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ANALYTIC PRICING OF AMERICAN PUT OPTIONS

by

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Abstract

American options are the most commonly traded financial derivatives in the market. Pricing these options fairly, so as to avoid arbitrage, is of paramount importance. Closed form solutions for American put options cannot be utilised in practice and so numerical techniques are employed. This thesis looks at the work done by other researchers to find an analytic solution to the American put option pricing problem and suggests a practical method, that uses Monte Carlo simulation, to approximate the American put option price. The theory behind option pricing is first discussed using a discrete model. Once the concepts of arbitrage-free pricing and hedging have been dealt with, this model is extended to a continuous-time setting. Martingale theory is introduced to put the option pricing theory in a more formal framework. The construction of a hedging portfolio is discussed in detail and it is shown how financial derivatives are priced according to a unique risk-neutral probability measure. Black-Scholes model is discussed and utilised to find closed form solutions to European style options. American options are discussed in detail and it is shown that under certain conditions, American style options can be solved according to closed form solutions. Various numerical techniques are presented to approximate the true American put option price. Chief among these methods is the Richardson extrapolation on a sequence of Bermudan options method that was developed by Geske and Johnson. This model is extended to a Repeated-Richardson extrapolation technique. Finally, a Monte Carlo simulation is used to approximate Bermudan put options. These values are then extrapolated to approximate the price of an American put option. The use of extrapolation techniques was hampered by the presence of non-uniform convergence of the Bermudan put option sequence. When convergence was uniform, the approximations were accurate up to a few cents difference.

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Contents

List of Tables

List of Figures

Chapter 1

Introduction

Derivative Securities are financial instruments that promise some payment on a future date depending on the price movement of the underlying stock. An option is a type of derivative security that gives the holder the right and not the obligation to exercise the claim at maturity. The two most common forms of vanilla options are European and American options.

European Options

A European call (put) option is an option that entitles the owner of the option to buy (sell) a financial asset for a fixed price K, the strike price, at the expiry date of the contract. The option holder may choose whether or not to exercise the option.

American Options

An American call (put) option is an option that entitles the owner of the option to buy (sell) a financial asset for a fixed price K at any time up until the expiry date of the contract. The option holder may choose whether or not to exercise the option.

1.1 Pricing American Put Options

European options are fairly easy to price since they can only be exercised at one point in time, namely the expiration date. American options on the other hand can be exercised at any point up to expiration. It is this ability to be exercised at any point that makes American options more difficult to price than their European counterparts. It will be shown that it is never optimal to exercise an American call option before expiration. As a result an American and European call option for the same underlying stock with the same strike price and expiration date are identical in value. The difficult task is trying to find a price for the American put option. Because American options are the most commonly traded options in the market, a solution to the pricing problem is of great importance.

The basic idea of risk-neutral pricing is dealt with in chapter 2. In order to develop the ideas behind the pricing process a discrete-time model is set up. This is extended to a continuous-time model in chapter 3 where geometric Brownian motion is used to model the stock price process. The highly celebrated Black-Scholes model is set up in chapter 4. This model is used as the basis for trying to solve the American pricing problem. In chapter 5 American options are discussed in detail as well as the conditions under which early expiration is beneficial. Numerical methods are currently used to approximate the true value of American put options. Geske and Johnson [10] developed a technique that was modified by Chang, et al. [6] and uses Richardson extrapolation to solve the pricing problem. The theory behind this technique is discussed in detail in chapter 6. A simple Monte Carlo simulation is run to obtain estimates of the true values of American put options.

Chapter 2

Discrete-Time Risk-Neutral Valuation Model

In order to price a derivative claim on some underlying stock that expires at time t, $t > 0$, we set up a simple binary model to track the movement of the stock price process. The basic assumptions are that the stock can only be observed at two time points, namely 0 and t. Starting at time 0 the stock's value can become one of two possible values at time t. The derivative claim on this stock promises some payoff that depends on the value of the stock at time t. The question that now arises is how much this derivative security should be worth at time 0. An initial guess would be to determine the expected value of the payoff of the claim under the market probability measure (discounting to bring the value back to time 0). However, this would not be a fair estimate of the discounted price of the claim because it may be possible for the holder to make a risk-free profit regardless of the stock's value. A situation in which a profit can be extracted with no risk is known as an arbitrage opportunity. Since market forces move to eliminate arbitrages we need a model with no arbitrage opportunities as a basic assumption.

Thus, in order to price a claim we construct a hedging portfolio that consists of units of the underlying security and cash bonds. A hedging portfolio on a derivative security replicates the payoff of the derivative for all possible values of the stock at expiration. In order to avoid an arbitrage, the time-zero value of the derivative must be the same as the value of the portfolio at time 0. If they were different, then buying the cheaper one and selling the more expensive one would result in a risk free profit. It will be shown that if a claim can be hedged, then the time-zero value of the claim is equal to the discounted expected value of the payoff of the claim under a new probability measure. This new probability measure is equivalent to the market measure and is known as the risk-neutral (or equivalent martingale) probability measure. Even though we will use expectation to price derivative claims, it is important to remember that hedging is the underlying mechanism that ensures an arbitrage-free price.

The basic binary model assumed is very crude and so it will be extended to a multiperiod binomial model. This model assumes that the stock process can be observed at a finite number of time points and can be thought of as a number of binary trees strung together with branches that recombine. By letting the number of observable time points increase and the time between observations decrease, the continuous-time stock process model can be constructed as the limit of the binomial model.

2.1 Pricing Strategy Using a Simple Binary Model

Consider a simple model where the price of a financial asset is observed at time 0 and at time t and interest rates are zero. Furthermore, assume that the asset can only be in one of two possible states at time t. Let the asset be priced as S_0 at time 0 and as either S_u or S_d at time t (where $S_d < S_0 < S_u$). Let C be the time-0 value of a European call option to purchase the asset at time t for strike price K. What is the value of C?

C can be calculated as follows:

If S_t is the value of the asset at time t then $S_t = S_u$ or $S_t = S_d$. If the option is exercised then the holder of the option realises a payoff of $S_t - K$. The option will only be exercised if $S_t > K$ since the option holder will want to pay the cheapest price for the asset. Thus the payoff will be 0 if $S_t \leq K$.

$$Payoff = (S_t - K)^+ = max(S_t - K, 0)$$

Now suppose that you are told that $S_t = S_u$ with probability p and $S_t = S_d$ with probability $1-p$. One might think that the fair value of C would simply be the expected payoff of the option under the probability measure $(p, 1-p)$. This will be shown to be incorrect. The reason for this is that taking such an expectation may result in an arbitrage opportunity.

Arbitrage Opportunity

An arbitrage opportunity is an opportunity to obtain a risk-free profit (Etheridge [9]). It can be achieved by taking advantage of the price difference between identical or similar financial instruments in the market. Suppose $S_t = \$100$, $S_u = \$200$, $S_d = \$50$, $p = 0.5$, $r = 0$ and $K = \$100$. Let C be the value of a call option at time 0.

$$\begin{aligned} E[Payoff] &= E[(S_t - K)^+] \\ &= (\$200 - \$100)^+ \times 0.5 + (\$50 - \$100)^+ \times 0.5 \\ &= \$100 \times 0.5 \\ &= \$50 \end{aligned}$$

Thus if $C = \$50$ then an arbitrage opportunity is available. To see this suppose that at time 0 you borrow \$50 from the bank and sell a call option for \$50. You then have \$100 which you use to buy one unit of stock. At this time you owe the bank \$50. At time t if the value of the security is \$200 then the option is exercised and you sell the security for \$100. You then realise a profit of \$50. If the value of the security is \$50 then the option is not exercised and you sell the security for \$50. You use the \$50 to repay your debt. Either way no loss is incurred which means an arbitrage opportunity is present.

The reason for avoiding arbitrage opportunities in the pricing of our options is that market forces move to eliminate such opportunities. If it is discovered that such an opportunity is available then everyone will start enjoying these risk-free profits and the market will not be in a state of equilibrium. The question now is how to price a financial derivative in order to avoid an arbitrage opportunity? The answer is to construct a hedging portfolio (also known as a replicating portfolio).

Present Value Analysis

In the previous examples it was assumed for the sake of simplicity that interest rates were zero. We now need to incorporate interest rates into the model. We need a model for the time value of money. If the value of a currency is 1 now then in t time units it will be worth e^{rt} (where the constant r ($r > 0$) is the continuously compounded interest rate).

The Law of One Price

If the present value payoff of two investments are identical then either the investments have the same cost or there is an arbitrage opportunity (Ross [16]). This is intuitive since if two investments with the same payoff function differed in their initial cost then an arbitrage could be obtained by buying the cheaper investment and selling the more expensive one.

There is no risk because when the investments mature the respective payoffs will cancel each other out. Thus our pricing problem is solved by constructing a hedging portfolio that has the same payoff at each time step as the financial derivative we wish to price (the portfolio consists of units of the underlying security and units of a cash bond). Now, because an investment that perfectly mimics the payoff of our desired financial derivative can be constructed, by the Law of One Price we know that the two financial instruments must have the same present value cost.

Hedging Strategy

A portfolio is said to hedge, or replicate, a derivative claim if the value of the underlying security coincides with the value of the derivative no matter what the value of the underlying asset (Kijima [12]). A claim C at expiration time T is attainable if it can be hedged, i.e. if there is a portfolio with the same value as C at time T. A market is complete if every claim is attainable. It will be shown that using a hedging strategy will result in a probability measure under which the discounted expected payoff of the derivative claim will be a fair price for the derivative security. There are basic assumptions that are taken for granted when hedging a claim. These underlying assumptions are (Shreve [18]):

- Unlimited short selling of stock
- Unlimited borrowing of cash bonds
- Zero transaction costs
- Trading does not affect the market.

Consider a derivative security with time-zero value D_0. In order to hedge the claim against us, we construct a portfolio (ϕ, ω) at time 0, where ϕ is the number of units of the underlying stock and ω is the number of units of the cash bond. Let V_0 be the value of the portfolio at time 0. Letting S_0 and B_0 be the time-zero values of the stock and cash bond respectively, the value of the portfolio at time 0 is:

$$V_0 = \phi S_0 + \omega B_0$$

At time t the stock is valued as one of two possible stock prices, namely S_u or S_d. A derivative claim on the stock will deliver a payoff that is contingent on the value of the stock (Baxter & Rennie [2]). Let $f(S_u)$ and $f(S_d)$ be the payoff functions associated with the two possible stock values. In order to hedge our claim we set up the following two equations:

$$f(S_u) = \phi S_u + \omega B_0 e^{rt}$$

$$f(S_d) = \phi S_d + \omega B_0 e^{rt}$$

The portfolio must exactly replicate the payoff of the claim regardless of the movement of the stock. Solving the two simultaneous equations we can determine the cost of the portfolio.

$$\phi = \frac{f(S_u) - f(S_d)}{S_u - S_d}$$

$$\omega = \frac{e^{-rt}}{B_0}(f(S_u) - \frac{f(S_u) - f(S_d)}{S_u - S_d} S_u)$$

The time zero-value of the portfolio is:

$$V_0 = \phi S_0 + \omega B_0 \quad = \quad \frac{f(S_u) - f(S_d)}{S_u - S_d} S_0 + e^{-rt}(f(S_u) - \frac{f(S_u) - f(S_d)}{S_u - S_d} S_u)$$

In order for the derivative security to have an arbitrage-free price, $D_0 = V_0$. Because the two investments have the same payoff function for all values of the underlying stock, by the Law of One Price they must have the same present value cost.

Hedging Portfolio Example

Using the call option example used earlier we will construct a portfolio that will have the same present value payoff as the call option we are trying to price. In our previous example consider selling the option and using the money to purchase ϕ stocks and ω cash bonds. If there is not enough money to finance the portfolio then more will have to be borrowed from the bank. In order to meet the claim against us we need to have

$$200\phi + \omega = 100 \quad if \quad S_t = 200$$

$$50\phi + \omega = 0 \quad if \quad S_t = 50$$

Solving for ϕ and ω results in $\phi = \frac{2}{3}$ and $\omega = -\frac{100}{3}$. Using the Law of One Price we can conclude that the cost of the portfolio is identical to the cost of the option. The time-zero value of the portfolio is thus

$$\begin{aligned} 100\phi + \omega &= 100 \times \frac{2}{3} + (-\frac{100}{3}) \\ &= 33.\dot{3} \end{aligned}$$

This is the fair price of the option and will result in no arbitrage opportunities. The claim was thus hedged by the replicating portfolio and is said to be attainable.

Risk-Neutral Probability Measure

Simply taking the present value of the expected payoff from a claim did not produce a fair price for the claim. However, expectations will in fact be used to determine the fair price of the claim. The market probability measure must be discarded in favour of another equivalent probability measure that will be instrumental in constructing a hedging portfolio. To demonstrate the validity of this approach, consider the hedging portfolio discussed earlier. The time-zero value of the portfolio (and the derivative security) is:

$$V_0 = \frac{f(S_u) - f(S_d)}{S_u - S_d} S_0 + e^{-rt}(f(S_u) - \frac{f(S_u) - f(S_d)}{S_u - S_d} S_u)$$

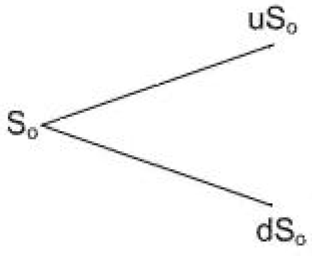
Suppose $S_u = uS_0$ and $S_d = dS_0$, where $d < e^{rt} < u$. Then

$$\begin{aligned} V_0 &= \frac{f(S_u) - f(S_d)}{u - d} + e^{-rt}(f(S_u) - \frac{f(S_u) - f(S_d)}{u - d} u) \\ &= \frac{f(S_u) - f(S_d)}{u - d} + e^{-rt} \frac{f(S_u)u - f(S_u)d - f(S_u)u + f(S_d)u}{u - d} \\ &= e^{-rt}(\frac{e^{rt} f(S_u) - e^{rt} f(S_d) - f(S_u)d + f(S_d)u}{u - d}) \\ &= e^{-rt}(\frac{e^{rt} - d}{u - d} f(S_u) + \frac{u - e^{rt}}{u - d} f(S_d)) \end{aligned}$$

Let $q = \frac{e^{rt} - d}{u - d}$. Then $1 - q = \frac{u - e^{rt}}{u - d}$. Thus $q + 1 - q = 1$ and

$$\begin{aligned} d < e^{rt} < u &\Rightarrow 0 < e^{rt} - d < u - d \\ &\Rightarrow 0 < \frac{e^{rt} - d}{u - d} < 1 \\ &\Rightarrow 0 < q < 1 \\ &\Rightarrow 0 < 1 - q < 1 \end{aligned}$$

Figure 2.1: Simple Binary Model



Therefore $(q, 1-q)$ satisfies the conditions of a probability measure and is known as the risk neutral probability measure. The price of the derivative security at time 0 is

$$V_0 = e^{-rt}(qf(S_u) + (1-q)f(S_d))$$

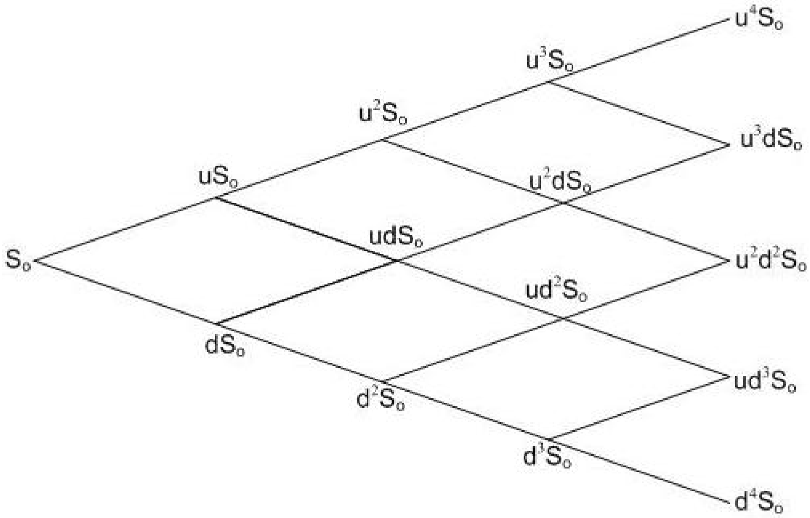
In other words, the discounted expected payoff of the derivative security under the risk-neutral probability measure is the fair price of the derivative security. It must be noted that it is because of our hedging strategy that this process works. The arbitrage free price of a claim will now be expressed as an expectation under an equivalent probability measure but it must be remembered that the hedging portfolio is the driving force behind the solution to our pricing problem.

2.2 Pricing Strategy Using a Binomial Model

Before we assumed a very simple model where there were only two time periods and only two possible values at which the security could be priced. We need to extend this model to make it more realistic. In order to hedge a derivative security on such a simple model we came up with a portfolio that consisted of two assets, namely the underlying security and a cash bond. Two simultaneous equations were set up to hedge the claim. This resulted in solving two equations with two unknown values. Now if we assume that the stock may be priced as one of three possible values then it can be shown that three tradable assets are required to hedge the derivative claim[1]. We run into problems as the number of possible stock values increases as we need as many tradable assets as possible stock values. However, this problem is easily overcome by using a binomial model that has recombinant branches.

[1]For more details see Etheridge [9]

Figure 2.2: Binomial Tree



Multiperiod Binary Model and Multiperiod Binomial Model

We assume as before that we have two tradable assets, namely the cash bond and underlying stock. The same basic assumptions are still maintained as before. Let $t = \frac{T}{N}$. We assume that the market is observable at times $0 < t < 2t < \ldots < (N-1)t < T$. Now over each time period $[it, (i+1)t]$ $(i = 1, 2, \ldots, N-1)$ we use the simple binary model as before. Thus after i periods there are 2^i possible stock values. A much simpler model is a multiperiod Binomial model. In this model, branches of the tree can recombine, and if u, d and r remain constant then the risk-neutral probability measure will remain constant on each upward branch and so the stock price is determined by a Binomial distribution (Etheridge [9]).

Binomial Model Hedging Strategy

The same basic assumptions that were present in the simple binary model hold true in the binomial case as well. A claim can be hedged with this model by constantly readjusting our portfolio by buying and selling units of the security and cash bond at each time point. The only constraint is that no additional funds may be introduced when buying or selling these units. This is known as the Self-Financing Property.

Starting with the initial price S_0, the stock price can become one of two values at the time instance t, namely uS_0 or dS_0. Again, at $2t$ the stock price can only become one of two values (even though there are three different stock prices at $2t$), but these values depend on the time-t value of the stock. For example, if $S_t = uS_0$ then either $S_{2t} = u^2 S_0$ or $S_{2t} = udS_0$. Carrying on in this fashion, the number of stock prices increases with the number of time instances. However, dependent on the value of the stock at a given time instance there will be only two possible values that the stock price can assume at the next time instant. This idea is formalised later with the concept of a filtration in a filtered probability space.

In order to hedge a claim on some underlying stock we work backwards. Consider a binomial tree with N periods. By observing a node at time $(N-1)t$ that branches out to two nodes at T, a portfolio (ϕ, ω) can be constructed in exactly the same way as for the simple binary model. The value of this portfolio at time $(N-1)t$ is treated like the new payoff function for the claim. Thus at each $(N-1)t$ node the value of the hedging portfolio is calculated. Taking one step backwards a new portfolio is constructed exactly the same as before. This process is repeated until the time-zero value of the portfolio is determined. The stock and cash bond holdings of the portfolio at time nt, $0 \leq n \leq N-1$, is:

$$\phi = \frac{f(uS_{nt}) - f(dS_{nt})}{uS_{nt} - dS_{nt}}$$

$$\omega = \frac{e^{-rt}}{B_{nt}}(f(uS_{nt}) - \frac{f(uS_{nt}) - f(dS_{nt})}{u-d}u)$$

The portfolio is thus constantly adjusted at each time point. No new funds are introduced into the portfolio. The portfolio is adjusted by buying and selling the units of the stock and cash bond.

Binomial Model Risk-Neutral Probability Measure

Using the simple binary model we could price a derivative security by calculating its discounted expected payoff with respect to a risk-neutral probability measure. The same can be done with the binomial model. At each time point the value of the stock can increase by u times the current value or decrease by d times the current value. Thus at each node of the tree, the risk-neutral probability measure specifies an increase in the value of the stock with probability $q = \frac{e^{rt}-d}{u-d}$ and a decrease with probability $1-q$. The probability that the stock price follows a particular path through the tree is the product of the probabilities of each branch taken multiplied by the number of paths that reach the particular stock price (Baxter & Rennie [2]). For example, $S_{2t} = u^2 S_0$ with probability q^2, $S_{2t} = udS_0$ with probability $2q(1-q)$ and $S_{2t} = d^2 S_0$ with probability $(1-q)^2$.

The expectation of some claim on the final nodes of the tree can be expressed as the sum of the final claim values weighted by the path probabilities that the values follow (Baxter & Rennie [2]).

2.3 Filtered Probability Space and Discrete Parameter Martingales

The ultimate goal is to be able to price financial derivatives in a continuous framework. The simple binary and binomial models were constructed in order to gain a greater understanding of the concept of risk-neutral (or arbitrage free) pricing. To formalise the work done so far we need to discuss the idea of discrete parameter martingales. A martingale is the mathematical equivalent of a fair game. We can also talk about supermartingales and submartingales. A supermartingale (submartingale) is the mathematical equivalent of an unfavourable (favourable) game. Previously, we priced a financial derivative by calculating the discounted expected payoff of the claim under a risk-neutral probability measure. It will be shown that this risk-neutral probability measure is the probability measure that makes the discounted stock price a martingale. As a result, any claim on that stock will be priced according to the same risk-neutral probability measure, i.e. the probability measure is unique. It is stressed that the hedging portfolio process is the driving force behind this pricing approach. It must be possible that every possible derivative claim can be hedged in order for this martingale measure to be unique.

Filtered Probability Space

Consider the recombinant binomial tree discussed earlier. At each time point we can determine the value of the derivative security and its hedging portfolio based on the value of the underlying stock at that time point. The value of the stock depends on the upward or downward movement of the stock process. For example, if the time-zero value of the stock is 100, $u = 2$, $d = \frac{1}{2}$ and the value of the stock at time point $2t$ is 100 then one of two stock movements are possible. Either the stock value increased to 200 at time t and then decreased to 100 at time $2t$, or the stock first dropped in value to 50 at time t and then increased to 100 at time $2t$. This is the information that is known up to time $2t$. We need to formalise this concept of information up to a certain time point.

Consider the probability space[2] (Ω, Σ, P). That is, consider the probability measure P that specifies the probability for each $A \in \Sigma$ as well as an increasing sequence of σ-algebras, $\Sigma_n \subseteq \Sigma_{n+1} \subseteq \cdots \subseteq \Sigma$. Now, the collection of σ-algebras $\{\Sigma_n\}_{n\geq 0}$ is called a filtration and $(\Omega, \Sigma, \{\Sigma_n\}_{n\geq 0}, P)$ is called a filtered probability space (Etheridge [9]).

The σ-algebra Σ_n can be thought of as a way to contain all of the information of the stock movement up to time point nt. To simplify notation it will be assumed that $t = 1$. A real-valued random variable X is Σ_n-measurable if $\{x_1 < X \leq x_2\} \in \Sigma_n$ for all $x_1 < x_2$. A stochastic process $\{X_n\}_{n\geq 0}$ is adapted to the filtration if X_n is $\{\Sigma_n\}_{n\geq 0}$-measurable for each n. X is said to be integrable if $E[\mid X \mid] < \infty$. The stochastic process $\{X(t)\}$ is integrable if $E[\mid X(t) \mid] < \infty$ for all values of t (Kijima [12]).

Discrete Parameter Martingales

Let $(\Omega, \Sigma, \{\Sigma_n\}_{n\geq 0}, P)$ be a filtered probability space. The stochastic process $\{X_n\}_{n\geq 0}$ is a martingale with respect to the probability measure P and the filtration $\{\Sigma_n\}_{n\geq 0}$ if

1. $E[\mid X_n \mid] < \infty$
2. $E[X_{n+1} \mid \Sigma_n] = X_n$ for all $n \geq 0$

If $E[X_{n+1} \mid \Sigma_n] \leq X_n$ ($E[X_{n+1} \mid \Sigma_n] \geq X_n$) for all n then $\{X_n\}_{n\geq 0}$ is said to be a supermartingale (submartingale) with respect to the probability measure P and the filtration $\{\Sigma_n\}_{n\geq 0}$.

