RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS

REAL OPTIONS VALUATION FOR SOUTH AFRICAN NUCLEAR WASTE MANAGEMENT USING A FUZZY MATHEMATICAL APPROACH

by

Obakeng Montsho

A thesis submitted in fulfillment of the requirements for the degree of

MASTER OF SCIENCE in **APPLIED MATHEMATICS**

December 2012

ABSTRACT

The feasibility of capital projects in an uncertain world can be determined in several ways. One of these methods is real options valuation which arose from financial option valuation theory. On the other hand fuzzy set theory was developed as a mathematical framework to capture uncertainty in project management. The valuation of real options using fuzzy numbers represents an important refinement to determining capital projects' feasibility using the real options approach.

The aim of this study is to determine whether the deferral of the decommissioning time (by a decade) of an electricity-generating nuclear plant in South Africa increases decommissioning costs. Using the fuzzy binomial approach, decommissioning costs increase when decommissioning is postponed by a decade whereas use of the fuzzy Black-Scholes approach yields the opposite result. A python code was developed to assist in the computation of fuzzy binomial trees required in our study and the results of the program are incorporated in this thesis.

KEYWORDS: Fuzzy Sets, Real Options, Capital Project Valuation.

A.M.S 2010 SUBJECT CLASSIFICATION: 03E72, 91G20, 91G50, 91G80.

DECLARATION

Except for the references that have been accurately cited and discussed herein, the content of this thesis represents my own efforts. The entire thesis has neither been nor is concurrently being submitted to any other academic institution for the purpose of obtaining a degree.

....

Obakeng Montsho

Grahamstown, R.S.A.

12/2012

ACKNOWLEDGEMENTS AND DEDICATION

I acknowlege the assistance received from my supervisor, Prof Murali (Dept. of Mathematics, Rhodes University) and co-supervisor Mr Ebrahim Seedat (Dept. of Economics, Rhodes University). I would also like to thank Ghislain Fouodji Tasse for his assistance in writing the Python code.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at are those of the author, and are not necessarily to be attributed to the NRF.

I dedicate this work to my parents Mmaselata Montsho, Keoamogetse Thekoeng and my aunt Mmarakau Kwamongwe.

Contents

List of Figures

List of Tables

LIST OF ABBREVIATIONS AND NOTATION

max	Maximum
min	Minimum
X	Set
$\mathbb{R}$	Set of real numbers
R	Binary relation
T	Time to maturity
r	Risk-free interest rate
U	Universe of discourse
supp A	Support of a fuzzy number A
NPV	Net Present Value
PP	Payback Period
ARR	Accounting Rate of Return
IRR	Internal Rate of Return
NCF	Net Cash Flow (cash inflows - cash outflows) in period t
DCF	Discounted Cash Flow
ROV	Real Option Valuation
FROV	Fuzzy Real Option Value
FENPV	Fuzzy Expanded Net Present Value
RTP	Reference Technical Plan
LOPP	Life of Plant Plan

LIST OF ABBREVIATIONS AND NOTATION (CONTINUED)

SF	Spent fuel
SFA	Spent fuel assembly
SFMD	Spent Fuel Management and Disposal
CV	Coefficient of Variation
$\mu_A(x)$	Characteristic Function
A_α	Alpha-cut set
C	Call option price
S	Value of the underlying asset
$N(d_i)$	Cumulative Normal Distribution
σ	Volatility
MW	Megawatt(s)
GWh	Gigawatt Hour
EV	Expected Value
MPV	Most possible values
DP	Downward potential
UP	Upward potential
$\vee$	Maximum
$\wedge$	Minimum

Subscripts not related to notation appearing earlier in this list

T	Terminal/expiry date of an option

0 Date on which option transaction is initiated

t A date between 0 and T ($0 < t < T$)

Superscripts not related to notation appearing earlier in this list

+ Positive part (e.g. a^+ is the positive part of some real-valued number a)

PREFACE

The feasibility of capital projects in an uncertain world can be determined in several ways. One of these methods is real options valuation which arose from financial option valuation theory. Real options valuation consists of the determination of an option price using several methods such as the parametric approach of Black and Scholes[5], and the binomial approach of Cox, Ross and Rubinstein[18]. Both of the aforementioned methods use probability theory in their treatment of uncertainty, but fuzzy logic and fuzzy set theory can be used more effectively to treat uncertainty or imprecise statements predicting future estimates than that of using probability theory [16].

Zadeh [55] introduced the word *fuzzy* as a formalization of uncertainty or vagueness in complex systems. Fuzzy set theory employs fuzzy numbers to quantify subjective fuzzy estimates. The valuation of real options using fuzzy numbers has been studied by numerous researchers such as Carlsson & Fuller [12], Collan, Carlsson & Majlender [17] and Liao & Ho [31].

The main aim of this study is to estimate the costs of deferring the decommissioning of an electricity-generating nuclear power plant which operates in South Africa. This will require the values of a real option to be estimated with the fuzzy mathematical approach.

The organisation of the remainder of this study is as follows. In Chapter 1, the necessary operations of fuzzy sets will be introduced, and we will then present ideas pertaining to fuzzy relations and fuzzy numbers. Chapter 2 contains the discussion of real options theory and traditional valuation methods. Given that the real option valuation method is considered superior to traditional capital project valuation methods, we explain these traditional methods first. The central theme of this study is treated in Chapter 3, which comprises the discussion of fuzzy set theoretic real option valuation methods. The fuzzy risk-neutral-approximation-based binomial models and the fuzzy Black-Scholes formula will be discussed in this chapter. Chapter 4 consists of the presentation of data and methods used in the study, and also the discussion of the results obtained. Chapter 5 contains a conclusion together with the discussion of future research prospects and topics.

For ease of reference, in each chapter the numbers of all definitions and examples are assigned serially. For example, 1.3.1 refers to the first entity of the third section of the study's first chapter.

Chapter 1

Fuzzy Set Theory

Most of the traditional tools used for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. Crisp means dichotomous, that is, yes-or-no or true-or-false type rather than more-or-less or somewhere-in-between type. We elaborate as follows: in traditional dual logic, for instance, a statement can either be true or false and nothing in between, nearly true or nearly false. Similarly, in set theory, an element can either belong to a set or not and in optimization a solution can be feasible or not [56]. In reality, certain of our knowledge is of a fuzzy nature, for instance today is a cool day or this man is tall. This knowledge is amorphous, vague, imprecise, inexact, or possibilistic in nature. Due to the existence of the knowledge that has the afore mentioned nature, Zadeh [55] introduced fuzzy sets as an extension of classical set theory. The use of fuzzy sets enables us to find solutions to many real-world problems. It assists systems to work efficiently with the inaccurate or inexact information and give expert opinions.

This chapter is organised in the following way. In Section 1.1, we introduce the concept of classical set theory whereas in Section 1.2 fuzzy sets will be defined together with the basic properties of fuzzy sets. Then in Section 1.3, we will discuss the difference between classical sets that are also known as crisp sets and Zadeh's (1965) fuzzy sets. In Section 1.4, we discuss the types of membership functions in a fuzzy set. The necessary operations of fuzzy sets will be introduced, and we will then present ideas pertaining to fuzzy relations and fuzzy numbers in Section 1.5 and 1.6, respectively.

1.1 Classical Set Theory

A collection of objects that are well defined is called a set. The objects in a set are called elements or members of the set [7] . Suppose that we let X be a non-empty fixed set called the Universal Discourse. For example, if we want to characterize students in the mathematics department according to the modules they are registered for, we will do so by enlisting their names and then our Universal Discourse would be the collection of students with the names in the mathematics department. From that collection of students we will have students who are registered for different modules, for instance the student who is doing second year would not do the same modules with the student who is registered for third year.

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition: an element either belongs to or does not belong to the set. For instance, if A is a subset of X, we express it by saying every element of X either belongs to only A or does not belong to A but not both. Using the aforementioned example, the collection of students would be X and the student registered for the third year modules would be represented by the subset A, then lastly the names of the students in A are the elements or members of our subset. Clearly, the students who are not part of the list of third year students do not belong to A.

Chakraborty [13] states that the classical set theory enumerates elements of a set $A = \{a_1, a_2, a_3, \cdots, a_n\}$. where A is a subset of a universal set X. Then the set A can be represented by its characteristic function given by, for all $x \in X$,

$$\mu_A(x) = \begin{cases} 1 & if \quad x = a_i \in A, \\ 0 & if \quad x \notin A. \end{cases}$$

If $\mu_A(x) = 1$ then we say x **belongs** to μ_A absolutely and if $\mu_A(x) = 0$, we say x **does not** belong to μ_A absolutely.

It can be observed that in classical set theory there are only two values that exist, those are 1 or 0 which represent "true" or "false" respectively. Such sets are called crisp sets.

Classical Set Theory Formulated in Terms of Membership Functions

One way of defining a set A is in terms of its membership function $\mu_A(x)$, taking values in $\{0, 1\}$. An element x belongs to set A if and only if $\mu_A(x) = 1$. A characteristic function is a function from some universal set U to the binary set $\{0,1\}$. Using the characteristic function, we can define the following well-known operations for $x \in X$:

Set Inclusion:

$$A \subset B \quad if \quad and \quad only \quad if \quad \forall x \in X$$

$$\mu_A(x) = 1 \quad implies \quad \mu_B(x) = 1$$

Set Equality:

$$A = B \quad if \quad and \quad only \quad if \quad \forall x \in X$$

$$\mu_A(x) = \mu_B(x)$$

Union: $A \cup B$

$$\mu_{A \cup B}(x) = max(\mu_A(x), \mu_B(x))$$

Intersection: $A \cap B$

$$\mu_{A \cap B}(x) = min(\mu_A(x), \mu_B(x))$$

Complement:

$$\mu_A^c(x) = 1 - \mu_A(x)$$

1.2 Fuzzy Set Theory

The extension of the classical set theory is a fuzzy set theory where elements have varying degrees of membership to allow modelling real-world problems.

The truth values that are used to represent the crisp sets are sometimes insufficient to describe the human reasoning. Human reasoning often uses vague predicates, individuals cannot be classified into two groups (either true or false). Fuzzy sets accommodate the values in between 0(false) and 1(true) to describe the human reasoning such as nearly true (0.9), nearly false (0.1). Fuzzy sets allow the elements to belong partially to a set.

Definition 1.2.1 *(Nguyen & Walker [41]) A fuzzy subset of a set X is simply a function from $X \to [0,1]$.*

From Definition 1.2.1 we can simply say that a fuzzy set is any set that allows its elements to have degrees of membership. The function that describes the degrees of membership is called **membership function**, in the interval [0,1] [13].

Definition 1.2.2 *If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:*

$$A = \{(x, \mu_A(x)) | x \in X\}, \tag{1.1}$$

$\mu_A(x)$ is called the membership value of x to μ_A . μ_A is called the membership function (generalized characteristic function).

1.3 Crisp versus Fuzzy Sets

Crisp sets are the ones that we have been using in most of our life. The most important rule that is used for these sets is that elements either belong or do not belong to a set. Suppose that we want to classify objects based on their characteristics, then we normally use crisp sets. For example, we obviously know that a banana is a fruit so it will fall in the set of fruits. However, it becomes a challenge when we have to classify a tomato as a fruit or as a vegetable. There are several examples that can be given, e.g. when we need to classify tall people. It will not be easy to do that because the term tall can be interpreted in different ways. Crisp set theory is not capable of representing descriptions and classifications in many cases; In fact, crisp set does not provide adequate representation for most cases [13]. In classical logic there are words that have only two values, examples are true or false; yes or no; black or white and lastly start or stop. From these arguments, it is clear that we need to introduce fuzzy sets in the picture.

Fuzzy sets are sets that possess vague boundaries, and the membership of x in A is a matter of the degree to which x is in A. In a fuzzy set, it is not necessary that x is a full member of the set. It can also be a partial member of the set. Fuzzy sets differ from crisp sets in the sense that fuzzy sets can capture uncertainty and they can also represent conceptual entities such as far, cold and expensive.

There are variables that describe crisp sets and fuzzy sets, they are called crisp and fuzzy variables respectively. The crisp variables represent the precise quantities only, whereas fuzzy variables represent the degree to which a quantity is possessed. The main difference between the two sets is that the crisp sets take either the value of 1 or 0 but the fuzzy sets can take any value between 0 and 1. It also follows that the crisp variable is measured with its uniform probability distribution but for fuzzy variable its membership function is associated with its domain.

From the table below we are able to depict the main difference between the crisp and fuzzy sets. It has been shown that crisp sets have only 2 values, 1 and 0, that represent fully in and fully out, respectively. On the other hand we have three different memberships to fuzzy sets that have been shown to possess different values. We notice that the values vary, such that they do not only have two truth values. However, the truth values are included in the fuzzy sets. Since $\{0, 1\} \subset [0, 1]$, every crisp subset is also a fuzzy subset as a special case. It is possible to have fuzzy subsets of X without any element having absolutely membership values 0 and/or 1. Thus $0 < \mu_A(x) < 1$ is possible for all $x \in X$.

Crisp set	Three-value fuzzy set	Four-value fuzzy set	Six-value fuzzy set
1 = fully in	1 = fully in	1 = fully in	1 = fully in
			0.9 = mostly but not fully in
		0.67 = more in than out	
			0.6 = more or less in
	0.5 = neither fully in nor out		
			0.4 = more or less out
		0.33 = more out than in	
			0.1 = mostly but not fully out
0 = fully out	0 = fully out	0 = fully out	0 = fully out

Table 1.1: *Crisp versus Fuzzy Sets*

Definition 1.3.1 *(Dubois [21]) A fuzzy subset A of X is a collection of ordered pairs $(x, \mu_A(x))$ where x is an element of X and $\mu_A(x)$ is a real number between 0 and 1 inclusive, with $\mu_A(x)$ representing the degree of membership of x to the fuzzy set A.*

Thus we get a function associated with a fuzzy subset given by:

$$\mu_A : X \to [0,1] \tag{1.2}$$

The subset of X consisting of those $x \in X$ for which $\mu_A(x) = 1$ is called the **core** of μ_A. Similarly the subset of X consisting those x in X for which $\mu_A(x) = 0$ is called the **co-support**.

Further by **support** of μ_A we mean the subset of elements $x \in X$ for which $\mu_A(x) > 0$.

Clearly core and co-support may or may not be empty. If the core and the co-support are empty, then the fuzzy subset μ_A is truly fuzzy without any crisp pairs.

On the other hand, the whole set can be the core of a fuzzy set

$$\mu_{core}(x) = 1 \qquad \text{for all } x \in X.$$

Similarly, the whole set X may be the co-support of a fuzzy set

$$\mu_{empty}(x) = 0 \qquad \text{for all } x \in X.$$

Example 1.3.1 *Suppose someone wants to describe the class of cars having the property of being expensive by considering BMW, Rolls Royce, Mercedes, Ferrari, Fiat, Honda and Renault. Some cars like Ferrari and Rolls Royce are definitely expensive and some like Fiat and Renault are cheaper in comparison and do not belong to the set of expensive cars. Using a fuzzy set, the fuzzy set of expensive cars can be described as:*

{*(Ferrari, 1), (Rolls Royce, 1), (Mercedes, 0.8), (BMW, 0.7), (Honda, 0.4), (Fiat, 0), (Renault, 0)* }.

Obviously, Ferrari and Rolls Royce have membership values of 1 whereas BMW, which is cheaper, has a membership value of 0.7, Honda 0.4 and Fiat and Renault each having a membership value of 0.

1.4 Types of Membership Functions

A membership function is a mathematical function which defines the degree of an element's membership in a fuzzy set. For a fuzzy set $A : X \to [0, 1]$, the function A is called the **membership function**, and the value $\mu_A(x)$ is called the **degree of membership** of x in the fuzzy set A [41].

The previous example about the expensive cars can be used to model the notion of "expensive" with a fuzzy set. For the following fuzzy set, the unit of measurement is in thousands of Rands:

$$\mu(x) = \begin{cases} 1 & if \quad 500 < x, \\ \frac{x-150}{350} & if \quad 150 \leq x \leq 500, \\ 0 & if \quad x < 150. \end{cases}$$

It can be observed from the above equation that all the cars that worth more than R500 000 are expensive and their degree of membership is 1. The other cars that cost between R150 000 and R500 000 are cheaper than the other group that we just described, and their degrees of membership are between 0 and 1. The last group of cars are cheap and worth less than R150 000, therefore they are given 0 as their degree of membership.

