FIXED POINTS OF SINGLE - VALUED AND MULTI - VALUED MAPPINGS WITH APPLICATIONS

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SIMFUMENE STOFILE

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Abstract

The relationship between the convergence of a sequence of self mappings of a metric space and their fixed points, known as the stability (or continuity) of fixed points has been of continuing interest and widely studied in fixed point theory. In this thesis we study the stability of common fixed points in a Hausdorff uniform space whose uniformity is generated by a family of pseudometrics, by using some general notions of convergence. These results are then extended to 2-metric spaces due to S. Gähler. In addition, a well-known theorem of T. Suzuki that generalized the Banach Contraction Principle is also extended to 2-metric spaces and applied to obtain a coincidence theorem for a pair of mappings on an arbitrary set with values in a 2-metric space. Further, we prove the existence of coincidence and fixed points of Ćirić type weakly generalized contractions in metric spaces. Subsequently, the above result is utilized to discuss applications to the convergence of modified Mann and Ishikawa iterations in a convex metric space. Finally, we obtain coincidence, fixed and stationary point results for multi-valued and hybrid pairs of mappings on a metric space.

Keywords: Fixed points, coincidence points, endpoints, stability, metric and 2-metric spaces, uniform spaces and multi-valued mappings.

American Mathematical society subject classification(2010): 47H10, 54H25.

Declaration

Except for the references that have been accurately cited and discussed herein, the content of this thesis represents my own work. The entire thesis has neither been nor is concurrently being submitted to any other academic institution for the purpose of obtaining a degree.

Simfumene Stofile

Grahamstown R.S.A

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Table of Contents

Symbols and Notations

$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of non-negative reals
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{N}$	The set of natural numbers
$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$	The set of naturals including infinity
$A \setminus B$	Complement of set B in set A
(X, d)	A metric space with a metric d
(X, ρ)	A 2-metric space with a 2-metric ρ
$(X, \mathcal{U})$	A uniform space with a uniformity $\mathcal{U}$
$CL(X)$	The set of all non-empty and closed subsets of X
$B(X)$	The set of all non-empty and bounded subsets of X
$CB(X)$	The set of all non-empty, closed and bounded subsets of X
$\forall$	For all
$\exists$	There exists
sup	Supremum
inf	Infimum
lim sup	Limit supremum
lim inf	Limit infimum
$\delta(A)$	$\sup\{d(x, y) : x, y \in A\}$
$\prod_{n\in\mathbb{N}} X_n$	is a sequence in X such that $x_n \in X_n \;\; \forall n \in \mathbb{N}$

Dedication

In memory of my father Wellington Oliver Stofile.

Introduction

0.1 General Background

Let X be a non-empty set and T a mapping of X into X (resp. into a collection of non-empty subsets of X). Then a point $z \in X$ is called a fixed point of T if $Tz = z$ (resp. $z \in Tz$). A topological space is said to have a fixed point property if every continuous self mapping on it has a fixed point. The problem of investigating sufficient conditions for the existence of a fixed point is one of the most vigorous among the fundamental branches of topology and functional analysis. In particular, fixed point theorems have extensive applications in proving existence and uniqueness of solutions of various functional equations. These theorems have found applications in the theory of differential and integral equations, dynamical systems, theory of games and mathematical economics among others. The relevance of fixed point theorems, with those of existence theorems for functional equations could be found in Banach [20], Nemyckii [95], Saaty [110] and Zeidler [145] while applications to game theory and mathematical economics could be found in Kim C and Border [73], Neumann and Morgenstern [96] and Efe A. Ok [97].

Broadly speaking, the study of fixed points (or fixed point theory) may be classified into two categories, namely, the Topological Fixed Point Theory and the Metric

Fixed Point Theory. However, the two classes are not mutually exclusive in a true sense due to the proof techniques involved. The former one largely involves the study of spaces with the fixed point property while the latter one involves the study of fixed points depending on the mapping conditions on the spaces under consideration.

Regarding the topological fixed point theory, it began with the classical fixed point theorem of Brouwer [31] in 1912 which states that every continuous mapping from a closed unit ball in $\mathbb{R}^n$ to itself has at least one fixed point. In $\mathbb{R}$ the above theorem means that every continuous mapping on unit interval has a fixed point. This one dimensional case of the above theorem is an easy consequence of the familiar mean value theorem. For various generalizations of Brouwer's theorem, including the infinite dimensional case due to Schauder [111, 112, 113, 114] and others, we refer to Smart [127]. The above celebrated result has tremendous applications in game theory and mathematical economics and the famous works of Neumann and Morgenstern [96] and John F. Nash Jr. [93, 94] fall in this area.

Regarding the metric fixed point theory, we recall an early work of Banach [20] in 1922 which states that every contraction mapping T of a complete metric space (X, d) into itself has a unique fixed point (recall that T is a contraction if $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ and $0 \leq k < 1$). The above theorem is constructive in its nature and provides a mechanism to arrive at the required fixed point. This is essentially done by using the convergence of Picard iterates. Again, this theorem has been extensively used in the study of solutions of various operator equations, including numerical approximations cf. Agarwal et al. [6, 7], Kirk and Sims [76] and Zeidler

[145]. Some notable generalizations of the above theorem are contractive mappings (mappings T satisfying the condition that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X,\ x \neq y$) by Edelstein [42] and nonexpansive mappings (mappings T satisfying the condition that $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$) by Browder [32]. These mappings have again found a wide range of applications to the theory of monotone operators and variational inequalities (cf. Zeidler [146, 147]). Another interesting generalization in this direction is due to Boyd and Wong [30] where the right hand side in each of the above inequalities is a function ϕ, from positive reals into itself satisfying certain properties. These mappings are called nonlinear contractions or ϕ-contractions. The minimum common denominator for the above classes of mappings is that they are all continuous. For an excellent discussion on metric fixed point theory, we refer to Goebel and Kirk [47].

Contractive fixed point theory falls within the area of the metric fixed point theory which is guided by the following fact that was noticed by R. Kannan [64]. Requiring a mapping to be a contraction mapping, amounts to demanding a strong continuity condition. Based on this fact, R. Kannan [64] - [68] obtained a series of results in which the mappings under investigation were not necessary required to be continuous and a fixed point can again be approximated by using the same convergence procedure as by Banach [20]. The above results of Kannan [64] - [68] lead to several contractive conditions in metric fixed point theory which are captured in Rhoades [104]. Out of 125 contractive conditions listed by Rhoades [104], 25 are independent and one of the most general contractive condition included in these 25 conditions is due to Ćirić [34]. Subsequent refinements can be found in J. Kincses and V. Totic

[74], Jachymski [56] and Collaco and Carvalho E silva [36]. However, regarding the continuity requirement, it is now settled that all contractive mappings as listed in Rhoades [104] and others are, in fact, continuous at the fixed point (see Rhoades [108]).

There have been several extensions of the known fixed point theorems for single-valued mappings to the case of point to set mappings or multi-valued mappings. An analogue of the Banach contraction principle was obtained by Nadler [90] which states that every multi-valued contraction mapping on a complete metric space with closed bounded values has a fixed point. This result still occupies an important place in fixed point theory of multi-valued mappings. For a comprehensive collection of results we note Nadler [90]. The corresponding fixed point theory using various contractive conditions as mention earlier for multi-valued mappings is now well-developed. For useful references we refer to Hicks and Rhoades [49] among others. Hybrid fixed point theory for nonlinear mappings is relatively a new development within the ambit of the fixed point theory of point to set mappings (multi-valued mappings) with a wide range of applications (see for instance Granas and Dugundji [48]).

0.2 The Present Thesis

The problem of investigating sufficient conditions under which the convergence of a sequence of mappings on a metric space implies the convergence of the sequence of their fixed points has been of continuing interest. In fixed point theory the problem is known as stability (or continuity) of fixed points. The origin of this problem seems into a result of Bonsall [28] where he proved that pointwise convergence of a sequence of contraction mappings on a complete metric space to a contraction mapping implies convergence of the sequence of their fixed points to a fixed point of the limit mapping, which is again a contraction mapping. This result has been applied to solve certain initial value problems. For a related result, we refer to Sonnenschein [129]. Subsequently, Nadler [89] (see also Fraser and Nadler [44]) by replacing the completeness of the space by the existence of fixed points proved a similar result under uniform convergence. For related results in this direction using various contractive conditions on different settings we refer to [78, 116, 124, 126].

It is well known that fixed points can be viewed as solutions of various operator equations and in many cases a localized version (where the domain of definition of a given operator is a nonempty subset of the given space) of a particular theorem is found to be more useful. In respect of stability, uniform convergence and pointwise convergence play an important role. However, when the domain of definition of all mappings in question is not the same space nor a unique nonempty subset of it, the above notions do not work. This difficulty has recently been overcomed by Barbet and Nachi [22] where some new notions of convergence have been introduced and utilized to obtain stability results in a metric space. These results generalize the earlier results of Bonsall [28] and Nadler [89].

Uniform spaces form a natural extension of metric spaces and include locally convex spaces as special cases. Therefore, it is interesting to investigate the stability problem in uniform spaces. Here, our intention is to study the stability of common fixed points for a pair of sequence of mappings on a uniform space using the Barbet - Nachi type convergence. *In Chapter 1 of this thesis, using the above ideas of Barbet and Nachi [22] and a result of Jungck [60] on common fixed points of commuting continuous mappings, we obtain stability results for common fixed points in a Hausdorff uniform space whose uniformity is generated by a family of pseudometrics. These results generalize the results of Mishra and Kalinde [86] among others and include the results of Barbet and Nach [22], Bonsall [28] and Nadler [89] as special cases when the space under consideration turns out to be metrizable. For related references on stability of fixed points in uniform spaces, we refer to [5, 13, 82, 84, 86, 105, 118, 119].*

In 1928 Menger [81] introduced the notion of generalized metric spaces. Many mathematicians had not paid much attention to Menger theory about the generalized metric spaces. A new development began in 1962 when S. Gähler [46] introduced the notion of 2-metric spaces. The concept of a metric abstracts the properties of distance function, while the concept of a 2-metric abstracts the properties of area function for a triangle determined by a triplet in Euclidean spaces. This notion has been considered by several authors (see Freese and Cho [45]), who have notably generalized Banach's contraction principle to obtain fixed point theorems, for example Abd El-Monsef, Abu-Donia, Abd-Rabou [4], Ahmed [8], Iseki [51, 52, 53, 54], Khan [71], Mishra [83], Naidu [91], Naidu and Prasad [92], Rhoades [106], Singh, Tiwari and Gupta [122], White [141], Zhang [148] and others. The basic philosophy is that since a 2-metric measures area, a contraction should send the space towards a configuration of zero area, which is to say a line (see Aliouche and Simpson [11]). *In Chapter 2, we further extend the results of Barbet and Nachi to 2-metric spaces due to Gähler*

[46]. We note the results so obtained are significant in the sense that 2-metric spaces differ substantially in terms of their topological properties from those of metric spaces.

Now, we turn our attention again back to the Banach contraction principle. Suzuki [134] proved a fixed point theorem which is a generalization of the above principle. The novelty in his theorem is that a contractive condition is assumed to hold not for all elements of the domain of the mapping under consideration, but only for elements satisfying an additional condition. The above result of Suziki [134] has been generalized further by several authors (see, for instance [10], [38], [40], [99], [135]) with some interest in applications on the existence of common solutions of certain functional equations (see Singh and Mishra [120]). *In Section 2.4 of Chapter 2 we proved an analogue of the Suzuki theorem in a 2-metric space. As an application we obtain a coincidence theorem for a pair of mappings in an arbitrary set with values in a 2-metric space.*

Weakly contractive mappings introduced by Alber and Guerre-Delabriere [9], forms a wider class of mappings which contains the classical Banach contraction as a special case and is closely related to the nonlinear contractions of Boyd and Wong [30]. Alber and Guerre-Delabriere [9] obtained certain fixed point theorems in Hilbert spaces for weakly contractive mappings and acknowledged that their results were true at least for uniformly smooth and uniformly convex Banach spaces. Subsequently, Rhoades [109] extended some of their results to complete metric spaces under less restrictive conditions and thus establishing that their results are still valid for arbitrary Banach spaces.