A martingale has been described as the mathematical equivalent of a fair game. To see this, consider $\{X_n\}_{n\geq 0}$, the process that tracks the total winnings of some arbitrary game. Thus, at time n, the total winnings are X_n. If the expected total winnings at time $n + 1$ (given that we know all the information up until time n) is simply the total winnings at time n then $\{X_n\}_{n\geq 0}$ is a martingale. In this fair game, the player is expected to break even at every time point. For a supermartingale (submartingale) the player is expected to make a loss (profit).

[2]See the appendix for definitions of measure spaces and probability spaces.

Risk-Neutral (Martingale) Probability Measure

Now that the concept of a martingale has been established, we can proceed with trying to solve the pricing problem. This is done by discarding the market probability measure in favour of an equivalent risk-neutral probability measure.

Let P denote the market probability measure and let Q denote the risk-neutral probability measure. The measures P and Q on the same probability space Ω are equivalent if for all $A \subseteq \Omega$

$$Q(A) = 0 \Longleftrightarrow P(A) = 0$$

Now we state and prove the following important result:

Under the risk-neutral probability measure, the discounted stock price process $\{e^{-rn}S_n\}$ *, where* S_n *is* Σ_n*-measurable, is a martingale.*

Proof:

Recall that the probability of an upward (downward) stock movement is $q = \frac{e^r - d}{u-d}$ $(1 - q = \frac{u-e^r}{u-d})$.

$$\begin{aligned} E^Q[e^{-r(n+1)}S_{n+1} \mid \Sigma_n] &= e^{-r(n+1)}(qu + (1-q)d)S_n \\ &= e^{-r(n+1)}(u(\frac{e^r - d}{u-d}) + d(\frac{u-e^r}{u-d}))S_n \\ &= e^{-r(n+1)}(\frac{e^r u - e^r d}{u-d})S_n \\ &= e^{-rn}(\frac{u-d}{u-d})S_n \\ &= e^{-rn}S_n \end{aligned}$$

The risk-neutral probability measure is also known as the martingale probability measure as it is the probability measure under which the discounted stock price process is a martingale.

Tower Property of Conditional Expectations

Consider two σ-algebras Σ_j and Σ_k such that $\Sigma_j \subseteq \Sigma_k$. The tower property of conditional expectations says that for any integrable random variable X:

$$E[E[X \mid \Sigma_k] \mid \Sigma_j] = E[X \mid \Sigma_j]$$

Let $\{X_n\}_{n\geq 0}$ be a $(P, \{\Sigma_n\}_{n\geq 0})$-martingale and let $j < k$. Then, using the tower property it can be shown that:

$$E[X_j \mid \Sigma_k] = X_j$$

Risk-Neutral Value of Attainable Claim

Let V_n be the time n value of a derivative claim with payoff V at its expiry date T. In the absence of arbitrage, the unique time n value of the derivative claim is:

$$V_n = e^{-r(T-n)} E^Q[V \mid \Sigma_n]$$

where the expectation is under the probability measure Q for which the discounted stock price is a martingale. The time zero price of the claim is:

$$\begin{aligned} V_0 &= e^{-rT} E^Q[V \mid \Sigma_0] \\ &= e^{-rT} E^Q[V] \end{aligned}$$

It turns out that the discounted claim process $\{\widetilde{V}_n\}_{n\geq 0}$, where $\widetilde{V}_n = e^{-rn} V_n = e^{-rT} E^Q[V \mid \Sigma_n]$, is also a $(Q, \{\Sigma_n\}_{n\geq 0})$-martingale. In fact, any hedgable claim on the underlying stock will be a $(Q, \{\Sigma_n\}_{n\geq 0})$-martingale.

Prove that $\{\widetilde{V}_n\}_{n\geq 0}$, *is also a* $(Q, \{\Sigma_n\}_{n\geq 0})$*-martingale:*

Let $X_n = E^Q[V \mid \Sigma_n]$. For $j < k$,

$$\begin{aligned} E^Q[X_k \mid \Sigma_j] &= E^Q[E^Q[V \mid \Sigma_k] \mid \Sigma_j] \\ &= E^Q[V \mid \Sigma_j] \qquad (\textit{by the tower property}) \\ &= X_j \end{aligned}$$

Therefore, $\{X_n\}_{n\geq 0}$ is a $(Q, \{\Sigma_n\}_{n\geq 0})$-martingale. Now,

$$\begin{aligned} E^Q[\widetilde{V}_k \mid \Sigma_j] &= E^Q[e^{-rk} e^{-r(T-k)} E^Q[V \mid \Sigma_k] \mid \Sigma_j] \\ &= e^{-rT} E^Q[X_k \mid \Sigma_j] \\ &= e^{-rT} X_j \\ &= \widetilde{V}_j \end{aligned}$$

Thus the discounted price process of a derivative claim is a $(Q, \{\Sigma_n\}_{n\geq 0})$-martingale and this implies that the discounted hedging process is also a $(Q, \{\Sigma_n\}_{n\geq 0})$-martingale.

Unique Martingale Measure

In a complete market every derivative claim is attainable, i.e. every claim can be hedged. A complete market is one of our main assumptions and will result in a unique martingale measure to price all financial derivatives on the same underlying stock. This result is now proved:

The market is complete if and only if the martingale measure, Q, is unique.

Proof:

Suppose the market is complete and arbitrage free and let Q and $\hat{Q}$ be two equivalent risk-neutral probability measures. Because the market is complete, every claim is attainable. Thus if V is the value of the claim at expiration, then the time zero price of the claim is $E^{Q}[\widetilde{V}]$ under Q and $E^{\hat{Q}}[\widetilde{V}]$ under $\hat{Q}$, where $\widetilde{V}$ is the discounted value of V. By the Law of One Price we have $E^{Q}[\widetilde{V}] = E^{\hat{Q}}[\widetilde{V}]$. Since the interest rate is the same for both processes we have: $E^{Q}[V] = E^{\hat{Q}}[V]$. Therefore $Q = \hat{Q}$.

Stopping Time

For a European derivative claim we know the expiration date of the claim and thus the date at which the claim is exercised. For American derivatives, early exercise means that the date at which exercising occurs is not known at time zero. Pricing these claims is more difficult because the time at which exercising occurs is a random variable. In order to deal with this complication the concept of stopping times needs to be discussed.

Given $(\Omega, \Sigma, \{\Sigma_n\}_{n\geq 0})$, a stopping time is defined as a random variable:

$$\tau : \Omega \to \mathbb{Z}_+, \; where \, \{\tau \leq n\} \in \Sigma_n, \; for \, all \, n \geq 0.$$

i.e. the event $\{\tau \leq n\}$ depends only on the history of the process up to and including time n. Thus by observing the movement of the stock process up until time n, we can determine whether or not the event has occured. For example, the date at which an American option is exercised is a stopping time. By observing the stock process up until time n we can determine if the option has been exercised or not.

Optional Stopping Theorem

In order to successfully price American derivatives we need to combine the concepts of martingales and stopping times. Let $(\Omega, \Sigma, \{\Sigma_n\}_{n\geq 0}, P)$ be a filtered probability space.

If τ is a bounded stopping time and $\tau \geq n$, where $n \geq 0$, then the Optional Stopping Theorem[3] says that:

$$E[X_\tau \mid \Sigma_n] \leq X_n \qquad if\ \{X_n\}_{n\geq 0}\ is\ a\ (P, \{\Sigma_n\}_{n\geq 0}) - supermartingale$$

$$E[X_\tau \mid \Sigma_n] \geq X_n \qquad if\ \{X_n\}_{n\geq 0}\ is\ a\ (P, \{\Sigma_n\}_{n\geq 0}) - submartingale$$

$\{X_n\}_{n\geq 0}$ is a $(P, \{\Sigma_n\}_{n\geq 0})$-martingale if it is both a $(P, \{\Sigma_n\}_{n\geq 0})$-supermartingale and $(P, \{\Sigma_n\}_{n\geq 0})$-submartingale, i.e.

$$E[X_\tau \mid \Sigma_n] = X_n$$

Previsible Hedging Process

When pricing a derivative claim, arbitrage is avoided by constructing a hedging portfolio that replicates the claim at each time point. Let $[\phi_n, \omega_n]$ be the amount of underlying stock and cash bond in the hedging portfolio over the time interval $[n-1, n]$. Recall that the stock holding in the hedging portfolio was $\phi_n = \frac{f(uS_{n-1})-f(dS_{n-1})}{uS_{n-1}-dS_{n-1}}$. Thus ϕ_n is known at time $n-1$. Regardless of the stock movement at time n, the value of the stock holding is known at time $n-1$. Thus ϕ_n is Σ_{n-1}-measurable for all $n \geq 1$. The process $\{\phi_n\}_{n\geq 1}$ is said to be $\{\Sigma_n\}_{n\geq 0}$-previsible (or predictable), for a given filtration $\{\Sigma_n\}_{n\geq 0}$.

2.4 The Binomial Representation Theorem

The Binomial Representation Theorem is the discrete counterpart to the Martingale Representation Theorem which will be defined later. It is of great importance since it tells us that we can express the discounted payoff of our derivative claim in terms of the discounted value of the underlying security. It also shows that a hedging strategy exists and gives us a way to determine the stock holdings in our replicating portfolio.

Let Q be the probability measure under which the discounted binomial price process, $\{\widetilde{S}_n\}_{n\geq 0}$ is a martingale and let $\{\widetilde{V}_n\}_{n\geq 0}$ be another Q-martingale with respect to the filtration $\{\Sigma_n\}_{n\geq 0}$. Then there is a $\{\Sigma_n\}_{n\geq 0}$-previsible process $\{\phi_n\}_{n\geq 1}$ such that:

[3] See Etheridge [9] for a proof of this theorem.

$$\widetilde{V}_n = \widetilde{V}_0 + \sum_{j=0}^{n-1} \phi_{j+1}(\widetilde{S}_{j+1} - \widetilde{S}_j)$$

The above equation[4] is the discrete form of a stochastic integral that will be discussed later. The work done so far to replicate a claim (and hence price the claim) can be summarised in three steps:

- Find a probability measure Q under which the discounted stock price is a martingale.
- Take the expected value, under the measure Q, of the discounted claim V at time T, i.e. $\widetilde{V}_n = e^{-rn}V_n = E^Q[e^{-rT}V \mid \Sigma_n]$.
- Find a previsible process $\{\phi_n\}_{n\geq 1}$ such that $\widetilde{V}_n = \widetilde{V}_0 + \sum_{j=0}^{n-1} \phi_{j+1}(\widetilde{S}_{j+1} - \widetilde{S}_j)$ (or $\triangle \widetilde{V}_n = \phi_n \triangle \widetilde{S}_n$).

Now that the key concepts of risk-neutral pricing have been established it is time to extend this model to a continuous-time setting. The continuous model can be thought of as the binomial model, except that there are infinitely many time points within the time interval $[0, T]$ and the distance between these time points tends to zero. As a result, the range of values for the stock price at any particular time point is infinite.

[4]For proof of the Binomial Representation Theorem see Etheridge [9].

Chapter 3

Continuous-Time Risk-Neutral Valuation Model

The discrete-time models were used to gain a deeper understanding of the inner workings of hedging and risk-neutral pricing. Now that this has been established a continuous-time model will be used. Geometric Brownian motion is used as a model of the fluctuations of stock prices. In order to study this model we need to use stochastic calculus. Similarly as in the discrete model, we discard the market probability measure in favour of a new probability measure which results in discounted asset prices being martingales. This measure is constructed using Girsanov's Theorem. In order to construct the hedging portfolio the Martingale Representation Theorem is employed.

3.1 Brownian Motion

Now that the concept of martingales has been developed, we are one step closer to formalising the continuous-time model for pricing financial derivatives in a risk-neutral framework. Geometric Brownian motion is used as the continuous-time model for our pricing problem. This section details Brownian motion and its properties, which will be instrumental in setting up the Black-Scholes financial model. Brownian motion can be thought of as an 'infinitesimal' random walk (Etheridge [9]). The concept of a simple random walk is first discussed to give greater insight into the definition of Brownian motion.

Simple Random Walk

Let $\xi_j \in \{-1, 1\}$, where $P[\xi_j = 1] = p$ and $P[\xi_j = -1] = 1 - p$ ($p \in [0, 1]$). $\{\xi_j\}_{j \geq 0}$ are independent, identically distributed random variables. Then the Markov process $\{S_n\}_{n \geq 0}$, where $S_0 = 0$ and $S_n = \sum_{j=1}^{n} \xi_j$, is a simple random walk. $-n \leq S_n \leq n$, since S_n can move at most n away from 0. Under what conditions will a random walk be a martingale?

$$-n \leq S_n \leq n \quad \Rightarrow \quad E[\mid S_n \mid] < \infty,$$

$$\begin{aligned} E[S_{n+1} \mid \Sigma_n] &= E[\sum_{j=1}^{n+1} \xi_j \mid \Sigma_n] \\ &= E[S_n + \xi_{n+1} \mid \Sigma_n] \\ &= S_n + E[\xi_{n+1} \mid \Sigma_n] \\ &= S_n + E[\xi_{n+1}] \end{aligned}$$

Now, $E[\xi_{n+1}] = p - (1 - p) = 2p - 1$. If $p = \frac{1}{2}$ then $E[\xi_{n+1}] = 0$. Thus a simple random walk will be a martingale if $p = \frac{1}{2}$. If $p < \frac{1}{2}$ ($p > \frac{1}{2}$) then the random walk will be a supermartingale (submartingale).

If $p = \frac{1}{2}$, then:

$$\begin{aligned} E[S_n] &= E[\sum_{j=1}^{n} \xi_j] \\ &= \sum_{j=1}^{n} E[\xi_j] \\ &= \sum_{j=1}^{n} (\frac{1}{2} - \frac{1}{2}) \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
Var(S_n) &= E[S_n^2] - 0 \\
&= E[(\sum_{j=1}^{n} \xi_j)^2] \\
&= E[\sum_{j=1}^{n} \xi_j^2 + 2\sum_{j<k} \xi_j \xi_k] \\
&= \sum_{j=1}^{n} E[\xi_j^2] + 2\sum_{j<k} E[\xi_j \xi_k] \\
&= \sum_{j=1}^{n} (\frac{1}{2} + \frac{1}{2}) + 2\sum_{j<k} E[\xi_j] E[\xi_k] \\
&= n + 0 \\
&= n
\end{aligned}
$$

$$
\begin{aligned}
cov(S_n, S_m) &= E[S_n S_m] - E[S_n]E[S_m] \\
&= E[S_n S_m] - 0 \\
&= E[E[S_n S_m] \mid \Sigma_{n\wedge m}] \\
&= E[S_{n\wedge m} E[S_{n\vee m}] \mid \Sigma_{n\wedge m}] \\
&= E[S_{n\wedge m} S_{n\wedge m}] \\
&= E[S_{n\wedge m}^2] = Var(S_{n\wedge m}) \\
&= n \wedge m
\end{aligned}
$$

Because $\{\xi_j\}_{j\geq 0}$ are independent random variables it means that, for $0 = j_0 \leq j_1 \leq j_2 \leq \cdots \leq j_n$, $\{S_{j_k} - S_{j_{k-1}}, k \in [1, n]\}$ are independent. Because $\{\xi_j\}_{j\geq 0}$ are identically distributed it means that, for $j - i = l - k$ $(0 \leq i \leq j \leq k \leq l)$, $S_j - S_i$ and $S_l - S_k$ have the same distribution. Thus $\{S_n\}_{n\geq 0}$ has stationary and independent increments. The same will be true for Brownian motion.

Definition of Brownian Motion

Now that the concept of a simple random walk has been discussed, we can now turn to Brownian motion. A real-valued stochastic process $\{W_t\}_{t\geq 0}$ is called a Brownian motion if the following properties are satisfied:

- $W_0 = 0$,

- W_t is a continuous function of $t \geq 0$,
- If $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$, then $\{W_{t_j} - W_{t_{j-1}}\}_{j=0,\ldots,n}$ are independent, i.e. Brownian motion has independent increments.
- For each $s \geq 0$ and $t > 0$, $W_{t+s} - W_s$ is normally distributed with mean zero and variance $\sigma^2 t$. If $\sigma^2 = 1$ then the process is known as standard Brownian motion.

Thus Brownian motion is characterised by having stationary and independent increments. From now on all Brownian motion processes will be assumed to be standard Brownian motion processes.

3.2 Continuous-Time Martingales

Discrete-time martingales have been already been discussed. The definition of continuous-time martingales is very similar to its discrete counterpart and the same logic is used, although now it is in a continuous sense.

Let Σ be a σ-algebra. $\{\Sigma_t\}_{t\geq 0}$ is a continuous-time filtration if

- $\Sigma_t \subseteq \Sigma$, for all t
- $\Sigma_s \subseteq \Sigma_t$, for all $s < t$

Let $(\Omega, \Sigma, \{\Sigma_t\}_{t\geq 0}, P)$ be a filtered probability space. The stochastic process $\{X_t\}_{t\geq 0}$ with $E[\mid X_t \mid] < \infty$ is a supermartingale (submartingale) with respect to the probability measure P and the filtration $\{\Sigma_t\}_{t\geq 0}$ if it is adapted to the filtration $\{\Sigma_t\}_{t\geq 0}$ and $E[X_t \mid \Sigma_s] \leq X_s$ $(E[X_t \mid \Sigma_s] \geq X_s)$ for all $t \geq s$.

$\{X_t\}_{t\geq 0}$ will be a $(P, \{\Sigma_t\}_{t\geq 0})$-martingale if it is both a supermartingale and a submartingale, i.e. $E[X_t \mid \Sigma_s] = X_s$ for all $t \geq s$.

Previsible (Predictable) Process

In the binomial model the stock holding in the hedging portfolio process was said to be previsible or predictable. The same is true in the continuous sense. Let Σ_{t-} be the $\sigma-$algebra generated by $\bigcup_{s<t} \Sigma_s$. The stochastic process $\{X_t\}_{t\geq 0}$ is $\{\Sigma_t\}_{t\geq 0}$-previsible (predictable) if X_t is Σ_{t-}-measurable for all t.

Optional Stopping Theorem

The Optional Stopping Theorem can be extended to a continuous-time setting. To do this, consider two bounded stopping times, τ_1 and τ_2, such that $\tau_1 \leq \tau_2$. The continuous version of the Optional Stopping Theorem says that if $\{X_t\}_{t\geq 0}$ is right continuous with left limits then:

$$E[X_{\tau_2} \mid \Sigma_{\tau_1}] \leq X_{\tau_1} \quad a.s., \quad if\ \{X_t\}_{t\geq 0}\ is\ a\ (P, \{\Sigma_t\}_{t\geq 0}) - supermartingale$$

$$E[X_{\tau_2} \mid \Sigma_{\tau_1}] \geq X_{\tau_1} \quad a.s., \quad if\ \{X_t\}_{t\geq 0}\ is\ a\ (P, \{\Sigma_t\}_{t\geq 0}) - submartingale$$

$$E[X_{\tau_2} \mid \Sigma_{\tau_1}] = X_{\tau_1} \quad a.s., \quad if\ \{X_t\}_{t\geq 0}\ is\ a\ (P, \{\Sigma_t\}_{t\geq 0}) - martingale$$

Further Properties of Standard Brownian Motion

These properties are stated without proof. Their justification comes from the idea that Brownian motion is an 'infinitesimal' random walk.

- $E[W_{t+s} - W_s \mid \{W_j\}_{0\leq j\leq s}] = 0$
- $cov(W_s, W_t) = s \wedge t$
- $\{W_t\}_{t\geq 0}$ is differentiable nowhere almost surely[1]
- $\{W_t\}_{t\geq 0}$ is a $(P, \{\Sigma_t\}_{t\geq 0})$-martingale.

Proof of martingale property:

$$\begin{aligned} E[W_t \mid \Sigma_s] &= E[W_s + (W_t - W_s) \mid \Sigma_s] \\ &= W_s + E[(W_t - W_s) \mid \Sigma_s] \\ &= W_s + 0 \\ &= W_s \end{aligned}$$

Brownian motion itself is insufficient as a model to track the stock price movement. The model that will be used is a function of Brownian motion, namely geometric Brownian motion. The reason that this model is used will be justified later, but before we proceed the concepts of stochastic calculus first need to be developed.

[1] $\mathrm{P}\{\{W_t\}_{t\geq 0}$ is differentiable nowhere$\} = 1$

3.3 Stochastic Calculus

Since Brownian motion is differentiable nowhere almost surely, we cannot use normal differential equations in setting up our pricing model. The reason for this is that a non-differentiable function has nonzero quadratic variation. Stochastic calculus takes quadratic variation into account when setting up stochastic differential equations. Brownian motion will be shown to have nonzero quadratic variation. After this result has been obtained, the Itô stochastic calculus is developed in order to study the continuous-time model that is being used to solve the pricing problem.

Quadratic Variation

Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of the interval $[0, T]$. $\| \Pi \| = \max_{k=0,\dots,n-1} (t_{k+1} - t_k)$ is called the mesh of the partition.

The quadratic variation, $[f]_t$, of a function $f(t)$ on the interval [0,T] is defined as

$$[f]_T = \lim_{\| \Pi \| \to 0} \sum_{k=0}^{n-1} | f(t_{k+1}) - f(t_k) |^2$$

If f is differentiable then in each subinterval $[t_k, t_{k+1}]$ there exists a point t_k^* such that $f(t_{k+1}) - f(t_k) = f^{'}(t_k^*)(t_{k+1} - t_k)$ (by the Mean Value Theorem). Thus for a differentiable function f:

$$\begin{aligned}
\lim_{\| \Pi \| \to 0} \sum_{k=0}^{n-1} | f(t_{k+1}) - f(t_k) |^2 &= \lim_{\| \Pi \| \to 0} \sum_{k=0}^{n-1} | f^{'}(t_k^*) |^2 (t_{k+1} - t_k)^2 \\
&\leq \lim_{\| \Pi \| \to 0} \| \Pi \| \sum_{k=0}^{n-1} | f^{'}(t_k^*) |^2 (t_{k+1} - t_k) \\
&= \lim_{\| \Pi \| \to 0} \| \Pi \| \, lim_{\| \Pi \| \to 0} \sum_{k=0}^{n-1} | f^{'}(t_k^*) |^2 (t_{k+1} - t_k) \\
&= \lim_{\| \Pi \| \to 0} \| \Pi \| \int_0^T | f^{'}(t) |^2 \, dt \\
&= 0
\end{aligned}$$

Thus the quadratic variation of a differentiable function is 0.

Quadratic Variation of a Brownian Motion Process

Brownian motion has a nonzero quadratic variation. This result is proved below:

Let $\{W_t\}_{t\geq 0}$ denote a P-Brownian motion. Then $[W]_T = T$.

Proof:

The basis for this proof comes from Etheridge [9]. Let $W(\Pi) = \sum_{k=0}^{n-1} \mid W_{t_{k+1}} - W_{t_k} \mid^2$. It must be shown that as $\parallel \Pi \parallel \to 0$:

$$E[\mid W(\Pi) - T \mid^2] \quad \to \quad 0.$$

Letting $\eta_{t_k} = \mid W_{t_{k+1}} - W_{t_k} \mid^2 - (t_{k+1} - t_k)$ we have:

$$\begin{aligned}
\mid W(\Pi) - T \mid^2 &= \mid \sum_{k=0}^{n-1} \mid W_{t_{k+1}} - W_{t_k} \mid^2 - T \mid^2 \\
&= \mid \sum_{k=0}^{n-1} [\mid W_{t_{k+1}} - W_{t_k} \mid^2 - (t_{k+1} - t_k)] \mid^2 \\
&= \sum_{k=0}^{n-1} \eta_{t_k}^2 + \sum_{j<k} \eta_{t_j}\eta_{t_k}
\end{aligned}$$

Since $E[W_{t_{k+1}} - W_{t_k}] = 0$ and $Var(W_{t_{k+1}} - W_{t_k}) = t_{k+1} - t_k$ we know that

$$E[(W_{t_{k+1}} - W_{t_k})^2 - (t_{k+1} - t_k)] = 0$$

Since Brownian motion has independent increments:

$$\begin{aligned}
E[\eta_{t_j}\eta_{t_k}] &= E[\eta_{t_j}]E[\eta_{t_k}] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
E[\eta_{t_k}^2] &= E[(W_{t_{k+1}} - W_{t_k})^4 - 2(W_{t_{k+1}} - W_{t_k})^2(t_{k+1} - t_k) + (t_{k+1} - t_k)^2] \\
&= E[(W_{t_{k+1}} - W_{t_k})^4] - 2(t_{k+1} - t_k)^2 + (t_{k+1} - t_k)^2 \\
&= E[(W_{t_{k+1}} - W_{t_k})^4] - (t_{k+1} - t_k)^2
\end{aligned}$$

If X is a normally distributed random variable with mean 0 and variance σ^2, then

$$E[X^4] = 3\sigma^4$$

Therefore:

$$\begin{aligned} E[\eta_{t_k}^2] &= 3(t_{k+1} - t_k)^2 - (t_{k+1} - t_k)^2 \\ &= 2(t_{k+1} - t_k)^2 \end{aligned}$$

Thus

$$\begin{aligned} E[\mid W(\Pi) - T \mid^2] &= 2\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2 \\ &\leq 2\sum_{k=0}^{n-1} \parallel \Pi \parallel (t_{k+1} - t_k) \\ &= 2 \parallel \Pi \parallel T \end{aligned}$$

So $E[\mid W(\Pi) - T \mid^2] \to 0$ as $\parallel \Pi \parallel \to 0$.

