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A STUDY OF FUZZY SETS AND SYSTEMS WITH APPLICATIONS TO GROUP THEORY AND DECISION MAKING.

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Abstract

In this study we apply the knowledge of fuzzy sets to group structures and also to decision-making implications. We study fuzzy subgroups of finite abelian groups. We set $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$. The classification of fuzzy subgroups of G using equivalence classes is introduced. First, we present equivalence relations on fuzzy subsets of X, and then extend it to the study of equivalence relations of fuzzy subgroups of a group G. This is then followed by the notion of flags and keychains projected as tools for enumerating fuzzy subgroups of G. In addition to this, we use linear ordering of the lattice of subgroups to characterize the maximal chains of G. Then we narrow the gap between group theory and decision-making using relations. Finally, a theory of the decision-making process in a fuzzy environment leads to a fuzzy version of capital budgeting. We define the goal, constraints and decision and show how they conflict with each other using membership function implications. We establish sets of intervals for projecting decision boundaries in general. We use the knowledge of triangular fuzzy numbers which are restricted field of fuzzy logic to evaluate investment projections.

KEYWORDS: Fuzzy logic and set, Fuzzy number, Equivalence relation, Fuzzy subgroup, Finite abelian group, Equivalence class, Decision-making, Net present value, Investment, Capital budgeting.

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PREFACE

Since the notion of fuzzy sets was introduced by Zadeh in 1965 [41], there have been attempts to extend useful mathematical notions to this wider setting, replacing sets by fuzzy sets. Another important development was the study of *finite abelian groups* in the field of algebra and also the application of the knowledge of fuzzy sets to *capital budgeting* in the field of economics. The concepts of fuzzy subgroups and fuzzy capital budgeting were put forward by Rosenfeld [37] and Buckley [4] respectively. In Chapter 1 we provide an introduction to *fuzzy logic* and *sets* theory which contains some basic facts necessary for dealing with the latter chapters.

Chapter 2 provides a background on *equivalence* for classifying the fuzzy subgroups of G. In this chapter, we look at the implications of equivalence for fuzzy subsets, and later on we extend it to fuzzy subgroups of G. We use the *flags* and *keychains* to characterize the properties of fuzzy subsets of X and later on we extend the notion of *flags* and *keychains* to the fuzzy subgroups of G. We give a definition of equivalence of *fuzzy points* belonging to a fuzzy subset and we look at its relation to equivalence of fuzzy subsets. Further, we analyse distinct *keychains* relative to an equivalence of *fuzzy points*. Then, we use *maximal chains* to construct a fuzzy subgroup-lattice diagram for these groups of G.
Chapter 3 discusses the important features of *arbitrary finite abelian groups* in classical case, that has its fuzzy analogous facts in Chapter 4.

Chapter 4 deals with fuzzy subgroups, notions such as the *definition* of fuzzy subgroups and also some of their basic properties. We define a *level-subgroup* and show how it can be used to study the fuzzy subgroups of G. The concept of *generated* fuzzy algebra that has a structure of immediate importance for the construction of various types of lattices is given in detail. Finally, we discuss the lattice of fuzzy subgroups that classify and illustrate the fuzzy subgroups of G. We also use the *equivalence classes* to classify these fuzzy subgroups of G.

Chapter 5 serves as an intermediate section between the study of group theory and that of decision-making. We performed a similarity relation, which is a generalization of the equivalence relation that links group theory with decision -making processes.

Chapter 6 and 7 concentrate on the *decision-making* process. This has application to the economics topic of *capital budgeting*. In Chapter 5, we discuss *decision-making* in a fuzzy environment in general that will provide a procedure for applying the knowledge of the mathematics of fuzzy sets to *capital budgeting*. Chapter 6 explores the fuzzy equivalents of the classical *capital budgeting* methods. It can be used to evaluate and compare projects in which the *cash flows*, *duration* and *required rate of return* are given imprecisely, in the form of fuzzy numbers. We rank these investments according to Buckley's theory [5], in the order of their priority. Lastly, we explore the use of *triangular fuzzy numbers* as a decision-making process on capital budgeting.

Chapter 1

Fuzzy logic and sets

1.1 Fuzzy sets

1.1.1 Introduction

We define *crisp sets* to distinguish them from the fuzzy sets that are the central topic of this section. Take M with elements $a_1, a_2, ..., a_{10}$, so that $M = \{a_1, a_2, ..., a_{10}\}$ or symbolically $M = \{a_i | 1 \leq i \leq 10\}$.

If the a_i's are members of some more comprehensive class X of objects, then M can be represented with a characteristic function,
i.e $\forall x \in X$

$$\chi_M(x) = \begin{cases} 1, & \text{if } x \in M \\ 0, & \text{otherwise} \end{cases} \quad (1.1.1)$$

The above information then allows us to define what is called a *fuzzy set* without any confusion.

1.1.1 Definitions
By a fuzzy set of X we mean a pair (X,μ), where μ: $X \rightarrow I$.

If we take $x \in X$, then $0\leq \mu_A(x) \leq 1$ is called the *degree of membership* of x to the fuzzy subset A of X.

If $\mu_A(x) = 1$, we say x belongs to A absolutely. If $\mu_A(x) = 0$, we say x does not belong to A absolutely.
If $\mu_A(x)$ (degree of membership) takes only $\{0,1\}$, $\forall x \in X$, then A is called a *crisp set.*

By collecting elements of Universe discourse X, of equal membership values, we get a partition of X whose blocks consist of elements with the same membership. This partition is called the *kernel of fuzzy sets.*

When we use the usual ordering of the unit I, we speak of an element $x_1 \in X$ as belonging to the fuzzy set A more than another element $x_2 \in X$ to the fuzzy set A, i.e
$x_1 \in X$ more than $x_2 \in X \leftrightarrow \mu_A(x_1) \geq \mu_A(x_2)$

A fuzzy set A of X, whose membership function is defined by:

$$\mu_A(\text{x}) = \begin{cases} \lambda, & \text{if x = a} \\ 0, & \text{otherwise,} \end{cases} \quad (1.1.2)$$

where $0< \lambda <1$

is called a *fuzzy point* of X. When $\lambda = 1$, we get a crisp point.

1.1.2 Note

In this discussion we denote a fuzzy point by a^{λ}.
An empty fuzzy set is one whose membership value is 0 $\forall x \in X$. In contrast to the above, when $\mu_A(\text{x})=1$, then the fuzzy set A is the *Universal set* X.
We say a fuzzy set A is normal, if its membership function assumes 1 at a

point x_0 in X for at least one x_0.
A normal $\rightarrow$ hgt(A) = 1 where hgt(A) is the highest membership value of A, but the converse is not true.
In the following example, take $X = N$ and $\mu(n) = 1$, now if you take any set of values from N, you can easily verify the above claim.

1.2 Alpha-cuts

Alpha-cuts are of considerable importance in many discussions of fuzzy sets. Let A be a fuzzy set of X, $\alpha \in I$ =[0,1], then we define a weak α-cut as:

$$\mathrm{A}^{\geq\alpha} = \{x \in X | \mu_A(x) \geq \alpha \}. \quad (1.2.3)$$

In a similar way we can define a strong α-cut as:

$$\mathrm{A}^{>\alpha} = \{\mathrm{x} \in X | \mu_A(x) > \alpha\}. \quad (1.2.4)$$

1.2.1 Note

- $\mathrm{A}^{>1} = \emptyset$.
- $\mathrm{A}^{>0} = \mathrm{Supp}(A)$.
- The strong α-cut $A^{>1}$ = core of the A, i.e $A^{>1} = \emptyset \longleftrightarrow$ hgt(A)= 1.

- If we take $\alpha, \beta \in$ [0,1] such that if $0 \leq \alpha \leq \beta \leq 1$, then $A^{\geq\alpha} \supseteq A^{\geq\beta}$.
- A fuzzy set can be characterized by a nested sequence of crisp sets.
- The fuzzy set's core is included in every set of the sequence.
- The Universal set X includes every set of the sequence.

1.3 Operations with fuzzy sets

1.3.1 The basic set algebraic operations

We only look at three basic set operations and later connect them with alpha-cuts.

The *union* of two fuzzy sets A, B is defined by:

$$\mu_{A\cup B}(\mathrm{x}) = max\{\ \mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x). \quad (1.3.5)$$

Similarly, the *intersection* of two fuzzy sets A, B is defined by:

$$\mu_{A\cap B}(\mathrm{x}) = min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x)\ \forall x \in X. \quad (1.3.6)$$

Finally, the compliment of a fuzzy set A, denoted by A^c (relative to the Universe of discourse X), is defined by:

$$\mu_{A^c}(\text{x}) = 1 \text{ - } \mu_A(\text{x}) \ \forall x \in X. \ (1.3.7)$$

We can connect these operations with weak α-cut as follows:

$$\begin{aligned}(\text{A} \cap B)^{\geq\alpha} &= A^{\geq\alpha} \cap B^{\geq\alpha}\\ (A \cup B)^{\geq\alpha} &= A^{\geq\alpha} \cup B^{\geq\alpha}\\ (A^c)^{\geq\alpha} = A^{\leq 1-\alpha} &= \{x \in X | \mu_A(x) \leq 1-\alpha\}. \ (1.3.8)\end{aligned}$$

Similarly, we can connect these operations with strong α-cut.

1.4 Inclusion for fuzzy sets

Given two fuzzy subsets A, B of X, we call A a subset of B, denoted by $A \subset B \Leftrightarrow$ the membership degree μ_A(x) is never greater than the membership degree μ_B(x), i.e

$$\text{A} \subseteq B \longleftrightarrow \mu_A(x) \leq \mu_B(x) \text{ for all } x \in X. \ (1.4.9)$$

1.5 Fuzzy numbers and their intervals

We will define fuzzy (real) numbers in a mathematical manner and establish their properties in order to institute a principle of working with them. We also develop a method of combining two entities that are stated to be imprecise. Once we adopt this method, it will be a guideline whenever we apply the procedure in real situations. Definitions and notations that are likely to be used in the later chapters will be introduced.

1.5.1 Note

We use $\mathbb{F}(X)$ to denote the class of all fuzzy subsets of the universe of discourse X.
Fuzzy (real) numbers are determined as imprecise values in the interval [0,1]. With the aid of [2], we define fuzzy (real) numbers below. We have:

1.5.2 Definition

A fuzzy set $A \in \mathbb{F}(\mathbb{R})$ is called a *fuzzy (real) number*, if and only if A is convex and if there exists exactly one real number a with $\mu_A(a) = 1$.

The shapes of membership functions of fuzzy numbers are *convex*. In general discourse, *convex* means a line that curves outwards. Below is a definition of the term convex with regard to fuzzy numbers.

A fuzzy number $A \in \mathbb{F}(\mathbb{R})$ is called *convex*, if and only if all (strong) α-cuts of A are intervals, i.e themselves convex sets in the usual sense.

In the case of "*betweenness*", we have a notion of interval $[a, b]$ with end points a, b, so that if $c \in [a, b]$, this implies that

$$\mu_A(c) \geq min\{\mu_A(a), \mu_A(b)\}. \quad (1.5.10)$$

The above inequality is also regarded as the definition of the convexity of A.

When A is only convex and normal, then A is called a *fuzzy interval.*

1.5.1 Fuzzy numbers and operations performed on them

Two things can only combine with each other if we apply an operation between them. The basic arithmetic operations involving fuzzy numbers are detailed below.
These operations can allow us to exercise the operations between the fuzzy numbers. These are the basic fuzzy arithmetics:

a) The sum $S{:=}\ A \oplus B$. So the membership function of a sum of two fuzzy numbers or intervals is determined as:

$\mu_S(a) = \text{supmin}_{x\in\mathbb{R}}\{\mu_A(x), \mu_B(a-x)\}$ (1.5.11)

for all $a \in \mathbb{R}$.

b) The difference $D{:=}\ A \ominus B$. The membership function of a difference of two fuzzy numbers or intervals is defined as:

$\mu_D(a) = \text{supmin}_{x\in\mathbb{R}}\{\mu_A(x), \mu_B(x-a)\}$ (1.5.12)

for all $a \in \mathbb{R}$.

c) The product $P{=}A \odot B$. The membership function of two fuzzy numbers or intervals is given as:

$\mu_P(a) :=\text{supmin}_{x,y\in\mathbb{R},a=xy}\{\mu_A(x),\mu_B(y)\}$ (1.5.13)

for all $a \in \mathbb{R}$.

d) The quotient of fuzzy numbers needs some restriction, for example exclude 0 from the support of the divisors set. If we use the above condition and assume that $0 \notin$ Supp (B), where B is a divisors set, then we define the quotient $Q := A \star B$ 'where $\star$ denotes the division operator', this yields the following membership function:

$\mu_Q(a) = \sup\ \min_{x,y\in\mathbb{R},a=x/y}\{\mu_A(x),\mu_B(y)\}$(1.5.14)
for all $a \in \mathbb{R}$.

Many laws that hold for the arithmetics of real numbers also hold for the fuzzy intervals but the distribution property does not hold unrestrictedly. We are not spelling out the procedures, but rather assuming that using the extension principle, we can apply many operations to this new mathematical system.

There are many forms of fuzzy numbers. These are associated with the following formats: *sine numbers, bell shape*, *polygonal, trapezoids, triangular* and so on.

The triangular fuzzy numbers are considered to be the most interesting for the financial field.

1.6 Triangular Fuzzy numbers(TFN)

The fuzzy numbers associated with a triangular shape are classed as triangular fuzzy numbers. In this section, we spell out the algebraic definition and the rough picture of the triangular fuzzy numbers.

1.6.1 Note

If the support of A is the interval (a_1, a_2), then the restrictions of the membership function μ_A: $\mathbb{R} \rightarrow$ [0,1] to the intervals (a_1, a_o) and (a_o, a_2) are denoted by μ_A^L, μ_A^R respectively. This we denote with $L|R$ - representation.

A fuzzy number with $L|R$ - representation is called a *triangular fuzzy number.* If we take $a_1, a_2, a_3 \in \mathbb{R}$ and Fig.5, then the membership function for the triangular fuzzy number takes the following shape:

$$\mu_A(\mathrm{x}) = \begin{cases} 0, & x = a_1 \\ \frac{x-a_1}{a_2-a_3},, & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2},, & a_2 \leq x \leq a_3 \\ 0, & x > a_3 \end{cases} \qquad (1.6.15)$$

The geometrical shape for the triangular fuzzy number in general is given in Fig.5 in the appendix to this thesis.

Sometimes, when certain aspects are fuzzified, we still need to look at them in normal circumstances, so we defuzzify these outputs. Below we state the difference between the fuzzification and defuzzification process. This will be useful in chapter 7 of this thesis.

Fuzzyfication is the mapping of the real numbers domain (generally discrete) to the fuzzy domain.

Defuzzification is the operation in which the value of the output linguistic value, inferred by the fuzzy rules, will be translated in to a discrete value (Shaw,1999).

1.7 The fuzzy logic

Fuzzy logic is a bridge that connects the human thinking to the machine's logic. Fuzzy logic was primarily designed for the representation and reasoning of knowledge that is inherently vague and linguistic in nature. Its fundamental character is that it is only to a certain degree that a property is characteristic of an object.

We pictorially show the nature of fuzzy logic in Fig.6 in the appendix . If we look at the diagram, there is an element "c" that partially belongs to set "A" that is $0 \leq$degree of belonging ≤ 1, so $mA(c) = 0.5$, where mA is the "*membership*" function. Fuzzy logic has a significant application in decision-making processes in financial fields.

Chapter 2

Equivalence relation on fuzzy subsets and fuzzy subgroups

2.1 Equivalence relation on fuzzy subsets

2.1.1 Introduction.

In this chapter we wish to discuss how to relate two things that are regarded as equal to some extent. We do this by using the notion of an equivalence relation. This notion has many applications in mathematics in general. Let us look at the general discussion on equivalence relation and also define the appropriate terms that arise in the context.

Further, we discuss the implication of equivalence on fuzzy sets and fuzzy subgroups. We will see later on that these equivalence relations provide a setting for classifying the fuzzy subgroups of G. Dengang [10], discusses the conditions under which the equivalence relation of fuzzy sets can be equivalently described by their level sets (see [10]).

In order to establish the required implications of equivalence relations on fuzzy subgroups, we will need to look at their effect on fuzzy subsets and consider some of the properties of subsets. Later we extend it to fuzzy subgroups.
Using equivalence relation, we classify fuzzy subgroups of finite abelian groups in some special cases. Many researchers consider isomorphism in the initial

stages but later they realize that equivalence is finer than isomorphism (see [37], [34], [18]).
A paper by Murali and Makamba [26], provides a comprehensive overview of equivalence of fuzzy subsets.
In their paper, a relation is defined in the following context:

2.1.1 Definition

The relation $\sim$ on I^X is defined as:

$$\mu \sim \nu \text{ if and only if}$$

(i) for all $x, y \in X$, $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$ and
(ii) $\mu(x) = 0$ if and only if $\nu(x) = 0$.
$iii)$ $\mu(x) = 1$ if and only if $\nu(x) = 1$.

2.1.2 Note

1. It is easy to check that this relation is indeed an equivalence relation on I^X.

2. When restricted to 2^X, this relation coincides with equality of subsets.
Part (ii) in the definition 2.1.1 is important for the equivalence relation application, as can been seen in the following example:

Consider $X_3 = \{e, g, g^2, h, gh, g^2h\}$, define two fuzzy subsets γ and β as:

$$\gamma(\mathrm{x}) = \begin{cases} 1, & \text{if } x = e \\ \frac{1}{2}, & \text{if } x = g, g^2 \\ \frac{1}{3}, & \text{otherwise} \end{cases} \quad (2.1.1)$$

$$\beta(\mathrm{x}) = \begin{cases} 1, & \text{if } x = e \\ \frac{1}{2}, & \text{if } x = g, g^2 \\ 0, & \text{otherwise} \end{cases} \quad (2.1.2)$$

The Supp $\gamma \neq$ Suppβ i.e $\frac{1}{3} \neq 0$, meaning that condition (ii) is not satisfied, while the rest of the conditions are satisfied for all $x \in X$.Thus $\gamma \nsim \beta$.

The equivalence class containing μ is denoted by $[\mu]$, with respect to the relation above.

