RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS

FUZZY IDEALS IN COMMUTATIVE RINGS

by

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A thesis submitted in partial fulfilment of the requirements for the degree of

MASTER OF SCIENCE in Mathematics

JANUARY 1995

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

ACKNOWLEDGEMENTS

I owe gratitude to a number of staff members of the Department of Mathematics of Rhodes University for their invaluable help during my course of study. I would like to express my sincere thanks to them.

Firstly Professor W.Kotze, who not only provided me with a scholarship but made me feel comfortable and at ease in Grahamstown. As a supervisor his attention extended to my professional development and his valuable comments made this thesis well readable. I am extremely grateful of his care shown for me.

Dr.V.Murali, my other supervisor has provided a great deal which enabled me to give a proper shape to this thesis. His advice proved to be invaluable in my research.

Dr.M.H.Burton's advice and moral support throughout my stay at Rhodes University, as well as his help in learning the $\mathcal{AMS}$-TEX package, proved crucial to the completion of my work.

Finally, the help of my friend Mr.I.Ikram (Ph.D student, Computer Science, Rhodes University) during my long hours of typing in Computer Lab, was of great comfort and support.

ABSTRACT

In this thesis, we are concerned with various aspects of fuzzy ideals of commutative rings. The central theorem is that of primary decomposition of a fuzzy ideal as an intersection of fuzzy primary ideals in a commutative Noetherian ring. We establish the existence and the two uniqueness theorems of primary decomposition of any fuzzy ideal with membership value 1 at the zero element. In proving this central result, we build up the necessary tools such as fuzzy primary ideals and the related concept of fuzzy maximal ideals, fuzzy prime ideals and fuzzy radicals. Another approach explores various characterizations of fuzzy ideals, namely, generation and level cuts of fuzzy ideals, relation between fuzzy ideals, congruences and quotient fuzzy rings. We also tie up several authors' seemingly different definitions of fuzzy prime, primary, semiprimary and fuzzy radicals available in the literature and show some of their equivalences and implications, providing counter-examples where certain implications fail.

Key-words: fuzzy ideal, fuzzy prime ideal, fuzzy primary ideal fuzzy maximal ideal, fuzzy radical and fuzzy primary decomposition.

AMS (1992) Subject classification:
Primary : 04A72, 13A15
Secondary: 03E72, 13A99 .

PREFACE

In the theory of commutative rings, every principal ideal domain is a unique factorization domain. Usually, the proof is based on elements of the ring rather than subsets. By viewing the same proof at the level of ideals , we get the well-known theorem of primary decomposition in a commutative Noetherian ring viz [Bar 1], [Sha 1],[Bur 1]. Since an elementary theory of fuzzy ideals is already developed, we consider in this thesis, the question of decomposing a fuzzy ideal in a commutative Noetherian ring into a finite intersection of fuzzy primary ideals. Thus we are led to define and study notions such as prime, primary, radical, maximal and irreducible ideals in fuzzy set theory of rings. Some authors have already made a beginning in this area.

The aim of this thesis is two-fold. One is to collect the available literature and tie up various authors' seemingly different theorems and definitions and show their equivalences and implications, providing counter-examples where certain implications fail. The second aim is to prove the analogues of the existence and the two uniqueness theorems of primary decomposition in a commutative Noetherian ring. As far as we know, the results are merely stated without proofs in the literature [Mal 4, 3.21, 3.22, 3.25, 3.26]. The proofs given in this thesis are our own [5.1.7, 5.1.8, 5.2.4] and they complement the existing literature. We now give detailed description of the contents of the thesis.

Chapter 1 introduces basic results in commutative rings and fuzzy set theory. In section 1.1 we first provide definitions of important specialised ideals and state main theorems in the crisp case which we fuzzify in the thesis. In section 1.2, we fix the notation for fuzzy subsets and introduce various operations such as addition, composition, multiplication and residual. One important property which is required through out the thesis namely the *Supremum property* is clearly stated.

We introduce the concept of fuzzy ideals in Chapter 2 and discuss the closure properties of fuzzy ideals under addition, multiplication and residual. A fuzzy ideal is defining its membership values at elements in the ring R characterized by the operations on fuzzy subsets in Proposition 2.2.3. Generation of fuzzy ideals and its generators are studied in section 2.3. Throughout this section, heavy use of the sup-property is made. As in the crisp case, distinct fuzzy subsets may generate the same fuzzy ideal. But we proved that different generators of the same fuzzy ideal must have the same image. Section 2.4 introduces fuzzy congruence relations. Here we prove that there is a one-to-one correspondence between fuzzy ideals and fuzzy congruences on R.

Chapters 3 and 4 explore the fundamental ideas of fuzzy prime and fuzzy primary ideals respectively. We proved that both fuzzy prime and primary ideals are two-valued fuzzy subsets and their base sets are prime and primary ideals respectively (the base set of μ is defined $\{r \in R : \mu(r) = \mu(0)\}$) and vice versa. In Chapter 3, we also discuss fuzzy radicals. If μ is a fuzzy ideal with $\mu(0) = 1$ then the fuzzy nil-radical is precisely the fuzzy prime radical of μ. Further, we introduce the weaker concept of fuzzy semiprime and derive some relationship between semiprime, prime and radical. If μ is a fuzzy primary ideal and if $\nu = \sqrt{\mu}$ is a fuzzy ideal, then μ is called ν-primary. Section 4.2 deals with such ν-primary ideals.
This section develops results which we need in Chapter 5. We end Chapter 4, with a discussion on fuzzy maximal ideals giving some results analogues to the crisp case.

Chapter 5 is central to the thesis. As mentioned in the Abstract, Chapters 3 and 4 prepare the results needed for Chapter 5. In this Chapter, we first prove an existence theorem and the first uniqueness theorem of fuzzy primary decomposition. In section 5.2, we associate a set of fuzzy prime ideals for a given primary decomposition of a fuzzy ideal. By using Zorn's Lemma, we arrive at the notion of fuzzy minimal prime ideal of a fuzzy ideal μ. We note that every fuzzy ideal μ has only finitely many fuzzy minimal prime ideals of μ. Also we prove in this section a second uniqueness theorem. The last section deals with irreducible ideals. We end this section with the important theorem that every fuzzy ideal μ with $\mu(0) = 1$ in a commutative Noetherian ring can be decomposed as a finite intersection of fuzzy primary ideals.

Throughout this thesis, acknowledgements to various authors are given where they are due, and as far as we know the following are our own results.

Proposition 2.3.3, Proposition 2.3.4, Theorem 2.3.6, Proposition 2.3.7, Theorem 2.3.11, Proposition 2.3.12(2), Proposition 2.3.13, Proposition 2.4.5, Theorem 2.4.6, Theorem 2.4.7, Theorem 2.4.8, Proposition 3.2.2, Proposition 3.2.4, Theorem 5.1.7, Proposition 5.1.8, Proposition 5.2.2, Proposition 5.2.3, and Theorem 5.2.4.
Some of the counter-examples are also our own.

CHAPTER I

INTRODUCTION

§ 1.1 Basics in commutative rings .

We collect some of the results of commutative rings which we study in the fuzzy-set theoretic setting. Throughout the thesis, a ring will mean commutative ring with identity. We restrict ourself to this case since most of our results are valied only in commutative rings.

Rings are denoted by R, R', S, etc.. and ideals are denoted by A, B, J, etc... If A is a non-empty subset of R, then by $< A >$ we mean ideal generated by A in R.

$$\text{i.e.} \quad < A >= \{\Sigma_{i=1}^{n} r_i a_i : r_i \in R, a_i \in A, n \in \mathbb{N}\}.$$

This is the smallest ideal of R containing A.
We use the usual operations on ideals such as $A + B, A \cap B, A \cup B, A \circ B, AB$ and $A : B$.
Where
$A + B = \{a + b : a \in A, b \in B\}$,
$A \circ B = \{ab : a \in A, b \in B\}$,
$AB = \{\Sigma_{i=1}^{n} r_i a_i b_i : r_i \in R, a_i \in A, b_i \in B, n \in \mathbb{N}\}$ and
$A : B = \{r \in R : rB \subseteq A\}$.
In general, $A \cup B$ and $A \circ B$ are not ideals, but $A + B, A \cap B, AB$ and $A : B$ are ideals of R.

Every ideal J of R induces a congruence relation '$\sim_J$ 'on R defined by $x \sim y$ if and only if $x - y \in J$. Conversely every congruence relation $\sim$ gives rise to an ideal of R given by $J = \{r \in R : r \sim 0\}$ where 0 is the additive zero element of R.
With each ideal A of R, we can associate the quotient ring R/A consisting of the additive cosets $\{r + A : r \in R\}$.

We collect for the sake of completeness the definitions of some specialised ideals.

(1) *Prime ideal:* An ideal A is a *prime ideal* if $A \subset R$ and whenever $a, b \in R$ with $ab \in A$, then either $a \in A$ or $b \in A$,
(2) *Primary ideal:* An ideal A is a *primary ideal* if $A \subset R$ and whenever $a, b \in R$ with $ab \in A$ either $a \in A$ or $b^n \in A$ for some $n \in \mathbb{N}$,
(3) *Nil radical:* Let A be an ideal of R. The *Nil radical* of A, $\sqrt{A}$ is defined as

$$\sqrt{A} = \{r \in R : r^n \in A \text{ for some } n \in \mathbb{N}\},$$

(4) *Prime radical:* The Intersection of all prime ideals of R containing the ideal A is called the *prime radical of* A and is denoted by $r(A)$,

(5) *B-primary:* Let A, B be ideals of R. A is said to be *B-primary* if A is primary and $\sqrt{A} = B$,

(6) *Semiprime:* An ideal A is called *semiprime* whenever $r \in R$ with $r^n \in A$, implies $r \in A$ for all $n \in \mathbb{N}$,

(7) *Semiprimary:* An ideal A of R is called *semiprimary* if $\sqrt{A}$ is a prime ideal of R,

(8) *Maximal:* An ideal A is called *maximal ideal* if $A \subset R$ and there exists no ideal B such that $A \subset B \subset R$,

(9) *Irreducible:* An ideal A is called *irreducible ideal* if $A \subset R$ and A cannot be expressed as the intersection two ideals of R properly containing A.

We now state some main theorems which we fuzzify in this thesis.

(1) Let A and B be ideals of R, then
 (1) $\sqrt{A}B = \sqrt{(A \cap B)} = \sqrt{A} \cap \sqrt{B}$,
 (2) $\sqrt{(\sqrt{A})} = \sqrt{A}$,
 (3) $A^k \subseteq B$ for some $k \in \mathbb{N}$ implies that $\sqrt{A} \subseteq \sqrt{B}$,
 (4) A is semiprime if and only if $\sqrt{A} = A$,

(2) The intersection of all prime ideals of R which contain a given ideal A is precisely the nil radical of A,

(3) Let M be the intersection of all maximal ideals of R, then
 (1) let A be an ideal of R, then $A \subseteq M$ if and only if each element of the coset $1 + A$ has an inverse in R,
 (2) $x \in M$ if and only if $1 - rx$ is invertible for each $r \in R$,
 (3) the quotient ring R/M is semisimple,

(4) Let I be an ideal of R. Then I is a primary ideal if and only if every zero divisor of the quotient ring R/I is nilpotent,

(5) If $Q_1, Q_2, \ldots, Q_n$ is a finite set of primary ideals of R, all of them having the same associated prime ideal P, then $Q = \cap_{i=1}^{n} Q_i$ is also primary, with $\sqrt{Q} = P$,

(6) Let P and Q be ideals of R. Then Q is primary for P if and only if
 (1) $Q \subseteq P \subseteq \sqrt{Q}$,
 (2) if $ab \in Q$ and $a \notin Q$, then $b \in P$,

(7) Every proper ideal of R possesses at least one minimal prime ideal,

(8) Let Q be an ideal R such that $\sqrt{Q} = M$ is a maximal ideal of R. Then Q is a primary (in fact M-primary) ideal of R. Consequently, all positive powers M^n $(n \in \mathbb{N})$ of maximal ideal M are M-primary,

(9) Let A be a decomposable ideal of R, and let

$$A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \quad \text{with} \quad \sqrt{Q}_i = P_i \quad \text{for } i = 1, 2, \ldots, n$$

be a reduced primary decomposition of A. Let P be a prime ideal of R. Then the following are equivalent:

(1) $P = P_i$ for some i with $1 \leq i \leq n$;

(2) there exists $a \in R$ such that $(A : a)$ is P-primary;

(3) there exists $a \in R$ such that $\sqrt{(A : a)} = P$,

(10) (The First Uniqueness Theorem)

Let A be a decomposable ideal of R, and let

$$A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \quad \text{with} \quad \sqrt{Q_i} = P_i \quad \text{for } i = 1, 2, \ldots, n$$

and

$$A = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_n \quad \text{with} \quad \sqrt{Q'_i} = P'_i \quad \text{for } i = 1, 2, \ldots, n'$$

be two reduced primary decompositions of A. Then $n = n'$, and we have

$$\{P_1, P_2, \ldots, P_n\} = \{P'_1, P'_2, \ldots, P'_n\},$$

(11) (The Second Uniqueness Theorem)

Let A be a decomposable ideal of R. Let

$$A = Q_1 \cap Q_2 \cap \ldots Q_n \quad \text{with} \quad \sqrt{Q_i} = P_i \quad \text{for } i = 1, 2, \ldots n$$

and

$$A = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_n \quad \text{with} \quad \sqrt{Q'_i} = P_i \quad \text{for } i = 1, 2, \ldots, n$$

be two reduced primary decompositions of A. Then, for each i with $1 \leq i \leq n$ for which P_i is a minimal prime ideal belonging to A, we have

$$Q_i = Q'_i,$$

(12) Let A be a proper ideal in a commutative Noetherian ring R. Then A has a primary decomposition .

§ 1.2 Fuzzy subsets and their basic properties .

The concept of a fuzzy subset μ [Zad 1] of a non-empty subset R involves degree of membership, measured by a real number between 0 and 1, of an element of R to μ. Thus we can represent μ as a function $\mu : R \longrightarrow I$ from R to I the unit interval, $\mu(x)$ is the degree to which x belongs to μ for $x \in R$. The set I^R denotes set of all fuzzy subsets of R. Elements of I^R are usually denoted by lower case Greek letters $\mu, \nu, \lambda, \omega$ etc.... We identify every crisp subset J of R with the characteristic function $\mathcal{X}_J$. Thus $\mathbf{2}^R \subset I^R$. There are two important properties which we use repeatedly throughout the thesis. We state them here.

(1) The *Supremum property* (*sup-property*) is defined as follows: A fuzzy subset μ of R is said to have the sup-property if for every non-empty subset A of R the supremum of $\{\mu(a) : a \in A\}$ is attained at a point of A.
i.e. there exists an $a_0 \in A$ such that $\mu(a_0) = \sup_{a \in A} \mu(a)$.
For example, all finite-valued fuzzy subsets or fuzzy subsets with decreasing sequence of membership values have the sup-property,

(2) Let $f : R \longrightarrow R'$ be a mapping. A fuzzy subset μ of R is said to be *f-invariant* if for all $x, y \in R$

$$f(x) = f(y) \quad \text{implies} \quad \mu(x) = \mu(y).$$

We now associate various crisp subsets of R with a given fuzzy subset μ of R.

(1) *Supp* $\mu = \{r \in R : \mu(r) > 0\}$, *supp* μ standing for the support of μ,
(2) $\mu_t = \{r \in R : \mu(r) \geq t\}$, for $0 \leq t \leq 1$. μ_t is called the *t-level subset* (or *t-level cut*) of μ,
(3) $\mu^t = \{r \in R : \mu(x) > t\}$, for $0 \leq t < 1$. μ^t is called the *strong t-level subset* (or *strong t-level cut*) of μ,
(4) If R is a ring, $\mu_0 = \{r \in R : \mu(r) = \mu(0)\}$. μ_0 is called the *base set of* μ .

We remark that in (4), the reader should not confuse μ_0 with the $0-$ level subset of μ. Which is also denoted by μ_0. But this $0-$level subset of μ which is the whole of R and is never used. So whenever the notation μ_0 appears, it only refers to the base set of μ in this thesis.

μ is contained in ν, denoted by $\mu \leq \nu$, means the point- wise ordering.
i.e. $\mu(x) \leq \nu(x)$ for all $x \in R$.

Further a fuzzy point r_t is defined as

$$r_t(x) = \begin{cases} t & \text{if} \quad x = r \\ 0 & \text{if} \quad x \neq r \end{cases}$$

Let $\mathcal{A}$ be a non-empty indexing set. Then

$$\left(\bigwedge_{i \in \mathcal{A}} \mu_i\right)(x) = \inf\{\mu_i(x) : i \in \mathcal{A}\}$$
$$= \bigwedge_{i \in \mathcal{A}} (\mu_i(x))$$

and

$$\left(\bigvee_{i\in\mathcal{A}} \mu_i\right)(x) = \sup\{\mu_i(x) : i \in \mathcal{A}\}$$
$$= \bigvee_{i\in\mathcal{A}} (\mu_i(x)).$$

The following facts describe the effect of level subset and strong level subset on taking arbitrary union and intersection as definedm above.

(1) $\left(\bigwedge_{i\in\mathcal{A}} \mu_i\right)_t = \cap_{i\in\mathcal{A}}(\mu_i)_t$,

(2) $\cup_{i\in\mathcal{A}}(\mu_i)_t \subseteq \left(\bigvee_{i\in\mathcal{A}} \mu_i\right)_t$, equality holds if $\mathcal{A}$ is finite,

(3) $\left(\bigwedge_{i\in\mathcal{A}} \mu_i\right)^t = \cup_{i\in\mathcal{A}}(\mu_i)^t$,

(4) $\left(\bigvee_{i\in\mathcal{A}} \mu_i\right)^t \subseteq \cap_{i\in\mathcal{A}}(\mu_i)^t$, equality holds if $\mathcal{A}$ is finite.

Remark. If R is a ring and $\mu \leq \nu$ in I^R it **does not** necessarily follow that $\mu_0 \subseteq \nu_0$.

We introduce the following definition of fuzzy subsets under mappings.

Definition 1.2.1[Mal 6,4.1].
Let $f : R \longrightarrow R'$ be a map and μ, μ' be fuzzy subsets of R and R' respectively. Then the fuzzy subset $f(\mu)$ of R' is defined as

$$f(\mu)(y) = \begin{cases} \sup\limits_{f(x)=y} \mu(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ \\ 0 & \text{if} \quad f^{-1}(y) = \emptyset \end{cases}$$

and

$$f^{-1}(\mu')(x) = \mu'(f(x)) \quad \text{for all } x \in R.$$

The above is known as " *fuzzy extension principle.* "
In the following Proposition we collect some results without proof for later use.

Proposition 1.2.2.
Let $f : R \longrightarrow R'$, $g : R' \longrightarrow R''$ be maps and $\mathcal{A}$ be an indexing set, then

(1) *If $\mu \leq \nu$, then $f(\mu) \leq f(\nu)$ for all $\mu, \nu, \in I^R$,*
(2) *If $\mu' \leq \nu'$, then $f^{-1}(\mu') \leq f^{-1}(\nu')$ for all $\mu', \nu' \in I^{R'}$,*
(3) *$f(f^{-1}(\mu')) \leq \mu'$ for all $\mu' \in I^{R'}$, equality holds if f surjective,*
(4) *$f^{-1}(f(\mu)) \geq \mu$ for all $\mu \in I^R$, equality holds if f is injective,*
(5) *$f(\mu) \leq \nu$ if and only if $\mu \leq f^{-1}(\nu)$ for all $\mu \in I^R, \nu \in I^{R'}$,*
(6) *$f(\bigvee_{i\in\mathcal{A}} \mu_i) = \bigvee_{i\in\mathcal{A}} f(\mu_i), \mu_i \in I^R$,*

(7) $f(\bigwedge_{i\in\mathcal{A}}\mu_i) \leq \bigwedge_{i\in\mathcal{A}} f(\mu_i), \mu_i \in I^R$,

(8) $f^{-1}(\bigvee_{i\in\mathcal{A}}\mu_i') = \bigvee_{i\in\mathcal{A}} f^{-1}(\mu_i'), \mu_i' \in I^{R'}$,

(9) $f^{-1}(\bigwedge_{i\in\mathcal{A}}\mu_i') = \bigwedge_{i\in\mathcal{A}} f^{-1}(\mu_i'), \mu_i' \in I^{R'}$,

(10) $g(f(\mu)) = g \circ f(\mu)$ *for all* $\mu \in I^R$,

(11) $f^{-1}(g^{-1}(\mu')) = (g \circ f)^{-1}(\mu')$ *for all* $\mu' \in I^{R'}$,

(12) $f(\mu_t) \leq (f(\mu))_t$ *for all* $\mu \in I^R, t \in [0,1]$, *equality holds if* μ *has the sup-property,*

(13) $f^{-1}(\mu_t') = (f^{-1}(\mu'))_t$ *for all* $\mu' \in I^{R'}$.

We provide some basic properties of supremum and infimum which are used later.

Proposition 1.2.3.

(1) *Let* $\mu, \nu \in I^R$ *and* A, B *be subsets of* R. *Then*

$$\sup_{x\in R}(\mu(x) \wedge \nu(x)) \leq \sup_{x\in R}\mu(x) \wedge \sup_{x\in R}\nu(x)$$

and

$$\sup_{x\in A}\mu(x) \wedge \sup_{y\in B}\mu(y) = \sup_{\substack{x\in A\\ y\in B}} \mu(x) \wedge \mu(y),$$

(2) *Let* A, B *be two subsets of the unit interval* I. *Then*

$$\sup A \wedge \sup B = \sup_{\substack{x\in A\\ y\in B}} x \wedge y$$

and

$$\inf A \vee \inf B = \inf_{\substack{x\in A\\ y\in B}} x \vee y.$$

We now extend the operations of addition(+), composition($\circ$), multiplication(.) and residual(:) on R to I^R.

Let $\mu, \nu, \lambda \in I^R$ and $x, y \in R$.

(1) $(\mu + \nu)(x) = \sup_{x=x_1+x_2} \mu(x_1) \wedge \nu(x_2)$,

(2) $(\mu \circ \nu)(x) = \sup_{x=x_1x_2} \mu(x_1) \wedge \nu(x_2)$,

(3) $(\mu\nu)(x) = \sup_{x=\Sigma_{i=1}^n x_i y_i} \bigwedge_i^n [\mu(x_i) \wedge \nu(y_i)]$,

(4) $\mu^n(x) = \sup_{x=\Sigma_{k=1}^p x_{k1}x_{k2}\ldots x_{kn}} \bigwedge_{k=1}^p \bigwedge_{i=1}^n \mu(x_{ki})$,

(5) $(-\mu)(x) = \mu(-x)$ for all $x \in R$,

(6) $(\mu : \nu)(x) = \sup\{\lambda(x) : \lambda \circ \nu \leq \mu\}$ for all $x \in R$.

For later use, we collect some properties of the above operations.

Proposition 1.2.4.
Let $\mu, \nu, \omega \in I^R$.

(1) $\mu + \nu = \nu + \mu$, $\mu \circ \nu = \nu \circ \mu$ *and* $\mu\nu = \nu\mu$,
(2) $-(-\mu) = \mu$ *and if* $\mu \leq \nu$ *then* $-\mu \leq -\nu$,
(3) $\mu + (\nu + \omega) = (\mu + \nu) + \omega$,
(4) $\mu \circ (\nu + \omega) \leq (\mu \circ \nu) + (\mu \circ \omega)$.

We provide proofs of the result similar to the above Proposition in Chapter 2.

Proposition 1.2.5.
Let $f : R \longrightarrow R'$ *be an epimorphism from a ring* R *to* R' *and* $\mu, \nu \in I^R$ *and* $\mu', \nu' \in I^{R'}$.

(1) $f(\mu) \circ f(\nu) \leq f(\mu \circ \nu)$,
(2) $f^{-1}(\mu') \circ f^{-1}(\nu') \leq f^{-1}(\mu' \circ \nu')$.

Proof.
We prove only (2) and refer to the Chapter 2, Proposition[2.2.10] for the proof of (1).

$$\begin{aligned} f^{-1}(\mu') \circ f^{-1}(\nu')(x) &= \sup_{x=x_1x_2} f^{-1}(\mu')(x_1) \wedge f^{-1}(\nu')(x_2) \\ &= \sup_{x=x_1x_2} \mu'(f(x_1)) \wedge \nu'(f(x_2)) \\ &\leq \sup_{x=x_1x_2} (\mu' \circ \nu')(f(x_1x_2)) \\ &= (\mu' \circ \nu')f(x) \\ &= f^{-1}(\mu' \circ \nu')(x). \end{aligned}$$

CHAPTER II

FUZZY IDEALS AND RELATIONS

In the following two sections we fuzzify the concept of subrings and ideals and prove that the operations on fuzzy ideals namely sum, product, infimum and residual are closed.

§ 2.1 Fuzzy subrings .

The usual definition for a subring of R can be expressed in terms of its characteristic function as follows:

A non-empty subset S of R is subring if and only if for all $x, y \in R$

$$\mathcal{X}_S(x-y) \geq \mathcal{X}_S(x) \wedge \mathcal{X}_S(y)$$

and

$$\mathcal{X}_S(xy) \geq \mathcal{X}_S(x) \wedge \mathcal{X}_S(y).$$

The basic idea behind the fuzzification of these concept is simply to replace the characteristic function by a fuzzy subset $\mu : R \longrightarrow I$. We therefore provide the following Definition for a fuzzy subring.

Definition 2.1.1. [Muk 2]
A fuzzy subset $\mu : R \longrightarrow I$ is called a *fuzzy subring* of R if for all $x, y \in R$

$$\mu(x-y) \geq \mu(x) \wedge \mu(y)$$

and

$$\mu(xy) \geq \mu(x) \wedge \mu(y).$$

We prove some properties of fuzzy subrings and its images under a homomorphism in the following Proposition:

Proposition 2.1.2.
Let μ and μ' be fuzzy subrings of R and R' respectively and let $f : R \longrightarrow R'$ be a homomorphism. Then

(1) $\forall x, y \in R, \quad \mu(xy - yx) \geq \mu(x) \wedge \mu(y)$,

(2) $\mu(0) = \sup_{x \in R} \mu(x)$, *where* 0 *is the zero element of* R,

(3) $\forall x \in R, \quad \mu(x) = \mu(-x)$,

(4) $f(\mu)(0) = \sup_{y\in R'} f(\mu)(y) = \sup_{x\in R}\mu(x) = \mu(0)$ *provided f is an epimorphism,*

(5) $f^{-1}(\mu')(0) = \sup_{x\in R} f^{-1}(\mu')(x) = \sup_{y\in R'} \mu'(y) = \mu'(0).$

Proof.

(1), (2), and (3) are straightforward. For part (4),we have

$$f(\mu)(y) = \sup_{f(x)=y} \mu(x) \leq \sup_{x\in R}\mu(x) \quad \text{for all } y \in R'.$$

Hence

$$\sup_{y\in R'} f(\mu)(y) \leq \sup_{x\in R}\mu(x)$$

i.e. $f(\mu)(0) \leq \mu(0)$. To show the converse, let $x \in R$. Then there exists $y \in R'$ such that $f(x) = y$. Since $f(\mu)$ is a fuzzy subring,

$$\begin{aligned} f(\mu)(0) &= \sup_{y\in R'} f(\mu)(y) \\ &\geq f(\mu)(y) = \sup_{f(x)=y} \mu(x) \\ &\geq \mu(x) \quad \text{for all } x \in R \end{aligned}$$

and hence $f(\mu)(0) \geq \sup_{x\in R}\mu(x) = \mu(0)$.

(5) can be proved similarly.

□

Proposition 2.1.3.

A fuzzy subset μ of R is fuzzy subring if and only if all of its level subsets are subrings of R.

Proof is straightforward.

§ 2.2 Fuzzy ideals .

As in the Definition[2.1.1], we can fuzzify the concept of an ideal of R as follows.

Definition 2.2.1.

A fuzzy subset μ of R is called *fuzzy ideal* of R
if $\forall x, y \in R$

$$\mu(x-y) \geq \mu(x) \wedge \mu(y)$$

and

$$\mu(xy) \geq \mu(x) \vee \mu(y).$$

$\mathcal{F}(R)$ denotes the set of all fuzzy ideals of R.

Example 2.2.2.
Let $R = (\mathbb{Z}_6, +, .)$ and define a fuzzy subset $\mu : \mathbb{Z}_6 \longrightarrow I$ by

$$\mu(0) = \mu(2) = \mu(4) = 1 \text{ and } \mu(1) = \mu(3) = \mu(5) = 0.8$$

then it easy to check that μ is a fuzzy ideal of R.

In the following proposition, we characterize a fuzzy ideal in terms of the operations on fuzzy subsets.

Proposition 2.2.3[Liu 2, 3.1].
A fuzzy subset μ of R is fuzzy ideal if and only if

(1) $\mu = -\mu$,
(2) $\mu + \mu \leq \mu$,
(3) $\mathcal{X}_R \circ \mu \leq \mu$ *and* $\mu \circ \mathcal{X}_R \leq \mu$.

Proof.
Let μ be a fuzzy ideal of R. We only prove (2). The rest is straightforward.
Let $x \in R$ then

$$\begin{aligned}(\mu + \mu)(x) &= \sup_{x=x_1+x_2} \mu(x_1) \wedge \mu(x_2) \\ &= \sup_{x=x_1+x_2} \mu(x_1) \wedge \mu(-x_2) \\ &\leq \sup_{x=x_1+x_2} \mu(x_1 - (-x_2)) \\ &= \mu(x).\end{aligned}$$

Conversely, let (1),(2),(3) hold for a fuzzy subset μ of R. Let $x, y \in R$ then

$$\begin{aligned}\mu(x - y) &\geq (\mu + \mu)(x - y) \\ &= \sup_{x-y=x_1+y_1} \mu(x_1) \wedge \mu(y_1) \\ &\geq \mu(x) \wedge \mu(-y) \\ &= \mu(x) \wedge \mu(y),\end{aligned}$$

$$\mu(xy) \geq (\mathcal{X}_R \circ \mu)(xy)$$

$$= \sup_{xy=x_1y_1} \mathcal{X}_R(x_1) \wedge \mu(y_1)$$

$$\geq \mathcal{X}_R(x) \wedge \mu(y) \quad = \mu(y)$$

Similarly, by considering $\mu \geq \mu \circ \mathcal{X}_R$ we can show $\mu(xy) \geq \mu(x)$.
Thus $\mu(xy) \geq \mu(x) \vee \mu(y)$.

□

Remark 1.
A non-constant fuzzy subset μ of R is a fuzzy ideal if and only if all of its level subsets are ideals of R.

($\Rightarrow$) Obvious.

($\Leftarrow$) Let $x, y \in R$ and $t = \mu(x) \wedge \mu(y)$ then $\mu(x) \geq t$ and $\mu(y) \geq t$ which implies $x, y \in \mu_t$. So $\mu(x-y) \geq t = \mu(x) \wedge \mu(y)$. Let now $t = \mu(x)$; then $x \in \mu_t$. Since μ_t is a ideal of R

$xy \in \mu_t$ and hence $\mu(xy) \geq \mu(x)$, similarly $\mu(xy) \geq \mu(y)$. Thus $\mu(xy) \geq \mu(x) \vee \mu(y)$.

Remark 2[Mal 5, 2.7 Bho 3, 2.2].
For later use, we collect some easy results which are consequences of the definition [2.2.1].

If μ is a fuzzy ideal of R, then

(1) For any $t, s \in Im(\mu)$, $\quad \mu_t = \mu_s$ if and only if $t = s$,
(2) If $\mu(x) < \mu(y)$, then $\mu(x - y) = \mu(x) = \mu(y - x)$,
(3) If $\mu(x - y) = \mu(0)$, then $\mu(x) = \mu(y)$,
(4) $\forall n \in \mathbb{N} \quad \mu(1) \leq \mu(x) \leq \mu^n(x^n) \leq \mu(x^n)$.

The following Corollary can be checked easily.

Corollary 2.2.4.
I is an ideal of R if and only if the characteristic function $\mathcal{X}_I$ of I is a fuzzy ideal of R.

Proposition 2.2.5.
If μ, ν are fuzzy ideals of R, then $\mu + \nu, \mu\nu, \mu \wedge \nu$ and $\mu : \nu$ are also fuzzy ideals of R.