Therefore $[W]_T = T = \int_0^T 1dt$.

Since Brownian motion has a nonzero quadratic variation then it cannot be a differentiable function.

The Itô Stochastic Integral

Before discussing Itô's formula, the Itô stochastic integral is first detailed. What makes this integral suitable for our purpose is that we can integrate a $\{\Sigma_t\}_{t\geq 0}$-previsible function with respect to a non-differentiable function, namely Brownian motion.

Consider the Brownian motion $\{W_t\}_{t\geq 0}$ with filtration $\{\Sigma_t\}_{t\geq 0}$. We assume the following properties:

- If $s \leq t$ then $\Sigma_s \subseteq \Sigma_t$,
- W_t is Σ_t-measurable for all t.

Consider the process $f(t,\omega)$, $t \geq 0$, which is adapted to the filtration $\{\Sigma_t\}_{t\geq 0}$ and is square-integrable, i.e. $E[\int_0^T f^2(t,\omega)dt] < \infty$, for all T.

The Itô integral[2] $I(t)$ is defined as:

$$I(t) = \int_0^t f(s, W_s)dW_s$$

Properties of Itô Integral

Let $f(t,\omega)$ be any adapted, predictable function. Then the Itô integral: $I(t) = \int_0^t f(s,\omega)dW_s$ has the following properties (see Shreve [19] for verification):

- $I(t)$ is $\{\Sigma_t\}_{t\geq 0}$-measurable for all values of t,
- If $I(t) = \int_0^t f(s,\omega)dW_s$ and $J(t) = \int_0^t g(s,\omega)dW_s$ then $I(t) \pm J(t) = \int_0^t (f(s,\omega) \pm g(s,\omega))dW_s$ and $mI(t) = \int_0^t mf(s,\omega)dW_s$ for any constant m,
- $I(t)$ is a $(P, \{\Sigma_t\}_{t\geq 0})$-martingale,
- $I(t)$ is a continuous function,
- Itô Isometry: $E[(\int_0^t f(s,\omega)dW_s)^2] = \int_0^t E[f(s,\omega)^2]ds$.
- $[I]_t = \int_0^t f(s,\omega)^2 ds$ $(dI(t)dI(t) = f(t,\omega)^2 dt)$

Example

Consider integrating the Brownian motion $\{W_t\}_{t\geq 0}$ with respect to itself:

$$\int_0^T W_s dW_s \quad = \quad \lim_{\| \Pi \| \to 0} \sum_{j=0}^{n-1} W_{t_j}(W_{t_j+1} - W_{t_j}),$$

[2]The construction of the Itô integral is similar to the construction of the Lebesgue integral and is discussed in the appendix.

$$
\begin{aligned}
T = [W]_T &= \lim_{\parallel \Pi \parallel \to 0} \sum_{j=0}^{n-1} (W_{t_j+1} - W_{t_j})^2 \\
&= \lim_{\parallel \Pi \parallel \to 0} \sum_{j=0}^{n-1} \{(W_{t_j+1}^2 - W_{t_j}^2) - 2W_{t_j}(W_{t_{j+1}} - W_{t_j})\} \\
&= \lim_{\parallel \Pi \parallel \to 0} (W_T^2 - W_0^2) - 2 \lim_{\parallel \Pi \parallel \to 0} \sum_{j=0}^{n-1} W_{t_j}(W_{t_{j+1}} - W_{t_j})\} \\
&= W_T^2 - W_0^2 - 2\int_0^t W_s dW_s \\
\Rightarrow 2\int_0^T W_s dW_s &= W_T^2 - W_0^2 - T \\
\Rightarrow \int_0^T W_s dW_s &= \frac{1}{2}W_T^2 - \frac{1}{2}T
\end{aligned}
$$

If f is differentiable with $f(0) = 0$ then:

$$\int_0^T f(s)df(s) = \int_0^T f(s)f^{'}(s)ds = \frac{1}{2}f^2(s)\mid_0^T = \frac{1}{2}f^2(T)$$

Thus, for non-differentiable Brownian motion we have an extra term, namely $-\frac{1}{2}T$. This is because Brownian motion has nonzero quadratic variation.

Because the Itô integral is a martingale we have $E[\int_0^T W_s dW_s] = 0 \Rightarrow E[\frac{1}{2}W_T^2] = \frac{1}{2}T$. This verifies that W_T is normally distributed with mean 0 and variance T.

Stochastic Differential Equations

Let $f(x)$ be a differentiable function. If W_t is differentiable then using the chain rule we have that: $\frac{d}{dt}f(W_t) = f^{'}(W_t)W_t^{'}$.

In differential form we have

$$df(W_t) = f^{'}(W_t)W_t^{'}dt = f^{'}(W_t)dW_t$$

Since Brownian motion is not a differentiable function we cannot use this expression. The correct formula is:

$$df(W_t) = f^{'}(W_t)dW_t + \frac{1}{2}f^{''}(W_t)dt$$

This equation is the differential form of Itô's formula which is discussed below.

Itô's Formula

Consider the function $f(t,x)$, where $x \in \mathbb{R}$ and $t \geq 0$ such that $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ exist and are continuous. Then Ito's stochastic chain rule formula says that

$$f(t,W_t) - f(0,W_0) = \int_0^t \frac{\partial f}{\partial x}(s,W_s)dW_s + \int_0^t \frac{\partial f}{\partial s}(s,W_s)ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,W_s)ds$$

In differential form we have: $df(t,W_t) = f^{'}(t,W_t)dW_t + \dot{f}(t,W_t)dt + \frac{1}{2}f^{''}(t,W_t)dt$

(where $\dot{f}(t,W_t) = \frac{\partial f}{\partial t}(t,W_t)$, $f^{'}(t,W_t) = \frac{\partial f}{\partial x}(t,W_t)$ and $f^{''}(t,W_t) = \frac{\partial^2 f}{\partial x^2}(t,W_t)$).

Justification and Proof of Itô's Formula

Using Taylor's Theorem on $f(t,x)$ we get:

$$\begin{aligned} f(t+\triangle t, W_{t+\triangle t}) - f(t,W_t) &= \triangle t\dot{f}(t,W_t) + O(\triangle t^2) + (W_{t+\triangle t} - W_t)f^{'}(t,W_t) \\ &\quad + \frac{1}{2!}(W_{t+\triangle t} - W_t)^2 f^{''}(t,W_t) + \ldots \end{aligned}$$

If W_t were replaced by a differentiable function X_t then $\frac{1}{2!}(X_{t+\triangle t} - X_t)^2 f^{''}(t,X_t)$ would be replaced by $O(\triangle t^2)$. This term cannot be ignored since $E[(W_{t+\triangle t} - W_t)^2] = \triangle t$ (Etheridge [9]). Thus it is reasonable to expect that the differential equation for $f(t,x)$ is:

$$df(t,W_t) = \dot{f}(t,W_t)dt + f^{'}(t,W_t)dW_t + \frac{1}{2}f^{''}(t,W_t)dt$$

Proof of Itô's formula:

The basis for this proof comes from Etheridge [9]. Let $\Pi = \{t_0 = 0, t_1, \ldots, t_n = t\}$ be a partition of the interval $[0,t]$.Then:

$$f(t,W_t) - f(0,W_0) = \sum_{j=0}^{n-1}(f(t_{j+1}, W_{t_{j+1}}) - f(t_j, W_{t_j}))$$

Using Taylor's Theorem on each interval we get:

$$\begin{aligned} f(t,W_t)-f(0,W_0) &= \sum_{j=0}^{n-1}\dot{f}(t_j,W_{t_j})(t_{j+1}-t_j) \\ &\quad +\sum_{j=0}^{n-1} f^{'}(t_j,W_{t_j})(W_{t_{j+1}}-W_{t_j}) \\ &\quad +\frac{1}{2}\sum_{j=0}^{n-1} f^{''}(\eta_j,W_{\eta_j})(W_{t_{j+1}}-W_{t_j})^2 \end{aligned}$$

where $\eta_j \in [t_j, t_{j+1}]$. Letting $\epsilon_j = f^{''}(\eta_j, W_{\eta_j}) - f^{''}(t_j, W_{t_j})$ we get:

$$\frac{1}{2}\sum_{j=0}^{n-1} f^{''}(\eta_j,W_{\eta_j})(W_{t_{j+1}}-W_{t_j})^2 = \frac{1}{2}\sum_{j=0}^{n-1}(f^{''}(t_j,W_{t_j})+\epsilon_j)(W_{t_{j+1}}-W_{t_j})^2$$

As $\parallel \Pi \parallel \to 0$, $\epsilon_j \to 0$. It can be shown that $E[|\ \sum_{j=0}^{n-1} f^{''}(t_j,W_{t_j})(W_{t_{j+1}}-W_{t_j})^2 - \sum_{j=0}^{n-1} f^{''}(t_j,W_{t_j})(t_{j+1}-t_j)\ |^2] \to 0$ as $\parallel \Pi \parallel \to 0$ (Etheridge [9]). Thus as $\parallel \Pi \parallel \to 0$

$$\begin{aligned} \sum_{j=0}^{n-1}\dot{f}(t_j,W_{t_j})(t_{j+1}-t_j) &\rightarrow \int_0^t \frac{\partial f}{\partial s}(s,W_s)ds \\ \sum_{j=0}^{n-1} f^{'}(t_j,W_{t_j})(W_{t_{j+1}}-W_{t_j}) &\rightarrow \int_0^t \frac{\partial f}{\partial x}(s,W_s)dW_s \\ \frac{1}{2}\sum_{j=0}^{n-1} f^{''}(\eta_j,W_{\eta_j})(W_{t_{j+1}}-W_{t_j})^2 &\rightarrow \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,W_s)ds \end{aligned}$$

Thus: $f(t,W_t)-f(0,W_0) = \int_0^t \frac{\partial f}{\partial x}(s,W_s)dW_s + \int_0^t \frac{\partial f}{\partial s}(s,W_s)ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,W_s)ds$

Stochastic Differential Equation

Let $X_t = f(t, W_t)$. Applying Itô's formula,

$$\begin{aligned} dX_t &= [\dot{f}(t,W_t)+\frac{1}{2}f^{''}(t,W_t)]dt + f^{'}(t,W_t)dW_t \\ &= \mu(t,X_t)dt + \sigma(t,X_t)dW_t \end{aligned}$$

This is known as a stochastic differential equation for $\{X_t\}_{t\geq 0}$, where $\mu(t,x)$ and $\sigma(t,x)$ are deterministic functions on $\mathbb{R}\times\mathbb{R}_+$.

3.4 Geometric Brownian Motion

Brownian motion by itself is insufficient as a model of the movement of stock prices for many reasons. For example, stock prices cannot fall below zero, which is allowed by a Brownian motion model. Another important reason is that Brownian motion has normally distributed increments, which is not a fair reflection on a price change. To illustrate this point suppose that there are two financial securities that are valued at \$100 and \$10 respectively at time t. At time $t+s$, both increase in value by \$10. The first security increased by 10% while the second increased by 100%. It is unreasonable to suggest that the distribution of the change in the two values is the same. By looking at a function of Brownian motion we can overcome this problem.

Let $\{W_t\}_{t\geq 0}$ be a standard P-Brownian Motion and let ν and $\sigma > 0$ be constants. Geometric Brownian motion is defined as

$$S_t = S_0 exp(\nu t + \sigma W_t)$$

Applying Itô's formula:

$$dS_t = \sigma S_t dW_t + (\nu + \frac{1}{2}\sigma^2)S_t dt$$

This equation is called the stochastic differential equation for S_t. Let $\mu = \nu + \frac{1}{2}\sigma^2$. μ is known as the drift and σ is known as the volatility of the process. This model is used to track stock movement as it only allows non-negative values of the stock as well as making the price changes lognormally distributed. This means that the ratio of price changes is taken into account. This is a more realistic way to compare price changes between different financial securities. The following result shows that, for a given probability measure P, a geometric Brownian motion process is a martingale if and only if the drift parameter is zero:

The geometric Brownian motion process is a $(P, \{\Sigma_t\}_{t\geq 0})$-martingale if and only if $\mu = 0$.

Proof:

$$\begin{aligned} dS_t &= \sigma S_t dW_t + \mu S_t dt \\ \Rightarrow S_t - S_0 &= \int_0^t \sigma S_s dW_s + \int_0^t \mu S_s ds \end{aligned}$$

Assume $\mu = 0$. Then:

$$S_t - S_0 = \int_0^t \sigma S_s dW_s$$

Because S_t is a function of Brownian motion it is continuous and is adapted to the filtration. An adapted and continuous function is predictable (Etheridge [9]). Thus $\int_0^t \sigma S_s dW_s$ is an Itô integral and must therefore be a $(P, \{\Sigma_t\}_{t\geq 0})$-martingale.

Conversely, assume that $\{S_t\}_{t\geq 0}$ is a $(P, \{\Sigma_t\}_{t\geq 0})$-martingale. Then $S_t - S_0 - \int_0^t \sigma S_s dW_s = \int_0^t \mu S_s ds$ is a martingale since the difference of two $(P, \{\Sigma_t\}_{t\geq 0})$-martingales is a $(P, \{\Sigma_t\}_{t\geq 0})$-martingale. Thus:

$$\begin{aligned} E[S_t - S_0 - \int_0^t \sigma S_s dW_s] &= 0 \\ \Rightarrow E[\int_0^t \mu S_s ds] &= 0 \\ \Rightarrow \int_0^t \mu E[S_s] ds &= 0 \end{aligned}$$

Since $S_t > 0$ for all values of t, $\Rightarrow E[S_t] > 0$ for all t. Therefore $\mu = 0$.

Remark

Thus if the drift parameter is zero then the geometric Brownian motion process, $\{S_t\}_{t\geq 0}$ is a $(P, \{\Sigma_t\}_{t\geq 0})$-martingale. If P is the market probability measure then it is very unlikely that $\{S_t\}_{t\geq 0}$ is a martingale. $\{S_t\}_{t\geq 0}$ will only be a martingale when the drift parameter is zero. Recall that calculating the discounted expected payoff of a financial claim under the market measure did not yield an arbitrage-free price. We found an equivalent probability measure that made the discounted stock price a martingale. Using this probability measure we successfully priced the claim to avoid arbitrage. This martingale measure will then result in a zero drift. Girsanov's Theorem guarantees the existence of the martingale measure and it will be shown how this measure results in a zero drift parameter in the next section.

3.5 Constructing the Martingale Measure and Hedging Portfolio Process

It is now possible to construct a probability measure that will result in the discounted underlying stock price being a martingale. This is achieved by Girsanov's Theorem. With this measure the value of any financial claim on the underlying stock can be determined. Furthermore, the existence of a hedging portfolio is guaranteed by the Martingale Representation Theorem.

The Radon-Nikodym Derivative

Recall the binomial model with market measure P. If the value of the underlying stock was S_n at time n, then, at time $n+1$, it would be worth uS_n with probability p, and dS_n with probability $1-p$. In order to price the claim we discarded P in favour of Q, the risk-neutral probability measure. Thus the stock would make an upward movement with probability $q = \frac{e^r - d}{u-d}$ and a downward movement with probability $1-q = \frac{u-e^r}{u-d}$. The question that arises is how the two probability measures are related. The answer is that Q is simply a weighted version of P. To see this consider $\{S_n\}_{0 \leq n \leq N}$. Let N^* denote the total number of upward movements of the stock price process. The probability of the stock movement under P is $p^{N^*}(1-p)^{N-N^*}$. The probability of the stock movement under Q is $q^{N^*}(1-q)^{N-N^*}$. Let

$$Z = (\frac{q}{p})^{N^*}(\frac{1-q}{1-p})^{N-N^*}$$

Then the probability of the stock movement under Q is $Z[p^{N^*}(1-p)^{N-N^*}]$. This weighting value Z is known as the Radon-Nikodym derivative of Q with respect to P. This can be extended to the continuous setting. The formal definition is stated below (Shreve [19]).

Consider two probability measures P and Q on a measurable space (Ω, Σ) such that for every $A \in \Sigma$ satisfying $P(A) = 0$ we also have $Q(A) = 0$. Then there exists a nonnegative random variable Z such that:

$$Q(A) = \int_A Z dP, \quad \forall A \in \Sigma$$

Z is the Radon-Nikodym derivative of Q with respect to P. Thus, starting with a probability measure P on a measurable space (Ω, Σ) we can construct an equivalent probability measure that will result in the discounted underlying stock price being a martingale. Pricing a derivative claim on the underlying stock is then a simple matter of taking the present value of the expectation of the payoff of the claim under the new martingale measure. The construction of the new martingale measure is a direct application of Girsanov's Theorem.

Girsanov's Theorem

Let $\{W_t\}_{t\geq 0}$ be a Brownian Motion on $(\Omega, \Sigma, \{\Sigma_t\}_{t\geq 0}, P)$ and let $\{\theta_t\}_{t\geq 0}$ be a $\{\Sigma_t\}_{t\geq 0}$-adapted process such that $E[exp\{\frac{1}{2}\int_0^T \theta_t^2 dt\}] < \infty$. Define:

$$\begin{aligned} Z(t) &= exp\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\}, \\ Q(A) &= \int_A Z(T)dP, \ \forall A \in \Sigma, \\ W_t^Q &= W_t + \int_0^t \theta_s ds. \end{aligned}$$

Then under the new probability measure Q, the process $\{W_t^Q\}_{t\geq 0}$ is a standard Brownian motion[3]. The following properties can be observed:

- $Z(T)$ is the Radon-Nikodym derivative of Q with respect to P. Now, $Z(t)$ is a P-martingale.

Proof:

$$\begin{aligned} dZ(t) &= -\frac{1}{2}\theta_t^2 Z(t)dt - \theta_t Z(t)dW_t + \frac{1}{2}\theta_t^2 Z(t)dW_t dW_t \\ &= -\frac{1}{2}\theta_t^2 Z(t)dt - \theta_t Z(t)dW_t + \frac{1}{2}\theta_t^2 Z(t)dt \\ &= -\theta_t Z(t)dW_t \end{aligned}$$

- $Z(0) = e^0 = 1$. Since $Z(t)$ is a P-martingale $\Rightarrow E^P[Z(t)] = Z(0) = 1$ for all nonnegative values of t. Now, $Q(\Omega) = \int_\Omega Z(T)dP = E^P[Z(T)] = 1$. So Q is indeed a probability measure.
- Let X be a random variable. Then $E^Q[X] = E^P[Z(T)X]$.
- If X is Σ_t-measurable $(0 \leq t \leq T)$, then $E^Q[X] = E^P[Z(t)X]$.

[3]For a proof of this theorem see Etheridge [9]

Proof:

$$\begin{aligned}
E^Q[X] &= E^P[Z(T)X] \\
&= E^P[E^P[Z(T)X \mid \Sigma_t]] \quad (\textit{by the tower property}) \\
&= E^P[XE^P[Z(T) \mid \Sigma_t]] \quad (\textit{since } X \textit{ is } \Sigma_t - \textit{measurable}) \\
&= E^P[XZ(t)] \qquad (\textit{since } Z(T) \textit{ is a } P - \textit{martingale})
\end{aligned}$$

Bayes' Rule

Another interesting property that will be useful later is Bayes' Rule:

If Σ_t-measurable random variable X with $0 \leq s \leq t \leq T$, then

$$E^Q[X \mid \Sigma_s] = \frac{1}{Z(s)} E^P[XZ(t) \mid \Sigma_s]$$

Proof:

$\frac{1}{Z(s)} E^P[XZ(t) \mid \Sigma_s]$ is clearly Σ_s-measurable. Thus

$$\begin{aligned}
\frac{1}{Z(s)} E^P[XZ(t) \mid \Sigma_s] &= E^Q[\frac{1}{Z(s)} E^P[XZ(t) \mid \Sigma_s] \mid \Sigma_s] \\
&= E^P[Z(s)\frac{1}{Z(s)} E^P[XZ(t) \mid \Sigma_s] \mid \Sigma_s] \\
&= E^P[XZ(t) \mid \Sigma_s] \qquad (\textit{by the tower property}) \\
&= E^Q[X \mid \Sigma_s] \qquad (\textit{since } X \textit{ is } \Sigma_s - \textit{measurable})
\end{aligned}$$

Equivalent Measures

Recall that the measures P and Q on the same probability space Ω are equivalent if for all $A \in \Sigma$ then $Q(A) = 0 \Longleftrightarrow P(A) = 0$. Now,

The probability measures P and Q defined in Girsanov's Theorem are equivalent:

Proof:

Assume that $P(A) = 0$. Then $\int_A Z(T)dP = 0$. Since $Q(A) = \int_A Z(T)dP$, $\forall A \in \Sigma$ it is true that $Q(A) = 0$.

Conversely, assume that $Q(A) = 0$. Then $\int_A \frac{1}{Z(T)} dQ = 0$. Since $P(A) = \int_A \frac{1}{Z(T)} dQ$, $\forall A \in \Sigma$ it is true that $P(A) = 0$.

Hedging Portfolio and Martingale Measure

Now that the martingale probability measure has been constructed we can price a financial derivative. In order to price a claim against us we need to construct a hedging portfolio that exactly replicates the payoff of the claim. Now, provided such a hedging portfolio exists, it will be shown that the construction of the portfolio will result in the discounted stock price process and the discounted portfolio process being martingales under the martingale measure.

Recall that if $X_t = f(t, W_t)$ then $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$ is the stochastic differential representation of X_t. We defined the stock price as:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with drift μ and volatility σ. Let r be the continuously compounded interest rate.

Consider a derivative security that has payoff V at expiration time T. To hedge the claim against us we construct a portfolio $[\phi_t, \omega_t]$ that consists of units of the underlying stock and riskless cash bond. Let V_t be the value of the portfolio at time t (We are assuming that a hedging portfolio actually exists. We will show later that this is a valid assumption). At time t we buy ϕ_t units of the stock which is worth S_t. The remainder is invested in a riskless cash bond. Thus

$$V_t = \phi_t S_t + [V_t - \phi_t S_t]$$

Now taking a tiny time-step forward the portfolio is worth:

$$V_{t+\triangle t} = \phi_t S_{t+\triangle t} + (1 + r(\triangle t))[V_t - \phi_t S_t]$$

Thus:

$$\begin{aligned}
dV_t &= \phi_t dS_t + r[V_t - \phi_t S_t]dt \\
&= rV_t dt + \phi_t[dS_t - rS_t dt] \\
&= rV_t dt + \phi_t[\mu S_t dt + \sigma S_t dW_t - rS_t dt] \\
&= rV_t dt + \phi_t[(\mu - r)S_t dt + \sigma S_t dW_t] \\
&= rV_t dt + \phi_t \sigma S_t[\frac{\mu - r}{\sigma} dt + dW_t]
\end{aligned}$$

Let $\widetilde{S}_t = e^{-rt}S_t$, $\widetilde{V}_t = e^{-rt}V_t$ and $\theta_t = \frac{\mu - r}{\sigma}$. Therefore:

$$\begin{aligned}
d\widetilde{S}_t &= -re^{-rt}S_t dt + e^{-rt}dS_t \\
&= [-r\widetilde{S}_t dt + e^{-rt}dS_t] \\
&= [-r\widetilde{S}_t dt + \mu\widetilde{S}_t dt + \sigma\widetilde{S}_t dW_t] \\
&= \sigma\widetilde{S}_t[\theta_t dt + dW_t]
\end{aligned}$$

$$\begin{aligned}
d\widetilde{V}_t &= -re^{-rt}V_t dt + e^{-rt}dV_t \\
&= [-r\widetilde{V}_t dt + e^{-rt}dV_t] \\
&= [-r\widetilde{V}_t dt + r\widetilde{V}_t dt + \phi_t\sigma\widetilde{S}_t[\theta_t dt + dW_t]] \\
&= \phi_t\sigma\widetilde{S}_t[\theta_t dt + dW_t] \\
&= \phi_t d\widetilde{S}_t
\end{aligned}$$

Using Girsanov's Theorem, let $W_t^Q = W_t + \int_0^t \theta_s ds$, where $Q(A) = \int_A Z(T)dP$, $\forall A \in \Sigma$, and $Z(t) = exp\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\}$.