The types of membership functions are:

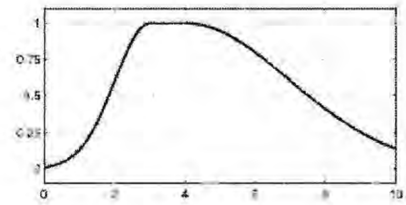


Figure 1.1: Gaussian distribution function

Source: MATLAB & Simulink Webpage, 2012

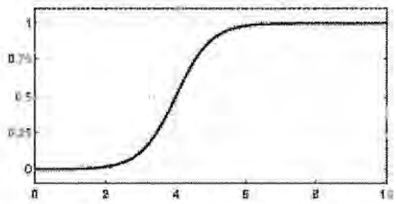


Figure 1.2: Sigmoid curve

Source: MATLAB & Simulink Webpage, 2012

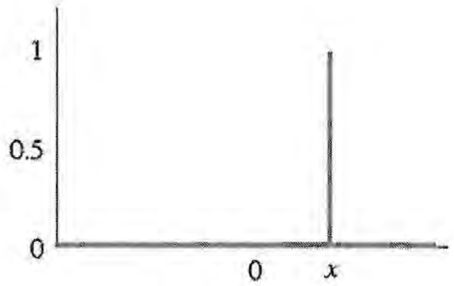


Figure 1.3: Singleton membership function

Source: MATLAB & Simulink Webpage, 2012

Finite and Infinite Universal Space

Universal sets in general can be finite or infinite. Any universal set is finite if it consists of a specific number of distinct elements, that is, if in counting the different elements of the set, the counting can come to an end, else the set is infinite [13].

When X is a finite set $\{x_1, \cdots, x_n\}$, a fuzzy set on X is expressed succintly as

$$A = \frac{\mu_A(x_1)}{x_1} + \cdots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i}.$$

In the infinite case we have:

$$A = \int_X \frac{\mu_A(x)}{x}.$$

Operations on Fuzzy Sets

Fuzzy sets also possess some theoretic operations like the classical set theory. Let A and B be fuzzy sets defined in the universal discourse X. There are several operations and relations on fuzzy sets and we list some of them below.

1. Inclusion

The fuzzy set A is included in the fuzzy set B if and only if for every x in the set X we have $\mu_A(x) \leq \mu_B(x)$.

2. Equality

$A = B$ if and only if $\forall x \in X$ $\mu_A(x) = \mu_B(x)$.

3. Comparability

Two fuzzy sets A and B are comparable if one is a subset of the other, or vice versa; that is $\mu_A \subset \mu_B$ or $\mu_B \subset \mu_A$. Fuzzy sets are incomparable if $\mu_A \not\subseteq \mu_B$ or $\mu_B \not\subseteq \mu_A$.

4. Complement

The membership function of the complement of fuzzy set A, is defined as $\mu_A^c(x) = 1 - \mu_A(x)$.

5. Intersection

The membership function of the intersection of two fuzzy sets A and B is defined in [56] as $\mu_{A\cap B}(x) = min(\mu_A(x), \mu_B(x))$.

6. Union

The membership function of the union of two fuzzy sets A and B is defined in [56] as $\mu_{A\cup B}(x) = max(\mu_A(x), \mu_B(x))$.

We now give three operations on fuzzy sets relating (in order) to inclusion, comparability and complementation as follows:

Example 1.4.1 *Suppose the fuzzy set μ_A represents the EXPENSIVE cars and the fuzzy set μ_B represents the VERY EXPENSIVE cars, then the inclusion operation can be represented by:*

$$\mu_A(x) = \{[1,1], [2,0.8], [3,0.7], [4,0.4], [5,0.2], [6,0.1], [7,0], [8,0]\}$$

$$\mu_B(x) = \{[1,1], [2,1], [3,0.9], [4,0.6], [5,0.4], [6,0.3], [7,0.1], [8,0]\}$$

It can be observed that $\mu_A(x) \leq \mu_B(x)$.

Example 1.4.2 *Let* $\mu_A = \{[a,1],[b,1],[c,0]\}$ *and* $\mu_B = \{[a,1],[b,1],[c,1]\}$. *Then* μ_A *is comparable to* μ_B, *since* μ_A *is a subset of* μ_B.

Example 1.4.3 *The complement of the set of the EXPENSIVE cars can be expressed as:*

$$\mu_A(x) = \{[1,1],[2,0.8],[3,0.7],[4,0.4],[5,0.2],[6,0.1],[7,0],[8,0]\}$$

$$\mu_A^c(x) = \{[1,0],[2,0.2],[3,0.3],[4,0.6],[5,0.8],[6,0.9],[7,1],[8,1]\}$$

Fuzzy Properties

Let μ_A, μ_B and μ_C be the fuzzy sets in the universal space X, then the following properties are satisfied by the fuzzy sets:

Identity:

$$\mu_A(x) \cup \mu_\emptyset(x) = max(\mu_A(x), \mu_\emptyset(x)) = \mu_A(x)$$

$$\mu_A(x) \cap \mu_U(x) = min(\mu_A(x), \mu_U(x)) = \mu_A(x)$$

Idempotence:

$$\mu_A(x) \cup \mu_A(x) = max(\mu_A(x), \mu_A(x)) = \mu_A(x)$$

$$\mu_A(x) \cap \mu_A(x) = min(\mu_A(x), \mu_A(x)) = \mu_A(x)$$

Commutativity:

$$\mu_A(x) \cup \mu_B(x) = max(\mu_A(x), \mu_B(x)) = \mu_B(x) \cup \mu_A(x)$$

$$\mu_A(x) \cap \mu_B(x) = min(\mu_A(x), \mu_B(x)) = \mu_B(x) \cap \mu_A(x)$$

Associativity:

$$\mu_A(x) \cup (\mu_B(x) \cup \mu_C(x)) = max\{\mu_A(x), \mu_B(x), \mu_C(x)\}$$

$$= (\mu_A(x) \cup \mu_B(x)) \cup \mu_C(x)$$

$$\mu_A(x) \cap (\mu_B(x) \cap \mu_C(x)) = min\{\mu_A(x), \mu_B(x), \mu_C(x)\}$$

$$= (\mu_A(x) \cap \mu_B(x)) \cap \mu_C(x)$$

Distributivity:

$$\mu_A(x) \cap (\mu_B(x) \cup \mu_C(x)) = min\{\mu_A(x), max\{\mu_B(x), \mu_C(x)\}$$

$$= max\{min\{\mu_A(x), \mu_B(x)\}, min\{\mu_A(x), \mu_C(x)\}$$

$$= (\mu_A(x) \cap \mu_B(x)) \cup (\mu_A(x) \cap \mu_C(x))$$

$$\mu_A(x) \cup (\mu_B(x) \cap \mu_C(x)) = max\{\mu_A(x), min\{\mu_B(x), \mu_C(x)\}$$

$$= min\{max\{\mu_A(x), \mu_B(x)\}, max\{\mu_A(x), \mu_C(x)\}$$

$$= (\mu_A(x) \cup \mu_B(x)) \cap (\mu_A(x) \cup \mu_C(x))$$

1.5 Fuzzy Relations

Crisp Relations

Let A and B be two nonempty crisp sets in the universal spaces X and Y. The Cartesian product $X \times Y$ is defined as the set of all ordered pairs (x, y) of elements where x is in X and y is in Y. The Cartesian product is defined as $X \times Y = \{(x, y) \in X \times Y | x \in X, y \in Y\}$. It should be noted that $X \times Y \neq Y \times X$.

Example 1.5.1 *Let $X = \{a_1, a_2, a_3\}$ and $Y = \{b_1, b_2\}$ then $X \times Y$ will be represented as $X \times Y = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$. Suppose R is a subset of $X \times Y$.*

$$\mu_R(x, y) = \begin{cases} 1 & if \quad (x, y) \in R, \\ 0 & if \quad (x, y) \notin R. \end{cases}$$

This membership function maps $X \times Y$ to set $\{0, 1\}$ taking 1 on R and 0 on outside of R.

$$\mu_R : X \times Y \to \{0, 1\}.$$

Definition 1.5.1 ***Binary Relation:*** *Let X and Y be two non-empty sets. R is a binary relation on $X \times Y$ if and only if $R = \{(x, y) | x \in X, y \in Y\}$.*

There are several ways of representing the binary relations, namely biparti-graph, coordinate diagram, matrix and directed graph (Digraph).

Below we have used matrix to represent the relations:

R	b_1	b_2	b_3
a_1	1	0	0
a_2	0	1	0
a_3	1	0	1
a_4	0	0	1

Fuzzy Relations

Fuzzy relations are very important because they describe interactions between variables [1]. Relations are used to compare the degree to which two variables or elements belong to a set. For example:

- x is older than y (Ages)
- x is much larger than y (Numbers)
- The product of x and y is approximately 16 (Numbers)

Fuzzy relations offer the capability to capture the uncertainty and vagueness in relations between sets and elements of a set [13]. A fuzzy relation represents the degree of presence or absence of association, interaction or interconnectedness between the elements of two or more sets [3].

Definition 1.5.2 *(Chakraborty [13]) Let X and Y be two universe of discourse. A fuzzy relation R(x,y) is a fuzzy set in the product space $X \times Y$ i.e., it is a fuzzy subset of $X \times Y$ and is characterized by the membership function $\mu_R(x,y)$. That is:*

$$R(x,y) = \{((x,y), \mu_R(x,y)) | (x,y) \in X \times Y\}.$$

Example 1.5.2 *The fuzzy relation on $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ is given by*

$$\begin{aligned} R = \{&((x_1,y_1),0), ((x_1,y_2),0.1), ((x_1,y_3),0.2), \\ &((x_2,y_1),0.7), ((x_2,y_2),0.2), ((x_2,y_3),0.3), \\ &((x_3,y_1),1), ((x_3,y_2),0.6), ((x_3,y_3),0.2)\} \end{aligned}$$

We can simply represent the above example in matrix form by

R	y_1	y_2	y_3
x_1	0	0.1	0.2
x_2	0.7	0.2	0.3
x_3	1	0.6	0.2

We notice that the main difference between the classical crisp relations and fuzzy relations is that crisp relation only focuses on the association of two or more elements in a dichotomous way, on the other hand fuzzy relation focus on the degree at which two or more are associated in the set.

A fuzzy relation R is a mapping from the Cartesian space $X \times Y$ to the interval [0, 1], where the strength of the mapping is expressed by the membership function of the relation $\mu_R(x, y)$

$$\mu_R : X \times Y \to [0, 1],$$

Thus, the fuzzy relation R can be expressed as:

$$R = \{((x, y), \mu_R(x, y)) | 0 \leq \mu_R(x, y) \leq 1, x \in X, y \in Y\}.$$

A **fuzzy binary relation** on X and Y is a fuzzy subset R on $X \times Y$. Our interest in this section is in the case in which Y = X. Thus, by a fuzzy relation, we mean a fuzzy binary relation given by $R : X \times X \to U$ [40].

Example 1.5.3 *A binary fuzzy relation on $U = \{1, 2, 3\}$, called "approximately equal" can be defined as*

$$R(1,1) = R(2,2) = R(3,3) = 1$$

$$R(1,2) = R(2,1) = R(2,3) = R(3,2) = 0.8$$

$$R(1,3) = R(3,1) = 0.3$$

The membership function of R is given by

$$R(x,y) = \begin{cases} 1 & if \quad x = y, \\ 0.8 & if \quad |x-y| = 1, \\ 0.3 & if \quad |x-y| = 2. \end{cases}$$

In matrix notation it can be represented as

R	y_1	y_2	y_3
x_1	*1*	*0.8*	*0.3*
x_2	*0.8*	*1*	*0.8*
x_3	*0.3*	*0.8*	*1*

Definition 1.5.3 *The notation of the fuzzy relation on $X \times Y$ is R or $R(x, y)$ and it is defined as the set*

$$R = \{((x,y), \mu_R(x,y)) | (x,y) \in X \times Y, \mu_R(x,y) \in [0,1]\},$$

where $\mu_R(x,y)$ is a function of two variables called the membership function.

The three projections that are associated with fuzzy relations are defined as follows:

1) First Projection of R:

$$R^{(1)} = \{(x), \mu_R^{(1)}(x,y)\} = \{((x), \max_Y \mu_R(x,y))|(x,y) \in X \times Y\}.$$

2) Second Projection of R:

$$R^{(2)} = \{(y), \mu_R^{(2)}(x,y)\} = \{((y), \max_X \mu_R(x,y))|(x,y) \in X \times Y\}.$$

3) Total Projection of R:

$$R^T = \max_X \max_Y \{\mu_R(x,y)|(x,y) \in X \times Y\}.$$

Note that in all these three expressions $\max_Y$ means max with respect to Y while x is considered fixed whereas $\max_X$ means max with respect to X while y is considered fixed.

Example 1.5.4 *An example of a fuzzy projection is the Fuzzy Relation R together with the First, Second and Total Projection of R is given by*

R	y_1	y_2	y_3	y_4	$R^{(1)}$
x_1	*0*	*0.1*	*0.2*	*0.5*	*0.5*
x_2	*0.7*	*0.2*	*0.3*	*0.6*	*0.7*
x_3	*1*	*1*	*0.2*	*0.8*	*1*
$R^{(2)}$	*1*	*1*	*0.3*	*0.8*	

From the given example our $R^{(T)}$ will be the combination of $R^{(1)}$ and $R^{(2)}$ as follows:

$$R^{(1)} = \{0.5, 0.7, 1.0\}$$

$$R^{(2)} = \{1.0, 1.0, 0.3, 0.8\}.$$

Operations on Fuzzy Relations

Let R and S be the fuzzy relations on $X \times Y$. The following operations exist:

1. Union of Relations

$\forall (x, y) \in X \times Y$

$$\mu_{R \cup S}(x, y) = max[\mu_R(x, y), \mu_S(x, y)]$$
$$= \mu_R(x, y) \vee \mu_S(x, y)$$

2. Intersection

$\forall (x, y) \in X \times Y$

$$\mu_{R \cap S}(x, y) = min[\mu_R(x, y), \mu_S(x, y)]$$
$$= \mu_R(x, y) \wedge \mu_S(x, y)$$

3. Complement

$\forall (x, y) \in X \times Y$

$$\mu^c_R(x, y) = 1 - \mu_R(x, y)$$

4. Inverse relation

$\forall (x, y) \in X \times Y$

$\mu_{R^{-1}}(y, x) = \mu_R(x, y)$.

5. Composition: In order to define composition, the following definition is required:

Definition 1.5.4 *(Murali, [40]) Suppose R and Q are two fuzzy relations on X. Then their composition, denoted by $R \circ Q$ is defined as, for $x, z \in X$*

$$(R \circ Q)(x, z) = \vee\{R(x, y) \wedge Q(y, z) | y \in X\},$$

Example 1.5.5 *Given the fuzzy sets M_R and M_S, we show the operations of union, intersection, complement and inverse relation below:*

M_R	y_1	y_2	y_3
x_1	*0.3*	*0.7*	*0.2*
x_2	*0.9*	*0.1*	*0.4*
x_3	*0.2*	*0.8*	*0.0*

M_S	y_1	y_2	y_3
x_1	*0.2*	*0.7*	*0.5*
x_2	*0.8*	*0.0*	*1.0*
x_3	*0.6*	*0.7*	*0.1*

$M_{R \cap S}$	y_1	y_2	y_3
x_1	*0.2*	*0.7*	*0.2*
x_2	*0.8*	*0.0*	*0.4*
x_3	*0.2*	*0.7*	*0.0*

$M_{R\cup S}$	y_1	y_2	y_3
x_1	*0.3*	*0.7*	*0.5*
x_2	*0.9*	*0.1*	*1.0*
x_3	*0.6*	*0.8*	*0.1*

M_{R^c}	y_1	y_2	y_3
x_1	*0.7*	*0.3*	*0.8*
x_2	*0.1*	*0.9*	*0.6*
x_1	*0.8*	*0.2*	*1.0*

Example 1.5.6 *Given the fuzzy sets R and Q, we show the operation of composition relation below:*

R	y_1	y_2	y_3	y_4
x_1	*0.3*	*0.7*	*0.2*	*0.2*
x_2	*0.9*	*0.1*	*0.4*	*0.8*
x_3	*0.2*	*0.8*	*0.0*	*0.6*

Q	y_1	y_2	y_3
x_1	*0.7*	*0.5*	*0.3*
x_2	*0.0*	*1.0*	*0.8*
x_3	*0.1*	*0.2*	*0.7*
x_4	*0.1*	*0.9*	*.06*

$R \circ Q$	y_1	y_2	y_3
x_1	0.3	0.3	0.7
x_2	0.7	0.8	0.6
x_3	0.2	0.8	0.8

In particular if R = Q then $(R \circ R) = \vee\{R(x,y) \wedge R(y,z)\}$. If $R \circ R = R$ then we say R is **transitive** i.e. $R(x,z) = \vee\{R(x,y) \wedge R(y,z) | y \in X\}$.

