TR92-37

RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS

Aspects of fuzzy spaces with special reference to cardinality, dimension, and order–homomorphisms

by

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A thesis submitted in fulfilment of the requirements for the Degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS

DECEMBER 1991

ABSTRACT

Aspects of fuzzy vector spaces and fuzzy groups are investigated, including linear independence, basis, dimension, group order, finitely generated groups and cyclic groups. It was necessary to consider cardinality of fuzzy sets and related issues, which included a question of ways in which to define functions between fuzzy sets.

Among the results proved, are the additivity property of dimension for fuzzy vector spaces, Lagrange's Theorem for fuzzy groups (the existing version of this theorem does not take fuzziness into account at all), a compactness property of finitely generated fuzzy groups and an extension of an earlier result on the order-homomorphisms.

An open question is posed with regard to the existence of a basis for an arbitrary fuzzy vector space.

AMS Classification : 03E72, 20K05, 20K99, 54A40, 15A03, 06B05, 04A05.

PREFACE

In 1965 L. A. Zadeh introduced the concept of a fuzzy set in his paper [49]. Since set theory is the basis for all mathematical constructions, it would seem at first that much of mathematics can be generalised by the use of fuzzy sets. Consequently a large number of branches of mathematics were and are being "fuzzified". Fuzzy set theory has encountered a kind of explosion with regard to the number of different definitions that extend particular "crisp" concepts to fuzzy ones. The main reason for this is that for any one particular crisp concept there are many different definitions extending it to the fuzzy situation. In the "fuzzy" literature there is also a number of inadequate definitions and results. To a large extent this work addresses some of these inadequacies, for instance the basis and the dimension of a fuzzy vector space and the order of a fuzzy group. In addition we consider and extend some of the results obtained by others. This includes the results on order–homomorphisms

Approximately half of this thesis has been written up for publication in [27], [28] and [29]. Our membership grade lattice is totally ordered with the exception of the section dealing with order–homomorphisms, where it is assumed to be completely distributive.

The seemingly diverse topics discussed in the thesis are unified on various levels e.g.

i) **Methodological unity:** General formulae are applied in formulating concepts and definitions. For example the same idea is used in defining compactness in fuzzy topological spaces as is used in compactness of fuzzy groups. In fact it is very instructive to apply the same concept to different fields and to find that they work correctly in both. This, in fact, is a further evidence their validity.

ii) **Relational unity**: Definitions and results in one chapter are necessary for others. Clearly, we need a notion of cardinality to define the dimension of vector space and order of a group. Functions and morphisms are fundamental. Chapter 2 deals with some aspects of these.

An exposition of each of the topics such as fuzzy cardinality, fuzzy groups and fuzzy vector spaces had to be sufficiently complete, and necessitated separate chapters. Naturally, we have also developed some of these topics further for their own sake.

In Chapter 1, some basic definitions and results are recalled and established which are required later. This chapter also fixes the notation to be used throughout this thesis. In Chapter 2 we consider order–homomorphisms (more particularly Erceg's functions between fuzzy sets introduced in [10]), then generalise and simplify the result of G.–J. Wang [41] on the order–homomorphisms fuzzy point invariance under certain conditions.

Chapter 3 deals with basic cardinality theory for fuzzy sets. We include a discussion on extending functions defined on crisp sets to ones on fuzzy sets, the relation between compactness and finiteness in fuzzy sets, and show how some cardinals for fuzzy sets can be obtained. In particular we use the definition of cardinality for fuzzy sets introduced by Blanchard [2] in the non–finite (fuzzy) case and display its various properties. Finally we give a new definition of cardinality, based on an observation that Blanchard's cardinals are like decreasing functions from the unit interval to a lattice as in Hutton's fuzzy unit interval [16]. We prove the additivity property for these cardinals and demonstrate their relation to Gottwald's and Blanchard's cardinals.

In Chapter 4 we consider arbitrary fuzzy vector spaces from an algebraic point of view. The definition of a fuzzy vector space was given by Katsaras and Liu in [18]. We generalise linear independence, basis and dimension from crisp vector spaces to fuzzy vector spaces. The most important results presented here concern the existence of a basis for a fuzzy vector space under some condition on the lattice and the additivity property of dimension. There remain a number of unanswered questions. For instance, do all fuzzy vector spaces possess a basis?
Chapter 5 deals with fuzzy groups. The concept of a fuzzy group was introduced by Rosenfeld in [38]. Since then various authors have given definition of various generalisations of concepts from group theory to fuzzy group theory. Some of these definitions and subsequent results are in our opinion inadequate. We put forward definitions of order, finitely generated fuzzy groups and cyclic fuzzy groups.
The definition of order is a fuzzy cardinal rather than a crisp cardinal as given in [32]. The resultant Lagrange's Theorem for fuzzy groups is therefore a proper extension of this theorem from crisp groups. (The existing Lagrange's Theorem for fuzzy groups states that if μ is a subgroup of a fuzzy group ν and $\mu(e) = \nu(e)$ then the order of μ divides order of ν. This is simply the crisp Lagrange's Theorem.).
We use a notion of compactness from fuzzy topology, namely that of Chadwick [3] and show that our definition of finitely generated fuzzy groups is equivalent to a fuzzy group being compact in the lattice of fuzzy groups. Finally we give a representation of cyclic fuzzy groups and prove that two cyclic fuzzy groups with the same order are strongly isomorphic, i.e. there exists a isomorphism $f : \text{supp}(\mu) \rightarrow \text{supp}(\nu)$ such that $f(\mu_{|\,\text{supp}(\mu)}) = \nu_{|\,\text{supp}(\nu)}$.

CONTENTS

5. FUZZY GROUPS

ACKNOWLEDGEMENTS

I would like to thank Prof. W. Kotzé, my supervisor, for his support, encouragement, for enabling me to undertake this study in a way that was probably most suitable for me and helping with the editing of my papers and this thesis.

Dr. T.Kubiak was involved in checking the drafts of this thesis at various stages. He has given me a lot of valuable pointers without which this thesis would be difficult to read. For this and more I must thank Dr. T. Kubiak.

Through useful conversation Dr. V. Murali and Dr. J. J. Chadwick have contributed to this thesis.

Financial assistance from the F.R.D. is greatly appreciated.

Finally, my parents must receive acknowledgement for their support on all fronts.

1. PRELIMINARY DEFINITIONS AND RESULTS

Denote by I the unit interval $[0,1] \subset \mathbb{R}$.

1.1. FUZZY SETS

Definition 1.1.1. (L.A.Zadeh [49])

A Fuzzy set μ is a function from some set X to I, i.e. $\mu : X \to I$.

We can find a short introduction to the theory of fuzzy sets in [49].

Definition 1.1.2. (iii,vi,vii L.A.Zadeh [49] , ix N.Blanchard [2])

i) Denote by I^X the set of all fuzzy sets on set X.

ii) If $A \subset X$ and $\alpha \in (0,1]$ then define fuzzy set $\alpha 1_A$ by $\alpha 1_A(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$.
In case $A = \{x\}$ is a singleton we will write $\alpha 1_x$ for $\alpha 1_{\{x\}}$.

iii) If $\mu, \nu \in I^X$ then ν is a subset of μ denoted $\nu \leq \mu$
if and only if for all $x \in X$, $\nu(x) \leq \mu(x)$.

iv) If $\mu \in I^X$ then we denote by $\mathcal{P}(\mu) = \{ \nu \in I^X : \nu \leq \mu \}$ the power set of μ.
Define $\quad \bar{\mathcal{P}}(\mu) = \{ \nu \in \mathcal{P}(\mu) : \forall x \in X , \nu(x) = \mu(x) \text{ or } \nu(x) = 0 \}$
and $\quad \underline{\mathcal{P}}(\mu) = \{ \nu \in \mathcal{P}(\mu) : \forall x \in X , \nu(x) < \mu(x) \text{ or } \nu(x) = 0 \}$.

v) If $\mu \in I^X$ then we denote by $p(\mu) = \{ \alpha 1_x : \alpha > 0 , x \in X \text{ and } \alpha 1_x \leq \mu \}$
the set of all fuzzy points in μ.
Define $\bar{p}(\mu) = p(\mu) \cap \bar{\mathcal{P}}(\mu)$ and $\underline{p}(\mu) = p(\mu) \cap \underline{\mathcal{P}}(\mu)$.
Note that $\bar{p}(\mu) = \{ \mu(x)1_x : \mu(x) > 0, x \in X \}$
and $\underline{p}(\mu) = \{ \alpha 1_x : \mu(x) > \alpha > 0, x \in X \}$.

vi) If $\{\mu_i\}_{i \in J}$ is a collection of subsets from I^X then the union $\bigvee_{i \in J} \mu_i \in I^X$ and the intersection $\bigwedge_{i\in J} \mu_i \in I^X$ of this collection is defined by:

$(\bigvee_{i\in J} \mu_i)(x) = \sup_{i\in J} \mu_i(x)$ and $(\bigwedge_{i\in J} \mu_i)(x) = \inf_{i\in J} \mu_i(x)$ respectively.

vii) If $\mu \in I^X$ then the complement of μ, μ' is defined by $\mu'(x) = 1 - \mu(x)$.

viii) If $\mathcal{C} = \{ \mu_i : X_i \to I : i \in J \}$ then the product $(\Pi\, \mathcal{C}) : \prod_{i\in J} X_i \to I$ is defined by $(\Pi\, \mathcal{C})(f) = \inf \{ \mu_i(f(i)) : i \in J \}$ for $f \in X^J$.

ix) If $\mu : X \to I$ and $\nu : Y \to I$ then $\mu^\nu : X^Y \to I$ is defined by $\mu^\nu(f) = \inf \{ \mu(f(y)) : \nu(y) > \mu(f(y)),\, y \in Y \}$ for $f \in X^Y$.

x) Two fuzzy sets μ, $\nu \in I^X$ are disjoint iff $\mu \wedge \nu = \phi = 1_\phi = 01_X$.

xi) If $\mu \in I^X$ then the range R_μ of μ is defined by $R_\mu = \{ \mu(x) \in I\setminus\{0\} : x \in X \}$.

Notes:

a) $\underline{p}(\mu)$ is the set of all fuzzy points belonging to μ in the sense of [40].

b) $p(\mu)$ is the set of all fuzzy points belonging to μ in C.K.Wong [46] sense ,

We write 1_A instead of 11_A and reserve the letters p and q to denote fuzzy points, i.e. elements from $p(1_X)$. The Greek letters μ , ν , ω , σ , ξ , ρ , λ will denote fuzzy sets, i.e. elements from I^X. Finally α , β , γ , δ will be from I.

J.A. Goguen [13] has introduced the L–fuzzy sets. The L–fuzzy sets have membership values in a lattice L . For our needs we consider only totally ordered complete lattices with an order reversing involution ′, with the exception of Chapter 2. Denote by 0 and 1 the bottom and the top of L. The definition 1.1.2. is exactly the same for the L–fuzzy sets with the exception of $\underline{\mathcal{P}}$ and $\underline{p}$. In our notation we also indicate which lattice is used. If $\alpha \in L\setminus\{0\}$ then α is called an isolated element iff $\bigvee \{ \beta \in L : \beta < \alpha \} \neq \alpha$. Denote by L_- the set of all the isolated elements in L.

For example : if $L = \{0,1\}$ then $L_- = \{1\}$ and if $L = [0,1]$ then $L_- = \phi$.

Definition 1.1.3.

Let L be a lattice as above (totally ordered and complete) and $\mu \in L^X$.

i) $\mathcal{P}(\mu,L) = \{ \nu \in L^X : \nu \leq \mu \}$

ii) $\bar{\mathcal{P}}(\mu,L) = \{ \nu \in L^X : \forall x \in X , \nu(x) = \mu(x) \text{ or } \nu(x) = 0 \}$

iii) $\underline{\mathcal{P}}(\mu,L) = \{ \nu \in L^X : \nu \leq \mu \text{ and if } \mu(x) \notin L_- \text{ then } \nu(x) < \mu(x) \text{ or } \nu(x) = 0 \}$

iv) $p(\mu,L) = \{ \alpha 1_x : \alpha 1_x \leq \mu, \alpha \in L\backslash\{0\} \text{ and } x \in X \}$

v) $\bar{p}(\mu,L) = \bar{\mathcal{P}}(\mu,L) \cap p(\mu,L)$.

vi) $\underline{p}(\mu,L) = \underline{\mathcal{P}}(\mu,L) \cap p(\mu,L)$.

In what follows we denote $p(\mu,[0,1])$ by $p(\mu)$ (e.t.c.) .

Definition 1.1.4. (L.A.Zadeh [49])

If $\mu : X \to L$ is a fuzzy set and $\alpha \in L$ then we denote by E^α_μ and T^α_μ the subsets $\mu^{-1}([\alpha,1])$ and $\mu^{-1}(\alpha)$ of X respectively.

Also we define $H^\alpha_\mu = \begin{cases} \mu^{-1}([\alpha,1]) & \text{if } \alpha \in L_- \\ \mu^{-1}((\alpha,1]) & \text{otherwise} \end{cases}$

E^α_μ and H^α_μ are called weak and strong α – cuts respectively.

If $\mathcal{C} \subset L^X$ then we extend the above notation to $\mathcal{C}$ as follows:

$E^\alpha_\mathcal{C} = \{ E^\alpha_\nu : \nu \in \mathcal{C} \}$ and similarly for $H^\alpha_\mathcal{C}$ and $T^\alpha_\mathcal{C}$.

We now show that $H^\alpha_{(\cdot)} : L^X \to \mathcal{P}(X,\{0,1\})$ and $E^\alpha_{(\cdot)} : L^X \to \mathcal{P}(X,\{0,1\})$ preserve some important basic set operations. Note that $\mathcal{P}(X,\{0,1\}) = 2^X$ and that we can identify $p(X,\{0,1\})$ with X. That means that if $x \in X$ then the p in the definition of $p(\mu,L)$ in 1.1.3 (iv) becomes $p = 1_x$ which is identified with the crisp point x. Similar identification can be done with sets.

Proposition 1.1.5.

Let $\mathcal{C} \subset L^X$, μ , $\nu \in L^X$ and $\alpha \in L$ then

i) $H^{\alpha}_{\vee\mathcal{C}} = \cup \{ H^{\alpha}_{\mu} : \mu \in \mathcal{C} \}$ and $E^{\alpha}_{\wedge\mathcal{C}} = \cap \{ E^{\alpha}_{\mu} : \mu \in \mathcal{C} \}$.

ii) $H^{\alpha}_{\mu \wedge \nu} = H^{\alpha}_{\mu} \cap H^{\alpha}_{\nu}$ and $E^{\alpha}_{\mu \vee \nu} = E^{\alpha}_{\mu} \cup E^{\alpha}_{\nu}$.

Proof

i) $$x \in H^{\alpha}_{\vee\mathcal{C}} \Leftrightarrow \begin{Bmatrix} (\vee\mathcal{C})(x) \geq \alpha & \text{if } \alpha \in L_- \\ (\vee\mathcal{C})(x) > \alpha & \text{otherwise} \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} \exists\, \mu \in \mathcal{C} \text{ such that } \mu(x) \geq \alpha \text{ since } \alpha \text{ is isolated} \\ \exists\, \mu \in \mathcal{C} \text{ such that } \mu(x) > \alpha \end{Bmatrix}$$

$$\Leftrightarrow x \in \cup \{ H^{\alpha}_{\mu} : \mu \in \mathcal{C} \}$$

The other identity is well known.

$$x \in H^{\alpha}_{\mu \wedge \nu} \Leftrightarrow \begin{Bmatrix} (\mu \wedge \nu)(x) \geq \alpha & \text{if } \alpha \in L_- \\ (\mu \wedge \nu)(x) > \alpha & \text{otherwise} \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} \mu(x) \geq \alpha \text{ and } \nu(x) \geq \alpha & \text{if } \alpha \in L_- \\ \nu(x) > \alpha \text{ and } \mu(x) > \alpha & \text{otherwise} \end{Bmatrix}$$

$$\Leftrightarrow x \in H^{\alpha}_{\mu \wedge \nu}$$

The other identity is also well known .

Proposition 1.1.6.

Let $\mathcal{C} = \{ \mu_i : X_i \to L : i \in J \}$ and $\alpha \in L$ then $E^{\alpha}_{\Pi\mathcal{C}} = \Pi \{ E^{\alpha}_{\mu_i} : i \in J \}$.

Also $H^{\alpha}_{\mu \times \nu} = H^{\alpha}_{\mu} \times H^{\alpha}_{\nu}$.

Proof

$f \in E^{\alpha}_{\Pi\mathcal{C}} \Leftrightarrow (\Pi\mathcal{C})(f) \geq \alpha \Leftrightarrow \inf \{ \mu_i(f(i)) : i \in J \} \geq \alpha$

$\Leftrightarrow \forall\, i \in J ,\ \mu_i(f(i)) \geq \alpha \Leftrightarrow \forall\, i \in J ,\ f(i) \in E^{\alpha}_{\mu_i} \Leftrightarrow f \in \Pi \{ E^{\alpha}_{\mu_i} : i \in J \}$

The other identity follows similarly .

Proposition 1.1.7.

i) If $\mu \in L^X$ then $\mu = \vee\, p(\mu,L) = \vee\, \underline{p}(\mu,L) = \vee\, \bar{p}(\mu,L) = \vee\, \bar{\mathcal{P}}(\mu,L) = \vee\, \underline{\mathcal{P}}(\mu,L)$.

ii) If $\mu \in L^X$ then $\mu = \vee \{ \alpha 1_{H^{\alpha}_{\mu}} : \alpha \in L\backslash\{0\} \} = \vee \{ \alpha 1_{E^{\alpha}_{\mu}} : \alpha \in L\backslash\{0\} \}$.

iii) If $\mathcal{C} \subset L^X$ then $\underline{p}(\vee \mathcal{C} , L) = \cup \{ \underline{p}(\omega,L) : \omega \in \mathcal{C} \}$

If $\mu , \nu \in \mathcal{P}(\mu,L)$ then $\underline{p}(\mu \wedge \nu , L) = \underline{p}(\mu,L) \cap \underline{p}(\nu,L)$.

Proof

i) The only non trivial result is the one involving $\underline{p}$ and $\underline{\mathcal{P}}$.

Let $x \in X$. If $\mu(x) \in L_-$ then $\mu(x)1_x \in \underline{p}(\mu,L)$ so $[\vee \underline{p}(\mu,L)](x) = \mu(x)$.

If $\mu(x) \notin L_-$ then $\forall\ \alpha < \mu(x)$, $\alpha 1_x \in \underline{p}(\mu,L)$. Since $\mu(x) \notin L_-$ we must have $\vee \{ \alpha \in L : \alpha < \mu(x) \} = \mu(x)$. Thus $[\vee \underline{p}(\mu,L)](x) = \mu(x)$.

It is important to note that the modified definition of $\underline{p}$ is necessary .

If we had $\underline{p}(\mu,L) = \{ \alpha 1_x : \alpha < \mu(x) , x \in X \}$ and $\mu(x)$ was isolated then $[\vee \underline{p}(\mu,L)](x) < \mu(x)$. We proceed similarly for $\underline{\mathcal{P}}$.

ii) The result for E^{α}_{μ} is well known.

The case $\mu(x) = 0$ is obvious, so let $x \in X$ and $\mu(x) > 0$.

If $\mu(x) \in L_-$ then $x \in H^{\mu(x)}_{\mu}$ and for all $\alpha > \mu(x)$, $x \notin H^{\alpha}_{\mu(x)}$

thus $[\vee \{ \alpha 1_{H^{\alpha}_{\mu}} : \alpha \in L\backslash\{0\} \}](x) = \mu(x)$.

If $\mu(x) \notin L_-$ then $x \notin H_\mu^{\mu(x)}$ and for all $\alpha < \mu(x)$, $x \in H_\mu^\alpha$. But since $\mu(x) \notin L_-$ we have $\vee \{ \alpha \in L : \alpha < \mu(x) \} = \mu(x)$. This proves the result. Note that the modification to the definition of $H_{(\cdot)}^\alpha$ was necessary for this result to hold.

iii) Here we proceed similarly as in i) .

We must point–out that Rodabaugh has proved 1.1.7.(ii) in [37], however our argument is much simpler. The symbols $\underline{p}(\cdot,\cdot)$ was defined as in Definition 1.1.3 (iv) in order for the result 1.1.7. (i) to hold.

Remark: We shall use the following convention:
if $\mu \in L^X$ and $\alpha \in L$ then $\mu \geq \alpha$ if and only if $\mu(x) \geq \alpha$ for all $x \in X$.
Similarly we define $\mu > \alpha$.

Definition 1.1.8. (L.A.Zadeh [49])
If $\mu : X \to L$ is a fuzzy set then we define:

i) $h(\mu) = \sup\{ \mu(x) : x \in X \}$,

ii) $\mathrm{supp}(\mu) = \{ x \in X : \mu(x) > 0 \}$.

Symbols $h(\mu)$ and $\mathrm{supp}(\mu)$ are called the height and the support of the fuzzy set μ respectively.
If $\Phi \subset L^X$ then we extend i) and ii) to Φ as follows:

i′) $h(\Phi) = \inf\{ h(\mu) : \mu \in \Phi \}$,

ii′) $\mathrm{supp}(\Phi) = \cap \{ \mathrm{supp}(\mu) : \mu \in \Phi \}$.

Note that $\mathrm{supp}(\mu) = H_\mu^0$.

We, now, introduce an idea which is going to become very important later in this work.

Definition 1.1.9.

A subset B, of a partially ordered set is said to be ***upper well ordered*** if for all non empty subsets $C \subset B$, $\sup C \in C$.

Lower well orderedness is defined dually.

The following proposition is going to be used in Chapter 3, when dealing with fuzzy cardinals.

Proposition 1.1.10

Let X and Y be partially ordered sets and $f : X \to Y$ be a decreasing function.

($f : X \to Y$ is decreasing if for all $a,b \in X$ with $a \leq b$, $f(a) \geq f(b)$.)

Then the image $f(B)$ of a lower well ordered set $B \subset X$ is upper well ordered.

Proof

Let $C \subset f(B)$, $C \neq \emptyset$ and $D = B \cap f^{-1}(C)$. Now, $f(D) = f(\ B \cap f^{-1}(C)\ ) =$ $f(B) \cap C = C$. Since B is lower well ordered and $D \subset B$, $x = \inf D \in D$.

Now it follows from the fact that $f(D) = C$ that $f(x) \in C$. So, since $f(x) \in C = f(D)$, $x = \inf D \in D$ and f is decreasing, it follows that $f(x) = \sup f(D)$. Thus $\sup C \in C$.

In particular we are interested in upper well ordered subsets of the unit interval.

Proposition 1.1.11. [27]

A set $B \subset [0,1]$ is upper well ordered if and only if B does not contain any increasing sequence.

The following result is an exercise 16B in [45].

Proposition 1.1.12.

All upper well ordered subsets of [0,1] are countable.

Proof

Suppose $B \subset [0,1]$ is an uncountable upper well ordered set.

Given $a \in B$, let $p(a) = \sup\{ c \in B : c < a \}$.

Since B is upper well ordered, $p(a) < a$.

Clearly if $a \neq b$, then $(p(a), a) \cap (p(b), b) = \emptyset$.

Indeed, if $a < b$, then $a \leq p(b)$.

We thus have an uncountable family of pairwise disjoint open intervals of [0,1]. This is impossible since [0,1] is second countable.

Clearly we can construct $B \subset [0,1]$ upper well ordered with an infinite number of decreasing limit points. For example consider:

$$B = \{ \frac{1}{n} + \frac{1}{m} : n, m \in \{2,3,4,\dots\}\}.$$

1.2. FUZZY TOPOLOGICAL SPACE

Definition 1.2.1. (C.L.Chang [4] , J. A. Goguen [12])

$\mathcal{T} \subset L^X$ is a L–fuzzy topology on X iff $\mathcal{T}$ satisfies the following:

i) $1_X , \phi \in \mathcal{T}$.

ii) If $\mathcal{C}$ is arbitrary subcollection from $\mathcal{T}$ then $\vee \mathcal{C} \in \mathcal{T}$.

iii) If $\mathcal{C}$ is finite subcollection from $\mathcal{T}$ then $\wedge \mathcal{C} \in \mathcal{T}$.

We call the triple (X , $\mathcal{T}$, L) a L–fuzzy topological space .

Note that Lowen's [24] definition of a fuzzy topology requires in addition to above, all the constant fuzzy sets to be open. Lowen [24] works only with case where L = I.

Definition 1.2.2. (C.L.Chang [4])

Let (X , $\mathcal{T}$, L) be a L–fuzzy topological space and $\nu \in L^X$ then define

i) The closure $cl(\nu) = \wedge \{ \sigma \in \mathcal{T}' : \sigma \geq \nu \}$ where $\mathcal{T}' = \{ \sigma : \sigma' \in \mathcal{T} \}$

ii) The interior $int(\nu) = \vee \{ \omega \in \mathcal{T} : \omega \leq \nu \}$

It can be shown that $[int(\nu)]' = cl(\nu')$ and $[cl(\nu)]' = int(\nu')$ see Pu–Liu [36] .

In [3] Chadwick has given a characterisation of compactness in terms of open coverings for the case where L = [0,1]. The definition below corresponds to Chadwick's definition in case L = [0,1]. However, for our needs we extend that definition to non–[0,1] case.

Definition 1.2.3. (c.f. Chadwick [3])

Given $(X , \mathcal{T} , L)$ and $\mu \in L^X$ then μ is f–compact if and only if

$\forall\ \alpha \in L$ and $\mathcal{C} \subset \mathcal{T}$ such that $\mu' \vee (\vee \mathcal{C}) \geq \alpha$ we have

$\alpha \notin L_-$ $\Longrightarrow \forall\ \beta \in L, \beta < \alpha\ \exists$ finite $\mathcal{D} \subset \mathcal{C}$ such that $\mu' \vee (\vee \mathcal{D}) \geq \beta$.

$\alpha \in L_-$ $\Longrightarrow \exists$ finite $\mathcal{D} \subset \mathcal{C}$ such that $\mu' \vee (\vee \mathcal{D}) \geq \alpha$

2. ORDER–HOMOMORPHISMS

In [49] Zadeh defines an image and a preimage of a fuzzy set under a crisp function as follows:

Definition 2.1 (Zadeh [49])
If $f : X \to Y$ and $\mu \in L^X$ then $f(\mu)(y) = \sup \{ \mu(x) : x \in f^{-1}(y) \}$
If $\nu \in L^Y$ then $f^{-1}(\nu)(x) = \nu(f(x))$.

We can find a rather complete investigation of the properties of these functions in [48]. In the literature there are various other proposals for functions between fuzzy sets. Some of them can be found in [35], [34] and [15]. First of all we are going to study functions between fuzzy sets as defined by Erceg [10]. This study is not going to be conducted for the case where L is totally ordered but for a more general lattice L. We are going to use a more general formulation of Erceg's functions, namely order-homomorphisms. We shall use the later name. Part of the results presented here can be found in the author's [28]. In essence a general result is established which leads to an extension and simplification of an earlier theorem by G.–J. Wang [41]. This result is required to draw conclusions about various other possible definitions of functions between fuzzy sets.
Let $L = (L, \wedge, \vee, ')$ be a completely distributive complete lattice with an order reversing involution $'$, containing at least two elements. The bottom and the top elements of a lattice will be denoted by 0 and 1 respectively. The symbol $\|$ will be used to denote non-comparability between elements in any lattice.
In what follows we assume that all lattices denoted by L, L_1 and L_2 are of the type defined above and where a sub–lattice is one which uses the same suprema and infima as in the lattice itself.

Definition 2.2

A mapping $F : L_1 \to L_2$ is called an order–homomorphism [10,41,21] if F satisfies the following:

(H1) $F(0) = 0$,

(H2) $F(\vee \mu_i) = \vee F(\mu_i)$,

(H3) $F^{-1}(\mu') = F^{-1}(\mu)'$,

where $F^{-1} : L_2 \to L_1$ is defined by

(H4) $F^{-1}(\mu) = \vee \{ \lambda \in L_1 : F(\lambda) \leq \mu \}$.

Remark 2.3 [41,21] Suppose $F : L_1 \to L_2$ is an order–homomorphism, $\mu \in L_1$ and $\nu, \nu_i \in L_2$ then,

(R1) $F^{-1}(F(\mu)) \geq \mu$,

(R2) $F(F^{-1}(\nu)) \leq \nu$,

(R3) $F^{-1}(\vee \nu_i) = \vee F^{-1}(\nu_i)$,

(R4) $F^{-1}(\wedge \nu_i) = \wedge F^{-1}(\nu_i)$,

(R5) $F^{-1}(1) = 1$,

(R6) $F^{-1}(0) = 0$.

We are interested in order–homomorphisms $F : L_1 \to L_2$ where $L_1 = L^X$ and $L_2 = L^Y$, in other words Erceg functions [10].

Lemma 2.4

If $x \in X$, $\nu_1 \neq 0$ and $\nu_2 \neq 0$ are two fuzzy sets in L^Y such that $\text{supp}(\nu_1) \cap \text{supp}(\nu_2) = \emptyset$ and $F^{-1}(\nu_1)(x), F^{-1}(\nu_2)(x) > 0$, then $F^{-1}(\nu_1)(x) \parallel F^{-1}(\nu_2)(x)$.

Proof.

Let $\alpha_1 = F^{-1}(\nu_1)(x)$ and $\alpha_2 = F^{-1}(\nu_2)(x)$. Clearly $\alpha_1, \alpha_2 > 0$.