Then:

$$dW_t^Q = dW_t + \theta_t dt$$

Therefore:

$$\begin{aligned}
d\widetilde{S}_t &= \sigma\widetilde{S}_t dW_t^Q \\
d\widetilde{V}_t &= \phi_t\sigma\widetilde{S}_t dW_t^Q
\end{aligned}$$

Thus, under Q, $\{\widetilde{S}_t\}_{t\geq 0}$ and $\{\widetilde{V}_t\}_{t\geq 0}$ are martingales. So, assuming a hedging portfolio actually exists it has been shown that a martingale measure is a probability measure that is equivalent to the market measure P and results in the discounted asset prices being martingales (Shreve [19]). The time zero value of the claim will be equal to the value of the hedging portfolio at time zero, i.e. V_0. This value is equivalent to the discounted expected payoff of the claim under the risk-neutral probability measure Q. Thus:

$$\begin{aligned}
V_0 &= E^Q[e^{-rT}V_T] \\
&= E^Q[e^{-rT}V]
\end{aligned}$$

European Call Option Example

Suppose $V = (S_T - K)^+$. If a portfolio that successfully hedges the claim exists then $V_0 = E^Q[\widetilde{V}_T] = E^Q[e^{-rT}V] = E^Q[e^{-rT}(S_T - K)^+]$. The question that now arises is whether a hedging portfolio actually does exist. This is shown to be a valid assumption as a result of the Martingale Representation Theorem.

The Market Price of Risk

$\theta = \frac{\mu - r}{\sigma}$ is known as the market price of risk (Etheridge [9]). It is also known as the Sharpe ratio. If μ is the rate of growth of the risky asset, r is the rate of growth of the riskless cash bond and σ is a measure of the risk of the asset, then the market price of risk is the excess rate of return per unit of risk. It is also the change in drift in the underlying Brownian motion when the market measure is replaced by the martingale measure.

3.6 The Martingale Representation Theorem

While it is very convenient to price the value of a claim as an expectation with respect to an equivalent martingale measure, this argument is irrelevant if a hedging strategy cannot be constructed. The market must be complete so that all claims are attainable. Fortunately a hedging strategy does exist and can be proved as a result of the Martingale Representation Theorem.

Let $\{W_t\}_{t\geq 0}$ be a P-Brownian Motion with natural filtration $\{\Sigma_t\}_{t\geq 0}$ and let $\{X_t\}_{t\geq 0}$ be a square-integrable $(P, \{\Sigma_t\}_{t\geq 0})$-martingale, i.e. $E^P[\mid X_t \mid^2 < \infty]$, $\forall t > 0$. Then the Martingale Representation Theorem says that there is an adapted process $\{\theta_t\}_{t\geq 0}$ such that:

$$X_t = X_0 + \int_0^t \theta_s dW_s, \quad a.s. \quad (dX_t = \theta_t dW_t)$$

In other words, a square-integrable martingale can always be expressed in terms of a standard Brownian motion.

Existence of Hedging Portfolio

Assuming the same information as before let V_t represent the value of a hedging portfolio at time t. We want to choose V_0 and ϕ_t, $0 \leq t \leq T$, such that $V_T = V$ (where V is the payoff of the claim at expiration time T). Define $Y_t = E^Q[e^{-rT}V \mid \Sigma_t]$. Now Y_t is clearly a Q-martingale. By the Martingale Representation Theorem there is an adapted process $\{\theta_t\}_{t\geq 0}$ such that:

$$Y_t = Y_0 + \int_0^t \theta_s dW_s^Q, \quad 0 \leq t \leq T$$

Setting $V_0 = Y_0 = E^Q[e^{-rT}V]$ and $\theta_s = \phi_s \sigma \widetilde{S}_s$ we have:

$$Y_t = V_0 + \int_0^t \phi_s \sigma \widetilde{S}_s dW_s^Q = \widetilde{V}_t$$

(Recall, $d\widetilde{V}_t = \phi_t \sigma \widetilde{S}_t dW_t^Q \Rightarrow \widetilde{V}_t = V_0 + \int_0^t \phi_s \sigma \widetilde{S}_s dW_s^Q$). Thus

$$\begin{aligned}
\widetilde{V}_t = Y_t &= E^Q[e^{-rT}V \mid \Sigma_t], \quad 0 \leq t \leq T \\
\Rightarrow \widetilde{V}_T &= E^Q[e^{-rT}V \mid \Sigma_T] \\
&= e^{-rT}V \qquad (since\ e^{-rT}V\ is\ \Sigma_T - measurable) \\
\Rightarrow e^{-rT}V_T &= e^{-rT}V \\
\Rightarrow V_T &= V
\end{aligned}$$

So, a hedging portfolio exists but the Martingale Representation Theorem does not produce an explicit expression for the portfolio (Shreve [19]). A solution to our pricing problem is obtained by constructing a replicating portfolio that hedges the derivative claim we are trying to price. By hedging the claim we are avoiding arbitrage opportunities. With hedging in mind, the price of the derivative claim has been shown to be the discounted expected payoff of the claim with respect to the risk-neutral (martingale) probability measure.

Thus, in order to find the risk-neutral price for a derivative security, three important steps must be taken to replicate the payoff of the claim and ultimately give the arbitrage-free price.

- Find a probability measure Q, equivalent to the market measure P, under which the discounted security price process $\{\widetilde{S}_t\}_{t\geq 0}$ is a martingale.
- Form the process $\widetilde{V}_t = E^Q[e^{-rT}V \mid \Sigma_t]$.
- Find a predictable process $\{\phi_t\}_{t\geq 0}$ such that $d\widetilde{V}_t = \phi_t d\widetilde{S}_t$

Now that the basic tools for pricing derivative securities have been discussed we can proceed with the Black-Scholes model. This model will form the basis for the solution to the American put option pricing formula.

Chapter 4

The Black-Scholes Model

The Black-Scholes model is one of the most important contributions to the field of mathematical finance. In this section the highly celebrated Black-Scholes partial differential equation will be derived. The basis for the proof comes from Shreve [19]. Black and Scholes derived their partial differential equation twice and in a completely different manner to the methods presented here. For further insight into their groundbreaking work see Black and Scholes [3]. As before it is assumed that there are only two tradable assets in the market, namely the underlying security and the cash bond. Using geometric Brownian motion as the model for the stock price movement, the risk-neutral price of a derivative security on the underlying stock must satisfy the Black-Scholes partial differential equation. It will be shown that the stock holding in the hedging portfolio can be expressed as the partial derivative of the derivative security with respect to the stock price. With the methods developed, closed form solutions of a European put option and a European call option will be derived to demonstrate the effectiveness of the model.

4.1 Basic Assumptions of Black-Scholes Model

To price a claim under this model we need to make some basic assumptions in order to make the mathematics bearable. Once the model has been set up, the restrictions can be relaxed to make the model more realistic. These assumptions are:

- The stock price process $\{S_t\}_{t\geq 0}$ pays no dividends and follows a geometric Brownian motion with constant drift μ and volatility σ, i.e. $dS_t = \mu S_t dt + \sigma S_t dW_t$.

- There are no arbitrage opportunities.
- The borrowing and lending of cash occurs at a constant risk-free interest rate (no bid-offer spread).
- There are no transaction costs.
- There is no risk of default on a derivative security.

It is assumed that the underlying stock pays no dividends to shareholders. This complication can be dealt with by adapting the derivative price process to a process with underlying stock that pays no dividends (Ross [16]).

Assuming that the interest rate is constant is a greatly simplifying assumption. We could define the interest rate as an adapted process $\{r(t) : t \in [0, T]\}$ but this complication could make us lose sight of the main goal of trying to price an American put option. Term-structure models deal with these interest rate processes (Shreve [19]).

4.2 Feynman-Kac Theorem

The Black-Scholes partial differential equation is a special case of the Feynman-Kac Theorem[1]. Define $\{X_t\}_{t\geq 0}$ such that $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$, $0 \leq t \leq T$, where $\{W_t\}_{t\geq 0}$ is a standard P-Brownian Motion. Now, assume the following for function F:

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) = 0 \quad , \quad 0 \leq t \leq T$$
$$F(T,x) = h(x)$$

If $\int_0^T E[(\sigma(t, X_t)\frac{\partial F}{\partial x}(t, X_t))^2]ds < \infty$ then the Feynman-Kac Theorem says that:

$$F(t,x) = E^P[h(X_T) \mid X_t = x].$$

4.3 Black-Scholes Partial Differential Equation

Let $V(t, x)$ be the discounted value of a derivative security at time t if $S_t = x$. In order for $V(t, x)$ to be the risk-neutral valuation of the derivative security at time t, the Black-

[1]See Kijima [12] for details regarding the Feynman-Kac Theorem.

Scholes partial differential equation must be satisfied:

$$\frac{\partial V}{\partial t}(t,x) + rx\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t,x) - rV(t,x) = 0 \quad , \quad 0 \leq t \leq T.$$

We can adapt the Feynman-Kac Theorem for our purposes by considering the terminal condition $F(T,x)$ as the payoff of our derivative claim and $F(t,x)$ as the value of the derivative at time t. By adjusting for interest rates the Black-Scholes partial differential equation can be formed under the risk-neutral probability measure Q. Thus, if this equation is shown to be zero then the value of the derivative security is simply the discounted expected payoff of the claim under Q.

Derivation of Black-Scholes Partial Differential Equation

The stock price process $\{S_t\}_{t\geq 0}$ satisfies the differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

The solution to this equation is $S_t = S_0 exp(\nu t + \sigma W_t)$ where $\nu = \mu - \frac{1}{2}\sigma^2$. Consider the discounted stock price process, $\{\widetilde{S}_t\}_{t\geq 0}$, where $\widetilde{S}_t = e^{-rt}S_t$. Now,

$$\begin{aligned} d\widetilde{S}_t &= -r\widetilde{S}_t dt + e^{-rt}dS_t \\ &= -r\widetilde{S}_t dt + e^{-rt}(\mu S_t dt + \sigma S_t dW_t) \\ &= -r\widetilde{S}_t dt + \mu\widetilde{S}_t dt + \sigma\widetilde{S}_t dW_t \\ &= \widetilde{S}_t((\mu - r)dt + \sigma dW_t) \end{aligned}$$

If we let $W_t^Q = W_t + \frac{(\mu-r)}{\sigma}t$, then $dW_t^Q = dW_t + \frac{(\mu-r)}{\sigma}dt$. Therefore,

$$d\widetilde{S}_t = \sigma\widetilde{S}_t dW_t^Q$$

By Girsanov's Theorem we know that $\{W_t^Q\}_{t\geq 0}$ is a standard Brownian motion under the probability measure Q and $\{\widetilde{S}_t\}_{t\geq 0}$ is a Q-martingale (The Radon-Nikadym derivative of

Q with respect to P is $Z(t) = exp\{-\theta dW_t - \frac{1}{2}\theta^2 dt\}$, where $\theta = \frac{(\mu - r)}{\sigma}$).

$$\begin{aligned}
d\widetilde{S}_t &= \sigma \widetilde{S}_t dW_t^Q \\
\Rightarrow d(e^{-rt}S_t) &= e^{-rt}\sigma S_t dW_t^Q \\
\Rightarrow -re^{-rt}S_t dt + e^{-rt}dS_t &= e^{-rt}\sigma S_t dW_t^Q \\
\Rightarrow -rS_t dt + dS_t &= \sigma S_t dW_t^Q \\
\Rightarrow dS_t &= rS_t dt + \sigma S_t dW_t^Q \\
\Rightarrow S_t &= S_0 exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t^Q)
\end{aligned}$$

Under the market measure P, $S_t = S_0 exp(\nu t + \sigma W_t)$. Under the martingale measure Q, $S_t = S_0 exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t^Q)$. The volatility σ is unaffected when changing probability measures. It is only the drift that is affected.

Define $F(t, S_t) = E^Q[h(S_T) \mid \Sigma_t]$, where $h(S_T)$ is the payoff function for the derivative security. So, for $0 \leq s \leq t \leq T$,

$$\begin{aligned}
E^Q[F(t, S_t) \mid \Sigma_s] &= E^Q[E^Q[h(S_T) \mid \Sigma_t] \mid \Sigma_s] \\
&= E^Q[h(S_T) \mid \Sigma_s] \qquad (\textit{by the tower property}) \\
&= F(s, S_s)
\end{aligned}$$

Therefore $F(t, S_t)$ is a Q-martingale. Using Itô's formula on $F(t, S_t)$ we get:

$$\begin{aligned}
dF(t, S_t) &= \frac{\partial F}{\partial t}(t, S_t)dt + \frac{\partial F}{\partial x}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t, S_t)dS_t dS_t \\
&= \frac{\partial F}{\partial t}(t, S_t)dt + \frac{\partial F}{\partial x}(t, S_t)(rS_t dt + \sigma S_t dW_t^Q) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t, S_t)(rS_t dt + \sigma S_t dW_t^Q)^2 \\
&= \frac{\partial F}{\partial t}(t, S_t)dt + rS_t\frac{\partial F}{\partial x}(t, S_t)dt + \sigma S_t \frac{\partial F}{\partial x}(t, S_t)dW_t^Q + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t)dt \\
&= [\frac{\partial F}{\partial t}(t, S_t) + rS_t\frac{\partial F}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t)]dt + \sigma S_t \frac{\partial F}{\partial x}(t, S_t)dW_t^Q
\end{aligned}$$

Since $F(t, S_t)$ is a Q-martingale this implies:

$$\frac{\partial F}{\partial t}(t, x) + rx\frac{\partial F}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t, x) = 0, \quad 0 \leq t \leq T, \quad x \geq 0$$

Applying Itô's formula to $V(t, x) = e^{-r(T-t)}F(t, x)$ yields the Black-Scholes partial differential equation.

$$\begin{aligned}
F(t,x) &= e^{r(T-t)}V(t,x) \\
\frac{\partial F}{\partial t}(t,x) &= -re^{r(T-t)}V(t,x) + e^{r(T-t)}\frac{\partial V}{\partial t}(t,x) \\
\frac{\partial F}{\partial x}(t,x) &= e^{r(T-t)}\frac{\partial V}{\partial x}(t,x) \\
\frac{\partial^2 F}{\partial x^2}(t,x) &= e^{r(T-t)}\frac{\partial^2 V}{\partial x^2}(t,x)
\end{aligned}$$

Thus:

$$-re^{r(T-t)}V(t,x) + e^{r(T-t)}\frac{\partial V}{\partial t}(t,x) + rxe^{r(T-t)}\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 e^{r(T-t)}\frac{\partial^2 V}{\partial x^2}(t,x) = 0$$

$$\Rightarrow -rV(t,x) + \frac{\partial V}{\partial t}(t,x) + rx\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t,x) = 0.$$

Alternate Derivation of Black-Scholes Partial Differential Equation

Since it is the existence of a hedging strategy that ensures that the price of a derivative security is arbitrage free, it makes sense that the hedging portfolio process can be used to verify the Black-Scholes partial differential equation. Recall that we can construct a portfolio $[\phi_t, \omega_t]$ that consists of units of the underlying stock and riskless cash bond. Let V_t be the value of the portfolio at time t. Then V_t satisfies the differential equation:

$$dV_t = \phi_t dS_t + r[V_t - \phi_t S_t]dt$$

Now,

$$\begin{aligned}
dV_t &= \phi_t(\mu S_t dt + \sigma S_t dW_t) + r(V_t - \phi_t S_t)dt \\
&= [rV_t + (\mu - r)\phi_t S_t]dt + \phi_t \sigma S_t dW_t
\end{aligned}$$

Let the derivative security have a payoff function $h(S_T)$ at time T. Let $V(t,x)$ be the value of this derivative security at time t if $S_t = x$. Using Itô's Formula:

$$\begin{aligned}
dV(t,S_t) &= \frac{\partial V}{\partial t}(t,S_t)dt + \frac{\partial V}{\partial x}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)dS_t dS_t \\
&= \frac{\partial V}{\partial t}(t,S_t)dt + \frac{\partial V}{\partial x}(t,S_t)(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)\sigma^2 S_t^2 dt \\
&= [\frac{\partial V}{\partial t}(t,S_t) + \frac{\partial V}{\partial x}(t,S_t)\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)\sigma^2 S_t^2]dt + \frac{\partial V}{\partial x}(t,S_t)\sigma S_t dW_t
\end{aligned}$$

In order for our hedging strategy to replicate the claim we need $V_t = V(t,S_t)$ for all t. Thus,

$$\begin{aligned}
\phi_t \sigma S_t &= \frac{\partial V}{\partial x}(t,S_t)\sigma S_t, \\
rV_t + (\mu - r)\phi_t S_t &= \frac{\partial V}{\partial t}(t,S_t) + \frac{\partial V}{\partial x}(t,S_t)\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)\sigma^2 S_t^2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \phi_t &= \frac{\partial V}{\partial x}(t,S_t) \\
\Rightarrow rV(t,S_t) + (\mu - r)\frac{\partial V}{\partial x}(t,S_t)S_t &= \frac{\partial V}{\partial t}(t,S_t) + \frac{\partial V}{\partial x}(t,S_t)\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)\sigma^2 S_t^2
\end{aligned}$$

$$\Rightarrow rS_t\frac{\partial V}{\partial x}(t,S_t) + \frac{\partial V}{\partial t}(t,S_t) - rV(t,S_t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)\sigma^2 S_t^2 = 0$$

Thus $V(t,x)$ is the solution to the partial differential equation $rx\frac{\partial V}{\partial x}(t,x) + \frac{\partial V}{\partial t}(t,x) - rV(t,x) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,x)\sigma^2 x^2 = 0$ that satisfies the condition $V(T,x) = h(x)$. To hedge the claim an investor starting with an initial wealth of V_0 must rebalance his portfolio to hold $\frac{\partial V}{\partial x}(t,S_t)$ units of stock at each time interval t. Then $V_t = V(t,S_t)$ for all t and $V_T = V(T,S_t) = h(S_T)$ (Shreve [19]).

The value of the stock holding was shown to be:

$$\phi_t = \frac{\partial V}{\partial x}(t,S_t)$$

This makes sense because in the discrete model the stock holding was equal to the ratio of the change in the value of the derivative security to the change in the value of the stock price over each time step. Since the continuous model can be thought of as having infinitesimal time steps, the stock holding should be the partial derivative of the value of the claim with respect to the stock price.

4.4 Pricing European Options Using Black-Scholes Formula

Now that the Black-Scholes partial differential equation has been derived, the solution to the pricing problem for European options can be solved with an explicit closed form solution. The derivation of this solution can be found in Etheridge [9]. *The value at time t of a European option with payoff at maturity of $h(S_T)$ is $V(t, S_t)$ where:*

$$V(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} h(xexp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma y\sqrt{T-t}))\frac{exp(-y^2/2)}{\sqrt{2\pi}}dy$$

Proof:

Recall that $S_T = S_t exp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^Q - W_t^Q)$. Then

$$\begin{aligned} V(t,S_t) &= E^Q[e^{-r(T-t)}h(S_T) \mid \Sigma_t] \\ &= E^Q[e^{-r(T-t)}h(S_t exp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^Q - W_t^Q)) \mid \Sigma_t] \end{aligned}$$

(We know that $W_T^Q - W_t^Q$ is a normally distributed random variable with mean 0 and variance $T - t$).

$$\begin{aligned} V(t,S_t) &= \int_{-\infty}^{\infty} e^{-r(T-t)}h(S_t exp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma x)\frac{1}{\sqrt{2\pi(T-t)}}exp(-x^2/2(T-t))dx \\ &= e^{-r(T-t)}\int_{-\infty}^{\infty} h(S_t exp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma y\sqrt{T-t}))\frac{exp(-y^2/2)}{\sqrt{2\pi}}dy \end{aligned}$$

(Letting $x = y\sqrt{T-t} \Rightarrow dx = dy\sqrt{T-t}$).

European Put Option

We are one step closer to pricing an American put option. A closed form solution to a European put option can be obtained using the Black-Scholes formula developed above. *For a European put option with strike price K that matures at time T we have:*

$$V(t,x) = Ke^{-r(T-t)}\Phi(-\omega + \sigma\sqrt{T-t}) - x\Phi(-\omega),$$

where $\omega = \frac{\log(x/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ *and* $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} exp(-y^2/2)dy$.

Proof:

For a European put with strike price K, $h(x) = (K-x)^+ = max\{K-x, 0\}$

$$\begin{aligned} V(t,x) &= e^{-r(T-t)} \int_{-\infty}^{\infty} h(xexp((r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t})) \frac{exp(-y^2/2)}{\sqrt{2\pi}}dy \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} max\left\{K - xexp((r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t}), 0\right\} \frac{exp(-y^2/2)}{\sqrt{2\pi}}dy \end{aligned}$$

If $e^{-r(T-t)}[K - xexp((r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t})] > 0$ then:

$$\begin{aligned} \Rightarrow Ke^{-r(T-t)} &> xexp(-\frac{1}{2}\sigma^2(T-t)+\sigma y\sqrt{T-t}) \\ \Rightarrow \frac{K}{x}e^{-r(T-t)} &> exp(-\frac{1}{2}\sigma^2(T-t)+\sigma y\sqrt{T-t}) \\ \Rightarrow \log\frac{K}{x} - r(T-t) &> -\frac{1}{2}\sigma^2(T-t)+\sigma y\sqrt{T-t} \\ \Rightarrow \log\frac{K}{x} &> (r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t} \\ \Rightarrow -\log\frac{x}{K} - (r-\frac{1}{2}\sigma^2)(T-t) &> \sigma y\sqrt{T-t} \\ \Rightarrow y &< \frac{-\log\frac{x}{K} - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = y^* \end{aligned}$$

Therefore,

$$
\begin{aligned}
V(t,x) &= e^{-r(T-t)} \int_{-\infty}^{y^*} [K - xexp((r - \frac{1}{2}\sigma^2)(T-t) + \sigma y\sqrt{T-t})] \frac{exp(-y^2/2)}{\sqrt{2\pi}} dy \\
&= Ke^{-r(T-t)} \int_{-\infty}^{y^*} \frac{exp(-y^2/2)}{\sqrt{2\pi}} dy \\
&\quad - \int_{-\infty}^{y^*} xexp(-\frac{1}{2}\sigma^2(T-t) + \sigma y\sqrt{T-t}) \frac{exp(-y^2/2)}{\sqrt{2\pi}} dy \\
&= Ke^{-r(T-t)} \int_{-\infty}^{y^*} \frac{exp(-y^2/2)}{\sqrt{2\pi}} dy \\
&\quad - \int_{-\infty}^{y^*} \frac{1}{\sqrt{2\pi}} xexp(-\frac{1}{2}\sigma^2(T-t) + \sigma y\sqrt{T-t} - y^2/2) dy \\
&= Ke^{-r(T-t)} \Phi(\frac{\log\frac{K}{x} - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}) \\
&\quad -x \int_{-\infty}^{y^*} \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}\sigma^2(T-t) + \sigma y\sqrt{T-t} - y^2/2) dy \\
&= Ke^{-r(T-t)} \Phi(-\omega + \sigma\sqrt{T-t}) \\
&\quad -x \int_{-\infty}^{y^*} \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}(y - \sigma\sqrt{T-t})^2) dy \\
&= Ke^{-r(T-t)} \Phi(-\omega + \sigma\sqrt{T-t}) \\
&\quad -x \int_{-\infty}^{y^* - \sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} exp(-z^2/2) dz \\
&= Ke^{-r(T-t)} \Phi(-\omega + \sigma\sqrt{T-t}) - x\Phi(y^* - \sigma\sqrt{T-t}) \\
&= Ke^{-r(T-t)} \Phi(-\omega + \sigma\sqrt{T-t}) - x\Phi(-\omega + \sigma\sqrt{T-t} - \sigma\sqrt{T-t}) \\
&= Ke^{-r(T-t)} \Phi(-\omega + \sigma\sqrt{T-t}) - x\Phi(-\omega)
\end{aligned}
$$

There is only one unknown parameter that is needed to solve this formula, namely the volatility σ. The rest of the parameters are known beforehand. This is also the case for the hedging portfolio. One way to estimate this parameter is to invert the Black-Scholes formula and determine the implied volatility for this derivative security.