Recently, Dutta and Choudhury [41] generalized the weak contractive condition and introduced the notion of (ψ, ϕ) weak contractive mappings, where ψ and ϕ are functions from positive reals into itself satisfying certain conditions. They proved a fixed point theorem for a self mapping, which in turn generalizes the above result of Rhoades [109]. Beg and Abbas [24] obtained a common fixed point theorem extending weak contractive condition for two mappings. In this direction, Zhang and Song [149] introduced the concept of a generalized ϕ-weak contraction condition and obtained a common fixed point for two mappings, while Doric [39] proved a common fixed point theorem for generalized (ψ, ϕ)-weak contractions . *In Chapter 3, we study the notion of Ćirić [34] type weakly generalized contraction mappings in a metric space and prove theorems concerning the existence of coincidence and fixed points of such mappings. Further, applications regarding the convergence theorems for modified Mann iterations [79] and modified Ishikawa iterations [55] in a convex metric space are also considered.*

The existence of stationary points (or endpoints) of multi-valued mappings have been studied by several authors (cf. [2, 12, 16, 17, 26, 50, 59, 88, 140, 142, 143, 144] and others). *In Chapter 4 we obtain fixed and stationary point theorems in metric spaces without using the completeness of the space and continuity conditions. These mappings satisfy the well-known* $(E.A)$ *property introduced and studied by Aamri and Moutawakil [1] for the first time. It is interesting to note that the above property presents a nice generalization of non-compatible mappings, for compatibility conditions we refer to Jungck [61]. In addition, we also obtain results on stationary points for generalized hybrid pairs of single-valued and multi-valued mappings. The results* *so obtained extend and generalize certain results of Amini-Harandi [12], Moradi et al. [88] and others.*

In Chapter 5 we introduce the notion of set-valued generalized asymptotic contraction of Meir-Keeler type, which includes the known notions of asymptotic contractions due to Fakhar [43], Kirk [75] and Suzuki [131]. Subsequently, this notion is utilized to obtain coincidence and fixed point theorems for such contractions which generalize and unify a number of known results due to Fakhar [43], Wlodarczyk et al. [142] among others.

Definitions, theorems, corollaries and remarks are numbered per Chapter and sequentially per section, for example, Definition 1.2.6 means the sixth definition of the second section of Chapter 1.

To the best of our knowledge, the results stated below are our own major results in this thesis:

Theorem 1.4.2, Theorem 1.4.3, Theorem 1.4.4, Theorem 2.3.1, Theorem 2.3.2, Theorem 2.3.4, Theorem 2.3.6, Theorem 2.3.7, Theorem 2.4.2, Theorem 3.3.1, Theorem 4.3.1, Theorem 4.3.4 and Theorem 5.3.3.

Chapter 1

Stability Results in Uniform Spaces

1.1 Introduction

In this Chapter stability results for a pair of sequences of mappings and their common fixed points in a Hausdorff uniform space using (G)-convergence and (H)-convergence are proved. We first present some preliminary notions and results that are needed in the sequel.

1.2 Preliminaries

We start this section by recalling Barbet - Nachi [22] convergence in metric spaces and some basic concepts from the theory of uniform spaces. We then present the above mentioned notions of convergence in the setting of uniform spaces.

1.2.1 Barbet - Nachi Convergence in Metric Spaces

Definition 1.2.1. [22] Let (X, d) be a metric space, $\{X_n\}_{n \in \overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n \in \overline{\mathbb{N}}}$ a sequence of mappings. Then:

(i) T_∞ is called a (G)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G), where

(G) $Gr(T_\infty) \subset \liminf Gr(T_n)$: $\forall x \in X_\infty$, $\exists \{x_n\}_{n\in\mathbb{N}}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_n d(x_n, x) = 0 \text{ and } \lim_n d(T_n x_n, T_\infty x) = 0,$$

and $Gr(T)$ stands for the graph of T.

The following notion of (G^-) convergence is weaker than (G)-convergence.

(ii) T_∞ is called a (G^-)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property $((G^-)$, where

(G^-) $Gr(T_\infty) \subset \liminf Gr(T_n)$: $\forall x \in X_\infty$, $\exists \{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ which has a subsequence $\{x_{n_j}\}$ such that

$$\lim_n d(x_{n_j}, x) = 0 \text{ and } \lim_n d(T_{n_j} x_{n_j}, T_\infty x) = 0.$$

(iii) T_∞ is called an (H)-limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H), where

(H) If $\forall \{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$, $\exists \{y_n\}_{n\in\mathbb{N}} \subset X_\infty$ such that

$$\lim_n d(x_n, y_n) = 0 \text{ and } \lim_n d(T_n x_n, T_\infty y_n) = 0.$$

1.2.2 Discussions and Examples

Remark 1.2.1. We note the following essential features of the above limits.

(i) pointwise convergence $\Rightarrow$ (G) - convergence. However, the above implication is not reversible unless $\{T_n\}_{n\in\mathbb{N}}$ is equicontinuous on a common domain of definition (see Example 1.2.1).

(ii) a (G)-limit need not be unique (see Example 1.2.2). However if T_n is a k-contraction (resp. k-Lipschits) for each $n \in \mathbb{N}$, then it is so (see Theorem 1.4.1 and its Corollary 1.4.1).

(iii) an (H)-limit need not be unique.

(iv) when T_∞ is continuous and the condition $X_\infty \subset \liminf X_n$ is satisfied, then the following implications hold ([22, Proposition 9]):

$$(H) \Rightarrow (G) \Rightarrow (G^-).$$

However, without the two restrictions above, we have the relationship.

$$(G) \Rightarrow (G^-), (H) \Rightarrow (G^-).$$

Further, a (G)-limit is not necessarily an (H)-limit (see Example 1.2.3).

(v) the interrelationship between the (H) convergence and uniform convergence is captured in [22, Proposition 10].

Example 1.2.1. *[22] Consider the family* $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ *defined by* $T_n x = \frac{nx}{1+nx}$ *and* $T_\infty(x) = 1$ *for all* $x \in \mathbb{R}_+$. *Then the map* T_∞ *is a (G)-limit of* $\{T_n\}$ *but pointwise convergence is not satisfied.*

Example 1.2.2. *[22] Consider* $X_n = \mathbb{R}(n \in \overline{\mathbb{N}})$ *and the sequence* $\{T_n : \mathbb{R} \to \mathbb{R}\}_{n\in\overline{\mathbb{N}}}$ *of mappings defined by* $T_n x = \frac{nx}{1+nx}$ *for all* $x \in \mathbb{R}$. *Then* $T_\infty(x) = 1$ *for any* $x \in \mathbb{R}_+$, $T_\infty(0) = 0$. *Clearly* T_∞ *is a (G)-limit of* $\{T_n\}$. *Let* $T'_\infty : \mathbb{R} \to \mathbb{R}$ *be defined by* $T'_\infty(x) = T_\infty(x)$ *if* $x \neq 0$ *and* $T'_\infty(0) = \frac{1}{2}$. *Then* T'_∞ *is also a (G)-limit of* $\{T_n\}$, *indeed the point* $x = 0$ *is the limit of the sequence* $\{x_n = \frac{1}{n}\}_{n\in\mathbb{N}}$ *such that* $\{T_n x_n\}$ *converges to* $T'_\infty(0)$.

Example 1.2.3. *[22] Let $\{T_n : \mathbb{R}_+ \to \mathbb{R}\}_{n\in\overline{\mathbb{N}}}$ be defined by $T_n x = \frac{nx}{1+nx}$ and $T_\infty x = 1$ for all $x \in \mathbb{R}_+$. Then T_∞ is a (G)-limit of $\{T_n\}$. But the property (H) is not satisfied, since for the null sequence $\{x_n\}$ we get $|T_n 0 - T_\infty y_n| = 1$ for any sequence $\{y_n\}$ converging to 0.*

1.2.3 Uniform Spaces

Definition 1.2.2. Let X be a set. A subset U of $X \times X$ is called a relation on X. i.e.

$$U = \{(x, y) : x, y \in X\}.$$

In particular

$$\Delta = \{(x, x) : x \in X\}$$

is called the diagonal relation. If

$$U = \{(x, y) : x, y \in X\}$$

is a relation on X, then

$$U^{-1} = \{(y, x) : (x, y) \in U\},$$

if $U = U^{-1}$, then U is said to be symmetric. For any two relations, U and V we define a composition by:

$$U \circ V = \{(x, z) \in X \times X : (x, y) \in V \text{ and } (y, z) \in U \text{ for some } y \in X\}.$$

To every subset $A \subset X$ we can assign the set

$$U[A] = \{y : (x, y) \in U \text{ for some } x \in A\},$$

and if x is a point of X, then $U[x] = U[\{x\}]$.

Definition 1.2.3. A uniformity on a Set X is a nonempty family $\mathcal{U}$ consisting of subsets of $X \times X$ which satisfy the following conditions:

1. every element $U \in \mathcal{U}$ contains Δ,

2. if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$,

3. if $U \in \mathcal{U}$ then $V \circ V \subset U$ for some $V \in \mathcal{U}$,

4. if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$,

5. if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$ then $V \in \mathcal{U}$.

The ordered pair $(X,\mathcal{U})$ is called a uniform space and the elements of $\mathcal{U}$ are called the entourages.

Every uniform space can be considered as a topological space with a natural topology induced by the uniformity. The usual uniformity of the real line is the family $\mathcal{U}$ of all subsets U of $X \times X$ such that $\{(x, y) : |x - y| < r\} \subset U$ for some positive number r. Other examples of uniform spaces are metric spaces, topological groups and topological vector spaces.

Definition 1.2.4. A uniform space $(X,\mathcal{U})$ is said to be Hausdorff if and only if $\cap\{U : U \in \mathcal{U}\} = \Delta$.

Definition 1.2.5. A subfamily $\mathfrak{B}$ of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ iff each member of $\mathcal{U}$ contains a member of $\mathfrak{B}$. A Subfamily $\mathcal{S}$ is a subbase for $\mathcal{U}$ iff the family of finite intersection of members of $\mathcal{S}$ is a base for $\mathcal{U}$. A base for $\mathcal{U}$ is said to be countable iff it is finite or it can be put into one-to-one correspondence with the set of natural numbers.

Definition 1.2.6. The collection $\{U[x] : x \in X\}$, where $U \in \mathcal{U}$ and $U[x] = \{y \in X : (x, y) \in U\}$ constitutes a neighborhood base for $\mathcal{U}$ that generates a unique topology τ_u on X, called the uniform topology on X.

Definition 1.2.7. A topological space (X, τ) is called uniformizable if there exists a uniformity $\mathcal{U}$ in X such that $\tau = \tau_u$.

Definition 1.2.8. A uniform space $(X,\mathcal{U})$ is said to be pseudo-metrizable (metrizable) if and only if there is a pseudo-metric (metric) ρ such that $\mathcal{U}$ is the uniformity generated by ρ.

1.2.4 Associated Families of Pseudo-Metrics

Let$(X,\mathcal{U})$ be a uniform space. A family $P = \{\rho_\alpha : \alpha \in I\}$ of pseudo-metrics on X, where I is an indexing set is called an associated family for the uniformity $\mathcal{U}$ if the family

$$\mathfrak{B} = \{V(\alpha, \epsilon) : \alpha \in I, \epsilon > 0\}$$

where

$$V(\alpha, \epsilon) = \{(x, y) \in X \times X : \rho_\alpha(x, y) < \epsilon\}$$

is a subbase for the uniformity $\mathcal{U}$. We may assume $\mathfrak{B}$ itself to be a base for $\mathcal{U}$ by adjoining finite intersections of members of $\mathfrak{B}$ if necessary. The corresponding family of pseudo-metrics is called an augmented associated family for $\mathcal{U}$. An augmented family for $\mathcal{U}$ will be denoted by P^*. (cf. Kelley [70] and Thron [139]). In view of Kelley [70], we note that each member $V(\alpha, \epsilon)$ of $\mathfrak{B}$ is symmetric and ρ_α is uniformly continuous on $X \times X$ for each $\alpha \in I$.

We note the following well known result (cf. Kelley [70]).

Theorem 1.2.1. *Every uniformity on a given set X is generated by a family of pseudo-metrics which are uniformly continuous on $X \times X$.*

Remark 1.2.2. The uniformity $\mathcal{U}$ is not necessary pseudo-metrizable (resp. metrizable) unless $\mathfrak{B}$ is countable, and in that case, $\mathcal{U}$ may be generated by a single pseudo-metric (resp. metric) ρ on X (see Kelley [70]). For an interesting motivation, we refer to Reilly [103, Example 2] (see also Kelley [70, Example C, p. 204]). For further details on uniform spaces and a systematic account of fixed point theory there in (including applications), we refer to Kelley [70] and Angelov [14] respectively.