Proof.
Let $x, y \in R$ then

$$(\mu + \nu)(x - y) = \sup_{x-y=x'+y'} \mu(x') \wedge \nu(y').$$

But

$$(\mu+\nu)(x) \wedge (\mu+\nu)(y) = \sup_{x=x_1+x_2} \mu(x_1) \wedge \nu(x_2) \wedge \sup_{y=y_1+y_2} \mu(y_1) \wedge \nu(y_2)$$

$$= \sup_{\substack{y=y_1+y_2 \\ x=x_1+x_2}} \mu(x_1) \wedge \mu(y_1) \wedge \nu(x_2) \wedge \nu(y_2)$$

$$\leq \sup_{\substack{x-y=x_1-y_1 \\ +x_2-y_2}} \mu(x_1-y_1) \wedge \nu(x_2-y_2)$$

$$\leq \sup_{x-y=x'+y'} \mu(x') \wedge \nu(y')$$

$$= (\mu+\nu)(x-y).$$

We now consider

$$(\mu+\nu)(xy) = \sup_{xy=x'+y'} \mu(x') \wedge \nu(y')$$

and

$$(\mu+\nu)(x) = \sup_{x=x_1+y_1} \mu(x_1) \wedge \nu(y_1)$$

$$\leq \sup_{xy=x_1y+y_1y} \mu(x_1y) \wedge \nu(y_1y)$$

$$\leq \sup_{xy=x'+y'} \mu(x') \wedge \nu(y')$$

$$= (\mu+\nu)(xy).$$

Similarly, $(\mu+\nu)(xy) \geq (\mu+\nu)(y)$

The proof of $\mu\nu$ is almost same as in the proof of $\mu+\nu$. It is easy to see that $\mu \wedge \nu$ is a fuzzy ideal of R.

Let us now prove $\mu : \nu$ is fuzzy ideal of R.

Let $x, y \in R$ then

$$(\mu : \nu)(x-y) = \sup\{\omega(x-y) : \omega \in \mathcal{F}(R), \nu \circ \omega \leq \mu\} \text{ and}$$

$$(\mu : \nu)(x) \wedge (\mu : \nu)(y) = \sup\{\omega(x) : \omega \in \mathcal{F}(R), \nu \circ \omega \leq \mu\} \wedge \sup\{\omega'(y) : \omega' \in \mathcal{F}(R), \nu \circ \omega' \leq \mu\}$$

$$= \sup\{\omega(x) \wedge \omega'(y) : \omega, \omega' \in \mathcal{F}(R), (\nu \circ \omega) \vee (\nu \circ \omega') \leq \mu\}.$$

Now, we have

$$\{\omega, \omega' \in \mathcal{F}(R) : (\nu \circ \omega) \vee (\nu \circ \omega') \leq \mu\} \subseteq \{\omega, \omega' \in \mathcal{F}(R) : \nu \circ (\omega + \omega') \leq \mu\}$$

Since if ω, ω' are in $\mathcal{F}(R)$ such that $(\nu \circ \omega) \vee (\nu \circ \omega') \leq \mu$, then

$$\nu \circ (\omega + \omega') \leq (\nu \circ \omega) + (\nu \circ \omega') \quad \text{by the Proposition[1.2.4,4]}$$

$$\leq \mu + \mu \quad \leq \mu.$$

Hence

$$(\mu : \nu)(x) \wedge (\mu : \nu)(y) \leq Sup\{\omega(x) \wedge \omega'(y) : \omega, \omega' \in \mathcal{F}(R), \nu \circ (\omega + \omega') \leq \mu\}$$

$$\leq Sup\{(\omega + \omega')(x - y) : \omega, \omega' \in \mathcal{F}(R),\ \nu \circ (\omega + \omega') \leq \mu\}$$

$$\leq (\mu : \nu)(x - y).$$

To show the other condition,

$$(\mu : \nu)(xy) = Sup\{\omega(xy) : \omega \in \mathcal{F}(R), (\nu \circ \omega) \leq \mu\}$$

$$\geq Sup\{\omega(x) : \omega \in \mathcal{F}(R), (\nu \circ \omega) \leq \mu\}$$

$$= (\mu : \nu)(x).$$

Similarly, we can show $(\mu : \nu)(xy) \geq (\mu : \nu)(y)$; , thus $\mu : \nu$ is a fuzzy ideal of R.

We now give an example to show $\mu \vee \nu$ and $\mu \circ \nu$ need not be fuzzy ideals.

Example 2.2.6.
Define fuzzy ideals $\mu, \nu : (\mathbb{Z}_{12}, +_{12}, \circ_{12}) \longrightarrow I$ by

$$\mu(0) = \mu(4) = \mu(8) = 0.9 \quad \text{and zero elsewhere}$$

and

$$\nu(0) = \nu(6) = 0.8 \quad \text{and zero elsewhere .}$$

then

$$(\mu \vee \nu)(6 + 4) = \mu(10) \vee \nu(10) = 0$$

$$(\mu \vee \nu)(6) = 0.8, (\mu \vee \nu)(4) = 0.9$$

hence $\quad (\mu \vee \nu)(6 + 4) \not\geq (\mu \vee \nu)(6) \wedge (\mu \vee \nu)(4).$

Example 2.2.7.
Let R be the ring $R[x_1, x_2, x_3, x_4], I = Rx_1 + Rx_2$ and $J = Rx_3 + Rx_4$. Then the characteristic functions $\mathcal{X}_I$ and $\mathcal{X}_J$ are fuzzy ideals of R. Since $x_1x_4 - x_2x_3 \notin \{xy : x \in I, y \in J\}$ $\mathcal{X}_I \circ \mathcal{X}_J(x_1x_4 - x_2x_3) = 0$ but $\mathcal{X}_I \circ \mathcal{X}_J(x_1x_4) \wedge \mathcal{X}_I \circ \mathcal{X}_J(x_2x_3) = 1$. Hence $\mathcal{X}_I \circ \mathcal{X}_J$ is not fuzzy ideal.

The following proposition deals with addition, multiplication, composition and infimum operations on fuzzy ideals.

Proposition 2.2.8.
Let μ, ν, ω be fuzzy ideals of R, then

(1) $\mu \circ \nu \leq \mu \wedge \nu$ *and* $\mu\nu = \nu\mu$,
(2) *If* $\mu \leq \nu$, *then* $\mu\omega \leq \nu\omega$,
(3) $\mathcal{X}_R\mu = \mu$,
(4) *For all* $k, r \in \mathbb{N}$ $\mu^k\mu^r = \mu^{k+r}$ *and* $(\mu^k)^r = \mu^{kr}$,
(5) *If* $\mu(0) = \nu(0) = \omega(0) = 1$, *then* $\mu \leq \mu + \nu, \mu(\nu + \omega) = \mu\nu + \mu\omega$ *and* $(\mu \wedge \nu)(\mu + \nu) \leq \mu\nu$,
(6) *If* $\mu + \nu = \mathcal{X}_R$, *then* $\mu \wedge \nu = \mu\nu$.

Proof.
(1) and (2) are obvious. (3) Clearly $\mathcal{X}_R\mu \leq \mu$. Let $x \in R$ then $\mathcal{X}_R\mu(x) = \mathcal{X}_R\mu(1x) \geq \mathcal{X}_R(1) \wedge \mu(x) = \mu(x)$.

(4). Proof is by induction on r. When $r = 1$

$$\mu^k\mu^1 = \sup_{x=\sum_1^p x_{1i}x_{2i}} \wedge_{i=1}^p \mu^k(x_{1i}) \wedge \mu(x_{2i})$$

$$= \sup_{x=\sum_1^p x_{1i}x_{2i}} \wedge_{i=1}^p \left[\sup_{x_{1i}=\sum_{j=1}^{p'} y_{1j}y_{2j}\ldots y_{kj}} \wedge_{j=1}^{p'} \mu(y_{1j}) \wedge \mu(y_{2j}) \wedge \ldots\wedge \mu(y_{kj}) \right] \wedge \mu(x_{2i})$$

$$= \sup_{\substack{x=\sum_{i=1}^p x_{1i}x_{2i} \\ x_{1i}=\sum_{j=1}^{p'} y_{1j}y_{2j}\ldots y_{kj}}} \wedge_{i=1}^p \wedge_{j=1}^{p'} \mu(y_{1j}) \wedge \mu(y_{2j}) \wedge \ldots\wedge \mu(y_{kj}) \wedge \mu(x_{2i})$$

$$= \sup_{x=\sum_{i=1}^n z_{1i}z_{2i}\ldots z_{(k+1)i}} \wedge_{i=1}^n \mu(z_{1i}) \wedge \mu(z_{2i}) \wedge \ldots\wedge \mu(z_{(k+1)i})$$

$$= \mu^{k+1}(x).$$

Suppose $\mu^{k+r}(x) = \mu^k\mu^r(x)$. We now consider

$$\begin{aligned} \mu^{k+r+1}(x) &= (\mu^{k+r}\mu^1)(x) \\ &= (\mu^k\mu^r)\mu^1(x) \\ &= \mu^k(\mu^r\mu^1)(x) \\ &= \mu^k(\mu^{r+1})(x) \\ &= \mu^k\mu^{r+1}(x). \end{aligned}$$

Similarly, we can show that $(\mu^k)^r = \mu^{kr}$.

(5). Let $x \in R$ then

$$\begin{aligned} (\mu+\nu)(x) = (\mu+\nu)(x+0) &\geq \mu(x) \wedge \nu(0) \\ &= \mu(x) \qquad \text{for all } x \in R. \end{aligned}$$

Consider

$$\begin{aligned} \mu(\nu+\omega)(x) &= \sup_{x=\sum_{i=1}^{p} x_{1i}x_{2i}} \wedge_1^p \mu(x_{1i}) \wedge (\nu+\omega)(x_{2i}) \\ &= \sup_{x=\sum_{i=1}^{p} x_{1i}x_{2i}} \wedge_1^p \mu(x_{1i}) \wedge \sup_{x_{2i}=y_{1i}+y_{2i}} \nu(y_{1i}) \wedge \omega(y_{2i}) \\ &= \sup_{\substack{x=\sum_{i=1}^{p} x_{1i}x_{2i} \\ x_{2i}=y_{1i}+y_{2i}}} \wedge_1^p \mu(x_{1i}) \wedge \nu(y_{1i}) \wedge \omega(y_{2i}) \\ &= \sup_{x=\sum_{i=1}^{p} x_{1i}y_{1i} + \sum_{i=1}^{p} x_{1i}y_{2i}} \wedge_1^p \mu(x_{1i}) \wedge \nu(y_{1i}) \wedge \omega(y_{2i}) \\ &\leq \sup_{x=x_1+x_2} \sup_{x_1=\sum_{i=1}^{p_1} s_{1i}t_{1i}} \wedge_1^{p_1} \mu(s_{1i}) \wedge \nu(t_{1i}) \wedge \sup_{x_2=\sum_{i=1}^{p_2} m_{2i}n_{2i}} \wedge_1^{p_2} \mu(m_{2i}) \wedge \omega(n_{2i}) \\ &= \sup_{x=x_1+x_2} (\mu\nu)(x_1) \wedge (\mu\omega)(x_2) \\ &= (\mu\nu + \mu\omega)(x). \end{aligned}$$

On the other hand, we have $\nu \leq \nu + \omega$ which implies $\mu\nu \leq \mu(\nu+\omega)$.Similarly , $\mu\omega \leq \mu(\nu+\omega)$. Hence $\mu\nu + \mu\omega \leq \mu(\nu+\omega)$ thus $\mu\nu + \mu\omega = \mu(\nu+\omega)$.

For the last part,

$$\begin{aligned} (\mu \wedge \nu)(\mu+\nu) &= (\mu\wedge\nu)\mu + (\mu\wedge\nu)\nu \\ &\leq \nu\mu + \mu\nu \\ &\leq \mu\nu. \end{aligned}$$

(6). By the above part,

$$\begin{aligned}\mu \wedge \nu = (\mu \wedge \nu)\mathcal{X}_R &= (\mu \wedge \nu)(\mu + \nu) \\ &\leq \mu\nu.\end{aligned}$$

Thus $\mu\nu = \mu \wedge \nu$.

Next we look at the residual operations on fuzzy ideals.

Proposition 2.2.9[Liu 2].
Let $\mu, \nu, \omega, \lambda \in \mathcal{F}(R)$, then

(1) $\mu : \mu = \mathcal{X}_R$, $\mu \leq \mu : \nu$, $(\mu : \nu)\nu \leq \mu$ *and* $(\mu : \nu)\omega = \mu : \nu\omega$,

(2) *If* $\mu \leq \nu$, *then* $\mu : \omega \leq \nu : \omega$ *and* $\omega : \nu \leq \omega : \mu$,

(3) *If* $\nu_1, \nu_2, \ldots, \nu_n$ *are fuzzy ideals of* R *such that* $\mu(0) = \nu_1(0) = \cdots = \nu_n(0) = 1$, *then*

$$\mu : (\nu_1 + \nu_2 + \cdots + \nu_n) = \wedge_{i=1}^{n}(\mu : \nu_i),$$

(4) *If* $\nu_i, i \in \mathcal{A}$ *($\mathcal{A}$ is an indexing set) are non-constant fuzzy ideals of* R, *then*

$$(\bigwedge_{i\in\mathcal{A}} \nu_i) : \mu \leq \bigwedge_{i\in\mathcal{A}}(\nu_i : \mu),$$ *equality holds if* $\mathcal{A}$ *is finite [Mal 4, 2.10].*

Proof.

(1) Since for all $\nu \in \mathcal{F}(R)$ $\mu \circ \nu \leq \mu$, $\mu : \mu = \mathcal{X}_R$.

Since $\mu \circ \nu \leq \mu$ we have $\mu \leq \mu : \nu$.

Let $x \in R$

$$\begin{aligned}(\mu : \nu)\nu(x) &= \sup_{x=\sum_{i=1}^{p} x_{1i}x_{2i}} \wedge_1^p (\mu : \nu)(x_{1i}) \wedge \nu(x_{2i}) \\ &= \sup_{x=\sum_{i=1}^{p} x_{1i}x_{2i}} \wedge_1^p \sup_{\nu\circ\lambda\leq\mu} \lambda(x_{1i}) \wedge \nu(x_{2i}) \\ &= \sup_{\substack{\nu\circ\lambda\leq\mu \\ x=\sum_{i=1}^{p} x_{1i}x_{2i}}} \wedge_1^p \lambda(x_{1i}) \wedge \nu(x_{2i}) \\ &\leq \sup_{\substack{\nu\circ\lambda\leq\mu \\ x=\sum_{i=1}^{p} x_{1i}x_{2i}}} \wedge_1^p \lambda \circ \nu(x_{1i}x_{2i}) \\ &\leq \sup_{x=\sum_{i=1}^{p} x_{1i}x_{2i}} \wedge_1^p \mu(x_{1i}x_{2i}) \\ &\leq \mu(x).\end{aligned}$$

We claim that

$$\{\lambda : \lambda \in \mathcal{F}(R), \omega \circ \lambda \leq \mu : \nu\} = \{\lambda : \lambda \in \mathcal{F}(R), \nu\omega \circ \lambda \leq \mu\}.$$

Let $\lambda \in \mathcal{F}(R)$ such that $\nu\omega \circ \lambda \leq \mu$ then $\nu \circ \omega\lambda \leq \mu$ [refer 2.3.12(3)] which implies $\omega \circ \lambda \leq \omega\lambda \leq \mu : \nu$.
To show the converse part, let $\lambda \in \mathcal{F}(R)$ such that $\omega \circ \lambda \leq \mu : \nu$ and $x \in R$ then

$$(\nu\omega \circ \lambda)(x) = \sup_{x=x_1x_2} \nu\omega(x_1) \wedge \lambda(x_2)$$

$$= \sup_{\substack{x=x_1x_2 \\ x_1=\Sigma_{i=1}^{p} y_{1i}y_{2i}}} \wedge_1^p \quad \nu(y_{1i}) \wedge \omega(y_{2i}) \wedge \lambda(x_2)$$

$$\leq \sup_{\substack{x=x_1x_2 \\ x_1=\Sigma_{i=1}^{p} y_{1i}y_{2i}}} \wedge_1^p \quad \nu(y_{1i})(\omega \circ \lambda)(y_{2i}x_2)$$

$$\leq \sup_{\substack{x=x_1x_2 \\ x_1=\Sigma_{i=1}^{p} y_{1i}y_{2i}}} \wedge_1^p \quad \nu(y_{1i})(\mu : \nu)(y_{2i}x_2)$$

$$= \sup_{x=\Sigma_{i=1}^{p} y_{1i}y_{2i}x_2} \wedge_1^p \quad \nu(y_{1i}) \wedge \sup_{\nu\circ\lambda' \leq \mu} \lambda'(y_{2i}x_2)$$

$$= \sup_{\substack{x=\Sigma_{i=1}^{p} y_{1i}y_{2i}x_2 \\ \nu\circ\lambda' \leq \mu}} \wedge_1^p \quad \nu(y_{1i}) \wedge \lambda'(y_{2i}x_2)$$

$$\leq \sup_{\substack{x=\Sigma_{i=1}^{p} y_{1i}y_{2i}x_2 \\ \nu\circ\lambda' \leq \mu}} \wedge_1^p \quad \nu \circ \lambda'(y_{1i}y_{2i}x_2)$$

$$\leq \sup_{x=\Sigma_{i=1}^{p} y_{1i}y_{2i}x_2} \wedge_1^p \quad \mu(y_{1i}y_{2i}x_2)$$

$$\leq \mu(x).$$

Thus $\nu\omega \circ \lambda \leq \mu$. Hence

$$Sup\{\lambda(x) : \lambda \in \mathcal{F}(R), \omega \circ \lambda \leq \mu : \nu\} = Sup\{\lambda(x) : \lambda \in \mathcal{F}(R), \nu\omega \circ \lambda \leq \mu\}.$$

$$i.e. \quad (\mu : \nu) : \omega = \mu : \nu\omega.$$

(2) Obvious.

(3) Since $\nu_i \leq \nu_1 + \nu_2 + \cdots + \nu_n$ for all $i = 1, 2, \ldots, n$, $\mu : \nu_i \leq \mu : (\nu_1 + \nu_2 + \cdots + \nu_n)$ for all $i = 1, 2, \ldots, n$ and hence $\wedge_{i=1}^{n} (\mu : \nu_i) \leq \mu(\nu_1 + \nu_2 + \cdots + \nu_n)$. To show the converse. let $x \in R$ then

$$\mu : (\nu_1 + \nu_2 + \cdots + \nu_n)(x) = \sup_{(\nu_1+\nu_2+\cdots+\nu_n)\circ\lambda \leq \mu} \lambda(x)$$

$$\leq \sup_{\nu_i \circ \lambda \leq \mu} \lambda(x) \quad \text{for all } i = 1, 2, \ldots, n.$$

$$= (\mu : \nu_i)(x) \quad \text{for all } i = 1, 2, \ldots, n$$

hence $\mu : (\nu_1+\nu_2+\cdots+\nu_n)(x) \leq \wedge_{i=1}^{n} (\mu : \nu_i)(x)$, thus $\mu : (\nu_1+\nu_2+\cdots+\nu_n) = \wedge_{i=1}^{n} (\mu : \nu_i)$.

(4) Obvious.

□

Corollary 2.2.10[Bho 4, 4.2, 4.3].
Let $f : R \longrightarrow R'$ be a homomorphism and μ, μ' be fuzzy ideals of R and R' respectively then

(1) *if f is epimorphism then $f(\mu)$ is fuzzy ideal of R,*
(2) *$f^{-1}(\mu')$ is fuzzy ideal of R',*
(3) *if μ is f-invariant then $f^{-1}(f(\mu)) = \mu'$.*

Hence if f epimorphism then one can easily establish a one-to-one correspondence between the set of all f-invariant fuzzy ideals of R and set of all fuzzy ideals of R'.

Proposition 2.2.11.
Let $f : R \longrightarrow R'$ be an epimorphism and μ, ν be fuzzy ideals of R and μ', ν' be fuzzy ideals of R' then

$$f(\mu)f(\nu) \leq f(\mu\nu)$$

and

$$f^{-1}(\mu')f^{-1}(\nu') \leq f^{-1}(\mu'\nu').$$

Proof.
Let $y \in R'$ then

$$f(\mu)f(\nu)(y) = \sup_{y=\Sigma_{i=1}^{p} y_{1i}y_{2i}} \wedge_1^p \quad f(\mu)(y_{1i}) \wedge f(\nu)(y_{2i})$$

$$= \sup_{y=\Sigma_{i=1}^{p} y_{1i}y_{2i}} \wedge_1^p \quad \sup_{x_{1i}\in f^{-1}(y_{1i})} \mu(x_{1i}) \wedge \sup_{x_{2i}\in f^{-1}(y_{2i})} \nu(x_{2i})$$

$$= \sup_{y=\Sigma_{i=1}^{p} y_{1i}y_{2i}} \wedge_1^p \quad \sup_{\substack{x_{1i}\in f^{-1}(y_{1i}) \\ x_{2i}\in f^{-1}(y_{2i})}} \mu(x_{1i}) \wedge \nu(x_{2i})$$

$$\leq \sup_{y=\Sigma_{i=1}^{p} y_{1i}y_{2i}} \wedge_1^p \quad \sup_{x_{1i}x_{2i}\in f^{-1}(y_{1i}y_{2i})} \mu\nu(x_{1i}x_{2i})$$

$$\leq \sup_{y=\Sigma_{i=1}^{p} y_{1i}y_{2i}} \wedge_1^p \quad \sup_{x\in f^{-1}(y_{1i}y_{2i})} \mu\nu(x)$$

$$= \sup_{y=\Sigma_{i=1}^{p} y_{1i}y_{2i}} \wedge_1^p \quad f(\mu\nu)(y_{1i}y_{2i}) \quad \leq f(\mu\nu)(y) \text{ since } f(\mu\nu) \text{ is a fuzzy ideal .}$$

We can prove the rest part by using the same technique as in the proof above.

□

§ 2.3 Fuzzy ideal generators.

In [Swa 1], U.M.Swamy and K.L.N.Swamy and in [Zah 3] M.M.Zahedi introduced the notion of a fuzzy ideal generated by a fuzzy subset. In this section we study how the generating ideal and its generators behave over some crisp subsets of R. From the fact that the intersection of an arbitrary collection of fuzzy ideals is a fuzzy ideal we give a definition for a fuzzy generating ideal as follows:

Definition 2.3.1.
Let μ be a non-constant fuzzy ideal of R. Then the intersection of all fuzzy ideals of R containing μ is called a *fuzzy ideal generated by* μ and is denoted by $<\mu>$.

Throughout this section we consider fuzzy subsets and fuzzy ideals with the sup-property, unless otherwise stated. We can express the ideal $<M>$ generated by a non-empty subset M in terms of the characteristic function as follows.

$$\mathcal{X}_{<M>}(x) = \sup_{x=\Sigma_{i=1}^{n} r_i x_i} \wedge_1^n \mathcal{X}_M(x_i).$$

Hence we have the following Proposition.

Proposition 2.3.2.
Let μ be a non-constant fuzzy subset of R and $\nu : R \longrightarrow I$ be a fuzzy subset of R defined by

$$\nu(x) = \sup_{x=\Sigma_{i=1}^{n} r_i x_i} \wedge_1^n \mu(x_i).$$

Then ν is a fuzzy ideal of R and $\nu = <\mu>$.

Proof.
Let $x, y \in R$ then

$$\nu(x) \wedge \nu(y) = \sup_{x=\Sigma_{i=1}^{n_1} s_i x_i} \wedge_1^{n_1} \mu(x_i) \wedge \sup_{y=\Sigma_{j=1}^{n_2} t_j y_j} \wedge_1^{n_2} \mu(y_j)$$

$$= \sup_{\substack{x=\Sigma_{i=1}^{n_1} s_i x_i \\ y=\Sigma_{j=1}^{n_2} t_j y_j}} \wedge_1^{n_1} \wedge_1^{n_2} \mu(x_i) \wedge \mu(y_j)$$

$$\leq \sup_{x-y=\Sigma_{i=1}^{n} r_i z_i} \wedge_1^n \mu(z_i)$$
$$= \nu(x-y)$$

and

$$\nu(x) = \sup_{x=\Sigma_{i=1}^{n} r_i x_i} \wedge_1^n \mu(x_i)$$

$$\leq \sup_{xy=\Sigma_{i=1}^{n}(yr_i)x_i} \wedge_1^n \mu(x_i)$$

$$\leq \sup_{xy=\Sigma_{i=1}^{n} s_i z_i} \wedge_1^n \mu(z_i)$$

$$= \nu(xy).$$

Similarly, we can show that $\nu(y) \leq \nu(xy)$. Thus ν is a fuzzy ideal. Since $x = x1$, we have $\mu(x) \leq \nu(x)$ for all $x \in R$, hence $\mu \leq \nu$.
Let ω be any fuzzy ideal of R such that $\mu \leq \omega$. Then for all $x \in R$ of the form $x = \Sigma_{i=1}^{n} r_i x_i$

$$\omega(x) \geq \wedge_1^n \omega(x_i) \geq \wedge_1^n \mu(x_i).$$

Since it is true for all $x = \Sigma_{i=1}^{n} r_i x_i$

$$\omega(x) \geq \sup_{x=\Sigma_{i=1}^{n} r_i x_i} \wedge_1^n \mu(x_i) = \nu(x).$$

Thus $\nu \leq \omega$; hence ν is the smallest fuzzy ideal containing μ, i.e. $\nu =< \mu >$.

Proposition 2.3.3.

Let μ be a non-constant fuzzy subset of R, then

$$\nu(0) = \sup_{x\in R}\nu(x) = \sup_{x\in R}\mu(x)$$

and

$$< \mu_t >= \nu_t \qquad \textit{for all } t \in [0,1]$$

where $\nu =< \mu >$.

Proof.
Since $\mu \leq \nu$ $\quad \sup_{x\in R}\mu(x) \leq \sup_{x\in R}\nu(x)$.
Let $y \in R$ be of the form $y = \Sigma_{i=1}^{n} r_i x_i$. Then

$$\sup_{x\in R}\mu(x) \geq \mu(x_i) \geq \wedge_1^n \mu(x_i).$$

Since it is true for all $y = \Sigma_{i=1}^{n} r_i x_i$

$$\sup_{x\in R}\mu(x) \geq \sup_{y=\Sigma_{i=1}^{n} r_i x_i} \wedge_1^n \mu(x_i) = \nu(y).$$

Hence $\sup_{x\in R}\mu(x) \geq \nu(y)$ for all $y \in R$. Therefore

$$\sup_{x\in R}\mu(x) \geq \sup_{x\in R}\nu(x) = \nu(0).$$

Let $t \in [0,1]$; then clearly $\mu_t \subseteq \nu_t$. So $<\mu_t> \subseteq \nu_t$, since ν_t is an ideal of R. On the other hand, if $x \in \nu_t$ then $\nu(x) \geq t$; so $\sup_{x=\Sigma_{i=1}^{n} r_i x_i} \wedge_1^n \mu(x_i) \geq t$. Since μ has the sup-property, for some $x = \Sigma_{i=1}^{n} r_i x_i$, $\wedge_1^n \mu(x_i) \geq t$ which implies $x_i \in \mu_t$ for all $i = 1, 2, \ldots n$. So $x \in <\mu_t>$.

□

By considering different level subsets of μ, we inductively define crisp subsets $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n, \ldots$ as follows

$$\mathcal{A}_0 = \{x \in R : \mu(x) = \sup_{r\in R}\mu(r)\}$$

and

$$\mathcal{A}_n = \{x \in R : \mu(x) = \sup_{r \in R \setminus \cup_0^{n-1}\mathcal{A}_k} \mu(r)\} \qquad \text{for all } n = 1, 2, \ldots$$

In the following Proposition, we prove that these subsets are mutually disjoint; further we prove that μ and ν have the same supremum value considered over some chosen crisp subsets. These results are not true if μ does not have the sup-property.

Theorem 2.3.4.
Let the family $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of crisp subsets of R be defined as above. Then

(1) $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ *for* $i \neq j$,

(2) $\sup_{x \in R \setminus <\cup_0^{n-1}\mathcal{A}_k>} \nu(x) = \sup_{x \in R \setminus <\cup_0^{n-1}\mathcal{A}_k>} \mu(x)$,

(3) *If* $<\cup_0^n \mathcal{A}_k> \setminus <\cup_0^{n-1}\mathcal{A}_k> \neq \emptyset$, *thenfor all* $x \in <\cup_0^n \mathcal{A}_k> \setminus <\cup_0^{n-1}\mathcal{A}_k>$

$$\nu(x) = \sup_{r \in R \setminus <\cup_0^{n-1}\mathcal{A}_k>} \nu(r) = \sup_{r \in R \setminus \cup_0^{n-1}\mathcal{A}_k} \mu(r).$$

Proof.

(1) Suppose there exists x such that $x \in \mathcal{A}_i \wedge \mathcal{A}_k$ for $i \neq j$, say $i > j$. Then

$$\mu(x) = \sup_{r \in R \setminus \cup_0^{i-1}\mathcal{A}_k} \mu(r) = \sup_{r \in R \setminus \cup_0^{j-1}\mathcal{A}_k} \mu(r).$$

Since μ has the sup-property

$$\mu(x) = \mu(r_1) = \mu(r_2) \text{ for some } r_1 \in R \setminus \cup_0^{i-1} \mathcal{A}_k, r_2 \in R \setminus \cup_0^{j-1} \mathcal{A}_k$$

which implies $r_1 \notin \mathcal{A}_k$ for all $k = 0, 1, \ldots, i-1$, hence

$$\mu(r_1) < \sup_{r \in R \setminus \cup_0^{i-2} \mathcal{A}_k} \mu(r).$$

Since $i > j$, $\quad i - 2 \geq j - 1$, we have

$$\mu(r_1) = \sup_{r \in R \setminus \cup_0^{j-1} \mathcal{A}_k} \mu(r) \geq \sup_{r \in R \setminus \cup_0^{i-2} \mathcal{A}_k} \mu(r)$$

which is a contradiction. Thus the result follows.

(2) By the Proposition 2.3.3 , the result is true for $n = 0$.
Let $n \geq 1$. Since $\mu \leq \nu$

$$\sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >} \mu(r) \leq \sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >} \nu(r).$$

To prove the other way, for all $r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >$,
such that $r = \Sigma_{i=1}^{p} s_i x_i$ $x_{i_0} \notin < \cup_0^{n-1} \mathcal{A}_k >$ for some i_0 with $1 \leq i_0 \leq p$. So,

$$\wedge_1^p \mu(x_i) \leq \mu(x_{i_0}) \leq \sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >} \mu(r).$$

Since it is true for all $r = \Sigma_{i=1}^{p} s_i x_i$

$$\nu(r) = \sup_{r = \Sigma_{i=1}^{p} s_i x_i} \wedge_1^p \mu(x_i) \leq \sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >} \mu(r)$$

and hence

$$\sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >} \nu(r) \leq \sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >} \mu(r).$$

(3) When $n = 0$, let $r \in < \mathcal{A}_0 >$. Then r is of the form $r = \Sigma_{i=1}^{p} s_i x_i$,
$x_i \in \mathcal{A}_0$ for all $i = 1, 2, \ldots, p$. So we have

$$\mu(x_i) = \sup_{x \in R} \mu(x) \quad \text{for all } i = 1, 2, \ldots, p$$

hence

$$\wedge_1^p \mu(x_i) = \sup_{x \in R} \mu(x)$$

which implies

$$\nu(r) = \sup_{r = \Sigma_{i=1}^{p} r_i x_i} \wedge_1^p \mu(x_i) = \sup_{x \in R} \mu(x) = \sup_{x \in R} \nu(x).$$

When $n \geq 1$, let $r \in < \cup_0^n \mathcal{A}_k > \smallsetminus < \cup_0^{n-1} \mathcal{A}_k >$ such that $r = \Sigma_{i=1}^p s_i x_i$ then

$$x_{i_0} \notin \cup_0^{n-1} \mathcal{A}_k \text{ and } x_{i_0} \in \mathcal{A}_n \text{ for some } i_0 \text{ with } 1 \leq i_0 \leq p.$$

Since $x_i \in \cup_0^n \mathcal{A}_k$ for all $i = 1, 2, \ldots, p$

$$\mu(x_i) \geq \mu(x_{i_0}) \text{ for all } i = 1, 2, \ldots, p.$$

So

$$\wedge_1^p \mu(x_i) = \mu(x_{i_0}) = \sup_{x \in R \smallsetminus \cup_{i=1}^{n-1} \mathcal{A}_k} \mu(x)$$

Since the above equality is true for all such $r = \Sigma_{i=1}^p s_i x_i$

$$\nu(r) = \sup_{r \doteq \Sigma_{i=1}^p s_i x_i} \wedge_1^p \mu(x_i) = \sup_{x \in R \smallsetminus \cup_0^{n-1} \mathcal{A}_k} \mu(x)$$

thus

$$\begin{aligned} \sup_{x \in R \smallsetminus \cup_0^{n-1} \mathcal{A}_k} \mu(x) = \nu(r) &\leq \sup_{x \in R \smallsetminus < \cup_0^{n-1} \mathcal{A}_k >} \nu(x) \\ &= \sup_{x \in R \smallsetminus < \cup_0^{n-1} \mathcal{A}_k >} \mu(x) \\ &\leq \sup_{x \in R \smallsetminus \cup_0^{n-1} \mathcal{A}_k} \mu(x). \end{aligned}$$

□

Remark.
[2.3.4 , (3)] will fail if $< \cup_0^n \mathcal{A}_k > \smallsetminus < \cup_0^{n-1} \mathcal{A}_k >= \emptyset$. The following example illustrates the fact.