Definition 1.5.5 *A fuzzy relation R on X is said to be* ***reflexive*** *if R(x, x) = 1 for all $x \in X$ and said to be* ***symmetric*** *if R(x, y) = R(y, x) for all $x, y \in X$.*

A fuzzy relation on U which is reflexive, symmetric and transitive is called a **fuzzy equivalence relation** on U or sometimes similarity relation on U. There is an extensive literature on both theoretical and practical aspects of the fuzzy equivalence relation.

1.6 Fuzzy Numbers

In order to discuss the fuzzy numbers, we recall the following. First, scalars are well-known or mathematically defined integers and real numbers. Second, intervals are numbers whose values are not known with certainty but about which bounds can be established. Finally, fuzzy numbers are numbers built on uncertainty for which, in addition to knowing a range of possible values, one can say that some values are more plausible or "more possible" than others. We need to define a normal set and convex set in order to introduce fuzzy numbers. These sets are defined in the following way.

Definition 1.6.1 *If Y is a normal set containing X, then it must contain the conjugates of every element of X.*

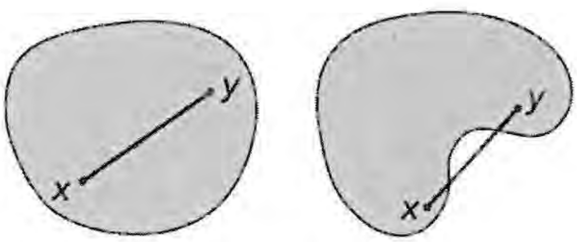
Definition 1.6.2 *A set Z in R^n is said to be convex if for each x, $y \in Z$, the line segment $\lambda x + (1-\lambda)y$ for $\lambda \in (0,1)$ belongs to Z.*

Panel (a) of Figure 1.1 depicts a convex set whereas a non-convex set with a line segment outside the set is depicted in Panel (b) of Figure 1.1.

Definition 1.6.3 *(Bansal [2]) A fuzzy set A, defined on the universal set of $\mathbb{R}$ is said to be a fuzzy number if its membership function has the following characteristics:*

1. A is normal.

2. A is convex.



(a) Panel A (b) Panel B

Figure 1.4: Convex and non-convex sets

3. There should be exactly one $x \in \mathbb{R}$ with membership function that is equal to one, $\mu_A(x) = 1$.

4. The membership function $\mu_A(x), x \in \mathbb{R}$ is at least piecewise continuous.

Definition 1.6.4 *(Gao [24]) A support of a fuzzy number A is an interval on $\mathbb{R}$ denoted by*

$$suppA = \{x|\mu_A(x) \geq 0, x \in \mathbb{R}\},$$

if its membership function $\mu_A(x)$ is continuous on real-valued and x is called a mean value of the fuzzy number, if and only if $\mu_A(x) = 1$.

Operations on Fuzzy Numbers

As we have learnt from classical set theory that to combine two things we need an operation between them, the same principle is practiced with fuzzy numbers. The principle that is applied in extending the classical operators (division, substraction) to their fuzzy counterparts, such that we can also handle intermediate degrees of membership is called the extension principle.

Suppose we have two fuzzy numbers A and B, then the basic fuzzy arithmetics are:

1. Fuzzy addition

$$\mu_{A \oplus B}(x) = max_{y,z \in \mathbb{R}}\{min\{\mu_A(y), \mu_B(z)\}|y + z = x\} \text{ for all } x \in \mathbb{R}$$

2. Fuzzy difference

$$\mu_{A \ominus B}(x) = max_{y,z \in \mathbb{R}}\{min\{\mu_A(y), \mu_B(z)\}|y - z = x\} \text{ for all } x \in \mathbb{R}$$

3. Fuzzy product

$$\mu_{A \otimes B}(x) = max_{y,z \in \mathbb{R}}\{min\{\mu_A(y), \mu_B(z)\}|yz = x\} \text{ for all } x \in \mathbb{R}$$

4. Quotient of fuzzy numbers

$$\mu_{A\star B}(x) = max_{y,z\in\mathbb{R}}\{min\{\mu_A(y),\mu_B(z)\}|\tfrac{y}{z} = x, z \neq 0\} \text{ for all } x \in \mathbb{R}$$

Unlike other fuzzy arithmetics, the division operator has some restrictions. The denominator need not to be equal to zero, therefore from the above equation of quotient of fuzzy numbers we assume that $0 \notin max(B)$, where B is a divisors set.

Types of Fuzzy Numbers

There are different kinds of fuzzy numbers that exists namely Gaussian, triangular, sine, bell shape, exponential, trapezoidal fuzzy numbers and so on. In our study we will only discuss triangular fuzzy numbers and trapezoidal fuzzy numbers.

A fuzzy number $A = (a_1, a_m, a_2)$ is said to be a triangular fuzzy number (TFN) if its membership function is in this form:

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_m-a_1} & \text{for } a_1 \leq x \leq a_m, \\ \frac{a_2-x}{a_2-a_m} & \text{for } a_m \leq x \leq a_2, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

where $[a_1, a_2]$ is the supporting interval and the point a_m is the peak.

In some applications there is a point $a_m \in (a_1, a_2)$ which is located in the middle of the supporting interval, i.e $a_m = \frac{a_1+a_2}{2}$ [7]. Therefore substituting the value of a_m into Equation (1.3) yields

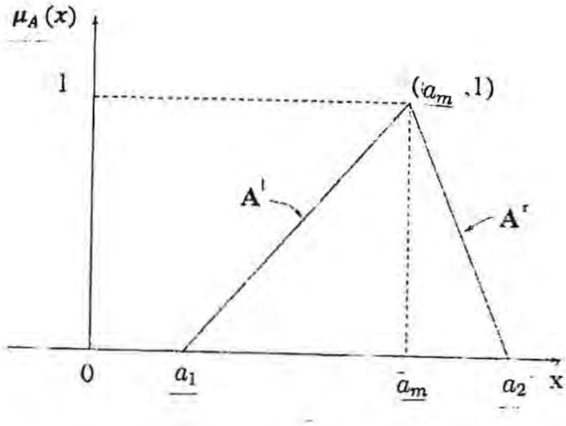


Figure 1.5: Triangular Fuzzy Number

Source: Bojadziev, 2007

$$\mu_A(x) = \begin{cases} 2\frac{x-a_1}{a_2-a_1} & \text{for } a_1 \leq x \leq \frac{a_1+a_2}{2}, \\ 2\frac{a_2-x}{a_2-a_1} & \text{for } \frac{a_1+a_2}{2} \leq x \leq a_2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Then Equation (1.4) is called a *central triangular fuzzy number.*

Equation 1.3 and Equation 1.4 are depicted in Figure 1.1 and 1.2, respectively.

A triangular fuzzy number is also known as linear fuzzy number, simply because of its linear type membership function that it possess; due to this characteristic TFN is widely used. Three important properties of the arithmetic of the triangular fuzzy numbers are:

1. The sum or difference of triangular fuzzy numbers gives us a triangular fuzzy number.

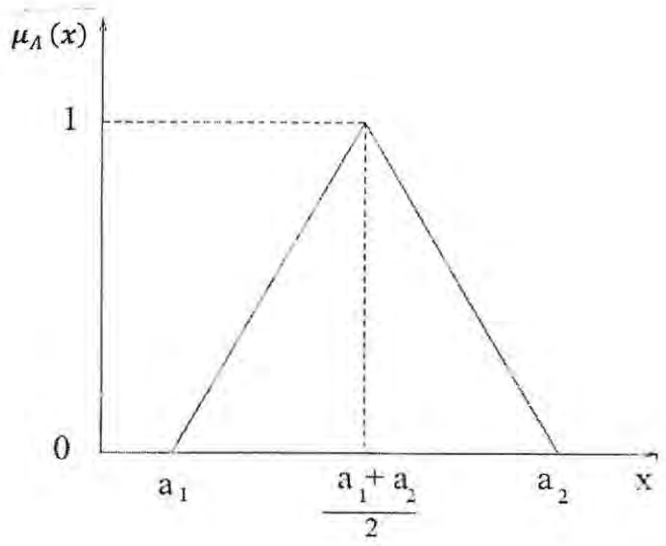


Figure 1.6: Central Triangular Fuzzy Number

Source: Bojadziev, 2007

2. The product or quotient of triangular fuzzy numbers does not yield triangular fuzzy numbers.

3. Max or min operations do not yield triangular fuzzy numbers.

Alpha-cuts

The classical set A_α, called alpha-cut set, is the set of elements whose degree of membership in A is no less than α. It is defined as:

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\},$$

The classical set $A*_\alpha$ is called *strong alpha-cut set.* It is defined as:

$$A*_\alpha = \{x \in X | \mu_A(x) > \alpha\}.$$

The alpha-cut operation is also applied at the fuzzy numbers. Consider two triangular fuzzy numbers $\tilde{K}$ = (a, b, c) and $\tilde{S}$ = (d, e, f) from which we obtain the following operations:

$$\tilde{K} \oplus \tilde{S} = (a+d, b+e, c+f)$$

$$\tilde{K} \ominus \tilde{S} = (a-f, b-e, c-d)$$

$$K_\alpha = [(b-a)\alpha + a, (b-c)\alpha + c]$$

$$S_\alpha = [(e-d)\alpha + d, (e-f)\alpha + f]$$

Example 1.6.1 *Consider the following example of two triangular fuzzy numbers that are defined as follows:*

$$K(x) = \begin{cases} \frac{x+2}{4} & \textit{for } -2 < x \leq 2, \\ \frac{5-x}{3} & \textit{for } 2 < x \leq 5, \\ 0 & \textit{otherwise.} \end{cases}$$

$$S(x) = \begin{cases} \frac{x-1}{2} & \textit{for } 1 < x \leq 3, \\ \frac{5-x}{2} & \textit{for } 3 < x \leq 5, \\ 0 & \textit{otherwise.} \end{cases}$$

Then, their alpha-cuts are:

$$K_\alpha = [4\alpha - 2, 5 - 3\alpha] \qquad for \quad \alpha \in (0, 1]$$

$$S_\alpha = [2\alpha + 1, 5 - 2\alpha] \quad for \quad \alpha \in (0, 1]$$

$$(K_\alpha + S_\alpha) = [6\alpha - 1, 10 - 5\alpha] \quad for \quad \alpha \in (0, 1]$$

$$(K_\alpha - S_\alpha) = [6\alpha - 7, 4 - \alpha] \quad for \quad \alpha \in (0, 1]$$

$$(K_\alpha * S_\alpha) = [-8\alpha^2 + 24\alpha - 10, 6\alpha^2 - 25\alpha + 25] \quad for \quad \alpha \in (0, 0.5]$$

$$= [8\alpha^2 - 2, 6\alpha^2 - 25\alpha + 25] \quad for \quad \alpha \in (0.5, 1]$$

$$(\frac{K_\alpha}{S_\alpha}) = [\frac{4\alpha - 2}{2\alpha + 1}, \frac{5 - 3\alpha}{2\alpha + 1}] \quad for \quad \alpha \in (0, 0.5]$$

$$= [\frac{4\alpha - 2}{5 - 2\alpha}, \frac{5 - 3\alpha}{2\alpha + 1}] \quad for \quad \alpha \in (0.5, 1]$$

The resulting fuzzy numbers are:

$$(K_\alpha + S_\alpha)(x) = \begin{cases} \frac{x+1}{6} & \text{for } -1 < x \leq 5, \\ \frac{10-x}{5} & \text{for } 5 < x \leq 10, \\ 0 & \text{otherwise.} \end{cases}$$

$$(K_\alpha - S_\alpha)(x) = \begin{cases} \frac{x+7}{6} & \text{for } -7 < x \leq -1, \\ 4 - x & \text{for } -1 < x \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.6.5 *(Bansal [2]) A fuzzy set $A = (a,\ b,\ c,\ d)$ is said to be a trapezoidal fuzzy number if its membership function is given by, where $a \leq b \leq c \leq d$*

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & \text{for } a \leq x \leq b, \\ 1 & \text{for } b \leq x \leq c, \\ \frac{d-x}{d-c} & \text{for } c \leq x \leq d, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

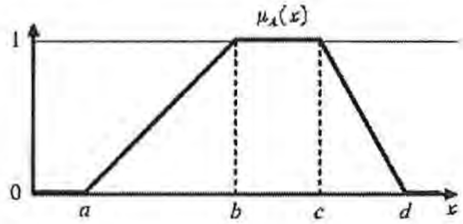


Figure 1.7: Trapezoidal Fuzzy Number

Source: Bojadziev, 2007

Equation 1.5 is depicted in Figure 1.3.

Definition 1.6.6 *The equality of two trapezoidal fuzzy numbers* $\tilde{K} = (a, b, c, d)$ *and* $\tilde{S} = (e, f, g, h)$ *exists if all the elements are equal component-wise, i.e.* $a = e$, $b = f$, $c = g$ *and* $d = h$.

The arithmetic operations for two trapezoidal fuzzy numbers $\tilde{K} = (a, b, c, d)$ and $\tilde{S} = (e, f, g, h)$ are:

1. **Fuzzy addition**

$$\tilde{K} \oplus \tilde{S} = (a, b, c, d) \oplus (e, f, g, h) = (a + e, b + f, c + g, d + h).$$

2. **Fuzzy subtraction**

$$\tilde{K} \ominus \tilde{S} = (a, b, c, d) \ominus (e, f, g, h) = (a - h, b - g, c - f, d - e).$$

3. **Fuzzy multiplication**

$$\tilde{K} \otimes \tilde{S} = (a, b, c, d) \otimes (e, f, g, h) \cong (ae, bf, cg, dh).$$

Chapter 2

Real Options Theory

In order to estimate the costs associated with deferring the decommissioning of the nuclear plant in question, the value of a real option must be determined. Real option valuation (ROV) is based on the observation that the possibilities financial options give their holders, resemble the possibilities to invest in real investments as well as those found within real investments [38]. ROV is intended to supplement, rather than replace, traditional capital project valuation methods.

The methods that have been introduced when pricing financial options are used when valuing real options. Since traditional capital project valuation methods were initially derived for use in pricing financial options, it is necessary to consider the difference between real and financial options. As mentioned earlier, real options are related to financial options and this relation will be explained in Section 2.2 and Section 2.3 of this chapter. Note that the differences between real options and financial options are that "real" assets are considered in the case of the former and real options are "valued" whereas financial options are "priced". In addition, real options can be used in valuing projects that have a long life-span (decades) but financial options' life-spans are mostly restricted to periods lasting less than a year [43].

Given that the real option valuation method is considered superior to traditional capital project valuation methods we first explain these traditional methods (in Section 2.1) which include Net Present Value (NPV), Payback Period (PP), Accounting Rate of Return (ARR) and Internal Rate of Return (IRR). The superioty of the real option approach compared to these traditional methods arises from the fact that it accommodates managerial flexibility and uncertainty.

We will discuss the most common financial option pricing methods which are most often applied in the real options context. These methods include the Binomial option pricing model and the Black-Scholes model.

2.1 Traditional Capital Project Valuation Methods

We now briefly describe traditional capital project valuation methods. The **Payback Period method** assumes that every capital project must pay for itself at the end of a certain period of time. That capital should be obtained from additional earnings generated from the capital assets. When managers decide whether to invest in the project, the payback period should be less than the one specified by the management. This method is often used by smaller companies, whereas larger companies would prefer to use the methods discussed below.