Equivalence relation on the fuzzy subsets can be studied further. We use these equivalence relations to study the equivalence of fuzzy subgroups of G. The group structures can be classified by assigning equivalence classes to its fuzzy subgroups. As the equivalence of fuzzy subgroups can be used in the classification of fuzzy groups of some special abelian groups, [39] determines whether the method of classification can be more generally applied.

We give more information on the fuzzy equivalence relations on the subsets of X below.

In their paper [32], B. Seselja and A. Tepavcevic have outlined a corresponding equivalence relation on the related fuzzy power set.
Furthermore the collection of classes in this relation can be ordered and it can also be verified that these classes are lattices. (For more information see [32]).

The discussion on a complete lattice in Gratzer [14], together with the definition in 2.1.1 are useful tools for the following proposition.
We examining whether a relation $'\sim'$ is an equivalence relation on $\Im_L(X)$, where $\Im_L(X)$ is the collection of all fuzzy subsets on X whose co-domain is L, the lattice.
According to [32], this has immediate application in the following proposition.

2.1.3 Proposition
The relation $'\sim'$ is an equivalence relation on $\Im_L$(X).
Proof : *See* [32].

The relation between fuzzy equivalence relation and crisp equivalence relation on X is called a *cut-relation* of R, where R stands for the relation. The cut-relation R is also called the support of the subset X.

Having talked about the equivalence relation, we now look at the correspondence between two related fuzzy subsets and their images. The following proposition is immediate:

2.1.4 Proposition

$If \gamma \sim \beta$, then $|Im(\gamma)| = |Im(\beta)|$.
Proof : *See* [26].

Remarks on the above propostion:
The converse to proposition 2.1.4 is not true, as shown by the following example:

Take $Y_3 = \{e, g, g^2, h, gh, g^2h\}$, the fuzzy subsets γ and β are defined as:

$$\gamma(\text{y}) = \begin{cases} 1, & \text{if } y = e \\ \frac{1}{2}, & \text{if } y = g \\ \frac{1}{3}, & \text{otherwise} \end{cases} \quad (2.1.3)$$

$$\beta(\text{y}) = \begin{cases} 1, & \text{if } y = e \\ \frac{1}{2}, & \text{if } y = gh \\ \frac{1}{3}, & \text{otherwise} \end{cases} \quad (2.1.4)$$

2.1.5 Note

The images and support of two fuzzy subsets are equal, however $\gamma(g) > \gamma(gh)$, but $\beta(g) \not> \beta(gh)$. So, $\gamma \nsim \beta$.

The α-cut represents the equivalence classes in some sense. The equivalence classes classify the fuzzy subsets of X. The classification determines the nature of the equality of two distinct fuzzy subsets. The proposition below helps to prove that, if the α-cuts of two distinct fuzzy subsets are equal, the relation defined in 2.1.1 holds between those two fuzzy subsets.

2.1.6 Proposition
(Murali and Makamba, [26])
Let γ and β be two fuzzy subsets of X. Suppose for each $l > 0$ there exists $k > 0$ such that $\gamma^l = \beta^k$. Then $\gamma \sim \beta$.

Proof:
If Supp(γ) $= \emptyset$, then Supp(β) must also be empty. Thus $\beta(y) = 0 = \gamma(y)$ for all $y \in Y$. Hence $\gamma \simeq \beta$ trivially. Now let $y \in Supp\gamma$. Then $y \in \gamma^l$ for some $l > 0$. By hypothesis, there is a $k > 0$, such that $\gamma^l = \beta^k$. This implies $\beta(y) \geq k > 0$. So $y \in Supp\beta$. Similarly, we can show that $Supp\beta \subseteq Supp\gamma$. Therefore $Supp\gamma = Supp\beta$.

Now let $l = \gamma(y) > \gamma(x)$. By hypothesis, $y \in \beta^k = \gamma^l$. If $\beta(y) \leq \beta(x)$, then $\beta(x) \geq k$, implying $x \in \beta^k = \gamma^l$, which is a contradiction. Hence $\beta(y) > \beta(x)$. Similarly , $\beta(y) > \beta(x)$ implies $\gamma(y) > \gamma(x)$. □

2.2 Flags and Keychains

The above equivalence relation on fuzzy subsets has wider implications and offers scope for further discussion. We introduce new terms to characterize the properties of fuzzy subsets of X, namely *flags* and *keychains*. We start with their definitions and elaborate some of their properties.

2.2.1 Note
Unless otherwise stated we fix n in this discussion.

2.2.1 Flags

2.2.2 Definition

A *flag* $\mho$ on a set X is a maximal chain of subsets of X such that $X_0 \subset X_1 \subset ... \subset X_n = X$.

The maximal chains are given by permutations of the elements of X. We permute the subsets of X in $n!$. By permutation P of a set X, we mean a one to one mapping of X onto itself.
The length of this maximal chain of X is $(n + 1)$.

Consider an example, take $X = \{a, b, c, d, e\}$. We obtain maximal chains by permuting the elements of X; in this case we permute them $5! = 120$ times. The chain has a length $(5 + 1) = 6$.
The notion of flags can be extended easily to the study of vectors and ideals.

2.2.2 Keychains

A finite n - chain is a collection of numbers on [0,1] of the form $1 > \beta_1 > \beta_2 > ... > \beta_n$, where β_n may or may not be zero.

2.2.3 Definition

An n-chain is called a *keychain* if $1 = \beta_0 \geq \beta_1 \geq \beta_2 . . . \geq \beta_n \geq 0$.

We denote a keychain by "$\jmath$".

The β_is are called *pins*. The pins that are interlocked are called *components*. But the value 1 standing alone is not regarded as a component. A keychain with k-distinct components is called a *k-pad*, where $(1 \leq k \leq n)$. We can obtain distinct fuzzy subsets of X by interchanging the β_is of a flag. So the above β_is are classified either by " = " or " > ".

In general pins are allocated with their positions. The length of the chain is equal to the number of positions available in an n-chain, which in our case is $n + 1$.

The following examples illustrate some of the above situations.

2.2.4 Examples

(1) $1 > \beta_1 = \beta_2 > \beta_3 = \beta_4 = \beta_5 > \beta_6 > 0$,

In this example we have a 4-pad keychain of a 7-chain.

(2) $1 = \beta_1 = \beta_2 = \beta_3 > \beta_4 > \beta_5 = \beta_6 = \beta_7$,

This has a 3-pad keychain of a 7-chain.

The *paddity* of the component is the number of pins found in the interlocked position forming a component.
We can see from the first example above, that the paddities of the components are 2, 3 and 1.

The next thing is to examine the role played by the index of a k-pad , but first we define it.
The index of a k-pad keychain is the set of paddities of various components of the keychain in which singleton components are ignored for the sake of simplicity.
The index of a keychain forms a partition of the number n. Looking at the example (1), the partition of 7 given by 2 + 3 + 1+ 1 corresponds to a 4-pad keychain whose index is (2,3). We can tell that the index and the set of paddities determine each other.

2.2.5 Definition

We define a *pinned-flag* to be a pair consisting of a flag $\mho$ and a keychain $\jmath$.

We represent the α-cuts ($0 \leq \alpha \leq 1$) of fuzzy subsets belonging to the same equivalence class as a pinned-flag. We can construct a fuzzy subset μ on X corresponding to a pinned-flag on X, given by

$$0^1 \subset X_1^{\beta_1} \subset ... \subset X_n^{\beta_n}, \text{ in the following manner,}$$

$$\mu(\text{y}) = \begin{cases} 1, & \text{if } X = 0 \\ \beta_1, & \text{if } x \in X_1 \backslash \{0\} \\ \vdots & \\ \beta_n, & \text{if } x \in X_n \backslash X_{n-1} \end{cases} \quad (2.2.5)$$

The following proposition by Murali and Makamba in [26], discusses some conditions of pinned-flags corresponding to two fuzzy subsets with equivalent relations.

2.2.6 Proposition

Suppose the pinned-flags corresponding to two fuzzy subsets γ and β, are

$$(\mho_\gamma, \jmath_\gamma):\ P_0^1 \subset P_1^{\lambda_1} \subset \cdots \subset P_n^{\lambda_n} \ (2.2.6)$$

and

$$(\mho_\beta, \jmath_\beta):\ Z_0^1 \subset Z_1^{\sigma_1} \cdots \subset Z_m^{\sigma_m} \ (2.2.7)$$

where the λ_i and the β_i are all distinct.

Then $\gamma \sim \beta$ on X if and only if:
(i) $n = m$;
(ii) $P_i = Z_i$ for i = 0,1, ...,n;
(iii) $\lambda_i > \lambda_j$ if and only if $\sigma_i > \sigma_j$ for $1 \leq i, j \leq n$ and $\lambda_k = 0$ if and only if σ_k = 0 for some 1$\leq k \leq n$.

Proof : *See* [26].

This clarifies that two pin-flags are comparable. The above conditions clarify the equivalence relation between $(\mho_\gamma, \jmath_\gamma)$ and $(\mho_\beta, \jmath_\beta)$.

2.2.7 Note

1. If $\lambda_n = 0$, then the Supp $\mu \subset X$, otherwise ($\lambda_n \neq 0$), then Supp $\mu = X$.

2. If $\lambda_1 = \lambda_0 = 1$, the $\mu^1 = X_1$.

3. If $\lambda_0 = \lambda_1 = \lambda_2 = 1$, then $\mu^1 = X_2$ but if $\lambda_2 \neq 1$ then $\mu^1 = X_1$.

2.3 Equivalence points of equivalent fuzzy subsets

2.3.1 Equivalence of fuzzy points

The definition of fuzzy points is given in chapter 1 (def(1.1.1)). We determined the relationship between equivalent fuzzy subsets and equivalent fuzzy points. In this section we state the definition of a fuzzy point belonging to a fuzzy subset. Finally we look at the results given by distinct keychains relative to equivalence of fuzzy points. We take a chain of fuzzy subsets by specifying a crisp point from the chain and obtain distinct keychains that are being hosted by the crisp point.

Consider an example, by taking $g \in \mathbb{Z}$ such that

$$\mathrm{Z}_0 \subset \{g\} \subset \mathbb{Z}. \quad (2.3.8)$$

From the above chain we can deduce seven different keychains. These keychains represent the equivalence classes of fuzzy subsets. Table 1 contains the results for these keychains and their descriptions relative to the crisp point "g".

The example below discusses the nature of a fuzzy point belonging to a fuzzy subset under the defined equivalence above.

2.3.1 Example

Take $Y = \{y_1, y_2, y_3\}$, while μ and ν are defined as below:

$$\mu(\mathrm{y}) = \begin{cases} 1, & \text{if } y = y_1 \\ \frac{1}{2}, & \text{if } y = y_2 \\ \frac{1}{3}, & \text{otherwise} \end{cases} \quad (2.3.9)$$

$$\nu(\mathrm{y}) = \begin{cases} 1, & \text{if } y = y_1 \\ \frac{1}{4}, & \text{if } y = y_2 \\ \frac{1}{5}, & \text{otherwise} \end{cases} \quad (2.3.10)$$

We can tell from the example above that $\mu(y) > \mu(x)$ if and only if $\nu(y) > \nu(x)$ for $x, y \in X$ and $\mathrm{Supp}\mu = \mathrm{Supp}\nu$. So, $\mu \sim \nu$, but the fuzzy point y_2^λ for $\frac{1}{2} > \lambda > \frac{1}{3}$ belongs to μ and fails to belong to ν.

2.3.2 Note

1. The equivalent fuzzy point may belong to a given equivalent fuzzy subset, but may not belong to a given inequivalent fuzzy subset.

2. The equivalent fuzzy point may not belong to a given equivalent fuzzy subset, but may belong to a given inequivalent fuzzy subset.

The notion of fuzzy points and fuzzy points belonging to fuzzy subsets is not very useful to the study of equivalence of fuzzy subsets, as we illustrated in the example above.
So, Murali and Makamba [26] proposed a definition of a fuzzy point belonging to an equivalent class of fuzzy subsets in terms of pinned-flags, which is more useful to the study of equivalence of fuzzy subsets of X.

2.3.3 Definition
Suppose $(\mho_\mu, \jmath_\mu)$ is the pinned-flag corresponding to an equivalence class of fuzzy subset $[\mu]$ given by

$$(\mho_\mu, \jmath_\mu):\ X_0^1 \subset X_1^{\lambda_1} \subset . \ . \ . \subset X_n^{\lambda_n},\ (2.3.11)$$

Then we say an equivalence class of fuzzy point $[a^\lambda]$ belongs to $[\mu]$ if and only if $0 < \lambda \leq \lambda_i < 1$, where $0 < i \leq n$ is the least index with the property $a \in X_i$, but $a \notin X_{i-1}$.

The discussion on equivalence of fuzzy subsets of X, allows us to use this knowledge to study the equivalence of fuzzy subgroups of G. The equivalence of fuzzy subgroups will be discussed in detail in the next section.

2.4 Equivalence of fuzzy subgroups of G

2.4.1 Note
(1). In this thesis, the letter G will always denote a *group*, unless otherwise stated and the letter 'e' will always denote the *identity* of G.

(2). Unless otherwise stated, we fix $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, where p, q are distinct primes and n, m are any natural numbers.

We recall from sections 2.1, 2.2 and 2.3, where we discussed the equivalence relation on fuzzy subsets of X. This study of equivalence relations on fuzzy subsets of X, creates a basis for the study of the equivalence of fuzzy subgroups of a group G.

In this section, we take up the study of fuzzy subgroups of finite abelian groups. Because the number of crisp subgroups of a finite group G is also finite, it is possible from the point of view of equivalence to count the number of fuzzy subgroups of G up to equivalence. To be specific, we take a cyclic group $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, where p and q are two distinct primes, while n and m are fixed positive integers.

The goal is to classify all fuzzy subgroups of G, using equivalence classes of fuzzy subgroups. The fuzzy subgroups will be enumerated by flags and keychains (definitions given in 2.2.2 and 2.2.3 respectively).
Lastly, we construct fuzzy subgroups using lattice diagrams. We have already defined the appropriate terms for this study in the context of sets. Now we have to show how these satisfy the group structures together with their equivalence relations.

The equivalence of fuzzy subgroups is discussed in various papers by Murali and Makamba. In their first paper [26], they considered a finite abelian group of the form $G = \mathbb{Z}_p$, where p is a prime. In this case they classify fuzzy subgroups of this p-group.
Further they considered groups of the form $G = \mathbb{Z}_{p^n} + \mathbb{Z}_q$, where p and q are distinct primes and n is any natural number. This information serves as the fundamental basis for the equivalence of fuzzy subgroups of G. In another direction, Ngcibi [29] conducted research on fuzzy subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_{p^m}$.

We define the equivalence relation on the fuzzy subgroups of G as follows:

2.4.2 Definition

[34] Two fuzzy subgroups μ and ν of G are equivalent and we

write $\mu \sim \nu$ if and only if

(i) for all $x, y \in G$, $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$ and
(ii) $\mu(x) = 0$ if and only if $\nu(x) = 0$.

Some of the terms such as keychain, flag and maximal chain are already defined in set context. We just have to attach a group structure to such a fuzzy set to make them relevant to the present discussions.
In the first instance, we take the group G as a general case and later on consider particular examples for illustration.
Consider $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$. From an algebraic point of view, it is known that G is a cyclic group of rank 2. It has an order $u = p^n q^m$, where p and q are two distinct primes and n and m are any two natural numbers.

We can determine the subgroups of a group G if n and m are known. Since G is a finite cyclic group of order $\mathrm{o}(G) = p^n q^m$, for every divisior d of $\mathrm{o}(G)$, there is a unique subgroup of G of order d by a proposition in [15], page 92. Now, clearly there are $(n+1)(m+1)$ number of divisors of $\mathrm{o}(G)$. Therefore there are $(n+1)(m+1)$ many subgroups of G. So, in general $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ has $(n + 1)(m + 1) = nm + n + m + 1$ subgroups.

The following proposition can guide us to characterize the maximal chains of the fuzzy subgroups of a group G.

2.4.3 Proposition

[44]
If, in a group G, the length of the longest maximal subgroup chain with endpoints $\{e\}$ and G is n, the order of any fuzzy subgroup of G cannot be greater than n+1.

The above proposition is relevant when we come to consider the structure of maximal chains of the fuzzy subgroups of G.

2.5 Maximal chains of subgroups of G

One way to characterize maximal chains in G is to study the linear orderings of the lattice of subgroups. This study reveals that the maximal chains are obtained by permutations of $n+m$ objects with n of them identical objects say $p \quad p \quad . \quad . \quad .p$ and m of them are identical objects say $q \quad q \quad . \quad . \quad .q$.
We take a note from the characterizing maximal chain of subgroups from a subgroup lattice of groups of known order.
Here we consider a general case. Now $p^n q^m$ can be listed as follow:

$$0\underbrace{p \quad p \quad p \quad . \quad . \quad . \quad p}_{n} \qquad \underbrace{q \quad q \quad q \quad . \quad . \quad . \quad q}_{m} \quad (2.5.12)$$

The above listing gives rise to the following proposition.

2.5.1 Proposition
The list of orders of subgroups of G gives rise to the following maximal chain:

$$\{0\} \subset \mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset ... \subset \mathbb{Z}_{p^n} \subset \mathbb{Z}_{p^n q} \subset ... \subset \mathbb{Z}_{p^n q^m} \cong G, \quad (2.5.13)$$

where p and q are distinct primes while, m and n are two fixed positive integers.
Proof:

Suppose there is a subgroup H of G whose order is d, such that $\mathbb{Z}_{p^i} \subset H \subset \mathbb{Z}_{p^{i+1}}$, for some $0 \leq i \leq n-1$. Then, by Langrage theorem, $d|p^{i+1}$. This means

$d = p^j$ for some $j < i+1$.
Also, $\mathbb{Z}_{p^i} \subset H$ implies $p^i|d$. This implies that $i < j < i+1$. This is a contradiction.

Similarly, if we insert H anywhere else in the equation, then the claim given by equation 2.5.13, holds the same conclusion. So, the list of orders of subgroups of G gives rise to the above mentioned maximal chain. □

Now when we permute the subgroup orders of G, we obtain different maximal chains, for example:
$0\ p\ \ p\ \ p\ \ .\ \ .\ \ .\ \ p\ q\ \ q\ \ q\ \ .\ \ .\ \ .\ \ q$ can give rise to a maximal chain by shifting p in the second position of $p's$ to the second position of $q's$ and vice versa as follows:
$0\ \ p\ \ q\ \ .\ \ .\ \ .\ \ p\ \ q\ \ p\ \ .\ \ .\ \ .\ \ q.$

Other maximal chains can be obtained by doing the same scenario to all the $p's$ and $q's$.
By continuing swopping $p's$ and $q's$ we obtain different maximal chains. So, by symmetry we can swop $p's$ and $q's$ without affecting the number of maximal chains.