Example 2.3.5.
Define a fuzzy subset $\mu : \mathbb{Z}_6 \longrightarrow I$ by

$$\mu(0) = 1,\ \mu(1) = 0.7,\ \mu(2) = 0.9,\ \mu(3) = 0.6,\ \mu(4) = 0.8 \text{ and } \mu(5) = 0.6.$$

Then the fuzzy ideal ν generated by μ is

$$\nu(0) = 1,\ \nu(2) = \nu(4) = 0.9 \text{ and } \nu(1) = \nu(3) = \nu(5) = 0.7$$

and

$$\mathcal{A}_0 = \{0\},\ \mathcal{A}_1 = \{2\},\ \mathcal{A}_2 = \{4\},\ \mathcal{A}_3 = \{1\} \text{ and } \mathcal{A}_4 = \{3, 5\}$$

which implies

$$< \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 >=< \mathcal{A}_0 \cup \mathcal{A}_1 >$$

hence we have

$$\sup_{r\in\mathbb{Z}_6\setminus<\mathcal{A}_0\cup\mathcal{A}_1>}\nu(r)=0.7\neq\sup_{r\in\mathbb{Z}_6\setminus\mathcal{A}_0\cup\mathcal{A}_1}\mu(r)=0.8$$

and

$$\sup_{r\in\mathbb{Z}_6\setminus\mathcal{A}_0\cup\mathcal{A}_1}\nu(r)=0.9\neq\sup_{r\in\mathbb{Z}_6\setminus\mathcal{A}_0\cup\mathcal{A}_1}\mu(r).$$

We now prove that the converse of Proposition[2.3.4] is also true under a certain condition.

Theorem 2.3.6.
Let the family $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ be defined as earlier and $<\cup_0^n\mathcal{A}_k>\setminus<\cup_0^{n-1}\mathcal{A}_k>$ be a non-empty set. Then

$$\textit{if}\quad \nu(x)=\sup_{r\in R\setminus<\cup_0^{n-1}\mathcal{A}_k>}\nu(r),\quad x\in<\cup_0^n\mathcal{A}_k>\setminus<\cup_0^{n-1}\mathcal{A}_k>\quad \textit{for all}\quad n=0,1,2,\ldots.$$

Proof.
Let us prove first for $n=0$. i.e. for the case $<\cup_0^{n-1}\mathcal{A}_k>=\emptyset$. Let $\nu(x)=\sup_{r\in R}\nu(r)$.
Suppose $x\notin<\mathcal{A}_0>$; then for all x such that $x=\Sigma_{i=1}^p r_ix_i,\quad x_{i_0}\notin<\mathcal{A}_0>$ for some i_0 with $1\le i_0\le p$. So $\mu(x_{i_0})<\sup_{r\in R}\mu(r)$, and hence $\wedge_1^P\mu(x_i)\le\mu(x_{i_0})<\sup_{r\in R}\mu(r)$.Since it is true for all such $x=\Sigma_{i=1}^p r_ix_i$ and μ has sup-property

$$\nu(x)=\sup_{x=\Sigma_{i=1}^p r_ix_i}\wedge_1^p\mu(x_i)<\sup_{r\in R}\mu(r)=\sup_{r\in R}\nu(r)$$

which is a contradiction.
We now consider for $n\ge1$. Let $\nu(x)=\sup_{r\in R\setminus<\cup_0^{n-1}\mathcal{A}_k>}\nu(r)$.
Suppose $x\notin<\cup_0^n\mathcal{A}_k>\setminus<\cup_0^{n-1}\mathcal{A}_k>$, then there are two cases:

Case 1.
Suppose $x\in<\cup_0^{n-1}\mathcal{A}_k>$. Then x is of the form $x=\Sigma_{i=1}^p r_ix_i,\quad$ where $r_i\in R$, $x_i\in\cup_0^{n-1}\mathcal{A}_k$ and $p\le n-1$. Hence we have $\sup_{r\in R\setminus<\cup_0^{n-1}\mathcal{A}_k>}\mu(r)\le\sup_{r\in R\setminus\cup_0^{n-1}\mathcal{A}_k}\mu(r)<\mu(x_i)$ for all $i=1,2\ldots,p$. By Proposition [2.3.4], $\sup_{r\in R\setminus<\cup_0^{n-1}\mathcal{A}_k>}\nu(r)<\mu(x_i)$ for all $i=1,2,\ldots,p$.
i.e. $\nu(x)<\mu(x_i)$ for all $i=1,2\ldots,p$.
So $\nu(x)<\wedge_1^p\mu(x_i)$. But

$$\nu(x)=\sup_{x=\Sigma_{i=1}^t s_iy_i}\wedge_1^t\mu(y_i)\ge\wedge_1^p\mu(x_i)>\nu(x).$$

This is a contradiction.

Case 2.

Suppose $x \notin < \cup_0^n \mathcal{A}_k >$. Then for all x such that $x = \Sigma_{i=1}^t r_i x_i$, $\quad x_{i_0} \notin \cup_0^n \mathcal{A}_k \quad$ for some i_0

with $1 \leq i_0 \leq t$, therefore $\wedge_1^t \mu(x_i) \leq \mu(x_{i_0}) < \sup\limits_{r \in R \setminus \cup_0^{n-1} \mathcal{A}_k} \mu(r)$. Since it is true for all x of the form $x = \Sigma_{i=1}^t r_i x_i$ and μ has a sup-property,

$$\nu(x) = \sup_{x=\Sigma_{i=1}^t r_i x_i} \wedge_1^t \mu(x_i) < \sup_{r \in R \setminus \cup_0^{n-1} \mathcal{A}_k} \mu(r).$$

Thus

$$\nu(x) = \sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >} \nu(r) < \sup_{r \in R \setminus \cup_0^{n-1} \mathcal{A}_k} \mu(r)$$

which is a contradiction by Proposition[2.3.4 (2)].

□

Remark.

V.N.Dixit et al. [Dix2,4.2] defined a fuzzy subring generated by a fuzzy subset μ by our Theorems 2.3.4(3) and 2.3.6.

The above proposition turns out be false if μ does not have the sup-property.

Example 2.3.7.

We define a fuzzy subset $\mu : \mathbb{Z} \longrightarrow I$ of a ring $(\mathbb{Z}, +, .)$ by

$$\mu(x) = \begin{cases} 1/2 & \text{if} \quad x = 2 \\ 1/2 - (1/3)^{|x|} & \text{if} \quad x \text{ is odd} \\ 0 & \text{if} \quad x \text{ is even , } x \neq 2 \end{cases}$$

then $\sup\limits_{r \in \mathbb{Z}} \mu(r) = 1/2 \quad$ and $\mathcal{A}_0 = \{2\}$.

We now consider the following combinations and their corresponding membership values:

$$\begin{aligned} 3 &= 1.3 + (-0).2 & \mu(3) \wedge \mu(2) &= 1/2 - (1/3)^3 \\ &= 1.5 + (-1).2 & \mu(5) \wedge \mu(2) &= 1/2 - (1/3)^5 \\ &= 1.7 + (-2).2 & \mu(7) \wedge \mu(2) &= 1/2 - (1/3)^7 \\ &\quad . \quad . \quad . \\ &\quad . \quad . \quad . \\ &= 1.a + (-k).2 & \mu(a) \wedge \mu(2) &= 1/2 - (1/3)^a. \end{aligned}$$

Hence

$$\sup_{3=1.a+(-k).2} \mu(a) \wedge \mu(2) = 1/2.$$

Since for all $x \in \mathbb{Z}$, $\mu(x) \leq 1/2$ and $\nu(3) = \sup_{3=\Sigma_{i=1}^{n} r_i x_i} \wedge_1^n \mu(x_i)$, $\nu(3) = 1/2$ but $3 \notin < \mathcal{A}_0 >$, it is clear that μ does not have the sup-property.

In the following section, we consider fuzzy ideals $\bar{\mu}, \bar{\nu}$ generated by two distinct fuzzy subsets μ and ν respectively. As in the crisp case, distinct fuzzy subsets may generate the same fuzzy ideal; but we prove here that these fuzzy subsets coincide on a suitably chosen mutually disjoint collection of subsets of R. From this we are able to conclude that different generators of the same fuzzy ideal must have the same image. Thus the image of generator is an invariant for the fuzzy ideal.

Proposition 2.3.8.
Let μ, ν be fuzzy subsets of R.

(1) *If $\mu \leq \nu$, then $\bar{\mu} \leq \bar{\nu}$,*
(2) *If $\mu = \nu$, then $\bar{\mu} = \bar{\nu}$.*

The proof is straightforward.

Generally the converse of the above two results are not true. The examples follow.

Example 2.3.9.
In the ring $(\mathbb{Z}_4, +, \circ)$, we define two fuzzy subsets $\mu, \nu :\longrightarrow I$ by

$$\mu(0) = \mu(1) = 0, \ \mu(2) = 1/2, \ \mu(3) = 1/3$$

and

$$\nu(0) = \nu(1) = 0, \ \nu(2) = 1/3, \ \nu(3) = 3/4.$$

Then

$$\bar{\mu}(0) = \bar{\mu}(2) = 1/2, \ \bar{\mu}(1) = \bar{\mu}(3) = 1/3$$

and

$$\bar{\nu}(0) = \bar{\nu}(1) = \bar{\nu}(2) = \bar{\nu}(3) = 3/4$$

hence $\bar{\mu} < \bar{\nu}$, but clearly $\mu \nleq \nu$.

Example 2.3.10.
Define $\mu, \nu : \mathbb{Z}_4 \longrightarrow I$ by

$$\mu(0) = \mu(1) = 0, \ \mu(2) = 1/3, \ \mu(3) = 3/4$$

and

$$\nu(0) = \nu(3) = 0, \ \nu(2) = 1/3, \ \nu(1) = 3/4.$$

Then

$$\bar{\mu}(x) = \bar{\nu}(x) \quad \text{for all } x = 0, 1, 2, 3 \quad \text{but } \mu \neq \nu.$$

We have seen that $\bar{\mu} = \bar{\nu}$ does not necessarily imply $\mu(x) = \nu(x)\ \forall x \in R$, but we prove in the following $\mu(x) = \nu(y)\ \ \forall x \in \mathcal{A}_k,\ y \in \mathcal{A}'_k$ under a certain condition.

Theorem 2.3.11.

Let μ, ν be distinct fuzzy subsets of R such that $\bar{\mu} = \bar{\nu}$.
If $< \cup_0^n \mathcal{A}_k > \setminus < \cup_0^{n-1} \mathcal{A}_k >$ and $< \cup_0^n \mathcal{A}'_k > \setminus < \cup_0^{n-1} \mathcal{A}'_k >$ are non-empty sets, then

$$\mu(x) = \nu(y) \quad \text{for all } x \in \mathcal{A}_n, y \in \mathcal{A}'_n \text{ and } < \cup_0^n \mathcal{A}_k > = < \cup_0^n \mathcal{A}'_k > \quad \text{for all } n = 0, 1, \ldots, .$$

Proof.
This proof is by induction on k.
Let $x \in \mathcal{A}_0, y \in \mathcal{A}'_0$ then by Theorem[2.3.4],

$$\mu(x) = \sup_{r \in R} \mu(r) = \sup_{r \in R} \bar{\mu}(r) = \sup_{r \in R} \bar{\nu}(r) = \sup_{r \in R} \nu(r) = \nu(y).$$

$$i.e. \quad \forall x \in \mathcal{A}_0, y \in \mathcal{A}'_0 \quad \mu(x) = \nu(y).$$

Let $x \in \mathcal{A}_0$ then by Theorem[2.3.4]

$$\bar{\mu}(x) = \sup_{r \in R} \bar{\mu}(r) = \sup_{r \in R} \bar{\nu}(r) = \bar{\nu}(x)$$

from which follows that $x \in < \mathcal{A}'_0 >$. So $< \mathcal{A}_0 > \subseteq < \mathcal{A}'_0 >$. Similarly, we can show $< \mathcal{A}'_0 > \subseteq < \mathcal{A}_0 >$. Thus $< \mathcal{A}_0 > = < \mathcal{A}'_0 >$. Hence the result is true for $n = 0$.
Suppose the result is true for $n - 1$.

$$i.e. \quad \forall x \in \mathcal{A}_{n-1}, y \in \mathcal{A}'_{n-1} \quad \mu(x) = \nu(y) \text{ and } \quad < \cup_0^{n-1} \mathcal{A}_k > = < \cup_0^{n-1} \mathcal{A}'_k > .$$

Let $x \in \mathcal{A}_n, y \in \mathcal{A}'_n$ then by Theorem[2.3.4],

$$\mu(x) = \sup_{r \in R \setminus \cup_0^{n-1} \mathcal{A}_k} \mu(r) = \sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >} \bar{\mu}(r) = \sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}'_k >} \bar{\nu}(r) = \nu(y).$$

$$i.e. \quad \forall x \in \mathcal{A}_n, y \in \mathcal{A}'_n \quad \mu(x) = \nu(y).$$

To show the last part, let $x \in < \cup_0^n \mathcal{A}_k >$. Then two cases arise.

case 1.
This is the simpler case. Suppose $x \in < \cup_0^{n-1} \mathcal{A}_k >$; then by induction $x \in < \cup_0^{n-1} \mathcal{A}'_k >$, and hence $x \in < \cup_0^n \mathcal{A}'_k >$.

case 2.
Suppose $x \notin < \cup_0^{n-1} \mathcal{A}_k >$; then $x \in < \cup_0^n \mathcal{A}_k > \setminus < \cup_0^{n-1} \mathcal{A}_k >$.By Theorem[2.3.4], we have

$$\bar{\mu}(x) = \sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}_k >} \bar{\mu}(r) = \sup_{r \in R \setminus < \cup_0^{n-1} \mathcal{A}'_k >} \bar{\nu}(r) = \bar{\nu}(x).$$

Which implies by Theorem[2.3.6], $x \in < \cup_0^n \mathcal{A}'_k >$. So $< \cup_0^n \mathcal{A}_k > \subseteq < \cup_0^n \mathcal{A}'_k >$. Similarly, reversing the role of $\mathcal{A}_k$ and $\mathcal{A}'_k$, we get the reverse inclusion.

□

Remark 1.
By Theorem[2.3.4], $\{\mathcal{A}_n\}$ and $\{\mathcal{A}'_n\}$ are two distinct partitions of R and this gives rise to an equivalence relation on R. For each n i.e. for each equivalence class μ and ν have the same membership values. Thus both μ and ν must have the same image. It should be noted that $Im(\mu) = Im(\nu)$ does not imply $\mu = \nu$.

Remark 2.
We define a function $f : R \longrightarrow R$ by using the Axiom of Choice as follows: Let $x \in R$. Then there exists a unique $n \in \mathbb{N}$ such that $x \in \mathcal{A}_n$. Since $\{\mathcal{A}'_n : n \in \mathbb{N}\}$ is a non-empty collection of non-empty subsets, by the Axiom of Choice there exists a choice function Ψ such that for each $n \in \mathbb{N}$, $\Psi(n) \in \mathcal{A}'_n$. We therefore define a map $f : R \longrightarrow R$ by $f(x) = \Psi(n)$. Hence $\nu \circ f(x) = \nu(\Psi(n)) = \mu(x)$ by Theorem[2.3.11], *i.e.* $\nu \circ f = \mu$.

Remark 3.
A relation '$\sim$ 'on I^R, defined by $\mu \sim \nu$ if and only if $< \mu >=< \nu >$, is easily checked to be an equivalence relation on I^R. Thus all fuzzy subsets in any equivalence class with respect to '$\sim$ 'have the same image.

In the following Proposition we collect some results pertaining to union, product and composition of fuzzy ideals .

Proposition 2.3.12[Liu 1,3.1].

(1) *If $\mu, \nu, \mu_\alpha (\alpha \in I)$ are fuzzy ideals of R, then*
(a) $\mu\nu =< \mu \circ \nu >$,
(b) If $\mu(0) = \nu(0) = 1$, then $\mu + \nu =< \mu \vee \nu >$,
(c) $< \bigvee_{\alpha\in I} \mu_\alpha > \nu =< \bigvee_{\alpha\in I} \mu_\alpha \nu >$,

(2) *If μ, ν are fuzzy subsets of R, then* $< \mu\nu >=< \mu >< \nu >$,
(3) *Let μ, ν, ω be fuzzy ideals of R then*

$$\nu\omega \leq \mu \quad \textit{if and only if} \quad \nu \circ \omega \leq \mu,$$

(4) *Let x_r, y_s be any fuzzy points of R, then* $< x_r \circ y_s >=< x_r > \circ < y_s >$.

Proof.

(1) (a) By the definitions of composition and product, we have $\mu \circ \nu \leq \mu\nu$. Since $\mu\nu$ is a fuzzy ideal and $< \mu \circ \nu >$ is the smallest fuzzy ideal containing $\mu \circ \nu$, $< \mu \circ \nu > \leq \mu\nu$. To show the converse part, let ω be any fuzzy ideal of R such that $< \mu \circ \nu > \leq \omega$ then $\mu \circ \nu \leq \omega$. Let $x \in R$ then

$$\mu\nu(x) = \sup_{x=\Sigma_{i=1}^n x_i y_i} \wedge_1^n \mu(x_i) \wedge \nu(y_i) \leq \sup_{x=\Sigma_{i=1}^n x_i y_i} \wedge_1^n (\mu \circ \nu)(x_i y_i) \leq \omega(x).$$

Hence $\mu\nu \leq \omega$ thus $\mu\nu \leq <\mu \circ \nu>$.

(b) Since $\mu(0) = \nu(0) = 1$ by the Proposition[2.2.8,5], $\mu \leq \mu + \nu$ and $\nu \leq \mu + \nu$. So $\mu \vee \nu \leq \mu + \nu$ which follows that $<\mu \vee \nu> \ \leq \ \mu + \nu$.
Let ω be any fuzzy ideal of R such that $<\mu \vee \nu> \leq \omega$ then

$$(\mu+\nu)(x) = \sup_{x=y+z} \mu(y) \wedge \nu(z) \leq \sup_{x=y+z} \omega(y) \wedge \omega(z) \leq \omega(x)$$

hence the result follows .
(c) For each $\alpha \in I$ $\ \mu_\alpha \nu \ \leq <\vee_{\alpha\in I}\mu_\alpha> \nu$.
So $\vee_{\alpha\in I}(\mu_\alpha\nu) \ \leq \ <\vee_{\alpha\in I}\mu_\alpha> \nu$, thus $<\vee_{\alpha\in I}\mu_\alpha\nu> \leq \ <\vee_{\alpha\in I}\mu_\alpha> \nu$.
Let ω be any fuzzy ideal of R such that $<\vee_{\alpha\in I}\mu_\alpha\nu> \leq \omega$ then $\forall \alpha \in I \ \ \mu_\alpha\nu \leq \omega$.
Let $x \in R$ then

$$\begin{aligned} <\vee_{\alpha\in I}\mu_\alpha> \nu(x) &= \sup_{x=\Sigma_{i=1}^n y_i z_i} \wedge_1^n \sup_{y_i=\Sigma_{j=1}^p r_j y_{ij}} \wedge_1^p (\vee_{\alpha\in I}\mu_\alpha)(y_{ij}) \wedge \nu(z_i) \\ &= \sup_{x=\Sigma_{i=1}^n y_i z_i} \wedge_1^n \vee_{\alpha\in I} \sup_{y_i=\Sigma_{j=1}^p r_j y_{ij}} \wedge_1^p (\mu_\alpha(y_{ij}) \wedge \nu(z_i)) \\ &\leq \sup_{x=\Sigma_{i=1}^n y_i z_i} \wedge_1^n \vee_{\alpha\in I} \sup_{y_i=\Sigma_{j=1}^p r_j y_{ij}} \wedge_1^p (\mu_\alpha\nu)(y_{ij} z_i) \\ &\leq \sup_{x=\Sigma_{i=1}^n y_i z_i} \wedge_1^n \vee_{\alpha\in I} \sup_{y_i=\Sigma_{j=1}^p r_j y_{ij}} \wedge_1^p \omega(r_j y_{ij} z_i) \\ &\leq \sup_{x=\Sigma_{i=1}^n y_i z_i} \wedge_1^n \vee_{\alpha\in I}\omega(y_i z_i) \ \leq \omega(x). \end{aligned}$$

Hence $<\vee_{\alpha\in I}\mu_\alpha> \nu \leq \omega$.

(2) Since $\mu \ \leq \ <\mu>$ and $\nu \ \leq \ <\nu>$, we have $\mu\nu \ \leq \ <\mu><\nu>$. So $<\mu\nu> \ \leq \ <\mu><\nu>$.

For the converse part, let ω be any fuzzy ideal of R such that $<\mu\nu> \leq \omega$ and

$x \in R$. Then

$$< \mu >< \nu > (x) = \sup_{x=\Sigma_{i=1}^{n} x_i y_i} \wedge_1^n \sup_{x_i=\Sigma_{j=1}^{p1} r_j x_{ij}} \wedge_1^{p_1} \mu(x_{ij}) \quad \wedge \sup_{y_i=\Sigma_{k=1}^{p2} s_k y_{ik}} \wedge_1^{p_2} \nu(y_{ik})$$

$$= \sup_{x=\Sigma_{i=1}^{n} x_i y_i} \wedge_1^n \sup_{\substack{x_i=\Sigma_{j=1}^{p1} r_j x_{ij} \\ y_i=\Sigma_{k=1}^{p2} s_k y_{ik}}} \wedge_1^{p_1} \wedge_1^{p_2} (\mu(x_{ij}) \wedge \nu(y_{ik}))$$

$$\leq \sup_{x=\Sigma_{i=1}^{n} x_i y_i} \wedge_1^n \sup_{\substack{x_i=\Sigma_{j=1}^{p1} r_j x_{ij} \\ y_i=\Sigma_{k=1}^{p2} s_k y_{ik}}} \wedge_1^{p_2} \wedge_1^{p_2} \mu\nu(x_{ij} y_{ik})$$

$$\leq \sup_{x=\Sigma_{i=1}^{n} x_i y_i} \wedge_1^n \sup_{\substack{x_i=\Sigma_{j=1}^{p1} r_j x_{ij} \\ y_i=\Sigma_{k=1}^{p2} s_k y_{ik}}} \wedge_1^{p_1} \wedge_1^{p_2} \omega(r_j x_{ij} s_k y_{ik})$$

$$\leq \sup_{x=\Sigma_{i=1}^{n} x_i y_i} \wedge_1^n \quad \omega(x_i y_i)$$

$$\leq \omega(x).$$

Hence $< \mu >< \nu > \; \leq \; < \mu\nu >$, thus $< \mu >< \nu > = < \mu\nu >$.

(3) Straightforward.

(4)

$$\begin{aligned} < x_r \circ y_s > &= < (xy)_{r \wedge s} > \\ &= < xy >_{r \wedge s} \\ &= (< x >< y >)_{r \wedge s} \\ &= < x >_r \circ < y >_s \\ &= < x_r > \circ < y_s > . \end{aligned}$$

In the following Proposition we prove some effects of homomorphisms on a fuzzy generating ideal.

Proposition 2.3.13.
Let $f : R \longrightarrow R'$ be a homomorphism and μ, μ' be fuzzy ideals of R and R' respectively then

(1) *If f is an epimorphism ,then $< f(\mu) > = f(< \mu >)$*
(2) $< f^{-1}(\mu') > = f^{-1} < \mu' >$

Proof.

(1) Since $\mu \leq < \mu >$ and $f(< \mu >)$ is a fuzzy ideal, $\quad < f(\mu) > \leq f(< \mu >)$.

Let $y \in R'$ then

$$f(<\mu>)(y) = \sup_{x\in f^{-1}(y)} <\mu>(x)$$

$$= \sup_{x\in f^{-1}(y)} \sup_{x=\Sigma_{i=1}^n r_i x_i} \wedge_1^n \mu(x_i)$$

$$= \sup_{\Sigma_{i=1}^n r_i x_i \in f^{-1}(y)} \wedge_1^n \mu(x_i)$$

$$= \sup_{y=\Sigma_{i=1}^n f(r_i) f(x_i)} \wedge_1^n \mu(x_i)$$

$$= \sup_{y=\Sigma_{i=1}^n f(r_i) y_i} \sup_{y_i = f(x_i)} \wedge_1^n \mu(x_i)$$

$$\leq \sup_{y=\Sigma_{i=1}^n f(r_i y_i)} \wedge_1^n \sup_{x_i\in f^{-1}(y_i)} \mu(x_i)$$

$$= \sup_{y=\Sigma_{i=1}^n f(r_i) y_i} \wedge_1^n f(\mu)(y_i)$$

$$\leq \sup_{y=\Sigma_{i=1}^n s_i y_i} \wedge_1^n f(\mu)(y_i)$$

$$=< f(\mu) > (y).$$

Hence the result follows.

(2) Straightforward.

§ 2.4 Fuzzy equivalence and congruence relations of R.

Fuzzy equivalence and congruence relations are defined and studied on groups in [Mak 1] , [Nob 1]. Here we extend the same concepts in a ring- theoretical situation. We establish that there is a one-to-one correspondence between the set of all fuzzy ideals of R and the set of all fuzzy congruence relations on R. From these ideas; we define a fuzzy quotient ring with respect to a given fuzzy congruence relation.

Definition 2.4.1.
A function $\alpha : R \times R \longrightarrow I$ is called a *fuzzy relation* on R.

If $\forall x \in R \quad \alpha(x,x) = 1$, then we say α is *fuzzy reflexive,*
If $\forall x, y \in R \quad \alpha(x,y) = \alpha(y,x)$, then we say α is *fuzzy symmetric,*
If $\forall x, y \in R \quad \alpha(x,y) \geq \sup_{z \in R} [\alpha(x,z) \wedge \alpha(z,y)]$, then we say α is *fuzzy transitive* .

A fuzzy relation on R is a *fuzzy equivalence* if α is fuzzy reflexive, symmetric and transitive.

Let $\mathcal{FR}$ be the set of all fuzzy relations on R. We define three operations on $\mathcal{FR}$ which are addition, multiplication and composition as follows:

Let $\alpha, \beta \in \mathcal{FR}$ and $(x,y) \in R \times R$ then

(1)
$$(\alpha + \beta)(x,y) = \sup_{(x,y)=(x_1+x_2, y_1+y_2)} [\alpha(x_1,y_1) \wedge \beta(x_2,y_2)] ,$$

(2)
$$(\alpha\beta)(x,y) = \sup_{(x,y)=(x_1x_2, y_1y_2)} [\alpha(x_1,y_1) \wedge \beta(x_2,y_2)] ,$$

(3)
$$(\alpha \circ \beta)(x,y) = \sup_{z \in R} [\alpha(x,z) \wedge \beta(z,y)] .$$

The following can be readily checked:

(1) α is a fuzzy transitive if and only if $\alpha \geq \alpha \circ \alpha$,
(2) If α is a fuzzy equivalence relation, then $\alpha \circ \alpha = \alpha$ and for each $t \in [0,1]$ the level subsets $\alpha_t = \{(x,y) \in RXR : \alpha(x,y) \geq t\}$ is a crisp equivalence relation on R.

Definition 2.4.2.
A fuzzy equivalence relation on R is called a *fuzzy congruence* relation if $\forall (x,y) \in R \times R, r \in R$

$$\alpha(x+r, y+r) \geq \alpha(x,y) \qquad \text{and}$$

$$\alpha(xr, yr) \geq \alpha(x,y).$$

In the following Proposition we characterize the fuzzy congruence relation in terms of the operations addition and multiplication.

Proposition 2.4.3.
A fuzzy equivalence relation α on R is a fuzzy congruence if and only if $\alpha \geq \alpha + \alpha$ and $\alpha \geq \alpha\alpha$.

Proof.
The sufficient part is obvious. To prove the necessary part, let $(x,y) \in R \times R$, then

$$\alpha\alpha(x,y) = \sup_{(x,y)=(x_1x_2,y_1y_2)} [\alpha(x_1,y_1) \wedge \alpha(x_2,y_2)]$$

$$\leq \sup_{(x,y)=(x_1x_2,y_1y_2)} [\alpha(x_1x_2,y_1x_2) \wedge \alpha(y_1x_2,y_1y_2)]$$

$$\leq \sup_{(x,y)=(x_1x_2,y_1y_2)} \alpha(x_1x_2,y_1y_2)$$

$$= \alpha(x,y).$$

Thus $\alpha\alpha \leq \alpha$, similarly we can have $\alpha + \alpha \leq \alpha$.

□

The following Proposition can be easily proved.

Proposition 2.4.4.
A relation S on R is an equivalence relation (congruence) if and only if the characteristic function $\mathcal{X}_S$ is a fuzzy equivalence (congruence) relation on R.

It is easy to check that a fuzzy equivalence relation α on R is a congruence if and only if α is a fuzzy subring of $R \times R$.

Proposition 2.4.5.
Let α be a fuzzy congruence relation on R then for all $x, y, r \in R$

(1) $\alpha(r+x, r+y) = \alpha(x, y)$,
(2) *If r is a unit, then* $\alpha(rx, ry) = \alpha(x, y)$,
(3) $\alpha(x, y) = \alpha(-x, -y)$.

Proof.

(1) By the Definition [2.4.2] , $\alpha(x+r, y+r) \geq \alpha(x, y)$ and for the converse, $\alpha(x, y) = \alpha(x+r-r, y+r-r) \geq \alpha(x+r, y+r)$.
(2) Follows by the same technique as in (1).
(3) $\alpha(-x, -y) = \alpha(-1x, -1y) \geq \alpha(x, y)$. Since it is true for all $(x, y) \in R \times R$, we have $\alpha(-x, -y) = \alpha(x, y)$.

□

We now prove in the following Proposition that for a given fuzzy congruence relation α, there exists a unique fuzzy ideal μ of R. In other words fuzzy congruence relations can be characterized in terms of fuzzy ideals.

Theorem 2.4.6.
Let α be a fuzzy congruence relation on R, then there exists a unique fuzzy ideal μ of R such that $\alpha(x, y) = \mu(x-y)$.

Proof.
Define a fuzzy subset $\mu : R \longrightarrow I$ by

$$\mu(x) = \alpha(x, 0) \quad \text{for all} \quad x \in R.$$

Let $x, y \in R$ then

$$\begin{aligned} \mu(x-y) &= \alpha(x-y, 0) \\ &\geq (\alpha+\alpha)(x-y, 0) \\ &\geq \alpha(x, 0) \wedge \alpha(-y, 0) \\ &= \alpha(x, 0) \wedge \alpha(y, 0) \\ &= \mu(x) \wedge \mu(y). \end{aligned}$$

$$\begin{aligned} \mu(xy) = \alpha(xy, 0) &\geq \alpha\alpha(xy, 0) \\ &\geq \alpha(x, 0) \wedge \alpha(y, y) \\ &= \alpha(x, 0) = \mu(x). \end{aligned}$$

Similarly, we can show $\mu(xy) \geq \mu(y)$; so $\mu(xy) \geq \mu(x) \vee \mu(y)$. Hence μ is a fuzzy ideal of R.

Further

$$\mu(x-y) = \alpha(x-y,0) = \alpha(x-y+y, 0+y) \text{ by the Proposition 2.4.5}$$
$$= \alpha(x,y)$$

and $\mu(0) = 1$. It is easy to check the uniqueness part.

□

The converse of the above Proposition is also true. i.e. every fuzzy ideal determines a fuzzy congruence relation on R, as shown below.

Theorem 2.4.7.
Let μ be a fuzzy ideal of R with $\mu(0) = 1$. Then the fuzzy relation $\alpha : R \times R \longrightarrow I$ defined by $\alpha(x,y) = \mu(x-y)$ is a fuzzy congruence relation on R.

Proof.
Let $x, y \in R$ then

$$\alpha(x,x) = \mu(x-x) = \mu(0) = 1.$$

$$\alpha(x,y) = \mu(x-y) = \mu(-(x-y)) = \mu(y-x) = \alpha(y,x).$$

$$\sup_{z\in R}[\alpha(x,z) \wedge \alpha(z,y)] = \sup_{z\in R}[\mu(x-z) \wedge \mu(z-y)]$$

$$\leq \sup_{z\in R}\mu(x-y)$$

$$= \mu(x-y) = \alpha(x,y).$$

Thus α is a fuzzy equivalence relation on R.
Now consider

$$\alpha(r+x, r+y) = \mu(r+x-(r+y)) = \mu(x-y) = \alpha(x,y)$$

$$\alpha(rx, ry) = \mu(rx-ry) \geq \mu(x-y) = \alpha(x,y)$$

hence α is a fuzzy congruence relation on R.

Remark 1.
If α, β are two fuzzy congruence relations on R, then $\alpha \circ \beta = \beta \circ \alpha$.

Let $(x, y) \in R \times R$ then

$$\alpha \circ \beta(x, y) = \sup_{z \in R} [\alpha(x, z) \wedge \beta(z, y)]$$

$$= \sup_{z \in R} [\alpha(x - z, 0) \wedge \beta(0, y - z)]$$

$$= \sup_{z \in R} [\beta(x, x + y - z) \wedge \alpha(x + y - z, y)]$$

$$= \sup_{x+y-z \in R} [\beta(x, x + y - z) \wedge \alpha(x + y - z, y)]$$

$$= \beta \circ \alpha(x, y).$$

Remark 2.
If μ, ν are two fuzzy, ideals of R then $\alpha_\mu \circ \alpha_\nu = \alpha_{\mu+\nu}$ where α_μ, α_ν and $\alpha_{\mu+\nu}$ are fuzzy congruences relations induced by μ, ν and $\mu + \nu$ respectively.