The **Accounting Rate of Return (ARR)** mainly focuses on the returns or earnings from the investment over the whole life of the project. When management considers multiple projects, the ARR method is used to rank projects such that they choose the project yielding the greatest rate of return. One advantage of the ARR is that it can be easily understood and computed. Unlike the other valuation methods discussed above, the ARR is based on accounting numbers, rather than on cash flows which is considered to be a disadvantage [14].

Net Present Value (NPV) is the difference between the present value of a project's expected after-tax operating cash flows and the present value of its expected after-tax investment expenditures.

Managers using this method must predict the cash flow profiles during the life of a project. After that they should discount the net cash flow profile using an appropriate rate of return [26]. It is useful to consider the NPV to be the present value of all cash flows:

$$NPV = \sum_{t=0}^{n} \frac{NCF_t}{(1+k)^t}, \tag{2.1}$$

where NCF_t is Net Cash Flow[1] in period t where $t = 1, 2, 3, \cdots, n$, n is the project's estimated life and k is the cost of capital.

To apply Equation 2.1, the cash flow of the project should be known at the beginning of the project and it is expected to remain constant until the end of the project.

[1] Cash inflows - Cash outflows.

If NPV is positive (negative), the project must be selected (rejected) because it will augment (deplete) a company's capital. NPV is the most commonly used method for investment appraisal; however it has some flaws that affect its accuracy. Firstly, it excludes the uncertainties that arise during the life of a project. It is well-known that the market conditions are sometimes volatile, thus affecting the profitability. Secondly, it omits the presence of decision-makers or managers who are considered slow to react (if they do at all) to effect of uncertainty on the investment. For example, consider the estimated wage rates of the nuclear industry employees. Such employees may demand better wages. If such demands are accepted by managers, then the cash flow profile of that project would be affected.

Example 2.1.1 *Consider a capital project's valuation. The research and development team of a nuclear company has developed a new nuclear power plant that can be used to generate electricity. Sales of the electricity generated from this nuclear power plant would be R600 000 per year. According to the plant manager, the existing production line will be modified at a cost of R400 000. It has been estimated that the cost of producing electricity will be about R440 000 per year. Production is expected to occur for five years and the specialized equipment necessary for the project has an estimated salvage value of R60 000. The appropriate cost of capital is 15%. Using Equation 2.1 and the foregoing information we obtain:*

$$NPV = -400000 + \frac{160000}{1.15} + \frac{160000}{(1.15)^2} + \cdots + \frac{160000}{(1.15)^4} + \frac{220000}{(1.15)^5} = R166180$$

The NPV for the nuclear power plant project is therefore R166 180. Management will invest in this project because its NPV is positive. In other words, *the project will add value to the company.*

Internal rate of return (IRR) is similar to NPV in the sense that it also uses a decision rule. That is, if IRR is positive then the project is accepted and when it is negative the project is rejected. The only difference between IRR and NPV is the way we use the data. To calculate IRR, the NPV should be known and it should be equal to zero. If the NPV is not equal to zero, then it must be adjusted by using the discount rate. Should NPV be positive then discount rate applied must be greater until NPV becomes zero. Similarly, if it is negative, the rate of discount should be cut until NPV equals to zero. IRR and NPV are both discounted cash flow based methods and they do possess the same characteristics. While theoretically sound, there are unfortunately two major problems associated with the Discounted Cash Flow (DCF) based approaches."First, the expected future cash flows that are estimated typically do not properly reflect the flexibility that exists in the investment and operation of the assets producing those cash flows. Second, the cash flows at different points in time typically require different discount rates to appropriately reflect their risk. Using an average or blended discount rate for all cash flows, as it is typically done, may lead to significant error in the valuation of cash flows" [50].

2.2 Financial Options

In this section we will discuss financial options in order to understand the related concept of real options.

A standard European-style call (put) option on an underlying asset with price S_0 confers its holder the right without the obligation to buy (sell) this asset on a specific date T at a specific price X called the exercise price where $X \in R_+$.

Instead of holding such an option an economic agent can write (sell) it in what is referred to as an uncovered option trade. However, institutional rather retail traders do this given that the latter traders cannot maintain margin requirements.

Note that an American-style option can be exercised at any time before its expiry, unlike the European-style option.

Options would not exist if the future were known with certainty. In a risky environment, options remunerate the risk of an uncertain future [52]. Thus the reward for taking risk is the foundation of an option. Higham [27] states that options are extremely attractive to investors, both for *speculation* and *hedging*.

There is a systematic way to determine how much they are worth, thus they can be bought and sold with some confidence. Intrinsic value is the difference obtained between the price of the underlying asset and the predetermined price which is also called strike or exercise price with the following scenarios being possible:

- **out-of-the-money**: when the price of the underlying asset is greater than the exercise price.
- **at-the-money**: when the price of the underlying asset is equal to the exercise price.

Close	Strike	Jan	June	Dec
R446.49	R412.72	R36.70	R46.90	R56.28
R446.49	R450.24	R5.93	R20.19	R31.89
R446.49	R487.76	R3.81	R7.51	R15.01

Table 2.1: *Values of Call Option*

- **in-the-money**: when the price of the underlying asset is smaller than the exercise price.

An option price is influenced by the two components, namely the intrinsic value and time value. The intrinsic value depends on the relationship between exercise price and the value of the underlying asset. On the other hand, the time value of an option is a function of the underlying asset's volatility, or risk (σ); the current level of interest rate (r); and the option's maturity, or time to expiration (T) [15]. The time value can be simply obtained by subtracting the intrinsic value from the option price. To understand the aforementioned concepts, consider Table 2.1.

The salient facts deduced from Table 2.1 are as follows. First, the option is in the money when the strike price is R412.72 and that value is R33.77. The value can be obtained by calculating the difference between the close price and the strike price. Second, between January and December the price of the call option increases from R36.70 to R56.80. Thus, early exercise is sub-optimal if the option is American-style. Finally, greater strike prices are not associated with a significant increase of call option prices compared to the less strike prices. Notice that the value for the option at strike R412.72 has decreased to R3.81 at the strike price of R487.76.

The parameters used to price options are as follows:

Value of the underlying asset (S_0): the current price of an asset when the right of the option has been sold or bought.

Exercise price (X) : pre-determined price on which the option contract is struck.

Time to maturity (T): the predetermined time for exercising an option.

Volatility σ: is the degree of fluctuation (rate of change) in the price of the underlying asset and is expressed in terms of standard deviation [35].

Variable	Call option value	Put option value
If the value of the underlying asset increases then	Increase	Decrease
If exercise price is greater then	Less	Greater
If the volatility is greater then	Greater	Greater
If time to maturity is longer then	Greater	Less
Call option payoff function	$(S_0 - X)^+$	
Put option payoff function		$(X - S_0)^+$

Table 2.2: *Parameters for option valuation*

From Table 2.2, when the variables considered change, there are several effects. Suppose the state of the spot price or value of the underlying asset is increased then call option value will also increase but the put option's value will decrease. The opposite of what happened before prevails when the exercise price increases in that call option value decreases. Waiting much longer before exercising an American call option increases the call option's value whereas the American put option's value decreases. The value of a call option increases with the risk-free interest rate, while the value of a put option is an inverse function of the risk-free interest rate [52].

2.2.1 Financial Option Pricing Methods

The price of a financial option can be obtained by the application of the techniques that appear on Table 2.3.

Technique	Specific Method	Prominent Studies
	Closed form solutions using Black-Scholes and other similar equations	Black and Scholes [5], Merton [37].
Partial differential equations	Analytical approximations	Geske and Johnson [25], Bunch and Johnson [11].
	Numerical methods (e.g finite difference method)	Brennan and Schwartz [10].
Simulations	Monte Carlo	Boyle [8], Longstaff and Schwartz [32].
	Binomial	Cox, Ross and Rubinstein [18].
Lattices	Trinomial	Parkinson [44], Boyle [8].
	Multinomial	Kamrad and Ritchken [29].

Table 2.3: *Summary of financial option pricing methods*
Source: Enevoldse and Nordbaek, 2011

Binomial Lattice Model : Financial Option Applications

There are single-step and multi-steps binomial option pricing models, we will start first by discussing the simple single-step method. Suppose that the underlying asset's price (S_0) increases by an "up" factor

$$u = e^{r\sqrt{\delta t}} \tag{2.2}$$

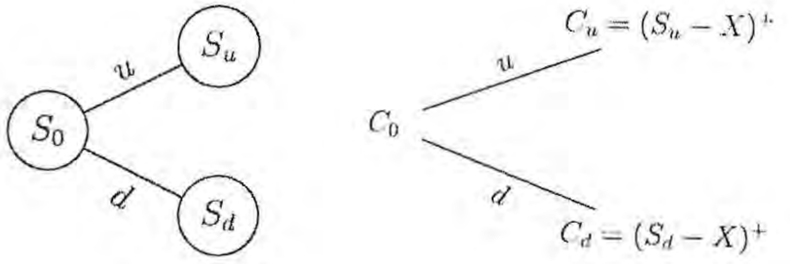
where δt is the chosen time interval, or decreases by a "down" factor

$$d = \frac{1}{u}, \tag{2.3}$$

then the designated prices will be S_u and S_d respectively. The call option's payoff when S_0 increases is $C_u = (S_u - X)^+$ and when S_0 decreases is $C_d =$

$(S_d - X)^+$. Note that payoffs have been expresssed in the form $max(a, 0) = a^+$ for some real-valued number a.

The single period binomial tree representing these concepts is:



(a) Panel A (b) Panel B

Figure 2.1: Simple Binomial Pricing Model

Panel (a) and (b) depicts the movement of underlying asset and call option's value, respectively.

Source: Clarke, 2000

During two time periods (t = 0 and t = 1), which are equivalent to one step of a binomial tree we can derive the risk-neutral probabilities which are the probabilities on the set of outcomes of the experiment that result in all bets being fair [45]. The risk-neutral probability $P_u(P_d)$ represents the increase (decrease) in S_0 to $S_u(S_d)$.

There are two types of interest rate compounding methods, namely, discrete and continuous compounding. Risk-neutral probabilities are deduced by assuming that discrete compounding is represented by (1 + r) whereas continuous compounding is represented by e^r.

The price of an option C_0 is determined by the following expression:

$$C_0 = \frac{1}{1+r}[P_u C_{1u} + P_d C_{1d}], \tag{2.4}$$

where r represents the risk-free interest rate, P_u and P_d are risk-neutral probabilities, which are:

$$P_u = \frac{(1+r)-d}{u-d} \tag{2.5}$$

$$P_d = \frac{(1+r)-u}{d-u} \tag{2.6}$$

Thus, the present value of the call option would be deduced from the discounted option values C_{1u} and C_{1d} with risk-neutral probabilities.

Equation (2.5) and Equation (2.6) were derived by solving the following system of equations:

$$\begin{cases} P_u + P_d = 1 \\ \frac{uP_u}{1+r} + \frac{dP_d}{1+r} = 1 \end{cases}$$

To obtain the solutions, we need to apply Cramer's Rule by arranging the coefficients of the foregoing system in the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ \frac{u}{1+r} & \frac{d}{1+r} \end{bmatrix}$$

where the determinant of A is denoted by $det(A) = \frac{d-u}{1+r}$. Then:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & \frac{d}{1+r} \end{bmatrix}$$

where $det(A_1) = \frac{d-(1+r)}{1+r}$. Thus, $P_u = \frac{det(A_1)}{det(A)} = \frac{d-(1+r)}{1+r} \cdot \frac{1+r}{d-u} = \frac{d-(1+r)}{d-u}$ and:

$$A_2 = \begin{bmatrix} 1 & 1 \\ \frac{u}{1+r} & 1 \end{bmatrix}$$

where $det(A_2) = \frac{(1+r)-u}{1+r}$. Then $P_d = \frac{det(A_2)}{det(A)} = \frac{(1+r)-u}{1+r} \cdot \frac{1+r}{d-u} = \frac{(1+r)-u}{d-u}$.

Note that the equivalent risk-neutral probabilities using continuous compounding are

$$P_u = \frac{e^r - d}{u - d} \tag{2.7}$$

$$P_d = \frac{u - e^r}{u - d}. \tag{2.8}$$

From Equations (2.4)-(2.6) we have observed that the jumping factors u and d play an important role in the value of the call option. Due to uncertainty of the underlying volatility, it is not trivial to estimate the values of these factors.

To find the value of an option we compute the values of the tree backward. The single period binomial model can be extended to consider multiple time periods. After n time periods, the call option's price will be

$$C_0 = \frac{1}{R^n} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (S_{u^k d^{n-k}} - K)^+ \tag{2.9}$$

where the gross return is $R = 1 + r$, n is the number of periods in the binomial tree, k is the number of upward price movements and p is the risk-neutral probability.

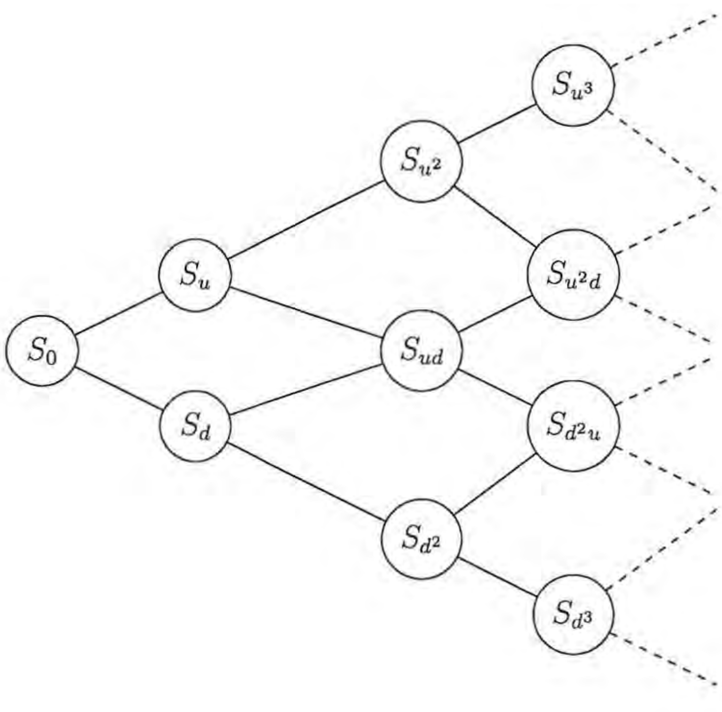


Figure 2.2: *Multi-Period Binomial Model*

Suppose that we have a timeline of three years and we divide it equally, then our binomial tree will have three different times. To understand the binomial tree, we have depicted it in Figure 2.2.

The diagram is divided into three steps, each step lasts for a year. From t = 0 at node S_0 the value of an asset can either go up or down, if it goes down we will get a value that is influenced by probability d and it will be S_d. Similarly, if the value goes up by the probability u, then we will obtain the node which is S_u. From S_u there are again chances that the value of underlying asset might increase by u or decrease by d, then S_{u^2} and S_{ud} will be obtained, respectively. We notice that from t = 2 the trees carry the special feature of recombining that cuts the number of computations. We can observe that node 5 can be obtained either when the value from node 2 decreases or when the one from node 3 increases.

Black-Scholes Model: Financial Option Applications

Black-Scholes [5] and Merton [37] give closed-form solutions to models assigning a price to the European-style call option only. These solutions are equivalent to that yielded by the binomial model. The Black-Scholes European call option price is:

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2), \tag{2.10}$$

where S_0 is the value of the underlying asset, X is the exercise price, r is the risk-free interest rate, T is the time to maturity, σ is the volatility and $N(d_i)$ is the cumulative normal distribution. Note that $d_1 = \frac{[ln(\frac{S_0}{X})+(r+\frac{\sigma^2}{2})T]}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Black and Scholes [5] assume (according to Wilmott [54]) that S_0 follows a lognormal random walk, r is a known function of time, no cash dividends are paid on the underlying, delta hedging is done continously, there are no transaction costs and there are no arbitrage opportunities.

2.3 Real Options

A real option can be defined in the same way as in Definition 2.2.1 except that a financial asset is replaced by a real asset and therefore the variables in Definition 2.2.1 are replaced by the capital project counterparts as will be discussed later. Managers of companies make decisions based on the circumstances at that moment or the consequences that might arise. Managers would prefer to take a decision that will benefit the company, by acting to counteract any uncertainty that may prevail during the life of a project. There are different options that management can take, namely option to wait, contract, expand, abandon, shut down, option to switch (e.g. outputs or inputs) and multiple interacting options.