Suppose without loss of generality that $\alpha_1 \leq \alpha_2$ then by (H2) we have,

(*) $F(\alpha_1 1_x) \leq F(\alpha_2 1_x)$.

Clearly $\alpha_1 1_x \leq F^{-1}(\nu_1)$ and $\alpha_2 1_x \leq F^{-1}(\nu_2)$. Thus by (H2)

(**) $F(\alpha_1 1_x) \leq F(F^{-1}(\nu_1))$ and $F(\alpha_2 1_x) \leq F(F^{-1}(\nu_2))$.

Using (R2) we obtain from (**),

$F(\alpha_1 1_x) \leq \nu_1$ and $F(\alpha_2 1_x) \leq \nu_2$.

Since $F^{-1}(0) = 0$, we have,

$0 < F(\alpha_1 1_x) \leq \nu_1$ and $0 < F(\alpha_2 1_x) \leq \nu_2$.

Since $\text{supp}(\nu_1) \cap \text{supp}(\nu_2) = \emptyset$ we must have $F(\alpha_1 1_x) \parallel F(\alpha_2 1_x)$.

This contradicts (*).

Lemma 2.5

For all $y \in \text{supp}(F(1_x))$ we have $F^{-1}(1_y)(x) > 0$.

Proof.

Suppose $y \in \text{supp}(F(1_x))$ and $F^{-1}(1_y)(x) = 0$. Clearly $1_y \vee 1_y' = 1$. Thus we have

$1 = F^{-1}(1_y \vee 1_y')(x) = F^{-1}(1_y)(x) \vee F^{-1}(1_y')(x) = F^{-1}(1_y')(x)$.

Since $1_x \leq F^{-1}(1_y')$ it follows that $F(1_x) \leq F(F^{-1}(1_y')) \leq 1_y'$.

This contradicts the fact that $y \in \text{supp}(F(1_x))$.

The following proposition gives us an insight into the structure of L from the behaviour of the order–homomorphism F.

Proposition 2.6

If $A = \text{supp}(F(1_x))$ then L contains a sub–lattice L_1 isomorphic to the lattice 2^A, such that the top and the bottom elements of L_1 are the top and the bottom elements of L. (We consider 2^A as a lattice ordered by set inclusion and set complement as an order reversing involution.)

Proof.

Let $L_1 = \{ F^{-1}(1_B)(x) : B \subset A \} \subset L$. We construct an isomorphism $G : 2^A \to L_1$ by letting $G(B) = F^{-1}(1_B)(x)$. By (R3) and (R4), G preserves $\vee$ and $\wedge$. Since $1_\emptyset = 0$ and $F(1_x) \leq 1_A$ we have that $G(\emptyset) = F^{-1}(0)(x) = 0$ and $G(A) = F^{-1}(1_A)(x) = 1$. To see that G preserves $'$ we use the fact that $G(A) = F^{-1}(1_A)(x) = 1$. Thus,

$$\begin{aligned} G(B') &= F^{-1}(\, 1_{A\setminus B}\,)(x) \\ &= F^{-1}(\, 1_A \wedge 1_{Y\setminus B}\,)(x) \\ &= F^{-1}(\, 1_A\,)(x) \wedge F^{-1}(\, 1_B'\,)(x) \\ &= 1 \wedge F^{-1}(\, 1_B{}'\,)(x) \\ &= F^{-1}(\, 1_B\,)(x)' \\ &= G(B)' \quad . \end{aligned}$$

In order to prove that G is 1–1 it is sufficient to show that for all $B_1, B_2 \subset A$, with $B_1 \neq B_2$, $G(B_1) \neq G(B_2)$. Suppose that $B_1, B_2 \subset A$, with $B_1 \neq B_2$. Since $B_1 \neq B_2$ then either $B_1\setminus B_2 \neq \emptyset$ or $B_2\setminus B_1 \neq \emptyset$. Assume that $B_1\setminus B_2 \neq \emptyset$, $B_2\setminus B_1 \neq \emptyset$ and $B_1 \cap B_2 \neq \emptyset$. Since $B_1\setminus B_2$ and B_2 are non–empty subsets of A, for each $y \in B_1\setminus B_2$, $F^{-1}(1_y)(x) > 0$ by Lemma 2.5. As F^{-1} is monotone, we have $F^{-1}(1_y)(x) < F^{-1}(1_{B_1\setminus B_2})(x)$ and thus $F^{-1}(1_{B_1\setminus B_2})(x) > 0$. Likewise $F^{-1}(1_{B_2})(x) > 0$. Since $B_1\setminus B_2$ and B_2 are disjoint it follows by Lemma 2.4 that $G(B_1\setminus B_2)$ and $G(B_2)$ are non–comparable. Similarly we obtain that $G(B_2\setminus B_1)$ and $G(B_1)$ are non–comparable. If $G(B_1) \leq G(B_2)$ then since $B_1\setminus B_2 \subset B_1$, we have $G(B_1\setminus B_2) \leq G(B_1) \leq G(B_2)$. This contradicts the fact that $G(B_1\setminus B_2)$ is

non–comparable with $G(B_2)$. Similarly, assuming that $G(B_2) \leq G(B_1)$ leads to a contradiction. Thus $G(B_1)$ and $G(B_2)$ are non–comparable. In particular they are not equal. If $B_1 \subset B_2$, $B_1 \neq B_2$ then B_1 and $B_2 \setminus B_1$ are disjoint then as above by Lemma 2.4 and Lemma 2.5, $G(B_1)$ and $G(B_2 \setminus B_1)$ are non–comparable.
Also $G(B_2) = G(B_1 \cup (B_2 \setminus B_1)) = G(B_1) \vee G(B_2 \setminus B_1)$. So $G(B_1) \neq G(B_2)$.
This proves that G is 1–1, and is by definition onto.
Thus G is the required isomorphism.

In [41] Lemma 2.2 Wang proves that if a lattice L is regular then order–homomorphisms take fuzzy points to fuzzy points. He calls, a lattice **regular** if and only if infimum of two non–zero elements is non–zero. With the use of the above proposition we obtain the following generalisation of his Lemma 2.2.
This generalisation is directly shown in Proposition 2.12.

Let $|A|$ denote the cardinality of set A.

Theorem 2.7

Suppose that L does not contain any sub–lattice L_1 isomorphic to the lattice 2^A, such that the top and the bottom elements of L_1 are the top and the bottom elements of L then, $|\ \mathrm{supp}(F(1_x))\ | < |\ A\ |$.

Proof.

Suppose that $|\ \mathrm{supp}(F(1_x))\ | \geq |\ A\ |$ for some $A \subset Y$. Let $B = \mathrm{supp}(F(1_x))$.
By Proposition 2.6 there exists a sub–lattice $L_1 \subset L$ isomorphic to 2^B such that the top and the bottom elements of L_1 are also the top and the bottom elements of L.
Now we shall construct a sub–lattice $\hat{L}$ of 2^B isomorphic to 2^A such that the top and the bottom elements of $\hat{L}$ are equal to the top and the bottom elements of 2^B.

Since $|A| < |B|$ there exists an injection $f: A \to B$. Choose any $x \in A$ and let $C = B \setminus f(A)$. Define

$$F(X) = \begin{cases} f(X) \cup C & \text{if } x \in X \\ f(X) & \text{otherwise} \end{cases}.$$

Let $\hat{L} = F(2^A)$. Now we shall show that $F: 2^A \to \hat{L}$ is an isomorphism.

Let $\{A_\alpha\}_{\alpha \in J} \subset 2^A$, $J_1 = \{\alpha \in J : x \in A_\alpha\}$ and $J_2 = J \setminus J_1$.

(A) If $x \in \cap A_\alpha$ then since f is 1–1 and $x \in A_\alpha$ for all $\alpha \in J$ we have,
$F(\cap A_\alpha) = f(\cap A_\alpha) \cup C = (\cap f(A_\alpha)) \cup C = \cap (f(A_\alpha) \cup C) = \cap F(A_\alpha)$.
If $x \notin \cap A_\alpha$ then $F(\cap A_\alpha) = f(\cap A_\alpha) = \cap f(A_\alpha) = [\cap \{f(A_\alpha) : \alpha \in J_1\}] \cap [\cap \{f(A_\alpha) : \alpha \in J_2\}] = [\cap \{f(A_\alpha) \cup C : \alpha \in J_1\}] \cap [\cap \{f(A_\alpha) : \alpha \in J_2] = [\cap \{F(A_\alpha) : \alpha \in J_1\}] \cap [\cap \{F(A_\alpha) : \alpha \in J_2\}] = \cap F(A_\alpha)$ since $J_2 \neq \emptyset$.

(B) Suppose that $x \in \cup A_\alpha$ then $F(\cup A_\alpha) = f(\cup A_\alpha) \cup C = \cup f(A_\alpha) \cup C$.
Clearly J_1 is non empty. Thus $\cup F(A_\alpha) = [\cup \{F(A_\alpha) : \alpha \in J_1\}] \cup [\cup \{F(A_\alpha) : \alpha \in J_2\}] = [\cup \{f(A_\alpha) \cup C : \alpha \in J_1\}] \cup [\cup \{f(A_\alpha) : \alpha \in J_2\}] = \cup f(A_\alpha) \cup C$. If $x \notin \cup A_\alpha$ then since $x \notin A_\alpha$ for all $\alpha \in J$ we have
$F(\cup A_\alpha) = f(\cup A_\alpha) = \cup f(A_\alpha) = \cup F(A_\alpha)$.

(C) If $x \in X$ then $x \notin A \setminus X$ thus since f is 1–1 we have $F(A \setminus X) = f(A \setminus X) = f(A) \setminus f(X)$. On the other hand $B \setminus F(X) = B \setminus (f(X) \cup C)$. By noting that $f(A) = B \setminus C$ we obtain $F(A \setminus X) = B \setminus F(X)$. In case $x \notin X$ the proof is similar. Also $F(A) = f(A) \cup C = B$ and $F(\emptyset) = f(\emptyset) = \emptyset$.

(D) The F is onto by definition of $\hat{L}$. Suppose $F(X_1) = F(X_2)$. If $x \notin X_1$ then $F(X_1) = f(X_1)$. Clearly $f(X_1) \cap C = \emptyset$ so $F(X_2) \cap C = \emptyset$. This means that $x \notin X_2$. Thus from definition $f(X_1) = f(X_2)$. Since f is 1–1, $X_1 = X_2$.
If $x \in X_1$ then $F(X_1) = f(X_1) \cup C$. By argument as previously we have $x \in X_2$ and $F(X_2) = f(X_2) \cup C$. Since $f(X_1 \setminus \{x\}) \cap C = \emptyset$ and $f(X_2 \setminus \{x\}) \cap C = \emptyset$ we must have $f(X_1 \setminus \{x\}) = f(X_2 \setminus \{x\})$. Since f is 1–1, $X_1 \setminus \{x\} = X_2 \setminus \{x\}$.
But $x \in X_1$ and $x \in X_2$ implies that $X_1 = X_2$.

So finally $F : 2^A \to \hat{L}$ is an isomorphism. This immediatelly implies that there exists a sublattice L_1 of L isomorphic to 2^A such that the top and the bottom elements of L_1 are the top and the bottom elements of L. This is a contradiction to our original assumption.

Corollary 2.8

If L does not contain any sub–lattice L_1 isomorphic to the power set of a two element set such that the top and the bottom elements of L_1 are the top and the bottom elements of L, then the order–homomorphism F takes fuzzy points with the same support to fuzzy points with the same support.

Proof.

By Theorem 2.7 $|\ supp(F(1_x))\ | < 2$. Thus $|\ supp(F(1_x))\ | \leq 1$. Since $F^{-1}(0)=0$ we must have $F(1_x) \neq 0$, so $|\ supp(F(1_x))\ | = 1$. In another words $F(1_x)$ is a fuzzy point. Furthermore, if $p \leq 1_x$ then $F(p) \leq F(1_x)$. Thus $F(p)$ is a fuzzy point with the same support as $F(1_x)$.

Example 2.9

Let $L = 2^{\{a,b\}}$. Consider an order–homomorphism $F : L^{\{x\}} \to L^{\{y,z\}}$ defined by $F(\{a,b\}1_x) = \{a\}1_y \vee \{b\}1_z$, $F(\{a\}1_x) = \{a\}1_y$, $F(\{b\}1_x) = \{b\}1_z$ and $F(0) = 0$. We treat $L = 2^{\{a,b\}}$ as a power set, boolean, lattice.

It is easy but tedious to check that the function F is an order–homomorphism which does not takes a fuzzy point 1_x to a fuzzy point.

Theorem 2.10

An order–homomorphism $F : L^X \to L^Y$ takes fuzzy points to fuzzy points if and only if there exist functions $f : X \to Y$ and $\kappa : X \times L \to L$ such that $\kappa(x,\cdot)$ is an order–homomorphism and for all $\mu \in L^X$ we have $F(\mu)(y) = \sup\{\ \kappa(x,\mu(x)) : x \in f^{-1}(y)\ \}$.

Proof.

Suppose $F : L^X \to L^Y$ takes fuzzy points to fuzzy points. For each $x \in X$ and $\alpha \in L\backslash\{0\}$, $F(\alpha 1_x)$ is a fuzzy point. Consequently we can write $F(\alpha 1_x) = \beta 1_y$ for some $y \in Y$ and some $\beta \in L\backslash\{0\}$. If $\gamma \in L\backslash\{0\}$ then $F((\alpha \vee \gamma)1_x) = F(\alpha 1_x \vee \gamma 1_x) = F(\alpha 1_x) \vee F(\gamma 1_x)$. Since $F((\alpha \vee \gamma)1_x)$ is a fuzzy point we must have $\mathrm{supp}(F(\alpha 1_x)) = \mathrm{supp}(F(\gamma 1_x))$. Thus $\mathrm{supp}(F(\alpha 1_x))$ is only dependent on x. This shows that there exist functions $\kappa : X \times L \to L$ and $f : X \to Y$ such that $F(\alpha 1_x) = \kappa(x,\alpha)1_{f(x)}$, where we allow α in $\alpha 1_x$ to be zero, i.e. $01_x = 0$. Moreover,

$$\begin{aligned} F(\mu)(y) &= F\left(\bigvee_{x\in X} \mu(x)1_x \right)(y) \\ &= \bigvee_{x\in X} F(\mu(x)1_x)(y) \\ &= \left(\bigvee_{x\in X} \kappa(x,\mu(x))1_{f(x)} \right)(y) \\ &= \bigvee \{ \kappa(x,\mu(x)) : f(x) = y \}. \end{aligned}$$

To see why $\kappa(x,\cdot)$ is an order–homomorphism consider the following:

(H1) Since $F(0) = 0$ we must have $\kappa(x,0) = 0$ for all $x \in X$.

(H2) Let $x \in X$ and $\{\alpha_i\}_{i\in J} \subset L$ then $\kappa(x, \bigvee_{i\in J} \alpha_i) = F(\bigvee_{i\in J} \alpha_i 1_x)(f(x)) =$

$= \bigvee_{i\in J} F(\alpha_i 1_x)(f(x)) = \bigvee_{i\in J} \kappa(x,\alpha_i)$.

(H3) Let $x \in X$, $\beta \in L$ and $\psi(\alpha) = \kappa(x,\alpha)$. Then,

$$\begin{aligned} \psi^{-1}(\beta') &= \bigvee \{ \alpha : \psi(\alpha) \leq \beta' \} \\ &= \bigvee \{ \alpha : \kappa(x,\alpha) \leq \beta' \} \\ &\qquad \text{since } \kappa(x,\alpha) \leq \beta' \text{ iff } \kappa(x,\alpha)1_{f(x)} \leq (\beta' 1_{f(x)})' \text{ then} \\ &= \bigvee \{ \alpha : \kappa(x,\alpha)1_{f(x)} \leq (\beta 1_{f(x)})' \} \\ &= \bigvee \{ \alpha : F(\alpha 1_x) \leq (\beta 1_{f(x)})' \} \\ &= [\bigvee \{ \alpha 1_x : F(\alpha 1_x) \leq (\beta 1_{f(x)})' \}](x) \\ &= F^{-1}((\beta 1_{f(x)})')(x) \end{aligned}$$

$= [F^{-1}(\beta 1_{f(x)})(x)]'$

$= [\vee \{ \alpha : \psi(\alpha) \leq \beta \}]'$

$= \psi^{-1}(\beta)'$.

The converse, follows similarly.

Compare this with $f(\mu)(y) = \sup\{ \mu(x) : x \in f^{-1}(y) \}$ which is the way a crisp function f extends to a fuzzy set μ, see Definition 2.1.

Example 2.11

Consider a lattice $L = \{ 0, \alpha, \beta, 1 \}$, where $\alpha' = \alpha$, $\beta' = \beta$ and α and β are non–comparable. The lattice L does not contain any sub–lattice isomorphic to the boolean lattice $2^{\{a,b\}}$. So, by Corollary 2.8 any order homomorphism $F : L^X \to L^Y$ will preserve fuzzy points. This lattice L is clearly not regular.

Proposition 2.12

Corollary 2.8 is an extension of Wang's Lemma 2.2.

Proof.

Suppose L_1 is a sub–lattice of a regular lattice L which is isomorphic to the boolean lattice $2^{\{a,b\}}$ such that L_1 contains the top and the bottom from L. Let $\alpha \in L_1 \backslash \{0,1\}$ then $\alpha \neq 0$ and $\alpha' \wedge \alpha = 0$. Consequently L is not regular. This means that if L is regular then the conditions of Corollary 2.8 are satisfied.

Note that now we revert back to our assumption that L is totally ordered.

3. FUZZY CARDINALS

3.1. INTRODUCTION

For our following discussion let us establish some notation.

Let K be the "set" of all cardinals. Of course the set of all cardinals is an ill defined concept. We can think of K as the set of all cardinals smaller than some cardinal k_∞, where k_∞ is chosen sufficiently large, so that all the operations that are considered on cardinals are internal to K.

The usage of the "set" K, will save on unnecessary repetitions. Let $|A|$ denote the cardinality of a crisp set A. Clearly $K = (K, +, \times, \leq)$.

In what follows we are going to deal with collections of fuzzy sets which do not necessarily have the same domains. Consequently, we require the following definitions. If $\{\mu_i : X_i \to L\}_{i \in J}$ is a collection of fuzzy sets, then their union $\vee\, \mu_i : \cup\, X_i \to L$ and intersection $\wedge\, \mu_i : \cup\, X_i \to L$ are given by,

$$(\vee\, \mu_i)(x) = \sup\{\, \mu_i^*(x) : i \in J \,\}$$

and

$$(\wedge\, \mu_i)(x) = \inf\{\, \mu_i^*(x) : i \in J \,\}$$

respectively, where $\mu_i^* : \cup\, X_i \to L$ are given by,

$$\mu_i^*(x) = \begin{cases} \mu_i(x) & \text{if } x \in X_i \\ 0 & \text{otherwise} \end{cases}.$$

Under this setup two fuzzy sets $\mu : X \to L$ and $\nu : Y \to L$ are disjoint iff $\mu \wedge \nu = \emptyset$.

In the future, unless otherwise stated, the fuzzy sets will not necessarily have the same domains.

For example, consider two fuzzy sets $\mu : [0,2] \to [0,1]$ and $\nu : [1,3] \to [0,1]$, defined by $\mu(x) = \frac{1}{2} x$ and $\nu(x) = \frac{1}{2}(x-1)$. Then, $\mu \vee \nu : [0,3] \to [0,1]$ and $\mu \wedge \nu : [0,3] \to [0,1]$ are defined by,

$$(\mu \vee \nu)(x) = \begin{cases} (1/2)x & \text{if } x \in [0,2] \\ (1/2)(x-1) & \text{if } x \in (2,3] \end{cases}$$

and

$$(\mu \wedge \nu)(x) = \begin{cases} (1/2)(x-1) & \text{if } x \in [1,2] \\ 0 & \text{otherwise} \end{cases}.$$

In the literature various definitions of cardinality of fuzzy sets have been given. A review of some of them can be found in [8]. Here we consider the definitions introduced by Gottwald [14], Blanchard [2] and Wygralak [47]. Each one of these definitions has associated with it some equipotence relation. The most obvious definition of equipotence can be given as follows:

(A) *Two fuzzy sets μ and ν are equipotent if and only if there exists an bijection* $f : \text{supp}(\mu) \to \text{supp}(\nu)$ *such that* $f(\mu_{|\,\text{supp}(\mu)}) = \nu_{|\,\text{supp}(\nu)}$.

This definition of equipotence results in Gottwald's cardinality of fuzzy sets. Gottwald's cardinality of a fuzzy set μ, denoted $\text{card}_G(\mu)$, is a function from L to cardinals (i.e. K), given by $\text{card}_G(\mu)(\alpha) = |\ \mu^{-1}(\alpha)\ |$. Clearly $\text{card}_G(\mu)$ is not a fuzzy set with membership grades belonging to L. However, it is possible to make cardinality based on the equipotence relation (A) such that it results in a fuzzy set with membership grades belonging to L. For instance let,

$$\text{card}_{G'}(\mu) : K \times L \to L$$

given by,

$$\text{card}_{G'}(\mu)((\kappa,\alpha)) = \begin{cases} \alpha & \text{if } |\ \mu^{-1}(\alpha)\ | = \kappa \\ 0 & \text{otherwise} \end{cases}.$$

However, the way in which this fuzzy set is constructed is very unnatural.

(What we mean by unnatural is going to be explained shortly.)

At this stage we shall investigate some basic properties of Gottwald's cardinality because it is used in later sections. Firstly, for any $g \in K^L$ there exists a fuzzy set μ such that $\text{card}_G(\mu) = g$. To see this we can construct the required fuzzy set μ as follows. Let $\{A_\alpha\}_{\alpha \in L\setminus\{0\}}$ be a collection of pairwise disjoint sets such that $|A_\alpha| = g(\alpha)$. Define $\mu : \cup A_\alpha \to L$ by letting $\mu(a) = \alpha$ if and only if $a \in A_\alpha$. Clearly $\text{card}_G(\mu) = g$. We now note that, two fuzzy sets μ and ν are equipotent **(A)** if and only if for all $\alpha \in L\setminus\{0\}$, $|\mu^{-1}(\alpha)| = |\nu^{-1}(\alpha)|$. This proves that card_G is well defined. We call any element from K^L ***Gottwald's cardinal***

Now suppose that $\text{Card}_G(\mu) = \text{Card}_G(\nu)$. This means that for all $\alpha \in L\setminus\{0\}$, $|\mu^{-1}(\alpha)| = |\nu^{-1}(\alpha)|$. So, there exist bijections $f_\alpha : \mu^{-1}(\alpha) \to \nu^{-1}(\alpha)$ for all $\alpha \in L\setminus\{0\}$. Define $f : \text{supp}(\mu) \to \text{supp}(\nu)$ by letting $f(x) = f_{\mu(x)}(x)$. Clearly f is a bijection and $f(\mu|_{\text{supp}(\mu)}) = \nu|_{\text{supp}(\nu)}$, i.e. μ and ν are equipotent in sense (A).

From a collection of fuzzy sets $\{\mu_i : X_i \to L\}$ we can construct a collection $\{\mu_i^* : X_i^* \to L\}$ of pairwise disjoint fuzzy sets such that μ_i^* and μ_i are equipotent by letting $X_i^* = \{i\} \times X_i$ and $\mu_i^*(i,x) = \mu_i(x)$.

Definition 3.1.1

Let $\{g_i\}_{i \in J}$ be a collection of Gottwald cardinals then the sum $\oplus$ and the product $\otimes$ are defined as follows:

(i) Let $\{\mu_i\}_{i \in J}$ be a collection of pairwise disjoint fuzzy sets such that

$\text{card}_G(\mu_i) = g_i$ then $[\bigoplus_i g_i](\alpha) = \text{card}_G(\bigvee_i \mu_i)(\alpha)$.

(ii) Let $\{\mu_i\}_{i \in J}$ be a collection of not necessarily disjoint fuzzy sets such that $\text{card}_G(\mu_i) = g_i$ then $[\bigotimes_i g_i](\alpha) = \text{card}_G(\prod_i \mu_i)(\alpha)$.

We now show that the operations $\oplus$ and $\otimes$ are well defined:–

(Subject to the conventions established in the first paragraph.)

(i) Let $\{\nu_i\}_{i\in J}$ be another collection of pairwise disjoint fuzzy sets such that $\text{card}_G(\nu_i) = \text{card}_G(\mu_i)$ and let $\{ f_i : \text{supp}(\mu_i) \to \text{supp}(\nu_i) \}_{i\in J}$ be a collection of bijections such that $f_i(\mu_{i|\,\text{supp}(\mu_i)}) = \nu_{i|\,\text{supp}(\nu_i)}$. Then,

$$[\oplus\, g_i](\alpha) = \text{card}_G(\bigvee_i \mu_i)(\alpha)$$

$$= |\ (\bigvee_i \mu_i)^{-1}(\alpha)\ |$$

$$= |\ \{ x \in \bigcup_i \text{supp}(\mu_i) : (\bigvee_i \mu_i)(x) = \alpha \}\ |$$

since μ_i are pairwise disjoint then

$$= |\ \bigcup_i \{ x \in \text{supp}(\mu_i) : \mu_i(x) = \alpha \}\ |$$

since f_i are bijections and $\nu_i(f_i(x)) = \mu_i(x)$ then

$$= |\ \bigcup_i \{ f_i(x) : x \in \text{supp}(\mu_i) \text{ and } \nu_i(f_i(x)) = \alpha \}\ |$$

$$= |\ \bigcup_i \{ y \in \text{supp}(\nu_i) : \nu_i(y) = \alpha \}\ |$$

since ν_i are pairwise disjoint then

$$= |\ \{ y \in \bigcup_i \text{supp}(\nu_i) : (\bigvee_i \nu_i)(y) = \alpha \}\ |$$

$$= |\ (\bigvee_i \nu_i)^{-1}(\alpha)\ |$$

$$= \text{card}_G(\bigvee_i \nu_i)(\alpha).$$

(ii) Let $\{\nu_i\}_{i\in J}$ be another collection of fuzzy sets such that $\text{card}_G(\nu_i) = \text{card}_G(\mu_i)$ and let $\{ f_i : \text{supp}(\mu_i) \to \text{supp}(\nu_i) \}_{i\in J}$ be a collection of bijections such that $f_i(\mu_{i|\,\text{supp}(\mu)}) = \nu_{i|\,\text{supp}(\nu_i)}$. Then,

$$[\underset{i}{\otimes} g_i](\alpha) = |\ (\prod_i \mu_i)^{-1}(\alpha)\ |$$

$$= |\ \{ (x_i) : x_i \in supp(\mu_i) \text{ and } \bigwedge_i \mu_i(x_i) = \alpha \}\ |$$

since f_i are bijections and $\mu_i(x) = \nu_i(f_i(x))$ then

$$= |\ \{ (f_i(x_i)) : x_i \in supp(\mu_i) \text{ and } \bigwedge_i \nu_i(f_i(x_i)) = \alpha \}\ |$$

$$= |\ \{ (y_i) : y_i \in supp(\nu_i) \text{ and } \bigwedge_i \nu_i(y_i) = \alpha \}\ |$$

$$= |\ (\prod_i \nu_i)^{-1}(\alpha)\ |$$

$$= card_G(\prod_i \nu_i)(\alpha).$$

<u>Proposition 3.1.2</u>

If g_1 and g_2 are two Gottwald cardinals then,

(i) $[g_1 \oplus g_2](\alpha) = g_1(\alpha) + g_2(\alpha)$

and

(ii) $[g_1 \otimes g_2](\alpha) = (g_1(\alpha) \times (\sum_{\gamma > \alpha} g_2(\gamma))) + (g_2(\alpha) \times (\sum_{\gamma \geq \alpha} g_1(\gamma)))$.

<u>Proof.</u>

(i) Let μ and ν are two disjoint fuzzy sets such that $card_G(\mu) = g_1$ and $card_G(\nu) = g_2$. Then,

$$[g_1 \oplus g_2](\alpha) = card_G(\mu \vee \nu)(\alpha)$$

$$= |\ (\mu \vee \nu)^{-1}(\alpha)\ |$$

since μ and ν are disjoint then

$$= |\ \mu^{-1}(\alpha) \cup \nu^{-1}(\alpha)\ |$$

$$= |\ \mu^{-1}(\alpha)\ | + |\ \nu^{-1}(\alpha)\ |$$

$$= card_G(\mu)(\alpha) + card_G(\nu)(\alpha)$$

$$= g_1(\alpha) + g_2(\alpha)$$

This argument is also valid for arbitrary collections of cardinals.