Let $(x, y) \in R \times R$ then

$$\alpha_\mu \circ \alpha_\nu(x, y) = \sup_{z \in R} [\alpha_\mu(x, z) \wedge \alpha_\nu(z, y)]$$

$$= \sup_{z \in R} [\mu(x - z) \wedge \nu(z - y)]$$

$$= \sup_{\substack{a = x - z \\ b = z - y}} \mu(a) \wedge \nu(b)$$

$$= \sup_{a + b = x - y} \mu(a) \wedge \nu(b)$$

$$= (\mu + \nu)(x - y)$$

$$= \alpha_{\mu+\nu}(x, y).$$

Let α be a fuzzy congruence relation on R, then the level subset α_t, $t \in Im(\alpha)$ gives rise to an equivalence relation on R. Thus the ring R can be divided into disjoint equivalence classes with respect to the equivalence relation α_t. Let us denote $\alpha_t[r]$ the equivalence class corresponding to r, $r \in R$, i.e. $\alpha_t[r] = \{x \in R : \alpha(x, r) \geq t\}$.
Now we define a fuzzy subset $\mu_r : R \longrightarrow I$ by

$$\mu_r(x) = \alpha(r, x)$$

and let $R/\alpha = \{\mu_r : r \in R\}$.

Theorem 2.4.8.
Let α be a fuzzy congruence relation on R then

(1) $\alpha(r,s) = 0$ *if and only if* $\mu_r \wedge \mu_s = 0$,

(2) $\bigvee_{r \in R} \mu_r = \mathcal{X}_R$,

(3) $\alpha(r,s) = 1$ *if and only if* $\mu_r = \mu_s$ *if and only if* $\alpha_t[r] = \alpha_t[s]$.

Proof.

(1) Since α is fuzzy transitive

$$\sup_{z \in R} \mu_r(z) \wedge \mu_s(z) = \sup_{z \in R} \alpha(r,z) \wedge \alpha(z,s) \leq \alpha(r,s) = 0,$$

hence $\mu_r(z) \wedge \mu_s(z) = 0$ for all $z \in R$, which implies $\mu_r \wedge \mu_s = 0$. For the converse part,

$$0 = \sup_{z \in R} \mu_r(z) \wedge \mu_s(z) = \sup_{z \in R} \alpha(r,z) \wedge \alpha(z,s) \geq \alpha(r,s) \wedge \alpha(s,s) = \alpha(r,s),$$

thus $\alpha(r,s) = 0$.

(2) Straightforward.

(3) Suppose $\alpha(r,s) = 1$ then

$$\mu_r(x) = \alpha(r,x) \geq \sup_{z \in R} \alpha(r,z) \wedge \alpha(z,x) \geq \alpha(r,s) \wedge \alpha(s,x) = \alpha(s,x) = \mu_s(x).$$

Hence $\mu_r \geq \mu_s$. Similarly we can show $\mu_s \geq \mu_r$.
Conversely suppose $\mu_r = \mu_s$ then for all $z \in R$, $\alpha(r,z) = \alpha(s,z)$; so $\alpha(r,s) \geq \sup_{z \in R} \alpha(r,z) \wedge \alpha(z,s) = \sup_{z \in R} \alpha(r,z) = 1$.

We now prove the rest of (3). Let $\mu_r = \mu_s$ and $x \in \alpha_t[r]$ then $\alpha(s,x) = \mu_s(x) = \mu_r(x) = \alpha(r,x) \geq t$; so $x \in \alpha_t[s]$. Similarly we can show the converse part and hence $\alpha_t[r] = \alpha_t[s]$. Now, suppose $\alpha_t[r] = \alpha_t[s]$. Let $t = \mu_r(x) = \alpha(r,x)$, then $x \in \alpha_t[r]$; so $x \in \alpha_t[s]$, hence $\mu_s(x) = \alpha(s,x) \geq t$. *i.e.*$\mu_s(x) \geq \mu_r(x)$. Similarly $\mu_r(x) \geq \mu_s(x)$. Hence the result follows.

□

We define two operations, sum and multiplication on R/α as follows:

$$\mu_r + \mu_s = \mu_{r+s} \text{ and } \mu_r \mu_s = \mu_{rs}$$

We show below that these operations are well-defined. If $\mu_r = \mu_{r_1}$ and $\mu_s = \mu_{s_1}$, then

$$\begin{aligned}\mu_{r+s}(x) &= \alpha(r+s, x) = \alpha(r, x-s) = \alpha(r_1, x-s)\\ &= \alpha(r_1 - x, -s) = \alpha(s, x - r_1) = \alpha(s_1, x - r_1)\\ &= \alpha(r_1 + s_1, x) = \mu_{r_1+s_1}(x)\end{aligned}$$

and since $\alpha(r, r_1) = \alpha(s, s_1) = 1, \quad \alpha(rs, r_1 s_1) = 1,$

$$\mu_{rs}(x) = \alpha(rs, x) \geq \alpha(rs, r_1 s_1) \wedge \alpha(r_1 s_1, x) = \mu_{r_1 s_1}(x).$$

Similarly, $\mu_{r_1 s_1}(x) \geq \mu_{rs}(x)$ for all $x \in R$ thus $\mu_{rs} = \mu_{r_1 s_1}$

Furthermore it can be easily checked that the set R/α together with the above two operations forms a ring with zero element μ_0, where 0 is the zero element of the ring R. This ring R/α is called the *quotient ring of R induced by the fuzzy congruence relation α* on R.

It is interesting to note that V.N.Dixit, et al [Dix 2,5.2] and H.V.Kumbhojker and M.S.Bapat[Bho 4,3] defined a fuzzy coset of a fuzzy ideal μ by $(x+\mu)(r) = \mu(x-r)$. If μ is a fuzzy ideal induced by α according to the Theorem [2.4.6], then we have $(x+\mu)(r) = \mu(x-r) = \alpha(x,r) = \mu_r(x)$ for all $r \in R$ *i.e.*$\forall x \in R, \quad (x+\mu) = \mu_x$. Hence the collection of all fuzzy cosets of μ, denoted by R/μ forms a ring under suitably defined binary operations on R/μ. It is easy to check that $R/\alpha \cong R/\mu$.

Since for each $r \in R$ there exists a fuzzy subset μ_r of R in R/α, we define a map $\sigma_\alpha : R \longrightarrow R/\alpha$ by $\sigma_\alpha(r) = \mu_r$. It is obvious that σ_α is a homomorphism. We call it the *natural homomorphism induced by the congruence relation α.*

Corollary 2.4.9.
If μ is a fuzzy ideal of R, then $R/\mu_0 \cong R/\mu$.

Proposition 2.4.10.
Let $f : R \longrightarrow R'$ be a homomorphism and μ, μ' be fuzzy ideals of R and R' respectively such that $\mu(0) = \mu'(0) = 1$ and $f(\mu) \leq \mu'$. Then there exists a homomorphism $\bar{f} : R/\mu \longrightarrow R'/\mu'$ such that $\sigma'_\mu \circ f = \bar{f} \circ \sigma_\mu$ where $\sigma_\mu : R \longrightarrow R/\mu$ is a natural homomorphism obtained by μ.
In other words, the diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle\sigma_\mu}\downarrow & & {\scriptstyle\sigma_{\mu'}}\downarrow \\ R/\mu & \xrightarrow{\bar{f}} & R/\mu' \end{array}$$

commutes.

Proof.
Define $\bar{f} : R/\mu \longrightarrow R'/\mu'$ by

$$\bar{f}(x+\mu) = f(x) + \mu'.$$

If $x + \mu = y + \mu$, then $\mu_x = \mu_y$, hence by the Theorem[2.4.8,3] $\alpha_\mu(x, y) = 1$.

$$\begin{aligned}\alpha_{\mu'}(f(x), f(y)) &= \mu'(f(x) - f(y)) \\ &= \mu'(f(x - y)) \\ &= f^{-1}(\mu'(x - y)) \\ &\geq \mu(x - y) \quad \text{by the hypothesis} \\ &= \alpha_\mu(x, y) = 1.\end{aligned}$$

So $\alpha_{\mu'}(f(x), f(y)) = 1$. Hence by the same Theorem[2.4.8,3] $\mu'_{f(x)} = \mu'_{f(y)}$; therefore $\bar{f}$ is well-defined.

$$\sigma_{\mu'} \circ f(x) = \sigma_{\mu'}(f(x)) = f(x) + \mu' = \bar{f}(x + \mu) = \bar{f}(\sigma_\mu(x)) = \bar{f} \circ \sigma_\mu(x).$$

Moreover, it is easy to see that if f is surjective then so is $\bar{f}$.

Corollary 2.4.11.
If $f : R \longrightarrow R'$ is an epimorphism and μ, μ' are fuzzy ideals of R and R' respectively such that $\mu = f^{-1}(\mu')$, then $R/\mu \cong R'/\mu'$.

Proof.
By Proposition[1.2.2,3] $f(\mu) = f(f^{-1}(\mu')) = \mu'$ so by above Proposition $\bar{f} : R/\mu \longrightarrow R/\mu'$ is epimorphism. If $f(x) + \mu' = f(y) + \mu'$, then by the Theorem[2.4.8,3] $\alpha_{\mu'}(f(x), f(y)) = 1$ which implies that $\mu'(f(x - y)) = 1$.
Hence $\mu(x - y) = f^{-1}\mu'(x - y) = 1$ so $x + \mu = y + \mu$. Thus $\bar{f}$ is injective.

□

Let $f : R \longrightarrow R'$ be a homomorphism. We define *fuzzy kernel* of f as FKer$f$ = $f^{-1}(\mathcal{X}_{0'})$ where $0'$ is the zero element of R'. If $x \in$ Ker$f$, then $f(x) = 0'$ so $\mathcal{X}_{0'}(f(x)) = 1$, if $x \notin$ Ker f, then $f(x) \neq 0'$ so $\mathcal{X}_{0'}(f(x)) = 0$. Hence FKer$f = \mathcal{X}_{\text{Ker } f}$ and FKer$\sigma_\mu = \mathcal{X}_{\mu_0}$.

Corollary 2.4.12.
If $f : R \longrightarrow R'$ is an epimorphism, then $R/$ FKer $f \cong R'$.

Proof.
Since FKer$f = f^{-1}(\mathcal{X}_{0'})$, by the Corollary[2.4.11] $R/$ FKer$f \cong R'/\mathcal{X}_{0'}$. But we have $R'/\mathcal{X}_{0'} = \{r' + \mathcal{X}_{0'} : r' \in R'\}$ which is isomorphic to R'.

Corollary 2.4.13.
Let $f : R \longrightarrow R'$ be a homomorphism and μ be a fuzzy ideal of R then $f^{-1}(f(\mu)) = \mu +$ FKer$f$.

Proof.
Let $x \in R$.

$$\begin{aligned}(\mu + \text{FKer}f)(x) &= \sup_{x=x_1+x_2} \mu(x_1) \wedge \text{FKer}f(x_2) \\ &= \sup_{\substack{x=x_1+x_2 \\ x_2 \in \mathcal{X}_{\text{Ker}f}}} \mu(x_1) \\ &= \sup_{f(x)=f(x_1)} \mu(x_1) \\ &= f(\mu)(f(x)) \\ &= f^{-1}(f(\mu))(x).\end{aligned}$$

□

Corollary 2.4.14.
If μ, ν are fuzzy ideals of R such that $\mu \leq \nu$, then $R/\nu \cong (R/\mu)/\sigma_\mu(\nu)$.

Proof.
Let $\sigma_\mu : R \longrightarrow R/\mu$ be the natural homomorphism. Then $\text{FKer}\sigma_\mu = \mathcal{X}_{\mu_0}$ and by the above Corollary $\sigma_\mu^{-1}(\sigma_\mu(\nu)) = \nu +$ $\text{FKer}\sigma_\mu = \nu + \mathcal{X}_{\mu_0} = \nu$ since $\mathcal{X}_{\mu_0} \leq \mu \leq \nu$. By the Corollary[2.4.11] $R/\nu \cong (R/\mu)/\sigma_\mu(\nu)$.

□

Corollary 2.4.15[Muk 1].
A ring R is regular if and only if for any fuzzy ideals μ, ν of R $\mu \wedge \nu = \mu\nu$.

Proof.
We defined

$$\mu\nu(x) = \sup_{x=\Sigma_{i=1}^n x_i y_i} \wedge_1^n \mu(x_i) \wedge \nu(y_i).$$

Since R is regular, there exists an $a \in R$ such that $x = xax$; , therefore $\mu\nu(x) \geq \mu(xa) \wedge \nu(x) \geq \mu(x) \wedge \nu(x) = (\mu \wedge \nu)(x)$. Thus $\mu\nu = \mu \wedge \nu$. For the converse part, suppose A, B be any two ideals of R then clearly $AB \subseteq A \cap B$. Let $x \in A \cap B$ then $\mathcal{X}_A\mathcal{X}_B(x) = \mathcal{X}_A \wedge \mathcal{X}_B(x) = 1 = \sup_{x=\Sigma_{i=1}^n x_i y_i} \mathcal{X}_A(x_i) \wedge \mathcal{X}_B(y_i)$. Hence for some $x_i \in A, y_i \in B, x = \Sigma_{i=1}^n x_i y_i$. So $x \in AB$. Thus $A \cap B = AB$.

□

Corollary 2.4.16.
A ring R is regular if and only if every fuzzy ideal of R is idempotent.

Proof.
We prove only the sufficient part. Let μ, ν be any fuzzy ideals of R then

$$\begin{aligned}(\mu \wedge \nu)(x) &= (\mu \wedge \nu)^2(x) \\ &= \sup_{x=\Sigma_{i=1}^n x_i y_i} (\mu \wedge \nu)(x_i) \wedge (\mu \wedge \nu)(y_i) \\ &\leq \sup_{x=\Sigma_{i=1}^n x_i y_i} \mu(x_i) \wedge \nu(y_i) \\ &= (\mu\nu)(x).\end{aligned}$$

Hence $\mu \wedge \nu = \mu\nu$.

□

CHAPTER III

FUZZY PRIME, RADICAL AND SEMIPRIME IDEALS

In this section we develop some properties of fuzzy prime ideals. In particular, the image of a fuzzy prime ideal is a two element set. Conversely if for a fuzzy ideal μ with $Im(\mu) = \{1, t\}, 0 \leq t < 1$ and the base set μ_0 a crisp prime ideal, then μ is a fuzzy prime ideal. Fuzzy point characterisation of a prime ideal is given in Proposition[3.1.8]. Various Definitions of fuzzy prime ideals available in the literature are studied and the relationships between these are considered.

§ 3.1 Fuzzy prime ideals.

Definition 3.1.1 [Muk 1].
A fuzzy ideal μ of R is called a *fuzzy prime ideal* of R if for any fuzzy ideals ν, ω of R

$$\nu \circ \omega \leq \mu \quad \text{implies either } \nu \leq \mu \text{ or } \omega \leq \mu.$$

Some authors use the product operation instead of composition in the above Definition. But it does not really a matter since we proved in Proposition[2.3.12,3] that $\nu \circ \omega \leq \mu$ if and only if $\nu\omega \leq \mu$. In the following Proposition we prove some important properties of a fuzzy prime ideal which are basic consequences of the above Definition.

Theorem 3.1.2 [Mal 1, 2.1].
Let μ be a non-constant fuzzy prime ideal of R then

(1) $Im(\mu) = \{1, t\}, \quad 0 \leq t < 1 \quad$ *and* $\quad \mu(0) = 1,$
(2) $\mu_0 = \{x \in R : \mu(x) = \mu(0) = 1\}$ *is a prime ideal of R.*

Proof.

(1) Let $x, y \in R$ with $0 \leq \mu(x) < 1, 0 \leq \mu(y) < 1$.
In the following discussion, x and y are kept fixed.
Define $\nu, \mu : R \longrightarrow I$ by

$$\nu(r) = \begin{cases} 1 & \text{if} \quad r \in <x> \\ 0 & \text{otherwise} \end{cases}$$

and

$$\omega(r) = \mu(x) \quad \text{for all } r \in R.$$

Then it is obvious that ν, ω are fuzzy ideals of R, and also for all $r, s \in R$ $\nu(r) \wedge \omega(s) \leq \mu(rs)$, since if $\nu(r) = 0$ then $\nu(r) \wedge \omega(s) = 0 \leq \mu(rs)$, and if $\nu(r) = 1$ then $r = r_1 x$ for some $r_1 \in R$ and $\nu(r) \wedge \omega(s) = \omega(s) = \mu(x) \leq \mu(r_1 x s) = \mu(rs)$. Hence $\sup_{t=rs} \nu(r) \wedge \omega(s) \leq \mu(t)$ from which follows that $\nu \circ \omega \leq \mu$. Since μ is a fuzzy prime ideal of R either $\nu \leq \mu$ or $\omega \leq \mu$. But $\nu(x) = 1 > \mu(x)$ and hence $\omega \leq \mu$. Therefore $\omega(y) = \mu(x) \leq \mu(y)$. Similarly we can prove that $\mu(y) \leq \mu(x)$. Thus $\mu(x) = \mu(y)$, hence $|Im(\mu)| = 2$.

Suppose $\mu(0) < 1$. Since μ in non-constant, there exists an $a \in R$ such that $\mu(a) < \mu(0)$.
Define the fuzzy ideals $\nu, \omega : R \longrightarrow I$ by

$$\nu(r) = \begin{cases} 1 & \text{if} \quad r \in \mu_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\omega(r) = \mu(0) \qquad \text{for all } r \in R.$$

For all $r, s \in R$, $\nu(r) \wedge \omega(s) \leq \mu(rs)$ since if $\nu(r) = 1$, then $\nu(r) \wedge \omega(s) = \omega(s) = \mu(0) = \mu(r) = \mu(rs)$, and if $\nu(r) = 0$ then clearly $\nu(r) \wedge \omega(s) = 0 \leq \mu(rs)$. So, $\nu \circ \omega \leq \mu$. But $\nu(0) = 1 > \mu(0)$ and $\omega(a) = \mu(0) > \mu(a)$. Hence we have $\nu \circ \omega \leq \mu, \nu \nleq \mu$ and $\omega \nleq \mu$. This is a contradiction, thus $\mu(0) = 1$ and $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$.

(2) Let I, J be any ideals of R such that $IJ \subseteq \mu_0$, then $\mathcal{X}_I \wedge \mathcal{X}_J = \mathcal{X}_{IJ} \leq \mathcal{X}_{\mu_0} \leq \mu$. Since μ is a fuzzy prime ideal, either $\mathcal{X}_I \leq \mu$ or $\mathcal{X}_J \leq \mu$, which implies $I \subseteq \mu_0$ or $J \subseteq \mu_0$. Hence μ_0 is a prime ideal of R.

□

The converse of the above Proposition is also true as we show below .

Theorem 3.1.3[Mal 1, 2.3].
Let μ be a fuzzy ideal of R such that $Im(\mu) = \{1, t\}$, $0 \leq t < 1$ and μ_0 a Prime ideal of R, then μ is a fuzzy prime ideal.

Proof.
Let ν, ω be fuzzy ideals of R with $\nu \circ \omega \leq \mu$. Suppose $\nu \nleq \mu$ and $\omega \nleq \mu$ then there exist $x, y \in R$ such that $\nu(x) > \mu(x)$ and $\omega(y) > \mu(y)$. Hence it is obvious that $x \notin \mu_0$ and $y \notin \mu_0$. Since μ_0 is a proper prime ideal of R, there exists $r \in R$ such that $xry \notin \mu_0$. So $\mu(x) = \mu(y) = \mu(xry) = t$. But $\nu \circ \omega(xry) \geq \nu(x) \wedge \omega(ry) \geq \nu(x) \wedge \omega(y) > \mu(x) \wedge \mu(y) = t = \mu(xry)$ which is a contradiction to fact that $\nu \circ \omega \leq \mu$. So either $\nu \leq \mu$ or $\omega \leq \mu$.

□

It is easy to check that if μ is a fuzzy prime ideal of R then all of its level subsets are prime ideals of R, but the converse is not true.

Example 3.1.4.
Define a fuzzy ideal $\mu : \mathbb{Z} \longrightarrow I$ by $\mu(x) = 0.9$ if $x \in< 2 >$ and $\mu(x) = 0$ otherwise. Then clearly all of its level subsets are prime ideals of $\mathbb{Z}$ but μ is not a fuzzy prime ideal since $\mu(0) \neq 1$.

Corollary 3.1.5.
Let I be a non-empty subset of R, then $\mathcal{X}_I$ is a fuzzy prime ideal of R if and only if I is a prime ideal of R.

Corollary 3.1.6.
Every non-constant fuzzy ideal of R with $\mu(0) = 1$ is prime if and only if R is a field.

Proof. ($\Leftarrow$)
Let $x \in R \setminus \{0\}$ then $\mu(x) = \mu(1x) \geq \mu(1) = \mu(xx^{-1}) \geq \mu(x,.$ $i.e. \mu(x) = \mu(1)$ for all $x \in R \setminus 0$ from which follows that $Im(\mu) = \{1, \mu(1)\}$ and $\mu_0 = \{0\}$ is a prime ideal of R. Hence μ is a fuzzy prime ideal of R. The converse is straightforward.

□

In general, the intersection of two fuzzy prime ideals need not be fuzzy prime. The same example as in the crisp case will fit here.

Proposition 3.1.7.
Let μ, ν be fuzzy ideals of R then $\mu \wedge \nu$ is a fuzzy prime ideal if and only if either $\nu \leq \mu$ or $\mu \leq \nu$.

The proof is omitted since $\mu\nu \leq \mu \wedge \nu$.

Proposition 3.1.8[Dix 1, 4.1].
If $\{\mu_i\}_{i \in I}$ is a chain of fuzzy prime ideals of R, then $\bigwedge\limits_{i \in I} \mu_i$ and $\bigvee\limits_{i \in I} \mu_i$ are fuzzy prime ideals of R.

Proof.
Since for all $i \in I$, $\mu_i(0) = 1$ we have $(\bigvee\limits_{i \in I} \mu_i)(0) = 1$. Since $\{(\mu_i)_0\}_{i \in I}$ is a chain, $\bigcup\limits_{i \in I}(\mu_i)_0$ is a prime ideal of R. Let ν, ω be fuzzy ideals of R such that $\nu \circ \omega \leq \bigvee\limits_{i \in I} \mu_i$. Suppose $\nu \nleq \bigvee\limits_{i \in I} \mu_i, \omega \nleq \bigvee\limits_{i \in I} \mu_i$, then there exist $x, y \in R$ such that $\nu(x) > \left(\bigvee\limits_{i \in I} \mu_i\right)(x)$ and $\omega(y) > \left(\bigvee\limits_{i \in I} \mu_i\right)(y)$ which implies $x, y \notin \left(\bigvee\limits_{i \in I} \mu_i\right)_0$, so $x, y \notin \bigcup\limits_{i \in I}(\mu_i)_0$. Since $\bigcup\limits_{i \in I}(\mu_i)_0$ is prime ideal, $xy \notin \bigcup\limits_{i \in I}(\mu_i)_0$. Therefore $\mu_i(x) = \mu_i(y) = \mu_i(xy) = t_i$ for all $i \in I$. Hence $\left(\bigvee\limits_{i \in I} \mu_i\right)(x) = \left(\bigvee\limits_{i \in I} \mu_i\right)(y) = \left(\bigvee\limits_{i \in I} \mu_i\right)(xy) = \sup\limits_{i \in I} t_i$, implying

$\nu \circ \omega(xy) \geq \nu(x) \wedge \omega(y) > \left(\bigvee_{i\in I} \mu_i\right)(xy)$, which is a contradiction. Thus the result follows. It is easy to show that $\bigwedge_{i\in I} \mu_i$ is a fuzzy prime ideal of R.

□

We now turn our attention to the equivalence of different definitions available in the literature for a fuzzy prime ideal.

Proposition 3.1.9[Bho 1, 3].
Let μ be a fuzzy ideal of R. Then

(1) *μ is a non-constant fuzzy prime ideal if and only if for all fuzzy points $x_r, y_s; r, s \in [0,1]$, $x_r \circ y_s \in \mu$ implies either $x_r \in \mu$ or $y_s \in \mu$,*

(2) *If μ is fuzzy prime ideal, then for all $x, y \in R$, either $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y)$,*

(3) *If μ is a fuzzy prime ideal of R, then for all $x, y \in R$, $\mu(xy) = \mu(0)$ implies either $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$.*

Proof.

(1) Let μ be a non-constant fuzzy prime ideal and x_r, y_s be fuzzy points with $x_r \circ y_s \in \mu$ then $< x_r \circ y_s > \leq \mu$. Hence by Proposition[2.3.12,4] $< x_r > \circ < y_s > \leq \mu$ hence either $< x_r > \leq \mu$ or $< y_s > \leq \mu$. Thus either $x_r \in \mu$ or $y_s \in \mu$. To show the converse part, let ν, ω be fuzzy ideals of R such that $\nu \circ \omega \leq \mu$. Suppose $\nu \nleq \mu$ and $\omega \nleq \mu$ then there exist $x, y \in R$ such that $\nu(x) > \mu(x)$ and $\omega(y) > \mu(y)$. Hence $x_{\nu(x)} \notin \mu$ and $y_{\omega(y)} \notin \mu$, but

$$\begin{aligned}\mu(xy) &\geq (\nu \circ \omega)(xy)\\ &\geq \nu(x) \wedge \omega(y)\\ &= (xy)_{(\nu(x)\wedge\omega(y))}(xy)\\ &= \left(x_{\nu(x)} \circ y_{\omega(y)}\right)(xy)\end{aligned}$$

i.e. $x_{\nu(x)} \circ y_{\omega(y)} \in \mu$. So either $x_{\nu(x)} \in \mu$ or $y_{\omega(y)} \in \mu$, which is a contradiction.

(2) Suppose μ is a fuzzy prime ideal of R. Then by Theorem[3.1.2], $Im(\mu) = \{1, t\}$, for some $t \in [0,1)$ and μ_0 is a prime ideal of R. Let $x, y \in R$. If $xy \in \mu_0$, then either $x \in \mu_0$ or $y \in \mu_0$. Hence either $\mu(xy) = \mu(x)$ or $\mu(y)$. If $xy \notin \mu_0$, then $\mu(xy) = t \geq \mu(x) \wedge \mu(y)$. If $\mu(x) = 1$ and $\mu(y) = 1$, then $t \geq 1$, which is a contradiction. Therefore either $\mu(x) = t$ or $\mu(y) = t$. Thus either $\mu(xy) = \mu(x)$ or $\mu(y)$.

(3) Straightforward.

□

Remark.
The converse of the parts (2),(3) in the above Proposition are not true in general:

Example 3.1.10.

1. Define a fuzzy ideal $\mu : \mathbb{Z} \longrightarrow I$ by

$$\mu(x) = \begin{cases} 0.9 & \text{if} \quad x = 0 \\ 0.7 & \text{for all} \quad x \in <2> \setminus \{0\} \\ 0 & \text{for all} \quad x \in \mathbb{Z} \setminus <2> . \end{cases}$$

Then clearly for all $x, y \in \mathbb{Z}$, either $\mu(xy) = \mu(x)$ or $\mu(y)$; but μ is not a fuzzy prime ideal since $\mu(0) \neq 1$.

2. Define a fuzzy ideal $\mu : \mathbb{Z} \longrightarrow I$ by

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0.7 & \text{for all} \quad x \in <4> \setminus \{0\} \\ 0 & \text{for all} \quad x \in \mathbb{Z} \setminus <4> . \end{cases}$$

Then for all $x, y \in \mathbb{Z}$, $\mu(xy) = \mu(0)$ implies either $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$; but $\mu(2.2) = \mu(4) = 0.7$ and $\mu(2) \neq \mu(4)$. This shows that converse of [3.1.9, (3)] is not true in general.

Remark.

It is interesting to note that (2),(3) in the above Proposition serve as definitions for a fuzzy prime ideal in Yue[Zha 1, 2.3] and H.V.Kumbhojkar[Bho 1, 4.1] respectively. Since a fuzzy prime ideal can take only two distinct values and one of these is 1, one cannot say this definition really fuzzifies the notion of prime ideals. It would be more appropriate if we take the condition (2) in Proposition[3.1.9] as a definition for a fuzzy prime ideal. But unfortunately there are not many results that one can obtain from such a definition. However one can develop the notion of fuzzy primary, semiprimary, primary decomposition and irreducibility under the weak definition as proposed in [3.1.1]. This is exactly what we find in papers by Kumar[Dix1], [Kum 1], [Kum 5], Malik and Mordeson[Mal 1], [Mal 2], [Mal 3], [Mal 4]. The weak definition was used extensively in their work.

It can be easily checked that if μ is a fuzzy ideal of R and $f : R \longrightarrow I$ is a homomorphism, then μ is $f-$ invariant if and only if μ is constant on Ker$f$ and equal to $\mu(0)$ on Ker$f$ since $0 \in$ Ker$f$.

Proposition 3.1.11.

Let $f : R \longrightarrow R'$ be a epimorphism and μ be a $f-$invariant fuzzy prime ideal of R then $f(\mu)$ is a fuzzy prime ideal of R'.

Proof.

Let μ be a fuzzy prime ideal of R then by Theorem[3.1.2], $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$ and μ_0 is prime ideal of R. Since μ is $f-$ invariant , Ker$f \subseteq \mu_0$ hence $f(\mu_0)$

is a prime ideal of R'. Let $y \in R'$. If $y \in f(\mu_0)$, then there exists $x' \in \mu_0$ such that $f(x') = y$. So $f(\mu)(y) = \sup_{f(x)=y} \mu(x) \geq \mu(x') = 1$, hence $f(\mu)(y) = 1$. If $y \notin f(\mu_0)$, then for all $x \in R$ such that $f(x) = y, x \notin \mu_0$. So $\mu(x) = t$. It follows from the definition of $f(\mu)$ that $f(\mu)(y) = t$. Thus we have $Imf(\mu) = \{1, t\}, t \in [0, 1)$ and $f(\mu_0) = (f(\mu))_0$ is a prime ideal of R'. By Theorem[3.1.3], $f(\mu)$ is a fuzzy prime ideal of R'.

□

The following Proposition can be proved by a similar kind of argument as in the Proposition above.

Proposition 3.1.12.
Let $f : R \longrightarrow I$ be a homomorphism and μ' be a fuzzy prime ideal of R' then $f^{-1}(\mu')$ is a fuzzy prime ideal of R.

Remark.
If f is an epimorphism and μ is $f-$invariant, then by Proposition[1.2.2, 3,4], $f^{-1}(f(\mu)) = \mu$ and $f(f^{-1}(\mu')) = \mu'$ which leads us to the conclusion that μ is a fuzzy prime ideal of R if and only if $f(\mu)$ is fuzzy prime ideal of R', and μ' is fuzzy prime ideal of R' if and only if $f^{-1}(\mu')$ is fuzzy prime ideal of R. Thus there is a one-to-one correspondence between the set of all $f-$invariant fuzzy prime ideals of R and the set of all fuzzy prime ideals of R'.

Proposition 3.1.13.
Let μ be a fuzzy ideal of R. For all $x, y \in R$, either $\mu(xy) = \mu(x)$ or $\mu(y)$ if and only if every level subset μ_t is a prime ideal of R.

Proof.
The necessary part is straightforward. For the sufficient part, suppose μ_t is a prime ideal for all $t \in Im(\mu)$. Let $x, y \in R$, then if $\mu(xy) = \mu(1)$ either $\mu(x) = \mu(1)$ or $\mu(y) = \mu(1)$, since $\mu(1) \leq \mu(r)$ for all $r \in R$. If $\mu(xy) = t$ for some $t \in (\mu(1), 1]$, then $xy \in \mu_t$, which implies either $x \in \mu_t$ or $y \in \mu_t$. So $\mu(x) \geq t = \mu(xy)$ or $\mu(y) \geq t = \mu(xy)$. Thus the result follows.

□

It should be noted that the above Proposition is not true for a fuzzy prime ideal defined as in [3.1.1].

Proposition 3.1.14.
Let μ be a non-constant fuzzy ideal of a PID R with $\mu(0) = 1$ and $\mu_0 \neq \{0\}$. Then for all $x, y \in R$, either $\mu(xy) = \mu(x)$ or $\mu(y)$ if and only if $\mu(xy) = \mu(0)$ implies either $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$.

Proof.
Suppose for all $x, y \in R$, $\mu(xy) = \mu(0)$ implies either $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$.