Now if we consider the following maximal chain $\{0\}\subset H_1 \subset H_2 \subset ... \subset H_{n+m} = G$, this has the following subgroup orders $1|d_1|d_2|...|d_{n+m} = p^n q^m$. Therefore each d_i must be of the form $p^{i_1}q^{i_2}$, where $0\leq i_1 \leq n$ and $0\leq i_2 \leq m$. This still gives rise to equation 2.5.13. The $\{0\}$ cannot be permuted with other subgroups because of containment, for example the following situation is impossible $\mathbb{Z}_p \subset \{0\}$.

All the subgroups with order power of p and power of q together can be permuted $(n+m)!$ times, while among the $p's$ only, they will be permuted $n!$. Similarly, among the $q's$, order will be permuted $m!$ times. Therefore, for $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, we permute it by $\frac{(n+m)!}{n!m!}$. We can interpret the above statement by saying G has $\frac{(n+m)!}{n!m!}$ maximal chains. They are all of the same length with $(n+m+1)$ components.

In [27], Murali and Makamba established a different formula and a technique for finding the number of maximal chains of G.

2.6 Maximal chains of subgroups of $\mathbb{Z}_{72}$

Now we take a particular case of G, to demonstrate the above discussions. Suppose $G = \mathbb{Z}_{72} = \mathbb{Z}_{2^3} + \mathbb{Z}_{3^2}$ with $p = 2$, $q = 3$ and $n = 3$, $m = 2$. This group has an order of $u = 72$.

By isomorphism, $\mathbb{Z}_{72}$ and $\mathbb{Z}_{2^3} + \mathbb{Z}_{3^2}$ are isomorphic to each other. So, these two groups share the same group properties. Not all subgroups of G are comparable with respect to containment or inclusion. Therefore the subgroups of G do not form a chain but a lattice.

The question we have to ask at this stage is how many subgroups of $\mathbb{Z}_{72}$ are possible.
We answer this by considering $72 = 3^2 2^3$, and take $p = 2$ and $q = 3$.
So, the subgroups are listed with respect to their orders as follows:

$\{0\}$, $\{p\}$,$\{p^2\}$, . . . $\{q\}$, $\{q^2\}$,. . ., $\{p^i q^j\}$,. . .,$\{p^n q^m\}$.
for $0 \leq i \leq j \leq n$

The possible orders of the subgroups that are divisors of 72 are {1,2,3,4,6,8,9,12, 18,24,36,72}. Using the general expression of finding the number of subgroups of G, we find that G has (2+1)(3+1) = 12 subgroups in numbers. The maximal chains are all of the same length with (2 + 3 + 1) = 6 components.

Proposition 2.4.3 throws some useful light on the study of maximal chains of fuzzy subgroups of G. For any natural number n, and m =0, it is obvious that G has only one maximal chain, namely $\mathbb{Z}_{p^n} \supset \mathbb{Z}_{p^{n-1}} \supset ... \supset \mathbb{Z}_p \supset 0$, which we depict in Fig. 1.

Fig. 2 shows a crisp subgroup lattice diagram of G. Figures (1) and (2) diagrammatically represent the subgroup lattice. We follow a certain procedure to represent the fuzzy subgroups on the lattice diagram.

The subgroup-lattice diagram of $G = \mathbb{Z}_{72}$ consists of different chains of fuzzy subgroups of G and is depicted in Fig. 3. The group $G = \mathbb{Z}_{72}$ has the following maximal chains with the following group inclusion:
$\{0\} \subset \mathbb{Z}_3 \subset \mathbb{Z}_9 \subset \mathbb{Z}_{18} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72}$.
$\{0\} \subset \mathbb{Z}_3 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72}$
$\{0\} \subset \mathbb{Z}_3 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72}$
$\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72}$
$\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72}$
$\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72}$
$\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72}$
$\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72}$
$\{0\} \subset \mathbb{Z}_3 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{18} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72}$
$\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_6 \subset \mathbb{Z}_{18} \subset \mathbb{Z}_{36} \subset \mathbb{Z}_{72}$

From the subgroup-lattice diagram of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, we calculate the number of distinct fuzzy subgroups of G, using the counting principle, for small values of n and m.

2.7 Pinned-flag of G

There are different ways of placing things in a particular order, either based on their size, colour etc. In our discussion, we classify the fuzzy subgroups of G using keychains. The definition of a keychain is given in 2.2.3. The keychains are written in descending order of magnitude. Various keychains are distinct from each other if they are made up of different pins.

A *flag* on a group G is a maximal chain of subgroups of G, such as $\mathbb{Z}_0 \subset \mathbb{Z}_1 \subset \mathbb{Z}_2 \subset ... \subset \mathbb{Z}_n = \mathbb{G}$.

Now we define a *pinned-flag* of G to be a pair$(\mho, \jmath)$, consisting of a flag of subgroups and a keychain.

Given a keychain $1\lambda_1\lambda_2...\lambda_n$, where the λ_i's are not all distinct, we can construct a fuzzy subgroup μ on G corresponding to a pinned-flag on G given by

$$0^1 \subset G_2^{\lambda_1} \subset ... \subset G_n^{\lambda_n}$$

as follows:

$$\mu(x) = \begin{cases} 1, & x = 0 \\ \lambda_1, & x \in \mathbb{Z}_1 \backslash \{0\} \\ \lambda_2 & x \in \mathbb{Z}_2 \backslash \mathbb{Z}_1 \\ \lambda_3 & x \in \mathbb{Z}_3 \backslash \mathbb{Z}_2 \\ \lambda_4 & x \in \mathbb{Z}_4 \backslash \mathbb{Z}_3 \\ \lambda_5, & x \in \mathbb{Z}_5 \backslash \mathbb{Z}_4 \end{cases}$$

In this case $\mathbb{Z}_5 = G$ stands for the whole group.
If we take a finite abelian group, then the Im$\mu = \mu(G)$ is a finite subset of I. For a fuzzy subgroup μ of G with α-cuts of μ, elements of Imμ give rise to a finite chain of subgroups of G.

If we consider a weak α-cut, the above statement reflects the following: for $\alpha \leq \beta$ in I, its membership function with α-cuts has the following implication $\mu^\alpha \supseteq \mu^\beta$.

Now suppose Im$\mu = \{\lambda_0 = 1, \lambda_1, \lambda_2, ..., \lambda_n \geq 0\} \subset I$. By taking the definition of the support of μ of G into consideration, we can deduce the following: if $\lambda_n = 0$ for some n, then the Supp$\mu \subset G$, while if $\lambda_n \neq 0$ for each n, then the Supp$\mu = G$.

If we assume that $\lambda_0 > \lambda_1 > ... > \lambda_n \geq 0$, its corresponding α-cuts form a maximal chain of subgroups of G and is denoted by:

$$\mu^{\lambda_0} \subset \mu^{\lambda_1} \subset ... \subset \mu^{\lambda_n} = G.$$

Here is an example:
We assume λ, β and γ to be the real numbers in the interval [0,1], such that $0 < \gamma < \beta < \lambda \leq 1$. One of the maximal chains of subgroups of $G = \mathbb{Z}_{72}$ is $\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \mathbb{Z}_{24} \subset \mathbb{Z}_{72}$. A fuzzy subgroup is obtained with membership values given by $1 \geq \lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 > \lambda_5$ where $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 =$

β, $\lambda_4 = \gamma$ and $\lambda_5 = 0$

Chapter 3

A brief introduction to finite abelian groups

In this section we discuss the structure of an arbitrary finite abelian group. We use the standard notation and terminology accepted in group theory. We present the main definitions as well as the theory necessary for what is to follow.

3.1 Cyclic groups of any finite order

Let G be a group and $a \in G$. The set $< a > = \{ a^n \in G | \ n \in \mathbb{Z} \}$ is called the *cyclic subgroup* generated by a. The group G is called a *cyclic group* if there exists an element $a \in G$ such that $G = < a >$. In this case a is called a *generator* of G.

In short, a *cyclic group* is a group generated by a single element a.

A group G is *abelian* if $ab = ba$ for all $a, b \in G$. These groups play a special role in group theory.
Cyclic groups are abelian, but the converse is not true, for example, the rational numbers under addition are not cyclic, but are abelian.

The following proof shows that all cyclic groups are abelian:

Proof:
Let G be a cyclic group and a be a generator of G. Take $c, d \in G$, then there exist $x, y \in Z$ such that $c = a^x$ and $d = a^y$.
Since $cd = a^x a^y = a^{x+y} = a^{y+x} = a^y a^x = dc$, then G is abelian. □

Let's look at the definition of the order of a finite group, the order of an element of a group and some examples.

The *order* of a finite group is the number of its elements, while the *order* of an element $a \in G$ is the smallest positive integer n such that $a^n = e$.

As an example, cyclic groups of finite order n are denoted by $\mathbb{C}_n$, $\mathbb{Z}_n$ etc.

If we take $a \in G$ and $\text{o}(G) = n$ then its generator satisfies that $a^n = e$.
In general the integers 0,1,2,...,n-1 that are denoted by $\mathbb{Z}_n$ form a cyclic group of order n under addition modulo n. In this case 0 is the identity element of $\mathbb{Z}_n$.

3.1.1 Examples

- Groups of order 1 are trivial cyclic groups i.e { e }is cyclic.
- Groups of order say 6 are $\mathbb{C}_6$ and $\mathbb{S}_3$ and $\mathbb{S}_3$ is non - abelian, hence non cyclic.

If the subgroup $H =< a >$ from the definition above is finite, and $a^n = a^m$ for some positive natural numbers $n, m \in \mathbb{N}$ and $n > m$, then $a^n.a^{-m} = a^m.a^{-m}$. Therefore $a^{n-m} = e$.

Furthermore, if $n > m$ and the subgroup $< a >$ is finite, the set of integers $\mathbb{Z} =\{n \in \mathbb{N} : a^n = e \}$ is non empty, by *Well Order Principle*, so there is at least a positive integer r such that $a^r = e$. Therefore $\text{o}(a) = r$ such that $< a > = \{a^0 = e ,a ,a^2,...,a^{r-1}\}$.

We list the following useful properties of order of an element of G.

3.1.2 Proposition
Let a be an element of the group G.
(a) If a has finite order and k is any integer, then $a^k = e$, if and only if $o(a)/k$.
(b) If a has finite order $o(a) = n$, and for integers k, m we have $a^k = a^m$, then $k \equiv m \pmod{n}$, furthermore $| < a > | = o(a)$.

3.1.3 Proposition
Let G be a finite cyclic group with $o(G) = n$ a finite number. If $d|n$, then there exists exactly one subgroup of G of order d.
Proof [15], page 92.

Subgroups of cyclic groups are cyclic and all groups of prime order are cyclic.

A *simple group* is a group whose only normal subgroups are the trivial group of order one and improper subgroup consisting of the entire original group.

Consistently we can now define a simple cyclic group and a simple abelian group.

A *cyclic group* is simple if and only if the number of its elements is a prime. The *simplest abelian groups* are the cyclic groups of order n=1 or of a prime order p.

3.2 Isomorphism

If we speak about groups with different elements in nature, but equal in numbers, then we talk about groups that are approximately equal to each other. The expression "approximately equal to" is what we call isomorphism. Isomorphism allows us to treat certain groups as being alike.

We defined isomorphism with the aid of J.R. Durbin's book [12] on Modern Algebra as shown in the definition.

3.2.1 Definition

Let G be a group with operation $*$, and let H be a group with $\bullet$. An *isomorphism* of G onto H is a mapping θ: $G \to H$ that is one-to-one, onto and also satisfies:

$$\theta\ (a * b) = \theta(a) \bullet \theta(b)$$

for all $a, b \in G$.

If there is an isomorphism of G onto H, then G and H are said to be isomorphic and can be written $G \simeq H$.
As an example, we define θ from the set of integers to the set of even integers. This is one-to-one, onto and it preserves addition:
$\theta(n) = 2n$, thus θ is an isomorphism and $\mathbb{Z} \simeq 2\mathbb{Z}$.
In general, isomorphic groups share significant properties.

We elaborate a couple of properties in the form of theorems and propositions to describe isomorphic implications on group structures. The following theorem in [12] shows that any group isomorphic to an Abelian group must also be Abelian.

3.2.2 Theorem

If G and H are isomorphic groups and G is Abelian, then H is Abelian.

Proof:
We define the operation on G to be $\star$ and on H to be $\cdot$, respectively. And we set $\theta : G \to H$ to be an isomorphism.
Then for all $x, y \in H$, there are elements $a, b \in G$ such that $\theta(a) = x$ and $\theta(b) = y$.
Since θ preserves the operation and G is abelian,

$$x \cdot y = \theta(a) \cdot \theta(b) = \theta(a \star b) = \theta(b \star a) = \theta(b) \cdot \theta(a) = x \cdot y.$$

This shows that H is abelian. $\square$

Two groups that are isomorphic must have the same order. The following theorem in [12] gives some technical facts about isomorphism properties.

3.2.3 Theorem

Assume that G and H are groups and that $G \simeq H$,
(a) $|G| = |H|$
(b) If G is Abelian, then H is Abelian.
(c) If G is cyclic, then H is cyclic.
(d) If G has a subgroup of order n, $n > 0$, then H has a subgroup of order n.
(e) If G has an element of order n, then H has an element of order n.
(f) If every element of G is its own inverse, then every element of H is its own inverse.

The above properties give intuitive understanding of isomorphism properties in general. This will be instrumental in the later chapters.

In the same book of Modern Algebra [12], *isomorphism class* is defined as an equivalence class for the equivalence relation imposed on groups by isomorphism, as described in the following theorem.

3.2.4 Theorem

Isomorphism is an equivalence relation on the class of all groups.

Proof : *See* [12].

The best situation in mathematics with balance and checking is a scenario with proofs and examples. We illustrate an example below to elaborate roughly the above discussion.

3.2.5 Examples

We can find all abelian groups of order 8, up to isomorphism, and these are:

$$\mathbb{Z}_8,$$

$$\mathbb{Z}_4 \text{ x } \mathbb{Z}_2,$$
$$\mathbb{Z}_2\text{x } \mathbb{Z}_2 \text{ x } \mathbb{Z}_2.$$

It is difficult to show that two groups are not isomorphic, so we can only show that one group has properties that the other group does not have. By computing the characteristic factors, any abelian group can be expressed as a direct sum of cyclic subgroups, for example, finite group Z_2 x $\mathbb{Z}_4$ or finite group $\mathbb{Z}_2$ x $\mathbb{Z}_2$ x $\mathbb{Z}_2$.
Cyclic groups of the same order are isomorphic.

We intend to provide some groundwork that will serve as a basis for what is to follow. Our discussion is purely dominated by finite abelian groups. These groups can be any finite group of prime order p.

3.2.6 Theorem

If p is a prime and G is a group of order p, then G is isomorphic to $\mathbb{Z}_p$.

Proof:
Let the operation on G be $*$, and let a be a non-identity element of G. Then $< a >$ is a subgroup of G and $< a > \neq \{a\}$, so $< a > = G$.
Thus $G = \{e,a,a^2,...,a^{p-1}\}$.

Define θ:$G \to \mathbb{Z}_p$ by $\theta(a^k) = [k]$, this is well defined and one-to-one because $a^{k_1} = a^{k_2}$ iff $a^{k_1-k_2} = e$ iff $p/(k_1 - k_2)$ iff $[k_1] = [k_2]$.
Also, θ is onto.

Finally, if $a^m, a^n \in G$, then $\theta(a^m * a^n) = \theta(a^{m+n}) = [m + n] = [m]+[n]= \theta(a^m)\oplus\theta(a^n)$, therefore, θ is an isomorphism and $G \simeq \mathbb{Z}_p$. $\square$

This theorem classified all groups of prime order, and the principal problem of finite group theory is to do the same for groups of all finite orders.
This means that , when we ask for "all" groups having a property (such as being Abelian and of order n, for example), we are really asking for one group from each isomorphism class of groups with that property.

Some of this information on isomorphic groups will have useful applications

later when we explore fuzzy subgroups of finite abelian groups, which are a product of two distinct primes.
If θ, cited in theorem 3.2.2, is not assumed to be one-to-one and onto only, so that it preserves the operation, then the mapping is said to be a *homomorphism*.

3.2.1 Homomorphism

The central idea which is common to all aspects of modern algebra is the notion of homomorphism . We mean a mapping from one algebraic system to another algebraic system which preserves structure. This we make precise for groups, in the next definition.

3.2.7 Definitions
A mapping ϕ from a group G into a group G' is said to be a *homomorphism* if:

$$\phi(ab) = \phi(a)\phi(b) \quad (3.2.1)$$

for all $a, b \in G$.

We noticed that the product on the left side of the equation 3.2.1 is formed by the elements of G and the other on the right side is formed by the elements of G'.
Every isomorphism is a homomorphism, but the converse is not true. We illustrate this with an example.

For example, we take $n \in \mathbb{Z}$, define $\phi:\mathbb{Z} \to \mathbb{Z}_n$ by $\phi(a) = [a]$ for each $a \in \mathbb{Z}$. Then $\phi(a + b) = [a + b] = [a]+[b] = \phi(a) \oplus \phi(b)$ for all $a, b \in G$, so that ϕ is a homomorphism, but not isomorphism, then we have verified the above claim.

3.3 Equivalence relations

A relation is basically the way two things are connected. In this study we discuss a connection between groups. This notion of equivalence relations occurs not only in algebra, but throughout mathematics. In his book [15], Herstein defined equivalence relations in general, to suit any situation.

3.3.1 Definition

The binary relation $\sim$ on A is said to be an *equivalence relation* on A if for all a, b, c in A
1.$a \sim a$,
2.$a \sim b \to b \sim a$,
3.$a \sim b$ and $b \sim c \to a \sim c$.

The first of these properties is called *reflexivity*, the second *symmetry* and the third *transitivity*.
The notion of equivalence relation plays an important role later on, when it is applied to fuzzy subgroups of G to determine the number and nature of fuzzy subgroups of G with respect to this equivalence.
When working with an equivalence relation on a set A, it is often useful to have a complete set of *equivalence class representatives*.