(ii) Let μ and ν be two fuzzy sets such that $\mathrm{card}_G(\mu) = g_1$ and $\mathrm{card}_G(\nu) = g_2$. Then,

$$\begin{aligned}
[g_1 \otimes g_2](\alpha) &= \mathrm{card}_G(\mu \times \nu)(\alpha) \\
&= |\ (\mu \times \nu)^{-1}(\alpha)\ | \\
&= |\ \mu^{-1}(\alpha) \times \nu^{-1}(\alpha,1] \cup \mu^{-1}[\alpha,1] \times \nu^{-1}(\alpha)\ | \\
&= |\mu^{-1}(\alpha)| \times |\nu^{-1}(\alpha,1]| + |\mu^{-1}[\alpha,1]| \times |\nu^{-1}(\alpha)| \\
&= |\mu^{-1}(\alpha)| \times \Big(\sum_{\gamma>\alpha} |\nu^{-1}(\gamma)| \Big) + \Big(\sum_{\gamma\geq\alpha} |\mu^{-1}(\gamma)| \Big) \times |\nu^{-1}(\alpha)| \\
&= \Big(g_1(\alpha) \times \Big(\sum_{\gamma>\alpha} g_2(\gamma) \Big) \Big) + \Big(\Big(\sum_{\gamma\geq\alpha} g_1(\gamma) \Big) \times g_2(\alpha) \Big).
\end{aligned}$$

If μ and ν are two not necessarily disjoint fuzzy sets then we have: $[\ \mathrm{card}_G(\mu) \oplus \mathrm{card}_G(\nu)\ ](\alpha) = |\ \mu^{-1}(\alpha)\ | + |\ \nu^{-1}(\alpha)\ |$. To see this we proceed as follows: Let $g_1 = \mathrm{card}_G(\mu)$ and $g_2 = \mathrm{card}_G(\nu)$. Then,

$$\begin{aligned}
[\ \mathrm{card}_G(\mu) \oplus \mathrm{card}_G(\nu)\ ](\alpha) &= [\ g_1 \oplus g_2\ ] \\
&= g_1(\alpha) + g_2(\alpha) \qquad \text{by Proposition 3.1.2 (i)} \\
&= |\ \mu^{-1}(\alpha)\ | + |\ \nu^{-1}(\alpha)\ |.
\end{aligned}$$

Now we shall prove the additivity property for Gottwald's cardinality.

Theorem 3.1.3

If $\mu, \nu \in L^X$ are two fuzzy sets then,

$\mathrm{card}_G(\mu) \oplus \mathrm{card}_G(\nu) = \mathrm{card}_G(\mu \wedge \nu) \oplus \mathrm{card}_G(\mu \vee \nu)$.

Proof.

We first note that $(\mu \wedge \nu)^{-1}(\alpha) = (\ \mu^{-1}(\alpha) \cap \nu^{-1}(\alpha,1]\ ) \cup (\ \mu^{-1}[\alpha,1] \cap \nu^{-1}(\alpha)\ )$ and $(\mu \vee \nu)^{-1}(\alpha) = (\ \mu^{-1}(\alpha) \cap \nu^{-1}[0,\alpha]\ ) \cup (\ \mu^{-1}[0,\alpha) \cap \nu^{-1}(\alpha)\ )$.

Since $(\mu \wedge \nu)^{-1}(\alpha)$ and $(\mu \vee \nu)^{-1}(\alpha)$ are written as unions of disjoint sets, $(\mu^{-1}(\alpha) \cap \nu^{-1}(\alpha,1]) \cap (\mu^{-1}(\alpha) \cap \nu^{-1}[0,\alpha]) = \emptyset$, $(\mu^{-1}[\alpha,1] \cap \nu^{-1}(\alpha)) \cap (\mu^{-1}[0,\alpha) \cap \nu^{-1}(\alpha)) = \emptyset$, $(\mu^{-1}(\alpha) \cap \nu^{-1}(\alpha,1]) \cup (\mu^{-1}(\alpha) \cap \nu^{-1}[0,\alpha]) = \mu^{-1}(\alpha)$ and $(\mu^{-1}[\alpha,1] \cap \nu^{-1}(\alpha)) \cup (\mu^{-1}[0,\alpha) \cap \nu^{-1}(\alpha)) = \nu^{-1}(\alpha)$ we have,

$$
\begin{aligned}
&[\,\mathrm{card}_G(\mu \wedge \nu) \oplus \mathrm{card}_G(\mu \vee \nu)\,](\alpha) \\
&= |\,(\mu \wedge \nu)^{-1}(\alpha)\,| + |\,(\mu \vee \nu)^{-1}(\alpha)\,| \quad \text{By Proposition 3.1.2 (i)} \\
&= |\,(\mu^{-1}(\alpha) \cap \nu^{-1}(\alpha,1]) \cup (\mu^{-1}[\alpha,1] \cap \nu^{-1}(\alpha))\,| \\
&\qquad + |\,(\mu^{-1}(\alpha) \cap \nu^{-1}[0,\alpha]) \cup (\mu^{-1}[0,\alpha) \cap \nu^{-1}(\alpha))\,| \\
&= |\,\mu^{-1}(\alpha) \cap \nu^{-1}(\alpha,1]\,| + |\,\mu^{-1}[\alpha,1] \cap \nu^{-1}(\alpha)\,| \\
&\qquad + |\,\mu^{-1}(\alpha) \cap \nu^{-1}[0,\alpha]\,| + |\,\mu^{-1}[0,\alpha) \cap \nu^{-1}(\alpha)\,| \\
&= (\,|\,\mu^{-1}(\alpha) \cap \nu^{-1}(\alpha,1]\,| + |\,\mu^{-1}(\alpha) \cap \nu^{-1}[0,\alpha]\,|\,) \\
&\qquad + (\,|\,\mu^{-1}[\alpha,1] \cap \nu^{-1}(\alpha)\,| + |\,\mu[0,\alpha) \cap \nu^{-1}(\alpha)\,|\,) \\
&= |\,\mu^{-1}(\alpha)\,| + |\,\nu^{-1}(\alpha)\,| \\
&= [\,\mathrm{card}_G(\mu) \oplus \mathrm{card}_G(\nu)\,](\alpha).
\end{aligned}
$$

In essence Gottwald's cardinality is equivalent to asking the following question about a fuzzy set: If the required level of fuzziness is α, what is the cardinality? This sort of reasoning is not used in fuzzy set theory. In fuzzy set theory we ask questions in the following way: To what degree is the cardinality of a given fuzzy set equal to some cardinal κ? This way of reasoning was followed by Wygralak and Blanchard in their definitions. Now we shall consider these definitions from a different perspective than those given by Blanchard and Wygralak.

Consider a set X and its power set 2^X. The cardinality function assigns to each element from 2^X an element from K. In another words, $|\cdot| : 2^X \to K$. More generally let us consider some function $F : 2^X \to Z$ where Z is a set.

The question now arises as to how can one extend in a natural way the function F on 2^X, to a function on L^X.

To answer this question we need the following concepts introduced by Diskin [7] and used extensively by Šostak [39]

(a) generalised inclusion $[\cdot \subset \cdot] : L^X \times L^X \to L$ where,

$[\mu \subset \nu] = \inf\{ \mu'(x) \vee \nu(x) : x \in X \}$

(b) generalised equality $[\cdot = \cdot] : L^X \times L^X \to L$ where,

$[\mu = \nu] = [\mu \subset \nu] \wedge [\nu \subset \mu]$

Note that $[\mu \subset \nu]$ and $[\mu = \nu]$ can be regarded as the degree to which μ is a subset of ν and the degree to which μ equals ν respectively.

The motivation for the above definitions comes from the following facts: $A \subset B$ if and only if $B \cup A' = X$; $A = B$ if and only if $A \subset B$ and $B \subset A$ where $A, B \subset X$.

Each set $A \in 2^X$ can be identified with the fuzzy set $1_{\{A\}} : 2^X \to \{0,1\}$.

This observation motivates us to introduce $\Phi : L^X \to L^{(2^X)}$ given by

$\Phi(\mu)(A) = [1_A = \mu]$. We shall show that Φ is an injection on fuzzy points:–

Let $\mu = \alpha 1_x$, $\nu = \beta 1_y$, $x \neq y$, and $A = \{x\}$. Note that in general $[\, 1_A = \mu \,] = \inf_{x \in A} \mu(x) \wedge \inf_{x \notin A} \mu'(x)$. So in our case we have $[\, 1_A = \mu \,] = [\, 1_A = \nu \,]$ is equivalent to $\mu(x) \wedge \inf_{y \neq x} \mu'(y) = \alpha \wedge 1 = \alpha = 0 = 0 \wedge \beta' = \nu(x) \wedge \inf_{z \neq x} \nu'(z)$. That means that we must have $x = y$ and also $\alpha = \beta$.

In the case $\mu = 1_B$, we have $\Phi(\mu)(A) = \Phi(1_B)(A) = [1_A = 1_B]$.

It is easy to check that $[1_A = {}_B] = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{otherwise} \end{cases}$. Therefore $\Phi(1_B) = 1_{\{B\}}$.

Recall that if $f : X \to Y$ and $\mu \in L^X$ then $f(\mu)(y) = \sup\{ \mu(x) : f(x) = y \}$ (c.f. 2.1)

Since $\Phi(\mu)$ is a fuzzy set on 2^X and $F : 2^X \to Z$ the symbol $F(\Phi(\mu))(z)$ is defined, and we are in a position to extend $F : 2^X \to Z$ to $\tilde{F} : L^X \to L^Z$, by

$\tilde{F}(\mu)(z) = F(\Phi(\mu))(z) = \sup\{ \Phi(\mu)(A) : F(A) = z \} = \sup\{ [1_A = \mu] : F(A) = z \}$.

Now $\tilde{F}(1_A)(z) = \sup\{ [1_A = 1_B] : F(B) = z \}$. Since $[1_A = 1_B]$ is non–zero only if $A = B$ and then equals 1, we have that, $\tilde{F}(1_A) = 1_{F(A)}$ which can be identified with $F(A)$ (c.f. page 3). This shows that $\tilde{F}$ is indeed an extension of F.

In the case of the crisp cardinality function $|\cdot|$ (i.e. $F = |\cdot|$), we obtain,

$$\widetilde{|\cdot|}(\mu)(\kappa) = \mathrm{card}_W(\mu)(\kappa) = \sup\{ [1_A = \mu] : |A| = \kappa \}$$

which is Wygralak's definition of cardinality for fuzzy sets. (Wygralak considers the case where $L = [0,1]$ only.). In fact, Wygralak has instead of $[1_A = \mu]$ the following:

$$\begin{aligned}
[\![1_A = \mu]\!] &:= \inf_{x\in X} \{ 1 - | \mu(x) - 1_A(x) | \} \\
&= \left[\inf_{x\in A} \mu(x) \right] \wedge \left[\inf_{x\in X\setminus A} (1-\mu(x)) \right] \\
&= [1_A \subset \mu] \wedge [1_{X\setminus A} \subset \mu'] \\
&= [1_A \subset \mu] \wedge [\mu \subset 1_A] \\
&= [1_A = \mu].
\end{aligned}$$

It is useful to consider a different injection $\Psi : L^X \to L^{(2^X)}$ given by $\Psi(\mu)(A) = [1_A \subset \mu]$. Note that $[1_A \subset \mu] = \inf\{ \mu(a) : a \in A \}$.
For convenience we introduce $\mu\langle\!\langle\cdot\rangle\!\rangle : 2^X \to L$ given by $\mu\langle\!\langle A\rangle\!\rangle = [1_A \subset \mu]$, thus $\Psi(\mu)(A) = \mu\langle\!\langle A\rangle\!\rangle$. To show that Ψ is an injection it is sufficient to note that $\Psi(\mu)(\{a\}) = \mu(a)$. The symbol $\mu\langle\!\langle A\rangle\!\rangle$ represents the degree to which A is a subset of μ. In the crisp case Ψ assigns to each set A the collection

of all its subsets. For our later investigations the injection Ψ is more useful than Φ since it gives simpler results without the loss of insight.

Let $\mathcal{L} \subset L^{(2^X)}$ be given by $\mathcal{L} = \{ \eta : \exists \mu \in L^X \text{ with } \Psi(\mu) = \eta \}$.

Define operations $\wedge$ and $\vee$ on $\mathcal{L}$ as follows:

$$(\eta_1 \wedge \eta_2)(A) = \eta_1(A) \wedge \eta_2(A)$$

$$(\eta_1 \vee \eta_2)(A) = \inf \{ \eta_1(\{a\}) \vee \eta_2(\{a\}) : a \in A \}$$

The map Ψ preserves $\vee$ and $\wedge$, so that $\mathcal{L}$ is closed under $\vee$, $\wedge$ and is a lattice. Further investigation of $\mathcal{L}$ could be embarked upon, however it is not necessary for what follows.

Recall that Blanchard's definition of cardinality $\text{card}_B : L^X \to L^K$ is given by, $\text{card}_B(\mu)(\kappa) = \sup\{ \alpha \in L : | \mu^{-1}([\alpha,1]) | \geq k \} = \sup\{ \alpha \in L : | E_\mu^\alpha | \geq \kappa \}$, [2].

Proposition 3.1.4

Under the injection Ψ, the crisp cardinality function $|\cdot|$ results in Blanchards cardinality. In another words $\text{card}_B(\mu) = |\cdot|(\Psi(\mu))$.

Proof.

Fix κ. If $\alpha \in L$ such that $| E_\mu^\alpha | \geq \kappa$, then there exists $A \subset E_\mu^\alpha$ such that $\mu\langle\langle A\rangle\rangle \geq \alpha$ and $|A| = \kappa$. So $\sup\{ \mu\langle\langle A\rangle\rangle : |A| = \kappa \} \geq \sup\{ \alpha \in L : | E_\mu^\alpha | \geq \kappa \}$.

Let $A \subset X$ such that $|A| = \kappa$ and $\alpha = \mu\langle\langle A\rangle\rangle$ then $| E_\mu^\alpha | \geq \kappa$.

Thus $\sup\{ \mu\langle\langle A\rangle\rangle : |A| = \kappa \} \leq \sup\{ \alpha \in L : | E_\mu^\alpha | \geq \kappa \}$. The result follows.

Now, we note that if we use Φ injection then we obtain Wygralak's cardinality, and if we use Ψ injection we obtain Blanchard's cardinality.

We shall use the definition of cardinals given by Blanchard to develop our own cardinals. Consequently the following sections deal with her cardinals.
The development of Blanchard's cardinals was also carried out for its on sake.

3.2 BLANCHARD'S FUZZY CARDINALS – BASIC RESULTS

Blanchard defined her cardinals for a finite case, but they make sense in the infinite case as well.

Recall the definitions of $E^{\alpha}_{(\cdot)}$ and $H^{\alpha}_{(\cdot)}$ from 1.1.4.

Definition 3.2.1 (Blanchard [2])

Let $\mu \in L^X$ then $\mathrm{card}_B(\mu) : K \to L$, is defined by :

$\mathrm{card}_B(\mu)(\kappa) = \sup \{ \alpha \in L : | E^{\alpha}_{\mu} | \geq \kappa \} = \sup \{ \alpha \in L : |\mu^{-1}([\alpha,1])| \geq \kappa \}$.

This means that $\mathrm{card}_B(\mu) \in L^K$, i.e $\mathrm{card}_B(\mu)$ is a fuzzy set on K .

In case μ is a crisp set , it is easy to see that $\mathrm{card}_B(\mu) = 1_{\{0,\ldots,\kappa\}}$ where $\kappa = |\ \mathrm{supp}(\mu)\ |$, thus $|\ \mathrm{supp}(\mu)\ | = \sup H^{0}_{\mathrm{card}_B(\mu)}$.

Proposition 3.2.2

$\mathrm{card}_B(\mu)(\kappa) = \sup \{ \alpha \in L : | H^{\alpha}_{\mu} | \geq \kappa \}$.

Proof

Let $\alpha_0 = \mathrm{card}_B(\mu)(\kappa) = \sup \{ \alpha \in L : | E^{\alpha}_{\mu} | \geq \kappa \}$.

Clearly for all $\alpha > \alpha_0$, $| H^{\alpha}_{\mu} | \leq | E^{\alpha}_{\mu} | < \kappa$.

If $\alpha_0 \in L_-$ then $H^{\alpha_0}_{\mu} = E^{\alpha_0}_{\mu}$, $| E^{\alpha_0}_{\mu} | \geq \kappa$. Hence $\alpha_0 = \sup \{ \alpha \in L : | H^{\alpha}_{\mu} | \geq \kappa \}$.

If $\alpha_0 \notin L_-$ then for all $\alpha < \alpha_0$ there exists $\beta \in L$ such that $\alpha < \beta < \alpha_0$ and $| E^{\beta}_{\mu} | \geq \kappa$. Clearly $| H^{\alpha}_{\mu} | \geq | E^{\beta}_{\mu} | \geq \kappa$. This proves the result.

In the crisp theory two sets A and B are equipotent (or have the same cardinality) if and only if there exist a bijection between them.

What equipotence relation exists between two fuzzy sets with the same card_B? We can define an equipotence relation $\sim_B$ given by $\mu \sim_B \nu$ if and only if for all $\alpha \in L$, $| H^\alpha_\mu | = | H^\alpha_\nu |$.

Proposition 3.2.3

Two fuzzy sets μ and ν are $\sim_B$ equipotent if and only if $\text{card}_B(\mu) = \text{card}_B(\nu)$.

Proof.

If $\mu \sim \nu$ then for all $\alpha \in L$, $| H^\alpha_\mu | = | H^\alpha_\nu |$. By Proposition 3.2.2 $\text{card}_B(\mu) = \text{card}_B(\nu)$. Conversely, suppose $\text{card}_B(\mu) = \text{card}_B(\nu)$ and for some $\alpha \in L$, $\kappa_1 = | H^\alpha_\mu | \neq | H^\alpha_\nu | = \kappa_2$. With out loss of generality we can assume that $\kappa_1 < \kappa_2$. Since $| H^\alpha_\nu | = \kappa_2$ and $| H^\alpha_\mu | = \kappa_1 < \kappa_2$ we have $\text{card}_B(\nu)(\kappa_2) \geq \alpha$ and $\text{card}_B(\mu)(\kappa_2) \leq \alpha$. Consequently we have $\text{card}_B(\mu)(\kappa_2) = \text{card}_B(\nu)(\kappa_2) = \alpha$. If $\alpha \in L_-$ then clearly $| H^\alpha_\mu | = | E^\alpha_\mu | = | E^\alpha_\nu | = | H^\alpha_\nu |$, which is a contradiction. Otherwise $\alpha \notin L_-$. If κ_2 is not a limit cardinal then there exists $\beta > \alpha$ such that $| E^\beta_\nu | \geq \kappa_2$ which contradicts the fact that $\text{card}_B(\nu)(\kappa_2) = \alpha$. If κ_2 is a limit cardinal then we must have $\sup\{ | H^\beta_\nu | : \beta > \alpha \} = \kappa_2$ for otherwise we would have $| H^\alpha_\nu | < \kappa_2$. The interval (κ_1, κ_2) is infinite, so we can pick $\kappa_3 \in (\kappa_1, \kappa_2)$. We must have $\text{card}_B(\mu)(\kappa_3) = \text{card}_B(\nu)(\kappa_3)$ and $\text{card}_B(\nu)(\kappa_3) > \alpha$, since $| H^\alpha_\nu | = \kappa_2$. On the other hand for all $\beta > \alpha$, $| H^\beta_\mu | \leq \kappa_1$ thus $\text{card}_B(\mu)(\kappa_3) \leq \alpha$, which is a contradiction. Thus $| H^\alpha_\mu | = | H^\alpha_\nu |$ for all $\alpha \in L$.

Example 3.2.4

Let $\mu, \nu : \mathbb{R} \to [0,1]$ be defined by : $\mu = 1_{\mathbb{R}}$ and $\nu(x) = \begin{cases} 1 - |x| & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$

Since $L_- = \phi$ we have for all $\alpha \in [0,1)$, $H^\alpha_\mu = \mathbb{R}$ and $H^\alpha_\nu = \begin{cases} (\alpha-1, 1-\alpha) & \text{if } \alpha \in (0,1) \\ \mathbb{R} & \text{if } \alpha = 0 \end{cases}$

and clearly $H^1_\mu = H^1_\nu = \phi$. So for all $\alpha \in [0,1]$, $| H^\alpha_\mu | = | H^\alpha_\nu |$.

This is however not true for $E^\alpha_{(\cdot)}$, since $| E^1_\mu | = | \mathbb{R} | \neq 1 = | \{0\} | = | E^1_\nu |$

Proposition 3.2.5

If $c = \text{card}_B(\mu)$ is a fuzzy cardinal then,

(a) c is a non–increasing function on K,

(b) $c(0) = 1$ and there exists $\kappa \in K$ such that $c(\kappa) = 0$,

(c) R_c is upper–well–ordered, where $R_c = c(K)\setminus\{0\}$

(d) $\sup c^{-1}([\alpha,1]) \in c^{-1}([\alpha,1])$,

(e) c is continuous with respect to the order topologies on L and K,

i.e. the topologies generated by the open intervals.

Proof.

Parts (a), (b) are taken from [2].

Since K is lower–well–ordered and c is a non–increasing function on K, by Proposition 1.1.10, we must have R_c upper–well–ordered.

To see (d) let $\kappa_1 = \sup c^{-1}([\alpha,1]) \in K$. If κ_1 is not a limit cardinal then $\kappa_1 \in c^{-1}([\alpha,1])$ trivially. So, suppose that κ_1 is a limit cardinal.

Clearly for all $\kappa < \kappa_1$, $c(\kappa) \geq \alpha$. In case $\alpha \in L_-$, for all $\kappa < \kappa_1$, $|\ \mu^{-1}([\alpha,1])\ | \geq \kappa$, thus $|\ \mu^{-1}([\alpha,1])\ | \geq \kappa_1$, which means that $c(\kappa_1) \geq \alpha$, or $\kappa_1 \in c^{-1}([\alpha,1])$.

In case $\alpha \notin L_-$, we have for all $\kappa < \kappa_1$ and $\beta < \alpha$, $|\ \mu^{-1}([\beta,1])\ | \geq \kappa$, thus for all $\beta < \alpha$, $|\ \mu^{-1}([\beta,1])\ | \geq \kappa_1$, which means that $c(\kappa_1) \geq \alpha$, or $\kappa_1 \in c^{-1}([\alpha,1])$.

Finally, we show that c is continuous. Firstly by (d), for all $\alpha \in L$, $\sup c^{-1}([\alpha,1]) \in c^{-1}([\alpha,1])$ and by (a) and (b), $c(0) = 1$ and c is non–increasing thus $c^{-1}([\alpha,1]) = [0, \sup c^{-1}([\alpha,1])]$ which is a closed set.

Secondly since K is lower-well-ordered $\inf c^{-1}([0,\alpha]) \in c^{-1}([0,\alpha])$.

Thus $c^{-1}([0,\alpha]) = [\inf c^{-1}([0,\alpha]), k_\infty]$ which is closed (Certainly $c^{-1}([0,\alpha])$ is not empty because there exists $\kappa \in K$ such that $c(\kappa) = 0$).

Consequently c is continuous.

Example 3.2.6

For a non totally ordered lattice it is possible that sup $E^{\alpha}_{card(\mu)} \notin E^{\alpha}_{card(\mu)}$ for $\mu \in L^X$. Consider the following lattice L :

$L = \{0, a_1, a_2, \ldots, 1\}$ where 0 and 1 are bottom and top respectively with $a_i \parallel a_j$, $a_i \vee a_j = 1$ and $a_i \wedge a_j = 0$ if $j \neq i$.

Let $X = \cup \{ A_i : i \in \mathbb{N} \}$ where A_i are disjoint sets with $| A_i | = i$.

Define $\mu : X \to L$ by $\mu(x) = a_i$ if $x \in A_i$. Clearly $| E^{a_i}_{\mu} | = i$ and thus $card_B(\mu)(\kappa) = 1$ for all $\kappa \in \mathbb{N}$ and $card_B(\mu)(\aleph_0) = 0$ since for all $\alpha \in L \setminus \{0\}$, $| E^{\alpha}_{\mu} | < \aleph_0$. Thus sup $E^1_{card_B(\mu)} = \sup \mathbb{N} = \aleph_0 \notin \mathbb{N}$.

It seems that distributivity is necessary for Blanchard's cardinals to be continuous.

Proposition 3.2.5 motivates the following definition .

Definition 3.2.7

A L–fuzzy cardinal c is a non–increasing continuous function $c : K \to L$ such that $c(0) = 1$ and there exists $\kappa \in K$ such that $c(\kappa) = 0$. Denote the set of all L–fuzzy cardinals on K by K(L).

The above definition makes sense if and only if the following proposition is true.

Proposition 3.2.8

If c is a L–fuzzy cardinal then there exists a L–fuzzy set μ such that $card_B(\mu) = c$.

Proof

We are going to construct $\mu : X \to L$ in the following way : Let $R = R_c$ if $c(1) = 1$ otherwise $R = R_c \setminus \{1\}$. For all $\alpha \in R$ let $A_{\alpha} = c^{-1}(\alpha) \setminus \{0\}$. Each A_{α} is nonempty, so let $s_{\alpha} = \sup A_{\alpha}$ and $i_{\alpha} = \inf A_{\alpha}$. Since cardinals are well–ordered, $i_{\alpha} \in A_{\alpha}$.

Since c is continuous, non–increasing and c(0)=1, we have $c^{-1}[\alpha,1] = \{0,\ldots,\kappa\}$ which is a closed set in the order topology on K. Thus $s_\alpha = \kappa$ and $s_\alpha \in A_\alpha$.

If s_α is a finite cardinal then let X_α be a set with cardinality $s_\alpha - i_\alpha + 1$.

If s_α is an infinite cardinal then let X_α be a set with cardinality s_α.

Clearly we can choose all X_α disjoint. Let $X = \cup\ \{ X_\alpha : \alpha \in R_c \}$ and define $\mu : X \to L$ by $\mu(x) = \alpha$ for all $x \in X_\alpha$. Now we shall show that $\text{card}_B(\mu) = c$.

If s_α is finite then $R_c \cap [\alpha,1] = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$.

Suppose further that $\alpha = \alpha_1$ and if $i < j$ then $\alpha_i < \alpha_j$. Consequently,

$$
\begin{aligned}
|\ \mu^{-1}([\alpha,1])\ | &= |\ \cup \{ \mu^{-1}(\alpha_i) : i = 1, \ldots, n \}\ | \\
&= |\ \cup \{ X_{\alpha_i} : i = 1, \ldots, n \}\ | \\
&= \Sigma \{\ |X_{\alpha_i}| : i = 1, \ldots, n \} \\
&= (s_{\alpha_1} - i_{\alpha_1} + 1) + (s_{\alpha_2} - i_{\alpha_2} + 1) \\
&\qquad + \cdots + (s_{\alpha_n} - i_{\alpha_n} + 1) \\
&= s_{\alpha_1} + (s_{\alpha_2} - i_{\alpha_1} + 1) + (s_{\alpha_3} - i_{\alpha_2} + 1) \\
&\qquad + \cdots + (i_{\alpha_n} - 1). \\
&= s_\alpha
\end{aligned}
$$

In the last equality we have used the fact that $i_{\alpha_n} = 1$ and $s_{\alpha_i} = i_{\alpha_{i-1}} + 1$.

If s_α is infinite then $|\ X_\alpha\ | = s_\alpha$, but also we have $|\ X_\beta\ | < s_\alpha$ for all $\beta > \alpha$ so $|\ \mu^{-1}([\alpha,1])\ | = |\ X_\alpha\ | = s_\alpha$. Thus for all $\alpha \in L$, $|\ \mu^{-1}([\alpha,1])\ | = s_\alpha$.

Finally, if $\kappa \in \text{supp}(c)\backslash\{0\}$ then since R is upper–well–ordered and c is non–increasing there exists $\beta \in R$ such that $s_\beta \geq \kappa$ and for all $\alpha \in R$ with $\alpha > \beta$, $s_\alpha < \kappa$. So $\sup\{ \alpha \in R : s_\alpha \geq \kappa \} = \beta$ and $c(\kappa) = \beta$. Thus,

$$
\begin{aligned}
\text{card}_B(\mu)(\kappa) &= \sup\{ \alpha \in L : |\ \mu^{-1}([\alpha,1])\ | \geq \kappa \} \\
&= \sup\{ \alpha \in R : s_\alpha \geq \kappa \} \\
&= \beta \\
&= c(\kappa).
\end{aligned}
$$

Clearly $\text{card}_B(\mu)(0) = 1 = c(0)$. This concludes the proof.

Note that in proposition 3.2.8 we have constructed a fuzzy set such that R_μ is upper well ordered . This gives us the following important corollary.

Corollary 3.2.9

If c is a fuzzy cardinal then there exists a fuzzy set $\mu : X \to L$ such that $\mathrm{card}_B(\mu) = c$ and $R_\mu = R_c$ is upper well ordered .

Call a fuzzy set $\mu : X \to L$ a ***box*** iff $\mu = \alpha 1_T$ for some $T \subset X$

Note that $\mathrm{card}_B(\mu) = \alpha 1_{\{0,\dots,|T|\}}$.

It is easy to see that $\mathrm{card}_B(\nu)(\kappa)$ is the " height of the highest box μ such that $|\mathrm{supp}(\mu)| = \kappa$ that can be put inside ν " . Obviously highest means sup .

We now discuss some further motivation for the cardinality function .

First define the following two fuzzy sets μ , $\nu : \mathbb{N} \to I$ by $\mu = 1_{\mathbb{N}}$ and $\nu = \frac{1}{2} 1_{\{1\}} \vee 1_{\mathbb{N}\setminus\{1\}}$. It is easy to check that $\mathrm{card}_B(\mu) = \mathrm{card}_B(\nu) = 1_{\{0,\dots,\aleph_0\}}$.

Let $f : \mathbb{N} \to \mathbb{N}$ be any bijection then clearly $f(\mu) \neq \nu$ since $f(\mu)(n) \neq \frac{1}{2}$ for all $n \in \mathbb{N}$.