Then μ_0 is a prime ideal of R. Since $\mu_0 \neq \{0\}, \mu_0 \neq R$ and R is PID, μ_0 is a maximal ideal of R. Since μ is non-constant there exists $r \in R$ such that $\mu(r) = t \neq \mu(0)$. If $\mu_t \neq R$, then $\mu_0 \subsetneq \mu_t$ which is a contradiction to the maximality of μ_0. So $\mu_t = R$, hence $Im(\mu) = \{1, t\}$. By Theorem[3.1.3] μ is a fuzzy prime ideal and thus the result follows by Proposition[3.1.9,(3)]. The converse part is straightforward.

□

Proposition 3.1.15.
Let μ be a fuzzy ideal of R. Then

(1) *If μ is a fuzzy prime ideal, then R/α_μ is an integral domain,*
(2) *If R is Noetherian (Artinian) ring, then so is R/α_μ.*

Proof.

(1) By Corollary[2.4.9] $R/\mu_0 \cong R/\alpha_\mu$. It is clear that if μ is fuzzy prime, then R/μ_0 is an integral domain, hence so is R/α_μ.
(2) The proof is straightforward since $\sigma_{\alpha_\mu} : R \longrightarrow R/\alpha_\mu$ is a homomorphism and any homomorphic image of a Noetherian (Artinian) ring is also Noetherian (Artinian) ring.

□

Remark.
In[Dix 1 4.8] V.N.Dixit et al. it is stated that the converse of part (1) in the above Proposition is true, but we disprove that it is not true in general as follows:

Let R denote the ring $\mathbb{Z}[x, y]$. Define a fuzzy ideal $\mu : R \longrightarrow I$ by

$$\mu(f) = \begin{cases} 0.9 & \text{if} \quad f \in <x> \\ 0.8 & \text{if} \quad f \in <x, xy> \setminus <x> \\ 0.7 & \text{otherwise} . \end{cases}$$

Then $\mu_0 = \{f \in R : \mu(f) = \mu(0)\} = <x>$ is a prime ideal of R. It follows that the quotient ring R/μ_0 is an integral domain. By Corollary[2.4.9], $R/\alpha_\mu \cong R/\mu_0$ and hence R/α_μ is an integral domain. But it obvious that μ is not a fuzzy prime ideal by Theorem[3.1.2] since $\mu(0) \neq 1$ and $|Im(\mu)| > 2$.

§ 3.2 R-fuzzy prime subsets .

Since Definition[3.1.1] fails to fuzzify the concept of a prime ideal as we mentioned earlier, we make an attempt to allow μ to take more than two values. We introduce a new terminology: *R- fuzzy prime subset* of R and prove that the fuzzy ideal generated by R-fuzzy prime subset is a fuzzy prime ideal. Hopefully the concept of R-fuzzy prime subset can be extended furthermore.

Definition 3.2.1.
A fuzzy subset μ is called an *R-fuzzy prime subset* of R if for any fuzzy subsets ν, ω of R

$$<\nu\omega> \leq <\mu> \quad \text{implies} \quad \text{either} \quad <\nu> \leq <\mu> \quad \text{or} \quad <\omega> \leq <\mu>.$$

Proposition 3.2.2.
A fuzzy subset μ is an R-fuzzy prime subset if and only if $<\mu>$ is a fuzzy prime ideal.

Proof.
For the necessary part, let ν, ω be fuzzy ideals such that $\nu\omega \leq <\mu>$. It follows that $<\nu\omega> \leq <\mu>$. Hence by the definition either $\nu = <\nu> \leq <\mu>$ or $\omega = <\omega> \leq <\mu>$. Thus $<\mu>$ is a fuzzy prime ideal. To prove the sufficient part, let ν, ω be fuzzy subsets of R such that $<\nu\omega> \leq <\mu>$. By Proposition[2.3.12(2)] $<\nu><\omega> \leq <\mu>$ So either $<\nu> \leq <\mu>$ or $<\omega> \leq <\mu>$. Thus the result follows.

□

The following example illustrates the fact that R-fuzzy prime subset μ can have more than two values but the fuzzy prime ideal generated by μ takes two values.

Example 3.2.3.
Define $\mu : \mathbb{Z}_8 \longrightarrow I$ by

$$\mu(0) = 0.5,\ \mu(1) = 0.4, \mu(2) = 1, \mu(3) = 0.7, \mu(4) = \mu(5) = 0.3 \text{ and } \mu(6) = \mu(7) = 0$$

then

$$<\mu>(0) = <\mu>(2) = <\mu>(4) = <\mu>(6) = 1$$
$$\text{and} \quad <\mu>(1) = <\mu>(3) = <\mu>(5) = <\mu>(7) = 0.7.$$

Hence $<\mu>_0 = \{0, 2, 4, 6\}$ is a prime ideal of $\mathbb{Z}_8$ and $Im(<\mu>) = \{1, 0.7\}$. So $<\mu>$ is a fuzzy prime ideal, and hence μ is $\mathbb{Z}_8$-fuzzy prime subset.

Proposition 3.2.4.

(1) *Let $x \in R$. Then x is a prime element if and only if $\mathcal{X}_{\{x\}}$ is an R- fuzzy prime subset of R,*

(2) *If μ is an R-fuzzy prime subset of R, then $\sup_{x \in R} \mu(x) = 1$,*

(3) *Every fuzzy subset μ of R with $\sup_{x \in R} \mu(x) = 1$ and $< \mu >$ non-constant is an R-fuzzy prime subset if and only if R is a field .*

(1),(2) are straightforward and (3)is an immediate consequence of Proposition[3.1.6]

§ 3.3 Fuzzy radicals.

In this section, we deal with two different kinds of radicals known as nil radicals and prime radical. For a fuzzy ideal μ, we prove equivalent characterisations of fuzzy radicals $\sqrt{\mu}$ in terms of fuzzy points, fuzzy ideals and values of $\mu(x^n)$. Secondly we show if μ is a fuzzy ideal then so is $\sqrt{\mu}$. Further we prove some fuzzy analogues of well-known results in the crisp case. The effect of homomorphisms on fuzzy radicals is also discussed in this section.

Definition 3.3.1.
Let μ be a fuzzy ideal of R. The *fuzzy nil radical* , denoted by $\sqrt{\mu}$ is defined as

$$\sqrt{\mu}(x) = \sup\{t \in Im(\mu) : x \in \sqrt{\mu_t}\}.$$

In the following Proposition we give other characterizations of a fuzzy nil radical.

Theorem 3.3.2.
Let μ be a fuzzy ideal of R then

$$\begin{aligned}\sqrt{\mu}(x) &= \sup\{\mu(x^n) : n > 0\}\\ &= \sup\{t : t \in [0,1], \exists n \in \mathbb{N} \text{ such that } \mu(x^n) \geq t\}\\ &= \sup\{r_t : r \in R, t \in [0,1], \exists n \in \mathbb{N} \text{ such that } (r_t)^n \leq \mu\}\\ &= \sup\{\nu : \nu \text{ is fuzzy ideal of } R , \exists n \in \mathbb{N} \text{ such that } \nu^n \leq \mu\}.\end{aligned}$$

Proof.
For $x \in R$, let $\nu(x) = \sup\{\mu(x^n) : n > 0\}$ and $\sqrt{\mu}(x) = s$. If for some $n \in \mathbb{N}$, $r = \mu(x^n) > s$, then $x \in \sqrt{\mu_r}$, which implies $\sqrt{\mu}(x) = \sup\{t \in Im(\mu) : x \in \sqrt{\mu}_t\} \geq r$, i.e. $s \geq r$. This is a contradiction and hence for all $n \in \mathbb{N}$ $\mu(x^n) \leq s$. Thus for all $x \in R$ $\nu(x) \leq \sqrt{\mu}(x)$. Now suppose $\nu(x) < \sqrt{\mu}(x)$. Since $(\sqrt{\mu})(x) = \sup\{t \in Im(\mu) : x \in \sqrt{\mu}_t\}$, there exists $m \in Im(\mu)$ such that $x \in \sqrt{\mu}_m$ and $\nu(x) < m \leq (\sqrt{\mu})(x)$. Hence there exists an integer n_1 such that $\mu(x^{n_1}) \geq m$. So $\nu(x) \geq \mu(x^{n_1}) \geq m$, which is a contradiction. Thus $\nu(x) = \sqrt{\mu}(x)$ for all $x \in R$. The second characterization is an immediate consequence of the definition.
For the third, let $r_t \in \mathcal{D} = \{r_t : t \in [0,1], \exists n \in \mathbb{N} \text{ such that } (r_t)^n \leq \mu\}$. Then $\mu(r^n) \geq t$ since $(r_t)^n = (r^n)_t$, implying $\sqrt{\mu}(r) = \sup\{\mu(r^n) : n > 0\} \geq \mu(r^n) \geq t$. So $r_t \in \sqrt{\mu}$, and hence $\sup \mathcal{D} \leq \sqrt{\mu}$. On the other hand, let $x \in R$ then

$$\begin{aligned}\sqrt{\mu}(x) &= \sup\{\mu(x^n) : n > 0\}\\ &= \sup\{x_{\mu(x^n)}(x) : n > 0\}\\ &\leq \sup\{r_t(x) : r \in R, t \in [0,1], \exists n \in N \text{ such that } (r_t)^n \leq \mu\}\\ &= \sup \mathcal{D}(x).\end{aligned}$$

Hence $\sqrt{\mu} \leq \sup \mathcal{D}$.
For the last part, let $\mathcal{E} = \{\nu : \nu$ is a fuzzy ideal of R , $\exists n \in \mathbb{N}$ such that $\nu^n \leq \mu\}$. Then it is is easy to check that $\sqrt{\mu} = \sup \mathcal{D} \leq \sup \mathcal{E}$. Let $\nu \in \mathcal{E}$ then $\nu(x) \leq \nu^n(x^n) \leq \mu(x^n) \leq \sqrt{\mu}(x)$. So $\nu \leq \sqrt{\mu}$. Hence $\sup \mathcal{E} \leq \sqrt{\mu}$. Thus the result follows.

□

Remark.
$(\sqrt{\mu})(x) = \sup\{\mu(x^n) : n > 0\}$ is the definition of a radical according to [Bho 1,7.1].

Proposition 3.3.3.
If μ is a fuzzy ideal of R, then so is $\sqrt{\mu}$.

Proof.
Let $x, y \in R$ then

$$\begin{aligned}(\sqrt{\mu})(x+y) &= \sup\{\mu((x+y)^n) : n > 0\} \\ &= \sup\{\mu((x+y)^{2n}) : n > 0\} \vee \sup\{\mu((x+y)^{2n+1}) : n > 0\}\end{aligned}$$

We now consider

$$\mu((x+y)^{2n}) = \mu(\Sigma_{i=0}^{2n} \binom{2n}{r} x^{2n-r} y^r)$$

$$= \mu\left(\binom{2n}{0} x^{2n} + \binom{2n}{1} x^{2n-1} y^1 + \cdots + \binom{2n}{n} x^n y^n + \binom{2n}{n+1} x^{n-1} y^{n+1} + \cdots + \binom{2n}{2n} y^{2n}\right)$$

$$\geq \mu(x^{2n}) \wedge \mu(x^{2n-1}y) \wedge \ldots \wedge \mu(x^n y^n) \wedge \mu(x^{n-1}y^{n+1}) \wedge \ldots \wedge \mu(y^{2n})$$

$$\geq \mu(x^{2n}) \wedge \mu(x^{2n-1}) \wedge \ldots \wedge \mu(x^n) \wedge \mu(y^{n+1}) \wedge \ldots \wedge \mu(y^{2n})$$

$$\geq \mu(x^n) \wedge \mu(y^{n+1})$$

$$\geq \mu(x^n) \wedge \mu(y^n).$$

Similarly we can show that $\mu((x+y)^{2n+1}) \geq \mu(x^n) \wedge \mu(y^n)$. Hence

$$\begin{aligned}(\sqrt{\mu})(x+y) &\geq \sup\{\mu(x^n) \wedge \mu(y^n) : n > 0\} \\ &= \sup\{\mu(x^n) : n > 0\} \wedge \sup\{\mu(y^n) : n > 0\} \\ &= (\sqrt{\mu})(x) \wedge (\sqrt{\mu})(y).\end{aligned}$$

Since $\mu(x) = \mu(-x)$, $\mu(x^n) = \mu((-x)^n)$ for all $n > 0$, it follows that $(\sqrt{\mu})(-x) = \sup\{\mu((-x)^n) : n > 0\} = \sup\{\mu(x^n) : n > 0\} = (\sqrt{\mu})(x)$. Furthermore

$$\begin{aligned}(\sqrt{\mu})(xy) &= \sup\{\mu((xy)^n) : n > 0\} \\ &\geq \sup\{\mu(x^n) : n > 0\} \\ &= (\sqrt{\mu})(x).\end{aligned}$$

Similarly, $(\sqrt{\mu})(xy) \geq (\sqrt{\mu})(y)$. So $(\sqrt{\mu})(xy) \geq (\sqrt{\mu})(x) \vee (\sqrt{\mu})(y)$, hence the result follows.

□

Proposition 3.3.4.
Let μ be a fuzzy ideal of R and $t \in (0,1]$. Then

(1) $\sqrt{\mu_t} \subseteq (\sqrt{\mu})_t$, *equality holds if μ has the sup-property,*
(2) $\sqrt{\mu^t} = (\sqrt{\mu})^t$, *where $\mu^t = \{x \in R : \mu(x) > t\}$.*

Proof.

(1) Let $x \in \sqrt{\mu_t}$; then for some $n_0 > 0$, $x^{n_0} \in \mu_t$. Hence $\mu(x^{n_0}) \geq t$. But $(\sqrt{\mu})(x) = \sup\{\mu(x^n) : n > 0\} \geq \mu(x^{n_0}) \geq t$, which implies $x \in (\sqrt{\mu})_t$. So $\sqrt{\mu_t} \subseteq (\sqrt{\mu})_t$. We now suppose that μ the has sup-property. Let $x \in (\sqrt{\mu})_t$, then $(\sqrt{\mu})(x) = \sup\{\mu(x^n) : n > 0\} \geq t$. It follows by the sup-property, for some $n_0 > 0$, $\mu(x^{n_0}) \geq t$, and hence $x \in \sqrt{(\mu_t)}$. Thus $(\sqrt{\mu})_t = \sqrt{(\mu_t)}$.
(2) By the same argument as in (1), we can show $\sqrt{\mu^t} \subseteq (\sqrt{\mu})^t$. To show the converse part, let $x \in (\sqrt{\mu})^t$. Then $(\sqrt{\mu})(x) = \sup\{\mu(x^n) : n > 0\} > t$. If for all $n > 0$, $\mu(x^n) \leq t$, then $\sup\{\mu(x^n) : n > 0\} \leq t$, which is not true. So for some $n_0 > 0$, $\mu(x^{n_0}) > t$ and hence $x \in \sqrt{\mu^t}$.

□

The following example illustrates that the level subsets $(\sqrt{\mu})_t$ and $\sqrt{\mu_t}$ may not be identical in general.

Example 3.3.5.
Let $\mu : \mathbb{Z} \longrightarrow I$ be a fuzzy ideal by

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ n/(n+1) & \text{if} \quad x \in <p^n> \setminus <p^{n+1}> \\ 0 & \text{if} \quad x \notin <p> \end{cases}$$

where p is a prime number.
Since for all $n > 0$, $p^n \in <p^n> \setminus <p^{n+1}>$, $\mu(p^n) = n/(n+1)$ for all $n = 1, 2, 3, \ldots$. Hence $(\sqrt{\mu})(p) = \sup\{\mu(p^n) : n > 0\} = \sup\{n/(n+1) : n > 0\} = 1$. So $p \in (\sqrt{\mu})_1$. But for all $n = 1, 2, \ldots$ $\mu(p^n) = n/(n+1) < 1$, which implies $p^n \notin \mu_1$ for all $n > 0$. Therefore $p \notin \sqrt{\mu_1}$. Thus $(\sqrt{\mu})_1 \neq \sqrt{\mu_1}$.

Proposition 3.3.6.
If μ is a fuzzy prime ideal of R, then $\sqrt{\mu} = \mu$.

Proof.
By Proposition[3.1.9] $\forall x \in R$ $\mu(x^n) = \mu(x)$ for all $n > 0$, and hence by Proposition[3.3.2] ,$\sqrt{\mu}(x) = \sup\{\mu(x^n) : n > 0\} = \mu(x)$.

□

In the following section we define a fuzzy prime radical and prove the relation between fuzzy nil and prime radicals.

Definition 3.3.7.
Let μ be a fuzzy ideal of R and $\mathcal{P}$ be the set of all fuzzy prime ideals of R containing μ. Then the *fuzzy prime radical* $r(\mu)$ *of* R is defined by

$$r(\mu) = \bigwedge\{\nu : \nu \in \mathcal{P}\}.$$

Proposition 3.3.8[Mal 2, 3.5].
Let μ be a fuzzy ideal of R, then in general $\sqrt{\mu} \leq r(\mu)$.

Proof.
Let ν be a fuzzy prime ideal of R such that $\mu \leq \nu$. Then $\sqrt{\mu} \leq \sqrt{\nu} = \nu$. Hence $\sqrt{\mu} \leq r(\mu)$.

□

The following example shows that $\sqrt{\mu} \neq r(\mu)$ in general.

Example 3.3.9.
Define a fuzzy ideal $\mu : \mathbb{Z} \longrightarrow I$ by

$$\mu(x) = \begin{cases} 1/2 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases}$$

Then $(\sqrt{\mu})(0) = \sup\{\mu(0^n) : n > 0\} = 1/2$ and $(\sqrt{\mu})(x) = 0$ for all $x \neq 0$. Since $\nu(0) = 1$ for all fuzzy prime ideals of R, $r(\mu)(0) = \bigwedge\{\nu(0) : \nu \in \mathcal{P}\} = 1$, and hence $(\sqrt{\mu}) \neq r(\mu)$

Theorem 3.3.10.
Let μ be a fuzzy ideal with $\mu(0) = 1$. Then $\sqrt{\mu} = r(\mu)$.

Proof.
By Proposition[3.3.8] $\sqrt{\mu} \leq r(\mu)$. To prove the converse, suppose $\sqrt{\mu} < r(\mu)$. Then there exists a in R such that $\sqrt{\mu}(a) < r(\mu)(a)$. Let $\sqrt{\mu}(a) = t$, then $a \notin (\sqrt{\mu})^t$. By Proposition[3.3.4] $a \notin \sqrt{\mu^t}$. Since $\sqrt{\mu^t} = \cap\{P : P$ is a prime ideal of R , $\mu^t \subseteq P\}$, there exists a prime ideal P of R such that $\mu^t \subseteq P$ and $a \notin P$. Define a fuzzy prime ideal $\nu : R \longrightarrow I$ by

$$\nu(x) = \begin{cases} 1 & \text{if} \quad x \in P \\ t & \text{if} \quad x \notin P. \end{cases}$$

Then only the following two cases arise.

Case 1. $x \in P$. Then $\mu(x) \leq \nu(x)$.

Case 2. $x \notin P$. Then $x \notin \mu^t$, hence $\mu(x) \leq t = \nu(x)$. In either case $\mu \leq \nu$. By the definition of a fuzzy prime radical of μ, $r(\mu) \leq \nu$. So $\sqrt{\mu}(a) < r(\mu)(a) \leq \nu(a) = t = \sqrt{\mu}(a)$ which is a contradiction. Hence the result follows.

□

Remark.
In the above Proposition, if $\mu(0) \neq 1$ then $\sqrt{\mu}^t$ may not be an ideal of R. This observation was overlooked in [Mal 9]; in particular Theorem[3.10] of [Mal 9]is not true as it stands. And also in [Mal 9, 3.9] Malik and Mordeson proved $\sqrt{\mu} = r(\mu)$ provided μ has the sup-property. But we prove the same without the sup-property.

The following Proposition can be proved easily.

Proposition 3.3.11.
Let μ be a fuzzy ideal of R with the sup-property. Then $Im(\sqrt{\mu}) \subseteq Im(\mu)$.

Example 3.3.12.
In general, $Im(\sqrt{\mu})$ need not be a subset of $Im(\mu)$ if μ does not have the sup-property. Define a fuzzy ideal $\mu : \mathbb{Z} \longrightarrow I$ by

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 3/4 - (1/2)^n & \text{if} \quad x \in <2^n> \setminus <2^{n+1}> \qquad \text{for all} \quad n = 1, 2, \ldots \\ 0 & \text{if} \quad x \notin <2>. \end{cases}$$

Then

$$\begin{aligned} \sqrt{\mu}(2) &= \sup\{\mu(2^n) : n > 0)\} \\ &= \sup\{3/4 - (1/2)^n : n > 0\} \qquad \text{since } 2^n \in <2^n> - <2^{n+1}> \quad \text{for all} \quad n = 1, 2, \ldots \\ &= 3/4. \end{aligned}$$

So $3/4 \in Im(\sqrt{\mu})$, but $\mu(x) \neq 3/4$ for all $x \in \mathbb{Z}$ hence $3/4 \notin Im(\mu)$.

The following example illustrates that even if μ has sup-property in general $Im(\sqrt{\mu}) \subsetneq Im(\mu)$.

Example 3.3.13.
Define a fuzzy ideal $\mu : \mathbb{Z}_8 \longrightarrow I$ by

$$\mu(0) = 1, \mu(2) = \mu(6) = 0.7, \mu(4) = 0.9 \text{ and } \mu(1) = \mu(3) = \mu(5) = \mu(7) = 0.5.$$

Then $Im(\mu) = \{1, 0.9, 0.7, 0.5\}$ and μ has the sup-property. We have $\sqrt{\mu}(0) = \sqrt{\mu}(2) = \sqrt{\mu}(4) = \sqrt{\mu}(6) = 1$ since $2^3 = 4^2 = 6^3 = 0$ and $\sqrt{\mu}(1) = \sqrt{\mu}(3) = \sqrt{\mu}(5) = \sqrt{\mu}(7) = 0.5$. Hence $Im(\sqrt{\mu}) = \{1, 0.5\} \subsetneq Im(\mu)$, i.e. $Im(\sqrt{\mu})$ need not be equal to $Im(\mu)$ even if μ has the sup-property.

Proposition 3.3.14.
Let μ be an ideal of R, then $\mathcal{X}_{\sqrt{I}} = \sqrt{\mathcal{X}_I}$.

The above Proposition is straightforward.

Proposition 3.3.15[Mal 3, 4.5].
Let μ, ν be fuzzy ideals of R then

(1) $\sqrt{\mu}(0) = \mu(0)$,
(2) $\mu \leq \sqrt{\mu}$,
(3) $\sqrt{(\sqrt{\mu})} = \sqrt{\mu}$,
(4) *If* $\mu \leq \nu$, *then* $\sqrt{\mu} \leq \sqrt{\nu}$,
(5) $\sqrt{(\mu \wedge \nu)} = \sqrt{\mu} \wedge \sqrt{\nu}$,
(6) *If* $\mu(0) = \nu(0) = 1$, *then* $\sqrt{(\mu \circ \nu)} = \sqrt{\mu} \wedge \sqrt{\nu}$,
(7) *If* $Im(\mu) = \{1, t\}$, *then* $Im(\sqrt{\mu}) = \{1, t\}$,
(8) *If μ is constant, then $\sqrt{\mu}$ is constant,*
(9) *If* $\nu^n \leq \mu$, *then* $\nu \leq \sqrt{\mu}$, *for all* $n \in \mathbb{N}$.

Proof.
(1),(2),(4), (8) and (9) are straightforward.
(3) Let $x \in R$ then

$$\begin{aligned}(\sqrt{(\sqrt{\mu})})(x) &= \sup\{(\sqrt{\mu})(x^n) : n > 0\} \\ &= \sup\{\sup\{\mu((x^n)^m) : m > 0\} : n > 0\} \\ &= \sup\{\mu(x^{mn}) : m > 0, n > 0\} \\ &= \sup\{\mu(x^k) : k > 0\} \\ &= (\sqrt{\mu})(x) \qquad \text{for all} \quad x \in R.\end{aligned}$$

(5) Let $x \in R$ then

$$\begin{aligned}\sqrt{(\mu \wedge \nu)}(x) &= \sup\{(\mu \wedge \nu)(x^n) : n > 0\} \\ &= \sup\{\mu(x^n) \wedge \nu(x^n) : n > 0\} \\ &= \sup\{\mu(x^n) : n > 0\} \wedge \sup\{\nu(x^n) : n > 0\} \\ &= (\sqrt{\mu})(x) \wedge (\sqrt{\nu})(x) \quad = \quad (\sqrt{\mu} \wedge \sqrt{\nu})(x).\end{aligned}$$

(6) Since $\mu \circ \nu \leq \mu \wedge \nu$, $\sqrt{(\mu \circ \nu)} \leq \sqrt{(\mu \wedge \nu)} = \sqrt{\mu} \wedge \sqrt{\nu}$. Let ω be a fuzzy prime ideal of R such that $\mu \circ \nu \leq \omega$. Then either $\mu \leq \omega$ or $\nu \leq \omega$, and hence $\mu \wedge \nu \leq \omega$. So $\sqrt{(\mu \wedge \nu)} \leq \sqrt{(\mu \circ \nu)}$ thus $\sqrt{(\mu \circ \nu)} = \sqrt{(\mu \wedge \nu)}$. Hence one can easily see that for each $n \in \mathbb{N}$, $\sqrt{(\mu^n)} = \sqrt{\mu}$.

(7) Let $x \in R$. If $x \in \sqrt{\mu}_0$, then $x^n \in \mu_0$ for some $n \in \mathbb{N}$. So $\mu(x^n) = 1$, hence $(\sqrt{\mu})(x) \geq \mu(x^n) = 1$ from which follows $(\sqrt{\mu})(x) = 1$. If $x \notin \sqrt{\mu}_0$, then for all

$n > 0$, $x^n \notin \mu_0$. So $\mu(x^n) = t$ for all $n > 0$, and hence $(\sqrt{\mu})(x) = \sup\{\mu(x^n) : n > 0\} = t$. Thus $Im(\sqrt{\mu}) = \{1, t\}$.

□

The converse of part (8) in the above Proposition is not true. The example follows:

Example 3.3.16.
Let R be the ring $(\{0, 2, 4, 6\}, +_8, \circ_8)$ and define a fuzzy ideal $\mu : R \longrightarrow I$ by

$$\mu(0) = 1, \mu(2) = \mu(4) = \mu(6) = 0.9.$$

Then

$$\sqrt{\mu}(0) = \sqrt{\mu}(2) = \sqrt{\mu}(4) = \sqrt{\mu}(6) = 1.$$

Hence $Im(\sqrt{\mu}) = \{1\}$ and $Im(\mu) = \{1, 0.9\}$.

Effects of homomorphism on fuzzy radicals are stated and studied in the following Propositions.

Proposition 3.3.17.
If $f : R \longrightarrow R'$ is an epimorphism and μ is a fuzzy ideal of R, then $f(\sqrt{\mu}) \leq \sqrt{(f(\mu))}$. Further, if μ is f-invariant then $f(\sqrt{\mu}) = \sqrt{(f(\mu))}$.

Proof.
Let $y \in R'$; then $f^{-1}(y)$ is a non-empty subset since f is an epimorphism. We now consider

$$\begin{aligned} f(\sqrt{\mu})(y) &= \sup\{\sqrt{\mu}(x) : x \in f^{-1}(y)\} \\ &= \sup\{\sup\{\mu(x^n) : n > 0\} : x \in f^{-1}(y)\} \\ &= \sup\{\sup\{\mu(x^n) : x \in f^{-1}(y)\} : n > 0\} \\ &\leq \sup\{\sup\{\mu(x^n) : x^n \in f^{-1}(y^n)\} : n > 0\} \\ &\leq \sup\{\sup\{\mu(z) : z \in f^{-1}(y^n)\} : n > 0\} \\ &= \sup\{f(\mu)(y^n) : n > 0\} \\ &= \sqrt{(f(\mu))}(y). \end{aligned}$$

Hence $f(\sqrt{\mu}) \leq \sqrt{(f(\mu))}$.
Since $f^{-1}(y)$ is non-empty, there exists x_0 in $f^{-1}(y)$. Let $x \in f^{-1}(y^n)$ then $f(x_0^n) = f(x)$ which implies $\mu(x_0^n) = \mu(x)$ i.e. for all $x \in f^{-1}(y^n)$, $\mu(x) = \mu(x_0^n)$. If $x \in f^{-1}(y)$, then $f(x^n) = f(x_0^n)$. Hence $\mu(x^n) = \mu(x_0^n)$. Thus we have

$$\begin{aligned} \sqrt{(f(\mu))}(y) &= \sup\{f(\mu)(y^n) : n > 0\} \\ &= \sup\{\sup\{\mu(x) : x \in f^{-1}(y^n)\} : n > 0\} \\ &= \sup\{\mu(x_0^n) : n > 0\} \\ &= \sqrt{\mu}(x_0) \end{aligned}$$

and

$$\begin{aligned} f(\sqrt{\mu})(y) &= \sup\{\sqrt{\mu}(x) : x \in f^{-1}(y)\} \\ &= \sup\{\sup\{\mu(x^n) : n > 0\} : x \in f^{-1}(y)\} \\ &= \sup\{\sup\{\mu(x^n) : x \in f^{-1}(y)\} : n > 0\} \\ &= \sup\{\mu(x_0^n) : n > 0\} \\ &= \sqrt{\mu}(x_0) \end{aligned}$$

hence $f(\sqrt{\mu}) = \sqrt{(f(\mu))}$.

□

The last part of the above Proposition also holds if μ is constant on Ker$f$ instead of μ f-invariant.

Proposition 3.3.18.
If $f : R \longrightarrow R'$ is a homomorphism and μ' is a fuzzy ideal of R', then $f^{-1}(\sqrt{\mu'}) = \sqrt{(f^{-1}(\mu'))}$.

Proof.
Let $x \in R$ then

$$\begin{aligned} f^{-1}(\sqrt{\mu'})(x) &= \sqrt{\mu'}(f(x)) \\ &= \sup\{\mu'((f(x))^n) : n > 0\} \\ &= \sup\{f^{-1}(\mu')(x^n) : n > 0\} \\ &= \sqrt{(f^{-1}(\mu'))}(x) \qquad \text{for all} \quad x \in R. \end{aligned}$$

□

§ 3.4 Fuzzy semiprime ideals.

In [Dix 2],[Kum 2], [Bho 2] and [Zah 1] the concept of fuzzy semiprime ideals was introduced and studied. We prove the equivalence of different definitions for a fuzzy semiprime ideal and some relations between fuzzy semiprime, prime and radical. Finally we end this section by considering the effect of a homomorphism on fuzzy semiprime ideals.

Theorem 3.4.1.
The following are equivalent for a fuzzy ideal μ of R.

(1) *For any fuzzy ideal ν of R, $\nu^n \leq \mu$ implies $\nu \leq \mu$, where $n \in \mathbb{N}$,*
(2) *For any fuzzy ideal ν of R, $\nu^2 \leq \mu$ implies $\nu \leq \mu$,*
(3) $\forall x \in R, n \in \mathbb{N}$, $\mu(x^n) = \mu(x)$,
(4) $\forall x \in R$, $\mu(x^2) = \mu(x)$.

Proof.
(1) ⇒(2) is obvious.

(2)⇒(1)
The Proof is by induction and the result is true for $n = 2$. Suppose it is true for $n = k$. Let $\nu^{k+1} \leq \mu$. If k is odd, then $\nu^{k+1} = \nu^{(k+1)/2}\nu^{(k+1)/2} \leq \mu$. Since $\nu^{k+1/2}$ is a fuzzy ideal, $\nu^{k+1/2} \leq \mu$ and hence $\nu \leq \mu$ by our supposition. If k is even, then $\nu^{k+2} \leq \nu^{k+1} \leq \mu$, $i.e.\nu^{(k+2)/2}\nu^{(k+2)/2} \leq \mu$, so $\nu^{k+2/2} \leq \mu$ and hence $\nu \leq \mu$.

(3)⇒(4) Straightforward.

(4)⇒(3)
Suppose $\mu(x^k) = \mu(x)$. If k is an odd integer, then $\mu(x^{k+1}) = \mu(x^{k+1/2}x^{k+1/2})$ and $(k+1)/2 < k$. So $\mu(x^{k+1}) = \mu(x^{k+1/2}) = \mu(x)$. If k is even then $k+1$ is odd. By the above $\mu(x^{k+2}) = \mu(x) \geq \mu(x^{k+1}) \geq \mu(x)$, hence $\mu(x^{k+1}) = \mu(x)$. Thus the result is true for $n = k+1$.

(2) ⇒ (4)
Let $x \in R$ and $\mu(x^2) = t$, then $x^2 \in \mu_t$. Since μ_t is an ideal, $< x^2 > \subseteq \mu_t$. We define a fuzzy ideal ν of R by

$$\nu(r) = \begin{cases} t & \text{if} \quad r \in < x > \\ 0 & \text{otherwise.} \end{cases}$$

Then if $r \notin < x^2 >$ for all $r = r_1 r_2$, either $r_1 \notin < x >$ or $r_2 \notin < x >$. So $\nu^2(r) = \sup_{r=r_1 r_2} \nu(r_1) \wedge \nu(r_2) = 0$ and if $r \in < x^2 >$, then there exists $s \in R$ such that $r = sx^2 = s.sx$ and $sx, x \in < x >$. So $\nu^2(r) = t$ and hence $\nu^2 \leq \mu$. By hypothesis $\nu \leq \mu$, which implies that $t = \nu(x) \leq \mu(x) \leq \mu(x^2) = t$. Thus $\mu(x^2) = \mu(x)$.