2.3.1 Basic Concepts

Real option analysis is useful when a contingent investment decision exists; when there are lot of uncertainties and the best option would be to wait till more information has been found; when the value seems to be captured in possibilities for future growth options rather than current cash flow; if there is a large uncertainty that will lead to consideration of management flexibility.

Variable	Value of real option
Increase in the PV of the project	Increase
A higher investment cost	Decrease
A longer time to maturity	Increase
Increase in uncertainity (volatility of cash flows)	Increase
Increase in risk free rate	Increase
Increase in cash flow lost	Decrease

Table 2.4: *The impact of changes in option variables on the option's value*
Source: Vishwanath, 2009

Real option analysis is the only method that can be able to value investments in flexibility; and if during the life of the project there are updates and strategy corrections, the real options approach would be relevant [47].

If the owner decides to exercise that option then that simply means he has lost the option and gained an asset. The amount of an asset is called the strike price [19]. A real option's holder can, for a definite period, either decides for or against making an investment decision, without being obliged to do so initially [12]. For example, suppose a nuclear company has bought a design for building a plant. Then it has a right, without the obligation to employ the design by constructing the nuclear plant. If the management of that nuclear company decides to implement that design work, then it has gained an option that is not available to rival companies. In this example, the strike price would be the cost of building the plant.

2.3.2 Types of Real Options

The different types of options discussed below are used depending on the scenario confronting decision-makers.

Option to wait

The option to wait, also called the option to defer, is embedded in virtually every project [30]. Decision-makers would prefer to wait if the market conditions at that moment are not favourable or they think that it is worth waiting to exercise the option mainly because at a later stage it will bring more income compared to if it can be exercised at that stage. Sometimes decision-makers are willing to wait knowing that the market conditions would stabilize at a later stage. Thus, the company's performance would barely be affected if they opt to wait.

Option to abandon

The decision to choose this option is normally introduced after valuation of the project where it has been realized that during the life of project the company will incur losses. Thus, income will be limited or they will waste money and time by continuing with the project. Market conditions would be declining severely in such circumstances. Due to such conditions, the company's cash flow would be affected; therefore the project should be abandoned because it will be useless.

Option to expand

The choice to expand the project could be considered if decision-makers realise that the choice will assist them in increasing the company's market share. The expansion can happen in different ways. For example, a retailer may expand the company by increasing the number of branches or they might use internet to advertise their products. Moreover, they can introduce online shopping. If the retailer has issued stock, the value of such stock would increase if the foregoing measures succeed in generating cash flow.

Option to contract

Due to information or data available, decision-makers may decide to choose an option to contract. The project will occur at the specified time and end on a specific date thereby maximising profits as opposed to continuing with a loss-making project for a longer period of time.

Option to switch

Management might decide to use different inputs to produce the same output or they might even decide to use a different technique to produce that output. That might benefit the company financially or reduce the amount of time used to produce the output in question. Management might also opt to produce output in such a way that costs of production will decrease but the quality of output would remain the same.

Multiple interacting options

During the life of a project there might exist different options that can benefit the company financially. The combined value of such options is slightly greater than value of each individual option [46]. This arises because multiple options introduce more managerial flexibility compared to single options. It is not always possible to combine other options. For example, the option to defer an investment would not be combined with the option to extend or expand the project [31].

2.3.3 Real Option Valuation Methods

Real options can be valued anologously to financial options, with financial option pricing methods discussed in Table 2.3. The most common real option valuation methods are the binomial model and Black-Scholes formula. The transition from financial options to real options which is equivalent to the transition from financial option pricing to real option valuation occurs via a change in variables as demonstrated in Table 2.5.

	Financial call option	**Real option to invest in a project**
Underlying	Share or asset, (S_0)	Present value of future cash flows, (S_0)
Strike price	Strike price, X	Capital investment/ Present value of expected costs, X
Maturity	Contract maturity, T	Opportunity expires, T
Uncertainty	Share price uncertain	Project value uncertain
Volatility	Stock price volatility, σ	Volatility of project's expected cash flows, σ
Discount rate	Risk-free interest rate, r	Risk-free interest rate, r
Dividend rate	Continuous dividend rate, δ	Leakage in value, δ

Table 2.5: *Difference between a financial option and a real option*
Source: Luerman, 1998; Bodén and Åhlén, 2007; and Crundwell, 2008

2.3.4 Applications to Real Options Theory

We demonstrate the concept of real option valuation with the example of the Binomial model and Black-Scholes model. In the following binomial example, we will demonstrate the **option to abandon**. The option to abandon is embedded in virtually every project. This option is especially valuable where the net present value (NPV) is marginal but there is a great potential for losses [30].

Example 2.3.1 *OB Company generates electricity. From those initiatives there is a project related to the solar power and they are not sure about whether to implement it. This uncertainity is due to, solar power representing one among several techniques to produce electricity. The discounted cash flow analysis for the market potential on solar power shows that the present value of the payoff discounted at an appropriate market would be R382 million. The management of OB company may implement the project or sell its intellectual property for R248 million (known as the salvage value). The annual volatility is calculated to be 35% and the continuous annual riskless interest rate over the next five years is 5%. From the above scenario, it can be deduced that S = R 382 million; X = R 248 million; T = 5 years;* σ *= 35%; r = 5% and* δt *= 1. Using Equation 2.2, Equation 2.3 and Equation 2.7* [2]*, we can calculate* $u = e^{0.35\sqrt{1}} = 1.419$; $d = \frac{1}{1.419}$ *= 0.705 and* $P_u = \frac{e^{0.05\sqrt{1}} - 0.705}{1.419 - 0.705} = 0.485$.

Starting with the last step of the tree (Figure 2.3) we observe that the expected asset value for node S_{u^5} *is R2198 million. This value is much greater than the salvage value of R248 million and since we want to maximize our return we would not choose to abandon the project at that stage.*

We can also observe that at node S_{d^5} *the salvage value is greater than the expected asset value, now the decision that would be taken at this stage is to sell off the asset and abandon the project.*

At the penultimate step there are some changes that we notice. The expected assets values are calculated using the risk-neutral probability as weights and that gives us the discounted weighted average of potential future option values. If the salvage value is greater than the computed expected asset value, the

[2] We multiplied the superscript attached to the Euler's e term by $\sqrt{\delta t}$.

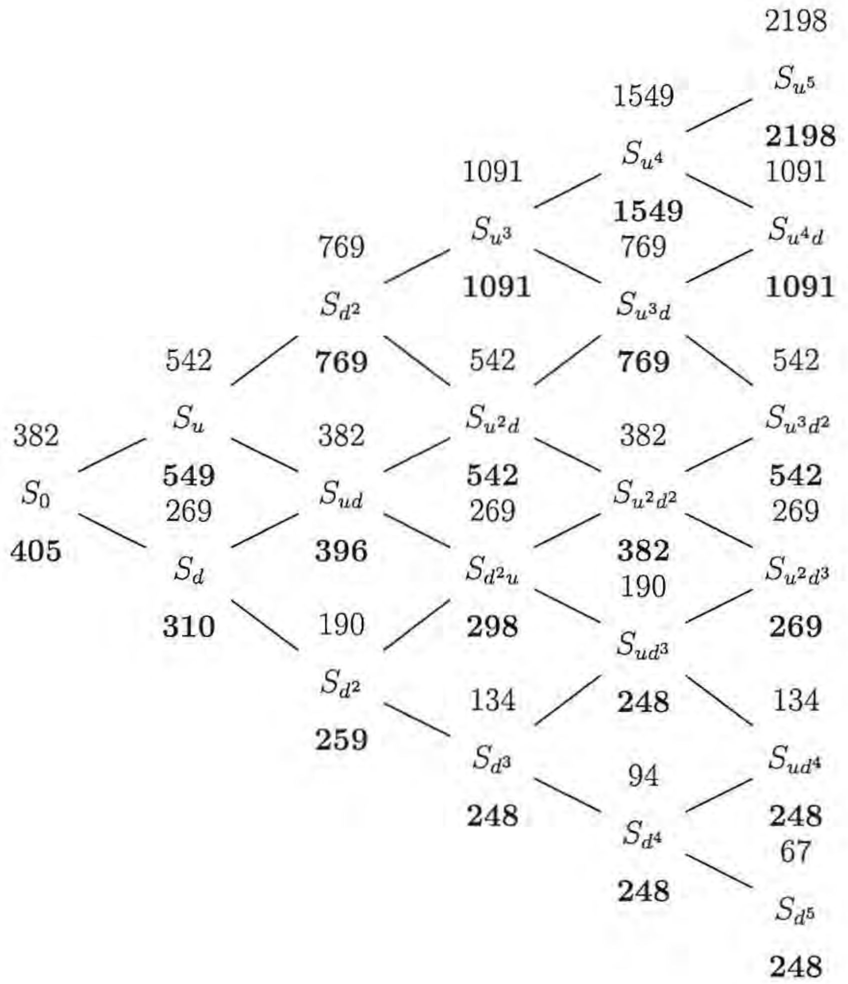


Figure 2.3: *Multi-Period Binomial Model for OB Company*

All numbers are expressed in R million, numbers appearing at the top are asset values and numbers appearing to the bottom in bold face are option values

option to abandon would be exercised.

The next example consists of the Black-Scholes formula being applied to an option to defer.

Example 2.3.2 (Valuing an Oil Reserve [20])

Consider an offshore oil property with an estimated oil reserve of 50 million barrels of oil, where the present value of the development cost is R12 per barrel and the development lag is two years. The firm has the rights to exploit this reserve for the next twenty years and the marginal value[3] *per barrel of oil is R12 per barrel currently. Once developed, the net production revenue each year will be 5% of the value of the reserves. The riskless rate is 8% and the variance is 0.03. The value of the developed reserve discounted back using the length of the development lag at the dividend yield is* $\frac{R12\times 50}{(1.05)^2} = R544.22$, *which is the stock price* (S_0). *The present value of development cost is R12* $\times$ *50 = R600 million. The time to expiration of the option (T) is 20 years and the dividend yield* (δ) *is 5%. Therefore the value of the oil reserve will be* $C_0 = 544.22e^{(-0.05)(20)}(0.8498) - 600e^{(-0.08)(20)}(0.6030) = R97.09$ *million where* $N(d_1) = 0.8498$ *and* $N(d_2) = 0.6030$ *which are computed from* $d_1 = \frac{[ln(\frac{544.22}{600})+(0.08-0.05+\frac{0.03}{2})20]}{\sqrt{0.03}\sqrt{20}} = 1.0359$ *and* $d_2 = 1.0359 - \sqrt{0.03}\sqrt{20} = 0.2613$, *respectively.*

If development starts today, the oil will be available for sale two years from

[3]Rand price per barrel - Marginal cost per barrel.

now. The estimated opportunity cost of this delay is the lost production revenue during the delay period [20].

Chapter 3

Fuzzy Set Theoretic Real Option Valuation Models

In this chapter we discuss the fuzzy set theoretic real options valuation models in the following way. In Section 3.1 the fuzzy risk-neutral-approximation-based binomial models will be discussed. In Section 3.2 the Black-Scholes formula will also be discussed.

As we have already discussed in the previous chapter, the NPV approach assumes that a stable project exits. It only favours the projects that start and end at a predetermined time without having any contingencies. However, this is not always the case. There are some contingencies that exist during the life of the project. For example, decision-makers would not force to start the project if the market conditions are unfavourable. Decision-makers can take a decision to postpone the starting time of the project or abandon the project completely; or they can also expand or extend the project when the market conditions are favorable. These flexibilities in the projects assist decision-makers in minimizing risk and also maximising profits. For this reason the flexibilities that are embedded as real options in investment projects, should be included in valuation.

3.1 Fuzzy Binomial Model

To evaluate investment projects that are embedded, the fuzzy binomial approach has been introduced by Liao and Ho [31]. The total value of the project can be obtained from its expanded NPV. Fuzzy numbers are used to estimate the parameters when the expanded NPV is estimated. Thus, this expanded NPV is called fuzzy expanded NPV (FENPV). Note that Liao and Ho [31] define expanded NPV or strategic NPV as

$$\text{Expanded NPV} = \text{Static NPV} + \text{Value of option.}$$

Most of the cash flow models used for financial decision making involve some degree of uncertainty. If the are deficiencies in the quality or quantity of data required for decision-making, then the decision is made based on the expert's knowledge of financial information. Using fuzzy set theory to rationalize the uncertainty, triangular fuzzy numbers are often used to test knowledge of profitability indices.

3.1.1 Triangular Fuzzy Numbers

If the jumping factors can be written as $\tilde{u} = [u_1, u_2, u_3]$ and $\tilde{d} = [d_1, d_2, d_3]$, the risk-neutral probabilities can be expressed in terms of fuzzy numbers as follows:

$$\begin{cases} \tilde{P_u} \oplus \tilde{P_d} = \tilde{1} \\ \frac{\tilde{u} \otimes \tilde{P_u}}{1+r} \oplus \frac{\tilde{d} \otimes \tilde{P_d}}{1+r} = \tilde{1} \end{cases}$$

where $\tilde{P_u} = [P_{u1}, P_{u2}, P_{u3}]$, $\tilde{P_d} = [P_{d1}, P_{d2}, P_{d3}]$.

Therefore, the above equation can be written as:

$$\begin{cases} P_{ui} + P_{di} = 1 \\ \frac{u_i \times P_{ui}}{1+r} + \frac{d_i \times P_{di}}{1+r} = 1 \end{cases}$$

for i = 1,2,3. Its solution is:

$$P_{ui} = \frac{(1+r) - d_i}{u_i - d_i}, \tag{3.1}$$

$$P_{di} = \frac{(1+r) - u_i}{d_i - u_i}. \tag{3.2}$$

In most cases the risk-free interest rate r and the exercise price X are clearly stated, thus they are crisp values. The fuzzy option values C_{1u} and C_{1d} are expressed in terms of the fuzzified jumping factors $\tilde{u}$ and $\tilde{d}$ which yields the fuzzy call option price:

$$\tilde{C_0} = \frac{1}{1+r}[\tilde{P_u} \otimes \tilde{C_{1u}} \oplus \tilde{P_d} \otimes \tilde{C_{1d}}] \tag{3.3}$$

3.1.2 Trapezoidal Fuzzy Numbers

If the trapezoidal fuzzy numbers $\tilde{u} = [u_1, u_2, u_3, u_4]$ and $\tilde{d} = [d_1, d_2, d_3, d_4]$ are used to represent the jumping factors of the underlying asset the risk-neutral probabilities can be written as:

$$\begin{cases} \tilde{P}_u \oplus \tilde{P}_d = \tilde{1} \\ \frac{\tilde{u} \otimes \tilde{P}_u}{1+r} \oplus \frac{\tilde{d} \otimes \tilde{P}_d}{1+r} = \tilde{1} \end{cases}$$

where $\tilde{P}_u = [P_{u1}, P_{u2}, P_{u3}, P_{u4}]$, $\tilde{P}_d = [P_{d1}, P_{d2}, P_{d3}, P_{d4}]$. Then the following equations arise:

$$\begin{cases} [P_{u1}, P_{u2}, P_{u3}, P_{u4}] \oplus [P_{d1}, P_{d2}, P_{d3}, P_{d4}] = [1, 1, 1, 1] \\ \frac{[u_1, u_2, u_3, u_4] \otimes [P_{u1}, P_{u2}, P_{u3}, P_{u4}]}{1+r} \oplus \frac{[d_1, d_2, d_3, d_4] \otimes [P_{d1}, P_{d2}, P_{d3}, P_{d4}]}{1+r} = [1, 1, 1, 1] \end{cases}$$

Thus:

$$\begin{cases} P_{ui} \oplus P_{di} = 1 \\ \frac{u_i \otimes P_{ui}}{1+r} \oplus \frac{di \otimes P_{di}}{1+r} = 1 \end{cases}$$

for i = 1, 2, 3, 4.

The risk-neutral probabilities are approximated by the following trapezoidal fuzzy number:

$$\tilde{P}_u = (\frac{1+r-d_4}{u_4-d_4}, \frac{1+r-d_3}{u_3-d_3}, \frac{1+r-d_2}{u_2-d_2}, \frac{1+r-d_1}{u_1-d_1}),$$

$$\tilde{P}_d = (\frac{u_1 - (1+r)}{u_1 - d_1}, \frac{u_2 - (1+r)}{u_2 - d_2}, \frac{u_3 - (1+r)}{u_3 - d_3}, \frac{u_4 - (1+r)}{u_4 - d_4}).$$

The call option price (C_0) is computed with Equation 3.3, noting that $\tilde{P}_u$ and $\tilde{P}_d$ are trapezoidal.