3.4 Equivalence class

3.4.1 Definition

If A is a set and $\sim$ is an equivalence relation on A, then the *equivalence class* of $a \in A$ is a set $\{x \in A | a \sim x\}$.

We denote the equivalence class containing a by $[a]$.

As a simple example, let S be any set and define $a \sim b$, for $a, b \in S$, if and only if $a = b$. This is an equivalence relation on S. In this example, the equivalence class of $[a]$ consists merely of a itself.

In general, equivalence relations generalize equality and also measure equality up to some property.

3.5 Cyclic groups of order corresponding to rank two

The concept of cyclic groups in general was discussed in a previous section, and now we introduce the notion of cyclic groups of order corresponding to rank one and then we look at ranks of higher order.

A *rank* is defined as the cardinality of the largest set of linearly independent elements of the group.

The integer and rational numbers have rank one, as well as every subgroup of the rationals.

3.6 Finite abelian group

A finite abelian group can be expressed as a direct product of its sylow p-subgroups.
For example, $\mathbb{Z}/15\mathbb{Z} \simeq \mathbb{Z}_{15}$ can be expressed as the direct sum of two cyclic subgroups of power order 3 and 5, i.e $\mathbb{Z}_{15} = \{0,5,10\} \oplus \{0,3,6,9,12\}$.
We can do the same to any abelian group of order 15, this leads us to the remarkable conclusion that all abelian groups of order 15 are isomorphic.
We formulate the above description in a more general perspective in the proposition below.

3.6.1 Proposition

Let G be a finite abelian group. Then G is isomorphic to a direct product of cyclic groups $\mathbb{Z}_{n_1}$ x $\mathbb{Z}_{n_2}$ x ... x $\mathbb{Z}_{n_k}$ such that n_i $/n_{i-1}$ for i = 2,3,...,k.

We have learnt that isomorphic groups share significant properties. The following example demonstrates a group's isomorphism. We are considering a group of order 108.

3.6.2 Examples

To illustrate with some practical examples, we can consider the problem of finding all abelian groups of order 108 (up to isomorphism).
Consider the prime factorization of $108 = 2^2 3^3$, then we have two possible groups of order 4, these are $\mathbb{Z}_4$ and $\mathbb{Z}_2$ x $\mathbb{Z}_2$ and also we have possible groups of order 27, they are:

$$\mathbb{Z}_{27},$$
$$\mathbb{Z}_9 \text{ x } \mathbb{Z}_3,$$
$$\mathbb{Z}_3 \text{ x } \mathbb{Z}_3 \text{ x } \mathbb{Z}_3.$$

Both cases of groups classed together give us the following isomorphism classes:

$$\mathbb{Z}_4 \text{ x } \mathbb{Z}_{27}$$
$$\mathbb{Z}_2 \text{ x } \mathbb{Z}_2 \text{ x } \mathbb{Z}_{27}$$
$$\mathbb{Z}_4 \text{ x } \mathbb{Z}_9 \text{ x } \mathbb{Z}_3$$
$$\mathbb{Z}_2 \text{ x } \mathbb{Z}_2 \text{ x } \mathbb{Z}_9 \text{ x } \mathbb{Z}_3$$
$$\mathbb{Z}_4 \text{ x } \mathbb{Z}_3 \text{ x } \mathbb{Z}_3 \text{ x } \mathbb{Z}_3$$
$$\mathbb{Z}_2 \text{ x } \mathbb{Z}_2 \text{ x } \mathbb{Z}_3 \text{ x } \mathbb{Z}_3 \text{ x } \mathbb{Z}_3.$$

If only finite Abelian groups are considered, then the problem of determining all isomorphism classes is completely settled by the following theorem, which has been known since at least the 1870s.

3.6.1 Fundamental theorem of finite abelian groups

For each non zero finite abelian group G, there is exactly one list $m_1, m_2, ..., m_k$ of integers $m_i > 1$, each a multiple of the next, for which there is an isomorphism $G = \mathbb{Z}_{m_1} \oplus ... \oplus \mathbb{Z}_{m_k}$.
In this description, the first integer m_1 is the least postive integer $m = m_1$ with $m\mathbb{G}$=0, while the product $m_1 m_2 ... m_k$ is the order of G.

Or equivalently:

Every *finite abelian group* is the direct product of cyclic groups.

The above result has been proven showing that any finite abelian group G is the direct product of its sylow subgroups [15] p.109 - 111; the proof is omitted.
We stated that a group $G = < a >$ is a finite cyclic group generated by $a \in G$ with o$(G) = n$, so G in this case can be written in the form $G = \{a, a^2, ..., a^n = e\}$ and $a^m \neq a^l$ where m, l are two distinct natural numbers, so $G \cong \mathbb{Z}_n$.

3.6.3 Note

Every finite cyclic group of integers is of the form $\mathbb{Z}_n$ for some n $\in \mathbb{N}$.

We discuss general group implications to support the above defined groups circumstances. The definition for a group being a simple finite group is given on page 31. The finite simple cyclic groups are of the form $\mathbb{Z}_p$, where p is a prime. The only subgroups for $\mathbb{Z}_p$(simple group) are $\{0\}$ and $\mathbb{Z}_p$ itself.
The results on groups by Langrange have aided significantly in many instances to verify whether a certain entity is a subgroup of a group G. Now, if $\mathbb{Z}_{p^n}$ has order p^n and say any subgroup has order d, where $d = p^i$, i= 0,1,...,n, then by Langrange Theorem d/p^n.

The isomorphism classes for $\mathbb{Z}_{p^n}$ with order p^n are described below:

$$\{0\},$$

$$\mathbb{Z}_p = \{0, p^{n-1}, 2p^{n-1}, ..., (p-1)p^{n-1}\},$$

$$\mathbb{Z}_{p^2} = \{0, 1.p^{n-2}, 2.p^{n-2}, ..., p.p^{n-2}, (p+1).p^{n-2}, ..., 2p.p^{n-2}, (2p+1).p^{n-2}, ..., 3p.p^{n-2}, ..., (p^2-1).p^{n-2}\},$$

$$\mathbb{Z}_{p^3},$$

$$\vdots$$

$$\mathbb{Z}_{p^{n-1}},$$

$\mathbb{Z}_{p^n}$. (3.6.2)

The trivial groups in the case of 3.6.3 are the whole group $\mathbb{Z}_{p^n}$ and $\{0\}$.
The above-mentioned subgroups of $\mathbb{Z}_{p^n}$ have the following inclusion relation: $\{0\}\subset\mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset \cdots \subset \mathbb{Z}_{p^{n-1}}\subset\mathbb{Z}_{p^n}$.
The group $\mathbb{Z}_{p^n}$ has p^n elements that make up its subgroups.
Each subgroup of $\mathbb{Z}_{p^n}$ has p^i elements for i between 0 and n. Cyclic groups $\mathbb{Z}_{p^n}$ for p a prime $n \in \mathbb{N}$ are called *cyclic of rank* 1.
In general, abelian groups have finite rank equal to the number 'n'. The rank is an isomorphism invariant.

However, the converse is not true, there are many abelian groups of finite rank that are not finitely generated, the rank 1 group $\mathbb{Q}$ is one example and the rank 0 group given by a direct sum of countably many copies of $\mathbb{Z}_2$ is another example.

There is a complete classification of rank 1 group. Large ranks are more difficult to classify, and no current system of classifying rank 2 groups is considered very effective.
A certain philosopher cited that larger ranks, especially infinite ranks, are often the source of entertaining paradoxical groups. For instance for every cardinal d, there are many abelian groups of rank d that cannot be written as a direct sum of any pair of their proper subgroups.
These groups are not built up from other small groups. This shows that rank 1 groups are not the building blocks of all abelian groups.

Given integer $n \geq k \geq l$, there is a group G of rank n, such that for any partition of n into $r_1 + r_2, ..., +r_k = n$, where $r_i > 0$, G is the direct sum of groups, the first with rank r_1, the second with rank r_2,..., and the k^{th} with rank r_k.

However, cyclic groups of rank 2 are of the form $\mathbb{Z}_u$ where $u = p^n q^m$, where p and q are distinct primes and m, n are natural numbers. So, $\mathbb{Z}_u = \mathbb{Z}_{p^n}$ x $\mathbb{Z}_{q^m}$, by the Fundamental theorem of finite abelian groups. In this regard, $u = p^n q^m$ where p and q are relatively prime to each other. The subgroups $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{q^m}$

are simple subgroups of $\mathbb{Z}_u$.

3.7 Lattice

3.7.1 Lattice of subgroups

In general when you order items, it is desirable to have a limit either below or above or both. This takes us to the stage of talking about least upper bound and greatest lower bound. The pattern in which these items are arranged is called a *lattice*. The notion of the lattice was introduced by Peirce and Schroder and this had some importance for the investigation of the axiomatics of boolean algebras.

In his book, George Gratzer [14] gives a definition of a *lattice* in the following way:

3.7.1 Definition
$(S, \leq)$ is a *lattice* if and only if $(S, \leq)$ is a partially ordered set and each pair of elements $x, y \in S$, $\{x, y\}$ has a least upper bound and a greatest lower bound.

When least upper bounds or greatest lower bounds exist, they are always unique. In [39], Weinstein justifies this in the form of a theorem.

Throughout this study we will use the notations:

$$\text{a } \wedge b = \inf\{a, b\} \quad (3.7.3)$$

and

$$a \vee b = \sup\{a, b\} \quad (3.7.4)$$

Call $\wedge$ the meet and $\vee$ the join.
These operations $\wedge$ and $\vee$ are idempotent, commutative, and associative.

Viewing lattices as a system with two binary operations leads us to the concepts of *sublattice, homomorphism*, and *isomorphism*.
The book by P. Crawley and R.P. Dilworth on the algebraic theory of lattices defined a *sublattice*, which is a non empty subset M of a lattice L, as a sublattice of L if $x \vee y$, $x \wedge y \in M$, where $x, y \in M$.

If we speak of a *homomorphism*, which is a mapping f of a lattice L to a lattice M, then for all $x, y \in L$, $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$.

So, a one-to-one homomorphism is called an *isomorphism*. Two lattices L and M are isomorphic (in symbol, $L \cong M$) if there exists an isomorphism of L onto M.

3.7.2 Complete lattice

3.7.2 Definition

A complete lattice L is a partially ordered set (poset) in which every subset has a least upper bound and a greatest lower bound.

Consequently, a complete lattice contains top ($1 = \bigvee L$) and bottom elements ($0 = \bigwedge L$).
Two examples of a complete lattice are the *real interval* ordered by $\leq$ and the power set of an arbitrary set ordered by inclusion.

3.7.3 Note

Any finite lattice is also complete.

In general some properties of groups are better investigated using subgroup lattices and congruence lattices.

Consistently, we find that a group $(G, +)$ is said to be partially ordered if it is equipped with a *reflexive, antisymmetric* and *transitive* relation $\leq$ which is compatible with $+$; that is $g \leq h$, then $g + k \leq h + k$ and $k + g \leq k + h$, for any $g, h, k \in G$.

Let us adopt the notation as follows: if G is a group then $L(G)$ denotes $\{H|H$ is a subgroup of $G\}$.

The following result describes some properties of subgroup lattices in general.
Let G be a group. Then $(L(G), \subseteq)$ is a lattice. Then the greatest lower bounds and least upper bounds in this lattice are described by $H \wedge K = H \cap K$ and $H \vee K = < H \cup K >$. Moreover, if G is abelian, then an alternative description for $H \vee K$ is $H \vee K = H + K$ but this is not true in general when $H \cap K \neq \{e\}$, where H and K are two subgroups of G.
We arrange subgroups on the lattice according to the number of elements they occupy.

The theorem below gives a subgroup-lattice characterization of groups.

3.7.4 Theorem
If G is a group such that

$$\text{L(G)} \cong L(\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}), \text{ then } G \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}. \quad (3.7.5)$$

Proof : *See* [39]

$L(G)$ is depicted in Fig. 3 of the appendix.

Chapter 4

Fuzzy subgroups of G

4.1 Fuzzy algebraic structure through elements

4.1.1 Introduction

Algebra is one of the focal areas in mathematics i.e areas of study where ideas such as group theory were developed. Intensive research has been conducted in the area of general and abstract thoughts.

In 1965, Zadeh introduced the notion of fuzzy sets. Later, Rosenfeld used these ideas to develop the notion of fuzzy subgroups. Moreover Yager [40], Nguyen [30], Zadeh [41] *et.al*, discussed fuzzy sets in general. These ideas significantly added to fuzzy group theory topics.

We define the term fuzzy subgroup of a group G.

4.1.1 Definition
Suppose G is a group, and $\mu : G \to I$ is a fuzzy subset of a group G, then μ is said to be a *fuzzy subgroup* of G if and only if $\mu(ab) \geq \mu(a) \wedge \mu(b)$ and $\mu(a^{-1}) \geq \mu(a)$ for all $a, b \in G$.

Some important properties of fuzzy subgroups are detailed in the following proposition.

4.1.2 Proposition

1. $\mu(a) = \mu(a^{-1})$ for all $a \in G$.
2. $\mu(e) \geq \mu(a)$ for all $a \in G$ and $e \in G$.
3. The α-cut μ^α of any fuzzy subgroup μ of G is a crisp subgroup of G for all α such that $0 \leq \alpha \leq \mu(e) \leq 1$.

Proofs:
(i) The proofs for (1) and (2) are straightforward.

(ii) We have to prove that $\mu^\alpha = \{\mu \in G : \mu(g) \geq \alpha\}$ is a crisp subgroup of G.
(1) Fix $g_1, g_2 \in \mu^\alpha$. Then $\mu(g_1) \geq \alpha$ and $\mu(g_2) \geq \alpha$.

Therefore $\mu(g_1 g_2) \geq \mu(g_1) \wedge \mu(g_2) \geq \alpha \wedge \alpha = \alpha$ for $g_1, g_2 \in G$.

$$\text{This implies } g_1 g_2 \in \mu^\alpha.$$

(2) Furthermore for any $\mu(g) \geq \alpha$, $\mu(g^{-1}) = \mu(g) \geq \alpha$, this implies that " $g \in \mu^\alpha$ if and only if $g^{-1} \in \mu^\alpha$".

Therefore μ^α is a crisp subgroup of G. □

In this thesis we assume that $\mu(e) = 1$, since $\mu(e) \geq \mu(x) \forall x \in G$.

Abelian groups arose out of the discussion by Neils Henrik Abel (1802 - 1829). This discussion received much attention in the 17^{th} century. Currently, researchers have introduced a concept of fuzzy abelian groups.

4.1.3 Definition

Suppose G is a group, and $\mu : G \to I$ is a fuzzy subgroup. Then μ is said to be *fuzzy abelian* if $\mu(ab) = \mu(ba)$ for all $a, b \in G$.

It is clear from the definition that any *fuzzy subgroup* of an abelian group is a fuzzy abelian.

4.1.4 Note

We only know that $\mu(ab) \geq \mu(a) \wedge \mu(b)$ and $\mu(ba) \geq \mu(b) \wedge \mu(a)$, but we cannot verify that these two are equal in general. Therefore the converse does not necessarily need to be true in general.

Also $\mu(a^2) \geq \mu(a.a) \geq \mu(a) \wedge \mu(a) = \mu(a)$ for all $a \in G$.
By induction on n, it follows that $\mu(a^n) \geq \mu(a)$ for all $a \in G$ and for all positive integers n and also for $n = 0$.
The two conditions in the definition 4.1.1, are equivalent.

4.1.5 Theorem

Let G be a group and $\mu : G \to I$ be a fuzzy subset. Then μ is a fuzzy subgroup if and only if
$\mu(ab^{-1}) \geq \mu(a) \wedge \mu(b)$.

Proof

By proposition 4.1.2(1), $\mu(a^{-1}) = \mu(a)$
$(ab^{-1}) \geq \mu(a) \wedge \mu(b^{-1})$

$= \mu(a) \wedge \mu(b)$

Conversely,
1) $\mu(ea^{-1}) = \mu(a^{-1})$
$\geq \mu(e) \wedge \mu(a)$
By proposition 4.1.2(2)
$\mu(a^{-1}) \geq \mu(a)$

2) $\mu(ab) = \mu(a(b^{-1})^{-1})$
$\geq \mu(a) \wedge \mu(b^{-1})$
$\geq \mu(a) \wedge \mu(b)$ since $\mu(b^{-1}) \geq \mu(b)$. □

We have learned that cyclic groups are abelian and that the converse is not true. If G is abelian then definition 4.1.1 can be rewritten using additive notation and we can write the conditions for fuzzy subgroups of $(G,+)$ as $\mu(a - b) \geq \mu(a) \wedge \mu(b)$.

From now on we will use the additive notation for groups.

4.2 Level subgroups

Shortly after fuzzy set theory was developed in general, mathematicians began to fuzzify various areas of classical mathematics. The concept of fuzzy subgroups follows naturally from the corresponding classical notion.

In [9], Das defined level subgroups and used his definition to study fuzzy subgroups of G. This study characterized several properties of fuzzy subgroups in terms of level subgroups of G. For instance fuzzy subgroups were represented in the form of a chain of level subgroups. Rosenfeld [37], started with the characterization of all fuzzy subgroups of a prime cyclic group in terms of the membership function.

Let us look at the notion of "*level subgroup*" in a general sense:

4.2.1 Note

In this section we replace α-cut with t-cut until further notice to be consistent with the literature.

4.2.2 Definition

Let G be a group and $\mu : G \to I$ be a fuzzy subgroup of G. The subgroup μ_t, $t \in$[0,1] and $t \leq \mu(e)$ is called a *level subgroup* of μ with respect to t.

4.2.3 Note

μ_t is a t-level subgroup of μ in G.

Suppose G is finite, then the number of its subgroups are finite as well, while the number of level subgroups of a fuzzy subgroup μ appears to be infinite. These level subgroups are not all distinct, since every level subgroup is indeed a subgroup of G.
This theorem clarifies this aspect.

4.2.4 Theorem

Let G be a group and $\mu : G \to I$ be a fuzzy subgroup of G. Two level subgroups μ_{t_1}, μ_{t_2} (with $t_1 < t_2$) of μ are equal if and only if there is no $x \in G$ such that $t_1 < \mu(x) < t_2$.

Proof :

Let $\mu_{t_1} = \mu_{t_2}$. Suppose there exists $x \in G$ such that $t_1 < \mu(x) < t_2$ then $\mu_{t_2} \subsetneq \mu_{t_1}$, since x belongs to μ_{t_1}, but not to μ_{t_2}, which contradicts the hypothesis.