(Note that in here we use the Definition 2.1 for an image of a fuzzy set)

Also it is easy to see that $|\ H^\alpha_\mu\ | = \aleph_0 = |\ H^\alpha_\nu\ |$ for all $\alpha \in [0,1)$. So $\mu \sim_B \nu$.

So there is no bijection between these two sets. Is this in contradiction with what is required in the fuzzy cardinality theory ? We say no. If we have a look at these two fuzzy sets clearly $1_{\mathbb{N}\setminus\{1\}}$ and $1_{\mathbb{N}}$ should have the same cardinality and from intuitive point of view addition of a single point of certainty $\frac{1}{2}$ should not change the cardinality. Is there maybe some other form of bijection between fuzzy sets (not Zadeh type) if and only if $\mathrm{card}_B(\mu) = \mathrm{card}_B(\nu)$?

This would lead us to the following formulation of such a bijection .

$F : L^X \to L^Y$ is an bijection if and only if for all $\mu \in L^X$ and for all $\alpha \in L$, $|\ H^\alpha_\mu\ | = |\ H^\alpha_{F(\mu)}\ |$ .This immediately implies that F takes fuzzy points to fuzzy points and also preserves the height of fuzzy sets.

We shall now investigate the finiteness of fuzzy sets.

Definition 3.2.10 [2]

A fuzzy set $\mu \in L^X$ is ***finite*** if and only if there does not exist a proper injection, i.e. non–bijective injection, $f : \text{supp}(\mu) \to \text{supp}(\mu)$ such that $f(\mu|_{\text{supp}(\mu)}) \leq \mu|_{\text{supp}(\mu)}$.

Proposition 3.2.11 [2]

A fuzzy set $\mu \in L^X$ is finite if and only if for all $\alpha \in L\backslash\{0\}$ the set $\mu^{-1}([\alpha,1])$ is finite.

It is possible for a finite set $\mu \in L^X$ to have an infinite support. Consider $\mu : \mathbb{N} \to [0,1]$ given by $\mu(n) = \frac{1}{n}$. Clearly the support of a finite fuzzy set must be countable.

In [2] Blanchard has formulated the following statement: $\mu \in L^X$ has finite support if and only if μ is a compact element of the lattice L^X. This statement is false, in general. It is enough to consider the case where $L = [0,1]$.
For any set $\mu \in [0,1]^X\backslash\{0\}$ we have $\mu = \vee \mathcal{C}$ where $\mathcal{C} = \{ \alpha 1_x : 0 < \alpha < \mu(x) \}$.
Clearly there does not exists a finite subcollection $\mathcal{F} \subset \mathcal{C}$ such that $\vee \mathcal{F} = \mu$.
Thus $0 \in [0,1]^X$ is the only (vacuously) compact fuzzy set. In fact this statement is only true if L is upper-well-ordered. A true fuzzy equivalent version of above theorem can be obtained when we use a different notion of compactness. We look towards fuzzy topology for a definition of compactness, in order to prove that a fuzzy set μ is finite if and only if μ is compact in the discrete fuzzy topological space $(X, \mathcal{T} = L^X)$. In fact we shall use f–compactness, refer to Definition 1.2.3.

Lemma 3.2.12

If $\mu \in L^X$ is finite and $1 \in L_-$ then supp(μ) is finite.

Proof.

Since L has an order reversing involution $'$ and $1 \in L_-$, we have $\inf \{ \alpha \in L : \alpha > 0 \} = \beta > 0$. Clearly $\mu^{-1}([\beta,1]) = \text{supp}(\mu)$ which is finite.

Theorem 3.2.13

A fuzzy set $\mu : X \to L$ is finite if and only if μ is f–compact in (X, L^X).

Proof.

Suppose μ is finite. Thus for all $\alpha \in L\setminus\{0\}$, $\mu^{-1}([\alpha,1])$ is finite.

Clearly for all $\alpha \in L\setminus\{1\}$, $(\mu')^{-1}([0,\alpha])$ is finite.

Suppose $\alpha \in L$ and $\mathcal{C} \subset L^X$ such that $(\mu' \vee (\vee \mathcal{C})) \geq \alpha 1_X$. In case $\alpha \in L_-$, for each $x \in X$, such that $\mu'(x) < \alpha$, we can select one $\nu_x \in \mathcal{C}$ such that $\nu_x(x) \geq \alpha$. Since the set $(\mu')^{-1}([0,\alpha))$ is finite (by above Lemma, α can also be 1) the above collection $\mathcal{F} = \{ \nu_x \}$ is finite and $(\mu' \vee (\vee \mathcal{F})) \geq \alpha 1_X$. In case $\alpha \notin L_-$, let $\beta < \alpha$. Clearly $\beta \neq 1$. Again the set $(\mu')^{-1}([0,\beta))$ is finite, and the proof follows similarly. The converse follows by backtracking.

Definition 3.2.14

We will follow Wygralak [47] in denoting fuzzy cardinals in vector form.

If c is a fuzzy cardinal such that $c(\kappa) = 0$ for some $\kappa < \aleph_0$ then we express c by : $(\alpha_1,\alpha_2,\ldots,\alpha_{\kappa_1})$ where $\alpha_i = c(i)$ and κ_1 is the highest cardinal for which $c(\kappa_1) \neq 0$. Note that since $\alpha_0 = c(0)$ is always 1 we do not include α_0 in our notation . If $c(1) = 0$ then we write $c = 0$, i.e. c is the zero cardinal.

Example 3.2.15

Let $\mu : \{1,2,3,4,5,6,7\} \to \{0,\frac{1}{4},\frac{1}{2},1\}$ be defined as follows :

$\mu(1) = 1$, $\mu(\{2,3\}) = \frac{1}{2}$ and $\mu(\{4,5,6,7\}) = \frac{1}{4}$ then

$\mathrm{card}_B(\mu) = (1,\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$.

We will denote by $K_f(L)$ the set of all finite fuzzy cardinals, i.e. $c \in K_f(L)$ if and only if there exits finite fuzzy set μ such that $\mathrm{card}_B(\mu) = c$. It is easy to see that finite fuzzy cardinals are themselves finite fuzzy sets with $c(\aleph_0) = 0$. Furthermore, if μ is finite fuzzy set then R_μ is either a finite set or a sequence decreasing to 0.

Proposition 3.2.16

Let μ and ν be finite fuzzy sets. Then $\mathrm{card}_B(\mu) = \mathrm{card}_B(\nu)$ if and only if there exists a bijection $f : \mathrm{supp}(\mu) \to \mathrm{supp}(\nu)$ such that $f(\mu|_{\mathrm{supp}(\mu)}) = \nu|_{\mathrm{supp}(\nu)}$.

Proof.

Let μ, ν be finite and $c = \mathrm{card}_B(\mu) = \mathrm{card}_B(\nu)$. By Proposition 3.2.3 for all $\alpha \in L$ $|\ H_\mu^\alpha\ | = |\ H_\nu^\alpha\ |$. Since μ is finite, R_μ is finite or a sequence $\{\ \alpha_1,\ \alpha_2,\ \ldots\ \}$ decreasing to zero. Consequently, we must have $|\ E_\mu^\alpha\ | = |\ E_\nu^\alpha\ |$ for all $\alpha \in R_\mu = R_\nu$.
Now it is easy to see that $|\ \mu^{-1}(\alpha_n)\ | = |\ E_\mu^{\alpha_n} \setminus E_\mu^{\alpha_{n-1}}\ | = |\ E_\mu^{\alpha_n}\ | - |\ E_\mu^{\alpha_{n-1}}\ | =$ $|\ E_\nu^{\alpha_n}\ | - |\ E_\nu^{\alpha_{n-1}}\ | = |\ E_\nu^{\alpha_n} \setminus E_\nu^{\alpha_{n-1}}\ | = |\ \nu^{-1}(\alpha_n)\ |$. This proves existence of the required bijection. The converse is clear.

From the above proposition we can conclude that card_B is sufficient for finite fuzzy sets. Consequently we shall use card_B for finite fuzzy sets, and use representation given in 3.2.14.

Now we shall investigate the operations on fuzzy cardinals.

Definition 3.2.17

Let $\{c_i\}_{i\in J}$ be a collection of fuzzy cardinals then by corollary 3.2.9 for each c_i we can find $\nu_i \in X_i^L$ for some set X_i such that X_i are disjoint and $\text{card}_B(\nu_i) = c_i$. Now, define,

i) $\oplus \{ c_i : i \in J \} = \text{card}_B(\vee \{ \nu_i : i \in J \})$

ii) $\otimes \{ c_i : i \in J \} = \text{card}_B(\Pi \{ \nu_i : i \in J \})$.

Proposition 3.2.18

The operations $\oplus$ and $\otimes$ on K(L) are well defined.

Proof.

The proposition 3.2.8. allows us to find the required collection $\{\mu_i\}_{i\in J}$ of disjoint fuzzy sets for the above definition. Suppose that $\{\mu_i\}_{i\in J}$, $\{\nu_i\}_{i\in J}$ are two collections of disjoint fuzzy sets such that $\text{card}_B(\mu_i) = \text{card}_B(\nu_i) = c_i$.

(i) To prove that $\text{card}_B(\vee \mu_i) = \text{card}_B(\vee \nu_i)$, by Proposition 3.2.3., it is enough to prove that $| H^{\alpha}_{\vee \mu_i} | = | H^{\alpha}_{\vee \nu_i} |$ for all $\alpha \in L$.

Using various properties of $| \cdot |$, $H^{\alpha}_{(\cdot)}$ and Proposition 3.2.3., we have,

$| H^{\alpha}_{\vee \mu_i} | = | \cup H^{\alpha}_{\mu_i} | = \Sigma | H^{\alpha}_{\mu_i} | = \Sigma | H^{\alpha}_{\nu_i} | = | \cup H^{\alpha}_{\nu_i} | = | H^{\alpha}_{\vee \nu_i} |$.

(ii) Let $\beta = \text{card}_B(\Pi \nu_i)(\kappa)$. If $\beta \in L_-$ then $E^{\beta}_{\nu_i} = H^{\beta}_{\nu_i}$, $E^{\beta}_{\mu_i} = H^{\beta}_{\mu_i}$ and $| E^{\beta}_{\Pi\nu_i} | \geq \kappa$. Consequently by Proposition 3.2.3.

$| E^{\beta}_{\Pi\mu_i} | = | \Pi E^{\beta}_{\mu_i} | = \Pi | E^{\beta}_{\mu_i} | = \Pi | E^{\beta}_{\nu_i} | = | \Pi E^{\beta}_{\nu_i} | = | E^{\beta}_{\Pi\nu_i} | \geq \kappa$.

Thus $\text{card}_B(\Pi \mu_i)(\kappa) \geq \beta$. By symmetry $\text{card}_B(\Pi \mu_i)(\kappa) = \text{card}_B(\Pi \nu_i)(\kappa)$.

If $\beta \notin L_-$ then for all $\alpha < \beta$, there exists $\gamma \in L$, such that $\alpha < \gamma < \beta$.

Thus $| E^{\alpha}_{\Pi\mu_i} | = \Pi | E^{\alpha}_{\mu_i} | \geq \Pi | H^{\alpha}_{\mu_i} | = \Pi | H^{\alpha}_{\nu_i} | \geq \Pi | E^{\gamma}_{\nu_i} | = | E^{\gamma}_{\Pi\nu_i} | \geq \kappa$, since $\text{card}_B(\Pi\, \nu_i)(\kappa) = \beta$. Thus for all $\alpha < \beta$, $|\, E^{\alpha}_{\Pi\mu_i} \,| \geq \kappa$.

Hence $\text{card}_B(\Pi\, \mu_i)(\kappa) \geq \beta$. By symmetry $\text{card}_B(\Pi\, \mu_i)(\kappa) = \text{card}_B(\Pi\, \nu_i)(\kappa)$.

Theorem 3.2.19

If $\{c_i\}_{i\in J}$ is a collection of fuzzy cardinals then

i) $[\oplus\, c_i](\kappa) = \sup\{ \inf\{ c_i(\kappa_i) : i \in J \} : \Sigma\, \kappa_i \geq \kappa \}$

ii) $[\otimes\, c_i](\kappa) = \sup\{ \inf\{ c_i(\kappa_i) : i \in J \} : \Pi\, \kappa_i \geq \kappa \}$

Proof.

Let $\{\nu_i\}_{i\in J}$ be any collection of disjoint fuzzy sets $\nu_i \in L^X$ such that $\text{card}_B(\nu_i) = c_i$.

i) Let $c(\kappa) = \sup\{ \inf\{ c_i(\kappa_i) \} : \Sigma\, \kappa_i \geq \kappa \}$

We first show that for all $\alpha \in L\backslash\{0\}$ we have $E^{\alpha}_{\oplus\, c_i} = E^{\alpha}_{c}$. We consider two cases :

If $\alpha \in L_-$ then :

($\Longrightarrow$) Let $\kappa \in E^{\alpha}_{\oplus\, c_i}$. By definition $\sup\{ \beta \in L : |\, H^{\beta}_{\vee\, \nu_i} \,| \geq \kappa \} \geq \alpha$

Since ν_i are disjoint we have $\sup\{ \beta \in L : \Sigma\, |\, H^{\beta}_{\nu_i} \,| \geq \kappa \} \geq \alpha$.

Since $\alpha \in L_-$ we must have $\Sigma\, |\, E^{\alpha}_{\nu_i} \,| \geq \kappa$. Let $\kappa_i = |\, E^{\alpha}_{\nu_i} \,|$.

Clearly $c_i(\kappa_i) \geq \alpha$ thus $\inf\{ c_i(\kappa_i) : i \in J \} \geq \alpha$ and $\Sigma\, \kappa_i \geq \kappa$.

Thus $c(\kappa) \geq \alpha$ i.e. $\kappa \in E^{\alpha}_{c}$.

($\Longleftarrow$) Let $\kappa \in E^{\alpha}_{c}$ i.e. $c(\kappa) \geq \alpha$. Since $\alpha \in L_-$ then there exists $\{\kappa_i\}_{i\in J}$ such that $\inf\{ c_i(\kappa_i) : i \in J \} \geq \alpha$ and $\Sigma\{ \kappa_i : i \in J \} \geq \kappa$. Thus for all $i \in J$ we have $c_i(\kappa_i) \geq \alpha$

Since $\alpha \in L_-$, $H^{\alpha}_{\nu_i} = E^{\alpha}_{\nu_i}$ so we must have $|\, H^{\alpha}_{\nu_i} \,| \geq \kappa_i$. Since ν_i are disjoint

$\Sigma\, |\, H^{\alpha}_{\nu_i} \,| = |\, H^{\alpha}_{\vee\, \nu_i} \,| \geq \Sigma\, \kappa_i \geq \kappa$. Thus $[\oplus\, c_i](\kappa) \geq \alpha$ i.e. $\kappa \in E^{\alpha}_{\oplus\, c_i}$.

If $\alpha \notin L_-$ then:

($\Longrightarrow$) Let $\kappa \in E^{\alpha}_{\oplus\, c_i}$ then $\sup\{ \beta \in L : |\, E^{\beta}_{\vee\, \nu_i} \,| \geq \kappa \} \geq \alpha$

Thus for all $\beta < \alpha$, $|\, E^{\beta}_{\vee\, \nu_i} \,| \geq \kappa$. Since ν_i are disjoint $\Sigma\, |\, E^{\beta}_{\nu_i} \,| \geq \kappa$

Let $\kappa_i^\beta = | E^\beta_{\nu_i} |$. Clearly $c_i(\kappa_i^\beta) \geq \beta$ and $\Sigma\, \kappa_i^\beta \geq \kappa$.

So for all $\beta < \alpha$, $\inf \{ c_i(\kappa_i^\beta) : i \in J \} \geq \beta$ and $\Sigma\, \kappa_i^\beta \geq \kappa$, thus $\kappa \in E^\alpha_c$.

($\Leftarrow$) Let $\kappa \in E^\alpha_c$ then for all $\beta < \alpha$ there exit κ_i^β such that

$\Sigma\, \kappa_i^\beta \geq \kappa$ and $\inf \{ c_i(\kappa_i^\beta) : i \in J \} \geq \beta$. Let $\gamma \in L$ such that $\beta < \gamma < \alpha$

then for all $i \in J$, $c_i(\kappa_i^\gamma) \geq \gamma$ thus for all $i \in J$, $| H^\beta_{\nu_i} | \geq \kappa_i^\gamma$ and since ν_i are disjoint

$\Sigma\, | H^\beta_{\nu_i} | = | H^\beta_{\vee\, \nu_i} |$ thus $\kappa \in E^\alpha_{\oplus\, c_i}$.

Finally we use proposition 1.1.7. (ii) to obtain the final result.

ii) The proof of this part follows exactly the same pattern as i) above.

This time we use the fact that $| E^\alpha_{\Pi\, \nu_i} | = \Pi\, | E^\alpha_{\nu_i} |$.

Definition 3.2.20 (N.Blanchard [2])

If c_1 and c_2 are two fuzzy cardinals then $c_1{}^{c_2} = \text{card}_B(\nu^\omega)$

where ν , $\omega \in L^X$ for some X such that $\text{card}_B(\nu) = c_1$ and $\text{card}_B(\omega) = c_2$.

The properties of the fuzzy sets μ^ν can be investigated further but it is not of interest to us here.

If we accept the definition of ν^ω in 1.1.2 (ix) then N.Blanchard [2] points out that if c_1 and c_2 are two finite fuzzy cardinals then $c_1{}^{c_2}$ might not be finite.
This is not desirable. We think that exponentiation of fuzzy sets should be defined as follows : Given $\mu : X \to L$ and $\nu : Y \to L$ then $\gamma = \mu^\nu : X^Y \to L$ is defined such that for all $\alpha \in L$ and $g \in {H^\alpha_\mu}^{H^\alpha_\nu}$ there exists unique $f \in H^\alpha_\gamma$ such that $f_{| H^\alpha_\nu} = g$ and conversely. Obviously in such a case μ^ν exists and is not necessarily unique .
From the construction of μ^ν we see that if we pick two different representatives of μ^ν say γ_1 and γ_2 then for all $\alpha \in L$, $| H^\alpha_{\gamma_1} | = | H^\alpha_{\gamma_2} |$ thus $\text{card}_B(\gamma_1) = \text{card}_B(\gamma_2)$. Thus $\text{card}_B(\mu^\nu)$ is independent from the choice of the representatives

but also,

(*) $|H^{\alpha}_{\mu}|^{|H^{\alpha}_{\nu}|} = |H^{\alpha}_{\mu^{\nu}}|$. Thus if c_1 and c_2 are two fuzzy cardinals then we define ${c_1}^{c_2}$ as follows let μ , ν be two fuzzy sets such that $\text{card}_B(\mu) = c_1$ and $\text{card}_B(\nu) = c_2$ then ${c_1}^{c_2} = \text{card}_B(\mu^{\nu})$. Using (*) we can proceed in the same way as in Theorem 3.2.19. to show that ${c_1}^{c_2}(\kappa) = \sup\{ c_1(\kappa_1) \wedge c_2(\kappa_2) : {\kappa_1}^{\kappa_2} \geq \kappa \}$. It is clear now that if c_1 and c_2 are finite fuzzy cardinals then ${c_1}^{c_2}$ is finite. Our definition of μ^{ν} is not very nice since μ^{ν} is not a unique fuzzy set on X^Y .

The additivity property for card was proved in [8] for sets with finite support. This is also true for arbitrary sets.

Theorem 3.2.21

If $\mu, \nu \in \dot{L}^X$ then $\text{card}_B(\mu) \oplus \text{card}_B(\nu) = \text{card}_B(\mu \vee \nu) \oplus \text{card}_B(\mu \wedge \nu)$

Proof.

Clearly we can choose two fuzzy sets μ_1 and ν_1 such that $c_1 = \text{card}_B(\mu_1) = \text{card}_B(\mu)$, $c_2 = \text{card}_B(\nu_1) = \text{card}_B(\nu)$ and $\mu_1 \wedge \nu_1 = 0$. By Proposition 3.2.3, $|H^{\alpha}_{\mu_1}| = |H^{\alpha}_{\mu}|$ and $|H^{\alpha}_{\nu_1}| = |H^{\alpha}_{\nu}|$ for all $\alpha \in L$. Consequently,

$$
\begin{aligned}
[\, \text{card}_B(\mu) \oplus \text{card}_B(\nu) \,](\kappa) \\
&= (c_1 \oplus c_2)(\kappa) \\
&= \text{card}_B(\mu_1 \vee \nu_1)(\kappa) \\
&= \sup\{ \alpha \in L : |H^{\alpha}_{\mu_1 \vee \nu_1}| \geq \kappa \} \\
&= \sup\{ \alpha \in L : |H^{\alpha}_{\mu_1} \cup H^{\alpha}_{\nu_1}| \geq \kappa \} \\
&= \sup\{ \alpha \in L : |H^{\alpha}_{\mu_1}| + |H^{\alpha}_{\nu_1}| \geq \kappa \} \\
&= \sup\{ \alpha \in L : |H^{\alpha}_{\mu}| + |H^{\alpha}_{\nu}| \geq \kappa \} \\
&= \sup\{ \alpha \in L : |H^{\alpha}_{\mu} \cap H^{\alpha}_{\nu}| + |H^{\alpha}_{\mu} \cup H^{\alpha}_{\nu}| \geq \kappa \} \\
&= \sup\{ \alpha \in L : |H^{\alpha}_{\mu \wedge \nu}| + |H^{\alpha}_{\mu \vee \nu}| \geq \kappa \}
\end{aligned}
$$

On the other hand, we can choose two fuzzy sets ω and σ such that

$c_{11} = \text{card}_B(\mu \wedge \nu) = \text{card}_B(\omega)$, $c_{22} = \text{card}_B(\mu \vee \nu) = \text{card}_B(\sigma)$ and

for all $\alpha \in L$, $|H^{\alpha}_{\omega}| = |H^{\alpha}_{\mu \wedge \nu}|$ and $|H^{\alpha}_{\sigma}| = |H^{\alpha}_{\mu \vee \nu}|$. Thus,

$$\begin{aligned}
[\,\text{card}_B(\mu \wedge \nu) \oplus \text{card}_B(\mu \vee \nu)\,](\kappa) \\
&= (\,c_{11} \oplus c_{22}\,)(\kappa) \\
&= \text{card}_B(\,\omega \vee \sigma\,)(\kappa) \\
&= \sup\{\,\alpha \in L : |H^{\alpha}_{\omega \vee \sigma}| \geq \kappa\,\} \\
&= \sup\{\,\alpha \in L : |H^{\alpha}_{\omega}| + |H^{\alpha}_{\sigma}| \geq \kappa\,\} \\
&= \sup\{\,\alpha \in L : |H^{\alpha}_{\mu \wedge \nu}| + |H^{\alpha}_{\mu \vee \nu}| \geq \kappa\,\}
\end{aligned}$$

It is important to note that the above theorem follows from Theorem 3.1.3 since,

$$\text{card}_B(\mu)(\kappa) = \sup\{\,\alpha \in L : \sum_{\gamma \geq \alpha} \text{card}_G(\mu)(\gamma) \geq \kappa\,\}.$$

<u>Proposition 3.2.22</u>

If $\{c_i\}_{i \in J}$ is a collection of fuzzy cardinals then :

$[\oplus\{\,c_i : i \in J\,\}](\kappa) = \sup\{\,\inf\{\,c_i(\kappa_i) : i \in J\,\} : \Sigma\{\,\kappa_i : i \in J\,\} = \kappa\,\}$

<u>Proof.</u>

Firstly, we note that, $\{\,\inf\{\,c_i(\kappa_i) : i \in J\,\} : \Sigma\,\kappa_i = \kappa\,\}$ is a subset of $\{\,\inf\{\,c_i(\kappa_i) : i \in J\,\} : \Sigma\,\kappa_i \geq \kappa\,\}$. Thus $\sup\{\,\inf\{\,c_i(\kappa_i) : i \in J\,\} : \Sigma\,\kappa_i = \kappa\,\} \leq \sup\{\,\inf\{\,c_i(\kappa_i) : i \in J\,\} : \Sigma\,\kappa_i \geq \kappa\,\}$.

Secondly, for each collection $\{\kappa_i\}_{i \in J}$ such that $\Sigma\,\kappa_i \geq \kappa$ we can choose a collection $\{\bar{\kappa}_i\}_{i \in J}$ such that for all $i \in J$, $\bar{\kappa}_i \leq \kappa_i$ and $\Sigma\,\bar{\kappa}_i = \kappa$. To see this we consider a few of cases. Let $J' = \{\,i \in J : \kappa_i > 0\,\}$. If there exists $i \in J'$ such that $\kappa_i \geq \kappa$ then let $\bar{\kappa}_i = \kappa$ and for all $j \neq i$ let $\bar{\kappa}_j = 0$. Then $\Sigma\,\bar{\kappa}_i = \kappa$ and for all $i \in J$, $\bar{\kappa}_i \leq \kappa_i$.

If there does not exist $i \in J'$ and $\kappa \geq \aleph_0$ then $|J'| \geq \kappa$. In this case let $J^* \subset J'$ such that $|J^*| = \kappa$ and $\bar{\kappa}_i = 1$ for all $i \in J^*$ and $\bar{\kappa}_i = 0$ otherwise. If κ is finite then

J′ must be finite. To construct the necessary $\bar{\kappa}_i$'s we can repeatedly take one from κ_i's until sum of them equals κ. Since c_i are non–increasing we have $c_i(\bar{\kappa}_i) \geq c_i(\kappa)$. So, sup { inf { $c_i(\kappa_i)$: i ∈ J } : $\Sigma\, \kappa_i = \kappa$ } ≥ sup{ inf { $c_i(\kappa_i)$: i ∈ J } : $\Sigma\, \kappa_i \geq \kappa$ }. This concludes the proof.

There is no equivalent of Proposition 3.2.22 for multiplication of fuzzy cardinals. To see this let L = [0,1], $c_1 = (1, \frac{1}{2}, \frac{1}{3})$ and $c_2 = (1, \frac{1}{4})$.
Then $c_1 \otimes c_2 = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Let $c \in L^K$ be given by,
$c(\kappa) = \sup\{ c_1(\kappa_1) \wedge c_2(\kappa_2) : \kappa_1\, \kappa_2 = \kappa \}$ then $c = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, \frac{1}{4})$.
So if "≥" is replaced by "=" in the definition of multiplication of fuzzy cardinals then it might not necessarily result in a fuzzy cardinal as is the case with c above. However we have the following restricted result.

Proposition 3.2.23

If k is an infinite cardinal and c_1 and c_2 are two fuzzy cardinals then,

$$[\, c_1 \otimes c_2 \,](\kappa) = \sup \{\, c_1(\kappa_1) \wedge c_2(\kappa_2) : \kappa_1\, \kappa_2 = \kappa \,\}$$

Proof.

By Proposition 3.2.19, $[\, c_1 \otimes c_2 \,](\kappa) = \sup \{\, c_1(\kappa_1) \wedge c_2(\kappa_2) : \kappa_1\, \kappa_2 \geq \kappa \,\}$.
Given $\kappa_1\, \kappa_2 \geq \kappa$, since κ is infinite we must have $\kappa_1 \geq \kappa$ or $\kappa_2 \geq \kappa$.
Without loss of generality we may assume that $\kappa_1 \geq \kappa$. We can write $\kappa_1 = \kappa + \bar{\kappa}_1$ and $\kappa_2 = 1 + \bar{\kappa}_2$. Thus we have $c_1(\kappa_1) \wedge c_2(\kappa_2) = c_1(\kappa + \bar{\kappa}_1) \wedge c_2(1 + \bar{\kappa}_2) \leq c_1(\kappa) \wedge c_2(1)$ because c_1 and c_2 are non–increasing functions. This gives the result because $\kappa\, 1 = \kappa$.

Corollary 3.2.24

If κ is an infinite cardinal then $[c_1 \otimes c_2](\kappa) = [c_1(\kappa) \wedge c_1(1)] \vee [c_2(\kappa) \wedge c_2(1)]$.

Proposition 3.2.25

If κ is an infinite cardinal then $[c_1 \oplus c_2](\kappa) = c_1(\kappa) \vee c_2(\kappa)$.

Proof.

From Proposition 3.2.22 we have $[c_1 \oplus c_2](\kappa) = \sup\{ c_1(\kappa_1) \wedge c_2(\kappa_2) : \kappa_1 + \kappa_2 = \kappa \}$.

Since κ is infinite if $\kappa_1 + \kappa_2 = \kappa$ then $\kappa_1 = \kappa$ or $\kappa_2 = \kappa$. Consequently,

if $\kappa_1 = \kappa$ then, $c_1(\kappa) = c_1(\kappa) \wedge c_2(0) \geq c_1(\kappa) \wedge c_2(\kappa_2)$ and

if $\kappa_2 = \kappa$ then, $c_2(\kappa) = c_1(0) \wedge c_2(\kappa) \geq c_1(\kappa_1) \wedge c_2(\kappa)$.

Thus $[c_1 \oplus c_2](\kappa) = c_1(\kappa) \vee c_2(\kappa)$.