Satisfying the Black-Scholes Partial Differential Equation

The stock holding in the replicating portfolio was shown to be:

$$\phi_t = \frac{\partial V}{\partial x}(t, S_t)$$

This partial derivative is known as the *delta* of the derivative security. This quantity belongs to a set of partial derivatives known as the *Greeks*. The *Greeks* measure the sensitivity of derivative securities with respect to parameter changes. Two other examples are the *gamma* ($\frac{\partial^2 V}{\partial x^2}(t,x)$) and the *theta* ($\frac{\partial V}{\partial t}(t,x)$). The *delta* measures the sensitivity with respect to a change in the underlying stock price, the *gamma* measures the rate of change of the *delta*, and the *theta* measures the sensitivity with respect to a change in time.

Now, under the Black-Scholes model, the risk-neutral valuation of a European option must satisfy the Black-Scholes partial differential equation. By calculating the following *Greeks*, we can verify if the formula does indeed satisfy the Black-Scholes partial differential equation. These formulas are derived in the appendix.

$$\begin{aligned}
\textit{delta} &= \frac{\partial V}{\partial x}(t,x) \\
&= -\Phi(-\omega) \\
\textit{gamma} &= \frac{\partial^2 V}{\partial x^2}(t,x) \\
&= \frac{e^{-\omega^2/2}}{\sigma x\sqrt{2\pi(T-t)}} \\
\textit{theta} &= \frac{\partial V}{\partial t}(t,x) \\
&= Kre^{-r(T-t)}\Phi(-\omega+\sigma\sqrt{T-t}) - \frac{\sigma x e^{-\omega^2/2}}{2\sqrt{2\pi(T-t)}}
\end{aligned}$$

Therefore:

$$\begin{aligned}
rx\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} - rV + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\sigma^2 x^2 &= -rx\Phi(-\omega) + Kre^{-r(T-t)}\Phi(-\omega+\sigma\sqrt{T-t}) \\
&\quad - \frac{\sigma x e^{-\omega^2/2}}{2\sqrt{2\pi(T-t)}} - rKe^{-r(T-t)}\Phi(-\omega+\sigma\sqrt{T-t}) \\
&\quad + rx\Phi(-\omega) + \frac{1}{2}\sigma^2 x^2 \frac{e^{-\omega^2/2}}{\sigma x\sqrt{2\pi(T-t)}} \\
&= -\frac{\sigma x e^{-\omega^2/2}}{2\sqrt{2\pi(T-t)}} + \frac{1}{2}\sigma^2 x^2 \frac{e^{-\omega^2/2}}{\sigma x\sqrt{2\pi(T-t)}} \\
&= -\frac{\sigma x e^{-\omega^2/2}}{2\sqrt{2\pi(T-t)}} + \frac{\sigma x e^{-\omega^2/2}}{2\sqrt{2\pi(T-t)}} \\
&= 0
\end{aligned}$$

So, the risk-neutral European put option formula satisfies the Black-Scholes partial differential equation.

European Call Option

For a European call option with strike price K that matures at time T we have:

$$V(t,x) = x\Phi(\omega) - Ke^{-r(T-t)}\Phi(\omega - \sigma\sqrt{T-t}),$$

where $\omega = \frac{\log(x/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ *and* $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} exp(-y^2/2)dy.$

The solution of the European call option is similar to the solution of the European put option and can be found in Etheridge [9]. This solution can also be shown to satisfy the Black-Scholes partial differential equation.

Because European options can only be exercised at maturity it is a straightforward matter to price them using the Black-Scholes model. American options, on the other hand, prove to be a greater challenge due to their ability to be exercised at any point up to and including maturity. In fact, closed form solutions exist for only a few American options. Certain American options can only be solved via numerical methods.

Chapter 5

American Options

American-style options are the most commonly traded options in the market. It is therefore very important to be able to price these options so that arbitrage is avoided. It will be shown that it is always optimal to exercise an American call option at expiration. As a result American call options are equivalent in value to their European counterparts and can be priced using Black-Scholes call option formula. The American put option is much harder to price due to the fact that an optimal exercise time may occur before expiration. It is only under certain conditions that the price of an American put option can be solved with a closed form solution. The perpetual American put option, which has no expiration date, can be solved as a solution to a free boundary problem. Geske and Johnson [10] showed that a closed form solution can be obtained by using a compound option pricing approach. However, their formula is impossible to determine practically and numerical methods must be employed to approximate the actual value.

Let $\Gamma_{[0,T]}$ be the set of all possible stopping times up to and including time T. In this case a stopping time is a time at which the option is exercised. Let $\{S_t\}_{t\geq 0}$ be a Q-martingale. The time-zero value of an American option with payoff $f(S_t)$ $(0 \leq t \leq T)$ is:

$$\sup_{\tau \in \Gamma_{[0,T]}} E^Q[e^{-r\tau} f(S_\tau) \mid \Sigma_0]$$

5.1 American Call Options

What distinguishes American options from European options is that American options can be exercised at any time up to the expiration time. As a result pricing these options appears to be more difficult than before. It can be shown that it is only optimal to exercise an American call option at the expiration date. This implies that American and European call options with the same strike price and expiration date are equivalent and are priced according to Black-Scholes risk-neutral pricing formula for European call options. In order to prove that the optimal exercise date of an American call option is at expiration we must first define Jensen's Inequality and show that a convex function of a martingale is a submartingale. The result is then straightforward to prove.

Jensen's Inequality for Conditional Expectations

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function, Σ is a $\sigma - algebra$ and X is a real-valued random variable, such that $E[\mid f(X) \mid] < \infty$, then:

$$E[f(X) \mid \Sigma] \geq f(E[X \mid \Sigma])$$

Proof: (See Shreve [19] for details)

We can express a convex function as a maximum over linear functions, i.e. $f(x) = \max_{h \leq f} h(x)$, where $h(x)$ is linear. Since $h(x)$ is linear it can be expressed as $h(x) = mx + c$. Therefore,

$$\begin{aligned} E[f(X) \mid \Sigma] &\geq E[mX + c \mid \Sigma] \\ &= mE[X \mid \Sigma] + c \\ &= h(E[X \mid \Sigma]) \\ \Rightarrow E[f(X) \mid \Sigma] &\geq \max_{h \leq f} h(E[X \mid \Sigma]) \\ \Rightarrow E[f(X) \mid \Sigma] &\geq f(E[X \mid \Sigma]) \end{aligned}$$

Lemma

If $\{X_t\}_{t \geq 0}$ is a P-martingale and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function then $\{f(X_t)\}_{t \geq 0}$ is a P-submartingale.

Proof:

$$E^P[f(X_t) \mid \Sigma_s] \geq f(E^P[X_t \mid \Sigma_s]) = f(X_s)$$

Optimal Exercise Time for an American Derivative Security

Consider an American derivative security with convex payoff function $f : [0, \infty) \to \mathbb{R}$ *where* $f(0) = 0$. *The value of this security at time zero is the same as that of a European derivative security with payoff function* $f(S_T)$.

Proof:

Let $\Gamma_{[0,T]}$ be the set of all possible stopping times up to and including time T. Let $\{e^{-rt}S_t\}_{t\geq 0}$ be a Q-martingale. It must be shown that:

$$\sup_{\tau \in \Gamma_{[0,T]}} E^Q[e^{-r\tau} f(S_T)] = E^Q[e^{-rT} f(S_T)]$$

A function $f(x)$ is convex if for all x and y and $0 < \lambda < 1$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

(See Ross [16] for details). Thus, letting $y = 0$, we have $f(\lambda x) \leq \lambda f(x) + (1-\lambda) f(0) = \lambda f(x)$. Therefore for $s < t$,

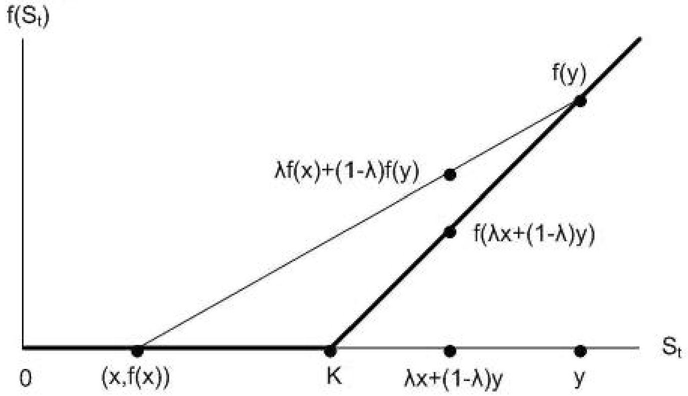
$$\begin{aligned}
f(e^{-r(t-s)}S_t) &\leq e^{-r(t-s)} f(S_t), && (since\ 0 < e^{-r(t-s)} < 1) \\
\Rightarrow E^Q[e^{-rt} f(S_t) \mid \Sigma_s] &= e^{-rs} E^Q[e^{-r(t-s)} f(S_t) \mid \Sigma_s] \\
&\geq e^{-rs} E^Q[f(e^{-r(t-s)}S_t) \mid \Sigma_s] && (since\ f\ is\ convex) \\
&\geq e^{-rs} f(E^Q[e^{rs}(e^{-rt}S_t) \mid \Sigma_s]) && (by\ Jensen's\ Inequality) \\
&= e^{-rs} f(e^{rs}e^{-rs}S_s) && (since\ \{e^{-rt}S_t\}_{t\geq 0}\ is\ a\ Q-martingale) \\
&= e^{-rs} f(S_s)
\end{aligned}$$

Thus, $\{e^{-rt} f(S_t)\}_{t\geq 0}$ is a Q-submartingale. Let τ be a stopping time ($0 \leq \tau \leq T$). By the Optional Stopping Theorem[1] we have:

$$\begin{aligned}
e^{-r\tau} f(S_\tau) &\leq E^Q[e^{-rT} f(S_T) \mid \Sigma_\tau] \\
\Rightarrow E^Q[e^{-r\tau} f(S_\tau)] &\leq E^Q[E^Q[e^{-rT} f(S_T) \mid \Sigma_\tau]] \\
&= E^Q[e^{-rT} f(S_T)] && (by\ the\ tower\ property) \\
\Rightarrow \sup_{\tau \in \Gamma_{[0,T]}} E^Q[e^{-r\tau} f(S_\tau)] &\leq E^Q[e^{-rT} f(S_T)]
\end{aligned}$$

[1] See the definition of the continuous Optional Stopping Theorem described earlier, where $\tau_1 = \tau$ and $\tau_2 = T$.

Figure 5.1: American Call Option Payoff Function



The payoff function, $f(S_t) = (S_t - K)^+$, is a convex function of S_t and is plotted as the thick dark line.

Since T is a stopping time (an in the money derivative security will definitely be exercised at T if it has not been exercised up until that point) we have that:

$$\sup_{\tau \in \Gamma_{[0,T]}} E^Q[e^{-r\tau} f(S_\tau)] \geq E^Q[e^{-rT} f(S_T)]$$

Therefore, by combining the two inequalities we can conclude that:

$$\sup_{\tau \in \Gamma_{[0,T]}} E^Q[e^{-r\tau} g(S_T)] = E^Q[e^{-rT} g(S_T)].$$

Optimal Exercise Time for an American Call Option

The above theorem says that for an American derivative security with a convex payoff function that is equal to zero when the stock price is zero, the expiration time is the optimal point at which to exercise .

For an American call option with strike price K, $f(S_t) = (S_t - K)^+$. Clearly $f(0) = 0$. From the plot of the payoff function it is clear that $(S_t - K)^+$ is a convex function of S_t. $f(S_t)$ is equal to 0 if $S_t < K$. This is because the option will not be exercised unless the strike price is less than the underlying stock price. Recall that options give the holder the right and not the obligation to exercise. Now, provided $0 < \lambda < 1$, then for any x and y it is true that $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.

Thus, for an American call option, the time-zero risk-neutral value of the option is equal to:

$$E^Q[e^{-rT}(S_T - K)^+]$$

5.2 American Put Options

For an American put option with strike price K, $f(S_t) = (K - S_t)^+$. $(K - S_t)^+$ is a convex function of S_t but $f(0) = K > 0$. Thus, the above argument is not valid for American put options since it is not true that $f(e^{-r(t-s)}S_t) \leq e^{-r(t-s)}f(S_t)$, which destroys the submartingale property of $\{e^{-rt}f(S_t)\}_{t\geq 0}$. As a result the optimal exercise time may be before expiration and so pricing this put option is more complicated than its European counterpart.

The time zero value of an American put option, $V(0, S_0)$ is:

$$V(0, S_0) = \sup_{\tau \in \Gamma_{[0,T]}} E^Q[e^{-r\tau}(K - S_\tau)^+ \mid \Sigma_0]$$

Remarks

Because $f(x) = (K - x)^+$ is a non-increasing function of x, it means that for $x_1 < x_2$, $f(x_1) \geq f(x_2)$.

$$\begin{aligned} 0 < \lambda < 1 &\Rightarrow \lambda x < x \\ &\Rightarrow f(\lambda x) \geq f(x) \\ f(x) \geq \lambda f(x) &\Rightarrow f(\lambda x) \geq \lambda f(x) \end{aligned}$$

If the continuous interest rate $r = 0$ then $\{e^{-rt}f(S_t)\}_{t\geq 0}$ is indeed a submartingale (since $f(e^0 S_t) = e^0 f(S_t)$) and results in the value of an American put option having an optimal exercise time at expiration. Thus, for zero interest rates, the value of an American put option is the same as that of the Black-Scholes valuation of a European put option.

Properties of American Put Options

Let $V(t, x)$ be the time-t value of an American put option with strike price K, expiration time T and riskless interest rate r. The payoff function $(K - x)^+$ is a decreasing function of x, for $0 \leq x < \infty$. Early exercise of the put option occurs at time t when its intrinsic value, $(K - S_t)^+$, is greater than the value of the put when not exercised. As the value of the underlying security decreases the intrinsic value of the put increases. There is a critical stock price S_t^* at which the intrinsic value of the put option will be greater than the value of the put option if not exercised. Any values under this stock price will yield the same result. Therefore, there exists a critical stock price $0 \leq S_t^* < \infty$ such that:

For $0 \leq x \leq S_t^*$ and $0 \leq t \leq T$:

- $V(t, x) = (K - x)$
- $\frac{\partial V}{\partial t}(t, x) + rx\frac{\partial V}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t, x) - rV(t, x) < 0$

The above inequality is true since for $V(t, x) = (K - x)$:

$$\begin{aligned}
\frac{\partial V}{\partial t}(t, x) + rx\frac{\partial V}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t, x) - rV(t, x) &= -rx + 0 - r(K - x) \\
&= -rK \\
&< 0
\end{aligned}$$

For $S_t^* < x < \infty$ and $0 \leq t \leq T$:

- $V(t, x) > (K - x)^+$
- $\frac{\partial V}{\partial t}(t, x) + rx\frac{\partial V}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t, x) - rV(t, x) = 0$

At the critical stock price boundary $x = S_t^*$ the following conditions must be met:

- $V(t, S_t^*) = (K - S_t^*)$
- $\frac{\partial V}{\partial x}(t, S_t^*) = -1$

The terminal condition must also be satisfied: $V(T, S_T) = (K - S_T)^+$

Remarks

$\frac{\partial V}{\partial x}(t, S_t^*) = -1$ since $V(t, S_t^*) = (K - S_t^*)$.

When $S_t^* < x < \infty$, the option will not be exercised early. The pricing formula must satisfy the Black-Scholes partial differential equation since the option price will be a Q-martingale. However, when $0 \leq x \leq S_t^*$, the option will be exercised early. The option price is thus a $Q-$supermartingale at this stage. Recall that a supermartingale is the mathematical equivalent of an unfavourable game. The change in the value of the option price when not exercised must result in a decrease and therefore $\frac{\partial V}{\partial t}(t,x) + rx\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t,x) - rV(t,x) < 0$.

Supermartingale Property of American Put Options

It has been claimed that the American put option process is a supermartingale under the martingale measure Q. To justify this claim we first assume the binomial model that was discussed earlier.

Assuming the binomial model used previously, let Q be the martingale measure which makes the discounted stock price $\{\widetilde{S}_n\}_{0\leq n\leq N}$ a martingale. Let $\widetilde{V}_n$ be the discounted value of an American put option with strike price K at the n^{th} time step. The payoff at the n^{th} time step is $(K - S_n)^+$. Let $\widetilde{Z}_n$ be the discounted payoff (intrinsic value) at the n^{th} time step.

Now, $\{\widetilde{V}_n\}_{0\leq n\leq N}$ is the smallest supermartingale (under Q) that dominates $\{\widetilde{Z}_n\}_{0\leq n\leq N}$.

Proof:

The value of the put option is equal to the maximum of the put's intrinsic value and the value of the put when not exercised. Now, for $0 \leq n \leq N$:

$$\widetilde{V}_{n-1} = max\{\widetilde{Z}_{n-1}, E^Q[\widetilde{V}_n \mid \Sigma_{n-1}]\},$$

Since an in the money put will be exercised at expiration time N it is true that $\widetilde{V}_N = \widetilde{Z}_N$. For $0 \leq n \leq N$:

$$\widetilde{V}_{n-1} \geq E^Q[\widetilde{V}_n \mid \Sigma_{n-1}]$$

Therefore $\{\widetilde{V}_n\}_{0\leq n\leq N}$ is a supermartingale. Let $\{\widetilde{U}_n\}_{0\leq n\leq N}$ be a supermartingale such that $\widetilde{U}_n \geq \widetilde{Z}_n$, $0 \leq n \leq N$. Therefore:

$$\widetilde{U}_N \geq \widetilde{V}_N$$

Assume $\widetilde{U}_n \geq \widetilde{V}_n$. We want to show that $\widetilde{U}_{n-1} \geq \widetilde{V}_{n-1}$.

$$\widetilde{U}_{n-1} \geq E^Q[\widetilde{U}_n \mid \Sigma_{n-1}] \geq E^Q[\widetilde{V}_n \mid \Sigma_{n-1}]$$

Then:

$$\widetilde{U}_{n-1} \geq max\{\widetilde{Z}_{n-1}, E^Q[\widetilde{V}_n \mid \Sigma_{n-1}]\} = \widetilde{V}_{n-1}$$

Using backwards induction the result is obtained (Etheridge [9]).

Remark

It can be shown that if $\{\widetilde{V}_n\}_{n\geq 0}$ is a $(Q, \{\Sigma_n\}_{n\geq 0})$-supermartingale then there is a previsible, non-decreasing process $\{\widetilde{A}_n\}_{n\geq 0}$ that results in the process $\{\widetilde{V}_n + \widetilde{A}_n\}_{n\geq 0}$ being a $(Q, \{\Sigma_n\}_{n\geq 0})$-martingale (See Etheridge [9] for proof of this statement). If $\widetilde{A}_0 = 0$ then the process $\{\widetilde{A}_n\}_{n\geq 0}$ is unique.

Let $\widetilde{X}_n = \widetilde{V}_n + \widetilde{A}_n$. Then $\{\widetilde{X}_n\}_{n\geq 0}$ is a $(Q, \{\Sigma_n\}_{n\geq 0})$-martingale. The option is exercised at the first time j when $\widetilde{A}_{j+1} \neq 0$.

If $\widetilde{A}_{j+1} = 0$ then $\widetilde{V}_j = E^Q[\widetilde{V}_{j+1} \mid \Sigma_j]$ since $\widetilde{V}_{j+1} = \widetilde{X}_{j+1}$ (which is a Q-martingale). It is better to hold onto the option as it is worth more than if it were to be exercised. If $\widetilde{A}_{j+1} \neq 0$ then $\widetilde{V}_j = \widetilde{Z}_j$. At this point, exercising the option is better than holding it.

American Put Option Example

To demonstrate the supermartingale property of the American put option process consider the following example.

Let $n = 2$, $S_0 = 100$, $u = 2$, $d = \frac{1}{2}$, $K = 100$ and $r = \frac{1}{4}$. Letting $p = \frac{1+r-d}{u-d}$ and $q = 1 - p$ we have $p = q = \frac{1}{2}$. Thus

$$\begin{aligned}
V_2(400) &= (100 - 400)^+ = 0 \\
V_2(100) &= (100 - 100)^+ = 0 \\
V_2(25) &= (100 - 25)^+ = 75 \\
V_1(200) &= max\{\frac{4}{5}[\frac{1}{2}(0) + \frac{1}{2}(0)], (100 - 200)^+\} = max\{0, 0\} = 0 \\
V_1(50) &= max\{\frac{4}{5}[\frac{1}{2}(0) + \frac{1}{2}(75)], (100 - 50)^+\} = max\{30, 50\} = 50 \\
V_0(100) &= max\{\frac{4}{5}[\frac{1}{2}(0) + \frac{1}{2}(50)], (100 - 100)^+\} = max\{20, 0\} = 20
\end{aligned}$$

Let us try and hedge this option using the hedging strategy used earlier. Starting with initial wealth $V_0 = 20$, we need to determine the stock holding process for our portfolio.

Therefore,

$$
\begin{aligned}
0 = V_1(uS_0) &= \phi_0(uS_0) + (1+r)(V_0 - \phi_0 S_0) \\
&= \phi_0(200) + \frac{5}{4}(20 - \phi_0(100)) \\
&= \phi_0(75) + 25 \\
\Rightarrow \phi_0 &= -\frac{1}{3} \\
50 = V_1(dS_0) &= \phi_0(dS_0) + (1+r)(V_0 - \phi_0 S_0) \\
&= \phi_0(50) + \frac{5}{4}(20 - \phi_0(100)) \\
&= \phi_0(-75) + 25 \\
\Rightarrow \phi_0 &= -\frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
0 = V_2(duS_0) &= \phi_{1(dS_0)}(duS_0) + (1+r)(V_1(dS_0) - \phi_{1(dS_0)}(dS_0)) \\
&= \phi_{1(dS_0)}(100) + \frac{5}{4}(50 - \phi_{1(dS_0)}(50)) \\
&= \phi_{1(dS_0)}(37.5) + 62.5 \\
\Rightarrow \phi_{1(dS_0)} &= -1\frac{2}{3} \\
75 = V_2(d^2S_0) &= \phi_{1(dS_0)}(d^2S_0) + (1+r)(V_1(dS_0) - \phi_{1(dS_0)}(dS_0)) \\
&= \phi_{1(dS_0)}(25) + \frac{5}{4}(50 - \phi_{1(dS_0)}(50)) \\
&= \phi_{1(dS_0)}(-37.5) + 62.5 \\
\Rightarrow \phi_{1(dS_0)} &= -\frac{1}{3}
\end{aligned}
$$

What went wrong? If we had been hedging a European put option then the value of $V_1(dS_0) = 30$ and this would result in $\phi_{1(dS_0)} = -1$ in both cases. When the stock is valued as 50 then it is optimal to exercise the put. If exercised then the payoff is 50. Now if the put is not exercised then the value of the option is 30. Thus in order to hedge the option the difference of 20 must be consumed.

Recall that for a European derivative security, the value of the hedging portfolio process $[\phi_j, \omega_j]$ at time $j+1$ was:

$$V_{j+1} = \phi_j S_j + (1+r)(V_j - \phi_j S_j)$$

For an American style option, the hedging formula needs to be adjusted for the amount consumed at every time point. Let A_j be the difference that must be consumed. Then the

American put option hedging portfolio at time $j+1$ is worth:

$$V_{j+1} = \phi_j S_j + (1+r)((V_j - A_j) - \phi_j S_j)$$

5.3 Perpetual American Put Options

Getting back to the continuous-time model, closed form solutions for American put options only exist if the option satisfies certain conditions. One such condition is having no expiration time, i.e. $T \to \infty$. This type of option is known as a perpetual American put option.

Consider an American put option that does not expire. The option holder can exercise the put option at any time t, $0 \le t < \infty$. Because the time to expiration is infinite, the value of the put option is a function of x alone, i.e. $V(t,x) = V(x)$ and for the exercise boundary we have $S_t^* = L$, for all $t > 0$ and some constant L (Etheridge [9]).

