Zeno Dynamics in Quantum Statistical Mechanics

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14th May 2003 (v2)
by

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Reviews:

- H Nakazato, M Namiki, S Pascazio (1996) Internat. J. Modern Phys. B **10** 247.
- M A B Whitaker (2000) Progress in Quantum Electronics **24** 1-106.

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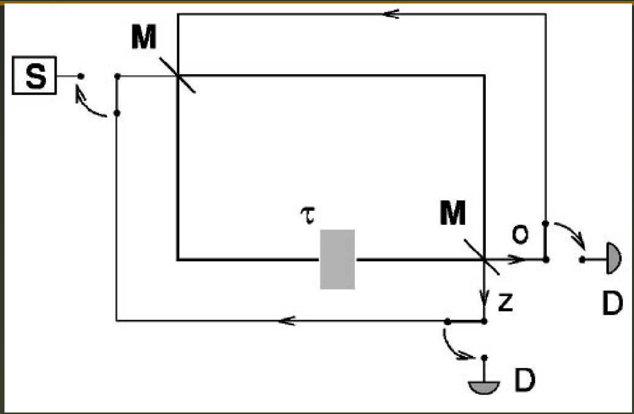

A Mach–Zehnder interferometer can be used to detect the presence of a black sample ($\tau = 0$), without absorbing the probing particle, if it is prepared to be in the Zeno channel (Z), and detected after L rounds in the interferometer, in the limit $L \to \infty$, see P Facchi *et al.* (2002) Phys. Rev. A **66** 012110 (pictures courtesy of S. Pascazio).

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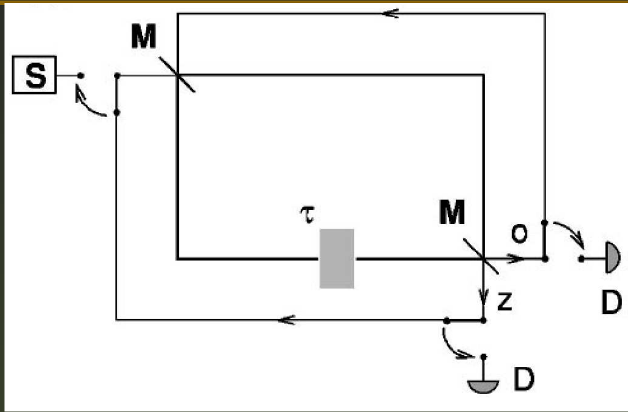

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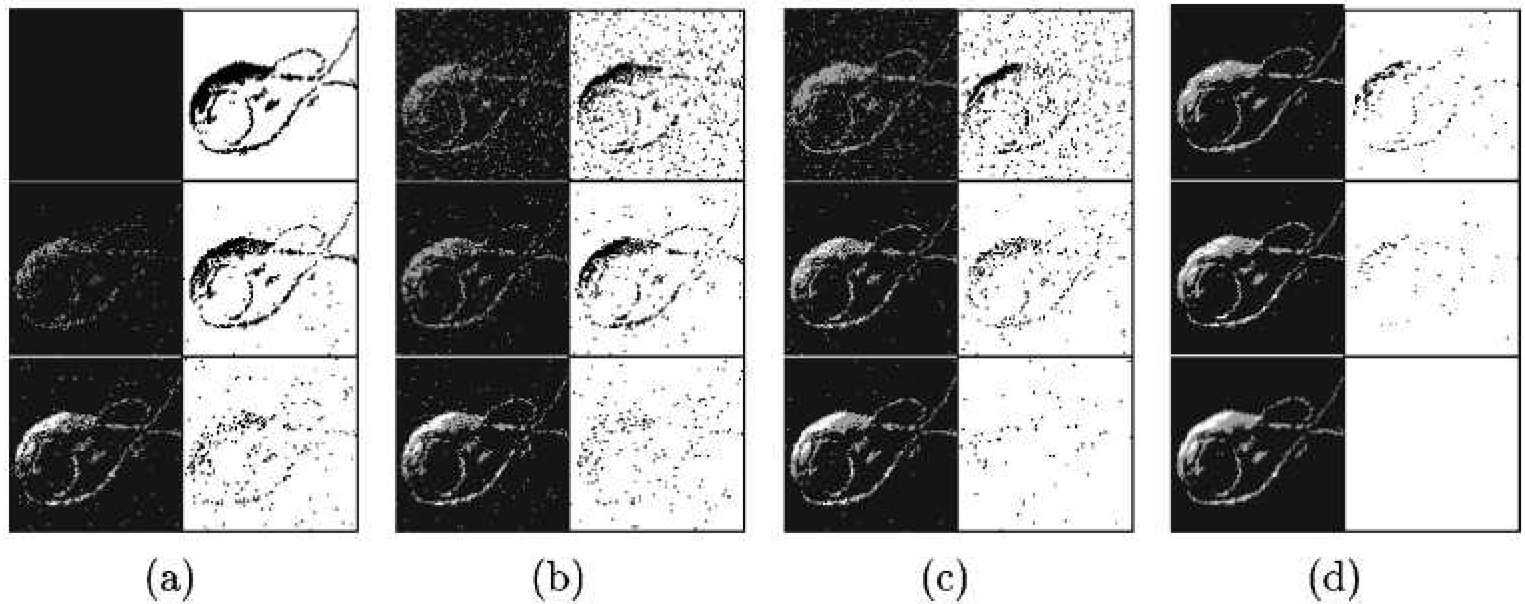