An equivalent of the Proposition 3.2.25 for $\otimes$ is not true. The reason for this is that $c_1(1)$ is not necessarily equal to $c_2(1)$. Consider two infinite cardinals c_1 and c_2 defined as follows: $c_1(1) = \frac{1}{2}$, $c_1(\kappa) = \frac{1}{3}$ for $\kappa \in \{2, 3, ..., \aleph_0\}$, $c_1(\kappa) = 0$ otherwise; $c_2(1) = \frac{1}{4}$, $c_2(\kappa) = \frac{1}{5}$ for $\kappa \in \{2, 3, ..., \aleph_0\}$, $c_2(\kappa) = 0$ otherwise. Then,

$[c_1 \otimes c_2](\aleph_0) = \sup\{ \frac{1}{2} \wedge \frac{1}{5}, \frac{1}{4} \wedge \frac{1}{3} \} = \frac{1}{4} \neq \frac{1}{5} \vee \frac{1}{3} = c_1(\aleph_0) \vee c_2(\aleph_0)$.

Definition 3.2.26

If $\{c_i\}_{i \in J}$ is a collection of fuzzy cardinals then we define a fuzzy cardinal

SUP $\{ c_i : i \in J \}$ by

SUP $\{ c_i : i \in J \}](\kappa) = \sup \{ \inf \{ c_i(\kappa_i) : i \in J \} : \sup \{\kappa_i\} \geq \kappa \}$.

Proposition 3.2.27

$[\,\mathrm{SUP}\ \{ c_1 , c_2 \}\,](\kappa) = c_1(\kappa) \vee c_2(\kappa)$.

Proof.

Fuzzy cardinals are non–increasing, so let $\kappa_1 = \kappa_2 = \kappa$ to obtain the result.

Example 3.2.28

Note that if $\{c_i\}_{i \in J}$ is an infinite collection of fuzzy cardinals then not necessarily

$[\,\mathrm{SUP}\,\{\, c_i : i \in J \,\}\,](\kappa) = \sup \{\, c_i(\kappa) : i \in J \,\}$.

Let $J = \mathbb{N}$ and $c_i = 1_{\{0,\dots,i\}}$ then $\vee \{\, c_i : i \in J \,\} = 1_{\mathbb{N}}$ and thus $\sup E^1_{1_{\mathbb{N}}} = \aleph_0 \notin E^1_{1_{\mathbb{N}}}$ and so $1_{\mathbb{N}}$ is not a fuzzy cardinal.

On the other hand $[\,\mathrm{SUP}\,\{\, c_i : i \in J \,\}\,] = 1_{\{0,\dots,\aleph_0\}}$ which is a fuzzy cardinal.

The above is equivalent of the following statement in the crisp setup :

$$\mathrm{SUP}\ \mathbb{N} = \aleph_0.$$

In the crisp case we can construct supremum of a collection $\{\kappa_i\}_{i \in J}$ of cardinals in the following fashion. Since cardinals are well–ordered we can well order J such that if $i,j \in J$ with $i \leq j$ then $\kappa_i \leq \kappa_j$. Now we can choose a collection of sets $\{A_i\}_{i \in J}$ such that $|A_i| = \kappa_i$ and if $\kappa_i \leq \kappa_j$ then $A_i \subset A_j$. Finally define,

$$\mathrm{SUP}\,\{\, \kappa_i : i \in J \,\} = |\, \cup A_i \,|.$$

Thus from the definition of SUP we have a partial ordering on the set of fuzzy cardinals on K. This ordering is clearly not a well ordering in the crisp sense.

3.3. A NEW APPROACH TO CARDINALITY OF FUZZY SETS

From Proposition 3.2.16 we see that Blanchard's cardinality is sufficient for finite fuzzy sets. However if we consider Blanchard's cardinality of infinite fuzzy sets we see how forgetful it is, i.e. two very much different infinite fuzzy sets can have exactly the same Blanchard cardinality. For instance consider two infinite fuzzy sets $\mu, \nu : [1, \infty) \to [0,1]$ given by, $\mu(t) = 1 - \frac{1}{t}$ and $\nu(t) = 1$. It is easy to check that $\text{card}_B(\mu) = \text{card}_B(\nu) = 1_{\{0, \ldots, \aleph_1\}}$, despite the fact that even $R_\mu \neq R_\nu$. Clearly the fuzzy sets μ and ν are markedly different. On the other hand one could argue that the fuzzy set μ should have the same cardinality as ν as follows: the set μ contains fuzzy sets, $t1_{[1/(1-t), \infty)}$ for all $t \in [0,1)$; each one having fuzzy cardinality $1_0 \vee t1_{\{0, \ldots, \aleph_1\}}$, consequently the fuzzy set μ should have cardinality equal to $1_{\{0, \ldots, \aleph_1\}}$.

In what follows we aim to establish a cardinality for fuzzy sets that gives a more accurate representation. Before we introduce our definition of cardinality we discuss what has lead us to it.

When we consider the Hutton's Urysohn Lemma for normal fuzzy topological spaces which uses the fuzzy unit interval I(L), see [16], we observe the following similarities between I(L) and K(L): a fuzzy real number ν is a non–increasing function from I to L such that for all $t < 0$, $\nu(t) = 1$ and for all $t > 1$, $\nu(t) = 0$ ($\nu(t)$ can be interpreted as the degree to which ν is larger than t).

On the other hand, a fuzzy cardinal c is a non–increasing function from cardinals to L which is eventually 0 (c(k) can be interpreted as degree to which cardinality is larger than k) and $c(0) = 1$. Hutton introduces an equivalence relation on the fuzzy reals identifying those which have the same left and right limits at all points in [0,1]. In our case such a treatment is not necessary since Blanchard's cardinals are already continuous with respect to the order topologies.

Finally the fuzzy topology on the fuzzy unit interval is a collection of fuzzy sets on it. Consequently we define a new fuzzy cardinal as a fuzzy set on Blanchard's fuzzy cardinals.

This is in line with our conclusion from Chapter 2 and Appendix which states that we should expand the range of functors between fuzzy objects rather than complicate them. Recall definition of $\mu\langle\!\langle\cdot\rangle\!\rangle$ from section 3.1.

For each $\mu \in L^X$, define the function $|\cdot|_\mu : 2^X \to K(L)$ by

$$|\cdot|_\mu(A) = \mathrm{card}_B(\mu_{|A}).$$

With the use of the injection Ψ from the Section 3.1 this function extends to the whole of L^X, denoted by Card : $L^X \to L^{K(L)}$, given in the following

Definition 3.3.1

$\mathrm{Card}(\mu)(c) = |\cdot|_\mu(\mu)(c) = \sup\{\ \mu\langle\!\langle A\rangle\!\rangle : \mathrm{card}_B(\mu_{|A}) = c\}$

For convenience sake let us denote a fuzzy cardinal of the form $1_0 \vee \alpha 1_{\{1,\ \ldots,\kappa\}}$ by $c(\kappa,\alpha)$. Define the functions $\kappa : K(L) \to K$ and $m : K(L) \to L$ by letting $\kappa(c) = \sup\{\mathrm{supp}(c)\}$ and $m(c) = c(\kappa(c))$.

Lemma 3.3.2

If μ is a fuzzy set then $\mu\langle\!\langle\mathrm{supp}(\mu)\rangle\!\rangle \leq m(\mathrm{card}_B(\mu))$.

Proof

Let $\beta = \mu\langle\!\langle\mathrm{supp}(\mu)\rangle\!\rangle$ and $\kappa = |\ \mathrm{supp}(\mu)\ |$. Thus for all $x \in \mathrm{supp}(\mu)$ we have $\mu(x) \geq \beta$. Clearly $|\ \mu^{-1}([\gamma,1])\ | = \kappa$ for all $0 < \gamma \leq \beta$. Thus $\mathrm{card}_B(\mu)(\kappa) \geq \beta$ and for all $\kappa_1 > \kappa$, $\mathrm{card}_B(\mu)(\kappa_1) = 0$. Consequently $\kappa(\mathrm{card}_B(\mu)) = \kappa$ and $m(\mathrm{card}_B(\mu)) \geq \beta = \mu\langle\!\langle\mathrm{supp}(\mu)\rangle\!\rangle$.

Proposition 3.3.3

If μ is a fuzzy set and c is a fuzzy cardinal then,

$$\mathrm{Card}(\mu)(c) = \begin{cases} m(c) & \text{if there exists } A \subset \mathrm{supp}(\mu) \text{ such that } \mathrm{card}_B(\mu_{|A}) = c \\ 0 & \text{otherwise} \end{cases}$$

Proof.

Assume that $L_- = \emptyset$. In case $L_- \neq \emptyset$ the proof is similar.

Suppose $\mathrm{Card}(\mu)(c) > m(c)$. This means that there exists $A \subset \mathrm{supp}(\mu)$ such that $\mu_{|A}\langle\langle A\rangle\rangle > m(c)$ and $\mathrm{card}_B(\mu_{|A}) = c$. Since $\mathrm{supp}(\mu_{|A}) = A$ we have a contradiction with Lemma 3.3.2. Consequently, $\mathrm{Card}(\mu)(c) \leq m(c)$.

Suppose that there exists $A \subset \mathrm{supp}(\mu)$ such that $\mathrm{card}_B(\mu_{|A}) = c$. Let $\nu = \mu_{|A}$. Then for all $\gamma < m(c)$, $|H_\nu^\gamma| = \kappa(c)$, $\mathrm{card}_B(\nu_{|H_\nu^\gamma}) = c$ and $\nu\langle\langle H_\nu^\gamma\rangle\rangle \geq \gamma$. Thus $\mathrm{Card}(\mu)(c) \geq m(c)$. Consequently $\mathrm{Card}(\mu)(c) = m(c)$.

Example 3.3.4

Let $\mu : \mathbb{N} \to [0,1]$ be a fuzzy set given by $\mu(n) = \frac{1}{n}$.

Clearly $c = \mathrm{card}_B(\mu) = 1_0 \vee (\bigvee_{i=1}^{\infty} \frac{1}{i} 1_{\{1, \ldots, i\}})$.

Note that $\mathrm{Card}(\mu)(c) = m(c) = 0$. This is not so disturbing if we observe that $\mathrm{Card}(\mu)(c(n,\frac{1}{n})) = \frac{1}{n}$.

Proposition 3.3.5

If μ is a fuzzy set and κ a cardinal such that $\mathrm{card}_B(\mu)(\kappa) > 0$ then $m(\mathrm{card}_B(\mu) \wedge 1_{\{0,\ldots,\kappa\}}) = \mathrm{card}_B(\mu)(\kappa)$.

Proof

Clearly $\mathrm{card}_B(\mu) \wedge 1_{\{0,\ldots,\kappa\}}$ is Blanchard cardinal. Since $\mathrm{card}_B(\mu)(\kappa) > 0$ we have $\{0,\ldots,\kappa\} \subset \mathrm{supp}(\mathrm{card}_B(\mu))$. Consequently $\kappa(\mathrm{card}_B(\mu) \wedge 1_{\{0,\ldots,\kappa\}}) = \kappa$. Finally, $m(\mathrm{card}_B(\mu) \wedge 1_{\{0,\ldots,\kappa\}}) = (\mathrm{card}_B(\mu) \wedge 1_{\{0,\ldots,\kappa\}})(\kappa) = \mathrm{card}_B(\mu)(\kappa)$.

Theorem 3.3.6

Let μ and ν be two fuzzy sets. Then,

(i) If $\text{card}_G(\mu) = \text{card}_G(\nu)$ then $\text{Card}(\mu) = \text{Card}(\nu)$.

(ii) If $\text{Card}(\mu) = \text{Card}(\nu)$ then $\text{card}_B(\mu) = \text{card}_B(\nu)$

Proof

(i) Since $\text{card}_G(\mu) = \text{card}_G(\nu)$ there exists a bijection $f : \text{supp}(\mu) \to \text{supp}(\nu)$ such that $f(\mu_{|\,\text{supp}(\mu)}) = \nu_{|\,\text{supp}(\nu)}$. Consequently,

$$\begin{aligned}\text{Card}(\mu)(c) &= \sup\{\ \mu\langle\!\langle A\rangle\!\rangle : \text{card}_B(\mu_{|\,A}) = c\ \} \\ &= \sup\{\ \mu\langle\!\langle A\rangle\!\rangle : \text{card}_B(\mu_{|\,A}) = c \text{ and } A \subset \text{supp}(\mu)\ \} \\ &= \sup\{\ \nu\langle\!\langle f(A)\rangle\!\rangle : \text{card}_B(\nu_{|\,f(A)}) = c\ \} \\ &= \sup\{\ \nu\langle\!\langle C\rangle\!\rangle : \text{card}_B(\nu_{|\,C}) = c\ \} \\ &= \text{Card}(\nu)(c).\end{aligned}$$

(ii) Assume that $L_- = \emptyset$. Suppose that for some cardinal κ $\text{card}_B(\mu)(\kappa) \neq \text{card}_B(\nu)(\kappa)$. Without loss of generality we may assume that $\text{card}_B(\mu)(\kappa) < \text{card}_B(\nu)(\kappa)$. From the definition of card_B we see that there must exist $\gamma, \delta \in L$ such that $\gamma > \delta$, $|\ H_\nu^\gamma\ | \geq \kappa$ and $|\ H_\mu^\delta\ | < \kappa$. Let $c = \text{card}_B(\nu) \wedge 1_{\{0,\ldots,\kappa\}}$.
Since $\nu\langle\!\langle H_\nu^\gamma\rangle\!\rangle \geq \gamma$ and $\text{card}_B(\nu_{|\,H_\nu^\gamma}) = \text{card}_B(\nu) \wedge 1_{\{0,\ldots,\kappa\}} = c$ we have $\text{Card}(\nu)(c) \geq \gamma$. Because $|\ H_\mu^\delta\ | < \kappa$, there does not exists $A \in \text{supp}(\mu)$ such that $\text{card}_B(\mu_{|\,A}) = c$. Consequently $\text{Card}(\mu)(c) = 0$. In case $L_- \neq \emptyset$ the argument is similar.

Corollary 3.3.7

In case μ is finite Card, card_G and card_B are equivalent.

Proof

Apply Proposition 3.2.16 and Theorem 3.3.6.

Example 3.3.8

Let the fuzzy sets μ, ν : $[0, 2] \to [0,1]$ be given as follows,

$$\mu(t) = \begin{cases} t & \text{if } t \in [0,1] \\ 1 & \text{if } t \in \mathbb{Q} \cap [1,2] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu = \begin{cases} t & \text{if } t \in [0, 1] \\ 1 & \text{otherwise} \end{cases}$$

One can check that $\mathrm{Card}(\mu) = \mathrm{Card}(\nu)$ and $\mathrm{card}_G(\mu)(1) = \aleph_0 \neq |\mathbb{R}| = \mathrm{card}_G(\nu)(1)$.

Example 3.3.9

Let the fuzzy sets μ, ν : $[0,1] \to [0,1]$ be given by $\mu(t) = t$ and $\nu(t) = 1$.

Clearly $\mathrm{card}_B(\mu) = \mathrm{card}_B(\nu) = c(|\mathbb{R}|,1) = 1_{\{0,\ldots,|\mathbb{R}|\}}$.

However, $\mathrm{Card}(\mu)(c(1,1/2)) = 1/2 \neq 0 = \mathrm{Card}(\nu)(c(1,1/2))$.

Lemma 3.3.10

Any cardinal $c \in K(L)$ can be written as

$$c = \bigoplus_{\gamma \in R_c} c(\kappa_\gamma, \gamma)$$

where for all $\gamma \in R_c$,

$$\kappa_\gamma = \begin{cases} \sup c^{-1}(\gamma) & \text{if } \sup c^{-1}(\gamma) \text{ is infinite} \\ \sup c^{-1}(\gamma) - \inf c^{-1}(\gamma) + 1 & \text{otherwise} \end{cases}.$$

Proof

By Proposition 3.2.8 and Corollary 3.2.9 there exists a fuzzy set $\mu : X \to L$ such that $c = \mathrm{card}_B(\mu)$ and $R_\mu = R_c$ is upper well–ordered. For all $\gamma \in R_\mu$ let $X_\gamma = \mu^{-1}(\gamma)$ and $\mu_\gamma = \mu_{|X_\gamma}$. Clearly X_γ are disjoint and so are μ_γ, and $\mathrm{card}_B(\mu_\gamma) = c(\kappa_\gamma, \gamma)$.

Since μ_γ are disjoint and $\mu = \bigvee_{\gamma \in R_c} \mu_\gamma$, we have

$$\mathrm{card}_B(\mu) = \mathrm{card}_B\Big(\bigvee_{\gamma \in R_c} \mu_\gamma\Big) = \bigoplus_{\gamma \in R_c} \mathrm{card}_B(\mu_\gamma) = \bigoplus_{\gamma \in R_c} c(\kappa_\gamma, \gamma).$$

This proves this proposition.

Lemma 3.3.11

If c, $c(\kappa,\alpha) \in K(L)$ such that $\kappa(c) < \kappa$ and $m(c) > \alpha > 0$ then

$$\mathrm{Card}(\mu)(\, c \oplus c(\kappa,\alpha)\,) = \mathrm{Card}(\mu)(c) \wedge \mathrm{Card}(\mu)(c(\kappa,\alpha)).$$

Proof

Let $\mathrm{Card}(\mu)(c) > 0$ and $\mathrm{Card}(\mu)(c(\kappa,\alpha)) > 0$. Then by Proposition 3.3.3 there exists sets A_1, $A_2 \subset \mathrm{supp}(\mu)$ such that $\mathrm{card}_B(\mu|_{A_1}) = c$, $\mathrm{card}_B(\mu|_{A_2}) = c(\kappa,\alpha)$, $\mathrm{Card}(\mu)(c) = m(c)$ and $\mathrm{Card}(\mu)(c(\kappa,\alpha)) = \alpha$. Let $B_1 = A_1 \cap H^{\alpha}_{\mu}$. Since $m(c) > \alpha$, $\mathrm{card}_B(\mu|_{B_1}) = c$. Clearly for all $x \in \mathrm{supp}(\mu|_{A_2})$, $\mu(x) \leq \alpha$. Also, B_1 and A_2 are disjoint, so $\mu|_{B_1}$ and $\mu|_{A_2}$ are disjoint. Thus $c \oplus c(\kappa,\alpha) =$
$\mathrm{card}_B(\mu|_{B_1}) \oplus \mathrm{card}_B(\mu|_{A_2}) = \mathrm{card}_B(\,\mu|_{B_1} \vee \mu|_{A_2}\,) = \mathrm{card}_B(\mu|_{(B_1 \cup A_2)})$.
Since, $m(\, c \oplus c(\kappa,\alpha)\,) = \alpha$ and $\mathrm{card}_B(\mu|_{(B_1 \cup A_2)}) = c \oplus c(\kappa,\alpha)$ by Proposition 3.3.3 we have $\mathrm{Card}(\, c \oplus c(\kappa,\alpha)\,) = \alpha = m(c) \wedge \alpha = \mathrm{Card}(\mu)(c) \wedge \mathrm{Card}(\mu)(c(\kappa,\alpha))$.
In the case $\mathrm{Card}(\mu)(c) = 0$, since $m(c) > 0$, by Proposition 3.3.3 there does not exist $A \subset \mathrm{supp}(\mu)$ such that $\mathrm{card}_B(\mu|_A) = c$. Consequently there does not exist $A \subset \mathrm{supp}(\mu)$ such that $\mathrm{card}_B(\mu|_A) = c \oplus c(\kappa,\alpha)$. Finally Proposition 3.3.3 implies that $\mathrm{Card}(\mu)(c \oplus c(\kappa,\alpha)) = 0 = \mathrm{Card}(\mu)(c) \wedge \mathrm{Card}(\mu)(c(\kappa,\alpha))$.
We proceed similarly in the case $\mathrm{Card}(\mu)(c(\kappa,\alpha)) = 0$.

Theorem 3.3.12

$\mathrm{Card}(\mu)(c) = \bigwedge_{\gamma \in R_c} \mathrm{Card}(\mu)(\, c(\kappa_\gamma,\gamma)\,)$ where κ_γ is as in Lemma 3.3.10.

Proof

Let $\beta \in R_c$ and $c_\beta = \bigoplus_{\gamma \geq \beta} c(\kappa_\gamma,\gamma)$. Observe that $m(c_\beta) = \beta$ and $\kappa(c_\beta) = \kappa_\beta$. Since R_c is upper–well–ordered we can find the predecessor α of β. Then c_β and $c(\kappa_\alpha,\alpha)$ satisfy the conditions of Lemma 3.3.11. Giving that $\mathrm{Card}(\mu)(\, c_\beta \oplus c(\kappa_\alpha,\alpha)\,) =$ $\mathrm{Card}(\mu)(c_\beta) \wedge \mathrm{Card}(\mu)(c(\kappa_\alpha,\alpha))$. By Lemma 3.3.10 we can write $c = \bigoplus_{\gamma \in R_c} c(\kappa_\gamma,\gamma)$.
Since R_c is upper well–ordered this theorem follows by transfinite induction.

Note that the above theorem states that $\mathrm{Card}(\mu)$ is entirely determined on the cardinals of the form $c(\kappa,\gamma)$.

We mention the following easily proved statement in passing.

Proposition 3.3.13

Suppose R_{μ} is upper–well–ordered. Then if $\mathrm{Card}(\mu) = \mathrm{Card}(\nu)$ then $\mathrm{card}_G(\mu) = \mathrm{card}_G(\nu)$.

Denote by $\mathcal{K}(L)$ the set of all $C \in L^{K(L)}$, such that there exists a fuzzy set μ with $\mathrm{Card}(\mu) = C$. Clearly $\mathcal{K}(L)$ can be endowed with operations $\oplus$ and $\otimes$ in the usual fashion.

Definition 3.3.14

Let $C_1, C_2 \in \mathcal{K}(L)$ and μ, ν be any two disjoint fuzzy sets such that $\mathrm{Card}(\mu) = C_1$ and $\mathrm{Card}(\nu) = C_2$. Then,

(i) $C_1 \oplus C_2 = \mathrm{Card}(\mu \vee \nu)$,

(ii) $C_1 \otimes C_2 = \mathrm{Card}(\mu \times \nu)$.

Proposition 3.3.15

The operation $\oplus$ and $\otimes$ on $\mathcal{K}(L)$ are well defined.

Proof

Let μ_1 and ν_1 be two disjoint fuzzy sets such that $\mathrm{Card}(\mu_1) = \mathrm{Card}(\mu)$, $\mathrm{Card}(\nu_1) = \mathrm{Card}(\nu)$ and $c \in K(L)$. (By Theorem 3.3.12 it is sufficient to prove the assertion on cardinals of the form $c(\kappa,\alpha)$.) Suppose that $\mathrm{Card}(\mu \vee \nu)(c) = \alpha > 0$. Then by Proposition 3.3.3 there exists $A \subset \mathrm{supp}(\mu \vee \nu)$ such that $c = \mathrm{card}_B((\mu \vee \nu)_{|A}) = \mathrm{card}_B(\mu_{|A} \vee \nu_{|A}) = \mathrm{card}_B(\mu_{|A}) \oplus \mathrm{card}_B(\nu_{|A}) =$

$c_\mu \oplus c_\nu$. Since $\mathrm{Card}(\mu_1)(c_\mu) = \mathrm{Card}(\mu)(c_\mu)$ and $\mathrm{Card}(\nu_1)(c_\nu) = \mathrm{Card}(\nu)(c_\nu)$ there exists $C_1 \subset \mathrm{supp}(\mu_1)$ and $C_2 \subset \mathrm{supp}(\nu_1)$ such that $\mathrm{card}_B(\mu_{1|C_1}) = c_\mu$ and $\mathrm{card}_B(\nu_{1|C_2}) = c_\nu$. Since μ_1 and ν_1 are disjoint $c = c_\mu \oplus c_\nu = \mathrm{card}_B(\mu_{1|C_1}) \oplus \mathrm{card}_B(\nu_{1|C_2}) = \mathrm{card}_B(\mu_{1|C_1} \vee \nu_{1|C_2}) = \mathrm{card}_B((\mu_1 \vee \nu_1)_{|(C_1 \cup C_2)})$.
So there exists $C = C_1 \cup C_2$ such that $\mathrm{card}_B((\mu_1 \vee \nu_1)_{|C}) = c$. The case where $\mathrm{Card}(\mu \vee \nu)(c) = 0$ is easier. This shows that $\mathrm{Card}(\mu_1 \vee \nu_1) = \mathrm{Card}(\mu \vee \nu)$, i.e. the well definition of $\oplus$ in $\mathcal{K}(L)$. We proceed similarly for $\otimes$.

The structure of $\mathcal{K}(L)$ can be investigated as in the section 3.2 for K(L). However in here we establish that our Card has the additivity property.

Recall the definition of the cartesian product of fuzzy sets. In particular if we have two fuzzy sets $\mu : X \to L$ and $\nu : Y \to L$ such that $\nu = 1_Y$ we can write $\mu \times \nu = \mu \times Y$. The fuzzy set $\mu \times Y : X \times Y \to L$ is given by $(\mu \times Y)(x,y) = \mu(x) \wedge 1_Y(y) = \mu(x)$.

Theorem 3.3.16

$\mathrm{Card}(\mu) \oplus \mathrm{Card}(\nu) = \mathrm{Card}(\mu \wedge \nu) \oplus \mathrm{Card}(\mu \vee \nu)$

Proof

By Theorem 3.1.3 we have,

$$\mathrm{card}_G(\mu) \oplus \mathrm{card}_G(\nu) = \mathrm{card}_G(\mu \wedge \nu) \oplus \mathrm{card}_G(\mu \vee \nu).$$

Since $(\mu \times \{1\}) \wedge (\nu \times \{2\}) = \emptyset$ and $((\mu \wedge \nu) \times \{1\}) \wedge ((\mu \vee \nu) \times \{2\}) = \emptyset$ it follows from the definition of the addition of fuzzy cardinals that,

$$\mathrm{card}_G((\mu \times \{1\}) \vee (\nu \times \{2\})) = \mathrm{card}_G(((\mu \wedge \nu) \times \{1\}) \vee ((\mu \vee \nu) \times \{2\})).$$

By Theorem 3.3.6 (i) we obtain,

$$\mathrm{Card}((\mu \times \{1\}) \vee (\nu \times \{2\})) = \mathrm{Card}(((\mu \wedge \nu) \times \{1\}) \vee ((\mu \vee \nu) \times \{2\})).$$

Again by the definition of the addition of fuzzy cardinals we have,

$$\mathrm{Card}(\mu) \oplus \mathrm{Card}(\nu) = \mathrm{Card}(\mu \wedge \nu) \oplus \mathrm{Card}(\mu \vee \nu).$$

4. FUZZY VECTOR SPACES

In this and the following chapter we apply fuzzy cardinals from Chapter 3 to other fuzzy objects. However the results in this chapter, for instance Theorem 4.5.7 and Theorem 4.5.10, do not require any specific cardinality as long as it has the additivity property. Throughout this chapter we are going to denote the cardinality by card which could be one of card_G, Card or card_B.

In this chapter we are mostly dealing with finite dimensional fuzzy vector spaces. Thus by Proposition 3.2.16 we see that in such cases Blanchard's cardinality is completely adequate and we use it.

This chapter is a revised version of the authors paper [27] on the fuzzy vector spaces. Now we use arbitrary fuzzy cardinals which possess the additivity property instead of the scalar cardinals as is the case in [27]. The results obtained here are more general than those in [27].

Let us preview the results in this chapter:

We define basis and dimension for a fuzzy vector space.

A class of fuzzy vector spaces having a finite fuzzy dimension is studied, and two standard results from the crisp theory are proved for this case, namely,

$\dim(\mu_1) \oplus \dim(\mu_2) = \dim(\mu_1 + \mu_2) \oplus \dim(\mu_1 \wedge \mu_2)$ and

$\dim(\mu) = \dim(f(\mu)) \oplus \dim(\mu_{|\ker f})$.

The sum of two fuzzy vector spaces is characterised under certain conditions.

4.1 INTRODUCTION

In this chapter we study purely algebraic properties of fuzzy vector spaces. The ideas presented here can easily be applied to other algebraic fuzzy objects. The concept of a basis is fundamental to the study of crisp vector spaces, in fact it gives us a very elegant representation of all crisp vector spaces. We define the concept of basis for a fuzzy vector space and show that a very wide class of fuzzy vector spaces possess it.

R. Lowen in [26] defines the dimension of a fuzzy vector space (on a finite dimensional space only) as a n–tuple; we define dimension for a fuzzy vector spaces as a fuzzy cardinal. Finally we investigate the properties of the introduced concepts.

4.2 PRELIMINARIES

If V_1 and V_2 are vector spaces and V_1 is a subspace of V_2 then we write $V_1 < V_2$. The concept of the fuzzy vector space was introduced by A.K. Katsaras and D.B. Liu in [18].

Definition 4.2.1

Fuzzy vector space μ over a vector space E is a function from a crisp vector space E to L, i.e. $\mu \in L^E$ with the property that for all $a,b \in \mathbb{R}$ and $x,y \in E$ we have $\mu(ax + by) \geq \mu(x) \wedge \mu(y)$.

Proposition 4.2.2

If μ is a fuzzy vector space over E then

i) $H_{\mu}^{\alpha} < E_{\mu}^{\alpha} < E$,

ii) $\forall\ a \in \mathbb{R} \setminus \{0\},\ \mu(ax) = \mu(x)$

iii) If u, v $\in$ E and $\mu(u) > \mu(v)$ then $\mu(u+v) = \mu(v)$.

Proof

We prove only (iii) since (i) and (ii) are well–known.

Since $\mu(u) > \mu(v)$ we have $\mu(u+v) \geq \mu(v)$.

Also $\mu\big[(u+v) - u\big] = \mu(v) \geq \mu(u+v) \wedge \mu(u)$.

Since $\mu(u) > \mu(v)$ we have $\mu(u+v) \leq \mu(v)$.