3.2 Fuzzy Black-Scholes Formula

3.2.1 Triangular Fuzzy Numbers

Nikookar et al., [42] introduced a real option rule in a fuzzy setting, where the present values of expected cash flows and expected costs are estimated by triangular fuzzy numbers. This method makes it possible for decision-makers to estimate the present values of expected cash flows by using the triangular possibility distribution of the form $\tilde{S}_0 = [s_l, s_m, s_r]$.

From the above triangular possibility distribution, s_m represents the core of the triangular fuzzy number $\tilde{S}_0$ which is the most possible value of the present value of expected cash flows, s_r and s_l represent the greatest and smallest values for the present values of expected cash flows, respectively.

Similarly, expected costs can also be estimated by the triangular possibility distribution which are of the form $\tilde{X} = [x_l, x_m, x_r]$.

It is obvious that the core of the triangular fuzzy number $\tilde{X}$ would be represented by x_m which is the most possible value of expected costs. The greatest and least for expected costs are represented by x_r and x_l respectively.

We shall consider the values that are influenced by the volatility of the cash inflow and riskless interest rate. To estimate them by the triangular possibility distribution, historic data is employed. Then, $\tilde{R} = [r_l, r_m, r_r], \tilde{\delta} = [\zeta_l, \zeta_m, \zeta_r]$ and $\tilde{\sigma} = [\sigma_l, \sigma_m, \sigma_r]$. The variables r_m, ζ_m and σ_m represent the most possible values of the discount rate, risk-adjusted discount rate and the volatility of cash inflows lie in the interval, respectively [42]. Similarly, the greatest values for discount rate, volatility of cash inflow, and risk-adjusted discount-rate are represented by r_r, σ_r and ζ_r respectively. Lastly, the smallest values for discount rate, volatility of cash inflow and risk-adjusted discount-rate are represented by r_l, σ_l and ζ_l.

In most cases, the exact values needed for capital budgeting are not known with certainty. There are many sources of uncertainty that exist in a capital project. Therefore, it is imperative to consider the following variables in valuing a capital project. The volatility $\tilde{\sigma}$ of capital project's cash inflows, the expected costs $\tilde{X}$ of a capital project, the project's expected cash flows $\tilde{S}_0$ and the discount rate $\tilde{R}$. The Black-Scholes formula that was extended by Merton [37] is:

$$ROV = \tilde{S}_0 \otimes e^{-\tilde{\delta} \otimes T} \otimes N(\tilde{d_1}) \ominus \tilde{X} \otimes e^{-\tilde{R} \otimes T} \otimes N(\tilde{d_2}),$$

where

$$\tilde{d_1} = \frac{ln(\tilde{S}_0 \oslash \tilde{X}) \oplus (\tilde{R} \ominus \tilde{\delta} \oplus \frac{\tilde{\sigma}^2}{2}) \otimes T}{\tilde{\sigma} \otimes \sqrt{T}} \tag{3.4}$$

$$\tilde{d_2} = \tilde{d_1} \ominus \tilde{\sigma} \otimes \sqrt{T}. \tag{3.5}$$

$\tilde{S}_0$ and $\tilde{\sigma}(\tilde{S}_0)$ denotes the possibilistic mean value and variance of the present value of expected cash flows, respectively. $\tilde{X}$ represents the possibilistic mean value of expected costs.

Considering all forms of the triangular possibility distribution, the above ROV equation will be expanded and expressed as:

$$FROV = (s_l, s_m, s_r) \otimes e^{-\tilde{\delta}\otimes T} \otimes N(\tilde{d_1}) \ominus (x_l, x_m, x_r) \otimes e^{-\tilde{R}\otimes T} \otimes N(\tilde{d_2}). \quad (3.6)$$

Using the fuzzy arithmetic operations defined in Chapter 1, Equation 3.4, Equation 3.5 and Equation 3.6 yields:

$$FROV = [s_l N(\tilde{d_1}) \otimes e^{-\tilde{\delta}\otimes T} - x_r N(\tilde{d_2}) \otimes e^{-\tilde{R}\otimes T}, s_m N(\tilde{d_1})$$
$$\otimes e^{-\tilde{\delta}\otimes T} - x_m N(\tilde{d_2}) \otimes e^{-\tilde{R}\otimes T}, s_r N(\tilde{d_1}) \otimes e^{-\tilde{\delta}\otimes T} - x_l N(\tilde{d_2}) e^{-\tilde{R}\otimes T}],$$

Note that:

$$\tilde{d_1} = [\frac{(ln\frac{s_l}{x_r} + r_l T - \zeta_r T + \frac{\sigma_l^2}{2})T}{\sigma_r\sqrt{T}}, \frac{(ln\frac{s_m}{x_m} + r_m T - \zeta_m T + \frac{\sigma_m^2}{2})T}{\sigma_m\sqrt{T}},$$

$$\cdots \frac{(ln\frac{s_r}{x_l} + r_r T - \zeta_l T + \frac{\sigma_r^2}{2})T}{\sigma_l\sqrt{T}}],$$

In order to find the maximum of the set $\{\tilde{C}_0, \tilde{C}_1, \cdots, \tilde{C}_T\}$ the elements of the set must be ranked. However, the function to order FROV which is defined as $\tilde{C}_{t*} = [c_l, c_m, c_r]$ is used [12]. The probabilistic decision rule for optimal investment is now generalized in a fuzzy setting. Where the maximum deferral time is T, make the investment (exercise the option) at time t^* for $0 \leq t^* \leq$ T, for which the option $\tilde{C}_{t^*}$ attains its maximum value [42]:

$$\tilde{C}_{t^*} = max_{t=0,1,\cdots T}\tilde{C}_t = \tilde{S}_0 \otimes e^{-\tilde{\delta}T} \otimes N(\tilde{d_1}) \ominus \tilde{X} \otimes e^{-\tilde{R}T} \otimes N(\tilde{d_2}).$$

Then

$$\tilde{S}_t = c\tilde{s}_0 + \sum_{j=1}^{T} \frac{c\tilde{s}_j}{(1+\tilde{\delta})^j} - c\tilde{s}_0 + \sum_{j=1}^{t} \frac{c\tilde{s}_j}{(1+\tilde{\delta})^j} = \sum_{j=t+1}^{T} \frac{c\tilde{s}_j}{(1+\tilde{\delta})^j},$$

where $c\tilde{s}_j$ denotes the expected (fuzzy) cash flow at time j, $\tilde{\delta}$ is the risk-adjusted discount rate (or required rate of return on the project).

Nikookar et al., [42] have employed the following function to determine the expected fuzzy real option values $\{\tilde{C}_l, \tilde{C}_m, \cdots, \tilde{C}_r\}$ of triangular form:

$$E(C_t) = \frac{[(\tilde{C}_r - \tilde{C}_l) + (\tilde{C}_m - \tilde{C}_l)]}{3} + \tilde{C}_l,$$

where $\tilde{C}_r, \tilde{C}_m$ and $\tilde{C}_l$ are the greatest, middle and least values of the fuzzy real option values.

The flaws in this method are as follows. It is unclear how the greatest values (s_r) and least values (s_l) for the present value of expected cash flows are computed. The same problem prevails in the computation of the greatest value $(\tilde{\sigma}_r)$ and least value $(\tilde{\sigma}_l)$ for volatility of cash inflow, it is not indicated how they are calculated.

3.2.2 Trapezoidal Fuzzy Numbers

It has not been easy to express the present value of the expected cash flow as a single number. The Waeno research project on giga-investments has shown that it is possible to estimate the present value expected of the cash flow as a trapezoidal possibility distribution of the form $\tilde{S}_0 = (s_1, s_2, \alpha, \beta)$, [12]. The interval $[s_1, s_2]$ represents the most possible values of the present value of expected cash flows and that makes the interval the core of the trapezoidal fuzzy number. The upward and downward potential for the present value of the expected cash flows are expressed as $(s_2 + \beta)$ and $(s_1 - \alpha)$ respectively.

Similarly, to estimate the value of expected costs the trapezoidal fuzzy numbers $\tilde{X} = (x_1, x_2, \alpha', \beta')$ are used. The interval $[x_1, x_2]$ represents the most possible values of expected costs and it is also a core of the trapezoidal number $\tilde{X}$. The upward and downward potential for expected costs are represented by $(x_2 + \beta')$ and $(x_1 - \alpha')$ respectively.

Thus, the formula for calculating fuzzy real option values is

$$FROV = \tilde{S}_0 e^{-\delta T} N(d_1) - \tilde{X} e^{-rT} N(d_2), \tag{3.7}$$

where

$$d_1 = \frac{[ln(\frac{E(\tilde{S}_0)}{E(\tilde{X})}) + (r - \delta + \frac{\sigma^2}{2})T]}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}, \tag{3.8}$$

and $E(\tilde{S}_0)$ denotes the possibilistic mean value of the expected cash flows which is computed as $\frac{s_1+s_2}{2} + \frac{\beta-\alpha}{6}$, $E(\tilde{X})$ stands for the possibilistic mean value of expected costs which is computed as $\frac{x_1+x_2}{2} + \frac{\beta'-\alpha'}{6}$ and $\sigma(\tilde{S}_0)$ is the possibilistic variance of the present value expected cash flows which is computed as $\sqrt{\frac{(s_2-s_1)^2}{4} + \frac{(s_2-s_1)(\alpha+\beta)}{6} + \frac{(\alpha+\beta)^2}{24}}$.

The above equation for computing fuzzy real option values can be expressed as

$$FROV = (s_1, s_2, \alpha, \beta)e^{-\delta T} N(d_1) - (x_1, x_2, \alpha', \beta')e^{-rT} N(d_2), \tag{3.9}$$

$= (s_1 e^{-\delta T} N(d_1) - x_2 e^{-rT} N(d_2), s_2 e^{-\delta T} N(d_1) - x_1 e^{-rT} N(d_2), \alpha e^{-\delta T} N(d_1) + \beta' e^{-rT} N(d_2), \beta e^{-\delta T} N(d_1) + \alpha' e^{-rT} N(d_2))$.

Chapter 4

Empirical Analysis

Currently, ESKOM operates two electricity-generating nuclear reactors at Koeberg. The construction of Koeberg's reactors began in 1976 and Unit 1 was synchronised to the grid in 1984, with Unit 2 following suit in 1985. The two units have a capacity of 900MW each and Koeberg's average annual production is 13 668GWh [23]. Both units were originally expected to operate for a period of forty years. This time-frame is now being reconsidered, with a view to defer it by a decade (decommissioning in 2035), given that the Reference Technical Plan (RTP) for Koeberg Spent Fuel Management and Disposal is revised from time to time. In the event of deferral, the total period of operation would be fifty years [22]. During the assigned time-schedule, numerous uncertainties will affect total operating costs.

The first form of uncertainty relates to the actual starting time for the disposal of spent fuel. If ESKOM is "allowed" to change the time-schedule for implementation, then it should also be allowed to change the timing of the related investments [48]. Thus, the change in time also affects the present value of costs. We assume that there exists flexibility regarding timing for the implementation of the deep disposal of spent fuel and hence flexibility regarding its investment timing. Second, uncertainties that are related to the RTP might arise from the unexpected problems such as the delay in the commencement of decommissioning. Finally, there might also be changes in labour and material costs due to the advancement of technology.

It can generally be expected that uncertainities in cost increase with longer time frames and that uncertainties will decrease with increasing accuracy and refinement of input data [4]. We recall that in Chapter 2 the different types of real options were discussed. In this study, the option to defer is relevant because the decision alternatives are related to investment timing.

The data and methods that are used in our study are presented in Section 4.1. Then in Section 4.2, we present and explain the results obtained using the fuzzy binomial model and fuzzy Black-Scholes formula.

4.1 Data and Methods

Important historic and current qualitative and quantitative information about Koeberg's operations are concealed by Eskom's management. Thus, variables relevant to the models used in this study must be estimated. These variables include the number of casks of Spent Fuel Assemblies (SFAs) produced annually, the expected annual production of SFAs till the decommissioning of Koeberg, Koeberg's expected cash flow and expected costs, the risk-free interest rates and the value lost over the duration of the option.

Note that some data SFA annual production data for 2001 - 2011 period was contained in Eskom's annual reports. This data was inadequate because it omitted relevant observations that were required for our study. Thus, we employed linear interpolation to find the historic values from the commissioning of Koeberg up until the year 2000, the same method was used to forecast the SFA's annual production from 2012 to 2035. Linear interpolation has been used by Lumby [34] and other researchers in estimating the missing values in the valuation methods.

We were not provided with the values of Koeberg's expected cash flow and expected costs. However, we referred to Morgan's [39] work about nuclear waste disposal and plant decommissioning costs. That enabled us to estimate the values of Koeberg's expected cash flows and expected costs. In [39] it is reported that the decommissioning cost for a nuclear power plant which operates for a minimum period of 40 years is \$0.5 billion. A typical nuclear power plant generates about 20 metric tons of used fuel per year and the average utility company's contribution per metric ton of SFAs is \$277 000.

The fuzzy binomial valuation approach and FROV using the Black-Scholes formula that were discussed in Section 3.1.1 and 3.2.2 repectively, will be used to assess a proposal to defer Koeberg's Life of Plant Plan (LOPP) from forty years to fifty years.

LOPP deferral from forty years to fifty years will cause changes in the production of spent fuel, dry storage casks, disposal canisters, transport requirements and costs. After the final shutdown at Koeberg, Spent Fuel Management and Disposal (SFMD) activities will occur [22].

The following assumptions are being considered in this project. The total number of Koeberg spent fuel assemblies (SFAs) generated over fifty years LOPP and to be finally disposed of, will be 3903. The mean of the SFAs is 73.41 per year and the standard deviation is 19.34 SFAs. Thus, the coefficient of variation is 0.26. After Koeberg's shutdown, the wet storage facilities will continue to operate for some time because the last batch of Spent Fuel (SF) from the reactor cyles should be allowed to cool off and there should also be sufficient time to transfer all the SFAs to an interim dry storage facility [22]. It is also assumed that the method of disposal to be used will be directly in a suitable deep geological repository built within the boundaries of South Africa, there will be no other method that will be employed [22].

The NPV of the direct disposal option is estimated to be R1.336 billion and the future cost is R7.680 billion for the period of fifty years since the commissioning. These values were computed as part of a study concerning spent nuclear fuel management options for South Africa [51].

The value of flexibility stems from the uncertainity of SFA production, hence we have to forecast the future production of SFAs. In this case, the uncertainty is characterized as possibility [31]. Thus, fuzzy numbers are introduced. We will employ triangular fuzzy numbers to represent the coefficient of variation (CV) of predicting the future SFAs production via the volatility of the SFAs production. Based on what has been done in [31] we estimate the CV to have a variation of $\pm30\%$ per year. Hence, the volatility of SFA production will be represented by the triangular fuzzy number $\tilde{\rho} = [(1 - 0.3)\times 0.26, 0.26, (1 + 0.3)\times 0.26] = [0.182, 0.26, 0.338]$.

Using $\tilde{\rho}$, the jumping factors $\tilde{u}$ and $\tilde{d}$ are computed with the use of $\tilde{u} = e^{\tilde{\rho}\sqrt{\tau}}$ and $\tilde{d} = \frac{1}{\tilde{u}}$, respectively; where τ is the chosen time interval size expressed in the same unit as $\tilde{\rho}$ [31]. In this case, annual volatility estimates are used, therefore the value of τ is 1. Thus, $\tilde{u} = [1.19961494, 1.29693007, 1.402140503]$ and $\tilde{d} = [0.7131952881, 0.7710515856, 0.8336013403]$.

From the given data, the binomial tree of project value can be constructed. Based on the studies done by Liao and Ho [31] the NPV of R1.336 billion will be employed to start the binomial tree.