(4) ⇒(2)
Let ν be a fuzzy ideal of R such that $\nu^2 \leq \mu$, then $\nu(x) \leq \nu^2(x^2) \leq \mu(x^2) = \mu(x)$ by Remark 2 after Proposition[2.2.3].

□

In the above Proposition, conditions (1), (2) and (4) were given as the definition for a fuzzy semiprime ideal by the authors Dixit et al.[Dix 2], M.M.Zahedi[Zah 1] and H.V.Kumbhojkar and M.S.Bapat[Bho 2] respectively.

Definition 3.4.2.
We say that a fuzzy ideal μ of R is *fuzzy semiprime* if μ satisfies any of the four conditions in Theorem[3.4.1].

Proposition 3.4.3.
A fuzzy ideal μ of R is semiprime if and only if each of its level subsets is a semiprime ideal of R.

Proof.
($\Leftarrow$) Let $x \in R$ and $\mu(x^2) = t$. Then $x^2 \in \mu_t$ hence $x \in \mu_t$ since μ_t is a semiprime ideal of R. It follows that $\mu(x) \geq t = \mu(x^2) \geq \mu(x)$ and hence $\mu(x^2) = \mu(x)$.

□

Corollary 3.4.4.
An ideal J of R is semiprime if and only if $\mathcal{X}_J$ is a fuzzy semiprime ideal of R.

It is easy to see that every fuzzy prime ideal is fuzzy semiprime, but the converse is not true. The same example as in the crisp case will fit in the fuzzy situation.

Proposition 3.4.5.
Let μ be a fuzzy ideal of R, then

(1) *μ is fuzzy semiprime if and only if $\sqrt{\mu} = \mu$,*
(2) *$\sqrt{\mu}$ is a fuzzy semiprime ideal ,*
(3) *The intersection of fuzzy semiprime ideals is fuzzy semiprime ,*
(4) *If μ is fuzzy semiprime and $\mu(0) = 1$, then μ is the intersection of fuzzy prime ideals of R.*

Proof.
(1) If μ is a fuzzy semiprime, then by Theorem[3.4.1], for all$n \in \mathbb{N}$ $\mu(x^n) = \mu(x)$. By Theorem[3.3.2], $\sqrt{\mu} = \mu$. Conversely suppose that $\sqrt{\mu} = \mu$. Then $\sqrt{\mu}(x) = \sup\{\mu(x^n) : n > 0\} \geq \mu(x^n) \geq \mu(x) = (\sqrt{\mu})(x)$ for all $n \in \mathbb{N}$, from which follows that for all $n \in \mathbb{N}$, $\mu(x^n) = \mu(x)$. Thus μ is fuzzy semiprime.

(2) is straightforward from (3) of Proposition[3.3.15].

(3) and (4) are also straightforward.

□

Proposition 3.4.6.
Let $f : R \longrightarrow R'$ be an epimorphism and μ be a fuzzy semiprime ideal such that μ is constant on $Kerf$. Then $f(\mu)$ is a fuzzy semiprime ideal of R'.

Proof.
Let $y \in R'$, then there exists $x \in R$ such that $f(x) = y$. Hence we have

$$\begin{aligned} f(\mu)(y^2) &= f(\mu)(f(x^2)) \\ &= \sup_{f(x_0)=f(x^2)} \mu(x_0) \\ &= \mu(x^2) \qquad \text{since } \mu \text{ is constant on Ker} f \\ &= \mu(x) \\ &= f(\mu)(f(x)) \qquad \text{since } \mu \text{ is constant on Ker} f \\ &= f(\mu)(y). \end{aligned}$$

□

Proposition 3.4.7.
Let $f : R \longrightarrow R'$ be a homomorphism and μ' be a semiprime fuzzy ideal of R'. Then $f^{-1}(\mu')$ is a fuzzy semiprime ideal of R.

The proof is straightforward.

We recall that a ring R is *regular* if for every $x \in R$ there exists an element $r \in R$ such that $x = xrx$.

Proposition 3.4.8.
A ring R is regular if and only if every fuzzy ideal of R is semiprime.

Proof.
($\Rightarrow$) Let ν be a fuzzy ideal of R such that $\nu^2 \leq \mu$. Then by Proposition[2.4.16], $\nu^2 = \nu$ and hence $\nu \leq \mu$.
($\Leftarrow$) Let $x \in R$, then $(\mathcal{X}_{<x>})^2 = \mathcal{X}_{<x^2>}$, since $\mathcal{X}_{<x^2>}$ is a fuzzy semiprime $\mathcal{X}_{<x>} \leq \mathcal{X}_{<x^2>}$. But we have $\mathcal{X}_{<x^2>} \leq \mathcal{X}_{<x>}$. So $\mathcal{X}_{<x^2>} = \mathcal{X}_{<x>}$, hence $x \in < x^2 >$, which implies there exists $r \in R$ such that $x = rx^2 = xrx$. Thus R is regular.

□

CHAPTER IV

FUZZY PRIMARY, SEMIPRIMARY AND MAXIMAL IDEALS

§ 4.1 Fuzzy primary ideals.

In the crisp case, primary ideal is a generalization of prime ideal. We fuzzify the same idea here. Series of Propositions describe different relationship between fuzzy prime and fuzzy primary. Also fuzzy primary ideals have been dealt with in terms of fuzzy points and level subsets. Several examples have been provided where certain converse implications are not true. Definitions of fuzzy primary ideals given by other authors [Zah 1], [Bho 1] have been stated and we proved their equivalences.

Definition 4.1.1[Kum 1, 4.1].
A fuzzy ideal μ of R is called *fuzzy primary* if for any fuzzy ideals ν, ω of R

$$\nu\omega \leq \mu \quad \text{implies either } \nu \leq \mu \text{ or } \omega \leq \sqrt{\mu}.$$

Remark 1.
The above Definition can also be stated as follows

$$\nu\omega \leq \mu \quad \text{implies either } \omega \leq \mu \text{ or } \nu \leq \sqrt{\mu} \text{ since } \nu\omega = \omega\nu \text{ in a commutative ring .}$$

Remark 2.
Let $a \in R$ and μ be a fuzzy ideal of R such that $\mu(1) < \mu(a)$ then $\sqrt{\mu}(1) < \mu(a)$.

Let $\mu(1) < \mu(a)$ for some $a \in R$ then by Theorem[3.3.2] $\sqrt{\mu}(1) = \sup\{\mu(1^n) : n > 0\} = \mu(1) < \mu(a)$.

Theorem 4.1.2.
If μ is a non-constant fuzzy primary ideal of R then $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$ and μ_0 is a primary ideal of R.

Proof.
First let us show $\mu(0) = 1$. Suppose $\mu(0) = n < 1, \sqrt{\mu}(1) = s$ and $\mu(1) = m$ then by the Remark 2, $m \leq s < n < 1$. Let u be a number in $(0, 1]$ such that $n < u \leq 1$. We now define fuzzy ideals $\nu, \omega : R \longrightarrow I$ by

$$\nu(x) = \begin{cases} u & \text{if} \quad x \in \mu_0 \\ m & \text{otherwise} \end{cases}$$

and

$$\omega(x) = n \quad \text{for all } x \in R.$$

If $x \in \mu_0$, then $\nu\omega(x) \leq \omega(x) = n = \mu(x)$ and if $x \notin \mu_0$ then $\nu\omega(x) \leq \nu(x) \wedge \omega(x) = m \wedge n = m = \mu(1) \leq \mu(x)$. Hence $\nu\omega \leq \mu$ but $\nu(0) = u > n = \mu(0)$ and $\omega(1) = n > s = \sqrt{\mu}(1)$. It follows that $\nu \nleq \mu$ and $\omega \nleq \sqrt{\mu}$ which is a contradiction to the fact that μ is fuzzy primary. Thus $\mu(0) = 1$. Since μ is non-constant $|Im(\mu)| \geq 2$. Suppose $|Im(\mu)| \geq 3$ then there exists $x \in R$ such that $\mu(x) = l$ and $m < l < 1$. By the Remark 2, $s < l$. We define fuzzy ideals $\nu_1, \omega_1 : R \longrightarrow I$ by

$$\nu_1(r) = \begin{cases} 1 & \text{if} \quad r \in \mu_l \\ m & \text{otherwise} \end{cases}$$

and

$$\omega_1(x) = l \quad \text{for all } x \in R.$$

Then it is easy to check that $\nu_1\omega_1 \leq \mu$. But $\nu_1(x) = 1 > l = \mu(x)$ and $\omega_1(1) = l > s = \sqrt{\mu}(1)$, hence $\nu_1 \nleq \mu$ and $\omega_1 \nleq \sqrt{\mu}$. This is a contradiction. Thus for some $t \in [0,1)$, $Im(\mu) = \{1, t\}$. To show μ_0 is primary, let A, B be ideals of R such that $AB \subseteq \mu_0$. Then we can easily see that $\mathcal{X}_A\mathcal{X}_B \leq \mu$, which implies either $\mathcal{X}_A \leq \mu$ or $\mathcal{X}_B \leq \sqrt{\mu}$, and hence either $A \subseteq \mu_0$ or $B \subseteq (\sqrt{\mu})_0 = \sqrt{\mu_0}$. Thus the result follows.

□

Corollary 4.1.3.
If μ is a fuzzy primary ideal of R, then all of its level subsets are primary ideals of R.

Proof.
Let $x, y \in R, t \in [0,1]$ and $xy \in \mu_t$. Then $\mathcal{X}^t_{<x>}\mathcal{X}^t_{<y>} \leq \mu$. Since μ is a fuzzy primary ideal , either $\mathcal{X}^t_{<x>} \leq \mu$ or $\mathcal{X}^t_{<y>} \leq \sqrt{\mu}$. Where

$$\mathcal{X}^t_{<x>}(y) = \begin{cases} t & \text{if} \quad y \in <x> \\ 0 & \text{if} \quad y \notin <x>. \end{cases}$$

Hence either $x \in \mu_t$ or $y \in (\sqrt{\mu})_t = \sqrt{(\mu)_t}$ by Proposition[3.3.4]. So either $x \in \mu_t$ or $y^n \in \mu_t$ for some $n \in \mathbb{N}$.

□

The following example illustrates that the converse of the above Corollary is not true.

Example 4.1.4.
Define fuzzy ideals $\mu, \nu, \omega : \mathbb{Z} \longrightarrow I$ by

$$\mu(x) = \begin{cases} 3/4 & \text{if} \quad x \in <4> \\ 1/2 & \text{if } x \in <2> \setminus <4> \\ 0 & \text{otherwise}, \end{cases}$$

$$\nu(x) = \begin{cases} 1 & \text{if} \quad x \in <4> \\ 0 & \text{otherwise ,} \end{cases}$$

and

$$\omega(x) = 3/4 \qquad \text{for all } x \in \mathbb{Z}.$$

Then

$$\nu \wedge \omega(x) = \begin{cases} 3/4 & \text{if} \quad x \in <4> \\ 0 & \text{otherwise.} \end{cases}$$

So $\nu\omega \leq \nu \wedge \omega \leq \mu$. But we have

$$\sqrt{\mu}(x) = \begin{cases} 3/4 & \text{if} \quad x \in <2> \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\nu(0) = 1 > \mu(0)$ and $\omega(1) = 3/4 > \sqrt{\mu}(1) = 0$ *i.e.*$\nu \nleq \mu$ and $\omega \nleq \sqrt{\mu}$, thus μ is not fuzzy primary but all level subsets $<4>, <2>$ are primary ideals of $\mathbb{Z}$.

Theorem 4.1.5.
If μ is a fuzzy ideal of R such that $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$ and μ_0 is a primary ideal of R, then μ is fuzzy primary ideal of R.

Proof.
Let ν, ω be any fuzzy ideals of R such that $\nu\omega \leq \mu$. Suppose $\nu \nleq \mu$ and $\omega \nleq \sqrt{\mu}$; then there exist $x, y \in R$ such that $\nu(x) > \mu(x)$ and $\omega(y) > \sqrt{\mu}(y)$. Hence $\mu(x) \neq 1$ and $\sqrt{\mu}(y) \neq 1$, so $\mu(x) = \sqrt{\mu}(y) = t, \nu(x) > t$ and $\omega(y) > t$. It follows that $\nu\omega(xy) \geq \nu(x) \wedge \omega(y) > t$. Since $x \notin \mu_0, y \notin \sqrt{\mu}_0$ and μ_0 is primary ideal, $xy \notin \mu_0$, hence $\mu(xy) = t < \nu\omega(xy)$ which is a contradiction. Thus the result follows.

□

Corollary 4.1.6.
An ideal J of R is primary if and only if $\mathcal{X}_J$ is fuzzy primary.

Remark.
The intersection of fuzzy primary ideals needs not be a fuzzy primary ideal. The same example in the crisp case will site in fuzzy setting as well.

We now turn our attention to the relationship between fuzzy prime and fuzzy primary ideals of R.

Proposition 4.1.7.
If μ is fuzzy primary, then $\sqrt{\mu}$ is fuzzy prime.

Proof.
Since μ is fuzzy primary, by Theoreem[4.1.2], $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$

and μ_0 is a primary ideal of R. Then by Proposition[3.3.15(7)] and Proposition[3.3.4] $Im(\sqrt{\mu}) = \{1, t\}$ and $(\sqrt{\mu})_0 = \sqrt{\mu_0}$ is a prime ideal of R. Hence by Theorem[3.1.3], $\sqrt{\mu}$ is a fuzzy prime ideal of R.

□

We give an example here to show that the converse of the above Proposition is not true.

Example 4.1.8.
Define a fuzzy ideal $\mu : \mathbb{Z} \longrightarrow I$ by

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x \in <4> \\ 1/2 & \text{if} \quad x \in <2> \setminus <4> \\ 1/3 & \text{otherwise} . \end{cases}$$

Let $x \in R$. If $x \in <2>$, then $x^n \in <4>$ for some $n > 0$, so $\mu(x^n) = 1$ and hence $\sqrt{\mu}(x) = \sup\{\mu(x^n) : n > 0\} = 1$. If $x \notin <2>$, then $x^n \notin <2>$ for all $n > 0$ and hence $\sqrt{\mu}(x) = 1/3$. It follows that $Im(\sqrt{\mu}) = \{1, 1/3\}$ and $(\sqrt{\mu})_0 = <2>$ is a prime ideal of $\mathbb{Z}$. By Theorem[3.1.3], $\sqrt{\mu}$ is a fuzzy prime ideal and it is easy to see that μ is not a fuzzy primary ideal.

Proposition 4.1.9.
If μ is a fuzzy primary ideal, then the quotient ring R/μ has the property that every zero divisor in R/μ is nilpotent.

Proof.
Since μ is a fuzzy primary ideal, μ_0 is a primary ideal of R. So every zero divisor in R/μ_0 is nilpotent. By Corollary[2.4.9], $R/\mu_0 \cong R/\mu$. Thus the result follows.

□

The example[4.1.8] shows that the converse of the above Proposition is not true in general.

If J is a prime ideal, then $J^n, n > 1$ need not be a primary ideal in the crisp case. So one cannot expect that if μ is a fuzzy prime ideal then $\mu^n, n > 1$ is a fuzzy primary ideal. But it is true under a certain condition.

Proposition 4.1.10.
Let μ be a fuzzy prime ideal of R then $\mu^n, n \geq 1$ is fuzzy primary if and only if $(\mu^n)_0$ is a primary ideal of R.

Proof.
Suppose μ is fuzzy prime, then $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$. We have

$$\mu^n(0) = \sup_{0=\Sigma_{k=1}^p x_{k1}x_{k2}\dots x_{kn}} \wedge_1^p \wedge_1^n \mu(x_{jk}) \geq \mu(0) = 1$$

hence $\mu^n(0) = 1$. Since for each j,k either $\mu(x_{jk}) = 1$ or t, $\mu^n(x)$ is either 1 or t. *i.e.* $Im(\mu^n) = \{1,t\}$ and also by hypothesis $(\mu^n)_0$ is a primary ideal. So by, Theorem[4.1.5] μ^n is fuzzy primary.

□

We now turn to the equivalence of different definitions for a fuzzy primary ideal of R.

Proposition 4.1.11.

Let μ be a non-constant fuzzy ideal of R then

(1) *μ is fuzzy primary if and only if for any fuzzy points x_r, y_s of R, $x_r y_s \in \mu$ implies either $x_r \in \mu$ or $y_s^n \in \mu$ for some $n \geq 1$,*

(2) *If μ is fuzzy primary, then for each $x,y \in R$ either $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y^n)$ for some $n \geq 1$,*

(3) *If μ is fuzzy primary then for each $x,y \in R$, $\mu(xy) = \mu(0)$ implies either $\mu(x) = \mu(0)$ or $\mu(y^n) = \mu(0)$ for some $n \geq 1$.*

Proof.

(1) Let x_r, y_s be any fuzzy points of R such that $x_r y_s \in \mu$. Then either $x_r \in \mu$ or $y_s \in \sqrt{\mu}$. Since μ is fuzzy primary, $Im(\mu) = \{1,t\}$ for some $t \in [0,1)$; hence by Proposition[3.3.15,7]

$$\sqrt{\mu}(x) = \begin{cases} 1 & \text{if } x \in \sqrt{\mu}_0 \\ t & \text{otherwise .} \end{cases}$$

If $y \in \sqrt{\mu}_0$, then $y^n \in \mu_0$ for some $n \geq 1$. So $\mu(y^n) = 1 > s$ and hence $y_s^n \in \mu$. If $y \notin \sqrt{\mu}_0$ then $y \notin \mu_0$ so $\mu(y) = \sqrt{\mu}(y) \geq s$, hence $y_s \in \mu$. Thus either $x_r \in \mu$ or $y_s^n \in \mu$ for some $n \geq 1$. Conversely suppose for any fuzzy points x_r, y_s $x_r y_s \in \mu$ implies either $x_r \in \mu$ or $y_s^n \in \mu$ for some $n \geq 1$. It is enough to show that $Im(\mu) = \{1,t\}$ for some $t \in [0,1)$ and μ_0 is a primary ideal of R. Suppose $\mu(0) < 1$. Since μ is non-constant there exists $x \in R - \{0\}$ such that $\mu(x) < \mu(0) < 1$. It follows that $x_{\mu(0)} 0_1 = 0_{\mu(0)} \in \mu$, but $x_{\mu(0)} \notin \mu$ and $0_1^n = 0_1 \notin \mu$ for all $n \in \mathbb{N}$. This is a contradiction to the supposition. To prove $Im(\mu) = \{1,t\}$, we have seen in the Remark 2 (e) after Proposition[2.2.3] that $\mu(1) \leq \mu(x)$ for all $x \in R$. Let $t = \mu(x) \neq \mu(0) = 1$ *i.e.*$\mu(x) < \mu(0)$. Then $x_1 1_{\mu(x)} = x_{\mu(x)} \in \mu$. Hence either $x_1 \in \mu$ or $1_{\mu(x)}^n = 1_{\mu(x)} \in \mu$ for some $n \geq 1$. Since $\mu(x) < 1$, $1_{\mu(x)} \in \mu$, i.e. $\mu(x) \leq \mu(1)$ and hence $\mu(x) = \mu(1)$, so $Im(\mu) = \{1,t\}$ for some $t \in [0,t)$. The fact that μ_0 is primary is straightforward.

(2) Suppose μ is fuzzy primary and let $x,y \in R$. If $xy \in \mu_0$ then either $x \in \mu_0$ or $y^n \in \mu_0$ for some $n \geq 1$ since μ_0 is a primary ideal of R, and hence

either $\mu(xy) = \mu(x)$ or $\mu(y^n)$. If $xy \notin \mu_0$ then $\mu(xy) = t \geq \mu(x) \wedge \mu(y)$ since $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$, which implies either $\mu(x) = t$ or $\mu(y) = t$ i.e. either $\mu(xy) = \mu(x)$ or $\mu(y)$.

(3) Follows in a straightforward manner from (2).

□

Remark.
We noted that the conditions (1),(2) were respectively given by Zahedi[Zah 1, 3.1] and Kumbhojkar and Bapat[Bho 1, 5.4] as definitions for a fuzzy primary ideal.

The example in [3.1.10] is enough to show that the converse of (2) in Proposition[4.1.11] is not true. The converse of (3) in Proposition[4.1.11] is also not true in general:

Example 4.1.12.
Define a fuzzy ideal $\mu : \mathbb{Z} \longrightarrow I$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.8 & \text{if } x \in <6> \setminus \{0\} \\ 0 & \text{otherwise .} \end{cases}$$

It is easy to check that for all $x, y \in \mathbb{Z}$ if $\mu(xy) = \mu(0)$ then either $\mu(x) = \mu(0)$ or $\mu(y^n) = \mu(0)$ for some $n \geq 1$. But $\mu(2.3) = 0.8$, $\mu(2) \neq 0.8$ and $\mu(3^n) \neq 0.8$ for all $n \geq 1$.

The following Proposition leads us to define the notion of fuzzy ν-primary ideal.

Theorem 4.1.13.
Let μ, ν be non-constant fuzzy ideals of R then μ is a fuzzy primary ideal and $\nu = \sqrt{\mu}$ if and only if $\mu \leq \nu \leq \sqrt{\mu}$ and if ω_1, ω_2 are fuzzy ideals of R such that $\omega_1\omega_2 \leq \mu$ then either $\omega_1 \leq \mu$ or $\omega_2 \leq \nu$.

Proof.
We prove only the sufficient condition. Let ω_1, ω_2 be fuzzy ideals such that $\omega_1\omega_2 \leq \mu$, then either $\omega_1 \leq \mu$ or $\omega_2 \leq \nu$, hence either $\omega_1 \leq \mu$ or $w_2 \leq \sqrt{\mu}$. Thus μ is a fuzzy primary ideal.By Theorem[4.1.2], $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$, hence by Proposition[3.3.15,7], $\sqrt{\mu}(x) = 1$ if $x \in \sqrt{\mu}_0$ and $\sqrt{\mu}(x) = t$ if $x \notin \sqrt{\mu}_0$. Let $x \in R$ if $x \notin \sqrt{\mu}_0$, Then $\sqrt{\mu}(x) = t = \mu(x)$ since $x \notin \mu_0$. Since $\mu \leq \nu \leq \sqrt{\mu}$ we have $\sqrt{\mu}(x) = \nu(x)$. If $x \in \sqrt{\mu}_0$, then let n be the least positive integer such that $x^n \in \mu_0$. If $n = 1$, then $x \in \nu_0$ since $\nu(0) = \sqrt{\mu}(0) = \mu(0), \mu_0 \subseteq \nu_0$. If $n > 1$ then $< x > \subseteq \sqrt{\mu}_0$, $< x^{n-1} > \nsubseteq \mu_0$ and $< x^n > \subseteq \mu_0$ which implies that $\mathcal{X}_{<x^{n-1}>}\mathcal{X}_{<x>} = \mathcal{X}_{<x^n>} \leq \mu$ and $\mathcal{X}_{<x^{n-1}>} \nleq \mu$. Hence by the supposition $\mathcal{X}_{<x>} \leq \nu$, so $x \in \nu_0$, i.e. $x \in \sqrt{\mu}_0$ implies $x \in \nu_0$. So $\sqrt{\mu}(x) = \sqrt{\mu}(0) = \nu(0) = \nu(x)$.

□

Proposition 4.1.14.
If $\mu = \bigwedge_{\alpha \in \mathcal{A}} \nu_\alpha$ is an intersection of fuzzy primary ideals, then μ_0 is an intersection of primary ideals, $|Im(\mu)| \leq |\mathcal{A}| + 1$ and $\mu(0) = 1$.

Proof.
Suppose $\mu = \bigwedge_{\alpha \in \mathcal{A}} \nu_\alpha$, where ν_α are fuzzy primary ideals . Then $\mu_0 = \bigwedge_{\alpha \in \mathcal{A}} (\nu_\alpha)_0$, and for each $\alpha \in \mathcal{A}$, $(\nu_\alpha)_0$ is primary ideal of R. Since $\forall \alpha \in \mathcal{A}$, $|Im(\nu_\alpha)| = 2$ and $\nu_\alpha(0) = 1$, we have $|\{\nu_\alpha(x) : \alpha \in \mathcal{A}, \nu_\alpha(x) \neq 1, x \in R\}| \leq |\mathcal{A}|$ and hence $|Im(\mu)| \leq |\mathcal{A}| + 1$ and $\mu(0) = 1$.

□

Corollary 4.1.15.
Let J be an ideal of R. If J is an intersection of primary ideals of R then for all $k \in [0, 1)$, λ_{J_k} is an intersection of fuzzy primary ideals of R where

$$\lambda_{J_k}(x) = \begin{cases} 1 & \text{if} \quad x \in J \\ k & \text{if} \quad x \notin J. \end{cases}$$

In the following Propositions we look at the effect of a homomorphism on a fuzzy primary ideal.

Proposition 4.1.16.
Let $f : R \longrightarrow R'$ be an epimorphism and μ be an f-invariant fuzzy primary ideal of R. Then $f(\mu)$ is a fuzzy primary ideal of R'.

Proof.
Let ν, ω be fuzzy ideals of R' such that $\nu\omega \leq f(\mu)$. Then by Proposition[2.2.10] $f^{-1}(\nu) f^{-1}(\omega) \leq f^{-1}(\nu\omega) \leq f^{-1}(f(\mu)) = \mu$. Since μ is fuzzy primary, either $f^{-1}(\nu) \leq \mu$ or $f^{-1}(\omega) \leq \sqrt{\mu}$. Hence by Proposition[3.3.17], $\nu \leq f(\mu)$ or $\omega \leq f(\sqrt{\mu}) = \sqrt{f}(\mu)$.

□

Proposition 4.1.17.
Let $f : R \longrightarrow R'$ be a homomorphism and μ' be a fuzzy primary ideal of R'. Then $f^{-1}(\mu')$ is a fuzzy primary ideal of R.

The proof is similar to the Proof of Proposition[4.1.16].

Remark.
Let $f : R \longrightarrow R'$ be an epimorphism, μ be an f-invariant fuzzy ideal and μ' be fuzzy ideal of R'. Then by [Proposition[1.2.2] $f(f^{-1}(\mu')) = \mu'$ and $f^{-1}(f(\mu)) = \mu$ and hence we can come to the following conclusions.

(1) μ' is fuzzy primary if and only if $f^{-1}(\mu')$ is fuzzy primary,
(2) μ is fuzzy primary if and only if $f(\mu)$ is fuzzy primary.

Thus there is a one-to-one correspondence between the set of all f-invariant fuzzy primary ideals of R and the set of all fuzzy primary ideals of R'.

§ 4.2 Fuzzy ν-primary ideals.

If it so happens that for a fuzzy primary ideal μ, the radical of μ $\sqrt{\mu} = \nu$, then we call μ, a fuzzy ν-primary ideal. If $\sqrt{\mu}$ is fuzzy prime then μ is called fuzzy semiprimary. This section deals with such cases, Yue [Zah 1] discussed these ideas in 1988. We point out some deficiencies in his approach.

Definition 4.2.1.
Let μ, ν be fuzzy ideals of R. We say that μ is *ν-primary* if μ is a fuzzy primary ideal and $\nu = \sqrt{\mu}$.

We prove the following Proposition for later use in Chapter 5

Theorem 4.2.2.
Let $\mu_1, \mu_2, \ldots \mu_n, \nu, \omega$ be fuzzy ideals of R.

(1) *If for each $i = 1, 2, \ldots n$, μ_i is ν-primary, then so is $\bigwedge_{i=1}^{n} \mu_i$,*
(2) *If the intersection $\bigwedge_{i=1}^{n} \mu_i$ is fuzzy ν-primary and $\omega \nleq \nu$, then $\bigwedge_{i=1}^{n} \mu_i : \omega$ is fuzzy ν-primary.*

Proof.

(1) By Proposition[2.2.5], $\bigwedge_{i=1}^{n} \mu_i$ is a fuzzy ideal of R. Let ν_1, ν_2 be fuzzy ideals such that $\nu_1\nu_2 \leq \bigwedge_{i=1}^{n} \mu_i$; then $\nu_1\nu_2 \leq \mu_i$ for all $i = 1, 2 \ldots, n$. Since each μ_i is fuzzy primary, for each $i = 1, 2 \ldots, n$ either $\nu_1 \leq \mu_i$ or $\nu_2 \leq \sqrt{\mu}_i$. If $\nu_1 \leq \mu_i$ for all $i = 1, 2 \ldots n$, then $\nu_1 \leq \bigwedge_{i=1}^{n} \mu_i$. If $\nu_1 \nleq \mu_{i_0}$ for some i_0 such that $1 \leq i_0 \leq n$, then $\nu_2 \leq \sqrt{\mu}_{i_0}$. Since $\sqrt{\bigwedge_{i=1}^{n} \mu_i} = \bigwedge_{i=1}^{n} \sqrt{\mu}_i = \bigwedge_{i=1}^{n} \nu = \nu = \sqrt{\mu}_{i_0}$, we have $\nu_2 \leq \sqrt{\bigwedge_{i=1}^{n} \mu_i}$. Thus $\bigwedge_{i=1}^{n} \mu_i$ is fuzzy ν-primary.
(2) By Proposition[2.2.9], $(\bigwedge_{i=1}^{n} \mu_i : \omega)\omega \leq \bigwedge_{i=1}^{n} \mu_i \leq \nu$. Since ν is fuzzy prime and $\omega \nleq \nu$, $\bigwedge_{i=1}^{n} \mu_i : \omega \leq \nu$. By the same Proposition[2.2.9], $\nu = \sqrt{\bigwedge_{i=1}^{n} \mu_i} \leq \sqrt{(\bigwedge_{i=1}^{n} \mu_i : \omega)}$ i.e. $\bigwedge_{i=1}^{n} \mu_i : \omega \leq \nu \leq \sqrt{(\bigwedge_{i=1}^{n} \mu_i : \omega)}$. Now, let ν_1, ν_2 be fuzzy ideals of R such that $\nu_1\nu_2 \leq \bigwedge_{i=1} \mu_i : \omega$, then $\nu_1\nu_2 \circ \omega \leq \bigwedge_{i=1}^{n} \mu_i$. It follows that $\nu_1(\nu_2\omega) \leq \bigwedge_{i=1}^{n} \mu_i$. Since $\bigwedge_{i=1}^{n} \mu_i$ is ν-primary by Theorem[4.1.13], either $\nu_1 \leq \bigwedge_{i=1}^{n} \mu_i$ or $\nu_2\omega \leq \nu$. Since ν is fuzzy prime and $\omega \nleq \nu$ we have $\nu_2 \leq \nu$, i.e. either $\nu_1 \leq \bigwedge_{i=1}^{n} \mu_i : \omega$ or $\nu_2 \leq \nu$. Hence by Theorem[4.1.13], $\bigwedge_{i=1}^{n} \mu_i : \omega$ is ν-primary.

□

Proposition 4.2.3.
Let μ be ν-fuzzy primary ideal of R and x_r, y_s $r, s \in [0,1]$ be fuzzy points. If $x_r y_s \in \mu$ and $x_r \notin \nu$, then $y_s \in \mu$.

Proof.
Suppose $x_r y_s \in \mu$, then $< x_r >< y_s > = < x_r y_s > \leq \mu$. Since R is a commutative ring, we can rewrite this as $< y_s x_r > \leq \mu$, and hence either $< y_s > \leq \mu$ or $< x_r > \leq \sqrt{\mu}$. Since $x_r \notin \nu = \sqrt{\mu}$, $< y_s > \leq \mu$. So $y_s \in \mu$.

□

According to Yue[Zah 1], a fuzzy ideal μ is fuzzy prime if for all $x, y \in R$, $\mu(xy) = \mu(0)$ implies either $\mu(x) = \mu(0)$ or $\mu(y) = \mu(0)$; and μ is fuzzy primary if $\mu(xy) = \mu(0)$, then either $\mu(x) = \mu(0)$ or $\mu(y^n) = \mu(0)$ for some $n \geq 1$. He defined fuzzy ν-primary ideal in terms of the following Proposition.

Proposition 4.2.4[Zah 1, 3.1].
Let ν be a fuzzy ideal of R and μ be a fuzzy primary ideal of R such that $x \in \nu_0$ if and only if $x^n \in \mu_0$ for some $n \geq 1$. Then

(1) *μ is a fuzzy prime ideal of R,*
(2) $\mu_0 \subseteq \nu_0$,
(3) *If $\mu_0 \subseteq \omega_0$ for any fuzzy prime ideal ω of R then $\nu_0 \subseteq \omega_0$.*

Remark.
Let μ be a fuzzy primary ideal of R and ν be a fuzzy prime ideal as defined in the above Proposition. Then according to Yue[Zah 1] μ is a fuzzy ν-primary ideal of R. One can easily check that Definition[4.2.1] implies Yue's Definition but the converse implication is not true in general. The following example illustrates this.

