infty} \left[\psi\left(\rho(x_{n_j}, y_{n_j}, x_\infty) + \rho(x_{n_j}, x_\infty, a) + \rho(x_\infty, y_{n_j}, a)\right)\right. \\
&\quad \left. - \phi\left(\rho(x_{n_j}, y_{n_j}, x_\infty) + \rho(x_{n_j}, x_\infty, a) + \rho(x_\infty, y_{n_j}, a)\right)\right] \\
&= \lim_{j\to\infty} \psi\left(\rho(x_{n_j}, x_\infty, a)\right) - \lim_{j\to\infty} \phi\left(\rho(x_{n_j}, x_\infty, a)\right).
\end{aligned}$$

Hence

$$\lim_{j\to\infty} \phi\left(\rho(x_{n_j}, x_\infty, a)\right) \le 0.$$

By the property of ϕ, we deduce that

$$\lim_{j\to\infty} \rho(x_{n_j}, x_\infty, a) = 0.$$

Thus $\{x_{n_j}\}$ converges to x_∞, the fixed point of T_∞. □

Corollary 4.3.7. *[53, Theorem 2.12] Let $\{X_n\}$ be a family of nonempty subsets of a 2-metric space X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of k-contraction mappings satisfying the property (G^-). If for any $n \in \mathbb{N}$, x_n is a fixed point of T_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. This comes from Theorem 4.3.4, when $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0, 1)$. □

4.4 Stability under (H)-convergence

The following theorem is our second stability result using the (H)-convergence in 2-metric spaces.

Theorem 4.4.1. *Let X be a 2-metric space, $\{X_n\}_{n\in\ \overline{\mathbb{N}}}$ a family of nonempty subsets of X, $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (H) and such that T_∞ is a (ψ, ϕ)-weakly contractive mapping. If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. By the property (H), there exists a sequence $\{y_n\}$ in X_∞ such that $\lim_{n\to\infty} \rho(x_n, y_n, a) = 0$ and $\lim_{n\to\infty} \rho(T_n x_n, T_\infty y_n, a) = 0$ for any $a \in X$. Hence for any $a \in X$,

$$\begin{aligned}
\rho(x_n, x_\infty, a) &= \rho(T_n x_n, T_\infty x_\infty, a) \\
&\le \rho(T_n x_n, T_\infty y_n, a) + \rho(T_\infty y_n, T_\infty x_\infty, a) + \rho(T_n x_n, T_\infty x_\infty, T_\infty y_n).
\end{aligned}$$

Since ψ is increasing,

$$\psi\left(\rho(x_n, x_\infty, a)\right) \leq \psi\left(\rho(T_n x_n, T_\infty y_n, a) + \rho(T_\infty y_n, T_\infty x_\infty, a) + \rho(T_n x_n, T_\infty x_\infty, T_\infty y_n)\right).$$

Taking the limit as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \psi\left(\rho(x_n, x_\infty, a)\right) &\leq \lim_{n\to\infty} \psi\left(\rho(T_\infty y_n, T_\infty x_\infty, a)\right) \\
&\leq \lim_{n\to\infty} \left[\psi\left(\rho(y_n, x_\infty, a)\right) - \phi\left(\rho(y_n, x_\infty, a)\right)\right] \\
&\leq \lim_{n\to\infty} \left[\psi\left(\rho(y_n, x_\infty, x_n) + \rho(y_n, x_n, a) + \rho(x_n, x_\infty, a)\right)\right. \\
&\quad \left. - \phi\left(\rho(y_n, x_\infty, x_n) + \rho(y_n, x_n, a) + \rho(x_n, x_\infty, a)\right)\right] \\
&= \lim_{n\to\infty} \psi\left(\rho(x_n, x_\infty, a)\right) - \lim_{n\to\infty} \phi\left(\rho(x_n, x_\infty, a)\right).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \phi\left(\rho(x_n, x_\infty, a)\right) \leq 0.$$

By the property of ϕ, we deduce that

$$\lim_{n\to\infty} \rho(x_n, x_\infty, a) = 0,$$

and the conclusion follows. □

Corollary 4.4.1. *[53, Theorem 3.4] Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of nonempty subsets of X, $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings satisfying the property (H) and such that T_∞ is a k-contraction. If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. This comes from Theorem 4.4.1, when $\psi(t) = t$ and $\phi(t) = (1-k)t$ and $k \in (0, 1)$. □

Chapter 5

Stability of fixed points in 2-metric spaces involving sequences of metrics

5.1 Introduction

In Chapter 4, we proved the existence of a unique (G)- limit for a sequence of (ψ, ϕ)-weakly contractive mappings in 2- metric spaces. We then extended the results of Barbet and Nachi [6] on the stability of fixed points in metric space using (G) convergence and (H) convergence to 2- metric spaces. In this chapter, we consider 2-metric spaces involving sequences of metrics and obtain a number of stability results for the class of (ψ, ϕ)-weakly contractive mappings. This chapter is based on the work of Fraser and Nadler [22].