FIG. 5. Comparison of standard and Zeno tomographic techniques. In each frame, top left = reconstruction by standard technique; top right = misinterpreted pixels by the standard technique; center left = reconstruction by Zeno technique with $L=10$; center right = misinterpreted pixels by the Zeno technique with $L=10$; bottom left = reconstruction by Zeno technique with $L=165$; bottom right = misinterpreted pixels by the Zeno technique with $L=165$. The mean number of absorbed particles per pixel (irradiation) is $N_a = 1.7$, 2.3, 4, and 13 for frames (a), (b), (c), and (d), respectively. The total number of particles N (total energy) scales approximately as $3N_a$, $1.8N_a$, and $6.5N_a$ for top, center, and bottom reconstructions, respectively. We used $\tau_w = 0.99$, $\tau_g = 0.96$, and $\tau_b = 0.8$. The sample consists of 10 000 ($=100\times 100$) pixels, where white, gray, and black occur with frequencies $\alpha_w = 0.02$, $\alpha_g = 0.07$, and $\alpha_b = 0.91$, respectively. The number of misinterpreted pixels are (top to bottom) (a) 968, 786, 315; (b) 942, 596, 212; (c) 717, 382, 68; (d) 205, 69, 0.

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- Allows to identify the Zeno subspace to which the evolution will be (approximately) confined.
- The Zeno dynamics is identified as ordinary quantum dynamics with boundary conditions [Facchi, Pascazio, Scardicchio, Schulman (2001) Phys. Rev. A **65**].

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$$\omega_\beta(A) \stackrel{\text{def}}{=} \frac{\mathrm{Tr}_{\mathcal{H}}\left(\mathrm{e}^{-\beta H} A\right)}{\mathrm{Tr}_{\mathcal{H}}\left(\mathrm{e}^{-\beta H}\right)}, \quad \text{for } A \in \mathcal{A},\ \dim \mathcal{H} < \infty.$$

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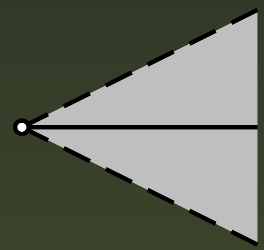

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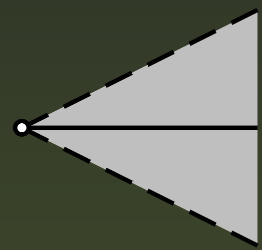

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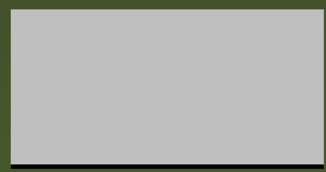