Example 4.2.5.
Define the fuzzy ideals $\mu, \nu : \mathbb{Z}_6 \longrightarrow I$ by

$$\mu(0) = \mu(2) = \mu(4) = 0.9, \mu(1) = \mu(3) = \mu(5) = 0.7 \quad \text{and}$$

$$\nu(0) = \nu(2) = \nu(4) = 1, \nu(1) = \nu(3) = \nu(5) = 0.8.$$

Then clearly μ is fuzzy primary and ν is fuzzy prime according to Yue, and also $x \in \nu_0$ if and only if $x^n \in \mu_0$ for some $n \geq 1$. But $\sqrt{\mu} \neq \nu$ since $\sqrt{\mu}(0) = 0.9 \neq \nu(0) = 1$.

Hence we can conclude that Yue's definition for fuzzy ν-primary is a more general definition than others but there are not many results developed with this definition for further study in primary, primary decomposition, irreducible, etc.

In this section we define and study the concept of fuzzy semiprimary ideal. Consider the following:

We proved in Proposition[4.1.7] that if μ is fuzzy primary then the radical of μ, $\sqrt{\mu}$, is fuzzy prime and the example[4.1.8] shows that the converse turns out to be false. Hence

Definition 4.2.6.
Let μ be a fuzzy ideal of R. We say that μ is *fuzzy semiprimary* if $\sqrt{\mu}$ is fuzzy prime ideal of R.

It is clear that every fuzzy primary ideal is fuzzy semiprimary and the same example [4.1.8] serves as an example to show that the converse not true.

Proposition 4.2.7.
Let J be an ideal of R and $k \in [0,1]$, then the fuzzy ideal λ_{J_k} is fuzzy semiprimary if and only if J is a primary ideal of R.

The proof is straightforward.

Proposition 4.2.8.
If μ is a fuzzy semiprimary ideal then all of its level subsets are semiprimary ideals of R.

Proof.
The result is obvious if μ is a constant fuzzy ideal. Otherwise, since $\sqrt{\mu}$ is fuzzy prime we have $Im(\sqrt{\mu}) = \{1, t\}$ for some $t \in [0,1)$. It follows that there are only two level subset $\sqrt{\mu}_t = R$ and $\sqrt{\mu}_0$ which are semiprimary ideals of R.

□

The following example shows that if all level subsets of a fuzzy ideal μ are semiprimary, μ need not be fuzzy semiprimary.

Example 4.2.9.
Define a fuzzy ideal $\mu : \mathbb{Z} \longrightarrow I$ by

$$\mu(x) = \begin{cases} 0.9 & \text{if} \quad x \in <4> \\ 0.8 & \text{if} \quad x \in <2> \setminus <4> \\ 0.7 & \text{otherwise} \end{cases}$$

Then its level subsets $<4>, <2>, \mathbb{Z}$ are all semiprimary ideals of $\mathbb{Z}$ but $\sqrt{\mu}$ is not a fuzzy prime ideal since $\sqrt{\mu}(0) \neq 1$ (i.e.) μ is not fuzzy semiprimary.

As in the previous Chapter, we now look at the effect of a homomorphism on fuzzy semiprimary ideals of R.

Proposition 4.2.10.
Let $f : R \longrightarrow R'$ be a homomorphism.

(1) *If f is an epimorphism and μ a f-invariant fuzzy semiprimary ideal of R then $f(\mu)$ is fuzzy semiprimary ideal of R',*
(2) *If μ' is a fuzzy semiprimary ideal of R' then $f^{-1}(\mu')$ is a fuzzy semiprimary ideal of R.*

Proof.

(1) Let ν_1, ν_2 be any fuzzy ideals of R such that $\nu_1\nu_2 \leq \sqrt{f}(\mu) = f(\sqrt{\mu})$ by the Proposition[3.3.17] then

$$\begin{aligned} f^{-1}(\nu_1)f^{-1}(\nu_2) &\leq f^{-1}(\nu_1\nu_2) && \text{by the Proposition[2.2.11]} \\ &\leq f^{-1}(f(\sqrt{\mu})) \\ &= \sqrt{f}^{-1}(f(\mu)) && \text{by the Proposition[3.3.18]} \\ &= \sqrt{\mu}. \end{aligned}$$

Since $\sqrt{\mu}$ is fuzzy prime , either $f^{-1}(\nu_1) \leq \sqrt{\mu}$ or $f^{-1}(\nu_2) \leq \sqrt{\mu}$. So $\nu_1 \leq f(\sqrt{\mu}) = \sqrt{f}(\mu)$ or $\nu_2 \leq f(\sqrt{\mu}) = \sqrt{f}(\mu)$ and hence $\sqrt{f}(\mu)$ is a fuzzy prime. Thus $f(\mu)$ is fuzzy semiprimary.
(2) can be proved in a similar way as in (1).

□

Hence there is a one-to-one correspondence between the set of all f-invariant fuzzy semiprimary ideals of R and the set of all fuzzy semiprimary ideals of R', where f is epimorphism.

§ 4.3 Fuzzy maximal ideals .

In the crisp case, we say that an ideal M of R is a maximal if M is maximal element in the set of all proper ideals of R under inclusion. Swamy and Swamy [Swa 1] fuzzified the above concept as follows: A fuzzy ideal μ is a fuzzy maximal ideal of R if μ is a maximal element in the set of all non-constant fuzzy ideals of R under the pointwise partial ordering. But Malik and Mordeson [Mal 3, 3.2] gave a slightly different but more general definition. We take up the study of these concepts in this section.

Definition 4.3.1.
Let μ be a non-constant fuzzy ideal of R. Then μ is said to be *fuzzy maximal* if for any fuzzy ideal ν of R, $\mu \leq \nu$ implies either $\mu_0 = \nu_0$ or $\nu = \mathcal{X}_R$.

Theorem 4.3.2.
Let μ be a fuzzy maximal ideal of R, then $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$ and μ_0 is a maximal ideal of R.

Proof.
Let us prove first $\mu(0) = 1$; Suppose $\mu(0) \neq 1$ then we can find a $t \in (0, 1)$ such that $\mu(0) < t < 1$. We now define a fuzzy ideal ν of R by $\nu(x) = t$ for all $x \in R$. Since $\forall x \in R, \quad \mu(x) \leq \mu(0)$, we have $\mu \leq \nu$ and further $\mu_0 \neq \nu_0$ and $\nu \neq \mathcal{X}_R$. It is a contradiction to the maximality of μ. Hence $\mu(0) = 1$. Next we prove $Im(\mu) = \{1, t\}$. Let $t \in Im(\mu)$ such that $0 \leq t < 1$ then $\mu_0 \subsetneq \mu_t$. Let ω be a fuzzy ideal of R such that $\omega(x) = 1$ if $x \in \mu_t$ and $w(x) = t$ if $x \notin \mu_t$. Then it can be easily checked that $\mu \leq \omega$. Since μ is fuzzy maximal either $\mu_0 = \omega_0$ or $\omega = \mathcal{X}_R$, but $\omega_0 = \mu_t \supsetneq \mu_0$ so $\omega = \mathcal{X}_R$ which implies $\omega_0 = R = \mu_t$. Hence for any $t \in Im(\mu)$ with $0 \leq t < 1$, $\mu_t = R$. Thus $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$. To prove the last part, since μ is fuzzy maximal, $\mu_0 \neq R$. Let J be an ideal of R such that $\mu_0 \subseteq J$. Define a fuzzy ideal ν' of R by $\nu'(x) = 1$ if $x \in J$ and $\nu'(x) = s$ if $x \notin J$ for some $t < s < 1$. Then it is clear that $\mu \leq \nu'$. By the maximality of μ either $\mu_0 = \nu'_0$ or $\nu' = \mathcal{X}_R$. Hence $\mu_0 = \nu'_0$, then $\mu_0 = J$ since $\nu'_0 = J$ and if $\nu' = \mathcal{X}_R$, then $J = \nu'_0 = R$. Thus μ_0 is a maximal ideal of R.

□

Theorem 4.3.3[Mal 3, 3.7].
Let μ be a fuzzy ideal of R. If μ_0 is a maximal ideal of R and $\mu(0) = 1$ then μ is a fuzzy maximal ideal of R.

Proof.
We first show that the image, $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$. Since μ_0 is a maximal ideal $\mu_0 \neq R$. So there exists $x \in R$ such that $\mu(x) < \mu(0)$. Hence μ has at least two distinct values. Let $t \in Im(\mu)$ such that $0 \leq t < 1$, then $\mu_0 \subsetneq \mu_t$. By the maximality of μ_0, $\quad \mu_t = R$. Hence $Im(\mu) = \{1, t\}$. We now let ν be any fuzzy ideal of R such that $\mu \leq \nu$. Then $\nu(0) = 1$ and $\mu_0 \subseteq \nu_0$. Since μ_0 is maximal, either $\mu_0 = \nu_0$ or $\nu_0 = R$ i.e. either $\mu_0 = \nu_0$ or $\nu = \mathcal{X}_R$. Thus the result follows.

□

One can easily check that if μ is a maximal element in the set of all non-constant fuzzy ideals of R under the pointwise partial ordering, then μ is a fuzzy maximal ideal, but the converse turns out to be false.

Example 4.3.4.
Define the fuzzy ideals $\mu, \nu : \mathbb{Z} \longrightarrow I$ by

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x \in <2> \\ 1/2 & \text{otherwise} \end{cases}$$

and

$$\nu(x) = \begin{cases} 1 & \text{if} \quad x \in < 2 > \\ 3/4 & \text{otherwise.} \end{cases}$$

Since $Im(\mu) = \{1, 1/2\}$ and $\mu_0 =< 2 >$ is a maximal ideal of $\mathbb{Z}$, μ is fuzzy maximal ideal of $\mathbb{Z}$ and $\mu \lneq \nu$, i.e. μ is not a maximal element in the set of all non-constant fuzzy ideals of $\mathbb{Z}$. Thus every fuzzy maximal ideal in the sense of Swamy and Swamy is a fuzzy maximal ideal according to [4.3.1], but not vice versa.

Remark.

In fact,the converse is true in a complete lattice with the infinite meet distributive law under a certain condition namely the existence of a dual atom [Swa 1, 3.1]. Let $\mu : R \longrightarrow L$ be a fuzzy maximal ideal of R. Then we have $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$ and μ_0 is a maximal ideal of R. If t is a dual atom of L then μ is a maximal element in the set of all non-constant fuzzy ideals of R. Suppose ν is a non-constant fuzzy ideal of R such that $\mu \leq \nu$, then $\mu_0 \subseteq \nu_0$. Since μ_0 is a maximal ideal and $\nu \neq \mathcal{X}_R$ we have $\mu_0 = \nu_0$ and hence $\forall x \in \mu_0, \quad \mu(x) = \nu(x)$. If $x \notin \mu_0$, then $t = \mu(x) \leq \nu(x) < 1$. Since t is a dual atom, $t = \nu(x)$. Thus $\mu = \nu$.

Corollary 4.3.5.

Every fuzzy maximal ideal of R is a fuzzy prime ideal.

Proposition 4.3.6.

Let μ be a non-constant fuzzy ideal of R. Then there exists a fuzzy maximal ideal ν of R such that $\mu \leq \nu$.

Proof.

Since μ is non-constant there exists $x \in R$ such that $\mu(x) < \mu(0)$. Let $t \in [0, 1)$ such that $\mu(x) < t < \mu(0)$, then μ_t is a proper ideal of R. So there is a maximal ideal M of R containing μ_t. We define a fuzzy ideal ν of R by $\nu(x) = 1$ if $x \in M$ and $\nu(x) = t$ if $x \notin M$. Then by Theorem[4.3.3], ν is a fuzzy maximal ideal and it is clear that $\mu \leq \nu$.

□

In the following Proposition, we characterize fuzzy a maximal ideal in terms of its membership values.

Proposition 4.3.7[Kum 6].

Let μ be a fuzzy ideal of R. Then μ is fuzzy maximal if and only if $\mu(0) = 1, \quad \mu(1) < \mu(0)$ and whenever $\mu(x) < \mu(0)$ for some $x \in R$, $\mu(1 - rx) = \mu(0)$ for some $r \in R$.

Proof.

Suppose μ is a fuzzy maximal ideal. Then by Proposition[4.3.2], $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$ and μ_0 is a maximal ideal. Since $\forall x \in R, \quad \mu(1) \leq \mu(x)$, we have

$\mu(1) < \mu(0)$. Let $x \in R$ such that $\mu(x) < \mu(0)$; then $x \notin \mu_0$. Since μ_0 is maximal $\mu_0 + < x >= R$. Hence there exists $r \in R$ such that $1 = y + rx$ for some $y \in \mu_0$. So $\mu(1 - rx) = \mu(y) = \mu(0)$. Conversely suppose μ satisfies the three hypotheses. It is enough to show that μ_0 is a maximal ideal of R. Let M be an ideal of R such that $\mu_0 \subsetneq M$. Then there exists $x \in M$ such that $x \notin \mu_0$. Hence $\mu(1 - rx) = \mu(0)$ for some $r \in R$, so $1 - rx \in \mu_0$. It follows that $1 \in M$ and hence $M = R$.

□

Theorem 4.3.8[Kum 3, 2.6,2.10,2.12].
Let ω be the intersection of all fuzzy maximal ideals of R then the following results are true.

(1) *If μ is any non-constant fuzzy ideal such that $\mu \le \omega$, then $x + 1$ is invertible for each $x \in \mu_0$,*
(2) *$x \in R$ is invertible if and only if ω_x is invertible, where ω_x is defined in just before Theorem[2.4.8],*
(3) *The ring R/ω is semisimple .*

Proof.

(1) Suppose there exists $x \in \mu_0$ such that $x + 1$ is not invertible then there is a maximal ideal M of R containing $x+1$. We now define a fuzzy ideal $\nu : R \longrightarrow I$ by
$$\nu(y) = \begin{cases} 1 & \text{if} \quad y \in M \\ t & \text{if} \quad y \notin M, \qquad \text{where } 0 \le t < \mu(0). \end{cases}$$
Then by Theorem[4.3.3.], ν is a fuzzy maximal ideal. Hence $t < \mu(0) = \mu(x) \le \omega(x) \le \nu(x)$ which implies $\nu(x) = 1$ i.e. $x \in M$. But $x + 1 \in M$, so $1 \in M$ which is a contradiction. Thus the result follows.
(2) Let $x \in R$ be invertible; then there exists $y \in R$ such that $xy = 1$. It is easy to check that $\omega_x\omega_y = \omega_1$ and hence ω_x is invertible, conversely suppose ω_x is invertible. Then there exists $y \in R$ such that $\omega_x\omega_y = \omega_{xy} = \omega_1$. By Theorem[2.4.8], $\omega(xy - 1) = \omega(0) = 1$. Let M be a any maximal ideal of R and we define a fuzzy ideal ν of R by
$$\nu(x) = \begin{cases} 1 & \text{if} \quad x \in M \\ t & \text{if} \quad x \notin M, \qquad \text{where } t \in [0, 1). \end{cases}$$
Then by Theorem[4.3.3], ν is a fuzzy maximal ideal of R. So $1 = \omega(xy - 1) \le \nu(xy - 1)$ and hence $\nu(xy - 1) = 1$. Therefore $xy - 1 \in M$. It follows $x \notin M$ for all maximal ideals M of R. By Corollary[Sha 1, 3.11] x is invertible.
(3) It is enough to show that the Jacobson radical of R/ω is zero, i.e. the Jacobson radical of $R/\omega = \{\omega_0\}$. Let ω_x be any element in the Jacobson radical of R/ω.

Then ω_x must be in all maximal ideals of R/ω, hence $\omega_1 - \omega_r\omega_x$ must not be in any maximal ideal of R/ω for all $r \in R$. By Corollary[Sha 1, 3.11] $\omega_1 - \omega_r\omega_x$ is invertible. So for some $y \in R$, $\omega_{1-rx}\omega_y = \omega_{(1-rx)y} = \omega_1$. By Proposition[2.4.8], $\omega((1-rx)y-1) = \omega(0) = 1$. It follows that for each fuzzy maximal ideal ν of R, $\nu((1-rx)y-1) = 1$. Hence for each maximal ideal M of R, $(1-rx)y - 1 \in M$, which implies $(1-rx)y \notin M$ for all maximal ideals M of R. By Corollary[Sha 1, 3.11], $(1-rx)y$ is invertible, implying $(1-rx)$ is invertible. By Lemma[Sha 1, 3.17], $x \in M$ for all maximal ideals of R. Hence $\nu(x) = \nu(0)$ for every fuzzy maximal ideal ν. Since ω is the intersection of all fuzzy maximal ideals $\omega(x) = \omega(0)$, which implies $\omega_x = \omega_0$.

□

We now prove some relations between fuzzy maximal and primary ideals of R.

Proposition 4.3.9.
Let μ be a fuzzy ideal of R with $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$. If $\sqrt{\mu}$ is fuzzy maximal then μ is a fuzzy primary ideal of R.

Proof.
Since $Im(\mu) = \{1, t\}$, by Proposition[3.3.15,7]

$$\sqrt{\mu}(x) = \begin{cases} 1 & \text{if} \quad x \in \sqrt{\mu}_0 \\ t & \text{otherwise.} \end{cases}$$

Since $\sqrt{\mu}_0$ is maximal, μ_0 is primary ideal of R. Hence by Theorem[4.1.5], μ is a fuzzy primary ideal of R.

□

Proposition 4.3.10.
If μ is a fuzzy maximal ideal of R then μ^n is a μ-primary ideal for all $n \in \mathbb{N}$.

Proof.
Since μ is fuzzy maximal, by Theorem[4.3.3], $Im(\mu) = \{1, t\}$ for some $t \in [0, t)$ and μ_0 a is maximal ideal of R. Let $x \in R$. We defined μ^n as

$$\mu^n(x) = \sup_{x = \Sigma_{k=1}^p x_{k1}x_{k2}\ldots x_{kn}} \wedge_{k=1}^p \wedge_{i=1}^n \mu(x_{ki}).$$

If $x \in (\mu^n)_0 = (\mu_0)^n$, then x is of the form $x = \Sigma_{k=1}^p x_{k1}x_{k2}\ldots x_{kn}$ where $x_{ki} \in \mu_0$ for all $i = 1, 2, \ldots, n$. Hence $\mu^n(x) \geq \wedge_{k=1}^p \mu(x_{k1})\mu(x_{k2})\ldots\mu(x_{kn}) = 1$, so $\mu^n(x) = 1$. If $x \notin (\mu^n)_0$, then for each $x = \Sigma_{k=1}^p x_{k1}x_{k2}\ldots x_{kn}$ there exists i_0 such that $1 \leq i_0 \leq n$ and $x_{i_0} \notin \mu_0$ which implies $\mu^n(x) = t$. So we have

$$\mu^n(x) = \begin{cases} 1 & \text{if} \quad x \in (\mu^n)_0 \\ t & \text{otherwise .} \end{cases}$$

Since μ_0 is maximal, by Proposition[Sha 1, 4.9], $(\mu_0)^n$ is a μ_0-primary ideal of R for all $n \in \mathbb{N}$, i.e. $\sqrt{(\mu^n)_0} = \mu_0$. But by Proposition[3.3.15,7] we have

$$\sqrt{\mu}^n(x) = \begin{cases} 1 & \text{if} \quad x \in \sqrt{\mu_0^n} = \mu_0 \\ t & \text{otherwise} \end{cases}$$

and hence $\sqrt{\mu}^n = \mu$. Thus the result follows.

□

Corollary 4.3.11[Dix 1, 5.2].
Let μ be a fuzzy ideal of R. If μ is a fuzzy maximal ideal then R/μ is a field.

Proof is straightforward since μ_0 is maximal ideal and $R/\mu_0 \cong R/\mu$.

Remark.
In[Dix 1, 5.2] V.N.Dixit et al. stated that the converse of the above Corollary is also true, but we provide a counter-example to disprove it.

Example 4.3.12.
Let R be the ring $F[x,y]$. We define a fuzzy ideal $\mu : R \longrightarrow I$ by

$$\mu(f) = \begin{cases} 0.9 & \text{if} \quad f \in <x, y, 2> \\ 0.8 & \text{otherwise.} \end{cases}$$

Then μ_0 is a maximal ideal [Bar 1, pg 125]. So R/μ_0 is field and hence R/μ is field. But μ is not a fuzzy maximal ideal since $\mu(0) \neq 1$.

Proposition 4.3.13[Kum 6, 3.20].
Let μ, μ' be distinct fuzzy maximal ideals of R such that $Im(\mu) = Im(\mu')$. Then $\mu\mu' = \mu \wedge \mu'$.

Proof.
Since μ, μ' are maximal by Theorem[4.3.2], $Im(\mu) = \{1, t\} = Im(\mu')$ for some $t \in [0, 1)$ and μ_0, μ'_0 are distinct maximal ideals of R. Hence we have

$$(\mu \wedge \mu')(x) = \begin{cases} 1 & \text{if} \quad x \in \mu_0 \cap \mu'_0 \\ t & \text{otherwise .} \end{cases}$$

We now prove that

$$\mu\mu'(x) = \begin{cases} 1 & \text{if} \quad x \in \mu_0\mu'_0 \\ t & \text{otherwise .} \end{cases}$$

Let $x \in R$. If $x \in \mu_0\mu'_0$, then x is of the form $x = \Sigma_{k=1}^{p} x_{k1}x_{k2}$, where $x_{k1} \in \mu_0, x_{k2} \in \mu'_0$.

$$\mu\mu'(x) = \sup_{x=\Sigma_{k=1}^{n} y_{k1}y_{k2}} \wedge_1^n \mu(y_{k1}) \wedge \mu'(y_{k2}) \geq \wedge_1^p \mu(x_{k1}) \wedge \mu'(x_{k2}) = 1$$

hence $\mu\mu'(x) = 1$. If $x \notin \mu_0\mu'_0$, then for each $x = \Sigma_{k=1}^{n} y_{k1}y_{k2}$ either $y_{k_01} \notin \mu_0$ or $y_{k_02} \notin \mu'_0$ for some k_0 with $1 \leq k_0 \leq n$. So $\mu\mu'(x) = t$. Since μ_0 and μ'_0 are maximal ideals of R, $\mu_0\mu'_0 = \mu_0 \cap \mu'_0$, hence $\mu\mu' = \mu \wedge \mu'$.

□

In the following Proposition, we consider the effect of homomorphisms on fuzzy maximal ideals

Proposition 4.3.14.

Let $f : R \longrightarrow R'$ be a homomorphism, μ and μ' be fuzzy maximal ideals of R and R' respectively then

(1) *If f is an epimorphism and μ is f-invariant, then $f(\mu)$ is a fuzzy maximal ideal of R',*
(2) *$f^{-1}(\mu')$ is a fuzzy maximal ideal of R.*

Proof.

(1) Let $y \in R'$, then $f(\mu)(y) = \sup\limits_{f(x)=y} \mu(x)$. Since μ_0 is maximal and $\ker f \subseteq \mu_0$, $f(\mu_0)$ is a maximal ideal of R'. If $y \in f(\mu_0)$ then there exists $x \in \mu_0$ such that $f(x) = y$, therefore $f(\mu)(y) = 1$. If $y \notin f(\mu_0)$ then for all $x \in R$ such that $f(x) = y$, $x \notin \mu_0$, hence $\mu(x) = t$, $t \in [0, 1)$. So $f(\mu)(x) = t$, i.e. $Imf(\mu) = \{1, t\}$, and then the result follows.
(2) Proof is similar to (1).

□

CHAPTER V

FUZZY PRIMARY DECOMPOSITION AND FUZZY IRREDUCIBLE IDEALS

§ 5.1 Fuzzy primary decompositions .

This section deals with fuzzy primary decompositions of a fuzzy ideal. The fuzzy case is dealt with by looking at the primary decomposition of the base set μ_0. In Theorem[5.1.7], we prove an analogue in the fuzzy setting to an important result in the crisp case which leads us to the First Uniqueness Theorem for fuzzy primary decomposition.

Definition 5.1.1[Mal 4, 3.8].
Let μ be a fuzzy ideal of R. A *fuzzy primary decomposition* or *representation* of μ means that μ is an intersection of finitely many fuzzy primary ideals ν_i of R i.e. $\mu = \bigwedge_{i=1}^{n} \nu_i$.

This primary decomposition is called *irredundant* or *reduced* if for $i = 1, 2, \ldots, n$ $\bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \nu_j \nleq \nu_i$ and $\sqrt{\nu}_i$'s are all distinct.

Example 5.1.2.
Let F be a field and $R = F[x, y]$ be the ring of polynomials in x and y over F. Define the fuzzy ideals μ, ν_k, ν_2 of R by

$$\mu(f) = \begin{cases} 1 & \text{if} \quad f \in < x^3, xy > \\ 2/3 & \text{otherwise,} \end{cases}$$

$$\nu_k f = \begin{cases} 1 & \text{if} \quad f \in < x^3, xy, y^k > \\ 2/3 & \text{otherwise} \end{cases}$$

and

$$\nu_2(f) = \begin{cases} 1 & \text{if} \quad f \in < x > \\ 2/3 & \text{otherwise.} \end{cases}$$

Then $\mu = \nu_k \wedge \nu_2$ is a reduced primary decomposition of μ for each $k = 3, 4, \ldots$. Hence the fuzzy reduced primary decomposition for a given fuzzy ideal needs not be unique.

The next Proposition illustrates a method of extracting a reduced primary decomposition from a given primary decomposition.

Proposition 5.1.3.
Let μ be a fuzzy ideal of R. If μ has a fuzzy primary decomposition then μ has a reduced fuzzy primary decomposition.

Proof.
Let $\mu = \bigwedge_{i=1}^{n} \nu_i$ be a fuzzy primary decomposition of R. Firstly if for some $\nu_{i1}, \nu_{i2}, \ldots \nu_{im} \in \{\nu_1, \ldots \nu_n\}$, $\sqrt{\nu}_{i1} = \sqrt{\nu}_{i2} = \ldots \sqrt{\nu}_{im}$ then take $\nu_i' = \bigwedge_{l=1}^{m} \nu_{il}$. By Proposition[4.2.2,1], ν_i' is a fuzzy primary ideal. Hence $\mu = \bigwedge_{i=1}^{p} \nu_i'$, $1 \leq p \leq n$ and all $\sqrt{\nu_i'}$ are distinct. Secondly we discard ν_1' if $\nu_1' \geq \bigwedge_{j=2}^{p} \nu_j'$ and then consider the remaining $\nu_2', \nu_3', \ldots, \nu_p'$; at the kth stage, discard ν_k' if $\nu_k' \geq \bigwedge_{\substack{j=1 \\ j \neq k}}^{p} \nu_j'$. By contuining in this way , we get a stage after which we have $\mu = \bigwedge_{i=1}^{q} \nu_i'$, $1 \leq q \leq p$ and for all $i = 1, 2, \ldots, q$, $\quad \nu_i' \not\geq \bigwedge_{\substack{j=1 \\ j \neq i}}^{q} \nu_j'$. Hence $\mu = \bigwedge_{j=1}^{q} \nu_j'$, $\quad 1 \leq q \leq p$ is a fuzzy reduced primary decomposition of μ.

□

Theorem 5.1.4.
Let μ be a fuzzy ideal of R. If μ has a fuzzy primary decomposition, then μ_0 has a primary decomposition. Further if the primary decomposition of μ_0 is reduced, then the fuzzy primary decomposition of μ is reduced.

Proof.
Let $\mu = \bigwedge_{i=1}^{n} \nu_i$ be a fuzzy primary decomposition of μ then $\mu_0 = \left(\bigwedge_{i=1}^{n} \nu_i\right)_0 = \cap_{i=1}^{n} (\nu_i)_0$. Since each ν_i is fuzzy primary, by Theorem[4.1.2], $(\nu_i)_0$ is a primary ideal of R for all $i = 1, 2, \ldots, n$. Hence $\mu_0 = \cap_{i=1}^{n} (\nu_i)_0$ is a primary decomposition of μ_0. Suppose $\mu_0 = \cap_{i=1}^{n} (\nu_i)_0$ is reduced and for some i, $\quad \nu_i \geq \bigwedge_{\substack{j=1 \\ j \neq i}} \nu_j$. Then

$(\nu_i)_0 \supseteq \left(\bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \nu_j\right)_0 = \cap_{\substack{j=1 \\ j \neq i}} (\nu_j)_0$ since $\nu_j(0) = 1$ for all $i = 1, 2, \ldots n$. This is a contradiction to the fact that $\mu_0 = \cap_{i=1}^{n} (\nu_i)_0$ is a reduced primary decomposition. Thus the result follows.

□

In general we believe that μ need not have a fuzzy primary decomposition even if μ_0 has a primary decomposition; but we do not have an example. However, we have an example to support the reduced case , i.e. if μ has a reduced fuzzy primary decomposition it does not necessarily imply μ_0 has one.

Example 5.1.5.
Let R denote the ring defined in example[5.1.2]. Define the fuzzy ideals

$\mu, \nu_1, \nu_2 : R \longrightarrow I$ by

$$\mu(x) = \begin{cases} 1 & \text{if} \quad f \in < x^3, xy > \\ 3/4 & \text{if} \quad f \in < x^3, xy, y^3 > \setminus < x^3, xy > \\ 1/4 & \text{otherwise} \end{cases}$$

$$\nu_1(x) = \begin{cases} 1 & \text{if} \quad f \in < x^3, xy > \\ 3/4 & \text{otherwise} \end{cases}$$

and

$$\nu_2(x) = \begin{cases} 1 & \text{if } f \in < x^3, xy, y^3 > \\ 1/4 & \text{otherwise.} \end{cases}$$

Then it is easy to check that $\mu = \nu_1 \wedge \nu_2$ and ν_1, ν_2 are fuzzy primary ideals of R. Since $\nu_1 \not\leq \nu_2, \nu_2 \not\leq \nu_1$ and $\sqrt{\nu_1} \neq \sqrt{\nu_2}$, $\mu = \nu_1 \wedge \nu_2$ is a fuzzy reduced primary decomposition . But $\mu_0 = (\nu_1)_0 \wedge (\nu_2)_0$ and $(\nu_1)_0 \subseteq (\nu_2)_0$ and hence $\mu_0 = (\nu)_0 \cap (\nu_2)_0$ is not a reduced primary decomposition.

Proposition 5.1.6.
Every finite valued fuzzy ideal with $\mu(0) = 1$ has a fuzzy primary decomposition if and only if every ideal of R has a primary decomposition.

Proof.
Let J be an ideal of R, then the characteristic function $\mathcal{X}_J$ is a fuzzy ideal of R with $\mathcal{X}_J(0) = 1$, so it has a fuzzy primary decomposition by the hypothesis. By Theorem[5.1.4], J has a primary decomposition.
For the sufficient part, let μ be any fuzzy ideal such that $Im(\mu) = \{t_0, t_1, \ldots, t_n\}$ and $t_n < t_{n-1} < \cdots < t_0 = \mu(0)$ then $\mu_{t_0} \subset \mu_{t_1} \subset \cdots \subset \mu_{t_n}$. For each $i = 1, 2, \ldots, n$ define a fuzzy ideal $\nu_i : R \longrightarrow I$ by

$$\nu_i(x) = \begin{cases} 1 & \text{if} \quad x \in \mu_{t_{i-1}} \\ t_i & \text{otherwise .} \end{cases}$$

Let $x \in R$ and $\mu(x) = t_k, \quad 1 \leq k \leq n$ then $x \in \mu_{t_i}$ for all $i \geq k$ and $x \notin \mu_i$ for all $i < k$. Hence $\nu_i(x) = 1$ for all $i \geq k+1$ and $\nu_i(x) = t_i$ for all $i \leq k$. Therefore $\bigwedge_{i=1}^n \nu_i(x) = t_k = \mu(x)$. Thus $\mu = \bigwedge_{i=1}^n \nu_i$ and since μ is a fuzzy ideal for each $i = 0, 1, 2, \ldots, n$, μ_{t_i} is an ideal of R. By the hypothesis μ_{t_i} has a primary decomposition (say) $\cap_{k=1}^{m_i} J_{ik}$. Hence it is easy to check that $\nu_{i+1} = \cap_{k=1}^{m_i} \lambda_{J_{ik}}^{t_i}$ for all $i = 0, 1, 2, \ldots, n-1$. Since each J_{ik} is a primary ideal by Proposition[4.1.5], $\lambda_{J_{ik}}^{t_i}$ is a fuzzy primary ideal of R implying that μ has a fuzzy primary decomposition, where

$$\lambda_{J_{ik}}^{t_i}(x) = \begin{cases} 1 & \text{if} \quad x \in J_{ik} \\ t_i & \text{if} \quad x \notin J_{ik}. \end{cases}$$

□

Theorem 5.1.7.
Let μ be a fuzzy ideal of R and $\mu = \bigwedge_{i=1}^{n} \nu_i$ be a reduced fuzzy primary decomposition of μ. Let ω be a fuzzy prime ideal of R. Then the following are equivalent.