Consequently $\mu(u+v) = \mu(v)$.

Proposition 4.2.3

If μ is a fuzzy vector space over E and v, w $\in$ E with $\mu(v) \neq \mu(w)$ then $\mu(v+w) = \mu(v) \wedge \mu(w)$.

Proof

Apply Proposition 4.2.2 (iii).

Proposition 4.2.4

If μ is a fuzzy vector space over E then $\mu(0) = h(\mu)$.

Proof

$x \in E \Longrightarrow \mu(0) = \mu(0x) \geq \mu(x)$.

4.3 LINEAR INDEPENDENCE IN FUZZY VECTOR SPACES

We find an attempt at the definition of fuzzy linear independence in [30] . However this reduces to normal linear independence. We give below an alternative definition.

Definition 4.3.1

Let μ be a fuzzy vector space over E. We say that a finite set of vectors $\{x_i\}_{i=1}^{n} \subset E$ is linear independent in μ if and only if $\{x_i\}_{i=1}^{n}$ is linearly independent in E and for all $\{a_i\}_{i=1}^{n} \subset \mathbb{R}$, $\mu(\sum_{i=1}^{n} a_i x_i) = \bigwedge_{i=1}^{n} \mu(a_i x_i)$.

Any set of vectors is linearly independent in μ if all finite subsets of that set are linearly independent in μ.

Example 4.3.2

Consider $\mu : \mathbb{R}^2 \to [0,1]$, where,

$$\mu((x,y)) = \begin{cases} 1 & \text{if } x = y = 0 \\ 1/2 & \text{if } x = 0 \text{ and } y \neq 0 \\ 1/4 & \text{if } x \neq 0 \end{cases} .$$

It is easily checked that vectors $x = (1,0)$ and $y = (-1,1)$ are linearly independent in E but are not linearly independent in μ. This example also illustrates a situation where $\mu(x) = \mu(y)$ and $\mu(x + y) > \mu(x)$.

Proposition 4.3.3

Let μ be any fuzzy vector space over E, then any set of vectors $\{x_i\}_{i=1}^{N} \subset E \setminus \{0\}$ which has distinct μ–values is linearly independent in μ.

Proof

We prove the proposition by induction on N.

In case N = 1 we have only one vector – clearly the statement is true.

Now suppose that the proposition is true for N.

Let $\{x_i\}_{i=1}^{N+1}$ be a set of vectors in $E\setminus\{0\}$ with distinct μ–values.

By inductive hypothesis we have: $\{x_i\}_{i=1}^{N}$ is linearly independent in μ. Suppose that $\{x_i\}_{i=1}^{N+1}$ is not linearly independent in E, thus $x_{N+1} = \sum_{i\in S} a_i x_i$ where $S \subset \{1, ..., N\}$, $S \neq \phi$ and for all $i \in S$, $a_i \neq 0$.

Therefore $\mu(x_{N+1}) = \bigwedge_{i\in S} \mu(a_i x_i) = \bigwedge_{i\in S} \mu(x_i)$

and then $\quad \mu(x_{N+1}) \in \{\mu(x_i)\}_{i=1}^{N}$.

This contradicts the fact that $\{x_i\}_{i=1}^{N+1}$ has distinct μ–values. Therefore $\{x_i\}_{i=1}^{N+1}$ is linearly independent in E. Finally Propositions 4.2.2 (ii), 4.2.3 and 4.2.4 clearly show that $\{x_i\}_{i=1}^{N+1}$ is linearly independent in μ.

<u>Remark:</u> If μ is a fuzzy vector space over E such that dim E = n , then $|\mu(E)| \leq n + 1$.

4.4 BASIS FOR A FUZZY VECTOR SPACE

Definition 4.4.1

A set $B \subset E$, is a basis for a fuzzy vector space $\mu : E \to L$ if and only if

i) B is linearly independent in μ,

and,

ii) B is a basis for E.

The following shows how we can construct a very wide class of fuzzy vector spaces with a basis.

Given a vector space E with a basis $B = \{v_\alpha\}_{\alpha \in A}$, constant $\mu_0 \in L \setminus \{0\}$ and any set of constants $\{\mu_\alpha\}_{\alpha \in A} \subset L \setminus \{0\}$ such that $\mu_0 \geq \mu_\alpha$ for all $\alpha \in A$. Let us construct a function $\mu : E \to L$ in the following way. Any $z \neq 0$, $z \in E$ can be uniquely written as $z = \sum_{i=1}^{N} a_i v_{\alpha_i}$ with $a_i \neq 0$. Define $\mu(z) = \bigwedge_{i=1}^{N} \mu(v_{\alpha_i}) = \bigwedge_{i=1}^{N} \mu_{\alpha_i}$ and $\mu(0) = \mu_0$. Clearly μ is defined for all $z \in E$ and is well–defined.

Theorem 4.4.2

The set μ is a fuzzy vector space with basis B.

Proof

We prove that $\mu(ax + by) \geq \mu(x) \wedge \mu(y)$. Let $a \neq 0$, $b \neq 0$, $x = \sum_{i=1}^{n} a_i v_{\alpha_i}$ and $y = \sum_{j=1}^{m} b_j v_{\beta_j}$. Of course, we may assume that $a_i \neq 0$ and $b_j \neq 0$. We may also write $ax + by = \sum_{k=1}^{s} c_k v_{\gamma_k}$ with $c_k \neq 0$. Now,

$$\mu(ax+by) = \bigwedge_{k=1}^{s} \mu(v_{\gamma_k}) \quad \text{by definition}$$

$$\geq \left(\bigwedge_{i=1}^{n} \mu(v_{\alpha_i}) \right) \wedge \left(\bigwedge_{j=1}^{m} \mu(v_{\beta_j}) \right)$$

since $\{ v_{\gamma_k} : k=1,\ldots,s \} \subset \{ v_{\alpha_i} : i=1,\ldots,n \} \cap \{ v_{\beta_j} : j=1,\ldots,m \}$

$= \mu(x) \wedge \mu(y)$.

The case where $a = 0$ and $b = 0$ is trivial as $\mu(0) \geq \mu(v_\alpha)$ for all $\alpha \in A$.

Thus μ is a fuzzy vector space and B is a basis for μ.

We were unable to prove that all fuzzy vector spaces have a basis or find an example of a fuzzy vector space without a basis. However, we have a simple condition under which a fuzzy vector space has a basis.

At this point we refer the reader to the Chapter 1, Section 1, where the concept of upper well orderness is introduced. Recall that if $\mu : X \to L$ then $R_\mu = \mu(X) \setminus \{0\}$.

Lemma 4.4.3

If μ is a fuzzy vector space over E such that R_μ is upper well ordered, and V is a proper subspace of E then there exists $w \in E \setminus V$ such that for all $v \in V$, $\mu(w+v) = \mu(w) \wedge \mu(v)$.

Proof

Since R_μ is upper well ordered we can find $w \in E \setminus V$ such that $\mu(w) = \sup[\mu(E \setminus V)]$. Let $v \in V$. If $\mu(v) \neq \mu(w)$ then by Proposition 4.2.3, $\mu(w+v) = \mu(w) \wedge \mu(v)$. If $\mu(v) = \mu(w)$ then by Definition 4.2.1, $\mu(w+v) \geq \mu(w) \wedge \mu(v)$. Since $w+v \in E \setminus V$ and $\mu(w) = \sup[\mu(E \setminus V)]$ we must have $\mu(w+v) \leq \mu(w) = \mu(v)$. Thus $\mu(w+v) = \mu(w) \wedge \mu(v)$.

Lemma 4.4.4

If μ is a fuzzy vector space over E such that R_μ is upper well ordered, and B* is a fuzzy basis for $\mu_{|V}$ where V is a proper subspace of E then there exists $w \in E\setminus V$ such that $B^+ = B^* \cup \{w\}$ is a basis for $\mu_{|W}$, where $W = \langle B^+ \rangle$ is the vector space spanned by B^+.

Proof

Pick $w \in E\setminus V$ such that $\mu(w) = \sup[\mu(E\setminus V)]$, then clearly by Lemma 4.4.3 w is linearly independent from B* in μ. Let $B^+ = B^* \cup \{w\}$.

Clearly B^+ is a basis for $\mu_{|W}$.

Theorem 4.4.5

All fuzzy vector spaces $\mu : E \to L$ for which R_μ is upper well ordered have a basis.

Proof

Let $\mu : E \to L$ be any fuzzy vector space for which R_μ is upper well ordered.

Let $\Omega = \{ B \subset E \mid B \text{ is linearly independent in } \mu \}$.

Partial order Ω by set inclusion. Let C be a totally ordered subset of Ω and let $A = \bigcup_{B \in C} B$. Clearly A is upper bound for C. Suppose $a_1, \ldots, a_n \in A$, then there exist $B_{\alpha(1)}, \ldots, B_{\alpha(n)} \in C$ such that $a_i \in B_{\alpha(i)}$.

Since C is totally ordered, one of the sets, say $B_{\alpha(k)}$ is a superset of the others.

Hence $a_1, \ldots, a_n \in B_{\alpha(k)}$. Since $B_{\alpha(k)}$ is linearly independent in μ, $a_1, \ldots, a_n$ are linearly independent in μ. Thus A is an upper bound of C in Ω.

By Zorn's Lemma there exists a maximal element B* in Ω.

Suppose $\langle B^* \rangle = V$ is a proper subspace of E then by Lemma 4.4.4 there exists $w \in E\setminus V$ such that $B^+ = B^* \cup \{w\}$ is a basis for $\mu_{|W}$, where $W = \langle B^+ \rangle$.

This contradicts the fact that B* is a maximal element in Ω.

Thus we must have $\langle B^* \rangle = E$ and B* is a basis for μ.

Corollary 4.4.6

If μ is a fuzzy vector space over E such that E is finite dimensional then μ has a basis.

Proof

Since E is finite dimensional $\mu(E)$ is finite and consequently upper well ordered. Thus by above theorem μ has a basis.

Definition 4.4.7 (Liu, Katsaras [18])

Let $\mu_1, \mu_2 \in L^E$ be two fuzzy vector spaces over E.

Define the intersection of μ_1 and μ_2 to be $\mu_1 \wedge \mu_2$ and

the sum $\mu_1 + \mu_2$ of μ_1 and μ_2 as a function $\mu_1 + \mu_2 : E \to L$ by:

$$(\mu_1 + \mu_2)(x) = \sup \{ \mu_1(x_1) \wedge \mu_2(x_2) \mid x = x_1 + x_2 \text{ and } x_1, x_2 \in E \}$$
$$= \sup \{ \mu_1 (x_1) \wedge \mu_2 (x-x_1) \mid x_1 \in E \}.$$

Proposition 4.4.8

Let μ_1 and μ_2 be two fuzzy vector spaces over E.

We have the following results:

1) $\mu_1 \wedge \mu_2$ is a fuzzy vector space over E,

2) $\mu_1 + \mu_2$ is a fuzzy vector space over E,

3) If R_{μ_1} and R_{μ_2} are upper well ordered then $\mu_1 \wedge \mu_2$ and $\mu_1 + \mu_2$ have a basis.

Proof

1) See [18] Proposition 3.4.

Proof of 2) was not presented in [18], thus we prove it here.

2) Suppose $(\mu_1 + \mu_2)(x+y) < (\mu_1 + \mu_2)(x) \wedge (\mu_1 + \mu_2)(y)$.

Thus there exists x_1 and x_2 such that for all x_3 we have:

(*) $\mu_1(x_3) \wedge \mu_2(x+y-x_3) < [\mu_1(x_1) \wedge \mu_2(x-x_1)] \wedge [\mu_1(x_2) \wedge \mu_2(y-x_2)]$

but $[\mu_1(x_1) \wedge \mu_2(x-x_1)] \wedge [\mu_1(x_2) \wedge \mu_2(y-x_2)]$

$$= \mu_1(x_1) \wedge \mu_1(x_2) \wedge \mu_2(x-x_1) \wedge \mu_2(y-x_2)$$

$$\leq \mu_1(x_1+x_2) \wedge \mu_2(x+y-x_1-x_2)$$

Therefore there exists $x_3 = x_1 + x_2$ for which (*) is false.

Thus we have a contradiction.

Therefore $(\mu_1 + \mu_2)(x+y) \geq (\mu_1 + \mu_2)(x) \wedge (\mu_1 + \mu_2)(y)$.

Also, if $a \neq 0$, then,

$$\begin{aligned}(\mu_1 + \mu_2)(ax) &= \sup\{ \mu_1(x_1) \wedge \mu_2(x_2) : x_1 + x_2 = ax \} \\ &= \sup\{ \mu_1(ax_1) \wedge \mu_2(ax_2) : x_1 + x_2 = x \} \\ &\geq \sup\{ \mu_1(x_1) \wedge \mu_2(x_2) : x_1 + x_2 = x \} \\ &= (\mu_1 + \mu_2)(x).\end{aligned}$$

If $a=0$,

$$\begin{aligned}(\mu_1 + \mu_2)(0x) &= (\mu_1 + \mu_2)(0) \\ &= \sup\{ \mu_1(x_1) \wedge \mu_2(x_2) : x_1 + x_2 = 0 \} \\ &= \mu_1(0) \wedge \mu_2(0), \text{ by Proposition 4.2.4.}\end{aligned}$$

3) Clearly $R_{(\mu_1 \wedge \mu_2)} = (\mu_1 \wedge \mu_2)(E) \subset \mu_1(E) \cup \mu_2(E) = R_{\mu_1} \cup R_{\mu_2}$

and $R_{(\mu_1 + \mu_2)} = (\mu_1 + \mu_2)(E) \subset \mu_1(E) \cup \mu_2(E) = R_{\mu_1} \cup R_{\mu_2}$.

Since R_{μ_1} and R_{μ_2} are upper well ordered the set $R_{\mu_1} \cup R_{\mu_2}$ is upper well ordered and all operations are done within this set, consequently, $R_{(\mu_1 \wedge \mu_2)}$ and $R_{(\mu_1 + \mu_2)}$ are upper well ordered.

By Theorem 4.4.5, $\mu_1 \wedge \mu_2$ and $\mu_1 + \mu_2$ have a basis.

Theorem 4.4.9

Let $\mu : E \to L$ be a fuzzy vector space then:

B is a basis for μ if and only if for all $\alpha \in L$, $E_\mu^\alpha \cap B$ is a basis for E_μ^α.

Proof

($\Rightarrow$) Suppose that B is a basis for μ and for some $\alpha \in L$, $B \cap E_\mu^\alpha$ is not a basis for E_μ^α. Thus span($B \cap E_\mu^\alpha$) is a proper subspace of E_μ^α. Let $x \in E_\mu^\alpha \setminus$span($B \cap E_\mu^\alpha$). So $\mu(x) \geq \alpha$. Since B is a basis for μ there exist $\{ b_1, b_2, \ldots, b_n \} \subset B$ and $\{ a_1, a_2, \ldots, a_n \} \subset \mathbb{R}\setminus\{0\}$ such that $x = \sum_{i=1}^{n} a_i b_i$ and $\mu(x) = \bigwedge_{i=1}^{n} \mu(b_i) \geq \alpha$. Since $x \notin$ span($B \cap E_\mu^\alpha$) there exists $j \in \{ 1, \ldots, n \}$ such that $b_j \in E \setminus E_\mu^\alpha$ i.e. $\mu(b_j) < \alpha$. This means that $\mu(x) = \bigwedge_{i=1}^{n} \mu(b_i) < \alpha$. This is contradiction.

($\Leftarrow$) Suppose that for all $\alpha \in L$, $B \cap E_\mu^\alpha$ is a basis for E_μ^α. Since $E_\mu^0 = E$, B is a basis for E. Let $\{ b_1, b_2, \ldots, b_n \} \subset B$, $\{ a_1, a_2, \ldots, a_n \} \subset \mathbb{R}\setminus\{0\}$, $\alpha_i = \mu(b_i)$ and $\alpha = \min(\alpha_i : i = 1, \ldots, n \}$. We can choose b_i such that $\alpha_i \leq \alpha_j$ iff $i \leq j$. Select $k \in \{ 1, \ldots, n \}$ such that $\alpha_k = \alpha$ and for $i > k$, $\alpha_i > \alpha$ and then let $x = \sum_{i=1}^{k} a_i b_i$. Now we show that $\mu(x) = \alpha$. Suppose that $\mu(x) = \beta > \alpha$, i.e. $x \in E_\mu^\beta$. Since $B \cap E_\mu^\beta$ is a basis for E_μ^β there exists an unique $\{ \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m \} \subset B \cap E_\mu^\beta$ and $\{ \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m \} \subset \mathbb{R}\setminus\{0\}$ such that $x = \sum_{i=1}^{m} \bar{a}_i \bar{b}_i$. So we have found two different expressions for x, namely $\sum_{i=1}^{m} \bar{a}_i \bar{b}_i$ and $\sum_{i=1}^{k} a_i b_i$. They are different because $\mu(b_i) = \alpha$ for $i = 1,\ldots,k$ and $\mu(\bar{b}_i) \geq \beta$ for all $i = 1,\ldots,m$. But since B is a basis there is only an unique representation for x. This leads to a contradiction. Thus $\mu(x) \leq \alpha$. But we also have $\mu(x) \geq \bigwedge_{i=1}^{m} \mu(b_i) = \alpha$. So $\mu(x) = \alpha$. Let $y = \sum_{i=k+1}^{n} a_i b_i$. By definition $\mu(y) > \alpha$. So $\mu(x) \neq \mu(y)$. Thus by Proposition 4.2.3, $\mu(x+y) = \mu(x) \wedge (y) = \alpha$. Finally, $\mu(\sum_{i=1}^{n} a_i b_i) = \mu(x) \wedge \mu(y) = \alpha = \bigwedge_{i=1}^{n} \mu(b_i)$. So B is a basis for μ.

From Theorem 4.4.9 we see that the problem of finding a basis for a fuzzy vector space is equivalent to the following :

Proposition 4.4.10

Suppose $\mathcal{V} = \{V_\alpha\}_{\alpha \in J}$ is a chain of vector spaces then there exists a basis B for the vector space $\cup V_\alpha$ such that for all $\alpha \in J$, $B \cap V_\alpha$ is a basis for V_α.

If we assume that the collection $\mathcal{V}$ is upper well ordered under set inclusions, then we can find the required basis. This is shown in Theorem 4.4.5. As we have stated previously, we were not able to prove or find a counter–example to this statement in general.

4.5 DIMENSION OF FUZZY VECTOR SPACES

Definition 4.5.1

The dimension of a fuzzy vector space μ over E is $\dim(\mu) = \mathrm{card}(\mu_{|B})$ where B is any basis for μ.

Note that the dimension of μ is a fuzzy cardinal.

Proposition 4.5.2

Given a fuzzy vector space μ over E and two basis B_1 and B_2 for μ, there exists a bijection $f : B_1 \to B_2$ with $\mu(b) = \mu(f(b))$ for all $b \in B_1$. Thus $\mathrm{card}(\mu_{|B_1}) = \mathrm{card}(\mu_{|B_2})$, i.e. the dimension of μ is independent from the choice of a basis for μ.

Proof

Since B_1 and B_2 are basis for E, it is well known that $|B_1| = |B_2|$. So there exists a bijection between B_1 and B_2. We must also have that $R_{\mu_{|B_1}} = R_{\mu_{|B_2}}$. Otherwise suppose W.L.O.G. $\mu(b) \notin R_{\mu_{|B_2}}$ for some $b \in B$ and $b \neq 0$, this means that $\mu(b) \neq \bigwedge_{i=1}^{N} \mu(a_i b_i)$ for all $\{a_i\}_{i=1}^{N} \subset \mathbb{R}$ and $\{b_i\}_{i=1}^{N} \subset B_2$, because the infs are taken over finite sets. That would mean that B_2 is not a basis for μ.

Thus we must have $R_{\mu_{|B_1}} = R_{\mu_{|B_2}}$. Let $\alpha \in R_{\mu_{|B_1}}$. Recall that E_μ^α and H_μ^α are both vector spaces with $H_\mu^\alpha < E_\mu^\alpha$. Now given $x \in T_\mu^\alpha = \mu^{-1}(\alpha)$ it is easy to see that we can write $x = \sum_{i=1}^{N_1} a_{i,1} b_{i,1} + \sum_{i=1}^{M_1} \bar{a}_{i,1} \bar{b}_{i,1} = \sum_{i=1}^{N_2} a_{i,2} b_{i,2} + \sum_{i=1}^{M_2} \bar{a}_{i,2} \bar{b}_{i,2}$ where, $a_{i,1}, \bar{a}_{i,1}, a_{i,2}, \bar{a}_{i,2} \in \mathbb{R}$, and $b_{i,1} \in E^\alpha_{\mu_{|B_1}}$, $\bar{b}_{i,1} \in T^\alpha_{\mu_{|B_1}}$, $b_{i,2} \in E^\alpha_{\mu_{|B_2}}$, $\bar{b}_{i,2} \in T^\alpha_{\mu_{|B_2}}$. This follows from the fact that B_1 and B_2 are both basis for μ.

If an element from $\mu^{-1}([0,\alpha))$ was in the linear combination expressing x then we should have $\mu(x) < \alpha$ and thus $x \notin T^{\alpha}_{\mu}$, a contradiction.
From all this we get that $E^{\alpha}_{\mu} = \text{span}(T^{\alpha}_{\mu|B_1}) + H^{\alpha}_{\mu} = \text{span}(T^{\alpha}_{\mu|B_2}) + H^{\alpha}_{\mu}$ and thus $|T^{\alpha}_{\mu|B_1}| = |T^{\alpha}_{\mu|B_2}|$, so there exists a bijection $f_{\alpha} : T^{\alpha}_{\mu|B_1} \to T^{\alpha}_{\mu|B_2}$.
We can construct such f_{α} for all $\alpha \in R_{\mu|B_1} = R_{\mu|B_2}$. Thus we can construct the required bijection $f : B_1 \to B_2$ by letting $f(b) = f_{\mu(b)}(b)$.
Clearly then $\mu_{|B_1}(b) = \mu_{|B_2}(f(b))$ for all $b \in B_1$.

We say that a fuzzy vector space μ over E for which a basis B exists is finite dimensional if and only if $\mu_{|B}$ is a finite fuzzy set or $\dim(\mu_{|B})$ is a finite fuzzy cardinal.

Proposition 4.5.3

All fuzzy vector spaces $\mu : E \to L$ for which E^{α}_{μ} is finite dimensional for all $\alpha \in L\setminus\{0\}$ have a basis and are finite dimensional. We can justify the converse as well.

Proof

Since for all $\alpha \in L\setminus\{0\}$, E^{α}_{μ} is finite dimensional, $\mu(E^{\alpha}_{\mu})$ is finite and thus we must have R_{μ}, a decreasing sequance to zero, which is upper well ordered. Thus by Theorem 4.4.5 μ has a fuzzy basis. Clearly μ is also finite dimensional. i.e. $\mu_{|B}$ is finite. Converse follows simply as well.

Proposition 4.5.4

Let μ be a fuzzy vector space over a finite dimensional vector space E with basis B for μ and B^* be any other basis for E, then there exists a bijection $f : B^* \to B$ such that $\mu(b) \leq \mu(f(b))$.

Proof

Since E is finite dimensional, $|\mu(E\setminus\{0\})| = k \leq \dim E$. Let $\mu(E\setminus\{0\}) = \{\mu_i\}_{i=1}^{k}$ such that $\mu_i > \mu_{i+1}$. Since B is a basis for μ, $B \cap E_\mu^{\mu_i}$ is a basis for the vector subspace $E_\mu^{\mu_i}$ and $B^* \cap E_\mu^{\mu_i}$ is an linearly independent subset of $E_\mu^{\mu_i}$.
Thus $|B^* \cap E_\mu^{\mu_i}| \leq |B \cap E_\mu^{\mu_i}|$ for all $i \in \{0, ..., k\}$. Define recursively a set of injections $\{f_1, f_2, ..., f_n\}$ as follows: let f_1 be any injection from $B^* \cap E_\mu^{\mu_1}$ to $B \cap E_\mu^{\mu_1}$. Such f_1 exists since $|B^* \cap E_\mu^{\mu_1}| \leq |B \cap E_\mu^{\mu_1}|$ and clearly $\mu(v) \leq \mu(f_1(v))$ for all $v \in B^* \cap E_\mu^{\mu_1}$. Given an injection f_{n-1} from $B^* \cap E_\mu^{\mu_{n-1}}$ to $B \cap E_\mu^{\mu_{n-1}}$ such that $\mu(v) \leq \mu(f_{n-1}(v))$ for all $v \in B^* \cap E_\mu^{\mu_{n-1}}$, let g_n be any injection from $B^* \cap T_\mu^{\mu_n}$ to $\mathcal{E}_n = (B \cap E_\mu^{\mu_n})\setminus f_{n-1}(B^* \cap E_\mu^{\mu_{n-1}})$ such that there exists $x \in B^* \cap T_\mu^{\mu_n}$ with $\mu(x) = \max(\mathcal{E}_n)$. Such g_n exists since $\mathcal{E}_n$ is finite and,

$$\begin{aligned} |B^* \cap T_\mu^{\mu_n}| &= |B^* \cap E_\mu^{\mu_n}| - |B^* \cap E_\mu^{\mu_{n-1}}| \\ &\leq |B \cap E_\mu^{\mu_n}| - |B^* \cap E_\mu^{\mu_{n-1}}| \\ &= |B \cap E_\mu^{\mu_n}| - |f_{n-1}(B^* \cap E_\mu^{\mu_{n-1}})| \\ &= |(B \cap E_\mu^{\mu_n}) \setminus f_{n-1}(B^* \cap E_\mu^{\mu_{n-1}})| . \end{aligned}$$

Define $f_n : B^* \cap E_\mu^{\mu_n} \to B \cap E_\mu^{\mu_n}$ as follows: If $v \in B^* \cap E_\mu^{\mu_{n-1}}$ then let $f_n(v) = f_{n-1}(v)$, otherwise let $f_n(v) = g_n(v)$. It is now clear that f_n is an injection and since $g_n(B^* \cap T_\mu^{\mu_n}) \subset E_\mu^{\mu_n}$, $n \in \{2, ..., k\}$, it follows that $\mu(v) \leq \mu(f_n(v))$ for all $v \in B^* \cap E_\mu^{\mu_n}$. Since $E_\mu^{\mu_k} = E$ and $|B^*| = |B|$ it follows that f_k is a bijection between B^* and B such that $\mu(v) \leq \mu(f_k(v))$ for all $v \in B^*$.

The above proposition extends to any finite dimensional fuzzy vector space μ. This can be seen by observing that if μ is finite dimensional then R_μ is either finite, in which case the above proposition applies directly, or a sequence decreasing to zero such that for all $\alpha \in L\setminus\{0\}$, E_μ^{α} is finite dimensional.
In the later case we define bijection $f : B^* \to B$ as follows: Let $R_\mu = \{\mu_1, \mu_2, ...\}$. If $v \in \text{supp}(\mu) \cap B^*$ then $x \in E_\mu^{\mu_n}$ for some $n \in \mathbb{N}$. So, let $f(v) = f_n(v)$. At this point we may assume that $E = \text{supp}(\mu)$. Clearly f is an injection. Let $b \in B$. Then for

some $n \in \mathbb{N}$, $b \in E_{\mu}^{\mu_n}$. From the definition in Proposition 4.5.4 of g_n's we see that there exists $m \in \mathbb{N}$ and $v \in B^*$ such that $g_m(v) = b$. Consequently f is a bijection.

Let us introduce the following equivalence relation $\sim$ with respect to μ on the set of all basis for E: $B_1 \sim B_2$ iff there exits a bijection $f : B_1 \to B_2$ such that $\mu(f(b)) = \mu(b)$ for all $b \in B_1$. It is easy to show that $\sim$ is indeed an equivalence relation. Let $C = \{ [B] : B \text{ is a basis for } E \}$, where [B] is the equivalence class containing B.
Furthermore we can introduce the following partial ordering $\preceq$ with respect to μ on C: $[B_1] \preceq [B_2]$ iff there exist a bijection $f : B_1 \to B_2$ such that for all $b \in B_1$, $\mu(f(b)) \geq \mu(b)$. It is reasonably easy to show that $\preceq$ is well defined and indeed is a partial ordering on C.

Proposition 4.5.4 states that a basis B for a finite dimensional fuzzy vector space μ is any element from the maximal equivalence class C under this order. In fact the maximum equivalence class.

We now investigate further the properties of dimension of fuzzy vector spaces.
In the example which follows we are going to use Blanchard's cardinality i.e. $\dim(\mu) = \text{card}_B(\mu)$. Recall notation setup in the Definition 3.1.12.
An important result from crisp theory that we would like to have in fuzzy setting is:
If μ_1 and μ_2 are two fuzzy vector spaces over E then:

(0) $\dim(\mu_1 + \mu_2) \oplus \dim(\mu_1 \wedge \mu_2) = \dim(\mu_1) \oplus \dim(\mu_2)$.

Unfortunately this is not always true.
Consider $E = \mathbb{R}$, $\mu_1 \equiv \frac{1}{2}$, $\mu_2 \equiv \frac{1}{4}$ then $\mu_1 \wedge \mu_2 = \mu_1 + \mu_2 \equiv \frac{1}{4}$. So we don't even have $\mu_1 \vee \mu_2 \leq \mu_1 + \mu_2$. Clearly $\dim(\mu_1) = (\frac{1}{2})$, $\dim(\mu_2) = (\frac{1}{4})$, $\dim(\mu_1 + \mu_2) = (\frac{1}{4})$ and $\dim(\mu_1 \wedge \mu_2) = (\frac{1}{4})$ thus

$\dim(\mu_1 + \mu_2) \oplus \dim(\mu_1 \wedge \mu_2) = (\frac{1}{4},\frac{1}{4}) \neq (\frac{1}{2},\frac{1}{4}) = \dim(\mu_1) \oplus \dim(\mu_2)$.