Conversely, suppose there is no $x \in G$ such that $t_1 < \mu(x) < t_2$, with t_1 and t_2 as above. Since $t_1 < t_2$, we have $\mu_{t_2} \subseteq \mu_{t_1}$. Let $x \in \mu_{t_1}$, then $\mu(x) \geq t_1$ and hence $\mu(x) \geq t_2$, since $\mu(x)$ can not lie between t_1 and t_2. Therefore $x \in \mu_{t_2}$. So $\mu_{t_1} \subseteq \mu_{t_2}$. Thus $\mu_{t_1} = \mu_{t_2}$. □

4.2.5 Corollary

Let G be a finite group of order n and μ be a fuzzy subgroup of G. Let Im(μ) = $\{t_i \in I : t_i = \mu(x)$ for some $x \in G$, $1 \leq i \leq n\}$. Then $\{\mu_{t_i}\}$ are the only level subgroups of μ for i = 1, 2, . . .,n.

4.3 Generation of fuzzy subgroups

The concept of generated fuzzy algebra has a structure of immediate importance for the construction of various types of lattices.

In his thesis, B.B Makamba [22], defined the fuzzy subgroup of G generated by μ as follows:

4.3.1 Definition

Let G be a group and μ a fuzzy subset of G, $\mu \neq 0$. The smallest fuzzy subgroup of G containing μ, denoted by $< \mu >$ is called the *fuzzy subgroup* of G *generated* by μ.

Kim [18] has characterized the fuzzy subgroup $\mu^\star$ of G generated by μ internally.

4.3.2 Proposition

Let μ be a fuzzy subset in a group G. Then the fuzzy subset $\mu^\star$ defined by

$$\mu^\star(x) = Sup_{t \leq sup\mu}\{t | x \in < \mu_t >\} \quad (4.3.1)$$

is the smallest fuzzy subgroup of G containing μ.

Proof : *See* [34]

Therefore $\mu^\star$ is indeed a *fuzzy subgroup generated* by μ in G, that is $\mu^\star = < \mu >$.
In particular if μ is a fuzzy point a^λ, then the fuzzy subgroup $\mu = a^\lambda$ generated by a^λ is denoted by $< a^\lambda >$.

4.3.3 Note

1. The subgroup generated by the empty subset of G is the trivial subgroup $\{e\}$.

2. The notion of fuzzy generation coincides with the notion of crisp generation when [0,1] replaced by the two element set {0,1}.

3. A fuzzy subgroup μ is finitely generated if it is generated by a fuzzy set with finitely many membership values.

4. In particular if $\mu = < a^\lambda >$ for some fuzzy point a^λ, then μ is said to be *cyclic*.

5. It is clear that a level subgroup of a finitely generated fuzzy subgroup is finitely generated.

4.3.1 Cyclic fuzzy subgroups

A cyclic fuzzy subgroup is based on the idea of cyclic groups. Sidky and Mishref [33] defined a fuzzy subgroup μ to be cyclic if and only if each level subgroup of μ is cyclic. Makamba [22] showed that if μ is fuzzy cyclic, then all the non zero level subgroups of μ are cyclic, but the zero level subgroup of μ need not be cyclic.
To discuss these ideas properly, we start with:

4.3.4 Definition

Let G be a group and a^λ a fuzzy point in G. A fuzzy subgroup μ is *cyclic* in G if there exists a fuzzy point a^λ such that $\mu = < a^\lambda >$.

4.3.5 Proposition

(Makamba [22])

Let $\mu = < a^\lambda >$ and

$$\nu(x) = \begin{cases} \lambda, & x \in < a > \\ 0, & \text{x} \notin < a > \end{cases} \quad \forall x \in G \qquad (4.3.2)$$

Then $\mu = \nu$.

Proof :
Let $b^\beta \in \nu$ if $\nu(b) \geq \beta$. If $b = e$, then $e \in < a >$. Hence $\nu(e) \geq \lambda \geq \beta$. Now

$\mu(a) \geq \lambda$ and $\mu(e) \geq \mu(a) \geq \lambda \geq \beta$. So $b^\beta = e^\beta \in \mu$.
Suppose $b \neq e$. $\nu(b) \geq \beta > 0$. Hence $\nu(b) = \lambda \geq \beta$ and $b \in < a >$. So $b = a^m$ for some $m \in \mathbb{Z}$.
Therefore, $\mu(b) \geq \mu(a) \geq \lambda \geq \beta$. So $b^\beta \in \mu$. Therefore $\nu \leq \mu$.
Now $a^\lambda \in \nu$ since $a \in < a >$. By the definition of μ, $\mu \leq \nu$.
Hence $\mu = \nu$. □

4.4 Fuzzy Abelian Subgroups

The notion of fuzzy abelian subgroups was pioneered by Bhattachary and Mukherjee [3]. They define fuzzy abelian subgroups as follows.

4.4.1 Definition
Let μ be a fuzzy subgroup of G. Let $H = \{x \in G | \mu(x) = \mu(e)\}$. Then μ is a *fuzzy abelian* if H is an abelian subgroup of G.

Makamba [22] showed that the above definition was too weak. For example, any fuzzy subgroup μ satisfying $\{x \in G\colon \mu(x) = \mu(e)\} = \{e\}$ is necessarily fuzzy abelian even if Supp μ is not abelian. The alternative proposed definition to strengthen 4.4.1 is:

4.4.2 Definition
Let μ be a fuzzy subgroup of G. μ is *fuzzy abelian* if μ^t is abelian for all $t \in (0, \mu(e)]$.
The above definition 4.4.1 by Bhattachary and Mukherjee [3] was corrected with the result below:

4.4.3 Theorem
(Makamba [22]).
μ is fuzzy abelian if and only if Supp μ is abelian.

Proof:
Suppose Supp μ is abelian. Now $\mu^t \subseteq$ Supp μ for all $t \in (0, \mu(e)]$, and so μ^t is abelian for all $t \in (0, \mu(e)]$. Hence μ is fuzzy abelian.

Conversely, suppose μ^t is abelian for all $t \in (0,\mu(e)]$. Let $a, b \in$ Supp μ. Since Supp$\mu = \cup\{\mu^t\colon\ 0 < t \leq \mu(e)\}$, $a \in \mu^t$ and $b \in \mu^t$ for some $t_1, t_2 \in (0,\mu(e)]$. Without loss of generality, assume $t_1 \leq t_2$. Then $\mu^{t_2} \subseteq \mu^{t_1}$. Hence $a, b \in \mu^{t_1}$. Therefore $ab = ba$. □

4.4.1 Lattice of fuzzy subgroups

We have seen in section 3.7 that by a *lattice* we mean a "partially ordered set". We intend to classify and illustrate the fuzzy subgroups of G on a subgroup-lattice diagram.
Seselja and Tepavcevic [32] consider a poset to be a collection of fuzzy subgroups under two different orderings.
One way is the ordering among functions, having values in a poset or in a lattice and the other is the set inclusion among collections of levels.

In this case, we consider the first method i.e the ordering among membership functions.

An example depicted in Fig. 3 shows a subgroup-lattice of fuzzy subgroups of finite abelian group $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$.

Ajmal and Thomas [1] suggest a way of joining the fuzzy substructures. This leads to the formation of various types of lattices and sublattices of fuzzy substructures of a group. The main aim is to ascertain the nature of meet and join of fuzzy subgroups.
The meet and join of two fuzzy subgroups A and B of G will be denoted by $A \wedge B$ and $A \vee B$ respectively.

4.4.2 The join of fuzzy substructures

Ajmal and Thomas [1] in their paper provided a universal fuzzy algebraic technique for the construction of join of a family of fuzzy subgroups of a given group.

These classes constitute complete lattices under the ordering of fuzzy set inclusion, and form a descending chain of fuzzy sets with its least member $F_6(G)$ of the class of all fuzzy normal subgroups.

That $F_6(G)$ is a modular lattice, can be shown by construction of the fuzzy set $\mu \cup \eta \cup \mu \cup \eta$, but this has failed to indicate for $F_i(G)$, $i \neq 1,5$, that each smaller class is a sublattice of the larger class. (For more information on this see [1].) We will discuss this concept in detail later on. The idea of a lattice is basically for illustration. The lattice will have more influence on further studies of fuzzy subgroups of finite abelian groups.

In section 4.1, we defined the concept of a *fuzzy subgroup* of a group G. So far we have discussed fuzzy subgroups using different notions such as *level subgroup*, *maximal chains* and *lattice diagram*.
In [9], Das characterized several properties of fuzzy subgroups in terms of level subgroups of G, while in [15] Murali and Makamba discussed fuzzy subgroups of G and classified the fuzzy subgroups using the notion of maximal chains.

The fuzzy subgroups of G can be represented by maximal chains. This representation is a way of classifying fuzzy subgroups of G using membership values. The following example illustrates the classification of fuzzy subgroups of G on the maximal chain. We have:

$$\{\mathrm{e}\} \subset (H_1^{\lambda_1}) \subset (H_2^{\lambda_2}) \subset ... \subset (H_{n-1}^{\lambda_{n-1}}) \subset (H_n^{\lambda_n}) \subset G \quad (4.4.3)$$

where $H_i's$ are subgroups of G, the $\lambda_i's$ are values between [0,1] where $i = 0,1, \ . \ . \ . \ ,n$.

4.5 Equivalence classes of fuzzy subgroups on G

We use the length[1] of the maximal chain of the subgroups of G, to determine the equivalence classes of fuzzy subgroups.

When $G = \mathbb{Z}_{p^1} + ... + \mathbb{Z}_{p^n}$, G has a length equal to $n + 1$. Hence the number of equivalence classes of fuzzy subgroups is $2^{n+1} - 1$.
At this stage, we considered a prime number of one kind only but our discussion is entirely dominated by two distinct primes p and q, whose power is n and m respectively.

In [27], Murali and Makamba provided the following proposition that gives an algebraic formula for determining the number of distinct equivalence classes of fuzzy subgroups of $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$.
In this case, we have seen that any maximal chain of subgroups of G has a length of $n + m$ +1 components. The number of fuzzy subgroups on any one of the maximal chain is 2^{n+m+1} - 1.

4.5.1 Proposition

For any $n, m \in \mathbb{N}$, there are $2^{n+m+1} \sum_{r=0}^{m} \frac{1}{2^{-r}} \binom{n}{n-r} \binom{m}{r} - 1$ where $m \leq n$ distinct equivalence classes of fuzzy subgroups on $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, where p and q are distinct primes.

Proof: *See* [27].

Some of the keychains on distinct maximal chains determine the same equivalence class of fuzzy subgroups. We illustrate this with the following example: let us take the following keychains $1\lambda\lambda\beta\gamma$ where $(1 > \lambda > \beta > \gamma > 0)$on the following two maximal chains:

$$0 \subset p \subset pq \subset p^2q \subset p^2q^2$$
$$0 \subset q \subset pq \subset p^2q \subset p^2q^2$$

[1] By the length of the maximal chain we mean the number of components of the fuzzy subgroups on the maximal chain.

determine the same fuzzy subgroup whose pinned-flag is given by either:

$$0^1 \subset p^\lambda \subset (pq)^\lambda \subset (p^2q)^\beta \subset (p^2q^2)^\gamma$$

or

$$0^1 \subset q^\lambda \subset (pq)^\lambda \subset (p^2q)^\beta \subset (p^2q^2)^\gamma,$$

which can be reduced to

$$0^1 \subset (pq)^\lambda \subset (p^2q)^\beta \subset (p^2q^2)^\gamma.$$

4.6 Classification of subgroups

The purpose is to group items of certain similar characters together. This process is a classification [2]. We demonstrated a way of classifying subgroups using the notion of maximal chains. Consider a maximal chain $M_\mu : \mathbb{Z}_{p^n} \supset \mathbb{Z}_{p^{n-1}} \supset ... \supset \mathbb{Z}_p \supset 0$. M_μ defines a fuzzy subgroup μ of G as follows:
μ assumes λ_n on $\mathbb{Z}_{p^n} \backslash \mathbb{Z}_{p^{n-1}}, \lambda_{n-1}$ on $\mathbb{Z}_{p^{n-1}} \backslash \mathbb{Z}_{p^{n-2}}$, . . ., λ_1 on $\mathbb{Z}_p \backslash \{0\}$ and 1 on $\{0\}$, where

$1 \geq \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{n-1} \geq \lambda_n \geq 0.$ (4.6.4)

Generally any fuzzy subgroup of $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ for specific small numbers n and m is equivalent to one of the keychains in Table 2, where their corresponding meaning are given in Table 1.

If the maximal chain in equation 4.6.4 has a length of four components, then we have 15 distinct equivalence classes of fuzzy subgroups on $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ for a specific m and n. This appears in column 3 of Table 2.

[2]Classification is a process of putting things similar to each other into a class.

4.7 Some other methods of characterization

Some researchers analysed fuzzy subgroups using different approaches. Below are two procedures, one used by Kim [19] *via fuzzy $p^\star$-subsets and fuzzy $p^\star$-subgroups* and the other by Zhang [44] using *classification of cosets* to characterize fuzzy subgroups of a group G.

Via fuzzy $p^\star$-subsets and fuzzy $p^\star$-subgroups

Kim instituted a notion A_p to characterize fuzzy p-subgroups of a group G. This study establishes that the structure of p-fuzzy subgroups is much simpler than that of ordinal fuzzy subgroup structures. This notion characterized a fuzzy subgroup as the intersection of all its minimal fuzzy p-subgroups. However, the notion of A_p can be applied only to fuzzy subgroups in which all elements have finite fuzzy orders. Further, he introduced a notion of $A_{p^\star}$ of a fuzzy subgroup A that is free from the limitation as stated above and can be regarded as an extension of the notion of A_p.
This notion of $A_{p^\star}$ characterized all fuzzy subgroups and showed that every commutative fuzzy subgroup can be characterized as the intersection of its all minimal fuzzy $p^\star$-subgroups (for more information see [19]).

Classification of cosets

In his paper [44] Zhang classified groups using membership values of fuzzy subsets, which is a classification of cosets about normal subgroup H. In normal circumstances, we classify fuzzy subgroups of a group G, by knowing the group structures beforehand. In [44], we are equipped with fuzzy subgroup knowledge and that enables us to classify the group G. The result proceeds as follows:
Let μ be a fuzzy subset of a group G and define $H = \{h \in G | \mu(xay) = \mu(xy), \forall x, y \in G\}$. Then $a \sim b$ if and only if $aH = bH$ for all $a, b \in G$.
The procedure is just to verify that H is a normal subgroup of a group G through the fuzzy subset μ.

Chapter 5

A case involving a relationship between groups and decision-making

In this thesis, we began with the study of finite abelian groups. We explored the fuzzy subgroups of these finite abelian groups. The above process provided the background for procedures that will enable us to apply the knowledge of the mathematics of fuzzy sets.
We intend to establish a relationship between the study of finite abelian groups and the following studies of decision-making. We have already defined a fuzzy relation in section 2.1. Now we give an example to illustrate the application of fuzzy relations.
Say we have a bucket, Z, containing fruit of one type, but shaded in various colours in the order given below.

raw *green* *red* *ripe* *rotten.*

We classify the chances of certain fruit bearing certain colours using membership values. When fruit is raw, it is equivalent to the membership value 0, while between red and green is 0.9, raw and red is 0.1, and green and ripe is 0.1 as well, while red and ripe is 0.9 and rotten is 1.
The above discussion can be assembled in a more condensed representation using the membership function.

$$\mu_R(a,b) = \begin{cases} 0, & if \quad raw \\ 0.9, & if \quad a = red, b = ripe \quad or \quad a = green, b = raw. \\ 0.1, & if \quad a = raw, b = red \quad or \quad a = green, b = red \\ 1, & if \quad rotten \end{cases} \quad (5.0.1)$$

The concept of similarity relation is essentially a generalization of the concept of an equivalence relation (Zadeh, 1971). In general, a similarity relation S for the equivalence relation X is the α-level set of S such that:

$\mathrm{S}_\alpha = \{(x,y)|\mu_S(x,y) \geq \alpha\}$, $0 \leq \alpha \leq 1$ (5.0.2)

So, S_α is the α-cut of the similarity relation S. Now we institute the equivalence between the following situations (a bucket and a group): the bucket Z of fruit is equivalent to the group G with fruit at the state of the fuzzy subgroups of G. The classification of colors using membership functions project the decision-making process.
In this situation of classification we merely use approximate reasoning. By *approximate reasoning* we mean the process or processes by which a possibly imprecise conclusion is deduced from a collection of imprecise premises.

The situation we have just elaborated has the capacity to enable us to use the knowledge of mathematics of fuzzy sets on a decision-making process, in particular capital budgeting in the following chapters.

The main goal of this chapter is to narrow the gap between the studies of group theory and decision-making. We used a technique to illustrate the way we shift from one situation to another. We can now successfully discuss capital budget decision-making without puzzling.

Chapter 6

Decision-making in a fuzzy environment

6.0.1 Introduction

This chapter will discuss the decision-making process in a fuzzy environment and the next chapter will be on capital budgeting under fuzzy environment. The information from chapter 6 will provide a procedure so that we can apply the knowledge of fuzzy decision-making to capital budgeting. Capital budgeting under fuzzy environment is a relatively new idea that was first researched about 10 years ago. The study of capital budgeting has far-reaching consequences for applications in economics and business. It is receiving research attention from both mathematicians and economists.

We institute a fundamental procedure to guide the decision-maker to make correct decisions. Usually, people use conventional methods such as voting to justify pending matters. However, we establish a technical procedure that employs a knowledge of fuzzy mathematics to influence decision-making.

We begin this study by outlining the basic definitions and essential terminologies used in the context. The necessary fuzzy set operations, definitions and other properties are given in Chapter 1.

Decision-making has become a demanding practice to decide on pending matters. So, decisions in real the world are taking place in situations where goals, constraints and the consequences of possible actions are not known precisely. In normal circumstances we could only deal with imprecision using probability theory and some other tools provided by decision theory, control theory and information theory.

The definition for a decision process is elaborated in more precise meaning below. By Zimmermann [43] we have:
A *decision process* is a process in which the goals and constraints, but not necessarily the system under control, are fuzzy in nature.