(1) $\omega = \sqrt{\nu_i}$ *for some i with $1 \leq i \leq n$,*

(2) *there exists a fuzzy ideal λ of R such that $\lambda \nleq \mu$ and $(\mu : \lambda)$ is a ω-fuzzy primary ideal,*

(3) *there exists a fuzzy ideal λ of R such that $\lambda \nleq \mu$ and $\sqrt{(\mu : \lambda)} = \omega$.*

Proof.
(1)⇒(2) Let ω be a fuzzy ideal of R such that $\omega = \sqrt{\nu_i}$ for some i with $1 \leq i \leq n$. Since $\mu = \bigwedge_{j=1}^{n} \nu_j$ is reduced, $\bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \nu_j \nleq \nu_i$, which implies there exists $x_i \in R$ such that $\bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \nu_j(x_i) > \nu_i(x_i)$. Let $t = \bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \nu_j(x_i)$ and define a fuzzy ideal $\lambda : R \longrightarrow I$ by

$$\lambda(x) = \begin{cases} t & \text{if} \quad x \in < x_i > \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mu(x_i) = \bigwedge_{j=1}^{n} \nu_j(x_i) = \nu_i(x_i) < \bigwedge_{\substack{j=1 \\ j \neq i}} \nu_j(x_i) = \lambda(x_i)$, $\lambda \nleq \mu$ and clearly $\lambda \leq \bigwedge_{\substack{j=1 \\ j \neq i}} \nu_j$; hence $\lambda \nleq \nu_i$ and $\lambda \leq \nu_j$ for all $j = 1, 2, \ldots, n$ with $j \neq i$. Further let ν be a fuzzy ideal of R such that $\nu \leq \nu_i : \lambda$ then $\lambda \circ \nu \leq \nu_i$. Since $\lambda \nleq \nu_i$ and ν_i is a fuzzy primary ideal, $\nu \leq \sqrt{\nu_i}$ and $\nu_i : \lambda \leq \sqrt{\nu_i}$. By Proposition[2.2.9(4)], $\nu_i \leq \nu_i : \lambda \leq \sqrt{\nu_i}$ from which follows that $\sqrt{\nu_i} = \sqrt{(\nu_i : \lambda)}$. Since for each $j \neq i$, $\lambda \leq \nu_j$ by Proposition[2.2.9,2],$\nu_j : \lambda = \mathcal{X}_R$. Therefore by the same Proposition[2.2.9,3], $(\mu : \lambda) = (\bigwedge_{j=1}^{n} \nu_j : \lambda) = \bigwedge_{j=1}^{n} (\nu_j : \lambda) = (\nu_i : \lambda)$ and hence $\sqrt{(\mu : \lambda)} = \sqrt{\nu_i} = \omega$.

(2)⇒(3) Straightforward.

(3)⇒(1) Suppose there exists a fuzzy ideal λ of R such that $\lambda \nleq \mu$ and $\sqrt{(\mu : \lambda)} = \omega$. Then $\omega = \sqrt{\bigwedge_{j=1}^{n} (\nu_j : \lambda)} = \bigwedge_{j=1}^{n} \sqrt{(\nu_j : \lambda)}$ by Proposition[3.3.15,5], hence $\omega \leq \sqrt{(\nu_j : \lambda)}$ for all $j = 1, 2, \ldots, n$. Since $\lambda \nleq \mu$ there exists i with $1 \leq i \leq n$ such that $\lambda \nleq \nu_i$. If $\lambda \nleq \nu_i$ for some i with $1 \leq i \leq 1$ then by Proposition[2.2.9,1], $\lambda \circ (\nu_i : \lambda) \leq \nu_i$ and ν_i is fuzzy primary, so we have $\nu_i : \lambda \leq \sqrt{\nu_i}$ hence $\omega \leq \sqrt{\nu_i}$ and $\sqrt{(\nu_i : \lambda)} = \sqrt{\nu_i}$. If $\lambda \leq \nu_i$ for some i with $1 \leq i \leq n$ then by Proposotion[2.2.9,1],$\nu_i : \lambda = \mathcal{X}_R$ from which follows that $\sqrt{(\nu_i : \lambda)} = \mathcal{X}_R$. Therefore $\sqrt{(\mu : \lambda)} = \bigwedge_{j=1}^{m} \sqrt{(\nu_j : \lambda)} = \bigwedge_{j=1}^{m} \sqrt{\nu_j} = \omega$ for some $m \leq n$. Hence $\sqrt{\nu_1}\sqrt{\nu_2} \ldots \sqrt{\nu_m} \leq \omega$, and since ω is fuzzy prime, $\sqrt{\nu_i} \leq \omega$ for some $i \leq m$. Thus $\omega = \sqrt{\nu_i}$ for some i with $1 \leq i \leq n$.

□

Proposition 5.1.8 (The First Uniqueness Theorem for Fuzzy Primary Decomposition.).
Let μ be a fuzzy ideal of R and let $\mu = \bigwedge_{i=1}^{n} \nu_i$ with $\sqrt{\nu}_i = \omega_i$ for $i = 1, 2, \ldots, n$ and $\mu = \bigwedge_{i=1}^{m} \nu'_i$ with $\sqrt{\nu}'_i = \omega'_i$ for $i = 1, 2, \ldots, m$ be fuzzy primary decompositions of μ. Then $n = m$ and $\{\omega_1, \omega_2, \ldots, \omega_n\} = \{\omega'_1, \omega'_2, \ldots, \omega'_n\}$.

Proof.
By Theorem[5.1.7], there exists a fuzzy ideal λ of R such that $\lambda \nleq \mu$ and $\sqrt{(\mu : \lambda)} = \omega_i$ for all $i = 1, 2, \ldots, n$, and hence by applying Theorem[5.1.7] to the second decomposition there exists j with $1 \leq j \leq m$ such that $\omega_i = \omega'_j$. Since it is true for each $i = 1, 2, \ldots, n$ we get $n \leq m$ and $\{\omega_1, \omega_2, \ldots \omega_n\} \subseteq \{\omega'_1, \omega'_2, \ldots \omega'_m\}$. By reversing the roles of ω_i and ω'_j we have $m \leq n$ and $\{\omega'_1, \omega'_2, \ldots \omega'_m\} \subseteq \{\omega_1, \omega_2, \ldots \omega_n\}$. Thus the result follows.

□

The above Proposition gives rise to the following:

Definition 5.1.9.
Let μ be a fuzzy ideal of R and $\mu = \bigwedge_{i=1}^{n} \nu_i$ be a reduced fuzzy primary decomposition of R with $\sqrt{\nu}_i = \omega_i$ for all $i = 1, 2, \ldots, n$. Then the n-element set $\{\omega_1, \omega_2, \ldots, \omega_n\}$ which is independent of the choice of the reduced primary decomposition of μ, is called the *set of associated fuzzy prime ideals of* μ, and we denote this set as $ass_R(\mu)$.

§ 5.2 Associated and minimal fuzzy prime ideals of μ.

In the last section we have derived a finite collection of fuzzy prime ideals associated with a given decomposition of a fuzzy ideal. In this section we introduce and study a partial ordering in the set of fuzzy prime ideals of a given fuzzy ideal. This gives rise to the notion of fuzzy minimal prime ideal. Using this we prove the Second Uniqueness Theorem.

Definition 5.2.1.
Let μ be a fuzzy ideal of R. A fuzzy prime ideal ν of R is called a *fuzzy prime ideal of* μ if $\mu \leq \nu$ and $\mu_0 \subseteq \nu_0$.

Proposition 5.2.2.
Every fuzzy ideal μ of R has at least a fuzzy prime ideal of μ,
i.e. for a given fuzzy ideal μ of R there exists a fuzzy prime ideal ν of R such that $\mu \leq \nu$ and $\mu_0 \subseteq \nu_0$.

Proof.
Let μ be a non-constant fuzzy ideal of R then $|Im(\mu)| \geq 2$.

Case 1.
Suppose $|Im(\mu)| = 2$, then for all $x \in R \setminus \mu_0$, $\mu(x) = t \neq \mu(0)$ and $\mu_t = R$ for some $t \in [0, 1)$. Since μ_0 is a proper ideal of R, there exists a prime ideal P of R which contains μ_0. We define a fuzzy ideal $\nu : R \longrightarrow I$ by

$$\nu(x) = \begin{cases} 1 & \text{if} \quad x \in P \\ t & \text{if} \quad x \notin P \end{cases}$$

then ν is a fuzzy prime ideal of R such $\mu \leq \nu$ and $\mu_0 \subseteq \nu_0$.

Case 2.
Suppose $|Im(\mu)| > 2$, then there exists $x \in R$ such that $\mu(x) = t < \mu(0)$ and $\mu_t \neq R$. Hence there exists a prime ideal P with $\mu_t \subseteq P$. We now define a fuzzy prime ideal of μ as in the Case 1. So, in any case there is a fuzzy prime ideal of R such that $\mu \leq \nu$ and $\mu_0 \subseteq \nu_0$.

□

Remark 1.
For a given fuzzy ideal μ of R we define a set $\mathcal{A}$ as

$$\mathcal{A} = \{\nu : \nu \text{ is a fuzzy prime ideal of } \mu\}.$$

Then by the above Proposition, $\mathcal{A} \neq \emptyset$. We define a relation $\preccurlyeq$ on $\mathcal{A}$ by

$$\nu_1 \preccurlyeq \nu_2 \quad \text{if and only if} \quad \nu_2 \leq \nu_1$$

then $(\mathcal{A}, \preccurlyeq)$ is a partially ordered set. Let $\mathcal{B}$ be a non-empty chain in $\mathcal{A}$ and let $\nu' = \bigwedge\limits_{\nu \in \mathcal{B}} \nu$. By Proposition[3.1.7], ν' is a fuzzy prime ideal of R such that $\mu \leq \nu'$ and $\mu_0 \subseteq \nu'_0$. Hence ν' is an upper bound for $\mathcal{B}$. By Zorn's Lemma, the partially ordered set $(\mathcal{A}, \preccurlyeq)$ has a maximal element (say) ω. In other words if there exists a fuzzy prime ideal ν of μ with $\nu \leq \omega$ then $\nu = \omega$. This fuzzy prime ideal ω is called a *fuzzy minimal prime ideal of* μ .

Remark 2.
For later use we prove that for a given fuzzy ideal μ and a fuzzy prime ideal ω of μ, there is fuzzy minimal prime ideal λ of μ such that $\lambda \leq \omega$. Let

$$\mathcal{A}_\omega = \{\nu : \nu \quad \text{is fuzzy prime ideal of } \mu \text{ and } \nu \leq \omega\}.$$

Then clearly $\mathcal{A}_\omega \neq \emptyset$ and $(\mathcal{A}_\omega, \preccurlyeq)$ is a partially ordered set. By a similar sort of argument as in Remark 1 above, we can show that $\mathcal{A}_\omega$ has a maximal element. Let λ be the maximal element of $\mathcal{A}_\omega$. We claim that λ is the fuzzy minimal prime ideal of μ such that $\lambda \leq \omega$. Let ν be a fuzzy prime ideal of μ such that $\nu \leq \lambda$, then since $\mu \leq \nu \leq \lambda \leq \omega$, $\nu \in \mathcal{A}_\omega$. Therefore $\nu = \lambda$. Thus λ is a fuzzy minimal prime ideal of μ such that $\lambda \leq \omega$.

Summerizing the above, we get the following:

Proposition 5.2.3.

Let μ be a fuzzy decomposable ideal of R and ω be any fuzzy prime ideal of R. Then ω is a fuzzy minimal prime ideal of μ if and only if ω is a maximal element of $(ass_R(\mu), \preccurlyeq)$.

Proof.

Let $\mu = \bigwedge_{i=1}^{n} \nu_i, \quad \sqrt{\nu}_i = \omega_i$ be a reduced fuzzy primary decomposition of μ and let ω be a fuzzy minimal prime ideal of μ. Then $\omega = \sqrt{\omega} \geq \sqrt{\mu} = \bigwedge_{i=1}^{n} \omega_i \geq \omega_1\omega_2 \ldots \omega_n$. Since ω_i is fuzzy prime for some i with $1 \leq i \leq n, \quad \omega \geq \omega_i$. Hence by the minimality of ω we have $\omega = \omega_i \in ass_R(\mu)$. It is easy to check that for each $\omega_i \in ass_R(\mu)$, if $\omega_i \leq \omega$ then $\omega_i = \omega$, thus ω is a maximal element of $ass_R(\mu)$.
Conversely suppose ω is a maximal element of $ass_R(\mu)$. Then ω is a fuzzy prime ideal such that $\mu \leq \omega$. By Remark 2 above there is a fuzzy minimal prime ideal λ of μ such that $\lambda \leq \omega$. Since $\lambda = \sqrt{\lambda} \geq \sqrt{\mu} \geq \omega_1\omega_2 \ldots \omega_n$, $\omega_i \leq \lambda \leq \omega$ for some $\omega_i \in ass_R(\mu)$. It follows that $\omega = \lambda = \omega_i$, hence ω is a fuzzy minimal prime ideal of μ.

□

Remark.

All fuzzy minimal prime ideals of μ belong to $ass_R(\mu)$, and hence μ has only finitely many fuzzy minimal prime ideals of μ since $ass_R(\mu)$ is a finite set. By Remark 2 and the Proposition above for any fuzzy prime ideal ν of μ, there exists a fuzzy ideal $\omega \in ass_R(\mu)$ such that $\omega \leq \nu$.

Theorem 5.2.4(The Second Uniqueness Theorem for Fuzzy Primary Decomposition).

Let μ be a fuzzy ideal of R and let $\mu = \bigwedge_{i=1}^{n} \nu_i$ with $\sqrt{\nu}_i = \omega_i$, $\mu = \bigwedge_{i=1}^{n} \nu_i'$ with $\sqrt{\nu}_i' = \omega_i, \quad i = 1, 2, \ldots n$, be reduced fuzzy primary decompositions of μ. Then if ω_j, $1 \leq j \leq n$ is a fuzzy minimal prime ideal of μ, then $\nu_j = \nu_j'$.

Remark.

Since every non-constant fuzzy ideal μ of R has a fuzzy minimal prime ideal of μ, $ass_R(\mu)$ has at least one fuzzy minimal prime ideal of μ. But not necessarily all $\omega_i \in ass_R(\mu)$ are fuzzy minimal prime ideals of μ.

Proof.

If $n = 1$ then $\mu = \nu_1 = \nu_1'$. Suppose $n > 1$. Let ω_i, $1 \leq i \leq n$ be a fuzzy minimal prime ideal of μ. Then $\omega_i \not\geq \bigwedge_{\substack{j=1 \\ j\neq i}}^{n} \omega_j$ since if $\omega_i \geq \bigwedge_{\substack{j=1 \\ j\neq i}}^{n} \omega_j$, then $\omega_i \geq \omega_1\omega_2 \ldots \omega_{i-1}\omega_{i+1} \ldots \omega_n$. Since ω_i is fuzzy prime, $\omega_i \geq \omega_j$ for some j with $1 \leq j \leq n, j \neq i$ and $\omega_i \neq \omega_j$. It is a

contradicition to the fact that ω_i is a fuzzy minimal prime ideal of μ. Therefore there exists $x_i \in R$ such that $\omega_i(x_i) < \bigwedge_{\substack{j=1 \\ j\neq i}}^{n} \omega_j(x_i)$. Let $t = \bigwedge_{\substack{j=1 \\ j\neq i}}^{n} \omega_j(x_i)$, then $x_i \in (\omega_j)_t$ for all $j = 1, 2, \ldots, n$ with $j \neq i$. This implies $x_i^{n_j} \in (\nu_j)_t$ for some non-negative integer $n_j, j = 1, 2, \ldots, n$ with $j \neq i$. Let $n_0 \in \mathbb{N}$ be such that $n_0 \geq \max\{n_j : j = 1, 2, \ldots, n, j \neq i\}$. Then $x_i^{n_0} \in (\nu_j)_t$ for all $j = 1, 2, \ldots, n$ with $j \neq i$. We now define a fuzzy ideal $\nu : R \longrightarrow I$ by

$$\nu(x) = \begin{cases} t & \text{if} \quad x \in < x_i^{n_0} > \\ 0 & \text{if} \quad x \notin < x_i^{n_0} > . \end{cases}$$

It is easy to check that $\nu \leq \nu_j$ for all $j = 1, 2, \ldots n$ with $j \neq i$, which implies that $(\nu_j : \nu) = \mathcal{X}_R$ for all $j = 1, 2, \ldots, n$ with $j \neq i$. If $x_i^{n_0} \in (\omega_i)_t$ then $x_i \in (\omega_i)_t$ since $(\omega_i)_t$ is a prime ideal of R. Hence $\omega_i(x_i) \geq t = \wedge_{\substack{j=1 \\ j \neq i}} \omega_j(x_i)$, which is a contradiction. So $x_i^{n_0} \notin (\omega_i)_t$. Therefore $\nu \nleq \omega_i$. We now claim that $(\nu_i : \nu) = \nu_i$. Clearly $\nu_i \leq (\nu_i : \nu)$. Let λ be a fuzzy ideal of R such that $\lambda \leq (\nu_i : \nu)$. Then $\lambda o \nu \leq \nu_i$ by Proposition[2.2.9]. Since ν_i is fuzzy primary and $\nu \nleq \sqrt{\nu}_i = \omega_i$, we have $\lambda \leq \nu_i$, and hence $\nu_i : \nu \leq \nu_i$. So $(\nu_i : \nu) = \nu_i$. We consider $(\mu : \nu) = (\bigwedge_{j=1}^n \nu_j : \nu) = \bigwedge_{j=1}^n (\nu_j : \nu) = (\nu_i : \nu) = \nu_i$. Similarly if $\mu = \bigwedge_{j=1}^n \nu_j'$, then by choosing sufficiently large n_0, we will have $(\mu : \nu) = \nu_i'$. Thus $\nu_i = \nu_i'$.

□

Corollary 5.2.5.
Let $\mu = \bigwedge_{i=1}^n \nu_i$ be a reduced fuzzy primary decomposition with $\sqrt{\nu}_i = \omega_i$ for $i = 1, 2, \ldots, n$. Then μ has a finite set of fuzzy minimal prime ideals $\{\omega_1, \omega_2, \ldots, \omega_m\}$, $1 \leq m \leq n$ of μ such that $\sqrt{\mu} = \bigwedge_{i=1}^m \omega_i$.

Proof.
The first part is obvious. Clearly $\sqrt{\mu} \leq \bigwedge_{i=1}^m \omega_i$. For the converse part, let ω be any fuzzy ideal such that $\omega \leq \bigwedge_{i=1}^m \omega_i$. Then $\omega \leq \omega_i$ for all $i = 1, 2, \ldots, m$. Suppose ω_j is not a maximal element in $(ass_R(\mu), \preccurlyeq)$, then there exists a maximal element ω_i, $1 \leq i \leq m$ in $(ass_R(\mu), \preccurlyeq)$ such that $\omega_i \leq \omega_j$. Hence $\omega \leq \omega_j$ for all $j = 1, 2, \ldots, n$ and so $\omega \leq \bigwedge_{i=1}^n \omega_i = \sqrt{\mu}$. Thus $\sqrt{\mu} = \bigwedge_{i=1}^m \omega_i$.

□

Proposition 5.2.6.
Let $\mu = \bigwedge_{i=1}^n \nu_i$ be a fuzzy primary decomposition of μ with $\sqrt{\nu}_i = \omega_i$ for $i = 1, 2, \ldots, n$, and $\mu_0 = \cap_{i=1}^n (\nu_i)_0$ be a reduced primary decomposition of μ_0. If $\sqrt{(\nu_j)}_0 = (\omega_j)_0$ is a minimal prime ideal of μ_0, then ω_j is a fuzzy minimal prime ideal of μ.

Proof.
Suppose $(\omega_j)_0$ is a minimal prime ideal of μ_0. Let ω be any fuzzy prime ideal of μ then $\bigwedge_{i=1}^n \omega_i \leq \omega$. Hence $\omega_i \leq \omega$ for some $1 \leq i \leq n$. If $\omega \leq \omega_j$, then $\omega_i \leq \omega \leq \omega_j$ which implies $(\omega_i)_0 \subseteq (\omega_j)_0$. This is a contradiction to the fact that $\mu_0 = \cap_{i=1}^n (\nu_i)_0$ is reduced. Hence there exists no fuzzy prime ideal of μ which is contained in ω_j. Therefore ω_j is a fuzzy minimal prime ideal of μ.

□

Remark.

By the example [5.1.2] ν_k, ν_2 are fuzzy minimal prime ideals of μ. But $(\omega_k)_0 =< x, y >$ is not a minimal prime ideal of $\mu_0 =< x^3, xy >$. Hence the converse of the above Proposition turns out to be false in general .

Proposition 5.2.8.

Let $\mu = \bigwedge_{i=1}^{n} \nu_i$ be a reduced fuzzy primary decomposition. Then $Im(\mu) = \cup_{i=1}^{n} Im(\nu_i)$.

Proof.

Clearly $Im(\mu) \subseteq \cup_{i=1}^{n} Im(\nu_i)$. Now we shall prove for each i with $1 \leq i \leq n$ there exists $x_i \in R \setminus (\nu_i)_0$ such that $\mu(x_i) = \nu_i(x_i)$. Suppose there exists i with $1 \leq i \leq n$ such that for all $x \in R \setminus (\nu_i)_0$, $\mu(x) \neq \nu_i(x)$. Since $\mu(x) = \bigwedge_{i=1}^{n} \nu_i(x)$ there exists $j, 1 \leq j \leq n$ such that $\nu_j(x) < \nu_i(x)$. Hence for all $x \in R \setminus (\nu_i)_0$ $\mu(x) = \bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \nu_j(x)$. If $x \in (\nu_i)_0$, then $\nu_i(x) = 1$; so $\mu(x) = \bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \nu_j(x)$. It follows that $\mu = \bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \nu_j$ which is a contradiction to the hypothesis. Thus for each $i = 1, 2, \ldots, n$ there exists $x_i \in R \setminus (\nu_i)_0$ such that $\mu(x_i) = \nu_i(x_i)$. Let $t \in \cup_{i=1}^{n} Im(\nu_i)$. Then $t \in Im(\nu_i)$ for some $i, 1 \leq i \leq n$. So there is $x \in R$ such that $\nu_i(x) = t$. If $t = 1$ then $t \in Im(\mu)$. If $t \neq 1$ then $x \in R \setminus (\nu_i)_0$. But we proved that there exists $x_i \in R \setminus (\nu_i)_0$ such that $\mu(x_i) = \nu_i(x_i)$. Since $|Im(\nu_i)| = 2$, $\nu_i(x) = \nu_i(x_i)$. Hence $\mu(x_i) = \nu_i(x_i) = \nu_i(x) = t$, implying $t \in Im(\mu)$.

□

§ 5.3.Fuzzy irreducible ideals .

In this section we study the fuzzy irreducibility of a fuzzy ideal and prove some relations between fuzzy prime, semiprimary and irreducible ideals. We first prove that every fuzzy ideal in a Noetherian ring can be written as a finite intersection of fuzzy irreducible ideals, where the fuzzy ideal takes only two values. From this, we prove the existence of such a decomposition in the general case.

Definition 5.3.1[Kum 4, 3.1].

Let μ be a fuzzy ideal of R. We say that μ is *fuzzy irreducible* if μ cannot be expressed as the intersection of two fuzzy ideals of R properly containing μ; otherwise μ is called *reducible* .

Thus μ is fuzzy irreducible if and only if whenever $\mu = \nu_1 \wedge \nu_2$ with ν_1, ν_2 fuzzy ideals of R, then either $\mu = \nu_1$ or $\mu = \nu_2$.

We prove a useful result in the following Proposition.

Theorem 5.3.2.

Let μ be a non-constant fuzzy ideal of R. Then μ is fuzzy irreducible if and only if $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$ and μ_0 is an irreducible ideal of R.

Proof.

Suppose μ is a fuzzy irreducible ideal. Let us prove first that $\mu(0) = 1$. Assume $\mu(0) = s < 1$. Since μ is non-constant , μ_s is a proper ideal of R. We define two fuzzy ideals $\nu_1, \nu_2 : R \longrightarrow I$ by

$$\nu_1(x) = \begin{cases} 1 & \text{if} \quad x \in \mu_s \\ \mu(x) & \text{otherwise} \end{cases}$$

and

$$\nu_2(x) = s \quad \text{for all } x \in R.$$

If $x \in \mu_s$ then $\nu_1(x) \wedge \nu_2(x) = s = \mu(x)$, if $x \notin \mu_s$ then $\nu_1(x) \wedge \nu_2(x) = \mu(x)$. So $\mu = \nu_1 \wedge \nu_2$ and clearly $\mu < \nu_1, \mu < \nu_2$ which is contradiction to the supposition. Hence $\mu(0) = 1$. Suppose $|Im(\mu)| \geq 3$, then there exists $s, t \in (0, 1), s > t$ such that $\mu_s \subset \mu_t \subset R$. We now define the fuzzy subsets $\omega_1, \omega_2 : R \longrightarrow I$ by

$$\omega_1(x) = \begin{cases} 1 & \text{if} \quad x \in \mu_t \\ \mu(x) & \text{if} \quad x \notin \mu_t \end{cases}$$

$$\omega_2(x) = \begin{cases} 1 & \text{if} \quad x \in \mu_s \\ \mu(x) & \text{if} \quad x \in \mu_t \setminus \mu_s \\ t & \text{if} \quad x \in R \setminus \mu_t. \end{cases}$$

Then clearly ω_1 is a fuzzy ideal of R such that $\mu < \omega_1$. Since $(\omega_2)_{t'} = \mu_{t'}$ for all t' with $t < t' < 1$ we can show that all level subsets of ω_2 are ideals of R. So ω_2 is a fuzzy ideal of R. It can be easily seen that $\mu < \omega_2$. Further $\mu = \omega_1 \wedge \omega_2$. To show this let $x \in R$. If $x \in \mu_s$ then $\omega_1(x) \wedge \omega_2(x) = 1 = \mu(x)$. If $x \in \mu_t \setminus \mu_s$ then $\omega_1(x) \wedge \omega_2(x) = 1 \wedge \mu(x) = \mu(x)$, and if $x \in R \setminus \mu_t$ then $\omega_1(x) \wedge \omega_2(x) = \mu(x) \wedge t = \mu(x)$. Hence we get a contradiction to the fact that μ is fuzzy irreducible. Thus $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$. To prove the last part, assume μ_0 is reducible. Then μ_0 can be expressed as the intersection of two ideals of R properly containing μ_0 (say) $\mu_0 = J_1 \cap J_2, \mu_0 \subset J_1$ and $\mu_0 \subset J_2$. Define the fuzzy ideals $\lambda_1, \lambda_2 : R \longrightarrow I$ by

$$\lambda_1(x) = \begin{cases} 1 & \text{if } x \in \mu_0 \\ t' & \text{if } x \in J_1 \setminus \mu_0 \\ t & \text{if } x \in R \setminus J_1 \end{cases}$$

and

$$\lambda_2(x) = \begin{cases} 1 & \text{if } x \in \mu_0 \\ t' & \text{if } x \in J_2 \setminus \mu_0 \\ t & \text{if } x \in R \setminus J_2 \end{cases}$$

where $t < t' < 1$. Then it is easy to check that $\mu = \lambda_1 \wedge \lambda_2, \mu < \lambda_1$ and $\mu < \lambda_2$ which is a contradiction to the irreducibility of μ. Thus μ_0 is an irreducible ideal of R.
Conversely suppose $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$, and μ_0 is an irreducible ideal of R. Assume that μ is not irreducible, then $\mu = \nu_1 \wedge \nu_2$ for some fuzzy ideals with $\mu < \nu_1$ and $\mu < \nu_2$. Hence there exist $x, y \in R$ such that $\mu(x) < \nu_1(x)$ and $\mu(y) < \nu_2(y)$. It follows that $x, y \notin \mu_0$. If $x = y$ then $\mu(x) < (\nu_1 \wedge \nu_2)(x)$ which is a contradiction. So $x \neq y$, implying $\mu_0 \subset <\mu_0, x>$ and $\mu_0 \subset <\mu_0, y>$, and hence $\mu_0 \subset <\mu_0, x> \cap <\mu_0, y>$. Let $z \in <\mu_0, x> \cap <\mu_0, y>$. Then $z = m + rx = n + sy$ for some $m, n \in \mu_0$ and $r, s \in R$. Therefore $\mu(m - n) = \mu(-rx + sy) = 1$. Hence $\nu_1(-rx + sy) = \nu_2(-rx + sy) = 1$. By Remark 2 (3) in §2.2 ,$\nu_1(rx) = \nu_1(sy)$ and $\nu_2(rx) = \nu_2(sy)$. But $\nu_1(rx) \geq \nu_1(x) > \mu(x) = t$, and similarly $\nu_2(rx) > t$. So $\mu(rx) > t$, i.e. $rx \in \mu_0$, therefore $z \in \mu_0$. Thus we have $\mu_0 = <\mu_0, x> \cap <\mu_0, y>$ with $\mu_0 \subset <\mu_0, x>$ and $\mu_0 \subset <\mu_0, y>$ which is a contradiction to the irreducibility of μ_0.

□

Remark.
The Proof of [Kum 4,3.2(ii)] is restricted to the case $|Im(\mu)| = 3$. His proof does not work if $|Im(\mu)| > 3$.

Corollary 5.3.3.
Let I be an ideal of R. Then I is an irreducible ideal if and only if $\mathcal{X}_I$ is a fuzzy irreducible ideal of R.

Corollary 5.3.4.
If μ is a fuzzy prime ideal of R then μ is fuzzy irreducible.

Proof.
By Theorem[3.1.2], $Im(\mu) = \{1, t\}$ for some $t \in [0, 1)$ and μ_0 is a prime ideal of R. Hence μ_0 is an irreducible ideal. Thus μ is fuzzy irreducible.

□

Corollary 5.3.5.
If μ is both a fuzzy semiprimary and irreducible ideal, then μ is fuzzy prime ideal of R.

Proof.
This is an immediate consequence of the fact that every ideal which is both semiprimary and irreducible in a commutative ring with identity, is prime.

□

Corollary 5.3.6.
If μ is a fuzzy irreducible ideal of a Noetherian ring R then μ is fuzzy primary.

The proof is straightforward since every irreducible ideal in a Noetherian ring is a primary ideal.

In the following Proposition we provide an analogue of the well- known result,namely, 'every ideal in a Noetherian ring is a finite intersection of irreducible ideals of R 'in the fuzzy case.

Proposition 5.3.7.
Let μ be a fuzzy ideal of a Noetherian ring with $Im(\mu) = \{1, t\}$, for some $t \in [0, 1)$. Then μ can be expressed as a finite intersection of fuzzy irreducible ideals of R.

Proof.
Suppose $\mu_0 = \cap_{i=1}^{n} J_i$, J_i irreducible ideals of R. Define the fuzzy ideals $\nu_1, \nu_2, \ldots, \nu_n : R \longrightarrow I$ by

$$\nu_i(x) = \begin{cases} 1 & \text{if} \quad x \in J_i \\ t & \text{if} \quad x \notin J_i. \end{cases}$$

Then by Theorem[5.3.2], for each $i = 1, 2, \ldots, n$, ν_i is a fuzzy irreducible ideal of R and it is easy to check that $\mu = \bigwedge_{i=1}^{n} \nu_i$

□

Remark.
Suppose μ is any fuzzy ideal of a Noetherian ring. It is easily seen by the ascending chain condition, that the $Im(\mu)$ must be a finite set. By arguments similar to 5.1.6, we conclude that μ can be expressed as a finite intersection of fuzzy ideals, each with two membership values. Now we can apply the above Proposition[5.3.7] to get the following existence Theorem.

Theorem 5.3.8(Existence Theorem).
Every fuzzy ideal μ with $\mu(0) = 1$ in a Noetherian ring R can be decomposed as a finite intersection of fuzzy primary ideals in R.

We now discuss the effect of homomorphisms on fuzzy irreducible ideals.

Proposition 5.3.9.
Let $f : R \longrightarrow R'$ be an epimorphism and μ be a f-invariant fuzzy irreducible ideal of R then $f(\mu)$ is fuzzy irreducible ideal of R'.

Proof.
Suppose $f(\mu)$ is reducible. Then there exist two fuzzy ideals ν_1', ν_2' of R' such that $f(\mu) = \nu_1' \wedge \nu_2'$ and $f(\mu) < \nu_1', f(\mu) < \nu_2'$. Since μ is f-invariant by Proposition[1.2.2]

$\mu = f^{-1}(f(\mu)) = f^{-1}(\nu_1' \wedge \nu_2') = f^{-1}(\nu_1') \wedge f^{-1}(\nu_2')$ and $\mu < f^{-1}(\nu_1'), \mu < f^{-1}(\nu_2')$ which is a contradiction to the irreducibility of μ.

□

Proposition 5.3.10.
Let $f : R \longrightarrow R'$ be a homomorphism, R be a Boolean ring and μ' be a fuzzy irreducible ideal of R'. Then $f^{-1}(\mu')$ is fuzzy irreducible ideal of R.

The proof is straightforward by Proposition[1.2.2].

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