It is easy to check that if $\mu_2 \equiv \frac{1}{4}$ in above example is modified to $\mu_2(\mathbb{R}\setminus\{0\}) = \frac{1}{4}$ and $\mu_2(0) = \frac{1}{2}$ then (0) holds true. This example is significant in that it points out the flaws in the definition of the sum of two fuzzy vector spaces, which is derived using Zadeh's extension principle.

We believe that $\mu_1 + \mu_2$ should be defined as follows:

$\mu_1 + \mu_2 = \wedge \{ \mu : \mu \text{ is a fuzzy vector space with } \mu_1 \vee \mu_2 \leq \mu \}$

It is easy to see that such sum makes (0) valid in case $\mu_1 \equiv \frac{1}{2}$ and $\mu_2 \equiv \frac{1}{4}$.

We note in passing the following.

Proposition 4.5.5

If μ is any fuzzy vector space over E where E is finite dimensional, then any basis B for μ can be constructed in the following way:

Let $\mu(E\setminus\{0\}) = \{\alpha_1,...,\alpha_k\}$. For each α_i, i=1,...,k define B_{α_i} recursively (starting with B_{α_1}) such that B_{α_i} is any maximal set of linearly independent vectors in $T_\mu^{\alpha_i}$ that extend the basis $\bigcup_{j<i} B_{\alpha_j}$ of $H_\mu^{\alpha_i}$ to a basis $\bigcup_{i\leq j} B_{\alpha_j}$ of $E_\mu^{\alpha_i}$

Then $B = \bigcup_{i\leq k} B_{\alpha_i}$ is a basis of E such that B is also a basis for μ..

Proof (Easy)

We now shall prove that under certain conditions (0) holds true, and give more interesting examples to illustrate the result.

Note that in the following theorem the conditions $\mu_1(0) \geq \sup [\mu_2 (E\setminus\{0\})]$ and $\mu_2(0) \geq \sup [\mu_1 (E\setminus\{0\})]$ are equivalent to $\mu_1(0) \wedge \mu_2(0) \geq \mu_1(x) \vee \mu_2(x)$ for all $x \in E\setminus\{0\}$.

Theorem 4.5.6

Let μ_1 and μ_2 be two fuzzy vector spaces over a finite dimensional vector space E such that $\mu_1(0) \geq \sup [\mu_2 (E\setminus\{0\})]$ and $\mu_2(0) \geq \sup [\mu_1 (E\setminus\{0\})]$. Then there exist a basis B for E, which is also a basis for μ_1, μ_2, $\mu_1 \wedge \mu_2$ and $\mu_1 + \mu_2$. In addition, if $A_1 = \{ x \in E \mid \mu_1(x) < \mu_2(x) \}$, $A_2 = E\setminus A_1$, then for all $v \in B \cap A_1$, $(\mu_1 \wedge \mu_2)(v) = \mu_1(v)$, $(\mu_1 + \mu_2)(v) = \mu_2(v)$ and for all $v \in B \cap A_2$, $(\mu_1 \wedge \mu_2)(v) = \mu_2(v)$, $(\mu_1 + \mu_2)(v) = \mu_1(v)$.

Proof (By induction on dim E).

In case dim E = 1 the statement is clearly true. Now suppose that the theorem is true for all the fuzzy vector spaces with the dimension of the underlying vector space equal to n.

Let μ_1 and μ_2 be two fuzzy vector spaces over E with dim E = n+1 > 1.
Let $B_1 = \{v_i\}_{i=1}^{n+1}$ be any basis for μ_1. The existence of such basis is is guaranteed by Corollary 4.4.6. We may assume that $\mu_1(v_1) \leq \mu_1(v_i)$ for all $i \in \{2, ..., n+1\}$.
Let $H = \text{span}(\{v_i\}_{1=2}^{n+1})$. Since n+1 > 1, $H \neq \{0\}$. Clearly dim H = n.
Define the following two fuzzy vector spaces : $\nu_1 = \mu_{1|H}$ and $\nu_2 = \mu_{2|H}$.
By inductive hypothesis there exists a basis B* for H which is also a basis for ν_1, ν_2, $\nu_1 \wedge \nu_2$ and $\nu_1 + \nu_2$.
Also for all $v \in B^* \cap A_1$, $(\mu_{1|H} \wedge \mu_{2|H})(v) = \mu_{1|H}(v)$ and $(\mu_{1|H} + \mu_{2|H})(v) = \mu_{2|H}(v)$ and for all $v \in B^* \cap A_2$, $(\mu_{1|H} \wedge \mu_{2|H})(v) = \mu_{2|H}(v)$ and $(\mu_{1|H} + \mu_{2|H})(v) = \mu_{1|H}(v)$.
We shall now show that B* can be extended to B such that B is a basis for μ_1, μ_2, $\mu_1 \wedge \mu_2$ and $\mu_1 + \mu_2$. Furthermore for all $v \in B \cap A_1$, $(\mu_1 \wedge \mu_2)(v) = \mu_1(v)$, $(\mu_1 + \mu_2)(v) = \mu_2(v)$ and for all $v \in B \cap A_2$ $(\mu_1 \wedge \mu_2)(v) = \mu_2(v)$, $(\mu_1 + \mu_2)(v) = \mu_1(v)$.
First we have to show that for all $x \in H$

(1) $(\mu_1 + \mu_2)_{|H}(x) = (\mu_{1|H} + \mu_{2|H})(x)$.

Let $x \in H\backslash\{0\}$, then we have:

$$(\mu_1 + \mu_2)_{|H}(x) = \sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) | \ x_1 \in E\}$$
$$= \sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) | \ x_1 \in H\}$$
$$\vee \ \sup\{\mu_1(x_2) \wedge \mu_2(x-x_2) | \ x_2 \in E\backslash H\}$$

Since $x \in H\backslash\{0\}$ we have

$\mu_1(x) \wedge \mu_2(x-x) = \mu_1(x) \wedge \mu_2(0) \leq \sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in H\}$

$\mu_1(0) \wedge \mu_2(x-0) = \mu_1(0) \wedge \mu_2(x) \leq \sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in H\}$.

Since $\mu_1(0) \geq \sup[\mu_2(H\backslash\{0\})]$ and $\mu_2(0) \geq \sup[\mu_1(H\backslash\{0\})]$

$\mu_1(x) \wedge \mu_2(0) = \mu_1(x)$ and $\mu_1(0) \wedge \mu_2(x) = \mu_2(x)$,

this leads to the following inequality:

(2) $\quad \mu_1(x) \vee \mu_2(x) \leq \sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in H\}$.

Suppose that:

(3) $\quad \sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in H\} < \sup\{\mu_1(x_2) \wedge \mu_2(x-x_2) \mid x_2 \in E\backslash H\}$.

This means that there exists $\tilde{x} \in E\backslash H$ such that

$\sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in H\} < \mu_1(\tilde{x}) \wedge \mu_2(x-\tilde{x})$.

In view of (2) we must have

(4) $\quad \mu_1(x) \vee \mu_2(x) < \mu_1(\tilde{x}) \wedge \mu_2(x-\tilde{x})$.

Since $\tilde{x} \in E\backslash H$ and $\mu_1(E\backslash H) = \mu_1(v_1) \leq \mu_1(v_i)$ for all $i \in \{2,\dots, n+1\}$

we must have $\mu_1(x) \geq \mu_1(\tilde{x})$.

Thus (4) becomes $\mu_1(x) \vee \mu_2(x) < \mu_1(x) \wedge \mu_2(x-\tilde{x})$.

It is easily checked that the last inequality never holds.

(Use the properties of $\wedge$, $\vee$ and $<$). This means that our assumption (3) is false.

Therefore we must have:

$\sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in H\} \geq \sup\{\mu_1(x_2) \wedge \mu_2(x-x_2) \mid x_2 \in E\backslash H\}$.

Clearly this is also true if $x = 0$. To conclude we have for all $x \in H$

$$(\mu_1 + \mu_2)_{|H}(x) = \sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in E\}$$
$$= (\sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in H\})$$
$$\vee (\sup\{\mu_1(x_2) \wedge \mu_2(x-x_2) \mid x_2 \in E\setminus H\}).$$

But since $\sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in H\} \geq \sup\{\mu_1(x_2) \wedge \mu_2(x-x_2) \mid x_2 \in E\setminus H\}$

$$= \sup\{\mu_1(x_1) \wedge \mu_2(x-x_1) \mid x_1 \in H\}$$
$$= \sup\{\mu_{1|H}(x_1) \wedge \mu_{2|H}(x-x_1) \mid x_1 \in H\}$$
$$= (\mu_{1|H} + \mu_{2|H})(x).$$ This establishes (1).

Clearly (1) implies that B^* is linearly independent in $\mu_1 + \mu_2$.

Let $v^* \in E\setminus H$ such that $\mu_2(v^*) = \sup[\mu_2(E\setminus H)]$.

Clearly such v^* exists since μ_2 assumes a finite number of values.

By Lemma 4.4.3 and Lemma 4.4.4, v^* is an extension of basis B^* for ν_2 to $B = B^* \cup \{v^*\}$ a basis for μ_2. Since $\mu_1(E\setminus H) = \mu_1(v_1)$ then v^* is also an extension of basis B^* for ν_1 to B a basis for μ_1.

Now we shall show that v^* is an extension of the basis B^* for $\nu_1 \wedge \nu_2$ to B a basis for $\mu_1 \wedge \mu_2$. If $v^* \in A_1$ then since μ_1 is constant on $E\setminus H$, $(\mu_1 \wedge \mu_2)(A_1 \cap (E\setminus H)) = \mu_1(v^*)$ and for all $z \in A_2 \cap (E\setminus H)$, $(\mu_1 \wedge \mu_2)(z) \leq \mu_1(v^*)$.

From this we may conclude that if $v^* \in A_1$ then

$(\mu_1 \wedge \mu_2)(v^*) = \sup[(\mu_1 \wedge \mu_2)(E\setminus H)]$.

If $v^* \in A_2$ then $\mu_2(v^*) \leq \mu_1(v^*)$. Since $\mu_2(v^*) = \sup[\mu_2(E\setminus H)]$ and μ_1 is constant on $E\setminus H$ we must have $A_1 \cap (E\setminus H) = \phi$.

Therefore we have that if $v^* \in A_2$ then $(\mu_1 \wedge \mu_2)(v^*) = \sup[(\mu_1 \wedge \mu_2)(E\setminus H)]$.

By Lemma 4.4.4 we may now conclude that v^* extends basis B^* for $\nu_1 \wedge \nu_2$ to B a basis for $\mu_1 \wedge \mu_2$.

Now we shall show that v^* is also an extension of B^* a basis for $\nu_1 + \nu_2$ to B a basis for $\mu_1 + \mu_2$. Suppose that there exists $z \in E\setminus H$

such that $(\mu_1 + \mu_2)(v^*) < (\mu_1 + \mu_2)(z)$. Clearly vector z can be written in the form $z = a(v^* + v)$ where $a \neq 0$ and $v \in H$. Therefore we have

$(\mu_1 + \mu_2)(v^*) < (\mu_1 + \mu_2)(z) = (\mu_1 + \mu_2)(a(v^* + v)) = (\mu_1 + \mu_2)(v^* + v)$.

This means that there exists $x_1 \in E$ such that for all $\tilde{x}$

(5) $\mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x}) < \mu_1(x_1) \wedge \mu_2(v^* + v - x_1)$.

In particular (5) is true if $\tilde{x} = 0$, i.e.

$\mu_1(0) \wedge \mu_2(v^*) < \mu_1(x_1) \wedge \mu_2(v^* + v - x_1)$.

But since $\mu_1(0) \geq \sup[\mu_2(E\setminus\{0\})]$ we have

(6) $\mu_2(v^*) < \mu_1(x_1) \wedge \mu_2(v^* + v - x_1)$.

If $x_1 \in H$ then since $v \in H$ we must have $v - x_1 \in H$.

Thus by Lemma 4.4.3., $\mu_2(v^* + v - x_1) = \mu_2(v^*) \wedge \mu_2(v - x_1)$

so (6) becomes $\mu_2(v^*) < \mu_1(x_1) \wedge \mu_2(v^*) \wedge \mu_2(v - x_1)$, which is impossible.

Thus $x_1 \in E\setminus H$. Let $\tilde{x} = v^*$ in (5) and since $\mu_2(0) \geq \sup\{\mu_1(E\setminus\{0\})\}$ we have

(7) $\mu_1(v^*) < \mu_1(x_1) \wedge \mu_2(v^* + v - x_1)$.

Recall that $\mu_1(E\setminus H) = \mu_1(v_1)$ and thus $\mu_1(v^*) = \mu_1(x_1)$.

This again means that inequality (7) is false. Thus for all $z \in E\setminus V$, $(\mu_1 + \mu_2)(v^*) \geq (\mu_1 + \mu_2)(z)$. Therefore by Lemma 4.4.3, v^* is an extension of the basis B^* for $\nu_1 + \nu_2$ to a basis B for $\mu_1 + \mu_2$.

Now we shall show that if $v^* \in A_1$ then $(\mu_1 + \mu_2)(v^*) = \mu_2(v^*)$ and if $v^* \in A_2$ then $(\mu_1 + \mu_2)(v^*) = \mu_1(v^*)$. From the definition we have:

$(\mu_1 + \mu_2)(v^*) = \sup\{\mu_1(x_1) \wedge \mu_2(v^* - x_1) \mid x_1 \in E\}$.

Let $\tilde{x}$ be such that $\sup\{\mu_1(x_1) \wedge \mu_2(v^* - x_1) \mid x_1 \in E\} = \mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x})$.

By substituting $x_1 = 0$ and then $x_1 = v^*$ and recalling that

$\mu_1(0) \geq \sup\{\mu_1(E\setminus\{0\})\}$ and $\mu_2(0) \geq \sup\{\mu_1(E\setminus\{0\})\}$, we obtain

$\mu_1(v^*) \vee \mu_2(v^*) \leq \mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x})$.

Suppose that:

(8) $\mu_1(v^*) \vee \mu_2(v^*) < \mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x})$.

If $\tilde{x} \in H$ then by Lemma 4.4.3,(as $B = B^* \cup \{v^*\}$ is a basis for μ_2)

(8) becomes

$\mu_1(v^*) \vee \mu_2(v^*) < \mu_1(\tilde{x}) \wedge \mu_2(v^*) \wedge \mu_2(\tilde{x})$.

This is never true, thus $\tilde{x} \in E \backslash H$. But now since $\mu_1(v^*) = \mu_1(\tilde{x})$ the inequality (8) never holds, thus

(9) $\mu_1(v^*) \vee \mu_2(v^*) = \mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x}) = (\mu_1 + \mu_2)(v^*)$.

Equation (9) clearly leads to the required result. This completes the proof.

The Theorem 4.5.6 is a valuable tool for fuzzy vector spaces. The following is one result which follows from it.

Theorem 4.5.7

If μ_1 and μ_2 are two fuzzy vector spaces over E such that the dimension of E is finite and $\mu_1(0) \geq \sup[\mu_2(E \backslash \{0\})]$ and $\mu_2(0) \geq \sup[\mu_1(E \backslash \{0\})]$ then $\dim(\mu_1) \oplus \dim(\mu_2) = \dim(\mu_1 + \mu_2) \oplus \dim(\mu_1 \wedge \mu_2)$.

Proof

Let B be the basis from Theorem 4.5.6.

Then it follows from Theorem 4.5.6 that $(\mu_1 + \mu_2)_{|B} = \mu_{1|B} \vee \mu_{2|B}$ and $(\mu_1 \wedge \mu_2)_{|B} = \mu_{1|B} \wedge \mu_{2|B}$. We know that cardinals have the additive property from Chapter 3 (whichever cardinality we choose), thus

$\dim(\mu_1 + \mu_2) \oplus \dim(\mu_1 \wedge \mu_2) = \mathrm{card}(\ (\mu_1 + \mu_2)_{|B}\ ) \oplus \mathrm{card}(\ (\mu_1 \wedge \mu_2)_{|B}\ ) =$
$\mathrm{card}(\ \mu_{1|B} \vee \mu_{2|B}\ ) \oplus \mathrm{card}(\ \mu_{1|B} \wedge \mu_{2|B}\ ) = \mathrm{card}(\ \mu_{1|B}\ ) \oplus \mathrm{card}(\ \mu_{2|B}\ ) =$
$\dim(\mu_1) \oplus \dim(\mu_2)$.

Theorem 4.5.6 and Theorem 4.5.7 can be extended to fuzzy vector spaces with finite dimension by process recursion as follows:
If μ is finite dimensinal then from Proposition 4.5.3 we have that R_μ is a decreasing sequence to zero such that for all $\alpha \in R_\mu$, E_μ^α is finite dimensional. Since each level E_μ^α is finite dimensional we can apply Theorem 4.5.6 to $\mu|_{E_\mu^\alpha}$ to obtain the result.

Example 4.5.8

Suppose $E = \mathbb{R}^2$. Define μ_1 and μ_2 as follows:

$$\mu_1(x,y) = \begin{cases} 5/6 & \text{if } x = y = 0 \\ 1/2 & \text{if } x = 0 \text{ and } y \neq 0 \\ 1/4 & \text{otherwise} \end{cases}$$

and

$$\mu_2(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 1/3 & \text{if } x = y \text{ and } x \neq 0 \\ 1/5 & \text{otherwise} \end{cases}.$$

It is easily checked that μ_1 and μ_2 are fuzzy vector spaces over E, and $\mu_1(0) \geq \sup\{\mu_2(E\setminus\{0\})\}$ and $\mu_2(0) \geq \sup\{\mu_1(E\setminus\{0\})\}$. It is also easy to check that :

$$(\mu_1 \wedge \mu_2)(x,y) = \begin{cases} 5/6 & \text{if } x = y = 0 \\ 1/4 & \text{if } x = y \text{ and } x \neq 0 \\ 1/5 & \text{otherwise} \end{cases},$$

$$(\mu_1 + \mu_2)(x,y) = \begin{cases} 5/6 & \text{if } x = y = 0 \\ 1/2 & \text{if } x = 0 \text{ and } y \neq 0 \\ 1/3 & \text{otherwise} \end{cases}$$

and $B = \{(0,1), (1,1)\}$ is a basis for μ_1, μ_2, $\mu_1 \wedge \mu_2$ and $\mu_1 + \mu_2$, thus $\dim(\mu_1 + \mu_2) = (\frac{1}{2},\frac{1}{3})$, $\dim(\mu_1 \wedge \mu_2) = (\frac{1}{4},\frac{1}{5})$, $\dim(\mu_1) = (\frac{1}{2},\frac{1}{4})$ and $\dim(\mu_2) = (\frac{1}{3},\frac{1}{5})$.
$\dim(\mu_1) \oplus \dim(\mu_2) = (\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}) = \dim(\mu_1 \wedge \mu_2) \oplus \dim(\mu_1 + \mu_2)$.

Definition 4.5.9

Let μ be a fuzzy vector space over E and $f : E \to F$ a linear map,

where F is another vector space, then we define the image and the kernel of a linear map f on μ as $f(\mu)$ and $\mu|_{\ker f}$ respectively.
Note that we use Zadeh's definiton of an image $f(\mu)$ of a fuzzy set μ under crisp function f.

Theorem 4.5.10

Let μ be a fuzzy vector space over E with E finite dimensional, $f : E \to F$ a linear map, then $\dim(\mu|_{\ker f}) \oplus \dim(f(\mu)) = \dim(\mu)$.

Proof

Suppose that $\ker f \neq \{0\}$. If $\ker f = \{0\}$ then the result follows trivially. Now let B_{ker} be a basis for $\mu|_{\ker f}$, and B_{Ex} be an extension of B_{ker} to a basis for μ (this is clearly possible by a repeated application of Lemma 4.4.4.). So $B = B_{ker} \cup B_{Ex}$ is a basis for μ with $B_{ker} \cap B_{Ex} = \phi$. We first show that $B_{Im} = f(B_{Ex})$ is a basis for $f(\mu)$. Clearly B_{Im} is a basis for $\operatorname{Im} f = f(E)$. Let $v_1,\dots, v_k \in B_{Ex}$ and $a_1,\dots, a_k \in \mathbb{R}$ not all zero.
By definition we have :

$$f(\mu)\left(\sum_{i=1}^{k} a_i f(v_i)\right) = \begin{cases} \sup\{\mu(x) \mid x \in f^{-1}(\sum_{i=1}^{k} a_i f(v_i))\} & \text{if } f^{-1}(\sum_{i=1}^{k} a_i f(v_i)) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

Since $\sum_{i=1}^{k} a_i f(v_i) \in \operatorname{Im} f$ we have

$$f(\mu)\left(\sum_{i=1}^{k} a_i f(v_i)\right) = \sup\{\mu(x) \mid x \in f^{-1}(\sum_{i=1}^{k} a_i f(v_i))\}$$

by linearity of f and by the properties of f^{-1} we get

$$f(\mu)\left(\sum_{i=1}^{k} a_i f(v_i)\right) = \sup\{\mu(x) \mid x \in \ker f + \sum_{i=1}^{k} a_i v_i\}$$

If $z \in \ker f$ then $z = 0$ or $z = \sum_{i=1}^{p} b_i u_i$, $u_i \in B_{\ker}$

where not all b_i are zero; so if $z \in \ker f + \sum_{i=1}^{k} a_i v_i$

then either $\mu(x) = \mu(0 + \sum_{i=1}^{k} a_i v_i)$ or $\mu(x) = (\sum_{i=1}^{p} b_i u_i + \sum_{i=1}^{k} a_i v_i)$,

thus $\mu(x) = \min (\bigwedge_{i=1}^{p} \mu(b_i u_i), \bigwedge_{i=1}^{k} \mu(a_i v_i))$

which is clearly smaller or equal to $\mu(\sum_{i=1}^{k} a_i v_i)$. Thus:

$f(\mu)(\sum_{i=1}^{k} a_i f (v_i)) = \bigwedge_{i=1}^{k} \mu(a_i v_i)$.

By the same argument we get that

$f(\mu) (f(v_i)) = \mu(v_i)$. Thus

$f(\mu) (\sum_{i=1}^{k} a_i f(v_i)) = \bigwedge_{i=1}^{k} f(\mu)(a_i v_i)$ and therefore

B_{Im} is a basis for $f(\mu)$.

Now by the definition of dimension, and since $\mu_{| B_{\ker f}} \wedge \mu_{| B_{Ex}} = \phi$, we get

$$\begin{aligned}
\dim \mu &= \mathrm{card}(\mu_{| B}) \\
&= \mathrm{card}(\mu_{| B_{\ker}} \vee \mu_{| B_{Ex}}) \\
&= \mathrm{card}(\mu_{| B_{\ker}}) \oplus \mathrm{card}(\mu_{| B_{Ex}}) \\
&= \mathrm{card}(\mu_{| B_{\ker}}) \oplus \mathrm{card}(\mu_{| B_{Im}}) \\
&= \dim(\mu_{| \ker f}) \oplus \dim(f(\mu)).
\end{aligned}$$

<u>Corollary 4.5.11</u>

The above theorem extends to finite dimensional fuzzy vector spaces, by an argument as before. (i.e. if $\alpha > 0$ then E_{μ}^{α} is finite dimensional).

5. FUZZY GROUPS

5.1. INTRODUCTION–PRELIMINARIES

In this chapter we would like to improve on some of the results already obtained in fuzzy group theory. Throughout this chapter L will be a complete chain. The definition of a fuzzy group was given by A. Rosenfeld [38] in 1971. Since then definitions of various concepts related to fuzzy groups have been introduced. Unfortunately large proportion of these concepts are not interesting or aren't really generalisations of the crisp concepts. Before we proceed with this let us recall the following:

If H is a subgroup of G (written H < G) then, [G:H] denotes the index of H in G. Denote by e and S(n) the identity of any group and the cyclic group consisting of the n, n–th roots of unity respectively.

Given a group G, a ***fuzzy group*** μ on G is a fuzzy set $\mu : G \to L$ satisfying [38],

$$\mu(gh) \geq \mu(g) \wedge \mu(h), \text{ and } \mu(g^{-1}) = \mu(g)$$

Recall that if μ is a fuzzy group then, for all $\alpha \in L$, $H_\mu^\alpha < E_\mu^\alpha < G$.

Two fuzzy groups $\mu : G_1 \to L$ and $\nu : G_2 \to L$ are ***strongly isomorphic*** if and only if there exists an isomorphism

$$f : \mathrm{supp}(\mu) \to \mathrm{supp}(\nu)$$

such that

$$f(\mu|_{\mathrm{supp}(\mu)}) = \nu|_{\mathrm{supp}(\nu)}$$

and they are ***weakly isomorphic*** if and only if for all $\alpha \in L\setminus\{0\}$, the level groups H_μ^α and H_ν^α are isomorphic

We define finitely generated fuzzy groups as those for which all non zero level groups (i.e. E_{μ}^{α}, $\alpha \in L \setminus \{0\}$) are finitely generated. Then, we show that the finitely generated fuzzy groups are precisely those which are f–compact in the lattice of fuzzy groups. If we were to treat this lattice as a crisp lattice of crisp points and apply crisp lattice compactness then the previously mentioned result would not hold. Instead, we think of the elements of this lattice as fuzzy objects and we use compactness that makes sense in the fuzzy situation. In the case when the lattice is equal to $\{0,1\}$ all the results reduce to the crisp case. In the chapter on cardinality we have proved that a fuzzy set is finite if and only if it is f–compact in the fuzzy discrete topology. Thus, we will, immediately, notice the unity between the two mentioned f–compactness relations.

In [32] P. Bhattacharya and N. P. Mukherjee define the order of a fuzzy group $\mu : G \to [0,1]$ as the crisp cardinality of the set $\{ x \in G : \mu(x) = \mu(e) \}$.
(Also the order of G is assumed to be finite.)
Using this definition they obtain the following "fuzzy" version of Lagrange's Theorem: If μ , $\nu : G \to [0,1]$ are two fuzzy groups such that $\mu(e) = \nu(e)$ and $\nu \leq \mu$ then the order of ν divides the order of μ.
It must be pointed out that this is not a fuzzy version of Lagrange's Theorem. All that this statement says is that the order of group $E_{\nu}^{\nu(e)}$ divides the order of $E_{\mu}^{\mu(e)}$. Thus, such a definition of order of a fuzzy group is not satisfactory.
How can the order of a FUZZY group be a crisp cardinal?
Surely, we must look to fuzzy cardinals for a proper definition of order of a fuzzy group. In what follows we shall give a satisfactory definition of order of a fuzzy group, prove Lagrange's Theorem using this definition, define and justify finitely generated fuzzy groups and finally define and classify cyclic fuzzy groups.
Most of the contents of this chapter come from [29].

5.2. FINITELY GENERATED FUZZY GROUPS

There have been various proposals for a finitely generated algebraic fuzzy objects. For instance see [51]. The authors did not consider the finiteness–compactness relation before introducing a definition of finitely generated objects.
This has resulted in the definitions being too restrictive.
In this section we look more closely at the finiteness–compactness relation.

If G is a group, then let $\mathcal{G}$ be the set of all fuzzy groups on G.
The supremum $\vee$ and infimum $\wedge$ in $\mathcal{G}$ are defined in the following way:
if $\mu, \nu \in \mathcal{G}$ then $\mu \wedge \nu = \mu \wedge \nu$ and $\mu \vee \nu = \wedge \{ \omega \in \mathcal{G} : \mu \vee \nu \leq \omega \}$. Thus $\mathcal{G} = (\mathcal{G}, \wedge, \vee)$ is a lattice; in fact it is complete.

Proposition 5.2.1.

If $\mathcal{C}$ is a collection of fuzzy groups from $\mathcal{G}$ then,
$\vee \mathcal{C} = \vee \{ \alpha 1_{A_\alpha} : \alpha \in L \setminus \{0\} \}$, where $A_\alpha = \vee \{ H_\nu^\alpha : \nu \in \mathcal{C} \}$.

Proof.

Since $H_\nu^\alpha \leq H_\nu^\beta$ for all $\alpha \geq \beta$ and $\nu \in \mathcal{C}$, $\vee \{ H_\nu^\alpha : \nu \in \mathcal{C} \} \leq \vee \{ H_\nu^\beta : \nu \in \mathcal{C} \}$ for all $\alpha \geq \beta$ and $\nu \in \mathcal{C}$. Consequently, the fuzzy set,

$$\mu = \vee \{ \alpha 1_{A_\alpha} : \alpha \in L \setminus \{0\} \}$$

is a fuzzy group and $\vee \mathcal{C} \leq \mu$. Let $\omega \in \mathcal{G}$ such that $\vee \mathcal{C} \leq \omega$.
Clearly, $\vee \mathcal{C} \leq \omega$ if and only if for all $\alpha \in L \setminus \{0\}$, $\cup \{ H_\nu^\alpha : \nu \in \mathcal{C} \} \subset H_\omega^\alpha$.
Since for all $\alpha \in L \setminus \{0\}$, $H_\mu^\alpha = \vee \{ H_\nu^\alpha : \nu \in \mathcal{C} \}$
is the smallest group containing $\cup \{ H_\nu^\alpha : \nu \in \mathcal{C} \}$ we must have $H_\mu^\alpha < H_\omega^\alpha$.
Thus $\mu \leq \omega$. which proves this proposition.