Example 1.2.4. *[70] Let Ω_0 be the set of all ordinals which are less than the first uncountable ordinals Ω, and for each member a of Ω_0 let*

$$U_a = \{(x, y) : x = y \text{ or } x \geq a \text{ and } y \geq a\}.$$

Then the family of all sets of the form U_a is a base for a uniformity $\mathcal{U}$ for Ω_0 (observe that $U_a = U_a \circ U_a = U_a^{-1}$). The topology of this uniformity is the discrete topology and hence metrizable, but the uniform space $(\Omega_0, \mathcal{U})$ is not metrizable.

1.2.5 Convergence and Completeness in Uniform Spaces

Definition 1.2.9. Let $(X, \mathcal{U})$ be a uniform space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to converge to $x \in X$ if for all $U \in \mathcal{U}$ there is a natural number M such that $(x_n, x) \in U$ for all $n \geq M$.

Definition 1.2.10. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for all member $U \in \mathcal{U}$, there exists $M > 0$ such that $(x_n, x_m) \in U$ for all $n, m \geq M$.

Definition 1.2.11. The space $(X, \mathcal{U})$ is said to be sequentially complete if all Cauchy sequence in X converges to a point in X.

Throughout this work, by completeness, we shall mean the sequential completeness.

1.2.6 Banach Contraction Principle in Uniform Spaces

Definition 1.2.12. [138] Let $(X, \mathcal{U})$ be a uniform space and $P^* = \{\rho_\alpha : \alpha \in I\}$. A mapping $T : X \to X$ is called a P^*-contraction or simply contraction if for each $\alpha \in I$, there exists a real $k(\alpha)$, $0 < k(\alpha) < 1$ such that $\rho_\alpha(T(x), T(y)) \leq k(\alpha)\rho_\alpha(x, y)$ for all $x, y \in X$.

Remark 1.2.3. It is well known that $T : X \to X$ is P^*-contraction if and only if it is P-contraction (see Tarafdar [138, Remark 1]). Hence, now onward, we shall simply use the term k-contraction (resp. contraction) to mean either of them. In case the above condition is satisfied for any $k = k(\alpha) > 0$, T will be called k-Lipschitz (of simply Lipschitz).

The following result due to Tarafdar [138] (see also Acharya [5]) presents an exact analog of the well-known Banach contraction principle.

Theorem 1.2.2. *Let $(X, \mathcal{U})$ be a Hausdorff complete uniform space and let $\{\rho_\alpha : \alpha \in I\} = P^*$. Let T be a contraction on X. Then, T has a unique fixed point $a \in X$ such that $T^n x \to a$ in τ_u (the uniform topology) for all $x \in X$.*

Definition 1.2.13. Let $(X, \mathcal{U})$ be a uniform space, $P^* = \{\rho_\alpha : \alpha \in I\}$ and $S, T : Y \subseteq X \to X$. Then the pair (S, T) will be called J-Lipschitz (Jungck Lipschitz) if for each $\alpha \in I$, there exists a constant $\mu = \mu(\alpha) > 0$ such that

$$\rho_\alpha(Sx, Sy) \leq \mu\rho_\alpha(Tx, Ty) \text{ for all } x, y \in Y \tag{1.2.1}$$

The pair (S, T) is generally called Jungck contraction (or simply J-contraction) when $0 < \mu < 1$, and the constant μ in this case is called a Jungck constant (see for instance [116]). Indeed, J-contractions and their generalized versions become popular because of the constructive approach of proof adopted by Jungck [60]. Now onwards, a J-Lipschitz map (resp. J-contraction) with Jungck constant μ will be called a J-Lipschitz (resp. J-contraction) with constant μ.

The following example illustrate the generality of J-Lipschitz maps.

Example 1.2.5. *Let $X = (0, \infty)$ with the usual uniformity induced by $\rho(x, y) = |x - y|$ for all $x, y \in X$. Define $S : X \to X$ by*

$$Sx = \frac{1}{x} \text{ for all } x \in X.$$

Then,

$$\rho(Sx, Sy) = \frac{1}{xy}\rho(x, y) \text{ for all } x, y \in X.$$

Since $\frac{1}{xy} \to \infty$ for small x or $y \in X$, S is not a Lipschitz map. However, if we consider the map $T : X \to X$ defined by

$$Tx = \frac{1}{Lx} \text{ for all } x \in X \text{ and some } L > 0,$$

then

$$\rho(Sx, Sy) = L\rho(Tx, Ty)$$

and S is Lipschitz with respect to T or the pair (S, T) is J-Lipschitz.

1.2.7 Barbet - Nachi Convergence in Uniform Spaces

Throughout, by a uniform space $(X, \mathcal{U})$ we shall mean that the uniformity is defined by the family of pseudo-metrics $P^* = \{\rho_\alpha : \alpha \in I\}$.

Now, we present the following extension of Barbet-Nachi convergence to uniform spaces by Mishra and Kalinde [86].

Definition 1.2.14. Let $(X,\mathcal{U})$ be a uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings. Then:

(i) S_∞ is called a (G)-limit of sequence $\{S_n\}_{n\in\mathbb{N}}$ or equivalently $\{S_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the Property (G), where

(G) $Gr(S_\infty) \subset \liminf Gr(S_n)$: for all $x \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that for all $\alpha \in I$,

$$\lim_n \rho_\alpha(x_n, x) = 0 \text{ and } \lim_n \rho_\alpha(S_n x_n, S_\infty x) = 0,$$

and $Gr(S)$ stands for the graph of S.

(ii) S_∞ is called an (H)-limit of sequence $\{S_n\}_{n\in\mathbb{N}}$ or, equivalently $\{S_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H), where

(H) for all sequences $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$, there exists a sequence y_n in X_∞ such that for any $\alpha \in I$,

$$\lim_n \rho_\alpha(x_n, y_n) = 0 \text{ and } \lim_n \rho_\alpha(S_n x_n, S_n y_n) = 0.$$

1.3 Stability of Fixed Points in Metric Spaces

In this section, we first recall some fundamental results in stability of fixed points by Bonsall [28] and Nadler [89] followed by their generalizations by Barbet and Nachi [22] for sequences of mappings in variable domains.

Theorem 1.3.1. *[28] Let (X, d) be a complete metric space and T and $T_n (n = 1, 2, ...)$ be contraction mappings of X into itself with the same Lipschitz constant $k < 1$, and with fixed points u and $u_n (n = 1, 2, ...)$, respectively. Suppose that $\lim_n T_n x = Tx$ for every $x \in X$. Then, $\lim_n u_n = u$.*

We have the following remarks with respect to Theorem 1.3.1:

(a) The condition that all the contraction mappings $T_n (n = 1, 2, ...)$ have the same Lipschitz constant k is too restrictive as one can easily see by the remarks and examples given in Nadler [89].

(b) The assumption that T is a contraction mapping is superfluous as this follows from the fact that $T_n (n = 1, 2, ...)$ is a contraction and d is continuous.

(c) the completeness condition may be replaced by the assumption of the existence of fixed points for the mapping T and $T_n (n = 1, 2, ...)$. Because there exist contraction mappings on spaces which are not complete and have a fixed point.

Under uniform convergence of the sequence $\{T_n\}$ to T and retaining the essence of (a), (b) and (c) the following stability result was obtained by Nadler [89].

Theorem 1.3.2. *Let (X, d) be a metric space and $T_n : X \to X$ be a mapping with at least one fixed point u_n, for each $n = 1, 2, ...$ and let $T : X \to X$ be a contraction mapping with fixed point u. If the sequence $\{T_n\}$ converges uniformly to T, then the sequence $\{u_n\}$ converges to u.*

The above theorems were generalized by Barbet and Nachi [22] using (G) and (H)-convergence where a number of supporting results were also obtained to arrive at the desired conclusions.

The following are the main stability results of Barbet and Nachi [22].

Theorem 1.3.3. *Let (X,d) be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that, for all $n \in \mathbb{N}$, S_n is a k-contraction from (X_n,d) into (X,d). If, for all $n \in \overline{\mathbb{N}}$, x_n is a fixed point of S_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Theorem 1.3.4. *Let (X,d) be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (H) and such that S_∞ is a k_∞-contraction. If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of S_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

1.4 Stability of Common Fixed Points in Uniform Spaces

In this section, we shall study the stability of common fixed points for a pair of mappings in a uniform space. The results so obtained extend Theorems 1.3.3 and 1.3.4 of Barbet and Nachi [22]. During the process a number of supporting results will be obtained.

1.4.1 G-Convergence and Stability in Uniform Spaces

As noted earlier in Example 1.2.2 a (G)-limit need not be unique. First, we shall prove the following theorem which gives a sufficient condition for the existence of a unique (G)-limit.

Theorem 1.4.1. *Let $(X,\mathcal{U})$ be a uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of J-Lipschitz maps relative to a continuous map $T : X \to X$ with Lipschitz constant μ. If $S_\infty : X_\infty \to X$ is a G-limit of the sequence $\{S_n\}$, then S_∞ is unique.*

Proof. Let $U \in \mathcal{U}$ be an arbitrary entourage. Then, since $\mathfrak{B}$ is a base for $\mathcal{U}$, there exists $V(\alpha,\epsilon) \in \mathfrak{B}$, $\alpha \in I$, $\epsilon > 0$ such that $V(\alpha,\epsilon) \subset U$. Suppose that $S_\infty : X_\infty \to X$ and $S^*_\infty : X_\infty \to X$ are G-limit maps of the sequence $\{S_n\}$. Then, for every $x \in X_\infty$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that for any $\alpha \in I$:

$$\lim_n \rho_\alpha(x_n, x) = 0 \text{ and } \lim_n \rho_\alpha(S_n x_n, S_\infty x) = 0,$$

$$\lim_n \rho_\alpha(y_n, x) = 0 \text{ and } \lim_n \rho_\alpha(S_n y_n, S^*_\infty x) = 0.$$

Further, since S_n is J-Lipschitz, for any $\alpha \in I$, there exists a constant $\mu = \mu(\alpha) > 0$ such that

$$\rho_\alpha(S_n x_n, S_n y_n) \le \mu\rho_\alpha(Tx_n, Ty_n).$$

Therefore, for any $n \in \mathbb{N}$ and $\alpha \in I$,

$$\begin{aligned}
\rho_\alpha(S_\infty x, S^*_\infty x) &\le \rho_\alpha(S_\infty x, S_n x_n) + \rho_\alpha(S_n x_n, S_n y_n) + \rho_\alpha(S_n y_n, S^*_\infty x)\\
&\le \rho_\alpha(S_\infty x, S_n x_n) + \mu\rho_\alpha(Tx_n, Ty_n) + \rho_\alpha(S_n y_n, S^*_\infty x)\\
&\le \rho_\alpha(S_\infty x, S_n x_n) + \mu[\rho_\alpha(Tx_n, Tx) + \rho_\alpha(Tx, Ty_n)] + \rho_\alpha(S_n y_n, S^*_\infty x).
\end{aligned}$$

Since T is continuous and $x_n \to x$ and $y_n \to x$ as $n \to \infty$, it follows that $Tx_n \to Tx$, $Ty_n \to Tx$. Hence the R.H.S of the above expression tends to 0 as $n \to \infty$ and so, $\rho_\alpha(S_\infty x, S^*_\infty x) < \epsilon$ for all $n \ge N(\alpha,\epsilon)$. Therefore $(S_\infty x, S^*_\infty x) \in V(\alpha,\epsilon) \subset U$ and since X is Hausdorff, it follows that $S_\infty x = S^*_\infty x$.

Corollary 1.4.1. *Let* $(X,\mathcal{U})$ *be a uniform space,* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *a family of nonempty subsets of* X *and* $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ *a sequence of* J*-contraction maps relative to a continuous map* $T : X \to X$ *with contraction constant* $0 < \mu < 1$*. If* $S_\infty : X_\infty \to X$ *is a G-limit of the sequence* $\{S_n\}$*, then* S_∞ *is unique.*

The following result is due to Mishra and Kalinde [86, Proposition 3.1], follows as a special case of Theorem 1.4.1.