Python code has been written to create the binomial tree of project value and the decision tree with the option to defer (see Appendix). One should note that in computing the decision tree, we follow Liao and Ho [31] in estimating:

$$\tilde{C_0} = \frac{\{\tilde{P_u} \otimes max(\tilde{V}^{+}, I - \tilde{V}^{+})\} \oplus \{\tilde{P_d} \otimes max(\tilde{V}^{-}, I - \tilde{V}^{-})\}}{(1+r)} \tag{4.1}$$

where $\tilde{P_u}$ and $\tilde{P_d}$ are computed using Equations 3.1 and 3.2 respectively, $\tilde{V}^{+}$ and $\tilde{V}^{-}$ are the top and bottom decision nodes repectively, I represents the future costs of direct disposal and r is the annual discount rate.

The existence of managerial flexibility introduces skewness in the possibilistic distribution, thus the actual distribution is skewed to the right. Following [31], the root value of the decision tree represents the FENPV of the project which is assumed to have the right-skew characteristic. The mean value is used to represent FENPV as a crisp value. Furthermore, the different FENPV's can be comparable using their mean values. The method of computing FENPV that possesses a right-skew characteristic has been proposed by Liao and Ho [31] as

$$E(FENPV) = \frac{(1-\lambda)c_1 + c_2 + \lambda c_3}{2} \tag{4.2}$$

where λ is $\frac{A_R}{A_L+A_R}$, A_R and A_L are the right-part area and left-part area of the FENPV respectively. The premium of the option to defer is the difference of E(FENPV) computed using Equation 4.2 and the NPV of the project. Figure below depicts a FENPV with a right-skew distribution.

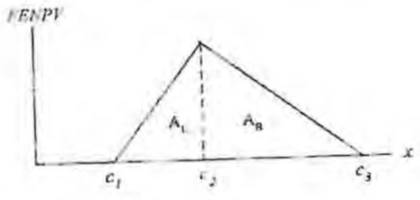


Figure 4.1: FENPV with right-skew distribution
Source: Liao and Ho, (2010)

The code required for assigning values to the options considered is supplied in, for instance when using the command "python Fuzzy_binomial_tree_python _code_15%.py -f7 -r5" the code will compute the values for binomial tree of project value up to 7 years and also the values for decision tree with the option to defer to start from the fifth year of the binomial tree of project value will be computed. The outputs of the above-mentioned command are presented in Figure 4.1 and 4.2.

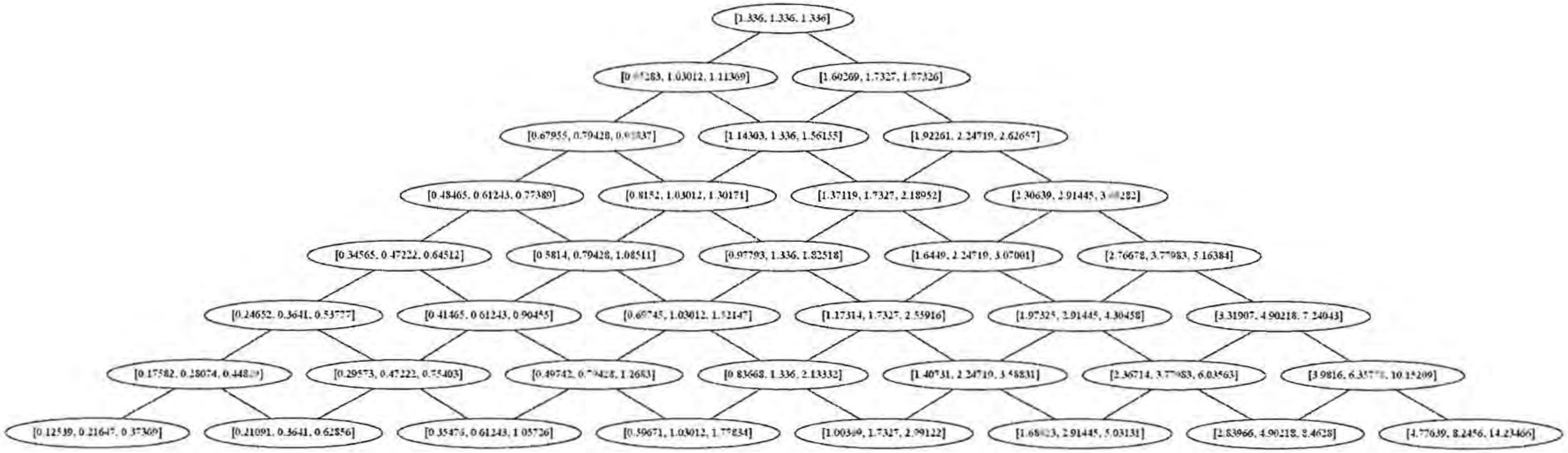


Figure 4.2: Binomial tree of project value for 7 years

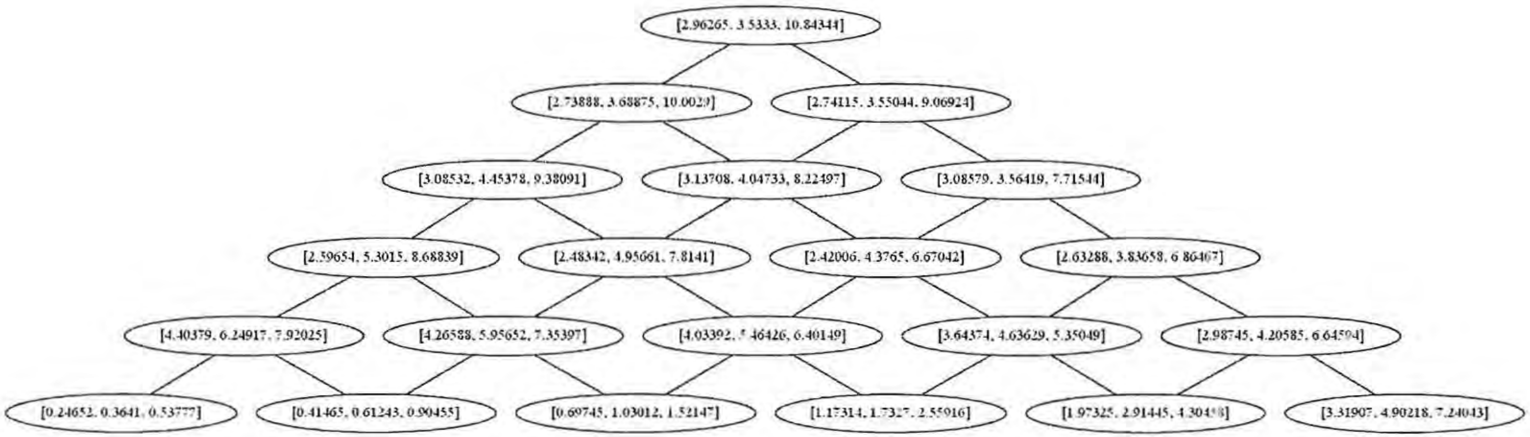


Figure 4.3: The decision tree with the option to defer for 5 years

To compute FROV we recall Equation 3.7 and Equation 3.8. In our case the variables of both Equation 3.7 and Equation 3.8 have the following meaning $\tilde{S}_0$ is the present value of Koeberg's expected free cash flows (fuzzy), $\tilde{X}$ is the present value of Koeberg's expected costs (fuzzy), $E(\tilde{S}_0)$ is the possibilistic mean value of the present value of expected cash flows (crisp), $E(\tilde{X})$ is the possibilistic mean value of expected costs (crisp), σ is the possibilistic standard deviation of the present value of Koeberg's expected cash flows (crisp), T is the time to expiry of the real option (crisp), δ is the value lost over the duration of the option (crisp) and r is the annualized continously compounded rate on a safe asset (crisp).

Referring to the extension of Koeberg's LOPP, the related parameters are estimated as follows. After fifty years of commissioning Koeberg's expected cash flow ($\tilde{S}_0$) would be worth between R891 billions and R1603.8 billions whereas the expected costs ($\tilde{X}$) are estimated to be worth between R6498 and R8447.4 billions. Using sensitivity analysis to assess the effect of uncertainty in forecasts, we will change some individual variables on a project's fuzzy real option value. The two variables to be changed are time (T), which will be 30, 40 and 50 years and the risk-free interest rate (r) which will take the 10%, 15% and 20% values. The value lost over the duration of the option (δ) is set at 0.03 following Carlsson and Fuller [12]. It is not trivial to identify the values of α and β, thus the parameters are estimated to be R338.58 billion. Similarly, the variables α' and β' are estimated to be R584.82 billion.

4.2 Results

We considered different cases to compute the expected FENPV and the option premium. We choose three different possible discount rates that could be applied together with three different years to exercise the option to defer and the results of these scenarios are discussed below. The maximal values for E(FENPV) and option premium are obtained when decision making is made at the fortieth year with the 10% risk-free interest rate. On the other hand, it can be observed that when decision to defer is exercised at the thirtieth year with the 20% risk-free interest rate, the minimal values are obtained as it is presented in Table 4.1. In all three cases, we notice that the delay of years to exercise the option to defer increases the values of E(FENPV) and the option premium. In our case, it means that the delay in decommissioning the Koeberg's nuclear plant increases the value of the decommissioning costs. It is observed that the binomial tree reveals that project value fluctuates with production of SFAs.

Attached on this thesis is the compact disc that contains the python codes and the binomial trees obtained. The python codes are similar, they only differ with the value of annual discount rate. The code used to compute the binomial trees with a 10% annual discount rate is labelled as "Fuzzy_binomial _tree_python_code_10%.py". The other python codes for 15% and 20% annual discount rates have been labelled in a similar way, except the changes in the value of annual discount rate. The outputs of the python codes are also embedded in the compact disc. We labelled them according to the different years and annual discount rates we considered in our study. For example, the output of decision making at 30 years with a 15% annual discount rate is labelled as "Binomial back 30 years 15%.pdf".

Due to the limitations in the python code, the root value of the decision trees with a 10% and 15% discount rates were unrealistic. For an example using Equation 4.2 and the root value of the decision tree labelled "Binomial back 40 years 10%.pdf" yielded an E(FENPV) of R95 687.99 billion. That is a very high value which is even greater than the gross domestic product (GDP) of South Africa. The gross domestic product at market prices during the 4th quarter of 2011 was R770 billion [49]. However, fuzzy Black-Scholes formula yields reasonable results which are illustrated in Table 4.5, Table 4.6 and Table 4.7. On the other hand, realistic results are obtained in the python code of fuzzy binomial approach when the percentage of annual discount rate is close to 20%. Thus, the values of E(FENPV) and option premium with 10% and 15% cannot be accepted. However, the obtained root value of the decision tree when the annual discount rate is 20% is acceptable and the results are illustrated in Table 4.1.

Year	E(FENPV) (R billion)	Option Premium (R billion)
30 years	365.7806	364.44463
40 years	2947.964	2946.628

Table 4.1: *Fuzzy binomial approach using discount-rate (r) = 20% per year*

Using Equation 3.9 we obtain:

$FROV = (R697.25 \quad billion, R866.27 \quad billion, R95.52 \quad billion, R95.52 \quad billion)$

Note that $N(d_1) = N(\frac{ln(\frac{1247.40}{7472.70})+(0.1-0.03+\frac{(0.3816)^2}{2})\times 30}{0.3816\times\sqrt{30}}) = N(1.1933) = 0.88357$

and $N(d_2) = d_1 - \sigma\sqrt{T} = -0.89681$ where $\sigma(\tilde{S}_0) = \sqrt{\frac{(712.8)^2}{4} + \frac{(712.8)(677.16)}{6} + \frac{(677.16)^2}{24}} = R476.00$ billion, $E(\tilde{S}_0) = \frac{891+1603.8}{2} + \frac{338.58-338.58}{6} =$ R1247.40 billion and $E(\tilde{X}_0) = \frac{6498+8447.40}{2} + \frac{781.76-781.76}{6} =$ R7472.70 billion. Then $\% \frac{\sigma(\tilde{S}_0)}{E(\tilde{S}_0)} = 0.3816 \times 100 = 38.16\%$.

The expected value of FROV is R781.76 billion and its most possible values are bracketed by the interval [R697.25 billion, R866.27 billion].

The downward potential is R601.73 billion and the upward potential is R961.79 billion. Thus, the maximal possible loss and gain in decommissioning the Koeberg nuclear plant 30 years after its commissioning would be R601.73 billion and R961.79 billion, respectively.

Year	d_1	$N(d_1)$	$N(d_2)$
30 years	1.1933	0.88357	-0.89681
40 years	1.6251	0.9479	-0.78835
50 years	1.9828	0.97628	-0.71552

Table 4.2: *Values of normal cumulative distribution with discount-rate (r) = 10% per year*

Year	d_1	$N(d_1)$	$N(d_2)$
30 years	1.9109	0.97136	-0.1792
40 years	2.4538	0.99298	-0.04035
50 years	2.9093	0.9981	0.2101

Table 4.3: *Values of normal cumulative distribution with discount-rate (r) = 15% per year*

Year	d_1	$N(d_1)$	$N(d_2)$
30 years	2.6286	0.99569	0.53849
40 years	3.2825	0.9995	0.86905
50 years	3.8358	0.99994	-1.69838

Table 4.4: *Values of normal cumulative distribution with discount-rate (r) = 20% per year*

The variables that change simultaneously with time and annual discount rates when computing FROV using Equation 3.9 are the normal cumulative distribution function. The values of this function appear in Table 4.2, Table 4.3 and Table 4.4 and were used to calculate the fuzzy real option values that are presented in Table 4.5, Table 4.6 and Table 4.7, respectively.

Year	FROV (R bil)	EV (R bil)	MPV (R bil)	DP (R bil)	UP (R bil)
30 years	[697.25, 866.27, 95.52, 95.52]	781.76	[697.25, 866.27]	601.73	961.79
40 years	[376.36, 551.71, 88.22, 88.22]	464.035	[376.36, 551.71]	288.14	639.93
50 years	[234.82, 380.70, 70.94, 70.94]	307.76	[234.82, 380.70]	163.88	451.64

Table 4.5: *FROV using discount-rate (r) = 10% per year*

Year	FROV (R bil)	EV (R bil)	MPV (R bil)	DP (R bil)	UP (R bil)
30 years	[368.70, 646.32, 132.55, 132.55]	507.51	[368.70, 646.32]	236.15	778.87
40 years	[265.64, 479.01, 101.32, 101.32]	372.325	[265.64, 479.01]	164.32	580.33
50 years	[197.45, 356.42, 75.47, 75.47]	276.94	[197.45, 356.42]	121.98	431.89

Table 4.6: *FROV using discount-rate (r) = 15% per year*

The Black-Scholes FROV values appear in Table 4.5, Table 4.6 and Table 4.7. The maximal value is obtained when the interest rate is set to be 10% and

Year	FROV (R bil)	EV (R bil)	MPV (R bil)	DP (R bil)	UP (R bil)
30 years	[349.42, 640.57, 137.84, 137.84]	494.995	[349.42, 640.57]	211.58	778.41
40 years	[265.77, 480.92, 102.10, 102.10]	373.345	[265.77, 480.92]	163.67	583.02
50 years	[199.45, 358.34, 75.50, 75.50]	278.90	[199.45, 358.34]	123.95	433.84

Table 4.7: *FROV using discount-rate (r) = 20% per year*

the year of exercising the option is 30 years. The expected value of FROV is minimal when the risk-free interest rate is 15% during the fortieth year. We deduce from Table 4.5, Table 4.6 and Table 4.7 that the expected value of FROV decreases as the number of years and risk-free interest rate increases. Thus, deferral of decommissioning is valuable because FROV decreases as the number of years increases. Expected values of FROV when r is 15% and r is 20% are similar especially during the fortieth and fiftieth years.

The two valuation methods used do not yield the same results and also behaves differently to the sensitivity analysis. However, we take note that the fuzzy real option value was computed using the trapezoidal fuzzy numbers whereas the triangular fuzzy numbers were used to compute fuzzy expanded net present value.

Chapter 5

Conclusion

We found that using the fuzzy triangular binomial approach, deferring the decommissioning time, increases the decommissioning costs, whereas use of fuzzy trapezoidal Black-Scholes formula yielded the opposite result. The traditional valuation methods are unable to capture the value of managerial flexibility or another alternatives that have an impact in estimating the value of a capital project. As a result, it is possible for a decision-maker who uses traditional valuation methods to reject or abandon the project that has a potential. However, valuation of real options take into account the managerial flexibilities and uncertainties that are embedded in the project and the entire value of an investment project can be revealed [31].