Corollary 5.2.2.

If $\mathcal{C}$ is a collection of fuzzy groups from $\mathcal{G}$, then:

$H^{\alpha}_{\vee\mathcal{C}} = \vee \{ H^{\alpha}_{\nu} : \nu \in \mathcal{C} \}$ and $E^{\alpha}_{\wedge\mathcal{C}} = \wedge \{ E^{\alpha}_{\nu} : \nu \in \mathcal{C} \}$.

Definition 5.2.3.

Given the lattice $\mathcal{G} = (\mathcal{G}, \wedge, \vee)$ of fuzzy groups on a crisp group G, then $\mu \in \mathcal{G}$ is ***f–compact in*** $\mathcal{G}$ if and only if for all $\alpha \in$ L and $\mathcal{C} \subset \mathcal{G}$ such that $(\mu' \vee (\vee \mathcal{C})) \geq \alpha 1_G$ then, in case $\alpha \in L_-$, there exists a finite subcollection $\mathcal{F} \subset \mathcal{C}$ such that $(\mu' \vee (\vee \mathcal{F})) \geq \alpha 1_G$, otherwise for all $\beta < \alpha$ there must exist a finite subcollection $\mathcal{F} \subset \mathcal{C}$ such that $(\mu' \vee (\vee \mathcal{F})) \geq \beta 1_G$.

Definition 5.2.4.

A fuzzy group $\mu \in \mathcal{G}$ is ***finitely generated*** if and only if for all $\alpha \in L\backslash\{0\}$ the group E^{α}_{μ} is finitely generated.

Theorem 5.2.5.

A fuzzy group $\mu \in \mathcal{G}$ is finitely generated if and only if μ is f–compact in $(G, \mathcal{G})$.

Proof.

Suppose $\mu \in \mathcal{G}$ is finitely generated, thus for all $\alpha \in L\backslash\{0\}$, $\mu^{-1}([\alpha,1])$ is a finitely generated crisp group. Clearly for all $\alpha \in L\backslash\{1\}$, $(\mu')^{-1}([0,\alpha])$ is a finitely generated crisp group. Let $\alpha \in$ L and $\mathcal{C} \subset \mathcal{G}$ be such that $(\mu' \vee (\vee \mathcal{C})) \geq \alpha 1_G$.
In case $\alpha \in L_-\backslash\{1\}$, by Corollary 5.2.2., $E^{\alpha}_{\vee\mathcal{C}} = H^{\alpha}_{\vee\mathcal{C}} = \vee \{ H^{\alpha}_{\nu} : \nu \in \mathcal{C} \}$ $= \vee \{ E^{\alpha}_{\nu} : \nu \in \mathcal{C} \}$ covers the finitely generated group $(\mu')^{-1}([0,\alpha])$.
Therefore there exists a finite subcollection $\mathcal{F} \subset \mathcal{C}$, such that $E^{\alpha}_{\vee\mathcal{F}} = \vee \{ E^{\alpha}_{\nu} : \nu \in \mathcal{F} \}$ covers $(\mu')^{-1}([0,\alpha])$. Consequently $(\mu' \vee (\vee \mathcal{F})) \geq \alpha 1_G$. If $\alpha = 1 \in L_-$ then, $(\mu')^{-1}([0,1))$ is finitely generated and $E^{1}_{\vee\mathcal{C}} = \vee \{ E^{1}_{\nu} : \nu \in \mathcal{C} \}$ certainly covers $(\mu')^{-1}([0,1))$. So there exists a finite subcollection $\mathcal{F} \subset \mathcal{C}$ such that $E^{1}_{\vee\mathcal{F}}$ covers

$(\mu')^{-1}([0,1))$. Consequently $(\mu' \vee (\vee \mathcal{F})) \geq 1_G$. In case $\alpha \notin L_-$, let $\beta < \alpha$. Clearly $\beta \neq 1$. Thus the group $(\mu')^{-1}([0,\beta])$ is finitely generated, and the proof follows similarly. The converse follows easily, by backtracking.

5.3. LAGRANGE'S THEOREM FOR FUZZY GROUPS

In this section we shall investigate the order of finite fuzzy groups.
For finite fuzzy sets it is sufficient to use Blanchard's fuzzy cardinality.
To proceed further we need to define when one fuzzy cardinal divides another and establish some other technical results.

Recall from Chapter 3 the notation used to express finite fuzzy cardinals.

Definition 5.3.1.

Let c_1 and c_2 be two finite fuzzy cardinals. We say that c_1 ***divides*** c_2 ***strongly*** if and only if there exists a finite fuzzy cardinal c such that $c_1 \otimes c = c_2$, and c_1 ***divides*** c_2 ***weakly*** if and only if for all $\alpha \in L\backslash\{0\}$, sup $E^{\alpha}_{c_1}$ divides sup $E^{\alpha}_{c_2}$.

Example 5.3.2.

Let $L = [0,1]$, $c_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ and $c_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ be two finite fuzzy cardinals. It is easily checked that c_1 divides c_2 weakly but c_1 does not divide c_2 strongly.

Proposition 5.3.3.

Let m, n $\in \mathbb{N}$, $\alpha, \beta \in L$ and $c_1, c_2, c_3 \in K_f(L)$ then,

(a) $\alpha 1_{\{0,\ldots,m\}} \otimes \beta 1_{\{0,\ldots,n\}} = (\alpha \wedge \beta) 1_{\{0,\ldots,mn\}}$,

(b) $(c_1 \vee c_2) \otimes c_3 = (c_1 \otimes c_3) \vee (c_2 \otimes c_3)$,

(c) $c_1 \otimes c_2 = \bigvee_{\gamma \in R_{c_1}} \bigvee_{\delta \in R_{c_2}} (\gamma \wedge \delta) 1_{\{0,\ldots,(\sup E^{\gamma}_{c_1})(\sup E^{\delta}_{c_2})\}}$,

(d) if $c_1(1) = c_2(1)$ then $R_{c_1 \otimes c_2} = R_{c_1} \cup R_{c_2}$ and,

$$c_1 \otimes c_2 = \bigvee_{\delta \in R_{c_1} \cup R_{c_2}} \delta 1_{\{0,\ldots,(\sup E^{\delta}_{c_1})(\sup E^{\delta}_{c_2})\}}.$$

Proof.

Part (a) is easy. To see part (b) we note,

$$\begin{aligned}
[(c_1 \vee c_2) \otimes c_3](k) &= \sup\{ (c_1 \vee c_2)(k_1) \wedge c_3(k_2) : k_1 k_2 \geq k \} \\
&= \sup\{ (c_1(k_1) \wedge c_3(k_2)) \vee (c_2(k_1) \wedge c_3(k_2)) : k_1 k_2 \geq k \} \\
&= \sup\{ c_1(k_1) \wedge c_3(k_2) : k_1 k_2 \geq k \} \vee \sup\{ c_2(k_1) \wedge c_3(k_2) : k_1 k_2 \geq k \} \\
&= (c_1 \otimes c_3)(k) \vee (c_2 \otimes c_3)(k).
\end{aligned}$$

Part (c) follows from the fact that we can write c_1 and c_2 in the following way:

$$c_1 = \bigvee_{\gamma \in R_{c_1}} \gamma 1_{\{0,\ldots,(\sup E^{\gamma}_{c_1})\}} \text{ and } c_2 = \bigvee_{\delta \in R_{c_2}} \delta 1_{\{0,\ldots,(\sup E^{\delta}_{c_2})\}},$$

and using parts (b) and (a) we obtain,

$$\begin{aligned}
c_1 \otimes c_2 &= \left(\bigvee_{\gamma \in R_{c_1}} \gamma 1_{\{0,\ldots,(\sup E^{\gamma}_{c_1})\}} \right) \otimes \left(\bigvee_{\delta \in R_{c_2}} \delta 1_{\{0,\ldots,(\sup E^{\delta}_{c_2})\}} \right) \\
&= \bigvee_{\gamma \in R_{c_1}} \left(\gamma 1_{\{0,\ldots,(\sup E^{\gamma}_{c_1})\}} \otimes \left(\bigvee_{\delta \in R_{c_2}} \delta 1_{\{0,\ldots,(\sup E^{\delta}_{c_2})\}} \right) \right) \\
&= \bigvee_{\gamma \in R_{c_1}} \bigvee_{\delta \in R_{c_2}} (\gamma 1_{\{0,\ldots,(\sup E^{\gamma}_{c_1})\}}) \otimes (\delta 1_{\{0,\ldots,(\sup E^{\delta}_{c_2})\}}) \\
&= \bigvee_{\gamma \in R_{c_1}} \bigvee_{\delta \in R_{c_2}} (\gamma \wedge \delta) 1_{\{0,\ldots,(\sup E^{\gamma}_{c_1})(\sup E^{\delta}_{c_2})\}}.
\end{aligned}$$

Finally we prove part (d). If $\gamma_1 \in R_{c_1 \otimes c_2}$. By part (c) and since $R_{c_1 \otimes c_2}$ is upper-well-ordered, $\gamma_1 = \gamma_2 \wedge \gamma_3$ for some $\gamma_2 \in R_{c_1}$ and $\gamma_3 \in R_{c_2}$.
Since L is totally ordered $\gamma_1 \in R_{c_1}$ or $\gamma_1 \in R_{c_2}$. In other words $\gamma_1 \in R_{c_1} \cup R_{c_2}$.
Hence $R_{c_1 \otimes c_2} \subset R_{c_1} \cup R_{c_2}$. Clearly $1 \in R_{c_1} \cap R_{c_2} \cap R_{c_1 \otimes c_2}$. If $\gamma_2 \in R_{c_1} \setminus \{1\}$ then let $\gamma_3 = \inf (R_{c_2} \cap [\gamma_2, 1])$. Since $c_1(1) = c_2(1)$ and the set $(R_{c_2} \cap [\gamma_2, 1])$ is finite, $E_{c_2}^{\gamma_3} \geq 1$ and $\gamma_3 \in R_{c_2}$. Consequently for all $\gamma_4 \in R_{c_1}$ such that $\gamma_4 > \gamma_2$ we have $(\sup E_{c_1}^{\gamma_4})(\sup E_{c_2}^{\gamma_3}) < (\sup E_{c_1}^{\gamma_2})(\sup E_{c_2}^{\gamma_3})$. Thus by part (c) there exists $k \in \mathbb{N}$ such that $(c_1 \otimes c_2)(k) = \gamma_3$. Hence $R_{c_1} \subset R_{c_1 \otimes c_2}$. Similarly $R_{c_2} \subset R_{c_1 \otimes c_2}$.
So $R_{c_1} \cup R_{c_2} \subset R_{c_1 \otimes c_2}$. The last identity follows easily.

Definition 5.3.4.

If μ is a finite fuzzy group we define the ***order*** $o(\mu)$ of this group by $o(\mu) = \text{card}(\mu)$.

In what follows we assume that for all fuzzy groups μ and ν, $\mu(e) = \nu(e)$. This requirement was imposed by other authors as well, see [1] and [32] p.87.
In fact in [1] $\mu(e) = 1$ for all fuzzy groups. A direct consequence of this assumption is that $o(\mu)(1) = o(\nu)(1)$. If one did not impose the above condition the following theorem would be meaningless. The definition of order in [31,32,33] is not satisfactory as we have already stated in the introduction.

Theorem 5.3.5. (***Weak Lagrange's Theorem for fuzzy groups***)

If ν is a finite fuzzy subgroup of μ, (i.e. $\nu \leq \mu$) then $o(\nu)$ divides $o(\mu)$ weakly.

Proof.

It is easy to verify that,

$$o(\mu) = \left(\bigvee_{\alpha \in R_\mu} \alpha 1_{\{0,\dots,|E_\mu^\alpha|\}} \right) \vee 1_{\{0\}} \text{ and } o(\nu) = \left(\bigvee_{\alpha \in R_\nu} \alpha 1_{\{0,\dots,|E_\nu^\alpha|\}} \right) \vee 1_{\{0\}}$$

Since $\nu \leq \mu$, for all $\alpha \in L\setminus\{0\}$, E^α_ν is a subgroup of E^α_μ and thus $\sup E^\alpha_{o(\nu)} = |\ E^\alpha_\nu\ |$ divides $|\ E^\alpha_\mu\ | = \sup E^\alpha_{o(\mu)}$. So $o(\nu)$ divides $o(\mu)$ weakly.

Definition 5.3.6.

If ν is a fuzzy subgroup of a finite fuzzy group μ; we say that ν is ***regular*** in μ if and only if for all $\alpha, \beta \in L\setminus\{0\}$ with $\alpha \leq \beta$, $[E^\alpha_\mu : E^\alpha_\nu] \geq [E^\beta_\mu : E^\beta_\nu]$.

Remark 5.3.7.

If ν is regular in a finite fuzzy group μ then $R_\nu \subset R_\mu$.

Proof.

Suppose $\alpha \in R_\nu\setminus R_\mu$. Let $\beta = \min\{\ \gamma \in R_\mu : \gamma > \alpha\ \}$. Clearly, $\beta > \alpha$, $E^\beta_\mu = E^\alpha_\mu$ and $E^\beta_\nu < E^\alpha_\nu$. Consequently, $[E^\alpha_\mu : E^\alpha_\nu] < [E^\beta_\mu : E^\beta_\nu]$. A contradiction.

Theorem 5.3.8. (***Strong Lagrange's Theorem for fuzzy groups***)

Let ν be a fuzzy subgroup of a finite fuzzy group μ then, $o(\mu)$ is strongly divisible by $o(\nu)$ if and only if ν is regular in μ.

Proof.

Suppose $o(\mu)$ is strongly divisible by $o(\nu)$. Thus there exists a finite fuzzy cardinal c such that $o(\mu) = c \otimes o(\nu)$. Since $o(\mu)(1) = o(\nu)(1)$, $c(1) \geq o(\mu)(1)$.

Without loss of generality we can assume that $c(1) = o(\nu)(1)$.

By part (d) from Proposition 5.3.3 we have:

$$o(\mu) = \bigvee_{\delta \in R_\mu} \delta 1_{\{0,\ldots,|\ E^\delta_\mu|\ \}} = c \otimes o(\nu) = \bigvee_{\delta \in R_c \cup R_\nu} \delta 1_{\{0,\ldots,(\sup E^\delta_c)|\ E^\delta_\nu|\ \}}$$

and since for $\alpha \leq \beta$, $\sup E^\alpha_c \geq \sup E^\beta_c$ we must have $[E^\alpha_\mu : E^\alpha_\nu] \geq [E^\beta_\mu : E^\beta_\nu]$.

So ν is regular in μ.

Conversely, let ν be regular in μ and $c \in K_f(L)$ be given by,

$$c = \left(\bigvee_{\alpha \in R_\mu} \alpha 1_{\{0,\ldots,[E_\mu^\alpha : E_\nu^\alpha]\}} \right) \vee 1_{\{0\}}$$

Clearly, $c(1) = o(\nu)(1)$, and thus by above Remark, part (d) of the previous Proposition, $o(\mu) = c \otimes o(\nu)$.

Example 5.3.9.

Let $L = [0,1]$, $\mu = \frac{1}{2}1_{S(4)} \vee \frac{1}{4}1_{S(8)}$ and $\nu = \frac{1}{3}1_{S(2)} \vee \frac{1}{3}1_{S(4)}$. Clearly ν is a fuzzy subgroup of μ, $o(\mu) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $o(\nu) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3})$.
From previous example we note that $o(\nu)$ divides $o(\mu)$ weakly and $o(\nu)$ does not divide $o(\mu)$ strongly. Hence the weak Lagrange Theorem holds but not the strong one. However if ν is redefined to $\nu = \frac{1}{2}1_{S(2)} \vee \frac{1}{4}1_{S(4)}$ then letting $c = (1,1)$ we obtain $o(\mu) = c \otimes o(\nu)$. Thus strong Lagrange Theorem is true in this case.

Definition 5.3.10.

If ν is a regular subgroup of a finite fuzzy group μ then we define the ***index*** of ν in μ, by, $[\mu{:}\nu] = \vee \{ c \in K_f(L) : o(\mu) = c \otimes o(\nu) \}$.
By looking at 5.3.8 it is easily checked that $[\mu{:}\nu] \in K_f(L)$ and $o(\mu) = [\mu{:}\nu] \otimes o(\nu)$.

5.4. CLASSIFICATION OF CYCLIC FUZZY GROUPS

Motivated by the theorem and the definition from the section on finitely generated fuzzy groups we introduce and characterise the concept of a fuzzy cyclic group. Note that this is different from other authors. They require the support to be a crisp cyclic group. We are less restrictive in the definition that follows.

Definition 5.4.1.

A fuzzy group $\mu : G \to L$ is *cyclic* if and only if for all $\alpha \in L\setminus\{0\}$ the crisp group E_{μ}^{α} is cyclic.

The following two technical lemmas from the crisp group theory are required in this section. These lemmas should be found somewhere in the literature, however we were unable to locate them so we include them here. The Lemma 5.4.3 is of a highly non–trivial nature since it uses Dirichlet's Theorem.

Lemma 5.4.2.

If G_1 is an infinite cyclic subgroup of an infinite cyclic group G_2 and $f_1 : G_1 \to \mathbb{Z}$ is an isomorphism, then there exits $n \in \mathbb{N}$ and an isomorphism $f_2 : G_2 \to \frac{1}{n}\mathbb{Z}$ such that $f_{2|G_1} = f_1$, where $\frac{1}{n}\mathbb{Z} = \{ \frac{1}{n} k : k \in \mathbb{Z} \}$ is a cyclic group generated by $\frac{1}{n}$.

Proof.

Clearly all cyclic super–groups (G is a super–group of H iff H is a subgroup of G) of $\mathbb{Z}$ are of the from $\frac{1}{n}\mathbb{Z}$. Let g_1 and g_2 be cyclic generators of G_1 and G_2 respectively. Clearly $g_1 = ng_2$ or $g_1 = -ng_2$ for some $n \in \mathbb{N}$. If $g_1 = ng_2$ then let $g_3 = g_2$ otherwise, let $g_3 = -g_2$. Now $g_1 = ng_3$ and g_3 is a cyclic generator of G_2. Define $f_2 : G_2 \to \frac{1}{n}\mathbb{Z}$ as follows: if $x \in G_2$ then $x = mg_3$ for some $m \in \mathbb{Z}$, let $f_2(x) = f_1(g_1)\frac{m}{n}$. It is easy to verify that f_2 is the required isomorphism.

Lemma 5.4.3.

If G_1 is a cyclic subgroup of a finite cyclic group G_2 and $f_1 : G_1 \to S(|G_1|)$ is an isomorphism, then there exists an isomorphism $f_2 : G_2 \to S(|G_2|)$ such that $f_{2|G_1} = f_1$.

Proof.

Let $n = |G_1|$ and $m = |G_2|$. Clearly $k = \frac{m}{n} \in \mathbb{N}$. Without loss of generality we can replace G_1 and G_2 by $S(n)$ and $S(m)$ respectively. Since f_1 is an isomorphism we can choose $t \in \{0, ..., n-1\}$ such that $f_1(e^{2\pi it/n}) = e^{2\pi i/n}$ and then $e^{2\pi it/n}$ is a generator of $S(n)$. Consequently we have $\gcd(t,n) = 1$.

The k–th roots of $e^{2\pi it/n}$ are given by $e^{2\pi i(t+nj)/nk}$ where $j = 0, 1, ..., k-1$.

Using Dirichlet's Theorem [20] we can find $j \in \mathbb{N}$ such that $t+nj$ is prime and $\gcd(t+nj,nk) = 1$. Let $q = (t+nj) \bmod nk$. We now show that $\gcd(q,kn) = 1$.

Firstly $t+nj = q + y(nk)$ for some $y \in \mathbb{N}$. If $\gcd(q,nk) = x > 1$, i.e. $q = xr_1$ and $nk = xr_2$ then $t+nj = xr_1 + yxr_2 = x(r_1 + yr_2)$. In case $y = 0$, $q = t + nj$ and so $\gcd(q,nk) = 1$, otherwise $r_1 + yr_2 > 1$, which contradicts the fact that $t+nj$ is prime. Clearly q can be written in the form $t+nr$ for some $r \in \{0, ..., k-1\}$.

Thus $e^{2\pi i(t+nr)/nk}$ is a k–th root of $e^{2\pi it/n}$ and a generator of $S(m)$.

Define $f_2 : S(m) \to S(m)$ as follows: given $e^{2\pi ia/nk} \in S(m)$ there exists unique $l \in \{0, ..., nk-1\}$ such that $e^{2\pi ia/nk} = e^{2\pi il(t+nr)/nk}$ let $f_2(e^{2\pi ia/nk}) = e^{2\pi il/nk}$.

This clearly gives a well–defined isomorphism. We now show that $f_{2|S(n)} = f_1$.

Let $e^{2\pi ib/n} \in S(n)$. Clearly $e^{2\pi ib/n} = e^{2\pi ist/n}$ for some $s \in \{0, ..., n-1\}$.

Also, $(e^{2\pi i(t+nr)/nk})^k = e^{2\pi it/n}$. So,

$$
\begin{aligned}
f_2(e^{2\pi ist/n}) &= f_2(e^{2\pi it/n})^s \\
&= f_2(e^{2\pi ik(t+nr)/nk})^s \\
&= (e^{2\pi ik/nk})^s \\
&= (e^{2\pi i/n})^s \\
&= f_1(e^{2\pi it/n})^s \\
&= f_1(e^{2\pi ist/n}).
\end{aligned}
$$

which completes the proof.

Theorem 5.4.4.

Suppose μ is a cyclic fuzzy group. Then,

(finite case)

If μ is finite then μ is strongly isomorphic to a finite cyclic fuzzy group ν given by:

$$\nu = \bigvee_{\alpha \in R_\mu} \alpha 1_{S(|E_\mu^\alpha|)}$$

where R_μ is either finite or a sequence converging to 0. The support of the finite cyclic fuzzy group ν is contained in a unit circle and is not necassarily a finite cyclic group.

(infinite case)

If μ is infinite then there exists $\beta \in R_\mu$ such that E_μ^β is infinite and μ is strongly isomorphic to an infinite cyclic fuzzy group ν given by:

$$\nu = \left(\bigvee_{\alpha \in R_\mu \cap (0,\beta)} \alpha 1_{(1/[E_\mu^\alpha : E_\mu^\beta])\mathbb{Z}} \right) \vee \left(\bigvee_{\alpha \in R_\mu \cap [\beta,\mu(e))} \alpha 1_{[E_\mu^\beta : E_\mu^\alpha]\mathbb{Z}} \right) \vee \mu(e) 1_{\{0\}}$$

where $R_\mu \cap (0,\beta)$ is finite or a sequence decreasing to 0 and $R_\mu \cap [\beta,\mu(e))$ is finite or an increasing sequence which does not contain its limit point. The support of the infinite cyclic group ν is contained in the real line $\mathbb{R}$ and is not necessarily an infinite cyclic group.

Proof.

(finite case) For all $\alpha \in L \backslash \{0\}$, E_μ^α is finite, consequently $[\alpha,1] \cap R_\mu$ is finite. This proves that R_μ is either finite or a sequence converging to zero.

Thus the set R_μ can be written as $R_\mu = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ or $R_\mu = \{\alpha_1, \alpha_2, ...\}$ where $\alpha_n < \alpha_{n-1}$. Let $f_1 : E_\mu^{\alpha_1} \to S(|E_\mu^{\alpha_1}|)$ be an isomorphism. Since $E_\mu^{\alpha_2}$ is isomorphic to $S(|E_\mu^{\alpha_2}|)$ and $S(|E_\mu^{\alpha_1}|) < S(|E_\mu^{\alpha_2}|)$, by Lemma 5.4.3. we can find an isomorphism $f_2 : E_\mu^{\alpha_2} \to S(|E_\mu^{\alpha_2}|)$ such that $f_{2|E_\mu^{\alpha_1}} = f_1$. Given an isomorphism $f_{n-1} : E_\mu^{\alpha_{n-1}} \to S(|E_\mu^{\alpha_{n-1}}|)$ we construct an isomorphism $f_n : E_\mu^{\alpha_n} \to S(|E_\mu^{\alpha_n}|)$

again with the use of Lemma 5.4.3. Define an isomorphism

$f : \mathrm{supp}(\mu) \to \cup \{ S(|E_{\mu}^{\alpha}|) : \alpha \in R_{\mu} \}$ by letting $f(g) = f_n(g)$ such that $g \in E_{\mu}^{\alpha_n}$. Since f_n's are restrictions, f is well defined. Clearly, $\nu_{|\mathrm{supp}(\nu)} = f(\mu_{|\mathrm{supp}(\mu)})$.

(infinite case) The existence of the required β is clear. To see that $R_{\mu} \cap (0,\beta]$ has the stated properties we suppose that $R_{\mu} \cap (0,\beta]$ is infinite.

Let $\alpha \in R_{\mu} \cap (0,\beta)$ then E_{μ}^{α} is an infinite cyclic super–group of the infinite cyclic group E_{μ}^{β}. Clearly there are only a finite number of subgroups between E_{μ}^{β} and E_{μ}^{α} consequently $R_{\mu} \cap [\alpha,\beta]$ is finite. Thus $R_{\mu} \cap (0,\beta]$ is either finite or a sequence converging to zero. Now suppose $R_{\mu} \cap [\beta,\mu(e))$ is infinite. First we show that $E_{\mu}^{\mu(e)}$ is isomorphic to the trivial group. If $E_{\mu}^{\mu(e)}$ is non–trivial then there exists $n \in \mathbb{N}$ such that $nE_{\mu}^{\beta} = E_{\mu}^{\mu(e)}$. However there is only a finite number of subgroups between E_{μ}^{β} and $E_{\mu}^{\mu(e)}$, which contradicts the fact that that $R_{\mu} \cap [\beta,\mu(e))$ is infinite. Thus $E_{\mu}^{\mu(e)}$ is a trivial group. Following a similar argument as in the finite case by the use of Lemma 5.4.2 instead of Lemma 5.4.3 we arrive at the remaining results.

Theorem 5.4.5.

Two finite cyclic fuzzy groups with the same order are strongly isomorphic.

Proof.

The proof is similar to the one in Theorem 5.4.4. Using Lemma 5.4.3, we are able to construct the required isomorphism by starting with the top level groups which are isomorphic.

Example 5.4.6.

Two infinite cyclic fuzzy groups are not necessarily strongly isomorphic.

To see that consider two infinite cyclic fuzzy groups $\mu, \nu : \mathbb{Z} \to \{0,\alpha,1\}$ given by, $\mu = \alpha 1_{\mathbb{Z}} \vee 1_{2\mathbb{Z}}$ and $\nu = \alpha 1_{\mathbb{Z}} \vee 1_{3\mathbb{Z}}$. Let $f_1 : 2\mathbb{Z} \to 3\mathbb{Z}$ be an isomorphism, then without loss of generality we may assume that $f_1(2n) = 3n$ for all $n \in \mathbb{Z}$.

It is clearly not possible to extend f_1 to an isomorphism from $\mathbb{Z}$ to $\mathbb{Z}$. Consequently μ and ν are not strongly isomorphic. Clearly there exits a bijection $g : \mathbb{Z} \to \mathbb{Z}$ such that $g(\mu) = \nu$.
This means that even if we consider two infinite cyclic fuzzy groups with the same order they may not be strongly isomorphic. (It may be easily verified that two cyclic fuzzy groups of the same order, using Blanchard's cardinals, are weakly isomorphic.).

Example 5.4.7.

The following are examples of finite and infinite cyclic fuzzy groups.
The membership grade L is taken to be the unit interval.

$$\nu = \bigvee_{n \in \mathbb{N}} \tfrac{1}{n} 1_{S(2^n)}$$

$$\mu = \left(\bigvee_{n \in \mathbb{N}} \tfrac{1}{2n} 1_{(2^{-n})\mathbb{Z}} \right) \vee \left(\bigvee_{n \in \mathbb{N}\setminus\{1\}} (1 - \tfrac{1}{2n}) 1_{(2^n)\mathbb{Z}} \right)$$

Note that the support of μ is given by $\cup \{ (2^{-n})\mathbb{Z} : n \in \mathbb{N} \}$ which is a non–cyclic group dense in $\mathbb{R}$. Similarly the support of ν is a dense non–cyclic subgroup of the unit circle.

In the crisp case every finitely generated abelian group G is a finite direct sum of cyclic groups. Unfortunately this is not so in the fuzzy situation. The following finitely generated abelian fuzzy group will suffice:

$$\nu = \bigvee_{n \in \mathbb{N}} \tfrac{1}{n} 1_{H^n}$$

where $H^n = \{ (a_i) : a_i \in \mathbb{Z}$ for $i \in \{1,\ldots,n\}$ and $a_i = 0$ otherwise $\}$ is a subgroup of $H = \{ (a_i) : a_i \in \mathbb{Z}, i \in \mathbb{N} \}$. The support of this fuzzy group is not finitely generated so we would require an infinite direct product of infinite cyclic groups to represent ν. However we have:

$$\nu = \prod_{n \in \mathbb{N}} (1_{\{e\}} \vee \tfrac{1}{n} 1_{H^n})$$

Maybe, an appropriate notion of fuzzy finite direct product would allow to give us a positive answer to the above problem.

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