Corollary 1.4.2. *Let* $(X,\mathcal{U})$ *be a Hausdorff uniform space,* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *a family of nonempty subsets of* X *and* $S_n : X_n \to X$ *a* k*-contraction(resp.* k*-Lipschitz) mapping for each* $n \in \overline{\mathbb{N}}$*. If* $S_\infty : X_\infty \to X$ *is a (G)-limit of* $\{S_n\}_{n\in\overline{\mathbb{N}}}$ *then* S_∞ *is unique.*

Proof. It comes from Theorem 1.4.1 when T is the identity map and $\mu \in (0,1)$ (resp. $\mu > 0$).

The following result of Barbet and Nachi [22] is also obtained as a consequence of Corollary 1.4.2 when X is metrizable.

Corollary 1.4.3. *Let* (X,d) *be a metric space,* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *a family of nonempty subsets of* X *and* $\{S_n : X_n \to X\}$ *a sequence of* k*-Lipschitz mappings. If* $S_\infty : X_\infty \to X$ *is a (G)-limit of* $\{S_n\}$ *then* S_∞ *is unique.*

Now, we present our first stability result.

Theorem 1.4.2. *Let* $(X,\mathcal{U})$ *be a uniform space,* $\{X_n\}_{n\in\overline{\mathbb{N}}}$ *a family of nonempty subsets of* X *and* $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ *two sequences of mappings each satisfying the property (G) and such that for all* $n \in \overline{\mathbb{N}}$*, the pair* (S_n, T_n) *is* J*-contraction with constant* μ *and* T_n *is continuous. If for all* $n \in \overline{\mathbb{N}}$*,* z_n *is a common fixed point of* S_n *and* T_n *then the sequence* $\{z_n\}$ *converges to* z_∞*, the common fixed point of* S_∞ *and* T_∞*.*

Proof. Let $W \in \mathcal{U}$ be arbitrary. Then, there exists $V(\lambda, \epsilon) \in \mathfrak{B}$, $\lambda \in I$, $\epsilon > 0$ such that $V(\lambda, \epsilon) \subset W$. Since z_n is a common fixed point of S_n and T_n for each $n \in \overline{\mathbb{N}}$, the property (G) holds and $z_\infty \in X_\infty$, there exists a sequence $\{y_n\}$ such that $y_n \in X_n$ (for all $n \in \overline{\mathbb{N}}$) such that for any $\lambda \in I$ and Condition 1.2.1,

$$\lim_n \rho_\lambda(y_n, z_\infty) = 0 \ , \ \lim_n \rho_\lambda(S_n y_n, S_\infty z_\infty) = 0 \text{ and } \lim_n \rho_\lambda(T_n y_n, T_\infty z_\infty) = 0.$$

Using the fact that the pair (S_n, T_n) is J-contraction, for any $\lambda \in I$, we have

$$\begin{aligned}
\rho_\lambda(z_n, z_\infty) &= \rho_\lambda(S_n z_n, S_\infty z_\infty) \\
&\leq \rho_\lambda(S_n z_n, S_n y_n) + \rho_\lambda(S_n y_n, S_\infty z_\infty) \\
&\leq \mu(\lambda)\rho_\lambda(T_n z_n, T_n y_n) + \rho_\lambda(S_n y_n, S_\infty z_\infty) \\
&\leq \mu(\lambda)\rho_\lambda(T_n z_n, T_\infty z_\infty) + \mu(\lambda)\rho_\lambda(T_n y_n, T_\infty z_\infty) + \rho_\lambda(S_n y_n, S_\infty z_\infty).
\end{aligned}$$

This gives

$$\rho_\lambda(z_n, z_\infty) \leq \frac{1}{1-\mu(\lambda)}[\mu(\lambda)\rho_\lambda(T_n y_n, T_\infty z_\infty) + \rho_\lambda(S_n y_n, S_\infty z_\infty)].$$

Since $\mu(\lambda) < 1$, it follows that $\rho_\lambda(z_n, z_\infty) \to 0$ as $n \to \infty$. Hence, $\rho_\lambda(z_n, z_\infty) < \epsilon$ for all $n \geq N(\lambda, \epsilon)$ and so $(z_n, z_\infty) \in V(\lambda, \epsilon) \subset W$ and the conclusion follows.

When for each $n \in \overline{\mathbb{N}}$, T_n is the identity map on X_n in Theorem 1.4.2, we have the following result due to Mishra and Kalinde [86, Theorem 3.3] as a special case.

Corollary 1.4.4. . *Let $(X, \mathcal{U})$ be a Hausdorff uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings satisfying the property (G) and S_n is a k-contraction for each $n \in \overline{\mathbb{N}}$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

When X is a metrizable uniform space, then we obtain Theorem 1.3.3 of Barbet and Nachi [22].

Again, when $X_n = X$, for all $n \in \overline{\mathbb{N}}$, we obtain, as a consequence of Theorem 1.4.2, the following result.

Corollary 1.4.5. . *Let $(X,\mathcal{U})$ be a uniform space and $S_n, T_n : X \to X$ be such that the pair (S_n, T_n) is J-contraction with constant μ and with at least one common fixed point z_n for all $n \in \overline{\mathbb{N}}$. If the sequences $\{S_n\}$ and $\{T_n\}$ converge pointwise respectively to $S, T : X \to X$, then the sequence $\{z_n\}$ converges to z_∞.*

We remark that under the conditions of Theorem 1.4.2 the pair (S_∞, T_∞) of G-limit maps is also a J-contraction. Indeed, we have the following stability result.

Theorem 1.4.3. *Let $(X,\mathcal{U})$ be a uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ two families of mappings each satisfying the property (G) and such that for all $n \in \mathbb{N}$, the pair (S_n, T_n) is J-contraction with constant $\{\mu_n\}_{n\in\mathbb{N}}$ a bounded (resp. convergent) sequence. Then, the pair (S_∞, T_∞) is J-contraction with constant $\mu = \sup\limits_{n\in\mathbb{N}} \mu_n$ (resp. $\mu = \lim\limits_n \mu_n$).*

Proof. Let $x, y \in X_\infty$. Then, by the property (G), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ such that the sequences $\{S_n x_n\}$, $\{S_n y_n\}$, $\{T_n x_n\}$ and $\{T_n y_n\}$ converge respectively to $S_\infty x$, $S_\infty y$, $T_\infty x$ and $T_\infty y$.

Therefore, for any $n \in \mathbb{N}$ and each $\alpha \in I$,

$$\begin{aligned}
\rho_\alpha(S_\infty x, S_\infty y) &\leq \rho_\alpha(S_\infty x, S_n x_n) + \rho_\alpha(S_n x_n, S_n y_n) + \rho_\alpha(S_n y_n, S_\infty y) \\
&\leq \rho_\alpha(S_\infty x, S_n x_n) + \mu_n \rho_\alpha(T_n x_n, T_n y_n) + \rho_\alpha(S_n y_n, S_\infty y).
\end{aligned}$$

Since

$$\limsup_n \mu_n \rho_\alpha(T_n x_n, T_n y_n) \leq \mu \rho_\alpha(T_\infty x, T_\infty y),$$

the above inequality yields $\rho_\alpha(S_\infty x, S_\infty y) \leq \mu\rho_\alpha(T_\infty x, T_\infty y)$ and the conclusion follows.

Theorem 1.4.3 includes, as a special case, the following result of Mishra and Kalinde [86, Proposition 3.5] for uniform spaces when $X_n = X$ and T_n is an identity mapping for each $n \in \overline{\mathbb{N}}$.

Corollary 1.4.6. *Let X be a Hausdorff uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings satisfying the property (G) and that T_n is a Lipschitz mapping with Lipschitz constant k_n for all $n \in \mathbb{N}$ and $\{k_n\}$ is bounded (resp. convergent). Then S_∞ is k-Lipschitz with $k = \limsup_n k_n$ (resp. $k = \lim_n k_n$).*

Now, when X is metrizable in the above Corollary, we have the following result of Barbet and Nachi [22, Proposition 4].

Corollary 1.4.7. *Let (X, d) be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings satisfying the property (G) and such that for all $n \in \mathbb{N}$, S_n is a k_n-Lipschitz with $\{k_n\}_{n\in\mathbb{N}}$ a bounded (resp. convergent) sequence. Then S_∞ is k-Lipschitz with $k = \sup_n k_n$ (resp. $k = \lim_n k_n$).*

1.4.2 H-Convergence and Stability in Uniform Spaces

In this section, we extend Theorem 1.3.4 to a pair of mappings in a uniform space.

Theorem 1.4.4. *Let $(X,\mathcal{U})$ be a uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be two families of mappings each satisfying the property (H). Further, let the pair (S_∞, T_∞) be a J-contraction with constant μ_∞. If, for every $n \in \overline{\mathbb{N}}$, z_n is a common fixed point of S_n and T_n, then the sequence $\{z_n\}$ converges to z_∞, the common fixed point of S_∞ and T_∞ .*

Proof. The property (H) implies that there exists a sequence $\{y_n\}$ in X_∞ such that for any $\alpha \in I$,

$$\lim_n \rho_\alpha(z_n, y_n) = 0,\ \lim_n \rho_\alpha(S_n z_n, S_\infty y_n) = 0 \text{ and } \lim_n \rho_\alpha(T_n z_n, T_\infty y_n) = 0.$$

Then

$$\begin{aligned}
\rho_\alpha(z_n, z_\infty) &= \rho_\alpha(S_n z_n, S_\infty z_\infty) \\
&\leq \rho_\alpha(S_n z_n, S_\infty y_n) + \rho_\alpha(S_\infty y_n, S_\infty z_\infty) \\
&\leq \rho_\alpha(S_n z_n, S_\infty y_n) + \mu_\infty \rho_\alpha(T_\infty y_n, T_\infty z_\infty) \\
&\leq \rho_\alpha(S_n z_n, S_\infty y_n) + \mu_\infty[\rho_\alpha(T_\infty y_n, T_n z_n) + \rho_\alpha(T_n z_n, T_\infty z_\infty)].
\end{aligned}$$

So, we get

$$\rho_\alpha(z_n, z_\infty) \leq \frac{1}{1-\mu_\infty}[\rho_\alpha(S_n z_n, S_\infty y_n) + \mu_\infty \rho_\alpha(T_\infty y_n, T_n z_n)].$$

Since the right hand side of the above inequality tends to 0 as $n \to \infty$, we deduce that $z_n \to z_\infty$ as $n \to \infty$.

As a consequence of Theorem 1.4.4, we have the following result due to Mishra and Kalinde [86, Theorem 3.13].

Corollary 1.4.8. *Let $(X,\mathcal{U})$ be a Hausdorff uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (H) and such that S_∞ is a k-contraction. If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. It comes from Theorem 1.4.4 by taking T_n to be the identity mappings for each $n \in \overline{\mathbb{N}}$.

If X is metrizable, then we get a stability result of Barbet and Nachi [22, Theorem 11] as follows.

Corollary 1.4.9. *Let (X,d) be a metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings satisfying the property (H) and such that S_∞ is a k_∞-contraction. If, for all $n \in \overline{\mathbb{N}}$, z_n is a fixed point of S_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Remark 1.4.1. We note that Theorem 1.3.2 follows as a direct consequence of Corollary 1.4.8 when $X_n = X$ for each $n \in \mathbb{N}$ with X being metrizable.

1.5 Extension to Locally Convex Spaces

Remark 1.5.1. Every locally convex topological vector space X is uniformizable being completely regular (cf. Kelley [70], Shaefer [115]) where the family of pseudometric $\{\rho_\alpha, \alpha \in I\}$ is induced by a family of seminorms $\{p_\alpha, \alpha \in I\}$ so that $\rho_\alpha(x,y) = p_\alpha(x-y)$ for all $x,y \in X$. Therefore, all the results proved previously for uniform spaces also apply to locally convex spaces.

Chapter 2

Fixed Point Theory in 2-Metric Spaces

2.1 Introduction

In Chapter 1 we extended to uniform spaces the results of Barbet and Nachi [22] on the stability of fixed points in metric space using new notions of convergence. In this chapter (Subsection 2.2.3 and Section 2.3), we extend these notions to 2-metric spaces and obtain the stability of common fixed points. We note that these results may be considered as significant in the sense that the 2-metric spaces differ topologically from metric spaces in many ways (see Remark 2.2.1).