The results of this chapter appear in *Advances in Fixed Point Theory, 3(2)(2013), 341-354.*

5.2 Preliminaries

The following classical results were obtained by Fraser and Nadler [22] in metric spaces.

Theorem 5.2.1. *[22, Theorem 2] Let (X, d) be a metric space and $\{d_n\}_{n\in\mathbb{N}}$ a sequence of metrics on X converging uniformly to d, where each d_n is equivalent to d. Let $\{T_n : X \to X\}_{n\in\mathbb{N}}$ be a sequence of contractive mappings on (X, d_n) converging pointwise to a mapping $T_\infty : X \to X$. If for each $n \in \mathbb{N}$, x_n is a fixed point of T_n, and if $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to x_∞, then x_∞ is a fixed point of T_∞.*

Theorem 5.2.2. *[22, Theorem 3] Let (X, d) be a metric space and $\{d_n\}_{n\in\mathbb{N}}$ a sequence of metrics on X converging uniformly to d. Let $\{T_n : X \to X\}_{n\in\mathbb{N}}$ be a sequence of $k-$contraction mappings on (X, d_n) converging pointwise to a mapping $T_\infty : X \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

We extend the above theorems to 2-metric spaces for a sequence of (ψ, ϕ)-weakly contractive mappings satisfying the property (G).

Following Nachi [58](see also [54]), we have the following convergence properties in 2-metric spaces.

Definition 5.2.1. Let (X, ρ) be a 2-metric space. $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X and $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X. Then $\{\rho_n\}_{n\in\mathbb{N}}$ is said to satisfy property:

(A) For all $x \in X_\infty$, $a \in X$ and $\{x_n\}_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} X_n$, $\lim\limits_{n\to\infty} \rho_n(x_n, x, a) = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho(x_n, x, a) = 0$.

(A$_0$) For all $x,\ a \in X$, and $\{x_n\}_{n\in\mathbb{N}} \subset X$, $\lim\limits_{n\to\infty} \rho_n(x_n, x, a) = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho(x_n, x, a) = 0$.

(B) For all sequences $\{x_n\}_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that $\lim\limits_{n\to\infty} \rho_n(x_n, y_n, a) = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho(x_n, y_n, a) = 0$ for all $a \in X$.

(B$_0$) For all sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset X$: $\lim\limits_{n\to\infty} \rho_n(x_n, y_n, a) = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho(x_n, y_n, a) = 0$.

5.3 (G)-convergence and sequences of metrics

In this section we present stability results for a sequence $\{T_n\}_{n\in\mathbb{N}}$ of (ψ, ϕ)- weakly contractive mappings in 2-metric spaces. We obtain the following analogue of Theorem 5.2.1 to 2 -metric spaces for (ψ, ϕ)- weakly contractive mappings.

Theorem 5.3.1. *Let (X, ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (A). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\mathbb{N}}$ a sequence of (ψ, ϕ)-weakly contractive mappings on (X_n, ρ_n) converging in the sense of (G) to a mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n and if the sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to a point $x_\infty \in X_\infty$, then x_∞ is a fixed point of T_∞.*

Proof. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ converging to $x_\infty \in X_\infty$. Then by the property (G) there exists a sequence $\{y_n\} \in \prod\limits_{n\in\mathbb{N}} X_n$ such that:

$$\lim_{n\to\infty} \rho(y_n, x_\infty, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n y_n, T_\infty x_\infty, a) = 0 \text{ for all } a \in X.$$

Therefore by the property (A),

$$\lim_{n\to\infty} \rho_n(y_n, x_\infty, a) = 0 \text{ and } \lim_{n\to\infty} \rho_n(T_n y_n, T_\infty x_\infty, a) = 0.$$

If we define a sequence $\{z_n\}$ such that

$$\begin{aligned} z_{n_j} &= x_{n_j} \text{ for all } j \in \mathbb{N}, \\ z_n &= y_n \text{ if } n \neq n_j, \text{ for any } j \in \mathbb{N}. \end{aligned}$$

Then $\lim_{n\to\infty} \rho(z_n, x_\infty, a) = 0$ and so $\lim_{n\to\infty} \rho_n(z_n, x_\infty, a) = 0$, by (A).