We look at the example of a fuzzy goal and a fuzzy constraint respectively:
Fuzzy goal: " The cost of A should not be *substantially* higher than α", where α is a specific constant.
Fuzzy constraint: " X should be in the *vicinity* of x_0", where x_0 is a constant.

So, *substantially, vicinity* are the sources of fuzziness in the above statements. We can view fuzziness as a class where there is no sharp transition from membership to non-membership. One thing we have noticed in the discourse of human fuzzy statements such as:

"X is much <u>*larger*</u> than Y",
" the stock market has suffered a <u>*sharp*</u> decline",

is that they send intelligent information, despite the imprecision of the meaning of the underlined words.

6.1 Goals, Constraints and Decisions

In the present note, our main concern is with introducing three basic concepts: fuzzy goal, fuzzy constraint and fuzzy decision; and exploring the application of these concepts to the multistage decision process in which the goals or the constraints may be fuzzy, while the system under control may be either deterministic or stochastic, but not fuzzy.

In normal discourse, the words goal, constraint and decision are often cited in frequent life interaction. We are attempting a new version of associating the fuzzy scenario to the above-mentioned terms to make them useful in a management scenario. So, we demonstrate the examples of fuzzy goal, fuzzy constraint and fuzzy decision in the mathematical expression.
Let $X = \{x\}$ be a given set of alternatives. The fuzzy goals or just simply a goal G will be identified with a given fuzzy set G in X.
The example we look at is when X is the real line, $X = \mathbb{R}$, the fuzzy goal is " x should be *substantially* larger that 10". This can be represented with the following membership function:

$$\mu_G(\text{x}) = \begin{cases} 0, & \text{if } x \leq 10 \\ ((1+(x-10)^{-1})^{-1}), & \text{if } x > 10. \end{cases} \quad (6.1.1)$$

Similarly, a constraint is a fuzzy set C in X. If X is the real line R, the constraint " x should be in the *vicinity* of 15", and here is the membership function representation:

$$\mu_C(\text{x}) = (1+(x-15)^{-15}). \quad (6.1.2)$$

Goals and constraints are connected together by '$\wedge$' which is ' *and* ' and this is an intersection of fuzzy sets. Therefore, $\mu_{G\wedge C}(x) = \mu_G(x) \wedge \mu_C(x)$ or more explicitly from equation 6.1.1 and 6.1.2

$$\mu_{G\cap C}(\text{x}) = \begin{cases} Min((1+(x-10)^{-2})^{-1}),(1+(x-15)^{-15})) & \text{for } x > 10 \\ 0, & \text{for } x \leq 10. \end{cases} \quad (6.1.3)$$

Fuzzy goals and fuzzy constraints can be defined precisely as fuzzy sets in the space.
Generally, a *decision* is basically a choice or a set of choices drawn from the available alternatives.
A *fuzzy decision* is defined as a fuzzy set of alternatives resulting from the intersection of goals and constraints.
A *maximum decision* is defined as a point in the space of alternatives at which the membership function of a fuzzy decision attains its maximum value.
With the aid of Zimmermann in [43], we can now formally define a fuzzy decision using the above facts.

6.1.1 Definition

Assume that we are given a fuzzy goal G and fuzzy constraint C in a space of alternatives X. Then, G and C combine to form a decision D, which is a fuzzy set resulting from the intersection of G and C.

Symbolically,
$D = G \cap C$ and the corresponding membership function is $\mu_D = \mu_G \wedge \mu_C$.
Fig. 4 of the appendix depicts the representation of fuzzy decision in graphic form. It shows the way goals and constraints conflict to form intersections that are regarded as the main basis for decision-making.

6.1.2 Note

If we have 'n' goals and 'm' constraints, the resultant decision is the intersection of the given goals and constraints.

We apply our human intuition to interpret a decision as the intersection of goals and constraints. If '*and*' is defined openly, then a decision is viewed as a fuzzy set of a confluence of the goals and the constraints. Finally we can say: Decision = confluence of goals and constraints.

We consider an example with X =$\{1, 2, ..., 10\}$ and G_1, G_2, C_1 and C_2 are defined with values in Table 3. If we establish the intersection (conjuction) of $\mu_{G_1}, \mu_{G_2}, \mu_{C_1}$ and μ_{C_2}, we obtain the result depicted in Table 4 of μ_D(x).

The fuzzy decision in this case is the following fuzzy set:

D= {(2,0.1),(4,0.7),(5,0.8),(6,0.6),(7,0.4),(8,0.2)}.(6.1.4)

The fuzzy set D from equation 6.1.4 can be read as : no x in X has a full grading; this shows how the specified goals and constraints conflict with one another. A fuzzy decision is regarded as an instruction that gained fuzziness from the imprecision of the given goals and constraints.
The linguistic variables for G_1, G_2, C_1 and C_2 can be interpreted as: " x should be close to 5","x should be close to 3","x should be close to 4" and " x should be close to 6".

The decision then, is to choose x close to 5. The exact meaning of '*close*' in each case is given by the values of the corresponding membership function.

6.2 Multistage decision process

We have defined several concepts and provided some properties to support them in the early sections of this chapter. These concepts have relevant applications contributing to problem solving models involving multistage decision-making in a fuzzy environment. Here we are not developing a new theory for the multistage decision-making process, but rather demonstrating the use of the fuzzy goal, fuzzy constraint and fuzzy decision in multistage decision-making.
In their journal [42], Bellmand and Zadeh have set up a procedure to reciprocate this process.

Here are some of the conditions we set for the process that we are dealing with: The system is taken to be A, which is a time invariant, finite state deterministic system. The two variables x_t and u_t at time t = 0,1,2,. . . are given with their respective order, x ={$\sigma_1, ..., \sigma_n$}, and u = {α_1, . . .,α_m}, where x_t is the output and u_t is the input.

At time $t+1$ we obtain A, defined by the following equation:

$x_{t+1} = f(x_t, u_t)$, t = 0,1,2, . . . (6.2.5)

where f:$X \times U \rightarrow X$.
So, $f(x_t, u_t)$ is a successor state of x_t for input u_t.

6.2.1 Note

If f is a random function, then the system is a stochastic one whose state at $t+1$ is a probability distribution over x, where $P(x_{t+1}|x_t, u_t)$ is a condition of x_t and u_t.
Similarly, if f is a fuzzy function on x_t and u_t, then A is a fuzzy system, which means that it is characterized by a membership function $\mu(x_{t+1}|x_t, u_t)$.
In this study f is a non-fuzzy function, unless specified in some cases.
We need to bear in mind that in a fuzzy environment only the goals and the constraints are fuzzy, but not necessarily the system itself. In this system the inputs from a fuzzy set U are highly subjected to the constraint C^t at time t, and this has a fuzzy analogy for the membership function $\mu_t(u_t)$. The fuzzy goal can exert some control over the process period, terminating it at $t = N$, so the goal in X is a fuzzy set $\mu_{G^N}(X_N)$.

Now we intend to introduce a model for multistage decision making in general in a fuzzy environment.
Before presenting the model, we thought it is better to investigate the three types of decision-making models. This was given a name as a *decision making model, strategies* and *theories.*

In his paper, Michael Richarme [36] decided to investigate the question of " how do consumers make decisions ?".

6.2.1 Three decision-making models

The three earliest economists who puzzled over the above question were Nicholas Bernoulli, John Von Neumann and Oskar Morgenstern.

This situation of Consumer Decision-making Models, Strategies, and Theories was initially investigated by Bernoulli and later the two gentlemen Neumann and Morgenstern extended it and called it a *Utility theory*.
It is believed that consumers make decisions as they are regarded to be rational actors who are able to estimate the probabilistic outcomes of uncertain decisions.
Consumers seem to be good on choosing which goods to buy and which brands to use or to ignore. However, it was found that consumers are not completely rational, but rather typically. In some cases, they could estimate how frequently events can happen. In the end it was found that this Utility Theory had some serious shortcoming that could not be solved by this model.
Our goal is to demonstrate the Multistage Decision-making Model. In their journal article [42] Zadeh and Bellman describe a procedure for this model.

6.3 A multistage decision-making model

.
We assume that f is a non-stochastic function with a system A characterized by equation 6.2.5. The terminating time and the initial state, x_0, of the process is assumed to be known. Finally we intend to find a maximal decision.
By a *maximal decision*, we mean a point in the space of alternatives at which the membership function of a fuzzy decision attains its maximum value.

With equation 6.1.4, we consider a decision to be a decomposable fuzzy set $U \times U \times ... \times U$, given in short by the following equation:

R=C$^0 \cap C^1 \cap ... \cap C^{N-1} \cap G'^N$ (6.3.6) where $G^{'N}$ is a fuzzy set in $U \times U \times ... \times U$, which induces G^N in X.

The corresponding membership function for the expression given in 6.3.6 is:

$\mu_D(u_0, ..., u_{N-1}) = \mu(u_0) \wedge ... \wedge \mu_{N-1}(u_{N-1}) \wedge \mu_{G^N}(x_N)$, (6.3.7)

where x_N is a function of x_0 and $u_0, ..., u_{N-1}$, through the iteration of 6.2.5. We aim to find tangible inputs for the model as given in explicit form in 6.3.7. Since the situation is a multistage model process, we expect the solution to be in the following terms:

u$_t = \pi_t(x_t)$, $t =$ 0,1,2,. . . ,$N - 1$ (6.3.8)

where π_t is called a *policy function*.
By dynamic programming, we obtain both the policy function π_t and a maximal decision $u_0^M, ..., u_{N-1}^M$. When we combine 6.3.6 and 6.3.7 we achieve the following membership function:

$\mu_D(u_0^M, ..., u_{N-1}^M)$=$Max_{u_0,...,u_{N-2}} Max_{u_{N-1}}(\mu_0(u_0)\wedge...\wedge\mu_{N-2}(u_{N-2})\wedge\mu_{N-1}(u_{N-1})\wedge$ $\mu_{G^N}(f(x_{N-1}, u_{N-1})))$ (6.3.9)

Now as a generic, let us pick a constant β and h to be any function of u_{N-1}, this produces the following important identity.

$\text{Max}_{u_{N-1}}(\beta \wedge h(u_{N-1})) = \beta \wedge Max_{u_{N-1}} h(u_{N-1})$.(6.3.10)
If we decide to rearrange 6.3.9, we obtain a much more formidable expression below:

$\mu_D(u_0^M, ..., u_{N-1}^M) = Max_{u_0,..,u_{N-1}}(\mu_0(u_0) \wedge ... \wedge \mu_{N-2}(u_{N-2}))\wedge\ \mu_{G^{N-1}}(x_{N-1}),$
(6.3.11)

where

$\mu_{G^{N-1}}(x_{N-1}) = Max_{u_{N-1}}(\mu_{N-1}(u_{N-1}) \wedge \mu_{G^N}(f(x_{N-1}, u_{N-1})))$. (6.3.12)

So, 6.3.11 is a membership function of a fuzzy goal at time $t = N - 1$, which is induced by the given goal G^N at t =N. By repeating this backward iteration, which is an example of dynamic programming, we achieve this recurrence equation:

$\mu_{G^{N-v}}(x_{N-v}) = Max_{u_{N-v}}(\mu(u_{N-v}) \wedge \mu_{G^{N-v+1}}(x_{N-v+1})),$
$x_{N-v+1} = f(x_{N-v}, u_{N-v})$,v= 1, ...,N.(6.3.13)

This provides the solution to the above problem.
Hence, a maximizing decision $u_0^M, ..., u_{N-1}^M$ is given by the successive maximizing values of u_{N-1} in 6.3.13, with u_{N-v}^M defined as a function of $x_{N-1}, v = 1, ..., N$.

Chapter 7

Capital Budgeting

7.0.1 Introduction

In this section we analyse the general structures of capital budgeting in crisp sense and then we fuzzify them using information from Chapters 1 and 6.
Capital Budgeting has become a critical matter around the world in different sectors of economic activity. Capital budgeting determines the future outcomes of investment projects. There are uncertainties about what will happen to a company's financial management. Nearly all of an organisation's profit is derived from the use of its capital investments.

In this study of capital budgeting we explore cases of budgeting: the first case is the fuzzy equivalent of the classical capital bugeting (investment choice) methods. It can be used to evaluate and compare projects in which the cash flows, duration and required rate of return are given imprecisely, in the form of fuzzy numbers. In the second case, we explore the use of triangular fuzzy numbers in capital budgeting. The triangular fuzzy numbers are restricted fields of fuzzy logic. We use the fuzzy logic knowledge to evaluate investment projections. This system provides a procedure for the decision-making process. Then, we demonstrate the method of ranking investments in the order of their priority.

In general, very short term projects do not pose the same degree of complexity found in projects that extend 10 or more years, but less than 20 years

[7].
Since the evaluation of both civilian and government projects is complex, the best criterion used is a method offered by the capital budgeting notion. This technique is what we referred to as *cost effectiveness analysis or system analysis*.

In their book Capital budgeting, planning and control of capital expenditures [7], John J. Clark, T. J Hindelang and Robert E. Pritchard defined *capital budgeting* as follows.

7.0.1 Definition

Capital Budgeting is a decision area in financial management that establishes goals and criteria for investing resources in long-term projects.

We found that a decision to continue in business or to terminate a project is also capital budgeting. A high rate of return on a project is not enough to justify its acceptance. So, the analysis for long-term asset management is made on a cash-flow basis. This leads us to give a description of cash flows.

7.0.2 Definition

Cash flows are simply the monetary value received and monetary value paid out by a firm at a particular point in time.

7.1 Evaluation of alternative investment

We try to find an economics method that would indicate whether a certain investment opportunity is vibrant or not. This analysis involves financial expenditures and human capacity. We are more concerned about the financial projection of the investment. Since funds in the organisation are limited, management is faced with dual problem. It is necessarly to establish some basic criteria. These criteria involve the acceptance, rejection or postponement of proposed investments and the ranking of projects that meet the criteria for acceptance in order of their value to the organisation.

Some methods have been established to guide management in the acceptance or rejection of proposed investments. Some of the methods are specifically intended to deal with questions of variability of risk among proposals. These methods are *payback*, *internal rate of return(IRR), net present value (NPV)* and *return on investment* etc. (Source: [7]).

In both cases, we demonstrate using the net present value method.
In the same book [7], the *net present value (NPV)* is defined as a method that requires discounting all expected after-tax cash flows to present value, and taking the difference between the sum of the discounted cash inflows and outflows.[1]

7.1.1 Discount cash flow - evaluation methods

The investments evaluation methods that are most commonly used and known involve the basic model of Discounted cash flow (DCF). This model can make numerous calculations better understood when associated with capital expenditures such as interest rate and compound rate etc. We intended to find a procedure to analyse the cash flows into various channels of a specified organisation. We can only do this through the machinery of investment methods. In this study, we only consider the net present value for demonstration, and assume that the notion extends to the other methods in same fashion.

The net present value is a discounted cash-flow procedure, because it considers the time value of the money by discounting expected cash flows to their present value.

Net present value

The net present value method is one of the discounted cash flow procedures. Some other methods are mentioned in section 6.1. This method is the chief preference in this study because of the simple nature of its formula. Also, when you know the present value of the sum to be received in the future, it

[1] We can assume that the investment decision made will not alter the firm's existing risk complexion, but this does not mean we are operating in a risk-free atmosphere. Rather, it means that projects accepted have the same average risk as is characteric of the firm.

has some future projection.
In summary of [7], we can say:
The *net present value (NPV)* is the difference in the present value of the inflows and outflows.
Algebraically we have:

NPV=$\sum_{t=0}^{n} \frac{s_t}{(1+k)^t} - A_o$ (7.1.1)

where
A_o = present value of the after-tax cost of the project.
s_t = cash inflow at each time.
k = discount rate (hurdle rate).
t = time period.
n = use life.

In many capital budgeting examples, funds are expected to be received at the end of every year. The net present value is a series of payments, the sum of the individual payments to be received each year. When we calculate the NPV, we judge based on the nature of the result of the NPV.
If the NPV is *positive*, it means the project is expected to yield a return in excess of the required rate.
If the NPV is *zero*, the yield is expected to be exactly equal to the required rate.
If the NPV is *negative*, the yield is expected to be less than the required rate.

Hence, only those projects that have *positive or zero* $NPVs$ meet the NPV criterion of acceptance.
The primary difficulty encountered with NPV is in deciding on the most appropriate rate of return (the hurdle rate) to use in discounting the cash flows.
In the first case we decided to explore the use of fuzzy equivalents to the classical investment methods to evaluate and compare projects. The following situation describes capital budgeting in the crisp case.

7.2 Capital budgeting methods in the classical crisp case

When experts are undertaking procedures to evaluate projects, the analysts obtain a numerical estimation of these investment projects. This allows the analysists to recommend whether or not to accept the investment and to compare it with alternative ones. We gather the data with possible uncertainties that are of serious concern for the investment projects in general.
So, the following data, according to Kuchta [20], are important for the project:
(i) the duration of the project , n (integer), in years.
(ii) $COF_i > 0$ is the cash outflow and $CIF_i > 0$ is the cash inflow and $CF_i = CIF_i - COF_i$ where $i = 1, . . ., n$.
(iii) the required rate of return on the investment or cost of capital is $r > 0$.

For this study, we expect the project to run over a duration of n years. If the project finishes before n years, we take the portion of n years, this we denote by α-part. So, these fractions will become αCOF_n, αCIF_n and αCF_n. Similarly, for the rate of return r, we obtained αr, where r is a rate of return for the whole year.

There are a variety of methods used in capital budgeting. Some of these methods are mentioned in section 6.1.
However, in this study we only discuss the net present value method, to evaluate a project with the above given data.
The letters EV will serve as the output of this method. It can be the difference between the revenue and all the input expenses of the project. Hence, we will use EV as a basis for the decision-making, whether to accept or reject the project.

The algebraic equation for the net present value is given by equation 7.1.1. We can establish the net present value method with EV as the evaluation output of its formula.

EV=CF$_0 + \sum_{i=1}^{n} \frac{CF_i}{(1+r)^i}$, (7.2.2)

The equation 7.2.2 can be interpreted as follows:
The interpretation is subjected to the situation of the project at that moment. The EV denote the project's initial value at its start. The enormous EV of the project means there is hope for the project. If there is some limit or minimal EV then this means the acceptance of the project (e.g 0).