In addition, in Section 2.4 we obtain an analogue in 2-metric spaces of the Suzuki contraction theorem [134], which is a generalization of the classical Banach contraction theorem in metric spaces and it characterizes the metric completeness. We then apply this theorem to prove a coincidence theorem for a pair of non-self mappings.

2.2 Preliminaries

In this section we present the notion of 2-metric spaces and some related properties to these spaces and extension of Barbet - Nachi Convergence in 2-Metric Spaces.

2.2.1 2-Metric Spaces

The following notion of 2-metric spaces is due to Gähler [46].

Definition 2.2.1. A 2-metric space is a space X with a real valued function ρ on $X \times X \times X$ satisfying the following conditions:

(G1) for two points $x, y \in X$ there is a point $z \in X$ such that $\rho(x, y, z) \neq 0$,

(G2) $\rho(x, y, z) = 0$ if at least two of the three points are equal,

(G3) $\rho(x, y, z) = \rho(z, x, y) = \rho(y, z, x)$ (symmetry about three variables),

(G4) $\rho(x, y, z) \leq \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z)$ (triangle area inequality or simply TA-inequality).

2.2.2 Convergence, Completeness and Continuity in 2-Metric Spaces

Definition 2.2.2. Let $\{x_n\}$ be a sequence in a 2-metric space (X, ρ). Then:

(i) $\{x_n\}$ is said to be convergent with limit $z \in X$ if

$$\lim_{n\to\infty} \rho(x_n, z, a) = 0 \text{ for all } a \in X.$$

Notice that if the sequence $\{x_n\}$ converges to z, then

$$\lim_{n\to\infty} \rho(x_n, a, b) = \rho(z, a, b) \text{ for all } a, b \in X.$$

(ii) $\{x_n\}$ is said to be Cauchy if

$$\lim_{m,n\to\infty} \rho(x_m, x_n, a) = 0 \text{ for all } a \in X.$$

(iii) (X, ρ) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.2.3. A 2-metric space (X, ρ) is said to be bounded if there is a constant K such that $\rho(a, b, c) \leq K$ for all $a, b, c \in X$.

Remark 2.2.1. The following remarks capture some distinct features of topological properties of 2-metric spaces which differ from those of metric spaces.

(i) Given any metric space which consist of more than two points, there always exists a 2-metric compatible with the topology of the space. But the converse is not always true as one can find a 2-metric space which does not have a countable basis associated with one of its arguments (see Gähler [46, page 123]).

(ii) It is known that a 2-metric ρ is continuous in any one of its arguments. Generally, we cannot however assert the continuity of ρ in all three arguments. But if it is continuous in any two arguments, then it is continuous in all the three arguments (see Gähler [46, Theorem 20 and example on page 145]).

(iii) In a complete 2-metric space a convergent sequence need not be Cauchy (see Example 2.2.1).

(iv) In a 2-metric space (X, ρ) every convergent sequence is Cauchy whenever ρ is continuous. However, the converse need not be true (see Example 2.2.2)

Example 2.2.1. *[92] Let* $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$*. Define* $\rho : X \times X \times X \to [0, \infty)$ *as*

$$\rho(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \text{ are distinct and } \{\frac{1}{n}, \frac{1}{n+1}\} \subset \{x, y, z\} \text{ for some positive integer } n \\ 0 & \text{otherwise.} \end{cases}$$

Then (X, ρ) *is a complete 2-metric space. The sequence* $\{\frac{1}{n}\}$ *converges to 0, but* $\{\frac{1}{n}\}$ *is not Cauchy.*

Example 2.2.2. *[92] Let* $X = \{a\} \cup \{a_n : n = 1, 2, ...\} \cup \{b\} \cup \{b_n : n = 1, 2, ...\}$*, where* $a = (1, 0), b = (0, 0), a_n = (1 + \frac{1}{n}, 0)$ *and* $b_n = (0, \frac{1}{n})$*. Define* $\rho : X \times X \times X \to [0, \infty)$ *as*

$$\rho(x, y, z) = \begin{cases} 1 & \text{if } \{x, y, z\} = \{a_n, b_n, a\} \text{ or } \{a_n, b_n, b\} \text{ for some } n \in \mathbb{N} \text{ or } \{a_n, b_n, a_m\} \\ & \text{or } \{a_n, b_n, b_m\} \text{ for some } m, n \in \mathbb{N} \text{ with } m \neq n \\ \Delta\ x\ y\ z & \text{otherwise,} \end{cases}$$

where $\Delta\ x\ y\ z$ *is the area of the triangle formed by the points* x, y *and* z*. Then* (X, ρ) *is a complete 2-metric space and every convergent sequence in it is Cauchy. But* ρ *is not continuous on* X*, for* $\{a_n\}$ *converges to* a*,* $\{b_n\}$ *converges to* b *and* $\{\rho(a_n, b_n, a)\}$ *does not converge to zero.*

Definition 2.2.4. Let (X, ρ) be a 2-metric space. A mapping $S : X \to X$ is called *k-Lipschitz* (or simply *Lipschitz*) if there exists a real $k > 0$ such that:

$$\rho(Sx, Sy, a) \leq k\rho(x, y, a) \text{ for all } x, y, a \in X. \tag{2.2.1}$$

In case the above condition is satisfied for $k \in (0, 1)$, T is called k-contraction (or simply contraction)(cf. [54], [77]).

Remark 2.2.2. It is well known that a contraction mapping on a 2-metric space X has a unique fixed point. Initially, an additional requirement of boundedness was placed on X by Iséki et al. [54] and which was dispensed with subsequently by Rhoades [106] and Lal and Singh [77] independently.

Definition 2.2.5. Let (X,ρ) be a 2-metric space, $S,T:Y\subseteq X\to X$. The the pair (S,T) will be called *J-Lipschitz* (Jungck Lipschitz) if there exists a constant $\mu>0$ such that:

$$\rho(Sx,Sy,a)\leq\mu\rho(Tx,Ty,a) \text{ for all } x,y,a\in Y. \tag{2.2.2}$$

If T is an identity mapping on X, then we recover definition 2.2.4.

The other comments made there in Subsection 1.2.6 apply.

2.2.3 Extension of Barbet - Nachi Convergence to 2-Metric spaces

We now extend the various notions of convergence due to Barbet and Nachi [22] mentioned in Subsection 1.2.1 to 2-metrics paces.

Definition 2.2.6. Let (X,ρ) be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a sequence of non-empty subsets of X and $\{S_n:X_n\to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings. Then:

(i) S_∞ is called a (G)-limit of sequence $\{S_n\}_{n\in\mathbb{N}}$, or equivalently $\{S_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G), where

(G) $Gr(S_\infty)\subset\liminf Gr(S_n)$: for all $x\in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod\limits_{n\in\mathbb{N}}X_n$ such that for all $a\in X$,

$$\lim_n\rho(x_n,x,a)=0 \text{ and } \lim_n\rho(S_nx_n,S_\infty x,a)=0,$$

and $Gr(S)$ stands for the graph of S.

(ii) S_∞ is called a (G^-)-limit of sequence $\{S_n\}_{n\in\mathbb{N}}$, or equivalently $\{S_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G^-), where

(G^-) $Gr(T_\infty) \subset \limsup Gr(T_n)$: for all $z \in X_\infty$, there exists a sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ and which has a subsequence $\{x_{n_j}\}$ such that

$$\lim_j \rho(x_{n_j}, z, a) = 0 \text{ and} \lim_j \rho(T_{n_j} x_{n_j}, T_\infty z, a) = 0, \text{ for all } a \in X.$$

(iii) T_∞ is called an (H)-limit of the sequence $\{S_n\}_{n\in\mathbb{N}}$ or equivalently $\{S_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H), where

(H) For all sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that for all $a \in X$

$$\lim_n \rho(x_n, y_n, a) = 0 \text{ and } \lim_n \rho(S_n x_n, S_\infty y_n, a) = 0.$$

2.3 Stability of Common Fixed Points in 2-Metric Spaces

In this section we generalize the results of Barbet and Nachi [22] to a pair of sequences of mappings in 2-metric spaces.

Throughout, unless stated otherwise, X will denote a 2-metric space (X, ρ) with ρ continuous.

2.3.1 (G)-Convergence and Stability

The following theorem gives a sufficient condition for the existence of a unique (G)-limit.

Theorem 2.3.1. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of non-empty subsets of X and $\{S_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of J-contraction mappings relative to a continuous mapping $T : X \to X$ with constant μ. If $S_\infty : X_\infty \to X$ is a (G)-limit of the sequence $\{S_n\}$, then S_∞ is unique.*

Proof. Suppose that $S_\infty, S_\infty^* : X_\infty \to X$ are (G)-limit mappings of the sequence $\{S_n\}$. Then for every $x \in X_\infty$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\prod\limits_{n\in\mathbb{N}} X_n$ such that for any $a \in X$,

$$\lim_n \rho(x_n, x, a) = 0 \text{ and} \lim_n \rho(S_n x_n, S_\infty x, a) = 0,$$

$$\lim_n \rho(y_n, x, a) = 0 \text{ and} \lim_n \rho(S_n y_n, S_\infty^* x, a) = 0.$$

Further, since S_n is a J-contraction for each $n \in \mathbb{N}$, there exist a constant $\mu \in (0,1)$ such that for any $a \in X$,

$$\rho(S_n x_n, S_n y_n, a) \leq \mu\rho(Tx_n, Ty_n, a).$$

Therefore for any $n \in \mathbb{N}$ and $a \in X$,

$$\begin{aligned}
\rho(S_\infty x, S_\infty^* x, a) &\leq \rho(S_\infty x, S_\infty^* x, S_n x_n) + \rho(S_\infty x, S_n x_n, a) + \rho(S_n x_n, S_\infty^* x, a) \\
&\leq \rho(S_\infty x, S_\infty^* x, S_n x_n) + \rho(S_\infty x, S_n x_n, S_n y_n) + \rho(S_\infty x, S_n y_n, a) \\
&\quad + \rho(S_n y_n, S_n x_n, a) + \rho(S_n x_n, S_\infty^* x, a) \\
&\leq \rho(S_\infty x, S_\infty^* x, S_n x_n) + \mu\rho(S_\infty x, T_n x_n, T_n y_n) + \rho(S_\infty x, S_n y_n, a) \\
&\quad + \rho(S_n y_n, S_n x_n, a) + \rho(S_n x_n, S_\infty^* x, a) \\
&\leq \rho(S_\infty x, S_\infty^* x, S_n x_n) + \mu[\rho(S_\infty x, T_n x_n, Tx) + \rho(S_\infty x, Tx, T_n y_n) \\
&\quad + \rho(Tx, T_n x_n, T_n y_n)] + \rho(S_\infty x, S_n y_n, a) + \rho(S_n y_n, S_n x_n, a) \\
&\quad + \rho(S_n x_n, S_\infty^* x, a).
\end{aligned}$$

Since T is continuous and $x_n \to x$, $y_n \to x$, it follows that $Tx_n \to Tx$, $Ty_n \to Tx$. Hence the R.H.S. of the above expression tends to 0 as $n \to \infty$. Therefore $S_\infty x = S_\infty^* x$.

Corollary 2.3.1. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of non-empty subsets of X and $\{S_n : X_n \to X\}_{n\in\mathbb{N}}$ a sequence of J-Lipschitz mappings relative to a continuous mapping $T : X \to X$ with constant μ. If $S_\infty : X_\infty \to X$ is a (G)-limit of the sequence $\{S_n\}$, then S_∞ is unique.*

The following result is an extension of Barbet and Nachi [22, Proposition 1] to 2-metric spaces and it also includes a result of Mishra et al. [87, Proposition 3.1]).

Corollary 2.3.2. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of non-empty subsets of X and $S_n : X_n \to X$ a k-contraction (k-Lipschitz) mappings relative to a continuous mapping $T : X \to X$ with constant μ. If $S_\infty : X_\infty \to X$ is a (G)-limit of $\{S_n\}_{n\in\mathbb{N}}$, then S_∞ is unique.*

Proof. it comes from Theorem 2.3.1 when T is the identity mapping and $\mu \in (0, 1)$ (resp. $\mu > 0$).

Now we present our first result on stability of common fixed points.