Now

$$\rho(z_n, y_n, a) \leq \rho(z_n, y_n, x_\infty) + \rho(z_n, x_\infty, a) + \rho(x_\infty, y_n, a) \to 0 \text{ as } n \to \infty,$$

and thus

$$\lim_{n\to\infty} \rho_n(z_n, y_n, a) = 0.$$

Further, we have

$$\begin{aligned} \rho_{n_j}\left(T_{n_j} z_{n_j}, T_\infty x_\infty, a\right) &\leq \rho_{n_j}\left(T_{n_j} z_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j}\right) \\ &\quad + \rho_{n_j}\left(T_{n_j} z_{n_j}, T_{n_j} y_{n_j}, a\right) + \rho_{n_j}\left(T_{n_j} y_{n_j}, T_\infty x_\infty, a\right). \end{aligned} \tag{5.3.1}$$

Since T_{n_j} is a (ψ, ϕ)-weakly contractive mapping on (X_{n_j}, ρ_{n_j}) for each $j \in \mathbb{N}$, we have

$$\begin{aligned} \psi(\rho_{n_j}(T_{n_j} z_{n_j}, T_{n_j} y_{n_j}, a)) &\leq \psi\left(\rho_{n_j}(z_{n_j}, y_{n_j}, a)\right) - \phi\left(\rho_{n_j}(z_{n_j}, y_{n_j}, a)\right) \\ &\leq \psi\left(\rho_{n_j}(z_{n_j}, y_{n_j}, a)\right). \end{aligned}$$

By the monotonicity of ψ, we obtain

$$\rho_{n_j}(T_{n_j} z_{n_j}, T_{n_j} y_{n_j}, a) \leq \rho_{n_j}(z_{n_j}, y_{n_j}, a). \tag{5.3.2}$$

From (5.3.1) and (5.3.2) we have

$$\rho_{n_j}\left(T_{n_j} z_{n_j}, T_\infty x_\infty, a\right) \leq \rho_{n_j}\left(T_{n_j} z_{n_j}, T_\infty x_\infty, T_{n_j} y_{n_j}\right) + \rho_{n_j}(z_{n_j}, y_{n_j}, a) + \rho_{n_j}\left(T_{n_j} y_{n_j}, T_\infty x_\infty, a\right).$$

On taking the limit as $j \to \infty$, we obtain

$$\lim_{j\to\infty} \rho_{n_j}\left(T_{n_j}z_{n_j}, T_\infty x_\infty, a\right) = 0. \tag{5.3.3}$$

Since $T_{n_j}z_{n_j} = T_{n_j}x_{n_j} = x_{n_j}$ and x_{n_j} converges to x_∞ as $j \to \infty$, (5.3.3) becomes $\rho_{n_j}(x_\infty, T_\infty x_\infty, a) = 0$ for all $a \in X$. Hence $T_\infty x_\infty = x_\infty$. □

In view of Remark 4.2.2 in Chapter 4, we have the following results as a direct consequence of Theorem 5.3.1.

Corollary 5.3.1. *Let (X, ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (A). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\mathbb{N}}$ a sequence of k-contraction mappings on (X_n, ρ_n) converging in the sense of (G) to a mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n and if the sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to a point $x_\infty \in X_\infty$, then x_∞ is a fixed point of T_∞.*

Corollary 5.3.2. *[54, Theorem 2.3] Corollary 5.3.1 with k-contraction replaced by nonlinear contraction.*

When $X_n = X$ for all $n \in \overline{\mathbb{N}}$ in Theorem 5.3.1, we have the following results:

Corollary 5.3.3. *Let X be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$, a sequence of 2-metrics on X satisfying the property (A_0). Let $\{T_n : X \to X\}_{n\in\mathbb{N}}$ be a sequence of (ψ, ϕ)-weakly contractive mappings on (X, ρ_n) converging pointwise to a mapping $T_\infty : X \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n and if the sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence converging to a point $x_\infty \in X$, then x_∞ is a fixed point of T_∞.*

Corollary 5.3.4. *Corollary 5.3.3 with (ψ, ϕ)- weakly contractive mapping replaced by k-contraction.*

The following theorem which is an extension of Theorem 5.2.2 to the context of 2- metric spaces is our first stability result in this chapter.