If the projects finishes when the α-part of the n^{th} year has elapsed, the respective evaluation becomes:

$$EV|_\alpha = NPV|_\alpha = \sum_{i=0}^{n-1} \frac{CF_i}{(1+r)^i} + \frac{\alpha.CR_n}{(1+r)^{n-1}(1+\alpha.r)}. \quad (7.2.3)$$

7.3 Fuzzy capital budgeting

Fuzzy capital budgeting is a technical procedure in the investment decision-making process. This study was pioneered by Buckley [4], Kuchta [20] *etal.*, from 1987 onwards. We aim to institute a practical tool for incorporating uncertainties into capital budgeting. This has an ability to inform us whether to accept or reject possible projects.
The classical capital budgeting methods mentioned in section 7.1, do not take into account the uncertainties there might be in the information used in these methods.

Many researchers used Buckley [4], as a basis. In his paper, he established mathematical tools associated with fuzzy meaning to interpret budgeting matters better. The fuzzy tools were created for future value and present value of a cash amount using interest rates over a defined number of periods. In some cases, we adopt a procedure of involving classical techiniques and then fuzzify it. Finally, we discuss methods for comparing fuzzy investment alternatives from the best to worst. This ranking open up avenues for accepting or rejecting the project. The essential features such as cash amount, interest rate and time period may all be fuzzy sets.

The fuzzy sets will be denoted with a squiggle over their letter. Therefore, $\tilde{n}$ is a number of fuzzy interest periods in years, $\tilde{A}$ is the fuzzy cash amount in an account (invested), $\tilde{r}$ is the fuzzy interest rate and $\tilde{S}$ is the fuzzy amount at the end of the period. Their membership functions will be denoted with $\mu(x|\tilde{A})$, $\mu(x|\tilde{S})$, $\mu(x|\tilde{r})$, $\mu(x|\tilde{n})$ *etc.*

Most of the economic circumstances in the mathematics of finance are better handled using fuzzy (real) numbers. The definition of fuzzy (real) numbers in general is given in 1.5.2. We intend to establish a principle for fuzzy real numbers. This give us a specific fuzzy number type to use in this study.

Suppose $\tilde{M}$ is a fuzzy number. With Buckley [4] we defined $\tilde{M}$ as below:

7.3.1 Definition

A fuzzy number $\tilde{M}$ is a special fuzzy subset of the real numbers. Its membership function is defined by (with the aid of graph 1),

$$\mu(x|\tilde{M}) = (m_1, f_1(y|\tilde{M})/m_2, m_3/f_2(y|\tilde{M}), m_4) \quad (7.3.4)$$

where $m_1 < m_2 \leq m_3 < m_4$.
Now,
-$f_1(y|\tilde{M})$ is a continuous monotone increasing function of y for $0 \leq y \leq 1$ with $f_1(0|\tilde{M}) = m_1$, and $f_1(1|\tilde{M}) = m_2$.

- $f_2(y|\tilde{M})$ is a continuous monotone decreasing function of y for $0 \leq y \leq 1$ with $f_2(0|\tilde{M}) = m_4$, and $f_2(1|\tilde{M}) = m_3$.

The procedure is to show that $\tilde{r}, \tilde{n}, \tilde{A}$ and $\tilde{S}$ *etc*, are fuzzy numbers. This can be done by verifying them using techniques from definition 7.3.1.

The following features are attainable from graph 1 :
(1) zero for $x \leq m_1$; (2) $x = f_1(y|\tilde{M})$ for $0 \leq y \leq 1$; (3) one for $m_2 \leq x \leq m_3$; (4) $x = f_2(y|\tilde{M})$ for $0 \leq y \leq 1$; and (5) zero for $x \geq m_4$. In this case we have x as a function of y.

If we consider y a function of x, then we have to employ the inverse. Then, the real number M is a special fuzzy number with $\mu(x|M) = 1$ if and only if $x = M$ and it is zero otherwise.
Sometimes we may employ straight line segments for $\mu(x|\tilde{M})$ on $[m_1, m_2]$ and $[m_3, m_4]$. Then

x=f$_1(y|\tilde{M}) = (m_2 - m_1)y + m_1$ (7.3.5)

x=f$_2(y|\tilde{M}) = (m_3 - m_4)y + m_4$ (7.3.6)

for $0 \leq y \leq 1$. So $\mu(x|\tilde{M})$ can simply be denoted by $(m_1/m_2, m_3/m_4)$.

In general the arithmetic for fuzzy numbers is easily done when we have x as a function of y. We have instituted adequate conditions to aid when we conduct the arithmetic of fuzzy numbers. We stick to the standard arithmetic operations defined in Chapter 1. The goal is to classify investments in the order of their priorities and to select the best investments. We will employ methods presented in [42] for the ranking of fuzzy numbers, a fuzzy number $\tilde{M}$ is positive if $m_1 \geq 0$ and negative when $m_4 \leq 0$.

In this study, fuzzy sets are usually fuzzy numbers. The fuzzy number of a defined duration will be a discrete positive fuzzy subset of the real numbers. The membership function $\mu(x|\tilde{n})$ is defined by a collection of positive integers n_i where $\mu(n_i|\tilde{n}) = \lambda_i$, $0 < \lambda_i \leq 1$, for $1 \leq i \leq K$, and $\mu(x|\tilde{n}) = 0$ otherwise. The value λ_i is interpreted as the possibility that the number of interest periods is n_i.

In the next section we define useful concepts and their arithmetic implications.

7.3.1 Future and present value

We try to piece together essential economical instruments which effectively assist in the study of ranking investment projects. The ranking give a better scope of investment projects in the order of their priorities. We first discuss the future value implications and then the present value's implications for economics.

The *future value* is the amount to which a present sum will grow at a future date, through the operation of interest.

If $\tilde{A}$ is the fuzzy present value invested and $\tilde{r}$ is a fuzzy interest rate of interest periods n, then $\tilde{S}$ is the fuzzy amount in the account after n fuzzy interest periods.
If we put all these variables together we obtain the fuzzy future value $\tilde{S}_n = \tilde{A} \odot (1 \oplus \tilde{r})^n$. Information in fuzzy sense gives a more precise meaning to undetermined situations.
The fuzzy future amount in the account projects the situation that certainly the account will bear in some future times. With the aid of Graph 2, we state the membership function for the fuzzy future value $\tilde{S}_n$ as:

$\mu(x|\tilde{S}_n) = (s_{n1}, f_{n1}(y|\tilde{s}_n)/s_{n2}, s_{n3}/f_{n2}(y|\tilde{s}_n), s_{n2})$. (7.3.7)

This is determined by

$\mathrm{f}_{ni}(y|\tilde{s}_n)= f_i(y|\tilde{A}).(1 + f_i(y|\tilde{r}))^n$, $i = 1{,}2$ (7.3.8) and $f_{n1}(0|\tilde{s}_n) = s_{n1}$, $f_{n1}(1|\tilde{s}_n) = s_{n2}$, $f_{n2}(0|\tilde{s}_n) = s_{n4}$, $f_{n2}(1|\tilde{s}_n) = s_{n3}$.

The possible functions included in the membership function of the fuzzy future value of the increasing part of $\mu(x|\tilde{S}_n)$ are $x = \mathrm{f}_1(y|\tilde{A})$, $x = f_1(y|\tilde{r})$ of $\mu(x|\tilde{A})$ and $\mu(x|\tilde{r})$ respectively. In a similar manner, we obtain the same information for the decreasing function. If the number of fuzzy interest periods is $\tilde{n}$(fuzzy number of periods) with $\tilde{S}$ as the fuzzy amount in the account, then the membership function for $\tilde{S}$ is:

$\mu(x|\tilde{S}) = \mathrm{Sup}_{\Gamma(x)}(\theta)$, (7.3.9)

where $\theta = min(\mu(u|\tilde{A}), \mu(v|\tilde{r}), \mu(w|\tilde{n}))$, $\Gamma(x) = \{(u, v, w)|u(1 + v)^w = x\}$.

In his study Buckley [4] has defined sufficient conditions for calculating various aspects of the mathematics of finance. So, continuous interest and effective rate are financial concepts that will effectively contribute to the future value derivation.
We intend to derive these parameters. In the continuous interest and effective rate investment, we assume $\tilde{r}$ to be the fuzzy interest rate per year, in decimal notation, while m is the number of the compoundings per year.
The above information leads us to $(\frac{1}{m}) \odot \tilde{r}$, which is a fuzzy rate per interest period and mn is the number of interest periods if the initial amount $\tilde{A}$ is left in the account for n years.
So $\tilde{S}_{mn} = \tilde{A} \odot (1 \oplus [(\frac{1}{m}) \odot \tilde{r}])^{mn}$. If we take the limit of the above expression we obtain:
$\tilde{S}_n = lim_{n\to\infty}\tilde{S}_{mn}$.

Since $\mu(x|\tilde{S}_n)$ is determined by the $lim_{n\to+\infty} f_{mni}(y|\tilde{S}_{mn})$ this is equal to $lim_{n\to+\infty}$ $f_i(y|\tilde{A}).(1 + (\frac{1}{m})f_i(y|\tilde{r}))^{mn}$, producing

$\mathrm{f}_{ni}(y|\tilde{S}_n) = f_i(y|\tilde{A})exp(nf_i(y|\tilde{r}))$, (7.3.10)

for i = 1,2 where $exp \quad y = e^y$.
This quantifies the fuzzy effective rate to be $\tilde{e}$.
According to Buckley [4], the definition of the fuzzy effective rate $\tilde{e}$ is given as below:

7.3.2 Definition

The fuzzy *effective rate* $\tilde{e}$ is the interest compounded once a year (m=1) that will make $\tilde{A}$ accumulated to the same amount $\tilde{S}_{mn}$ as compounding $\tilde{r}$ times m per year.

Therefore, $\tilde{e}$ solves

$$\tilde{S}_{mn} = \tilde{A} \odot (1 \oplus \tilde{e})^n. \quad (7.3.11)$$

Similarly, by definition 7.3.1, we can verify that $\tilde{e}$ is a fuzzy number.

It is important to know the present value of an amount S of n periods in the future if r is the interest rate per period. Let us denote $PV(S)$ the present value, if invested today at r it accumulates to S in n periods. So, if $PV(S)$ is the amount invested at the rate r for n periods, the amount to be received in the future is:

$$\text{PV(S)}(1 + \text{r})^n = S. \quad (7.3.12)$$

The expression in 7.3.12 can be rewritten as $PV(S) = S(1+r)^{-n}$.

The two definition(s) below describe how conditions of fuzzy present value influence the fuzzy future value at a fuzzy interest rate $\tilde{r}$ over the interest periods n. The result for the $PV(S)$ plays a significant role in the future amount S. Also the nature of the output has enourmous scope that will influence the decision-making process. The following definitions clarify the nature of the output of the fuzzy future value $\tilde{S}$. One definition will stand for positive $\tilde{S}$ and the other will stand for a negative $\tilde{S}$.

7.3.3 Definitions

(1)$PV_1(\tilde{S}, n) = \tilde{A}$ if and only if $\tilde{A}$ is a fuzzy number and $\tilde{A} \odot (1 \oplus \tilde{r})^n = \tilde{S}$.
(2)$PV_2(\tilde{S}, n) = \tilde{A}$ if and only if $\tilde{A}$ is a fuzzy number and $\tilde{A} = \tilde{S} \odot (1 \oplus \tilde{r})^{-n}$.

Now, if $PV_1(\tilde{S}, n)$ is invested today at a fuzzy rate $\tilde{r}$ over the interest period n, by definition (1) it will evaluate to $\tilde{S}$, because $PV_1(\tilde{S}, n) \odot (1 \oplus \tilde{r})^n = \tilde{S}$, but $PV_2(\tilde{S}, n) \odot (1 \oplus \tilde{r})^n$ will only be approximately equal to $\tilde{S}$. The membership functions for $PV_1(\tilde{S}, n)$ and $PV_2(\tilde{S}, n)$ are $\mu_1(x|\tilde{S}, n)$ and $\mu_2(x|\tilde{S}, n)$ respectively.
It is important to specify membership functions for the present value, this allows us to be able to quantify the cases described in the definitions above. The membership function $\mu_1(x|\tilde{S}, n)$ is determined by $f_i(y|\tilde{A}) = f_i(y|\tilde{S}).(1 + f_i(y|\tilde{r}))^{-n}$ for $i = 1{,}2$ and $a_1 = f_1(0|\tilde{A})$, $a_2 = f_1(1|\tilde{A})$, $a_3 = f_2(1|\tilde{A})$, $a_4 = f_2(0|\tilde{A})$. Thus $\tilde{A}$ will be a fuzzy number if and only if $f_1(y|\tilde{A})$ is increasing, $f_2(y|\tilde{A})$ is decreasing and $a_2 \leq a_3$. If any condition fail, the $\mathrm{PV}_1(\tilde{S}, n)$ is not defined. Similar discussion can be carried out for $\mu_2(x|(\tilde{S}, n))$.

7.4 Fuzzy annuity

In many capital budgeting problems, funds are expected to be received at the end of each year for a period of years. The future value and present value are two variables that can be associated with annuity. We only consider regular annuities when the payment period is equal to the interest period. We concentrate on the NPV method for this study.

In his paper Buckley defines a regular annuity according to the definition below.

7.4.1 Definition

A *regular annuity* is where equal periodic payments P are made at the end of each interest period for n periods.

If the interest rate r, in decimal notation, is per period and S_n is the amount in the account after the n^{th} payment, then $S_n = P\beta(n,r)$, where the actuarial function β is $\beta(n,r) = ((1+r)^n - 1)/r$.
With fuzzy annuity, we convert all the necessary variables into the fuzzy sense and convert the above expressions into a fuzzy version. Using definition 7.3.1 techniques, we can verify that $\tilde{S}_n$ is a fuzzy number with fuzzy variables $\tilde{P}$,$\tilde{n}$ and $\tilde{r}$ (for more information see [4] page 265).

7.5 The criteria for selecting the best investments

We establish a diversity of procedures that enable us to select the best investment projects. The required grounds are already defined in the introductory section of this chapter. This process was conducted by Kuchta [20] on capital budgeting. He defined the data required for the project information. The crucial data required for the meaning of the project information are: the duration of the project, cash flows and the rate of return.
We fuzzify these data to give a precise meaning that will guide us in making decisions. In a similar way, the fuzzified data have a squiggle placed over their letters. We describe these fuzzy data according to the λ-levels of the given $\lambda \in [0,1]$. This estimation of course will be based on the special knowledge of a defined $\lambda \in [0,1]$.
One description is, the 0-level may be the one with least certain information while the 1-level is the one with most definite information.

In section 7.1, we detailed the methods used to evaluate investment projects. We fuzzified these methods to obtain a procedure of interpreting project values.
Let M symbolize any capital budgeting method in crisp case and we denote the evaluation process of the project with $EV_M(n, CF_i, r)$.
The agenda is to give fuzzy equivalents of the methods mentioned in section 7.1. Now, for each crisp method M, we denote the corresponding fuzzy equivalent with $\tilde{M}$.

We give a mathematical interpretation that has decisive boundaries for projects values. The projects evaluation in a fuzzy capital budgeting environment is such that for each given $\lambda \in [0,1]$, we find a possible small closed interval denoted by $EV^{\star}_{\tilde{M}}{}^{\lambda}(n, \tilde{C}F_i, \tilde{r})$ for crisp duration of the project and by $EV^{\star}_{\tilde{M}}\lambda(\tilde{n}, \tilde{C}F_i, \tilde{r})$ for a fuzzy duration.
The following formulate the proposed picture of intervals, set for projection decision boundaries in general, which is in crisp and fuzzy case respectively. We have:

$\mathrm{EV}^{\star}_{\tilde{M}}\lambda(n, \tilde{C}F_i, \tilde{r}) \supset \{EV_M(n, y_i, Z)$ such that $y_i \in (CF_i)^{\lambda}(i = 0, ..., n), z \in r^{\lambda}\}$. (7.5.13)

and

$\mathrm{EV}^{\star}_{\tilde{M}}\lambda(\tilde{n}, \tilde{C}F_i, \tilde{r}) \supset \{EV_M(n, y_i, Z)$ s.t. $x \in n^{\lambda}, y_i \in (CF_i)^{\lambda}(i = 0, ...), [(n^{\lambda})_2] + 1, z \in r^{\lambda}$ }. (7.5.14)

where $[(n^{\lambda})_2]$ is the integer part of the upper of the interval n^{λ}.

This is the basis for a decision making platform. According to Kuchta [20], these intervals above, are the smallest closed ones. So, $E^{\star}_{\tilde{M}}{}^{\lambda}$ ($\lambda \in [0,1]$) or their set $E^{\star}_{\tilde{M}}$ will be taken as a basis for the investment decisions.

The two formulae in equation 7.5.13 and 7.5.14 determine, for a given λ, a lower and upper boundary for the values of the corresponding crisp evaluations.
We have set sufficient guidelines to frame the procedures when we fuzzify the classical capital budgeting methods mentioned in section 7.1.

The next criterion to consider is that of the study of capital budgeting using triangular fuzzy numbers. The information on capital budgeting in crisp sense and triangular fuzzy numbers is given in Chapter 7 and Chapter 1 respectively. We intend to establish a procedure that will provide a board of directors in an organization with good quality information on which to base

their decisions.
In this regard we still consider the present value method as our model. We first produce the NPV in deterministic case and then use software called *fuzzyinvest 1.0* to produce the *fuzzy* NPV in possibilistic case. Hence, we compare the two outputs. From Sanches [31], we use an experiment called quasi-experiment. According to Bryman (1989), the quasi-experiment is a research experiment where the researcher does not have total control over the input variables of the system and there's a non-random treatment of the experiment.
In this regard, Fig. 7 shows the triangular graph from the *fuzzyinvest 1.0* output and a field into which we fed the uncertainties values. The composition of the uncertainties is made up of fixed investment, working capital, monthly fixed cost, variable cost/unit, sales, price, planning horizon, residual value, ROR, income tax and depreciation.

In his study Sanches [31] realized that even if the uncertainties in the input variables are small they may result in great output variables, when it comes to *fuzzy* NPV.
The input variables are in the form of fuzzy numbers and we defuzzify the output to obtain *defuzzy* NPV so that we can be able to interpret it using the acceptance criteria of the organization (as indicated in [31], page 14).
With Fig. 7 showing the graphic output of triangular fuzzy numbers, we can interpret it as follows:
The indication of the failure possibility is on the negative side of the graph shown by the area under the triangle. The bigger the area on the left of the graph, the bigger the failure possibility of the project. The advice will be given that the project should be reviewed and further recommendations made.