Theorem 2.3.2. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of non-empty subsets of X and $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ two sequences of mappings, each satisfying the property (G) and such that for all $n \in \mathbb{N}$, the pair (S_n, T_n) is a J-contraction with constant μ and T_n continuous. If for all $n \in \overline{\mathbb{N}}$, z_n is a common fixed point of S_n and T_n, then the sequence $\{z_n\}$ converges to z_∞.*

Proof. Since $\{z_n\}$ is a common fixed point of S_n and T_n for each $n \in \overline{\mathbb{N}}$, the property (G) holds and $z_\infty \in X_\infty$, there exists a sequence $\{y_n\}$ with $y_n \in X_n$ (for all

$n \in \overline{\mathbb{N}}$) such that for any $a \in X$,

$$\lim_n \rho(y_n, z_\infty, a) = 0 \ , \ \lim_n \rho(S_n y_n, S_\infty z_\infty, a) = 0 \text{ and } \lim_n \rho(T_n y_n, T_\infty z_\infty, a) = 0.$$

Using the fact that the pair (S_n, T_n) is a J-contraction for all $n \in \mathbb{N}$, we have for any $a \in X$,

$$\begin{aligned}
\rho(z_n, z_\infty, a) &= \rho(S_n z_n, S_\infty z_\infty, a) \\
&\leq \rho(S_n z_n, S_\infty z_\infty, S_n y_n) + \rho(S_n z_n, S_n y_n, a) + \rho(S_n y_n, S_\infty z_\infty, a) \\
&\leq \rho(S_n z_n, S_\infty z_\infty, S_n y_n) + \mu\rho(T_n z_n, T_n y_n, a) + \rho(S_n y_n, S_\infty z_\infty, a) \\
&\leq \rho(S_n z_n, S_\infty z_\infty, S_n y_n) + \rho(S_n y_n, S_\infty z_\infty, a) + \mu[\rho(T_n z_n, T_n y_n, T_\infty z_\infty) \\
&\quad +\rho(T_n z_n, T_\infty z_\infty, a) + \rho(T_\infty z_\infty, T_n y_n, a)].
\end{aligned}$$

The R.H.S. of the above expression tends to 0 as $n \to \infty$ and the conclusion follows.

When for each $n \in \overline{\mathbb{N}}$, T_n is an identity mapping on X_n in Theorem 2.3.2, we have the following result as an extension of Barbet and Nachi [22, Theorem 2] to 2-metric spaces.

Corollary 2.3.3. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of non-empty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and S_n is a k-contraction for each $n \in \overline{\mathbb{N}}$. If x_n is a fixed point of S_n for each $n \in \overline{\mathbb{N}}$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞, the fixed point of S_∞ .*

The following result gives a comparison with Rhoades [106, Theorem 2] and presents a 2-metric space version of Theorem 1.3.1 of Bonsall [28].

Corollary 2.3.4. *Let X be a complete 2-metric space and $\{S_n : X \to X\}_{n\in\overline{\mathbb{N}}}$ a family of contraction mappings with the same Lipschitz constant $k < 1$ and such that the* *sequence $\{S_n\}_{n\in\mathbb{N}}$ converges pointwise to S_∞. Then, for all $n \in \overline{\mathbb{N}}$, S_n has a unique fixed point x_n and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. This comes from Corollary 2.3.3 and the fact that X is complete.

Again, when $X_n = X$, for all $n \in \overline{\mathbb{N}}$, we obtain, as a consequence of Theorem 2.3.2 the following result.

Corollary 2.3.5. *Let X be a 2-metric space, $S_n, T_n : X \to X$ be such that the pair (S_n, T_n) is a J-contraction with constant μ, T_n continuous and with at least one common fixed point z_n for all $n \in \overline{\mathbb{N}}$. If the sequences $\{S_n\}$ and $\{T_n\}$ converge pointwise respectively to $S_\infty, T_\infty : X \to X$ then the sequence $\{z_n\}$ converges to z_∞.*

We remark that under the conditions of Theorem 2.3.2 the pair (S_∞, T_∞) of (G)-limit mappings is also a J-contraction. Indeed, we have the following stability result.

Theorem 2.3.3. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of non-empty subsets of X and $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ two sequences of mappings, each satisfying the property (G) and such that for all $n \in \mathbb{N}$, the pair (S_n, T_n), with T_n continuous is a J-contraction with constant $\{\mu_n\}_{n\in\mathbb{N}}$, a bounded (resp. convergent) sequence. Then the pair (S_∞, T_∞) is a J-contraction with constant $\mu = \sup_{n\in\mathbb{N}} \mu_n$ (resp. $\mu = \lim_n \mu_n$)*

Proof. Let $x, y \in X_\infty$. Then by the property (G), there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that:

$$\lim_n \rho(x_n, x, a) = 0\ ,\ \lim_n \rho(S_n x_n, S_\infty x, a) = 0\ ,\ \lim_n \rho(T_n x_n, T_\infty x, a) = 0,$$

$$\lim_n \rho(y_n, y, a) = 0\ ,\ \lim_n \rho(S_n y_n, S_\infty y, a) = 0\ ,\ \lim_n \rho(T_n y_n, T_\infty y, a) = 0.$$

for all $a \in X$. Since for any $n \in \mathbb{N}$ and each $a \in X$,

$$\limsup_n \mu_n \rho(T_n x_n, T_n y_n, a) \leq \mu \rho(T_\infty x, T_\infty y, a),$$

the above inequality yields

$$\rho(S_\infty x, S_\infty y, a) \leq \mu\rho(T_\infty x, T_\infty y, a),$$

and the conclusion follows.

Corollary 2.3.6. *Theorem 2.3.3 with J-contraction replaced by J-Lipschitz.*

When for each $n \in \overline{\mathbb{N}}$, T_n is an identity mapping in Theorem 2.3.3, we have the following extension of Barbet and Nachi [22, Proposition 4] to 2-metric spaces.

Corollary 2.3.7. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings, satisfying the property (G) and such that for any $n \in \mathbb{N}$, S_n is J-Lipschitz (resp. J-contraction) with constant $\{k_n\}_{n\in\mathbb{N}}$ a bounded (resp. convergent) sequence. Then S_∞ is J-Lipschitz (resp. J-contraction) with constant $k := \sup\limits_{n\in\mathbb{N}} k_n$ (resp. $k := \lim\limits_n k_n$).*

The existence of a fixed point for a (G)-limit mapping is characterized by the following result when it is a contraction mapping.

Theorem 2.3.4. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of non-empty subsets of X and $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ two sequences of mappings, each satisfying the property (G) and such that for all $n \in \mathbb{N}$, the pair (S_n, T_n) is a J-contraction with constant μ and T_n continuous. Assume that for any $n \in \mathbb{N}$, x_n is a common fixed point of S_n and T_n. Then:*

S_∞ and T_∞ admit a common fixed point $\Leftrightarrow$ $\{x_n\}$ converges and $\lim\limits_n x_n \in X_\infty$
$\Leftrightarrow$ $\{x_n\}$ admits a subsequence converging to a point of X_∞ .

Proof. In view of Theorem 2.3.3, we only have to prove the sufficiency part. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\lim_j x_{n_j} = x_\infty \in X_\infty$. By (G), there exists a sequence $\{y_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ such that

$$\lim_n \rho(y_n, z_\infty, a) = 0 \ , \ \lim_n \rho(S_n y_n, S_\infty x_\infty, a) = 0 \text{ and } \lim_n \rho(T_n y_n, T_\infty x_\infty, a) = 0, \text{ for all } a \in X.$$

First we show that $S_\infty x_\infty = x_\infty$. For any $a \in X$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\rho(x_\infty, S_\infty x_\infty, a) &\leq \rho(x_\infty, S_{n_j}x_{n_j}, a) + \rho(S_{n_j}x_{n_j}, S_\infty x_\infty, a) + \rho(x_\infty, S_\infty x_\infty, S_{n_j}x_{n_j}) \\
&\leq \rho(x_\infty, S_{n_j}x_{n_j}, a) + \rho(S_{n_j}x_{n_j}, S_\infty x_\infty, S_{n_j}y_{n_j}) + \rho(S_{n_j}x_{n_j}, S_{n_j}y_{n_j}, a) \\
&\quad + \rho(S_{n_j}y_{n_j}, S_\infty x_\infty, a) + \rho(x_\infty, S_\infty x_\infty, S_{n_j}x_{n_j}) \\
&\leq \rho(x_\infty, S_{n_j}x_{n_j}, a) + \rho(S_{n_j}x_{n_j}, S_\infty x_\infty, S_{n_j}y_{n_j}) + \mu\rho(T_{n_j}x_{n_j}, T_{n_j}y_{n_j}, a) \\
&\quad + \rho(S_{n_j}y_{n_j}, S_\infty x_\infty, a) + \rho(x_\infty, S_\infty x_\infty, S_{n_j}x_{n_j}).
\end{aligned}$$

The R.H.S. of the above expression tends to zero as $n \to \infty$ and hence $S_\infty x_\infty = x_\infty$.

Next, by the triangle inequality we have:

$$\rho(x_\infty, T_\infty x_\infty, a) \leq \rho(x_\infty, T_{n_j}x_{n_j}, a) + \rho(T_{n_j}x_{n_j}, T_\infty x_\infty, a) + \rho(x_\infty, T_\infty x_\infty, T_{n_j}x_{n_j}).$$

The R.H.S. of the above expression tends to zero as $n \to \infty$ and hence $S_\infty x_\infty = x_\infty$. Therefore $S_\infty x_\infty = T_\infty x_\infty = x_\infty$ and x_∞ is a common fixed point of S_∞ and T_∞.

Remark 2.3.1. Under the assumptions of Theorem 2.3.4 and if

(i) $\liminf X_n \subset X_\infty$ (i.e, the limit of any convergent sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞), then:

$$S_\infty \text{ and } T_\infty \text{ admit a common fixed point } \Leftrightarrow \{x_n\} \text{ converges .}$$

(ii) $\limsup X_n \subset X_\infty$ (i.e, the cluster point of any sequence $\{x_n\}$ in $\prod_{n\in\mathbb{N}} X_n$ is in X_∞), then:

S_∞ and T_∞ admit a common fixed point $\Leftrightarrow \{x_n\}$ admits a convergent subsequence .

Under certain compactness assumptions, we have the following.

Theorem 2.3.5. *Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of non-empty subsets of a 2-metric space (X,ρ) , $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ two sequences of mappings, each satisfying the property (G) and such that for any $n \in \mathbb{N}$, the pair (S_n, T_n) is a J-contraction with constant μ and T_n continuous. Assume that, $\limsup X_n \subset X_\infty$ and $\cup_{n\in\mathbb{N}} X_n$ is relatively compact. If for any $n \in \mathbb{N}$, x_n is a common fixed point of S_n and T_n, then the pair (S_∞, T_∞) of (G)-limit mappings of S_n and T_n admits a common fixed point x_∞ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a common fixed point of S_n and T_n. Then by the compactness assumption,$\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{x_{n_j}\}$. Now by Remark 2.3.1, S_∞ and T_∞ admit a common fixed point x_∞ and by Theorem 2.3.2, $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.

Remark 2.3.2. By choosing X_n and T_n suitably in Theorem 2.3.4 and Theorem 2.3.5, we obtain the extensions of the corresponding results of Barbet and Nachi [22, Corollary 6 and Theorem 7]

2.3.2 (G^-)-Convergence and Stability

We shall establish in the next result that a fixed point of a (G^-)-limit map is a cluster point of the sequence of common fixed points of a pair of sequences of mappings (S_n, T_n).