We only consider the annual SFA production as the source of uncertainty although multiple uncertainties may occur in a practical case and that is the limitation of our study. Results were based on estimates given lack of disclosure of information about relevant variables. It will be useful to obtain more accurate information about Koeberg's operations and then estimate values using the binomial, trinomial and Black-Scholes formula considering both triangular and trapezoidal fuzzy numbers. Furthermore, the Koeberg's decommissioning costs can be estimated taking the multiple uncertainties or risks into consideration. These are some of the projects that one can undertake based on my present thesis, in future.

```
import os
import sys
import re
import optparse

#Choose to fuzzy with 3
fuzzy_size = 3

#Print formatted output for visualization
fuzzy_binomial_tree_forward = open('fuzzy_binomial_tree_forward.dot', 'w')
fuzzy_binomial_tree_reverse = open('fuzzy_binomial_tree_reverse.dot', 'w')
fuzzy_binomial_tree_forward.write('graph{')
fuzzy_binomial_tree_reverse.write('graph{')

#Parse command line options
parser = optparse.OptionParser()
parser.add_option('-f', '--forward_tree_age',
    help='Specify the max number of years for the forward tree',
    dest='forward_tree_age',
    action='store',
    type='int', default=50)
parser.add_option('-r', '--reverse_tree_age',
    help='Specify the starting year to use from the forward tree in order to
generate the reverse tree',
    dest='reverse_tree_age',
    action='store',
    type='int', default=50)
parser.add_option('-c', '--clean', help='Remove files created by this script',
    dest='clean', action='store_true', default=False)
parser.add_option('-v', '--verbose', help='Print output in terminal',
    dest='verbose', action='store_true', default=False)

google_chart_warning = """Please note that Google Chart will fail if the tree
is too big (~ >11 years)...
 We recommend installing the GraphViz software, obtainable
 from http://www.graphviz.org/Download.php
"""
parser.add_option('-g', '--use-google',
    help='Use Google Chart to automatically generate the
plots.'+google_chart_warning,
    dest='use_google_chart', action='store_true', default=False)

(opts, args) = parser.parse_args()


def my_print(*args):
    if opts.verbose:
        print(args)

def graphviz_label(my_node):
    my_str = str(map(lambda x: round(x,5), my_node))
    return 'N'+re.sub("[^0-9]", '', my_str)
def graphviz_node(my_node):
    my_str = str(map(lambda x: round(x,5), my_node))
    return ('N'+re.sub("[^0-9]", '', my_str)
            +"[label=\"%s\"]"%(my_str))

# Use Google Chart API to plot a tree from its .dot structure contained in a
file
def getGraphviz(file_name):
    try:
```

```
        import httplib, urllib
    except ImportError:
        import http.client as httplib
        import urllib.parse as urllib

    params = urllib.urlencode(
        {'cht': 'gv:dot', 'chl': open(file_name).read()})
    headers = {"Content-type": "application/x-www-form-urlencoded",
               "Accept": "text/plain"}
    conn = httplib.HTTPConnection("chart.googleapis.com:80")
    conn.request("POST", "/chart", params, headers)
    response = conn.getresponse()
    print(response.status, response.reason)
    if response.status == 200:
        open(file_name+'.png', 'wb').write(response.read())
    conn.close()

def forward_tree(age):
    my_print('-----forward_tree-----')

    up = [1.19961494, 1.29693007, 1.402140503]
    down = [0.7131952881, 0.7710515856, 0.8336013403]
    root_price = [1.336, 1.336, 1.336]
    tree = [[]]

    for age_step in range(age):    #go through each age step in the tree. the
root_price is at age_step 0
        my_print('age_step', age_step)
        age_values = []
        for k in range(age_step+1):  #there is (age_step + 1) values at each
age step
            price = []
            for i in range(fuzzy_size):
                price.append(
                    root_price[i] * (down[i]**(age_step - k)) * (up[i]**k)
#the binomial formula
                )

            age_values.append(price)
            my_print('S', 'd', (age_step - k), 'u', k, '=', price, ' ')

            graphviz_node_price = graphviz_node(price)
            graphviz_label_price = graphviz_label(price)
            fuzzy_binomial_tree_forward.write(
                graphviz_node_price+';')
            if tree[-1]:
                if not k:
                    fuzzy_binomial_tree_forward.write(
                        graphviz_label(tree[-1][0])+'--
'+graphviz_label_price+';')
                elif not age_step - k:
                    fuzzy_binomial_tree_forward.write(
                        graphviz_label(tree[-1][-1])+'--
'+graphviz_label_price+';')
                else:
                    fuzzy_binomial_tree_forward.write(
                        graphviz_label(tree[-1][k-1])+'--
'+graphviz_label_price+';')
                    fuzzy_binomial_tree_forward.write(
                        graphviz_label(tree[-1][k])+'--
'+graphviz_label_price+';')
```

```
        tree.append(age_values)
        my_print('\n')
    return tree

def reverse_tree(age, age_values):
    my_print('-----reverse_tree-----')

    up = [0.3718, 0.5209, 0.6821]
    down = [0.3179, 0.4791, 0.6282]
    I = 7.680
    gamma = 0.15

    for age_step in range(age-2, -1, -1): #go through each age step in the tree,
 in reverse.
        my_print('age_step', age_step)
        tmp_age_values = []

        for k in range(age_step+1):  #there is (age_step + 1) values at each
age step
            price = []
            for i in range(fuzzy_size):
                #((0.371*max(0.211, 7.680-0.211))+(0.3179*max(0.126, 7.680-
0.126)))/1.045 = 4.94968
                price.append(
                    ((up[i]*max(age_values[k+1][i],I-age_values[k+1][i]))
                      + (down[i]*max(age_values[k][i],I-age_values[k][i])))
                    fuzzy_binomial_tree_forward.write(
                )

            tmp_age_values.append(price)
            my_print('S', 'd', (age_step - k), 'u', k, '=', price, ' ')

            graphviz_node_price = graphviz_node(price)
            graphviz_label_price = graphviz_label(price)
            fuzzy_binomial_tree_reverse.write(
                graphviz_node_price+';')
            fuzzy_binomial_tree_reverse.write(
                graphviz_label_price+'--'+graphviz_label(age_values[k])+';')
            fuzzy_binomial_tree_reverse.write(
                graphviz_label_price+'--'+graphviz_label(age_values[k+1])+';')

        age_values = tmp_age_values
        my_print('\n')

if __name__ == '__main__':
    if opts.clean:
        try:
            os.remove('fuzzy_binomial_tree_forward.dot')
            os.remove('fuzzy_binomial_tree_reverse.dot')
            os.remove('fuzzy_binomial_tree_forward.dot.png')
            os.remove('fuzzy_binomial_tree_reverse.dot.png')
            print('Extra files removed')
        except OSError:
            print('Not all extra files were found')
        exit(0)

    if opts.reverse_tree_age > opts.forward_tree_age:
        my_print('Input error: reverse tree age should be less than the forward
tree age')
    else:
        tree = forward_tree(opts.forward_tree_age+1) #+1 to account for root at
age 0
```

```
        #For Graphviz, write the output of the choosen year in order to start
reversing the tree
        for node in tree[opts.reverse_tree_age+1]:
            fuzzy_binomial_tree_reverse.write(
                graphviz_node(node)+';')
        reverse_tree(opts.reverse_tree_age+1, tree[opts.reverse_tree_age+1])
#+1 to account for root at age 0

    fuzzy_binomial_tree_forward.write('}')
    fuzzy_binomial_tree_reverse.write('}')
    fuzzy_binomial_tree_forward.close()
    fuzzy_binomial_tree_reverse.close()

    #Generate the plots through Google Chart
    if opts.use_google_chart:
        print(google_chart_warning)
        if opts.forward_tree_age < 15:
            getGraphviz('fuzzy_binomial_tree_forward.dot')
            getGraphviz('fuzzy_binomial_tree_reverse.dot')
        else:
            print('Trees too large for Google chart to handle.')
```

Bibliography

[1] Alavala C. R., *Fuzzy logic and neural networks basic concepts and applications*, New Age International Publishers, New Delhi, (2008).

[2] Bansal A., *Trapezoidal fuzzy numbers (a, b, c, d): arithmetic behavior*, International Journal of Physical and Mathematical Sciences, **2** (1) (2011) 39 - 44.

[3] Bedzek J. C., *Fuzzy models - what are they, and why?*, IEEE Transactions on Fuzzy Systems, **1** (1) (1993) 1 - 6.

[4] Beyleveld J. F., *Cost estimate for the disposal of low level waste at Vaalputs*, VLP-REP-10/084, Rev. 0, South African Nuclear Energy Corporation (NECSA), (2010).

[5] Black F. and Scholes M., *"The pricing of options and corporate liabilities"*, Journal of Political Economy, **81** (1973) 637 - 654.

[6] Bodén B. and Åhlén A., *Real options analysis: a study of implementation impediments*, Bachelor's Thesis, Göteborg University, (2007).

[7] Bojadziev G. and Bojadziev M., *Fuzzy logic for business, finance, and management*, 2nd edition, Advances in Fuzzy Systems - Applications and Theory, World Scientific, Singapore, **23** (2007).

[8] Boyle P., *"Option valuation using a three-jump process"*, International Options Journal, **3** (1986) 7 - 12.

[9] Boyle P., *"Options: a Monte Carlo approach"*, Journal of Financial Economics, **4**(3) (1977) 323 - 338.

[10] Brennan M. and Schwartz E., *"The valuation of American put options"*, Journal of Finance, **32** (1977) 449 -462.

[11] Bunch D. S. and Johnson H., *"A simple and numerically efficient valuation method for American puts using a modified Geske-Johnson approach"*, Journal of Finance, **47** (1992) 809 -816.

[12] Carlsson C. and Fuller R., *A fuzzy approach to real option valuation*, Fuzzy Sets and Systems, **139** (2003) 297-312.

[13] Chakraborty R. C., *Fuzzy set theory: soft computing course*, Lecture Notes 29 - 34, (2010).

[14] Chance D. M., *Corporate finance and portfolio management in: CFA Institute Level I Program Curriculum*, CFA Institute and Pearson: Boston, **4** (2007).

[15] Clarke R. G., *Options and futures: a tutorial*, Blackwell Publishers, Virginia, (2000).

[16] Collan M., *A new method for real option valuation using fuzzy numbers*, IAMSR Research Report 7 (2008).

[17] Collan M., Carlsson C. and Majlender P., *Fuzzy Black and Scholes real option pricing*, Journal of Decision Systems, **12** (2003) 391 - 416.

[18] Cox J., Ross S. and Rubinstein M., *"Option pricing: a simplified approach"*, Journal of Financial Economics, **7** (1979) 229 - 263.

[19] Crundwell F. K., *Finance for engineers: evaluation and funding of capital projects*, Springer, London, (2008).

[20] Damodaran A., *Real Options*, Lecture Notes - Chapter 8, (2010).

[21] Dubois D. and Prade H., *Fuzzy Sets and Systems: Theory and Applications*, ¡athematics in Science and Engineering, Academic Press, New York, **144**.

[22] Eskom, *"Spent nuclear fuel management technical plan: reference technical plan for the long term management of Eskom's spent nuclear fuel"*, Eskom Document, Ref: NFD 990-0870 Rev 1, (2005).

[23] Eskom, *Koeberg nuclear power station*, http://www.eskom.co.za/c/74/koeberg-nuclear-power-station/, (last accessed on Nov 2012), (2012).

[24] Gao L. S., *The fuzzy arithmetic mean*, Fuzzy Sets and Systems, **107** (1999) 335 - 348.

[25] Geske R. and Johnson H. E., *"The American put valued analytically"*, Journal of Finance, **39** (1984) 1511 - 1524.

[26] Gilbert E., *Investment basics XLIX: an introduction to real options*, Investment Analysts Journal, **60** (2004) 49 - 52.

[27] Higham D. J., *An introduction to financial option valuation*, Mathematics, Stochastics and Computation, Cambridge University Press, New York, (2004).

[28] Huang J., Subrahmanyam M. and Yu G., *"Pricing and hedging American options: a recursive integration method"* Review of Financial Studies, **9** (1996) 277 - 300.

[29] Kamrad B. and Ritchken P., *Multinomial approximating models for options with k state variables*, Management Science, **37** (12) 1991 1640 - 1652.

[30] Kodukula P. and Papudesu C., *Project valuation using real options: a practitioner's guide*, J. Ross Publishing, Fort Lauderdale, (2006).

[31] Liao S. and Ho S., *Investment project valuation based on a fuzzy binomial approach*, Information Sciences, **180** (2010) 2124 - 2133.

[32] Longstaff F. A. and Schwartz E. S., *"Valuing American options by simulation: a simple least squares approach"*, Review of Financial Studies, **14**: 113

[33] Luerman T. A., *Investment opportunities as real options getting started on the numbers*, Harvard Business Review, Reprint 98404, Boston, (1998).

[34] Lumby S., *Investment appraisal & financial decisions*, 3rd edition, Van Nostrand Reinhold International Co. Ltd, Wokingham, (1988).

[35] Marx J., Mpofu R. T., De Beer J. S., Nortje A. and Van de Venter T. W. G., *Investment management*, 3rd edition, Van Schaik Publishers, Pretoria, (2010).

[36] MATLAB & Simulink, Foundation of Fuzzy Logic, http://www.mathworks.com/help/fuzzy/foundations-of-fuzzy-logic.html (last accessed on Nov 2012).

[37] Merton R. C., *Theory of rational option pricing*, The Bell Journal of Economics and Management Science, **4** (1) (1973) 141 - 183.

[38] Mezei J., *A quantitative view on fuzzy numbers*, Turku Centre for Computer Science, TUCS Dissertations, (2011).

[39] Morgan J., *Nuclear waste disposal and plant decommissioning costs*, http://nuclearfissionary.com/2010/03/22/nuclear-waste-disposal-and-plant-decommissioning-costs/, (last accessed on Nov 2012), (2010).

[40] Murali V., *A study of universal algebras in fuzzy set theory*, PhD Thesis, Rhodes University, (1987).

[41] Nguyen H. T and Walker E. A., *A first course in fuzzy logic*, 2nd edition, Chapman and Hall, Boca Raton, (2000).

[42] Nikookar G., Azarmsa S. A., Pooramazan F. and Shahvazian A., *A fuzzy real option valuation approach to capital budgeting under uncertainty*,

European Journal of Economics, Finance and Administrative Sciences, **45** (2012) 86 - 96.

[43] Ohama D., *Using design flexibility and real options to reduce risk in private finance initiatives: the case of Japan*, Master thesis, Massachusetts Institute of Technology, (2008).

[44] Parkinson M., *"Option pricing: the American put"*, Journal of Business, **50** (1977) 21 - 36.

[45] Ross S. M., *An elementary introduction to mathematical finance* , 3rd edition, Cambridge University Press, Cambridge, (2011).

[46] Schwartz S. and Trigeorgis L., *Real options and investment under uncertainty: classical readings and recent contributions*, MIT Press, Cambridge (2001).

[47] Schulmerich M., *Real options valuation: the importance of interest rate modelling in theory and practice*, 2nd edition, Springer, Berlin, (2010).

[48] Söderkvist J. and Jönsson K., *A real options approach to nuclear waste disposal in Sweden*, Handelshögskolan vid Göteborgs Universitet, (2004).

[49] Statistics South Africa, *Gross domestic product: fourth quarter (Report)*, (2011).

[50] Triantis A. J., *Real options in handbook of modern finance* ed. D.Logue and J. Seward, Research Institute of America, New York, D1 - D32 (2003).

[51] Twala V. G., *Evaluation of spent nuclear fuel management options for South Africa*, PhD Thesis, University of Johannesburg, (2007).

[52] Vernimmen P., Quiry P., Dallocchio M., Le Fur Y. and Salvi A., *Corporate finance theory and practice*, Wiley, Chichester, (2011).

[53] Wang T., *Real options "in" projects and systems design - identification of options and solution for path dependency*, PhD Thesis, Massachusetts Institute of Technology, (2005).

[54] Wilmott P., *Paul Wilmott introduces quantitative finance*, 2nd edition, John Wiley and Sons Inc, Chichester, (2007).

[55] Zadeh L. A., *Fuzzy sets*, Information and Control, **8** (1965) 338 - 353 .

[56] Zimmermann H. J., *Fuzzy set theory*, Wiley Interdisciplinary Reviews: Computational Statistics, **12**(3) (2010) 317 - 332.