The value of a perpetual American put option[2] $V(x)$ is:

$$V(x) = \{ \begin{array}{cc} (K-x) & if\ \ 0 \le x \le L \\ (K-L)(\frac{x}{L})^{-2r\sigma^{-2}} & if\ \ L < x < \infty \end{array}$$

where $L = \frac{2r\sigma^{-2}K}{1+2r\sigma^{-2}}$. This solution is subject to the following boundary conditions:

- $V(L) = (K-L)$
- $\lim\limits_{x \downarrow L} \frac{dV(x)}{dx} = -1$
- $\lim\limits_{x \to \infty} V(x) = 0$

For $0 \le x \le L$:

$$\begin{aligned} rx\frac{dV(x)}{dx} + \frac{1}{2}\sigma^2 x^2 \frac{d^2V(x)}{dx^2} - rV(x) &= -rx + 0 - r(K-x) \\ &= -rK \\ &< 0 \end{aligned}$$

[2]For an explicit proof of this solution see the appendix.

For $L < x < \infty$:

$$\begin{aligned}
rx\frac{dV(x)}{dx} + \frac{1}{2}\sigma^2 x^2 \frac{d^2V(x)}{dx^2} - rV(x) &= rx[\frac{-2r\sigma^{-2}}{L}(K-L)(\frac{x}{L})^{-2r\sigma^{-2}-1}] \\
&\quad + \frac{1}{2}\sigma^2 x^2 [\frac{(-2r\sigma^{-2})(-2r\sigma^{-2}-1)}{L^2}(K-L)(\frac{x}{L})^{-2r\sigma^{-2}-2}] \\
&\quad - r(K-L)(\frac{x}{L})^{-2r\sigma^{-2}} \\
&= (K-L)(\frac{x}{L})^{-2r\sigma^{-2}}[-2r^2\sigma^{-2} \\
&\quad + \frac{1}{2}\sigma^2(-2r\sigma^{-2})(-2r\sigma^{-2}-1) - r] \\
&= (K-L)(\frac{x}{L})^{-2r\sigma^{-2}}(-2r^2\sigma^{-2} + r(2r\sigma^{-2}+1) - r) \\
&= (K-L)(\frac{x}{L})^{-2r\sigma^{-2}}(-2r^2\sigma^{-2} + 2r^2\sigma^{-2} + r - r) \\
&= (K-L)(\frac{x}{L})^{-2r\sigma^{-2}}(0) \\
&= 0
\end{aligned}$$

The Linear Complementarity Problem

The free boundary problem can also be expressed as follows:

Let $BS_{V(x)} = rx\frac{dV(x)}{dx} + \frac{1}{2}\sigma^2 x^2 \frac{d^2V(x)}{dx^2} - rV(x)$.

Then

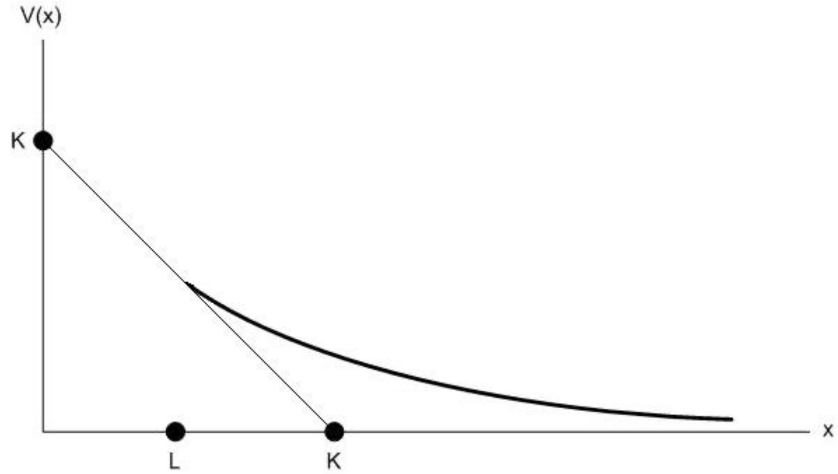
$BS_{V(x)}(V(x) - (K-x)^+) = 0$

subject to the conditions,

$$\begin{aligned}
BS_{V(x)} &\leq 0 \\
V(x) - (K-x)^+ &\geq 0
\end{aligned}$$

One of the above two inequalities will be an equality for all $x \in \mathbb{R}$, $x \neq L$.

Figure 5.2: Linear Complementarity



The solution to the perpetual American put option is plotted against the underlying stock price $x = S_t$. If the value of the underlying stock is below the critical stock price L, then the option is exercised and is worth $(K - x)$. The intrinsic value of the option is worth more than the value of the option if not exercised. However, when the underlying stock is worth more than L then the option is more valuable if it is not exercised. The option is thus worth $(K - L)(\frac{x}{L})^{-2r\sigma^{-2}}$.

Hedging Portfolio

In the binomial model, the hedging portfolio had to be adjusted for the difference consumed when the option was not exercised. When the difference is zero then the option will not be exercised. As soon as this difference becomes nonzero, the put option process becomes a supermartingale and early exercise is optimal. The hedging portfolio process is very similar in the continuous-time setting. Consider the portfolio process $[\phi_t, \omega_t]$. Let V_t be the value of the portfolio at time t. Then V_t satisfies the differential equation:

$$dV_t = \phi_t dS_t + r[V_t - \phi_t S_t]dt - A_t dt,$$

where A_t is the rate at which the difference is consumed. Thus:

$$\begin{aligned}
d(e^{-rt}V_t) &= -re^{-rt}V_t dt + e^{-rt}dV_t \\
&= -re^{-rt}V_t dt + e^{-rt}(\phi_t dS_t + r[V_t - \phi_t S_t]dt - A_t dt) \\
&= e^{-rt}\phi_t(rS_t dt + \sigma S_t dW_t^Q) - re^{-rt}\phi_t S_t dt - e^{-rt}A_t dt \\
&= -e^{-rt}A_t dt + e^{-rt}\phi_t \sigma S_t dW_t^Q
\end{aligned}$$

Using Itô's Formula on $V_t = V(t, S_t)$:

$$\begin{aligned}
dV(t,S_t) &= \frac{\partial V}{\partial t}(t,S_t)dt + \frac{\partial V}{\partial x}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)dS_t dS_t \\
&= \frac{\partial V}{\partial t}(t,S_t)dt + \frac{\partial V}{\partial x}(t,S_t)(rS_t dt + \sigma S_t dW_t^Q) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)\sigma^2 S_t^2 dt \\
&= [\frac{\partial V}{\partial t}(t,S_t) + \frac{\partial V}{\partial x}(t,S_t)rS_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)\sigma^2 S_t^2]dt + \frac{\partial V}{\partial x}(t,S_t)\sigma S_t dW_t^Q \\
\Rightarrow d(e^{-rt}V(t,S_t)) &= -re^{-rt}V(t,S_t)dt + e^{-rt}dV(t,S_t) \\
&= -re^{-rt}V(t,S_t)dt + e^{-rt}(\frac{\partial V}{\partial t}(t,S_t) + \frac{\partial V}{\partial x}(t,S_t)rS_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,S_t)\sigma^2 S_t^2)dt \\
&\quad + e^{-rt}\frac{\partial V}{\partial x}(t,S_t)\sigma S_t dW_t^Q \\
&= e^{-rt}(\frac{\partial V}{\partial t}(t,S_t) + rS_t\frac{\partial V}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial x^2}(t,S_t) - rV(t,S_t))dt \\
&\quad + e^{-rt}\frac{\partial V}{\partial x}(t,S_t)\sigma S_t dW_t^Q
\end{aligned}$$

Comparing the two differential equations we have:

$$\begin{aligned}
e^{-rt}\phi_t\sigma S_t &= e^{-rt}\frac{\partial V}{\partial x}(t,S_t)\sigma S_t \\
\Rightarrow \phi_t &= \frac{\partial V}{\partial x}(t,S_t)
\end{aligned}$$

$$\begin{aligned}
-e^{-rt}A_t &= e^{-rt}(\frac{\partial V}{\partial t}(t,S_t) + rS_t\frac{\partial V}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial x^2}(t,S_t) - rV(t,S_t)) \\
\Rightarrow A_t &= -(\frac{\partial V}{\partial t}(t,S_t) + rS_t\frac{\partial V}{\partial x}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial x^2}(t,S_t) - rV(t,S_t))
\end{aligned}$$

Since the Black-Scholes partial differential equation is equal to $-rK$ if $S_t \leq S_t^*$, and 0 otherwise, it is therefore true that

$$A_t = rK1_{\{S_t \leq S_t^*\}}$$

The stock holding in the hedging portfolio is once again equal to the partial derivative of the option price with respect to the underlying stock price. If $S_t \leq S_t^*$ then $V(S_t) = (K - S_t)$ and $\phi_t = -1$. By investing K in a cash bond and selling one share of stock short, the put option can be hedged. The amount that can be consumed is equal to the interest from the cash bond, i.e. rK.

5.4 Geske-Johnson Analytic Formula for American Put Options

Geske and Johnson [10] showed that under the Black-Scholes conditions, namely perfect markets, constant interest rate and volatility parameters as well as using geometric Brownian motion to track the movement of the stock price, an analytic solution to the American put problem exists and it satisfies Black-Scholes partial differential equation:

$$\frac{\partial P}{\partial t}(t,x) + rx\frac{\partial P}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2}(t,x) - rP(t,x) = 0 \quad , \quad 0 \leq t \leq T$$

The American put option can be exercised at any point up until maturity. It is optimal to exercise the put if the payoff from exercising is greater than the value of the put if it is not exercised. This means that the American put option must satisfy the following condition:

$$P(t, S_t) \geq (S_t - K)^+ \quad , \quad 0 \leq t \leq T$$

By treating each opportunity to exercise as a discrete event then an analytic formula is derived by letting the number of discrete exercise points go to infinity. Suppose that the put can be exercised at time dt, $2dt$, $3dt$, etc. At the first point of exercise, the put will be exercised if the payoff from exercising is at least as great as the value of the put if not exercised. Since the payoff is equal to the strike price minus the stock price and the strike price is fixed, the put will be exercised when the stock falls below a certain critical value. Let $\overline{S}_{dt}$be the critical stock price at the time point dt. The discounted payoff is integrated over all stock prices less than $\overline{S}_{dt}$. Thus a univariate normal integral is the result. At $2dt$, the put is exercised if it was not exercised previously and if the stock price at that time is below the critical stock price $\overline{S}_{2dt}$. A bivariate normal integral is obtained. The process carries on indefinitely and at each time a multivariate normal integral of a higher dimension is obtained. By combining all these integrals we obtain the Geske-Johnson analytic formula (Geske & Johnson [10]):

$$P = K\omega_2 - S_0\omega_1$$

where P is the American put value, K is the strike price, S_0 is the time zero value of the underlying stock and:

$$\begin{aligned}\omega_1 &= \{\Phi_1(-d_1(\overline{S}_{dt}, dt)) + \Phi_2(d_1(\overline{S}_{dt}, dt), -d_1(\overline{S}_{2dt}, 2dt); -\rho_{12}) \\ &\quad +\Phi_3(d_1(\overline{S}_{dt}, dt), d_1(\overline{S}_{2dt}, 2dt), -d_1(\overline{S}_{3dt}, 3dt); \rho_{12}, -\rho_{13}, -\rho_{23}) + \ldots\}\end{aligned}$$

$$\begin{aligned}\omega_2 &= \{e^{-rdt}\Phi_1(-d_2(\overline{S}_{dt}, dt)) + e^{-r2dt}\Phi_2(d_2(\overline{S}_{dt}, dt), -d_2(\overline{S}_{2dt}, 2dt); -\rho_{12}) \\ &\quad +e^{-r3dt}\Phi_3(d_2(\overline{S}_{dt}, dt), d_2(\overline{S}_{2dt}, 2dt), -d_2(\overline{S}_{3dt}, 3dt); \rho_{12}, -\rho_{13}, -\rho_{23}) + \ldots\}\end{aligned}$$

$$\begin{aligned}d_1(y,t) &= \frac{ln(S_0/y) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \\ d_2(y,t) &= d_1(y,t) - \sigma\sqrt{t}\end{aligned}$$

$\Phi_1(x)$ is a standard cumulative univariate normal distribution function.

$\Phi_j(x_1, x_2, \ldots, x_j; \rho_{12}, \rho_{13}, \ldots, \rho_{j-1,j})$ is a cumulative multivariate normal distribution function of dimension j, (where $j > 1$).

For the standard Brownian motions $\{W_{t_1}\}_{t_1 \geq 0}$ and $\{W_{t_2}\}_{t_2 \geq 0}$, where $t_2 > t_1 > 0$, recall that:

$$\begin{aligned}Corr(W_{t_1}, W_{t_2}) &= \frac{cov(W_{t_1}, W_{t_2})}{\sqrt{Var(W_{t_1})Var(W_{t_2})}} \\ &= \frac{t_1 \wedge t_2}{\sqrt{t_1 t_2}} \\ &= \sqrt{\frac{t_1}{t_2}}\end{aligned}$$

Thus:

$$\begin{aligned}
\rho_{ij} &= Corr(W_{idt}, W_{jdt}) \\
&= \sqrt{\frac{i}{j}} \\
\Rightarrow \rho_{12} &= \sqrt{\frac{1}{2}} \\
\rho_{13} &= \sqrt{\frac{1}{3}} \\
\rho_{23} &= \sqrt{\frac{2}{3}}
\end{aligned}$$

The correlation coefficient is negative when the correlation is between the time instant when exercising occurs and the previous times when exercising does not occur. This is because exercising occurs when the stock price falls below the critical exercise price. At the previous instants, the stock price was above the critical price and was not exercised. The correlation is positive when it is between the time instants at which exercising did not occur since at each of these time instants the stock price was above the critical stock value.

While the Geske-Johnson analytic formula solves the American put option problem, it cannot be implemented in a practical sense. As a result, numerical methods need to be implemented to approximate the solution. This can be done by thinking of the American put option as the limit of a sequence of put options with an increasing number of exercise opportunities. Then, by calculating a few of the put options in the sequence, we can extrapolate to the limit of the sequence .

Table 5.1: Comparison of American Put Option Prices

r	S	K	σ	T	$P(1)$	$P(An)$	$P(Num)$
0.125	1	1	0.5	1	0.1327	0.1476	0.148
0.08	1	1	0.4	1	0.117	0.1258	0.126
0.045	1	1	0.3	1	0.0959	0.1005	0.101
0.02	1	1	0.2	1	0.0694	0.0712	0.071
0.005	1	1	0.1	1	0.0373	0.0377	0.038
0.09	1	1	0.3	1	0.0761	0.0859	0.086
0.04	1	1	0.2	1	0.06	0.064	0.064
0.01	1	1	0.1	1	0.0349	0.0357	0.036
0.08	1	1	0.2	1	0.0442	0.0525	0.053
0.02	1	1	0.1	1	0.0304	0.0322	0.033
0.12	1	1	0.2	1	0.0317	0.0439	0.044
0.03	1	1	0.1	1	0.0263	0.0292	0.03

The values in this table are taken from Table 1 of Geske and Johnson [10]. The values in the first four columns are for the risk-free interest rate r*, the initial stock price* S*, the strike price* K *and the expiration time* T*.* $P(1)$ *contains the European put option values.* $P(An)$ *contains the analytic American put option values that are obtained by solving the equation* $P = K\omega_2 - S\omega_1$*.* $P(Num)$ *are numerical values obtained from Parkinson [15] as a benchmark to compare with the analytic values.*

Chapter 6

Numerical Methods for Valuing American Put Options

Thus far, under the Black-Scholes market economy, the arbitrage-free prices of European options and American call options can be determined with a closed form solution. The difficulty has been in trying to establish a risk-neutral valuation of an American put option. While Geske and Johnson [10] managed to determine a solution using a compound option pricing approach, their formula is unable to be determined for practical purposes. However, numerical methods that involve the calculation of a series of Bermudan-style options can be used in conjunction with a Richardson Extrapolation technique to yield an accurate approximation of the desired American put option price. The number of Bermudan puts necessary to determine an approximation for the American put option with a desired level of accuracy can be determined by what is known as the Repeated-Richardson Extrapolation technique. Other methods available combine the Binomial model with the Black-Scholes pricing formula and a two-point Richardson extrapolation. Using a spreadsheet, Monte Carlo simulation can be combined with Richardson Extrapolation to yield a fairly accurate approximation.

6.1 Bermudan Options

A Bermudan option is an option that may only be exercised at one of a discrete number of time points. The European put option is a Bermudan put option that has only one exercise time, i.e. at maturity time T. Let $P(j)$ be the value of a Bermudan put that can only be exercised at times t_i, $i = 1, 2, ..., j$, where $0 \leq t_1 \leq t_2 \leq .... \leq t_{j-1} \leq t_j = T$.

Recall that for an American put option, $P(0, S_0)$, with strike price K and maturity T we have the formula (Egloff, et al. [8]):

$$P(0, S_0) = \sup_{\tau \in \Gamma_{[0,T]}} E^Q[e^{-r\tau}(K - S_\tau)^+ \mid \Sigma_0]$$

For a Bermudan option with a finite number of exercise points we have the formula:

$$P(j) = \sup_{\tau \in \Gamma_{(0,t_1,\cdots,t_j=T)}} E^Q[e^{-r\tau}(K - S_\tau)^+ \mid \Sigma_0],$$

where $\Gamma_{(0,t_1,\cdots,t_j=T)}$ is the discrete set of all $(0, t_1, \cdots, t_j = T)$-stopping times up to and including time T. We can price these options by using the compound option approach of Geske and Johnson [10] and can arrive at an explicit solution. For example, consider $P(2)$, the value of a Bermudan put option with strike price K that can only be exercised at time t_1 or at time T, where $t_1 < T$. At time t_1, the option will be exercised if the payoff from exercising is as least as great as the value of the put if not exercised. This means that there is a critical stock value, $\overline{S}_{t_1}$, which results in early exercise. If the value of the underlying stock is below $\overline{S}_{t_1}$ then the put is exercised. $P(2)$ is priced as the discounted expected value of all future cash flows (Geske & Johnson [10]). The discounted payoff is integrated over all stock prices less than $\overline{S}_{t_1}$. At time T the option will only be exercised if the stock price is less than $\overline{S}_T$, the critical stock price at time T, and if the option has not already been exercised. Thus we have the formula:

$$\begin{aligned} P(2) &= [Ke^{-rt_1}\Phi_1(-d_2(\overline{S}_{t_1}, t_1)) - S_0\Phi_1(-d_1(\overline{S}_{t_1}, t_1))] \\ &\quad +[Ke^{-rT}\Phi_2(d_2(\overline{S}_{t_1}, t_1), -d_2(\overline{S}_T, T); -\sqrt{\frac{1}{2}}) \\ &\quad -S_0\Phi_2(d_1(\overline{S}_{t_1}, t_1), -d_1(\overline{S}_T, T); -\sqrt{\frac{1}{2}})] \end{aligned}$$

Pricing $P(1)$ requires the solving of one integral. Pricing $P(2)$ requires the solving of a single and a double integral. Carrying on in this fashion it can be shown that pricing $P(j)$ requires the solving of j integrals, the jth integral being of dimension j.

6.2 The Geske-Johnson Formula

The risk-neutral value of an American put option is the limit of a sequence of Bermudan put option values with the same strike price and maturity time, where the limit is with respect to an increasing number of exercise points, i.e.

$$P(0, S_0) = \lim_{j \to \infty} P(j) = P(\infty)$$

Determining $P(j)$ requires solving j integrals, the jth integral being of dimension j. Geske and Johnson [10] used the method of Richardson Extrapolation[1] in order to approximate the limit. The exercise times for the options were assumed to be separated by arithmetic time steps. For example, $P(2)$ could only be exercised at times $\frac{T}{2}$ and T and $P(3)$ could only be exercised at times $\frac{T}{3}$, $\frac{2T}{3}$ and T, etc. This technique requires the values of only a few Bermudan puts to approximate the limit. Their three-point extrapolation formula is:

$$P(\infty) = P(3) + \frac{7}{2}(P(3) - P(2)) - \frac{1}{2}(P(2) - P(1))$$

There are two main problems with this approach. Firstly, there is the possibility of non-uniform convergence. For uniform convergence we require $P(1) \leq P(2) \leq P(3)$. Acceleration techniques, such as Richardson Extrapolation, are best suited to approximating uniformly convergent sequences. Problems arise when sequences with oscillatory convergences are extrapolated. When the exercise times are separated by arithmetic time steps it is possible that $P(1) \leq P(2) > P(3)$. An example is a deep-in-the-money put option written on a low volatility, high-dividend paying stock going ex-dividend once during the term of the option at time $\frac{T}{2}$. There is a good chance that the option will be exercised at that time, immediately after the stock goes ex-dividend. Because $P(3)$ cannot be exercised at that time we have the possibility that $P(2) > P(3)$ (Omberg [14]).

Secondly, the number of Bermudan options needed in the extrapolation procedure to gain the desired level of accuracy is difficult to determine (Chang, et al. [6]). There is always the trade off between accuracy and efficiency. Including more options will increase the accuracy but may be too computationally intensive (including $P(4)$ in the extrapolation results in the solving of a $4th$ dimensional integral).

Modifications to the Geske-Johnson Formula

In order to overcome the problem of non-uniform convergence, Omberg [14] suggested that each successive Bermudan put option in the sequence must include the same exercise points as the previous Bermudan put option. This will ensure that subsequent Bermudan put options will be worth at least as much as previous put options. This can be obtained by

[1]For details on Richardson Extrapolation, see the appendix.

having the exercise points separated by geometric time steps. The result is the sequence $P(1)$, $P(2)$, $P(4)$, $P(8)$, ..., $P(2^j)$, etc. For example, $P(4)$ can be exercised at times $\frac{T}{4}$, $\frac{T}{2}$, $\frac{3T}{4}$ and T. $P(4)$ includes the exercise times of P(2) and thus $P(2) \leq P(4)$.

While the Geske-Johnson model uses arithmetic time steps to separate the exercise times, there is no reason why the exercise points need to be equally spaced. Bunch and Johnson [5] proposed that if the exercise points were chosen so as to maximise the value of the Bermudan put options then fewer put options would be needed in the Richardson Extrapolation procedure to maintain the desired accuracy. As a result of this maximizing procedure the problem of non-uniform convergence is overcome.

Let

$$P^{max}(j) = \max_{0 \leq t \leq T} P(j)$$

Clearly we have $P^{max}(i) \leq P^{max}(j)$, for all $0 < i \leq j$. $P^{max}(1) = P(1)$ since an out-of-the-money put would always have its exercise point at expiration (Bunch & Johnson [5]). The two-point formula is:

$$P(\infty) = P^{max}(2) - P(1)$$

6.3 Modifications to the Binomial Method

Broadie and Detemple [4] suggest modifying the Binomial method by replacing the 'continuation value' at the last time step before option maturity with the Black Scholes formula. This is known as the Binomial Black Scholes Method (BBS).

Another method is to combines the BBS method with two-point Richardson extrapolation (Broadie & Detemple [4]). The idea behind this method is that the accuracy of the binomial method increases as the number of time steps increases. Thus $P(\infty) = \lim_{n \to \infty} P_n$, where P_n is the binomial value of an American put option using n time steps. However, the Binomial method suffers from oscillatory convergence. Thus Richardson extrapolation might not be satisfactory when applied to the binomial method. Fortunately, we can apply Richardson extrapolation to the BBS method. Thus we compute P_m and P_n using the BBS method, where $m > n$, and use two-point Richardson extrapolation to obtain the final result of $P(\infty) = 2P_m - P_n$. This is known as BBSR (Binomial Black Scholes Method with Richardson Extrapolation).

6.4 Repeated-Richardson Extrapolation

Suppose you want to determine the value of a function $F(0)$, where $F(0) = \lim_{h\to 0} F(h)$. If the approximations $F(h)$ are available for $h > 0$ and the order of the approximations are known then we can express

$$F(h) = F(0) + a_1 h^{p_1} + a_2 h^{p_2} + \ldots + a_k h^{p_k} + O(h^{p_{k+1}}),$$

where a_1, a_2, a_3, etc. are unknown and $p_1 < p_2 < p_3 < \ldots$

Chang, et al. [6] proposed applying the numerical procedure known as the Aitken-Neville Algorithm to approximate the value of $F(0)$. The algorithm is as follows:

- Calculate $F(h)$ using a decreasing sequence of time steps, h_i, $i = 1, 2, 3, \ldots$. The result is the sequence of approximations $F(h_1), F(h_2), \ldots$
- Set $A_{i,0} = F(h_i)$ and $A_{i,m} = A_{i+1,m-1} + \frac{A_{i+1,m-1} - A_{i,m-1}}{(\frac{h_i}{h_{i+m}})^p - 1}$, when $p_j = jp$, $j = 1, \ldots k$.