Theorem 2.3.6. *Let $\{X_n\}$ be a family of nonempty subsets of a 2-metric space X and $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be two sequences of J-contraction mappings with constant μ and T_n continuous, each satisfying the property(G^-). If, for any $n \in \overline{\mathbb{N}}$, x_n is a common fixed point of S_n and T_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. By the property (G^-), there exists a sequence y_n in $\prod_n X_n$ which has a subsequence y_{n_j} such that:

$$\lim_j \rho(y_{n_j}, x_\infty, a) = 0,\ \lim_j \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) = 0 \text{ and } \lim_j \rho(T_{n_j} y_{n_j}, T_\infty x_\infty, a) = 0$$

for all $a \in X$. Since the pair (S_{n_j}, T_{n_j}) is a J-contraction for each $j \in \overline{\mathbb{N}}$, for any $a \in X$ we have:

$$\begin{aligned}
\rho(x_{n_j}, x_\infty, a) &= \rho(S_{n_j} x_{n_j}, S_\infty x_\infty, a) \\
&\leq \rho(S_{n_j} x_{n_j}, S_{n_j} y_{n_j}, a) + \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) + \rho(x_{n_j}, S_\infty x_\infty, S_{n_j} y_{n_j}) \\
&\leq \mu\rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, a) + \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) + \rho(x_{n_j}, S_\infty x_\infty, S_{n_j} y_{n_j}) \\
&\leq \mu[\rho(T_{n_j} x_{n_j}, T_{n_j} y_{n_j}, T_{n_j} x_\infty) + \rho(T_{n_j} x_{n_j}, T_{n_j} x_\infty, a) + \rho(T_{n_j} x_\infty, T_{n_j} y_{n_j}, a)] \\
&\quad + \rho(S_{n_j} y_{n_j}, S_\infty x_\infty, a) + \rho(x_{n_j}, S_\infty x_\infty, S_{n_j} y_{n_j})
\end{aligned}$$

The R.H.S of the above expression tends to zero as $j \to \infty$ and hence $\{x_{n_j}\}$ converges to x_∞, the common fixed point of S_∞ and T_∞.

When for all $n \in \overline{\mathbb{N}}$, $X_n = X$ and T_n is an identity mapping, we get the following analogue of Barbet and Nachi [22, Theorem 8] to 2-metric spaces.

Corollary 2.3.8. *Let$\{X_n\}$ be a family of nonempty subsets of a 2-metric space X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of k-contraction mappings satisfying the property (G^-). If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of S_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

2.3.3 (H)-Convergence and Stability

The following theorem presents another stability result.

Theorem 2.3.7. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of non-empty subsets of X and let $\{S_n, T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be two sequences of mappings each satisfying the property (H). Further, let the pair (S_∞, T_∞) be a J-contraction with constant μ_∞ and T_∞ continuous. If, for any $n \in \overline{\mathbb{N}}$, z_n is a common fixed point of S_n and T_n, then the sequence $\{z_n\}$ converges to z_∞.*

Proof. The property (H) implies that, there exists a sequence $\{y_n\}$ in X_∞ such that for any $a \in X$,

$$\lim_n \rho(z_n, y_n, a) = 0\ ,\ \lim_n \rho(S_n z_n, S_\infty y_n, a) = 0 \text{ and } \lim_n \rho(T_n z_n, T_\infty y_n, a) = 0. \text{ Then}$$

$$\begin{aligned}
\rho(z_n, z_\infty, a) &= \rho(S_n z_n, S_\infty z_\infty, a) \\
&\leq \rho(S_n z_n, S_\infty z_\infty, S_\infty y_n) + \rho(S_n z_n, S_\infty y_n, a) + \rho(S_\infty y_n, S_\infty z_\infty, a) \\
&\leq \rho(S_n z_n, S_\infty z_\infty, S_\infty y_n) + \rho(S_n z_n, S_\infty y_n, a) + \mu_\infty \rho(T_\infty y_n, T_\infty z_\infty, a) \\
&\leq \rho(S_n z_n, S_\infty z_\infty, S_\infty y_n) + \rho(S_n z_n, S_\infty y_n, a) + \mu_\infty[\rho(T_\infty y_n, T_\infty z_\infty, T_n z_n) \\
&\quad + \rho(T_\infty y_n, T_n z_n, a) + \rho(T_n z_n, T_\infty x_\infty, a)].
\end{aligned}$$

Since the right hand side of the above inequality tends to 0 as $n \to \infty$, we deduce that $z_n \to z_\infty$ as $n \to \infty$.

The following result of Mishra et al. [87, Theorem 3.4] which also extends a result of Barbert and Nachi [22] to 2-metric spaces follows as corollary of Theorem 2.3.7.

Corollary 2.3.9. *Let X be a 2-metric space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of non-empty subsets of X and $\{S_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (H)and* *such that S_∞ is a k_∞-contraction. If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of S_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. It comes from Theorem 2.3.7 when for all $n \in \overline{\mathbb{N}}$, T_n is an identity mapping.

When $X_n = X$, for all $n \in \overline{\mathbb{N}}$ in corollary 2.3.9, we get a special case of Rhoades [106, Theorem 3] which in turn presents a 2-metric space version of Nadler [89, Theorem 1].

Corollary 2.3.10. *Let X be a 2-metric space, $\{S_n : X \to X\}_{n\in\overline{\mathbb{N}}}$ a sequence of mappings which converges uniformly to a contraction mapping $\{S_\infty : X \to X\}$. If, for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of S_n then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

2.4 Suzuki Theorem in a 2-Metric Space

In this section, we first present Suzuki theorem in metric spaces and then obtain its analogue in 2-metric spaces.

Suzuki [134, Th. 2] obtained the following remarkable generalization of the Banach contraction theorem in a metric space.

Theorem 2.4.1. *Let (X, d) be a complete metric space and let T be a mapping on X. Define a non-increasing function θ from $[0, 1)$ onto $(\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \le r \le 2^{-\frac{1}{2}}, \\ (1+r)^{-1} & \text{if } 2^{-\frac{1}{2}} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that

$$\theta(r)d(x,Tx) \leq d(x,y) \text{ impies } d(Tx,Ty) \leq rd(x,y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

To extend the above theorem to 2-metric spaces we shall need the following result due to Singh [117].

Lemma 2.4.1. *Let $\{y_n\}$ be a sequence in a complete 2-metric space X. If there exists $h \in (0,1)$ such that $\rho(y_n, y_{n+1}, a) \leq h\rho(y_{n-1}, y_n, a)$ for all $n \in \mathbb{N}$ and all $a \in X$, then $\{y_n\}$ converges to a point in X.*

Theorem 2.4.2. *Let (X, ρ) be a complete 2-metric space and $T : X \to X$. Define a non-increasing function θ from $[0,1)$ onto $(\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq 2^{-\frac{1}{2}}, \\ (1+r)^{-1} & \text{if } 2^{-\frac{1}{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that

$$\theta(r)\rho(x,Tx,a) \leq \rho(x,y,a) \text{ impies } \rho(Tx,Ty,a) \leq r\rho(x,y,a) \tag{2.4.1}$$

for all $x, y, a \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim T^n x = z$ for any $x \in X$.

Proof. Pick $x \in X$. Construct a sequence $\{x_n\}$ in X such that

$$x_1 = Tx,\ x_2 = Tx_1, \ldots,\ x_{n+1} = Tx_n,\ n = 1, 2, \ldots$$

Note that

$$\theta(r)\rho(x_n, Tx_n, a) \leq \rho(x_n, x_{n+1}, a) \text{ for all } a \in X.$$

Therefore by (2.4.1),

$$\rho(Tx_n, Tx_{n+1}, a) \leq r\rho(x_n, x_{n+1}, a)$$

that is

$$\rho(x_{n+1}, x_{n+2}, a) \leq r\rho(x_n, x_{n+1}, a)$$

for all $a \in X$. Since this is true for all $n \geq 1$, by Lemma 2.4.1, the sequence $\{x_n\}$ converges to a point $z \in X$. By (2.4.1) for any $a \in X$, we have,

$$\rho(Tx, T^2x, a) \leq r\rho(x, Tx, a). \tag{2.4.2}$$

Next we show that

$$\rho(Tx, z, a) \leq r\rho(x, z, a) \tag{2.4.3}$$

for all $x \in X \setminus \{z\}$ and all $a \in X$.

For $x \in X \setminus \{z\}$, there exists $n_0 > 1$ such that

$$\rho(x_n, z, a) \leq \frac{1}{5}\rho(x, z, a)$$

for all $n \geq n_0$ and all $a \in X$. Then we have by TA-inequality,

$$\begin{aligned}
\theta(r)\rho(x_n, Tx_n, a) &\leq \rho(x_n, Tx_n, a) = \rho(x_n, x_{n+1}, a) \\
&\leq \rho(x_n, z, a) + \rho(x_{n+1}, z, a) + \rho(x_n, x_{n+1}, z) \\
&\leq \frac{3}{5}\rho(x, z, a) = \rho(x, z, a) - \frac{2}{5}\rho(x, z, a) \\
&\leq \rho(x, z, a) - \rho(x_n, z, a) - \rho(x, z, x_n) \\
&\leq \rho(x_n, x, a).
\end{aligned}$$

Hence by (2.4.1), $\rho(x_{n+1}, Tx, a) \leq r\rho(x_n, x, a)$ for all $n \geq n_0$ and all $a \in X$. As a 2-metric ρ is always continuous with respect to any one of the three arguments, making $n \to \infty$ yields (2.4.3).

We observe from (2.4.3) that if z is a fixed point of T then it is unique, since given two fixed points u and v with $u \neq v$ we have from (2.4.3), $\rho(u, v, a) = \rho(Tu, v, a) \leq r\rho(u, v, a)$, for $u \in X \setminus \{v\}$, then $\rho(u, v, a) = 0$, hence $u = v$.

Indeed, if $T^k z = z$ for some positive integer k then z is the unique fixed point of T. so, we can assume that $T^k z \neq z$ for all $k \in \mathbb{N}$. Then (2.4.3) yields

$$\rho(T^{k+1}z, z, a) \leq r^k \rho(Tz, z, a) \tag{2.4.4}$$

for all $k \in \mathbb{N}$ and all $a \in X$.

We consider three cases of $\theta(r)$ as in [134]

Case1: $0 \leq r \leq \frac{\sqrt{5}-1}{2}$. This implies $r^2 + r \leq 1$ and $2r^2 < 1$.

If we assume that that $\rho(T^2 z, z, a) < \rho(T^2, T^3, a)$ for some $a \in X$, then

$$\begin{aligned} \rho(z, Tz, a) &\leq \rho(z, T^2 z, a) + \rho(z, T^2 z, Tz) + \rho(T^2 z, Tz, a) \\ &< \rho(T^2 z, T^3 z, a) + \rho(T^2 z, T^3 z, Tz) + \rho(Tz, T^2 z, a). \end{aligned}$$

So by (2.4.4) and (2.4.3),

$$\begin{aligned} \rho(z, Tz, a) &\leq r^2 \rho(z, Tz, a) + r^2 \rho(z, T^2 z, Tz) + r\rho(z, Tz, a) \\ &= (r^2 + r)\rho(z, Tz, a) \leq \rho(z, Tz, a). \end{aligned}$$

a contradiction. So we have for all $a \in X$,

$$\rho(T^2 z, z, a) \geq \rho(T^2 z, T^3 z, a) = \theta(r)\rho(T^2 z, T^3 z, a) \tag{2.4.5}$$

Therefore by TA-inequality,(2.4.4) and (2.4.1),

$$\begin{aligned}\rho(z,Tz,a) &\leq \rho(T^3z,Tz,a)+\rho(z,T^3z,a)+\rho(T^3z,z,Tz)\\ &\leq r^2\rho(z,Tz,a)+r\rho(T^2z,z,a)+r^2\rho(Tz,z,Tz)\\ &\leq r^2\rho(z,Tz,a)+r^2\rho(Tz,z,a)+0=2r^2\rho(z,Tz,a)\\ &< \rho(z,Tz,a)\end{aligned}$$

for all $a\in X$. This contradiction proves that $Tz=z$.

Case2: $\frac{\sqrt{5}-1}{2}\leq r\leq 2^{-\frac{1}{2}}$. We note that $2r^2<1$. If we assume $\rho(T^2z,z,a)<\theta(r)\rho(T^2,T^3,a)$ for some $a\in X$, then we have by TA-inequality and (2.4.2),

$$\begin{aligned}\rho(z,Tz,a) &\leq \rho(z,T^2z,a)+\rho(T^2z,Tz,a)+\rho(z,Tz,T^2z)\\ &< \theta(r)\rho(T^2z,T^3z,a)+r\rho(Tz,z,a)+r\rho(z,z,Tz)\\ &\leq \theta(r)r^2\rho(z,Tz,a)+r\rho(z,Tz,a)=(1-r)\rho(z,Tz,a)+r\rho(z,Tz,a)\\ &= \rho(z,Tz,a),\end{aligned}$$

a contradiction, and so $\rho(T^2z,z,a)\geq\theta(r)\rho(T^2,T^3,a)$ for all $a\in X$. Hence by TA-inequality, (2.4.4) and (2.4.1),

$$\begin{aligned}\rho(z,Tz,a) &\leq \rho(z,T^3z,a)+\rho(T^3z,z,Tz)+\rho(T^3z,Tz,a)\\ &\leq r^2\rho(z,Tz,a)+r^2\rho(Tz,z,Tz)+r\rho(T^2z,z,a)\\ &= 2r^2\rho(z,Tz,a)<\rho(z,Tz,a)\end{aligned}$$

for all $a\in X$. This yields $Tz=z$.