Theorem 5.3.2. *Let* (X,ρ) *be a* 2*-metric space and* $\{\rho_n\}_{n\in\mathbb{N}}$ *a sequence of* 2*-metrics on* X *satisfying the property* (A)*. Let* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *be a family of nonempty subsets of* X *and* $\{T_n : X_n \to X_n\}_{n\in\mathbb{N}}$ *a sequence of* (ψ,ϕ)*-weakly contractive mappings on* (X_n,ρ_n) *converging in the sense of* (G) *to a mapping* $T_\infty : X_\infty \to X$*. If for each* $n\in\overline{\mathbb{N}}$*,* x_n *is a fixed point of* T_n*, then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x_∞*.*

Proof. Let $x_\infty \in X_\infty$ and by the property (G), there exists a sequence $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ such that

$$\lim_{n\to\infty}\rho(y_n,x_\infty,a)=0 \text{ and } \lim_{n\to\infty}\rho(T_ny_n,T_\infty x_\infty,a)=0$$

for all $a\in X$. By the property (A), we deduce that

$$\lim_{n\to\infty}\rho_n(y_n,x_\infty,a)=0 \text{ and } \lim_{n\to\infty}\rho_n(T_ny_n,T_\infty x_\infty,a)=0.$$

We have

$$\begin{aligned}\psi\left(\rho_n(x_n,x_\infty,a)\right) &= \psi\left(\rho_n(T_nx_n,T_\infty x_\infty,a)\right)\\ &\le \psi\left(\rho_n(T_nx_n,T_\infty x_\infty,T_ny_n)+\rho_n(T_nx_n,T_ny_n,a)+\rho_n(T_ny_n,T_\infty x_\infty,a)\right).\end{aligned}$$

Making $n\to\infty$ in the above inequality, we obtain

$$\begin{aligned}\lim_{n\to\infty}\psi\left(\rho_n(x_n,x_\infty,a)\right) &\le \lim_{n\to\infty}\psi\left(\rho_n(T_nx_n,T_\infty x_\infty,T_ny_n)+\rho_n(T_nx_n,T_ny_n,a)\right.\\ &\quad\left.+\rho_n(T_ny_n,T_\infty x_\infty,a)\right)\\ &= \lim_{n\to\infty}\psi\left(\rho_n(T_nx_n,T_ny_n,a)\right)\\ &\le \lim_{n\to\infty}\left[\psi\left(\rho_n(x_n,y_n,a)\right)-\phi\left(\rho_n(x_n,y_n,a)\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty} \psi\left(\rho_n(x_n,y_n,x_\infty)+\rho_n(x_n,x_\infty,a)+\rho_n(x_\infty,y_n,a)\right)\\
&\quad -\lim_{n\to\infty} \phi\left(\rho_n(x_n,y_n,x_\infty)+\rho_n(x_n,x_\infty,a)+\rho_n(x_\infty,y_n,a)\right)\\
&= \lim_{n\to\infty} \psi\left(\rho_n(x_n,x_\infty,a)\right)-\lim_{n\to\infty} \phi\left(\rho_n(x_n,x_\infty,a)\right).
\end{aligned}
$$

Hence

$$\lim_{n\to\infty} \phi\left(\rho_n(x_n,x_\infty,a)\right) \leq 0.$$

By the property of ϕ we get

$$\lim_{n\to\infty} \rho_n(x_n,x_\infty,a) = 0$$

and the conclusion holds. □

Corollary 5.3.5. *Let (X,ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (A). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\mathbb{N}}$ a sequence of k -contraction mappings on (X_n,ρ_n) converging in the sense of (G) to a mapping $T_\infty : X_\infty \to X$. If for each $n\in\overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Corollary 5.3.6. *[54, Theorem2.7] Corollary 5.3.5 with k-contraction replaced by nonlinear contraction.*

If $X_n = X$ for all $n\in\overline{\mathbb{N}}$ in Theorem 5.3.2, then we have the following results:

Corollary 5.3.7. *Let X be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (A_0). Let $\{T_n : X \to X\}_{n\in\mathbb{N}}$ be a sequence of (ψ,ϕ)-weakly contractive mappings on (X,ρ_n) converging pointwise to a mapping $T_\infty : X\to X$. If for each $n\in\overline{\mathbb{N}}$, x_n is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

The following result can be compared with Theorem 5.2.2.

Corollary 5.3.8. *Corollary* 5.3.7 *with* (ψ, ϕ)*- weakly contractive mapping replaced by* k*-contraction.*

5.4 (H)-convergence and sequences of metrics

The following theorem is our second stability result in this chapter using the (H) -convergence in 2-metric spaces.