An important fact about fuzzy logic, confirmed by Yager(1980), is the non-inversion of the operations, which can often lead to mistakes. For instance: $A + B = C$ does not imply that $C - B = A$, which is the case with real numbers.
In conclusion, triangular fuzzy numbers are a restricted field of fuzzy logic and thus simplify a lot the operations. So, the fuzzy logic capacity to provide resources in the decision-making process is unchallenged.

Now that we have set out the procedures for selecting the best investments, we will create an environment for ranking these investments, based on the criteria defined on page 14 of Sanches [31]. This is the case in the following discussion of *Fuzzy cash flows.*

7.6 Fuzzy cash flows

We are discussing the procedures for comparing investment projects. The most commonly used methods of comparing investment alternatives are net present value (NPV) and the internal rate of return (IRR).

Let us denote the cash flow of a proposed investment project by $\mathcal{A} = A_0, A_1, ..., A_n$ over the period of n years. These A_i may be positive or negative. This we defined as below:
We denote the total investment minus total return with - A_i if $A_i < 0$, that results from the project at the end of the i^{th} period. Simiraly, A_i will stand for the total return minus total investment if $A_i > 0$ for the i^{th} period. We are making two assumptions, one is $A_0 < 0$ and the second one, the time periods are equal to the interest period. This ascertain that the investment project starts off with an initial investment. We are only modeling with NPV method and assume that this applies to other methods in 7.1 in the manner. The classical part of NPV has already been discussed in section 7.1.1.
We rank the above proposal using $NPV > 0$. The procedure for ranking projects can be interpreted in the following order: " the highest NPV to the lowest NPV". The selection of the proposals proceeds in that format until we exhaust all of them.
Suppose $\tilde{A} = \tilde{A}_0, ..., \tilde{A}_n$ is a fuzzy net cash flow, $\tilde{r}_0$ is a fuzzy interest rate standing for the cost of capital of the organization. The membership function $\mu(x|\tilde{A})$ for the fuzzy number $\tilde{A}_i$ is defined by

$\mu(x|\tilde{A}_i) = (a_{i1}, f_{i1}(y|\tilde{A}_i)/a_{i2}, a_{i3}/f_{i2}(y|\tilde{A}_i), a_{i4})$ (7.6.15) for i = 0,1,2,. . .,n.

We need to start with a small amount of money and receive a large output. So, A_0 has to be a negative fuzzy number and other A_i 's can be either negative or positive fuzzy numbers. If we assemble the above parameters together, we obtain the net present value of $\tilde{A}$ as:

NPV(Ã,n)=Ã$_0 \oplus \sum_{i=1}^{n} PV_{k(i)}(\tilde{A}_i, i)$ (7.6.16)

where $\sum$ is the fuzzy addition and $k(i) = 1$ when A_i is negative, $k(i) = 2$ for positive A_i. The membership function $\mu(x|\tilde{A}, n)$ for $NPV(\tilde{A}, n)$ is defined by:

$\mu(x|\tilde{A}, n) = (\alpha_{n1}, f_{n1}(y|\tilde{A})/\alpha_{n2}, \alpha_{n3}/f_{n2}(y|\tilde{A}), \alpha_{n4})$, (7.6.17)

where $f_{ni}(y|\tilde{A}) = \sum_{i=0}^{n} f_{ji}(y|\tilde{A}_j)[1 + f_{k(j)}(y|\tilde{r}_0)]^{-j}$ for i = 1,2, where $k(j)$ =i for the negative $\tilde{A}_j$ and $k(j)$ = 3 - i for positive $\tilde{A}_j$.

We decided earlier in the introduction to denote the output for the net present value with EV, so the equation 7.6.16 will turn out to be:

EV(Ã,n)=Ã$_0 \oplus \sum_{i=1}^{n} PV_{k(i)}(\tilde{A}_i, i)$. (7.6.18)

Now consider the fuzzy cash flows discussed above $\tilde{A}, \tilde{B}$, . . . and so on. We find their fuzzy NPV such as $EV(\tilde{A}, n_a)$, $EV(\tilde{B}, n_b)$, . . . and so on. We rank these fuzzy NPV as defined by Buckley in [4]. This will be given as a

partition of the set of fuzzy proposals into sets $H_1, H_2, \ldots$ and so on. The proposal ranked highest belongs to H_1, all the proposals ranked second highest are in H_2 *etc.* The criterion we employ is to consider proposals that exceed $\tilde{0}$, where $\tilde{0}$ is a fuzzy zero. Other criteria are given in section 7.1.1. They can also be applied in the same sense and lead to a precise conclusion. The output can also be better visualized using the graphic output.
Now, suppose $\tilde{n}$ is a positive fuzzy number. Then, the membership for the net present value for a project $\tilde{A}$ is:

$\mu(x|\tilde{A}) = \text{Sup}_{\gamma(x)}(\theta)$, (7.6.19)

where $\theta = \min(\mu(u_0|\tilde{A}_0),...,\mu(u_w|\tilde{A}_w),\mu(v|\tilde{r}),\mu(w,|\tilde{n}))$, and $\gamma(x) = \{(u_0, ..., u_w, v, w)| \sum_{i=0}^{w} u_i(1+v)^{-i} = x\}$.
The following result by Buckley [4], in the form of a theorem, serves as a fundamental principle to indicate whether to rank the projects or not.

7.6.1 Theorem

If $\tilde{n} = n$ and $\tilde{A}_i$ is positive for $1 \le i \le n$, then $\mu(x|\tilde{A}) = \mu(x|\tilde{A}, n)$.
Proof : *See* [4].

With the net present value method, if the initial amount $A_0 < 0$ and the rest of the A_i i = 1,2,... are positive then by the previous theorem we can find the $NPV(\tilde{A})$ and rank the projects (investments), otherwise we cannot.
One of the major problem with IRR is that it cannot generalize to the fuzzy cash flows. That is why we cannot discuss it now.

We have established mathematical instruments that have the capacity to solve very dynamical economics models using the fuzzy approach. In this regard, we discussed decision-making in general, we defined terminologies that certainly have a meaning in the dynamic system we dealt with. We set up a process that assist in the multistage decision-making process. In capital budgeting endevours we demonstrated with fuzzy mathematics the use of some economics models such as net present value method for the decision-making process. We outlined procedures that lead us to remarkable conclusions in selecting the best investment projects.

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Appendix

Appendix A: Tables

Key chains	Interpretation
111	Set X
11λ	{a}
110	Empty set
1λλ	Fuzzy point a^{λ}
1λβ	The constant fuzzy subset whose degree membership is λ for all x in X.
1λ0	The fuzzy subset that distinguishes the crisp point 'a' among all points with the same degree of membership, namely λ.
100	The fuzzy subset that says that the distinguished point 'a' belongs to the fuzzy subset more than the other points in the set X.

Table 1

All the keychains with length 2	All the keychains with length 3	All the keychains with length 4
1λ 11 10	111 11λ 110 1λλ 1λβ 1λ0 100	1111 111λ 1110 11λλ 11λβ 11λ0 1100 1λλλ 1λλβ 1λλ0 1λββ 1λβγ 1λβ0 1λ00 1000

Table 2

x	1	2	3	4	5	6	7	8	9	10
μ_{G_1}	0	0.1	0.4	0.8	1.0	0.7	0.4	0.2	0	0
μ_{G_2}	0.1	0.6	1.0	0.9	0.8	0.6	0.5	0.3	0	0
μ_{C_1}	0.3	0.6	0.9	1.0	0.8	0.7	0.5	0.3	0.2	0.1
μ_{C_2}	0.2	0.4	0.6	0.7	0.9	1.0	0.8	0.6	0.4	0.2

Table 3 (Source:[40]).

x	1	2	3	4	5	6	7	8	9	10
μ_D	0	0.1	0.4	0.7	0.8	0.6	0.4	0.2	0	0

Table 4 (Source:[40]).

Appendix B: Diagrams

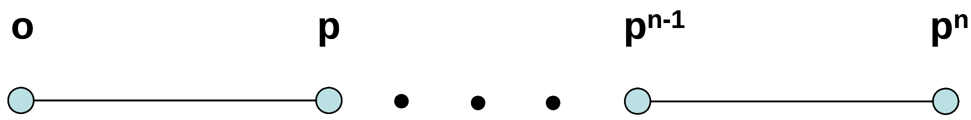


Figure 1.

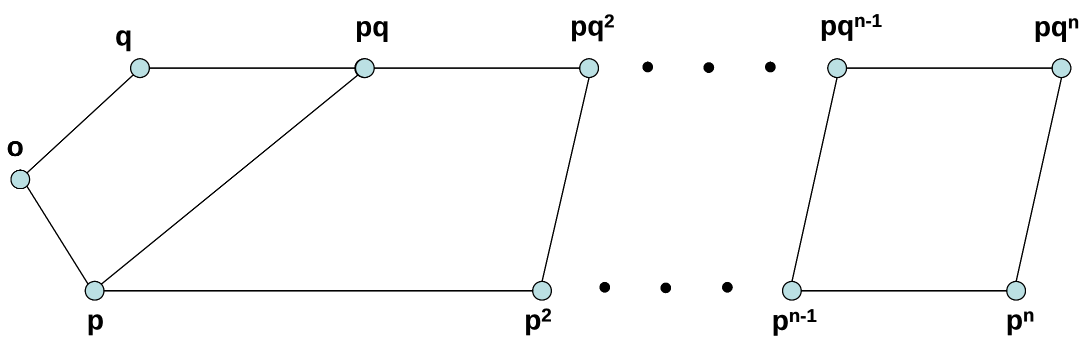


Figure 2.

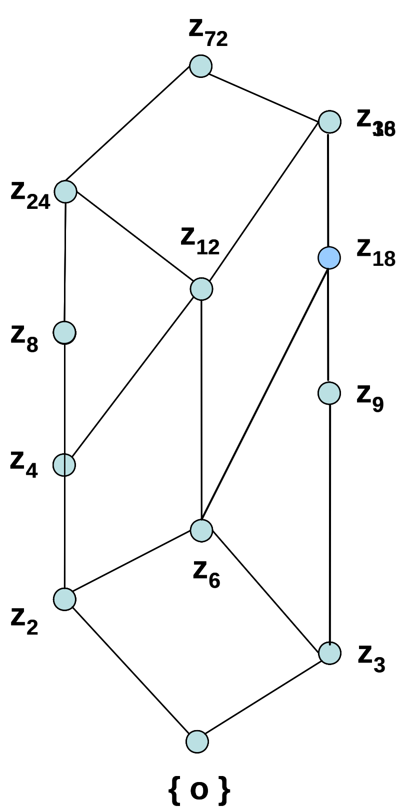


Figure 3.

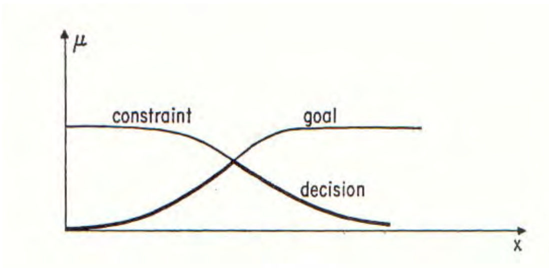
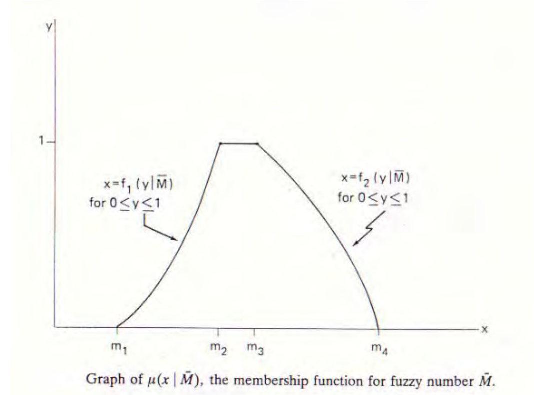
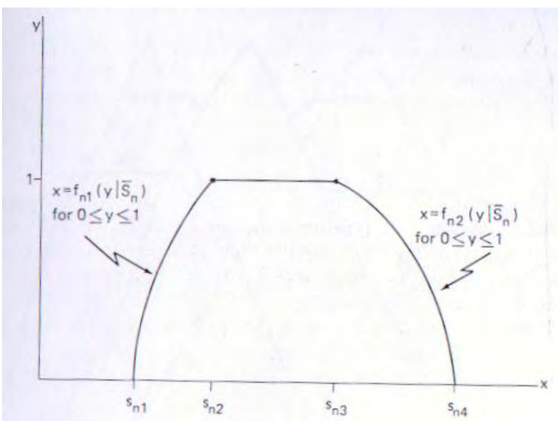


Figure 4: The decision-making process (Source [43]).



Graph 1 (Source[4]).



Graph 2: Graph of the membership function for the fuzzy future amount (Source [4]).

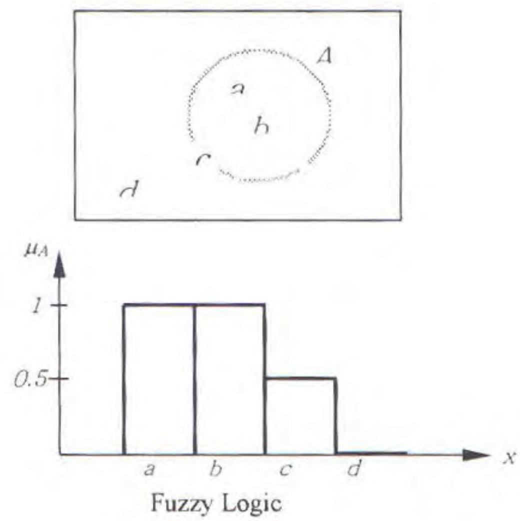


Figure 6 (Source [31]).

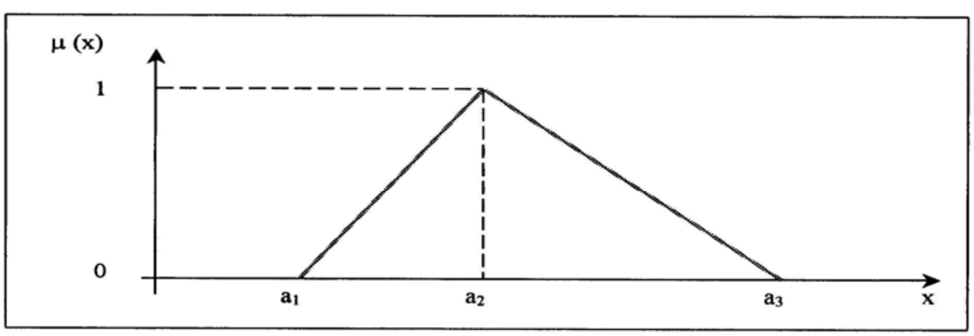


Figure 5 : Triangular Fuzzy Number Structure (Source [31]).

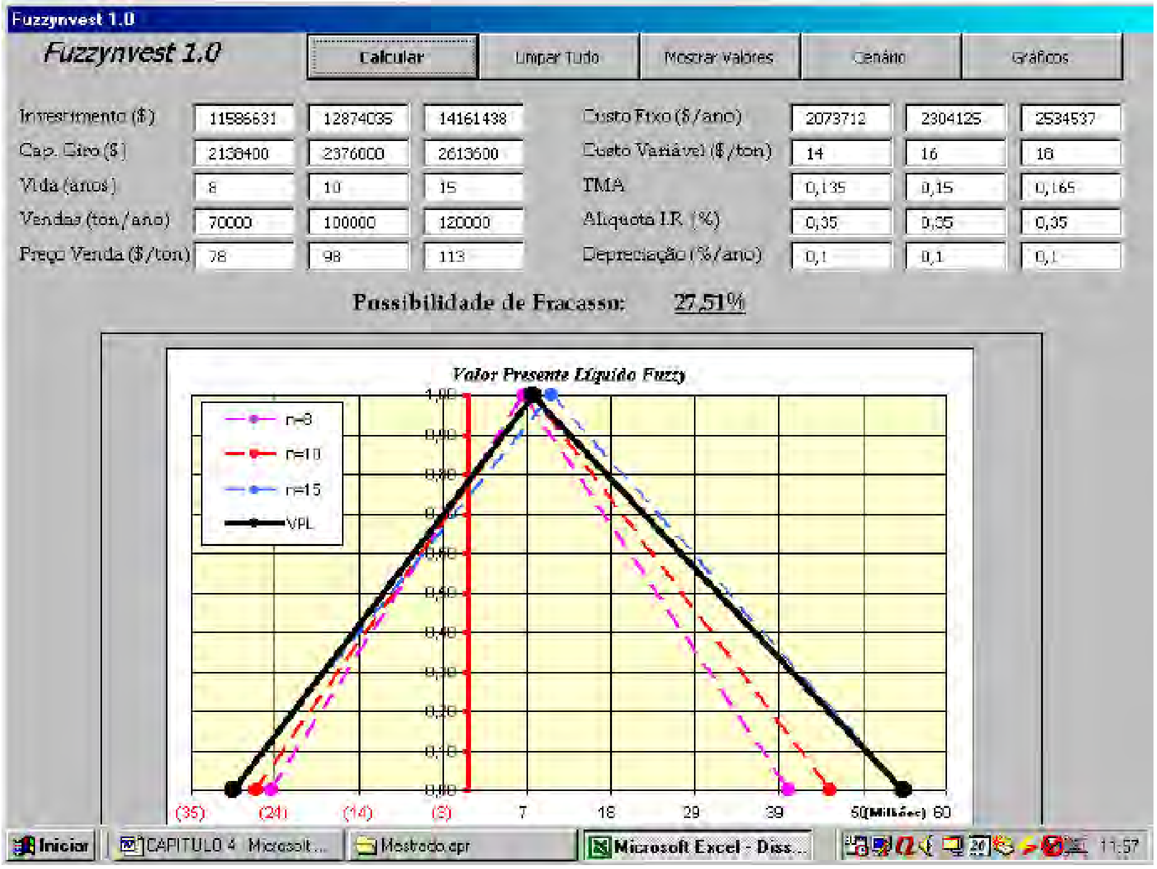


Figure 7. The fuzzy invest 1.0 version (Source[31]).

Fig.4 (Source: [40]) *The decision making process*

Graph .1 (Source: [41]).

Graph.2 (Source: [41]).

Fig.5 Triangular Fuzzy Number structure. (Source: [33]).

Fig.6 “Membership” function (Source:[33])