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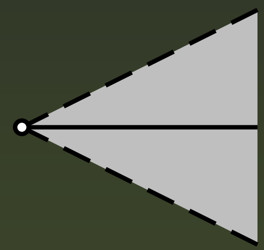

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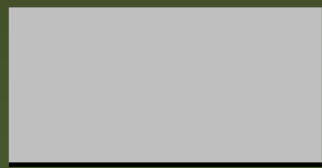
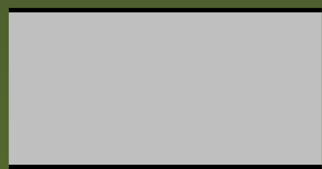


- **Quantum Statistical Mechanics (AUS):** The Zeno limit is needed on the boundary of a strip in $\mathbb{C}$. No sectorial result can be used.



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iii) *and the group* $W(t)$ *induces an automorphism group* τ^E *of* $\mathcal{A}_E$*.*

iv) *The vectors* $W(z) A_E \Omega$*,* $A_E \in \mathcal{A}_E$*, extend analytically to the strip* $0 < \operatorname{Im} z < \beta/2$ *and are continuous on its boundary.*

- The (AZC) renders applicable the methods of perturbation theory!

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$\Longrightarrow$ This effect generalizes to projections onto τ^P-invariant subspaces ($\tau^P(E) = E$).

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From perturbation theory follows the uniform estimate

$$U_{\Lambda'}(t) = U_{\Lambda' \setminus \Lambda}(t) U_\Lambda(t) + \int_0^t U_{\Lambda' \setminus \Lambda}(\tau) U_\Lambda(\tau) W_\Phi(\Lambda;\Lambda') \mathrm{d}\tau + O(t^2). \qquad (\dagger)$$

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Let $H_E \stackrel{\mathrm{def}}{=} (H^{1/2}E)^*(H^{1/2}E)$ be the self-adjoint operator associated with the quadratic form $H^{1/2}E$ on the form domain $D(H^{1/2}E)$. H_E is a self-adjoint extension of the (generally non-closed) operator EHE. H_E is defined, and acts nontrivially, on a closed subspace of $E\mathcal{H}$.

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Theorem. *If* ***H_E is densely defined on the whole space $\mathcal{H}$*** *then*

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Although only applicable to semibounded Hamiltonians, this result yields a much sharper condition for the existence of Zeno dynamics than AZC, and a more general characterization of the Zeno dynamics: We had to pose (besides AZC) the stronger condition that $\mathcal{A}_E\Omega$ contains a dense set of analytic elements for *the original Hamiltonian H*.

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Corollary. *Let $\beta > 0$. Let $\Lambda_\alpha \to \infty$ be such that the local dynamics converges uniformly, and the net of local Gibbs states ω_{Λ_α} has a thermodynamic limit point.*
If E_{Λ_α} converges in norm to a projection E in $\mathcal{A}$ satisfying (AZC),
then $\omega_E(A_E) \stackrel{\text{def}}{=} \lim_\alpha \omega^G_{E_{\Lambda_\alpha}}(A_E)$ is a (τ^E, β)-KMS state on $\mathcal{A}_E$.

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This state is also the strong coupling ($\lambda \to \infty$) limit of the Gibbs equilibria for

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[M Fannes, R F Werner (1995) Helv. Phys. Acta **68** 635]

Example: The X-Y Model 1/3

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A spin chain over $\mathbb{X} = \mathbb{Z}$, with $\mathcal{H}_x = \mathbb{C}^2$, with observable algebras $\mathcal{A}_{[n,m]}$, $n \leq m \in \mathbb{Z}$ generated by **creation** and **annihilation** operators a_x, a_x^*, $n \leq x \leq m$, satisfying CAR

$$[a_x, a_y] = 0 = [a_x, a_y^*],\ x \neq y, \quad \{a_x, a_x^*\} = 1,\ \{a_x, a_x\} = 0.$$

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