Theorem 5.4.1. *Let* (X, ρ) *be a* 2*-metric space and* $\{\rho_n\}_{n\in\mathbb{N}}$ *a sequence of* 2*-metrics on* X *satisfying the property* (B)*. Let* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *be a family of nonempty subsets of* X *and* $\{T_n : X_n \to X_n\}_{n\in\overline{\mathbb{N}}}$ *a sequence of mappings on* (X_n, ρ_n) *converging in the sense of* (H) *to a* (ψ, ϕ)*-weakly contractive mapping* $T_\infty : X_\infty \to X$*. If for each* $n \in \overline{\mathbb{N}}$*,* $x_n \in X_n$ *is a fixed point of* T_n*, then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x_∞*.*

Proof. By the property (H), there exists a sequence $\{y_n\}$ in X_∞ such that

$$\lim_{n\to\infty} \rho(x_n, y_n, a) = 0 \text{ and } \lim_{n\to\infty} \rho(T_n x_n, T_\infty y_n, a) = 0 \quad \text{for any } a \in X.$$

Therefore using the property (B), we have

$$\lim_{n\to\infty} \rho_n(x_n, y_n, a) = 0 \text{ and } \lim_{n\to\infty} \rho_n(T_n x_n, T_\infty y_n, a) = 0 \text{ for any } a \in X.$$

By the triangular area inequality,

$$\psi\left(\rho_n(x_n, x_\infty, a)\right) \leq \psi\left(\rho_n(T_n x_n, T_\infty x_\infty, T_\infty y_n) + \rho_n(T_n x_n, T_\infty y_n, a) + \rho_n(T_\infty y_n, T_\infty x_\infty, a)\right).$$

Taking the limit as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \psi\left(\rho_n(x_n, x_\infty, a)\right) &\leq \lim_{n\to\infty} \psi\left(\rho_n(T_\infty y_n, T_\infty x_\infty, a)\right) \\
&\leq \lim_{n\to\infty} \left[\psi\left(\rho_n(y_n, x_\infty, a)\right) - \phi\left(\rho_n(y_n, x_\infty, a)\right)\right] \\
&\leq \lim_{n\to\infty} \left[\psi\left(\rho_n(y_n, x_\infty, x_n) + \rho_n(y_n, x_n, a) + \rho_n(x_n, x_\infty, a)\right)\right] \\
&\quad - \lim_{n\to\infty} \left[\phi\left(\rho_n(y_n, x_\infty, x_n) + \rho_n(y_n, x_n, a) + \rho_n(x_n, x_\infty, a)\right)\right] \\
&= \lim_{n\to\infty} \left[\psi\left(\rho_n(x_n, x_\infty, a)\right) - \lim_{n\to\infty} \phi\left(\rho_n(x_n, x_\infty, a)\right)\right].
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \phi\left(\rho_n(x_n, x_\infty, a)\right) = 0$$

and the conclusion follows. □

Corollary 5.4.1. *Let (X, ρ) be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (B). Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings on (X_n, ρ_n) converging in the sense of (H) to a k-contraction mapping $T_\infty : X_\infty \to X$. If for each $n \in \overline{\mathbb{N}}$, $x_n \in X_n$ is a fixed point of T_n, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Corollary 5.4.2. *[54, Theorem2.11] Corollary* 5.4.1 *with k-contraction replaced by nonlinear contraction.*

When $X_n = X$ for all $n \in \overline{\mathbb{N}}$ in Theorem 5.4.1 , we obtain the following result:

Corollary 5.4.3. *Let X be a 2-metric space and $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of 2-metrics on X satisfying the property (B_0). Let $\{T_n : X \to X\}$ be a sequence of mappings on (X, ρ_n) converging uniformly to a (ψ, ϕ)- weakly contractive mapping $T_\infty : X \to X$. If for each $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Corollary 5.4.4. *Corollary* 5.4.3 *with* (ψ, ϕ)*- weakly contractive mapping replaced by* k*-contraction.*

Appendix

List of Published papers

1. *Some existence and convergence theorems for nonexpansive type mappings*, International Journal of Analysis (2013), Art. ID 539723 (with S. N Mishra and Rajendra Pant).

2. *Sequences of* (ψ, ϕ)*-weakly contractive mappings and stability of fixed Points*, International Journal of Mathematical Analysis, 7(22)(2013), 1085-1096 (with S. N Mishra and Rajendra Pant).

3. *Some general convergence theorems on sequences of fixed points*, Advances in Fixed Point Theory, 3(2)(2013), 341-354 (with S. N Mishra and Rajendra Pant).

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