$A_{i,m}$ is an m times Repeated-Richardson extrapolation approximation of $F(0)$ using the values, $F(h_i), F(h_{i+1}), \ldots, F(h_{i+m})$, where $0 < m \le k - 1$. The diagram illustrates the procedure.

$$\begin{array}{ccccccc}
A_{1,0} & \rightarrow & A_{1,1} & \rightarrow & A_{1,2} & \rightarrow & A_{1,3} \\
 & \nearrow & & \nearrow & & \nearrow & \\
A_{2,0} & \rightarrow & A_{2,1} & \rightarrow & A_{2,2} & & \\
 & \nearrow & & \nearrow & & & \\
A_{3,0} & \rightarrow & A_{3,1} & & & & \\
 & \nearrow & & & & & \\
A_{4,0} & & & & & & \\
 & & & & & & \\
\vdots & & & & & &
\end{array}$$

This procedure is applied to the Geske-Johnson formula by letting $P(\infty) = F(0)$, and $P(i) = F(h_i) = A_{i,0}$. In the appendix it is shown that for a two-point Richardson Extrapolation we have:

$$F(0)_R = F(h_2) + \frac{F(h_2) - F(h_1)}{(\frac{h_1}{h_2})^{p_1} - 1}$$

The Repeated-Richardson Extrapolation Procedure repeatedly extrapolates on two approximations for $F(0)$ that were, themselves, extrapolations on two other approximations.

For example, $A_{1,1}$ approximates $P(\infty)$ by extrapolating $P(1)$ and $P(2)$. $A_{2,1}$ approximates $P(\infty)$ by extrapolating $P(2)$ and $P(3)$. Now, $A_{1,2}$ approximates $P(\infty)$ by extrapolating $A_{1,1}$ and $A_{2,1}$. Thus, using Arithmetic time steps we have $h_1 = T$, $h_2 = \frac{T}{2}$, $h_3 = \frac{T}{3}$, etc. The following results are obtained:

$$\begin{aligned} A_{1,1} &= 2P(2) - P(1) \\ A_{2,1} &= \tfrac{3}{2}P(3) - \tfrac{1}{2}P(2) \\ A_{1,2} &= \tfrac{9}{2}P(3) - 4P(2) + \tfrac{1}{2}P(1) \end{aligned}$$

It can be seen that $A_{1,1}$ and $A_{1,2}$ are equivalent to the Geske-Johnson two-point and three-point Richardson Extrapolation approximation for $P(\infty)$ respectively. As before, the problem of non-uniform convergence arises when arithmetic time steps are used in determining the exercise times for the Bermudan puts. As with the modified Geske-Johnson formula we introduce geometric time steps to eliminate the problem of non-uniform convergence. Letting $P(\infty) = F(0)$, and $P(2^{i-1}) = A_{i,0}$, $i = 1, 2, \ldots$, the following results are obtained:

$$\begin{aligned} A_{1,1} &= 2P(2) - P(1) \\ A_{2,1} &= 2P(4) - P(2) \\ A_{1,2} &= \tfrac{8}{3}P(4) - 2P(2) + \tfrac{1}{3}P(1) \end{aligned}$$

It can be seen that $A_{1,1}$ and $A_{1,2}$ are equivalent to the modified geometric time step Geske-Johnson two-point and three-point Richardson Extrapolation approximation for $P(\infty)$ respectively.

Table 6.1: Comparison of American Put Option Prices

K	σ	T	$PR(1)$	$PR(\infty)$	$PR(1,2,3)$	$PR(1,2,4)$
35	0.2	0.0833	0.0062	0.0062	0.0062	0.0062
35	0.2	0.3333	0.1999	0.2004	0.1999	0.1999
35	0.2	0.5833	0.417	0.4329	0.4326	0.4325
40	0.2	0.0833	0.8404	0.8523	0.8521	0.8522
40	0.2	0.3333	1.5222	1.5799	1.576	1.5772
40	0.2	0.5833	1.8813	1.9906	1.9827	1.9847
45	0.2	0.0833	4.8399	5	4.9969	4.9973
45	0.2	0.3333	4.7805	5.0884	5.1053	5.1027
45	0.2	0.5833	4.8402	5.2671	5.2893	5.285
35	0.3	0.0833	0.0771	0.0775	0.0772	0.0773
35	0.3	0.3333	0.6867	0.6976	0.6973	0.6972
35	0.3	0.5833	1.189	1.2199	1.2199	1.2197
40	0.3	0.0833	1.2991	1.3102	1.3103	1.3103
40	0.3	0.3333	2.4276	2.4827	2.4801	2.4811
40	0.3	0.5833	3.0636	3.1698	3.1628	3.1651
45	0.3	0.0833	4.9796	5.0598	5.0631	5.0623
45	0.3	0.3333	5.529	5.7058	5.7019	5.7017
45	0.3	0.5833	5.9725	6.2438	6.2368	6.2367
35	0.4	0.0833	0.2458	0.2467	0.2463	0.2464
35	0.4	0.3333	1.3298	1.3462	1.3461	1.3459
35	0.4	0.5833	2.1129	2.155	2.1553	2.155
40	0.4	0.0833	1.7579	1.7685	1.7688	1.7687
40	0.4	0.3333	3.3338	3.3877	3.3863	3.3869
40	0.4	0.5833	4.2475	4.3529	4.3475	4.3496
45	0.4	0.0833	5.2362	5.287	5.2848	5.2851
45	0.4	0.3333	6.3769	6.51	6.5015	6.5035
45	0.4	0.5833	7.1657	7.3832	7.3696	7.3726

The values in this table are taken from Table 2 of Chang, et al. [6]. The values in the first three columns are for the strike price K, the volatility σ and the expiration time T. The stock price and risk-free interest rate are 40 *and* 0.05 *respectively.* $PR(1)$ *contains the European put option values.* $PR(\infty)$ *shows the BBSR approximations of American put options with* 10,800 *steps. The last two columns contain the values for the three point Geske Johnson Richardson extrapolation approximation using arithmetic time steps and geometric time steps.*

American Put Prediction Intervals Using Repeated-Richardson Extrapolation

Chang, et al. [6] showed that the advantage of using Repeated-Richardson extrapolation is that the error bounds of the American put approximation can be determined. Thus two important questions are answered. Firstly, the accuracy of the approximation can be determined. Secondly, the number of Bermudan put options needed to approximate the American put option within a given accuracy can also be determined. This is done by applying Schmidt's inequality (Schmidt [17]).

Schmidt's Inequality[2]:

$$\mid A_{i,m+1} - F(0) \mid \leq \mid A_{i,m+1} - A_{i,m} \mid,$$

for sufficiently large i, and $0 < m \leq k - 1$.

Letting, $F(0) = P(\infty)$, the American put value, and $A_{i,m}$ be the m-times Repeated-Richardson extrapolation approximation of $P(\infty)$, using Schmidt's inequality we know that the error of the approximation will be at most $\mid A_{i,m+1} - A_{i,m} \mid$. Thus, for the desired accuracy, ϵ, we can determine the smallest values of i and m such that $\mid A_{i,m+1} - A_{i,m} \mid \leq \epsilon$. If $i*$ and $m*$ are the smallest values of i and m respectively, such that $\mid A_{i,m+1} - A_{i,m} \mid \leq \epsilon$ then the American put can be approximated with the desired accuracy by calculating $A_{i*,m*+1}$. This requires $m * +2$ Bermudan puts with step sizes $h_{i*}, h_{i*+1}, \ldots, h_{i*+m*+1}$.

6.5 Monte Carlo Simulation with Richardson Extrapolation

In an attempt to quickly price a Bermudan put option, a Monte Carlo simulation was run using Pop Tools (Hood [11]) in a simple spreadsheet. Four types of Bermudan options were simulated:

- $P(1)$, the value of a Bermudan put option which can only be exercised at expiration time T (i.e. a European put option).
- $P(2)$, the value of a Bermudan put option which can only be exercised at time points $\frac{T}{2}$ and T.
- $P(3)$, the value of a Bermudan put option which can only be exercised at time points $\frac{T}{3}$, $\frac{2T}{3}$ and T.

[2] For a detailed proof of Schmidt's Inequality, see Chang, et al. [6].

- $P(4)$, the value of a Bermudan put option which can only be exercised at time points $\frac{T}{4}$, $\frac{T}{2}$, $\frac{3T}{4}$ and T.

Monte Carlo Simulation

Monte Carlo simulation can be used to estimate the expected value of some random variable X. This is done by generating a sequence of n independent random variables, $X_1, X_2, X_3, \ldots, X_n$, with the same probability distribution as X and then determining $\bar{X}$, their arithmetic average. A simulation run occurs every time a new value is generated (Ross [16]).

If $\mu = E[X]$ and $\sigma^2 = Var(X)$ then:

- $E[\bar{X}] = \frac{1}{n}\sum_{i=i}^{n} E[X_i] = \mu$
- $Var(\bar{X}) = Var(\frac{1}{n}\sum_{i=i}^{n} X_i) = \frac{1}{n^2}\sum_{i=i}^{n} Var(X_i) = \frac{\sigma^2}{n}$

As a result of the central limit theorem, $\bar{X}$ is approximately normally distributed with mean μ and variance $\frac{\sigma^2}{n}$ for large values of n. Thus for large values of n, the variance will be small and $\bar{X}$ will be close to μ. As a result, $\bar{X}$ will be a good estimator for μ. The greater the value of n, the better the estimate will be.

Procedure for Simulating Bermudan Put Options

In order to perform a Monte Carlo simulation on $P(j)$, a Bermudan put option with j exercise time points, the following procedure was adopted:

- Generate j standard normal random variables $(Z_1, Z_2, \ldots, Z_j)$ using the random variable generator from Pop Tools (Hood [11]).
- Adjust for drift and volatility by multiplying random variables by standard deviation $\sigma\sqrt{\frac{T}{j}}$ and adding risk-neutral mean $(r - \frac{1}{2}\sigma^2)\frac{T}{j}$, i.e. $W_i = (r - \frac{1}{2}\sigma^2)\frac{T}{j} + Z_i\sigma\sqrt{\frac{T}{j}}$, $i = 1, \ldots, j$.
- Calculate stock prices at each exercise point, $S_{\frac{T}{j}}, S_{\frac{2T}{j}}, \ldots S_{\frac{(j-1)T}{j}}, S_T$, with the following equation: $S_{\frac{iT}{j}} = S_{\frac{(i-1)T}{j}} exp(W_i)$, $i = 1, \ldots, j$, $S_0 =$ initial stock price. Thus at each exercise point, if the put option is not exercised then the stock price at this point is treated as the new initial stock price at the next exercise point.

- At the first exercise point, the intrinsic value $(K - S_{\frac{T}{j}})$ is calculated. This value is compared with the value of the put option if not exercised. To calculate the non-exercise value, 1000 stock prices are generated in the same procedure as before. The discounted average payoff for each of the thousand values is determined.

- If $(K - S_{\frac{T}{j}})$ is greater than the non-exercise value than the procedure stops. If not, then the intrinsic value $(K - S_{\frac{2T}{j}})$ is calculated and compared to the new non-exercise value of the put option. This new non-exercise value is computed the same as before by generating 1000 stock prices and determining the discounted average payoff for each of the thousand values.

- The procedure carries on until the intrinsic value at an exercise point is greater than the non-exercise value or until expiration when the put option will be exercised if it is in the money. Thus $j + 1000 \times j$ random variables are generated in total.

- Using Pop Tools (Hood [11]), a Monte Carlo simulation with 30000 iterations is run on the stopping value of the option, discounted to time zero. The mean of the simulation run is the approximation for the Bermudan put option.

Table 6.2: Monte Carlo Simulation of Bermudan Put Options

K	σ	T	$P(1)$	$P(2)$	$P(3)$	$P(4)$	$P(1,2)$	$P(1,2,3)$	$P(1,2,4)$	$PR(\infty)$
35	0.2	0.0833	0.0055	0.0061	0.0064	0.0067	0.0068	0.0069	0.0073	0.0062
35	0.2	0.3333	0.1983	0.1918	0.1993	0.2028	0.1853	0.2286	0.2234	0.2004
35	0.2	0.5833	0.4082	0.4099	0.4184	0.4308	0.4117	0.447	0.465	0.4329
40	0.2	0.0833	0.8393	0.8246	0.8433	0.8465	0.8099	0.916	0.8877	0.8523
40	0.2	0.3333	1.506	1.5504	1.5491	1.5062	1.5948	1.5223	1.4178	1.5799
40	0.2	0.5833	1.8593	1.9079	1.9416	1.9271	1.9565	2.0351	1.9429	1.9906
45	0.2	0.0833	4.8468	4.9074	4.9334	4.9564	4.9679	4.9944	5.0181	5
45	0.2	0.3333	4.8056	4.9369	4.9898	4.9568	5.0682	5.1094	4.9463	5.0884
45	0.2	0.5833	4.8137	5.0692	5.126	5.0905	5.3248	5.1971	5.0406	5.2671
35	0.3	0.0833	0.0762	0.0751	0.0758	0.0779	0.0742	0.0785	0.0827	0.0775
35	0.3	0.3333	0.6928	0.6752	0.6738	0.6866	0.6575	0.6778	0.7116	0.6976
35	0.3	0.5833	1.1814	1.1842	1.209	1.1913	1.187	1.2944	1.2021	1.2199
40	0.3	0.0833	1.2961	1.2876	1.3072	1.2849	1.2791	1.38	1.2831	1.3102
40	0.3	0.3333	2.43	2.4385	2.4532	2.4354	2.447	2.5006	2.4273	2.4827
40	0.3	0.5833	3.0486	3.1174	3.0825	3.0866	3.1861	2.9259	3.0123	3.1698
45	0.3	0.0833	4.9784	5.0037	5.0112	5.0278	5.029	5.0249	5.0595	5.0598
45	0.3	0.3333	5.4758	5.6386	5.629	5.5543	5.8013	5.5141	5.3596	5.7058
45	0.3	0.5833	5.9222	6.1998	6.1067	6.0692	6.4774	5.642	5.7589	6.2438
35	0.4	0.0833	0.2447	0.2483	0.2468	0.2459	0.2519	0.2399	0.2406	0.2467
35	0.4	0.3333	1.3274	1.3512	1.3318	1.3452	1.3751	1.2518	1.3273	1.3462
35	0.4	0.5833	2.0933	2.1063	2.1221	2.1305	2.1194	2.1706	2.1676	2.155
40	0.4	0.0833	1.7501	1.7596	1.7551	1.7296	1.769	1.7346	1.6765	1.7685
40	0.4	0.3333	3.3311	3.3171	3.3336	3.3172	3.3031	3.3983	3.322	3.3877
40	0.4	0.5833	4.2408	4.3008	4.2845	4.2726	4.3608	4.1975	4.2055	4.3529
45	0.4	0.0833	5.2523	5.2805	5.2344	5.2222	5.3087	5.059	5.1157	5.287
45	0.4	0.3333	6.4468	6.4744	6.42	6.3621	6.502	6.2156	6.1657	6.51
45	0.4	0.5833	7.1613	7.2111	7.269	7.1904	7.2609	7.4469	7.1393	7.3832

The values in the first three columns are taken from Table 1 of Geske and Johnson [10]. These are values for the strike price K, the volatility σ and the expiration time T. The stock price and risk-free interest rate are 40 and 0.05 respectively. The next four columns contain the values of the Monte Carlo simulation of four Bermudan put options ranging from $P(1)$, a Bermudan put option with only one exercise time to $P(4)$, a Bermudan put option with 4 exercise times. The next three columns contain the values of the $P(1,2)$ (the two point Richardson Extrapolation of $P(1)$ and $P(2)$), $P(1,2,3)$ (the three point Richardson Extrapolation of $P(1)$, $P(2)$ and $P(3)$) and $P(1,2,4)$ (the three point Richardson Extrapolation of $P(1)$, $P(2)$ and $P(4)$. $PR(\infty)$ shows the BBSR approximations of American put options with 10,800 steps taken from table 2 of Chang, et al. [6]. These values are used as a benchmark to compare the extrapolated approximations.

Table 6.3: Monte Carlo Simulation of Bermudan Put Options Continued

r	σ	$P(1)$	$P(2)$	$P(3)$	$P(4)$	$P(1,2)$	$P(1,2,3)$	$P(1,2,4)$	$P(Num)$
0.125	0.5	0.1333	0.1396	0.1438	0.1419	0.1459	0.1555	0.1438	0.148
0.08	0.4	0.1168	0.1213	0.1218	0.1215	0.1257	0.1216	0.1205	0.126
0.045	0.3	0.0966	0.0975	0.0983	0.0978	0.0983	0.1009	0.0981	0.101
0.02	0.2	0.0692	0.0696	0.0709	0.0704	0.0701	0.0751	0.0715	0.071
0.005	0.1	0.038	0.0369	0.0374	0.037	0.0358	0.0394	0.0375	0.038
0.09	0.3	0.0767	0.0815	0.0821	0.0823	0.0862	0.0822	0.0822	0.086
0.04	0.2	0.0607	0.0611	0.0622	0.0612	0.0615	0.0658	0.0617	0.064
0.01	0.1	0.0347	0.0347	0.0352	0.0348	0.0346	0.0371	0.035	0.036
0.08	0.2	0.0437	0.0486	0.0496	0.0496	0.0536	0.0506	0.05	0.053
0.02	0.1	0.0306	0.0311	0.0318	0.0313	0.0316	0.0339	0.0314	0.033
0.12	0.2	0.0321	0.0379	0.0396	0.0404	0.0436	0.0428	0.0427	0.044

The values in the first two columns are taken from Table 1 of Geske and Johnson [10]. This table is similar to the previous table except that interest rates are varied and the stock price, strike price and time to expiration are equal to 1. $P(Num)$ *contains numerical values from Parkinson [15]. These values are used as a benchmark to compare the extrapolated approximations.*

Results from Simulation

The purpose of the simulation was to quickly establish a value for an American put option with minimal input. However, due to the nature of the procedure, problems associated with non-uniform convergence arose. The estimation of the four Bermudan put options lacked the precision necessary to successfully ensure that uniform convergence was achieved. Using estimates to estimate the American put option resulted in a great deal of error creeping into the procedure. As a result, applying the Richardson extrapolation procedures would not always produce a valid solution to the pricing problem.

More advanced methods can be used to successfully price Bermudan and American put options using Monte Carlo simulation[3] . The point of the procedure was to quickly establish a solution that is straightforward and relatively simple to implement. Thus, a spreadsheet was chosen to implement the procedure. More advanced computer programs can be used to help solve the pricing problem. For example, using the programming language C++ is one of the most effective ways to price financial derivatives (Nhongo [13]).

[3] See Egloff, et al. [8] for a more involved analysis.

Chapter 7

Conclusion

The American put option pricing problem is solved analytically by calculating a sequence of Bermudan option prices and extrapolating to the limit of the sequence (the American put option price) using a Richardson extrapolation technique. The solution to the pricing problem must be arbitrage free. Since the sequence of Bermudan puts that are constructed must satisfy the Black-Scholes partial differential equation, the limit of the sequence must also satisfy the Black-Scholes partial differential equation. Thus, the main assumption of zero arbitrage is maintained with this approach.

Using Repeated-Richardson extrapolation to solve the pricing problem has the advantage of being able to determine the number of Bermudan put options necessary to obtain a desired accuracy. Pricing American options is thus the simple matter of determining the extrapolation formula using the values of a few Bermudan put options. The key to making this procedure more efficient is being able to price Bermudan options more accurately and quickly.

7.1 Future Work

When applying a Richardson extrapolation technique on a sequence of Bermudan options, the results will only be accurate when the sequence converges uniformly. This problem is overcome by using geometric time steps. When estimating the Bermudan options themselves via a Monte Carlo simulation, the problem of non-uniform convergence arises once more. The literature suggests that this problem can be overcome and Bermudan options can be successfully priced using simulation techniques (Egloff, et al. [8]). Thus, future developments in financial derivative simulation, especially with respect to Bermudan options, can be investigated with the use of more dynamic computer programs.

Chapter 8

Appendix

8.1 Measure Spaces

A σ-algebra (or σ-field) is a collection of subsets of an arbitrary space with the properties that the arbitrary space belong to the collection of subsets, and that the collection of subsets are closed under countable unions and closed under complements (De Barra [7]). This can be expressed more formally as follows:

Let Ω be a nonempty set. A collection Σ of subsets of Ω is called a σ-algebra in Ω if:

1. $\emptyset \in \Sigma$;
2. if $A \in \Sigma$ then the complement of A, A^c, is also in Σ;
3. if (A_n) is a sequence of sets in Σ, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$;

A measurable space is a pair (Ω, Σ), where Ω is a set and Σ is σ-algebra in Ω. The elements of the σ-algebra are called measurable sets (Aggoun & Elliot [1]). To discuss measure spaces, the concept of a measure needs to be defined. A measure is a generalization of the concept of length (De Barra [7]).

Let (Ω, Σ) be a measurable space. A measure on (Ω, Σ) is a function $\mu : \Sigma \to [0, 1]$ such that

1. $\mu(\emptyset) = 0$;

2. If (A_n) is a sequence of mutually disjoint sets in Σ then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

A measure space is a triple (Ω, Σ, μ), where Ω is a set, Σ a σ-algebra in Ω and μ a measure on Σ.

Probability Measure

Let (Ω, Σ, μ) be a measure space. if $\mu(\Omega) = 1$, then (Ω, Σ, μ) is called a probability space and μ is called a probability measure (Aggoun & Elliot [1]).

8.2 Constructing the Itô Integral

Simple Function

The construction of the Itô integral mimics that of the Lebesgue integral. The first step is to define the integral in terms of simple functions. the Itô integral is then equal to the limit of a sequence of these simple functions.

Let $f(s, \omega) = \sum_{i=1}^{n} a_i(\omega) 1_{I_i}(s)$, where $I_i = (s_i, s_{i+1}]$, $\bigcup_{i=1}^{n} I_i = (0, T]$, $I_i \bigcap I_j = \{\emptyset\}$ if $i \neq j$ and, for every $i = 1, \ldots, n$, $a_i : \Omega \to \mathbb{R}$ is a Σ_{s_i}-measurable random variable with $E[a_i^2(\omega)] < \infty$.

Define:

$$\int_0^t f(s, \omega) dW_s = \int f(s, \omega) 1_{[0,t]}(s) dW_s$$

Thus,

$$\int_0^t f(s, \omega) dW_s = \sum_{i=1}^{n} a_i(\omega) 1_{[0,t]}(s_i)(W_{s_{i+1} \wedge t} - W_{s_i})$$

Lemma

Given a simple function f:

- $\int_0^t f(s, \omega) dW_s$ is a continuous $(P, \{\Sigma_t\}_{t \geq 0})$-martingale,
- Itô Isometry: $E[(\int_0^t f(s, \omega) dW_s)^2] = \int_0^t E[f(s, \omega)^2] ds$,
- $E[\sup_{t \leq T} (\int_0^t f(s, \omega) dW_s)^2] \leq 4 \int_0^T E[f(s, \omega)^2] ds$.

Proof of Itô Isometry:

$$\int_0^t f(s,\omega)dW_s = \sum_{i=1}^n a_i(\omega)1_{[0,t]}(s_i)(W_{s_{i+1}\wedge t} - W_{s_i})$$

To simplify notation we assume: $\int_0^t f(s,\omega)dW_s = \sum_{i=1}^n a_i(\omega)(W_{s_{i+1}} - W_{s_i})$

$$\begin{aligned}
E[(\int_0^t f(s,\omega)dW_s)^2] &= E[(\sum_{i=1}^n a_i(\omega)(W_{s_{i+1}} - W_{s_i}))^2] \\
&= E[\sum_{i=1}^n a_i^2(\omega)(W_{s_{i+1}} - W_{s_i})^2] \\
&\quad +2E[\sum_{i<j} a_i(\omega)a_j(\omega)(W_{s_{i+1}} - W_{s_i})(W_{s_{j+1}} - W_{s_j})] \\
&= E[E[\sum_{i=1}^n a_i^2(\omega)(W_{s_{i+1}} - W_{s_i})^2 \mid \Sigma_{s_j}]] \\
&\quad +2E[E[\sum_{i<j} a_i(\omega)a_j(\omega)(W_{s_{i+1}} - W_{s_i})(W_{s_{j+1}} - W_{s_j}) \mid \Sigma_{s_j}]] \\
&= E[\sum_{i=1}^n a_i^2(\omega)E[(W_{s_{i+1}} - W_{s_i})^2 \mid \Sigma_{s_i}]] \\
&\quad +2E[\sum_{i<j} a_i(\omega)a_j(\omega)E[(W_{s_{i+1}} - W_{s_i})(W_{s_{j+1}} - W_{s_j}) \mid \Sigma_{s_j}]] \\
&= E[\sum_{i=1}^n a_i^2(\omega)](s_{i+1} - s_i) \\
&\quad +2E[\sum_{i<j} a_i(\omega)a_j(\omega)(W_{s_{i+1}} - W_{s_i})E[(W_{s_{j+1}} - W_{s_j}) \mid \Sigma_{s_j}]] \\
&= \sum_{i=1}^n E[a_i^2(\omega)](s_{i+1} - s_i) + 0 \\
&= \int_0^t E[f(s,\omega)^2]ds
\end{aligned}$$

Itô Integral of a General Integrand

Let f be a function such that:

- $f(t,\omega)$ is $\{\Sigma_t\}_{t\geq 0}$-measurable for $0 \leq t \leq T$,

- $\int_0^T E[f(s,\omega)^2]ds < \infty$.

Let $\{f_n\}_{n\geq 1}$ be a sequence of simple functions and define:

$$\int_0^t f(s,\omega)dW_s = \begin{array}{c} lim \\ n \to \infty \end{array} \int_0^t f_n(s,\omega)dW_s$$

The Itô integral is the limit of a sequence of integrals, each with simple functions as their integrand.

8.3 The Greeks

The *Greeks* measure the sensitivity of derivative securities with respect to parameter changes. Before calculating the *delta*, the *gamma* and the *theta* of a European put option, the following two lemmas need to be derived:

Lemma

Consider the stock price process $\{S_t\}_{0\leq t\leq T}$, where $x = S_t$, and interest rate r. Let Y be a standard normal random variable with characteristic equation:

$$I_{Y\leq y^*} = \{ \begin{array}{ccc} 1 & if & Y \leq y^* \\ 0 & if & Y > y^* \end{array}$$

Then (with $y^* = \frac{-\log\frac{x}{K} - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$*):*

$$e^{-r(T-t)}E[I_{Y\leq y^*}S_T] = x\Phi(-\omega)$$

Proof:

$$\begin{aligned}
e^{-r(T-t)}E[I_{Y\leq y^*}S_T] &= e^{-r(T-t)}\int_{-\infty}^{y^*} xexp((r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t})\frac{exp(-y^2/2)}{\sqrt{2\pi}}dy \\
&= \int_{-\infty}^{y^*}\frac{1}{\sqrt{2\pi}}xexp(-\frac{1}{2}\sigma^2(T-t)+\sigma y\sqrt{T-t}-y^2/2))dy \\
&= x\int_{-\infty}^{y^*}\frac{1}{\sqrt{2\pi}}exp(-\frac{1}{2}(y-\sigma\sqrt{T-t})^2)dy \\
&= x\int_{-\infty}^{y^*-\sigma\sqrt{T-t}}\frac{1}{\sqrt{2\pi}}exp(-z^2/2)dz \\
&= x\Phi(y^*-\sigma\sqrt{T-t}) \\
&= x\Phi(-\omega)
\end{aligned}$$

Lemma

Assume the same conditions as above. *Then:*

$$e^{-r(T-t)}E[I_{Y\leq y^*}S_TY] = x[-\frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}+\sigma\sqrt{T-t}\Phi(-\omega)]$$

Proof:

$$\begin{aligned}
e^{-r(T-t)}E[I_{Y\leq y^*}S_TY] &= e^{-r(T-t)}\int_{-\infty}^{y^*} yxexp((r-\frac{1}{2}\sigma^2)(T-t)+\sigma y\sqrt{T-t})\frac{exp(-y^2/2)}{\sqrt{2\pi}}dy \\
&= \frac{1}{\sqrt{2\pi}}x\int_{-\infty}^{y^*} yexp(-\frac{1}{2}\sigma^2(T-t)+\sigma y\sqrt{T-t}-y^2/2)dy \\
&= \frac{1}{\sqrt{2\pi}}x\int_{-\infty}^{y^*-\sigma\sqrt{T-t}}(z+\sigma\sqrt{T-t})exp(-z^2/2)dz \\
&= x[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\omega} zexp(-z^2/2)dz+\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}\int_{-\infty}^{-\omega}exp(-z^2/2)dz] \\
&= x[-\frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}+\sigma\sqrt{T-t}\Phi(-\omega)]
\end{aligned}$$

The three *Greeks* can be calculated by applying the two lemmas derived above.

The *Delta*:

$$\begin{aligned}
\frac{\partial V}{\partial x}(t,x) &= e^{-r(T-t)}\frac{\partial}{\partial x}E[(K-S_T)^+] \\
&= e^{-r(T-t)}E[I_{Y\le y^*}\frac{\partial}{\partial x}(K-S_T)] \\
&= -e^{-r(T-t)}E[I_{Y\le y^*}\frac{S_T}{x}] \\
&\quad -\frac{1}{x}e^{-r(T-t)}E[I_{Y\le y^*}S_T] \\
&= -\Phi(-\omega)
\end{aligned}$$

The *Gamma*:

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2}(t,x) &= -\frac{\partial}{\partial x}\Phi(-\omega) \\
&= -\frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}\frac{\partial(-\omega)}{\partial x} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}\frac{1}{x\sigma\sqrt{T-t}} \\
&= \frac{e^{-\omega^2/2}}{\sigma x\sqrt{2\pi(T-t)}}
\end{aligned}$$

The *Theta*:

$$
\begin{aligned}
\frac{\partial V}{\partial t}(t,x) &= \frac{\partial}{\partial t} e^{-r(T-t)} E[I_{Y\le y^*}(K-S_T)] \\
&= E[I_{Y\le y^*}\frac{\partial}{\partial t} e^{-r(T-t)}(K-S_T)] \\
&= E[I_{Y\le y^*}[re^{-r(T-t)}(K-S_T) \\
&\quad + e^{-r(T-t)}\frac{\partial}{\partial t}(K - xexp((r-\frac{1}{2}\sigma^2)(T-t)+\sigma Y\sqrt{T-t}))]] \\
&= E[I_{Y\le y^*}[re^{-r(T-t)}(K-S_T) \\
&\quad + e^{-r(T-t)}(-S_T)(-r+\frac{1}{2}\sigma^2 - \frac{\sigma}{2\sqrt{T-t}}Y)]] \\
&= E[I_{Y\le y^*}[re^{-r(T-t)}K + e^{-r(T-t)}S_T(-\frac{1}{2}\sigma^2 + \frac{\sigma}{2\sqrt{T-t}}Y)]] \\
&= Kre^{-r(T-t)}E[I_{Y\le -\omega+\sigma\sqrt{T-t}}] - \frac{\sigma^2}{2}e^{-r(T-t)}E[I_{Y\le y^*}S_T] \\
&\quad + \frac{\sigma}{2\sqrt{T-t}}e^{-r(T-t)}E[I_{Y\le y^*}S_T Y] \\
&= Kre^{-r(T-t)}\Phi(-\omega+\sigma\sqrt{T-t}) - \frac{\sigma^2}{2}x\Phi(-\omega) \\
&\quad + \frac{\sigma}{2\sqrt{T-t}}x[-\frac{1}{\sqrt{2\pi}}e^{-\omega^2/2} + \sigma\sqrt{T-t}\Phi(-\omega)] \\
&= Kre^{-r(T-t)}\Phi(-\omega+\sigma\sqrt{T-t}) - \frac{\sigma x e^{-\omega^2/2}}{2\sqrt{2\pi(T-t)}}
\end{aligned}
$$

To calculate the *Greeks* for a European call option, see Ross [16].

8.4 Perpetual American Put Options

The value of a perpetual American put option $V(x)$, with strike price K and riskless interest rate r is:

$$
V(x) = \{ \begin{array}{cc} (K-x) & if \;\; 0 \le x \le L \\ (K-L)(\frac{x}{L})^{-2r\sigma^{-2}} & if \;\; L < x < \infty \end{array}
$$

where $L = \frac{2r\sigma^{-2}K}{1+2r\sigma^{-2}}$

Proof:

This proof comes from Shreve [19]. Let the stock price process $\{S_t\}_{t\geq 0}$ satisfy the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q,$$

where $\{W_t^Q\}_{t\geq 0}$ is a standard Brownian motion under the martingale measure Q.

Letting $S_0 = x$ and $\theta = \frac{r-\frac{1}{2}\sigma^2}{\sigma}$, then:

$$\begin{aligned} S_t &= xexp\{(r - \frac{1}{2}\sigma^2)t + \sigma W_t^Q\} \\ &= xexp\{\sigma[\frac{r-\frac{1}{2}\sigma^2}{\sigma}t + W_t^Q]\} \\ &= xexp\{\sigma[\theta t + W_t^Q]\} \end{aligned}$$

Let $L \in [0, K]$. Then for all $x \geq L$, define:

$$\begin{aligned} \tau^* &= \inf\{t \geq 0; S_t = L\} \\ &= \inf\{t \geq 0; \theta t + W_t^Q = \frac{1}{\sigma}\log\frac{L}{x}\} \\ &= \inf\{t \geq 0; -W_t^Q = -\frac{1}{\sigma}\log\frac{L}{x} + \theta t\} \end{aligned}$$

Therefore,

$$\begin{aligned} V(L) &= E[e^{-r\tau^*}(K - L)] \\ &= (K - L)E[e^{-r\tau^*}] \end{aligned}$$

proposition:

Let $\tau = \inf\{t \geq 0; W_t^Q = a + bt\}$. Then for $\eta > 0$, $a > 0$ and $b > 0$:

$$E[e^{-\eta\tau}] = exp(-a(b + \sqrt{b^2 + 2\eta}))$$

For proof of this statement see Etheridge [9].

Now, $x > L \Rightarrow \log\frac{L}{x} < 0$. Thus $-\frac{1}{\sigma}\log\frac{L}{x} > 0$. $\{-W_t^Q\}_{t\geq 0}$ is also a standard Brownian motion under the martingale measure Q. Letting $a = -\frac{1}{\sigma}\log\frac{L}{x}$, $b = \theta$ and $\eta = r$ then:

$$\begin{aligned} V(L) &= (K-L)exp(\frac{1}{\sigma}\log\frac{L}{x}(\theta + \sqrt{\theta^2+2r})) \\ &= (K-L)exp(\frac{\theta}{\sigma}\log\frac{L}{x} + \frac{1}{\sigma}\log\frac{L}{x}\sqrt{\theta^2+2r})) \\ &= (K-L)(\frac{L}{x})^{\frac{\theta}{\sigma}+\frac{1}{\sigma}\sqrt{\theta^2+2r})} \\ &= (K-L)(\frac{x}{L})^{-\frac{\theta}{\sigma}-\frac{1}{\sigma}\sqrt{\theta^2+2r})} \end{aligned}$$

$$\begin{aligned} -\frac{\theta}{\sigma} - \frac{1}{\sigma}\sqrt{\theta^2+2r} &= -\frac{r}{\sigma^2} + \frac{1}{2} - \frac{1}{\sigma}\sqrt{(\frac{r-\frac{1}{2}\sigma^2}{\sigma})^2 + 2r} \\ &= -\frac{r}{\sigma^2} + \frac{1}{2} - \frac{1}{\sigma}\sqrt{\frac{r^2 - r\sigma^2 + \frac{1}{4}\sigma^4}{\sigma^2} + 2r} \\ &= -\frac{r}{\sigma^2} + \frac{1}{2} - \frac{1}{\sigma}\sqrt{\frac{r^2 + r\sigma^2 + \frac{1}{4}\sigma^4}{\sigma^2}} \\ &= -\frac{r}{\sigma^2} + \frac{1}{2} - \frac{1}{\sigma}\sqrt{(\frac{r+\frac{1}{2}\sigma^2}{\sigma})^2} \\ &= -\frac{r}{\sigma^2} + \frac{1}{2} - (\frac{r+\frac{1}{2}\sigma^2}{\sigma^2}) \\ &= -\frac{2r}{\sigma^2} \end{aligned}$$

Thus,

$$V(x) = \{ \begin{array}{cc} (K-x) & if\ \ 0 \le x \le L \\ (K-L)(\frac{x}{L})^{-2r\sigma^{-2}} & if\ \ L < x < \infty \end{array}$$

For $L < x < \infty$, $V(x) = (K-L)(\frac{x}{L})^{-2r\sigma^{-2}} = Cx^{-2r\sigma^{-2}}$ where $C = (K-L)(L)^{2r\sigma^{-2}}$. Since K, r, σ are fixed values, in order to maximise C, we determine the derivative of C

with respect to L and equate it to zero.

$$\begin{aligned}
0 = \frac{\partial C}{\partial L} &= -(L)^{2r\sigma^{-2}} + 2r\sigma^{-2}(K-L)(L)^{2r\sigma^{-2}-1} \\
&= (L)^{2r\sigma^{-2}}[-1 + 2r\sigma^{-2}(K-L)L^{-1}] \\
\Rightarrow 0 &= -1 + 2r\sigma^{-2}(K-L)L^{-1} \\
\Rightarrow L &= 2r\sigma^{-2}(K-L) \\
\Rightarrow L(1+2r\sigma^{-2}) &= 2r\sigma^{-2}K \\
\Rightarrow L &= \frac{2r\sigma^{-2}K}{(1+2r\sigma^{-2})}
\end{aligned}$$

Now

$$\frac{dV(x)}{dx} = \{ \begin{array}{cc} -1 & if \;\; 0 \le x \le L \\ -2r\sigma^{-2}(K-L)(L)^{2r\sigma^{-2}}(x)^{-2r\sigma^{-2}-1} & if \;\; L < x < \infty \end{array}$$

Therefore,

$$\begin{aligned}
\lim_{x \downarrow L} \frac{dV(x)}{dx} &= -2r\sigma^{-2}(K-L)(L)^{2r\sigma^{-2}}(L)^{-2r\sigma^{-2}-1} \\
&= -2r\sigma^{-2}(K-L)\frac{1}{L} \\
&= -2r\sigma^{-2}\frac{K}{(1+2r\sigma^{-2})}\frac{(1+2r\sigma^{-2})}{2r\sigma^{-2}K} \\
&= -1
\end{aligned}$$

8.5 Richardson Extrapolation

Suppose you want to determine the value of a function $F(0)$, where $F(0) = \lim_{h\to 0} F(h)$. If the approximations $F(h)$ are available for $h > 0$ and the order of the approximations are known then

$$F(h) = F(0) + a_1 h^{p_1} + a_2 h^{p_2} + \ldots + a_k h^{p_k} + O(h^{p_{k+1}}),$$

where a_1, a_2, a_3, etc. are unknown and $p_1 < p_2 < p_3 < \ldots$.

Suppose $p_1 = p$, $p_2 = 2p$ and $F(h) = F(0) + a_1 h^{p_1} + a_2 h^{p_2} + O(h^{p_3})$. We have three unknown quantities, $F(0)$, a_1 and a_2. let $F(0)_R$ be the Richardson approximation for

$F(0)$. We can solve for the three unknowns with the three equations:

$$\begin{aligned} F(h_3) &= F(0) + a_1 h_3^{p_1} + a_2 h_3^{p_2} + O(h^{p_3}) \\ F(h_2) &= F(0) + a_1 h_2^{p_1} + a_2 h_2^{p_2} + O(h^{p_3}) \\ F(h_1) &= F(0) + a_1 h_1^{p_1} + a_2 h_1^{p_2} + O(h^{p_3}) \end{aligned}$$

Using matrix notation and dropping the higher order terms we get:

$$\left(\begin{array}{ccc|c} 1 & h_3^{p_1} & h_3^{p_2} & F(h_3) \\ 1 & h_2^{p_1} & h_2^{p_2} & F(h_2) \\ 1 & h_1^{p_1} & h_1^{p_2} & F(h_1) \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & h_3^{p_1} & h_3^{p_2} & F(h_3) \\ 0 & h_3^{p_1} - h_2^{p_1} & h_3^{p_2} - h_2^{p_2} & F(h_3) - F(h_2) \\ 0 & h_2^{p_1} - h_1^{p_1} & h_2^{p_2} - h_1^{p_2} & F(h_2) - F(h_1) \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & h_3^{p_1} & h_3^{p_2} & F(h_3) \\ 0 & 1 & \frac{h_3^{p_2} - h_2^{p_2}}{h_3^{p_1} - h_2^{p_1}} & \frac{F(h_3) - F(h_2)}{h_3^{p_1} - h_2^{p_1}} \\ 0 & 1 & \frac{h_2^{p_2} - h_1^{p_2}}{h_2^{p_1} - h_1^{p_1}} & \frac{F(h_2) - F(h_1)}{h_2^{p_1} - h_1^{p_1}} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & h_3^{p_1} & h_3^{p_2} & F(h_3) \\ 0 & 1 & \frac{h_3^{p_2} - h_2^{p_2}}{h_3^{p_1} - h_2^{p_1}} & \frac{F(h_3) - F(h_2)}{h_3^{p_1} - h_2^{p_1}} \\ 0 & 0 & \frac{h_3^{p_2} - h_2^{p_2}}{h_3^{p_1} - h_2^{p_1}} - \frac{h_2^{p_2} - h_1^{p_2}}{h_2^{p_1} - h_1^{p_1}} & \frac{F(h_3) - F(h_2)}{h_3^{p_1} - h_2^{p_1}} - \frac{F(h_2) - F(h_1)}{h_2^{p_1} - h_1^{p_1}} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & h_3^{p_1} & h_3^{p_2} & F(h_3) \\ 0 & 1 & \frac{h_3^{p_2} - h_2^{p_2}}{h_3^{p_1} - h_2^{p_1}} & \frac{F(h_3) - F(h_2)}{h_3^{p_1} - h_2^{p_1}} \\ 0 & 0 & 1 & \frac{(F(h_3) - F(h_2))(h_2^{p_1} - h_1^{p_1}) - (F(h_2) - F(h_1))(h_3^{p_1} - h_2^{p_1})}{(h_3^{p_2} - h_2^{p_2})(h_2^{p_1} - h_1^{p_1}) - (h_2^{p_2} - h_1^{p_2})(h_3^{p_1} - h_2^{p_1})} \end{array}\right)$$

Therefore,

$$a_2 = \frac{(F(h_3) - F(h_2))(h_2^{p_1} - h_1^{p_1}) - (F(h_2) - F(h_1))(h_3^{p_1} - h_2^{p_1})}{(h_3^{p_2} - h_2^{p_2})(h_2^{p_1} - h_1^{p_1}) - (h_2^{p_2} - h_1^{p_2})(h_3^{p_1} - h_2^{p_1})}$$

$$
\begin{aligned}
a_1 &= \frac{F(h_3)-F(h_2)}{h_3^{p_1}-h_2^{p_1}} \\
&\quad -\left(\frac{h_3^{p_2}-h_2^{p_2}}{h_3^{p_1}-h_2^{p_1}}\right)\frac{(F(h_3)-F(h_2))(h_2^{p_1}-h_1^{p_1})-(F(h_2)-F(h_1))(h_3^{p_1}-h_2^{p_1})}{(h_3^{p_2}-h_2^{p_2})(h_2^{p_1}-h_1^{p_1})-(h_2^{p_2}-h_1^{p_2})(h_3^{p_1}-h_2^{p_1})} \\
F(0)_R &= F(h_3)-a_1h_3^{p_1}-a_2h_3^{p_2} \\
&= F(h_3)-\frac{h_3^{p_1}(F(h_3)-F(h_2))}{h_3^{p_1}-h_2^{p_1}} \\
&\quad +h_3^{p_1}\left(\frac{h_3^{p_2}-h_2^{p_2}}{h_3^{p_1}-h_2^{p_1}}\right)\frac{(F(h_3)-F(h_2))(h_2^{p_1}-h_1^{p_1})-(F(h_2)-F(h_1))(h_3^{p_1}-h_2^{p_1})}{(h_3^{p_2}-h_2^{p_2})(h_2^{p_1}-h_1^{p_1})-(h_2^{p_2}-h_1^{p_2})(h_3^{p_1}-h_2^{p_1})} \\
&\quad -h_3^{p_2}\frac{(F(h_3)-F(h_2))(h_2^{p_1}-h_1^{p_1})-(F(h_2)-F(h_1))(h_3^{p_1}-h_2^{p_1})}{(h_3^{p_2}-h_2^{p_2})(h_2^{p_1}-h_1^{p_1})-(h_2^{p_2}-h_1^{p_2})(h_3^{p_1}-h_2^{p_1})} \\
&= F(h_3)+(F(h_3)-F(h_2))\left(\frac{h_3^{p_1}(h_2^{p_2}-h_1^{p_2})-h_3^{p_2}(h_2^{p_1}-h_1^{p_1})}{(h_3^{p_2}-h_2^{p_2})(h_2^{p_1}-h_1^{p_1})-(h_2^{p_2}-h_1^{p_2})(h_3^{p_1}-h_2^{p_1})}\right) \\
&\quad -(F(h_2)-F(h_1))\left(\frac{h_2^{p_1}h_3^{p_2}-h_2^{p_2}h_3^{p_1}}{(h_3^{p_2}-h_2^{p_2})(h_2^{p_1}-h_1^{p_1})-(h_2^{p_2}-h_1^{p_2})(h_3^{p_1}-h_2^{p_1})}\right)
\end{aligned}
$$

Using Arithmetic Time Steps

Let $P(1) = F(h_1)$, $P(2) = F(h_2)$ and $P(3) = F(h_3)$. Then we have $h_1 = T$, $h_2 = \frac{T}{2}$, $h_3 = \frac{T}{3}$.

The Taylor series of $F(h)$ around $F(0)$ is:

$F(h) = F(0) + F^{'}(0)h + F^{''}(0)h^2+$ terms of order three and higher.

Thus, we have $p_1 = 1$ and $p_2 = 2$. Therefore, the Geske and Johnson formula for an

American put option is:

$$
\begin{aligned}
P(\infty) &= P(3) + \left(\frac{\frac{1}{3}(\frac{1}{4}-1) - \frac{1}{9}(\frac{1}{2}-1)}{(\frac{1}{9}-\frac{1}{4})(\frac{1}{2}-1) - (\frac{1}{4}-1)(\frac{1}{3}-\frac{1}{2})}\right)(P(3)-P(2)) \\
&\quad - \left(\frac{(\frac{1}{2})(\frac{1}{9}) - (\frac{1}{4})(\frac{1}{3})}{(\frac{1}{9}-\frac{1}{4})(\frac{1}{2}-1) - (\frac{1}{4}-1)(\frac{1}{3}-\frac{1}{2})}\right)(P(2)-P(1)) \\
&= P(3) + \frac{7}{2}(P(3)-P(2)) - \frac{1}{2}(P(2)-P(1)) \\
&= \frac{9}{2}P(3) - 4P(2) + \frac{1}{2}P(1)
\end{aligned}
$$

If we only want to use two Bermudan style options in our extrapolation then we solve:

$$
\begin{aligned}
F(h_1) &= F(0) + a_1 h_1^{p_1} + O(h^{p_2}) \\
F(h_2) &= F(0) + a_1 h_2^{p_1} + O(h^{p_2})
\end{aligned}
$$

Thus,

$$
\begin{aligned}
h_1^{p_1}F(h_2) - h_2^{p_1}F(h_1) &= F(0)_R(h_1^{p_1} - h_2^{p_1}) \\
\Rightarrow F(0)_R &= \frac{h_1^{p_1}F(h_2) - h_2^{p_1}F(h_1)}{(h_1^{p_1} - h_2^{p_1})} \\
&= \frac{(\frac{h_1}{h_2})^{p_1}F(h_2) - F(h_1)}{(\frac{h_1}{h_2})^{p_1} - 1} \\
&= F(h_2) + \frac{F(h_2) - F(h_1)}{(\frac{h_1}{h_2})^{p_1} - 1}
\end{aligned}
$$

Let $P(1) = F(h_1)$ and $P(2) = F(h_2)$. Then we have $h_1 = T$, $h_2 = \frac{T}{2}$. Therefore (with $p_1 = 1$),

$$
P(\infty) = 2P(2) - P(1)
$$

Using Geometric Time Steps

Let $P(1) = F(h_1)$, $P(2) = F(h_2)$ $P(4) = F(h_4)$. Then we have $h_1 = T$, $h_2 = \frac{T}{2}$, $h_4 = \frac{T}{4}$. Therefore, the Modified Geske and Johnson formula for an American put option is:

$$\begin{aligned} P(\infty) &= P(4) + \frac{5}{3}(P(4) - P(2)) - \frac{1}{3}(P(2) - P(1)) \\ &= \frac{8}{3}P(3) - 2P(2) + \frac{1}{3}P(1) \end{aligned}$$

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