Case3: $2^{-\frac{1}{2}}\leq r<1$. By TA-inequality and (2.4.3),

$$\begin{aligned}\rho(x,Tx,a) &\leq \rho(x,z,a)+\rho(z,Tx,a)+\rho(z,Tx,x)\\ &\leq \rho(x,z,a)+r\rho(x,z,a)+r\rho(x,z,x).\end{aligned}$$

so that $\frac{1}{1+r}\rho(x, Tx, a) \leq \rho(x, z, a)$ for all $a \in X$.

This implies by (2.4.1) that

$$\rho(Tx, Tz, a) \leq r\rho(x, z, a) \tag{2.4.6}$$

for any $x \in X$ and all $a \in X$. Taking $x = x_n$ in (2.4.6) and making $n \to \infty$, we obtain $\rho(z, Tz, a) \leq 0$. Consequently $Tz = z$. This completes the proof.

As an application of Theorem 2.4.2, we have the following coincidence theorem for a pair of non-self mappings.

Theorem 2.4.3. *Let Y be an arbitrary set, (X, ρ) a 2-metric space and $S, T : Y \to X$ such that $T(Y)$ is contained in $S(Y)$, and $T(Y)$ or $S(Y)$ is a complete subspace of X. Assume that there exists $r \in [0, 1)$ such that*

$$\theta(r)\rho(Sx, Tx, a) \leq \rho(Sx, Sy, a) \textit{ impies } \rho(Tx, Ty, a) \leq r\rho(Sx, Sy, a) \tag{2.4.7}$$

for all $x, y, a \in Y$. Then there exists a point $v \in Y$ such that $Sv = Tv$

Proof. Let $Ha = T(S^{-1}a)$ for each $a \in S(Y)$, where S^{-1} denotes the inverse image of a under S. Therefore $Ha \subset S(Y)$ for every $a \in S(Y)$. Suppose $b_1, b_2 \in Ha$. Then there exist $x_1, x_2 \in S^{-1}a$ such that $b_1 = Tx_1$ and $b_2 = Tx_2$. Now

$$\rho(b_1, b_2, z) = \rho(Tx_1, Tx_2, z) \leq r\rho(Sx_1, Sx_2, z).$$

Hence $b_1 = b_2$, thus the set Ha contains only one point. Moreover if $a, b \in S(Y)$and $x \in S^{-1}a$, $y \in S^{-1}b$ then:

$$\rho(Ha, Hb, z) = \rho(Tx, Ty, z) \leq r\rho(Sx, Sy, z) = rd(a, b, z).$$

This means H is a contraction on $S(Y)$. By Banach contraction principle (with the assumption that $S(Y)$ is a complete subspace of X), there exists $k \in S(Y)$ such that

$k = Hk$. Finally if v is an arbitrary point in $S^{-1}k$,

$$Tv = T(S^{-1}k) = Hk = k = Sv.$$

Corollary 2.4.1. *Theorem 2.4.3 with $Y = X$. If T and S commute at (one of) their coincidence points then T and S have a unique common fixed point.*

Proof. Suppose $v \in X$ is such that $Sv = Tv$ and $SSv = STv = TSv$. Then for all $a \in X$,

$$\theta(r)\rho(Sv, Tv, a) = 0 \leq \rho(Sv, SSv, a),$$

and, by the assumption (2.4.7),

$$\theta(r)\rho(Sv, SSv, a) = \rho(Tv, TSv, a) \leq r\rho(Sv, SSv, a).$$

Consequently, T and S have a common fixed point. It is easy to see that the common fixed point is unique.

Chapter 3

Fixed Point Theorems with Weak Contractive Conditions

3.1 Introduction

In this Chapter, we study the notion of Ćirić type weakly generalized contraction mappings in metric spaces and prove theorems concerning the existence of coincidence and fixed points of such mappings.

We further discuss applications regarding the convergence theorems for modified Mann iterations and modified Ishikawa iterations in a convex metric space.
In all that follows X will represent a metric space (X, d).

3.2 Preliminaries

Alber and Guerre-Delabriere [9] introduced the notion of weakly contractive as follows:

Definition 3.2.1. A self mapping T of X is weakly contractive if for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \tag{3.2.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$.

Further, they proved that every weakly contractive mapping on a Hilbert space has a unique fixed point. They noted that this result is still true for uniformly smooth and uniformly convex Banach spaces (see [109]). The above result of Alber and Guerre-Delabriere [9] was generalized by Rhoades [109] where he showed that in fact the result holds in an arbitrary complete metric space. Precisely, he obtained the following result.

Theorem 3.2.1. *Let (X, d) be a complete metric space, T a weakly contractive mapping. Then T has a unique fixed point in X.*

Notice that weakly contractive mappings contain the Banach contraction as the special case when $(\phi(t) = (1 - k)t)$. Taking another self mapping into consideration, the concept of weakly contractive mappings was extended to two self mappings independently by Beg and Abbas [24] and Song [128].

Definition 3.2.2. [128]. A self-map T of X is f-weakly contractive if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(fx, fy) - \phi(d(fx, fy)), \tag{3.2.2}$$

where f is a self mapping of X and $\phi : [0, \infty) \to [0, \infty)$ is lower semi-continuous from the right such that ϕ is positive on $(0, \infty)$ and $\phi(0) = 0$.

We remark that Definition 3.2.2 with ϕ continuous is due to Beg and Abbas [24]. Further, if f is the identity map on X, and ϕ is a continuous and nondecreasing function in Definition 3.2.2 then a f-weakly contractive map reduces to a weakly contractive mapping and if $\phi(t) = (1-k)t$, for a constant k with $0 < k < 1$, then a f-weakly contractive map is an f-contraction(see [19] and [128]). Notice that if $f = I$ (the identity map on X) and ϕ is a lower semi-continuous map from the right then $\varphi(t) = t - \phi(t)$ is upper semi-continuous from the right, then (3.2.2) becomes

$$d(Tx, Ty) \leq \varphi(d(x, y)). \tag{3.2.3}$$

Therefore, weakly contractive mappings for which ϕ is lower semi-continuous from the right are of Boyd and Wong [30] type (see also [19] and [128]). Further, notice that with $f = I$, if we define $k(t) = 1 - \frac{\phi(t)}{t}$ for $t > 0$ and $k(0) = 0$ then the condition (3.2.2) is reduced to the following Reich type [101] contractive condition:

$$d(Tx, Ty) \leq k(d(fx, fy))d(fx, fy). \tag{3.2.4}$$

Indeed, for a suitable choice of $k(t)$, for $t > 0$ and $k(0) = 0$, weakly contractive mappings reduce to (3.2.4). The following result is due to Beg and Abbas [24].

Theorem 3.2.2. *Let T and f be self mappings of X such that $TX \subseteq fX$ and (3.2.2) holds for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that ϕ is positive on $(0, \infty)$ and $\phi(0) = 0$. If $f(X)$ is a complete subspace of X then T and f have a common fixed point provided that T and f are commuting at their coincidence points.*

Azam and Shakeel [18] obtained a similar result for three self mappings of X satisfying the following condition for each $x, y \in X$,

$$d(Sx, Ty) \leq d(fx, fy) - \phi(d(fx, fy)), \tag{3.2.5}$$

where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that ϕ is positive on $(0, \infty)$ and $\phi(0) = 0$.

The following generalized contraction for a self mapping T of X is essentially due to Ćirić [34]:

$$d(Tx, Ty) \leq km(x, y), \tag{3.2.6}$$

where $0 < k < 1$ and

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$$

Now, let us consider the following Ćirić type weakly generalized contraction for a self mapping T of X

$$d(Tx, Ty) \leq m(x, y) - \phi(m(x, y)), \tag{3.2.7}$$

where $\phi : [0, \infty) \to [0, \infty)$ is lower semi-continuous from the right such that ϕ is positive on $(0, \infty)$ and $\phi(0) = 0$. Zhang and Song [149], have shown the existence of a unique common fixed point of self mappings S and T of a complete metric space X satisfying the following condition for each $x, y \in X$:

$$d(Sx, Ty) \leq m'(x, y) - \phi(m'(x, y)), \tag{3.2.8}$$

where ϕ is as in (3.2.7), and

$$m'(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\}.$$

See also the remark following Theorem 3.3.2 below. Notice that (3.2.8) with $S = T$ is the condition (3.2.7).

Using an additional function $\psi : [0, \infty) \to [0, \infty)$ and a quadruplet of maps Abbas and Doric [3] obtained the following common fixed point theorem in a complete metric space.

Theorem 3.2.3. *Let S, T, f and g be self mappings of a complete metric space (X, d) such that* *(i)* $TX \subseteq fX, SX \subseteq gX$,

(ii) one of the ranges SX, TX, fX *and* gX *is closed, and if for each* $x, y \in X$,

(iii) $\psi(d(Sx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y))$ *where*

$$M(x, y) := \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)]\},$$

ϕ *is as in (3.2.7) and* $\psi : [0, \infty) \to [0, \infty)$ *is continuous and nondecreasing with* $\psi(t) = 0$ *if and only if* $t = 0$.

Then S, T, f *and* g *have a unique common fixed point in* X *provided that the pairs* (S, f) *and* (T, g) *are weakly compatible.*

We remark that completeness of the space X in Theorem 3.2.3 implies that closed ranges considered in (ii) are also complete.

In the next section, first we obtain two variants of Theorem 3.2.3. Indeed, we obtain coincidence and common fixed point theorems for a pair, triplet and quadruplet of Ćirić type weakly generalized contractions on a metric space, wherein the completeness of the metric space X is replaced by much weaker alternative hypotheses. This is shown, as another alternative, that if the space X is complete in Theorem 3.2.3, the requirement (ii) is not needed provided that one of the maps S, T, f and g is continuous (cf. Theorem 3.3.2). Usefulness of the results is illustrated by examples to clarify underlying strength and distinctions.

3.3 Coincidence and Fixed Point Theorems

Throughout this section, let $C(S,f) = \{v : Sv = fv\}$ denote the collection of coincidence points of self mappings S and f of a metric space X:

$$M(x,y) := \max\{d(fx,gy), d(fx,Sx), d(gy,Ty), \frac{1}{2}[d(fx,Ty) + d(gy,Sx)]\},$$

$$M(f,T) := \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}[d(fx,Ty) + d(fy,Tx)]\},$$

$$m(f,T) := \max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\},$$

$\Phi = \{\phi | \phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous from the right such that ϕ is positive on $(0,\infty)$ and $\phi(0) = 0\}$ and

$\Psi = \{\psi | \psi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with $\psi(t) = 0$ if and only if $t = 0\}$.

The following is our first result for a quadruplet of maps on a metric space.

Theorem 3.3.1. *Let S, T, f and g be self mappings of a metric space X such that*

$$TX \subseteq fX \text{ and } SX \subseteq gX. \tag{3.3.1}$$

If for each $x, y \in X$,

$$\psi(d(Sx,Ty)) \le \psi(M(x,y)) - \phi(M(x,y)), \tag{3.3.2}$$

where $\phi \in \Phi$ and $\psi \in \Psi$. If one of fX, gX, SX or TX is a complete subspace of X then:

(I) $C(S,f)$ and $C(T,g)$ are nonempty. Further,

(II) S and f have a common fixed point provided that they commute just at a coincidence point;

(III) T and g have a common fixed point provided that they commute just at a coincidence point;

(IV) S, T, f and g have a unique common fixed point provided (II) and (III) are true.

Proof. Pick $x_0 \in X$. As $TX \subseteq fX$, we can choose a point $x_1 \in X$ such that $Tx_0 = fx_1$, and for this point x_1, there exists a point x_2 in X such that $Sx_1 = gx_2$. Continuing in this manner, we define two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n} = fx_{2n+1} \text{ and } y_{2n+1} = Sx_{2n+1} = gx_{2n+2},\ n = 0, 1, 2, ...$$

In view of Abbas and Doric [3], the sequence $\{y_n\}$ is Cauchy. Assume that fX is complete. Notice that the sequence $\{y_{2n}\}$ is contained in fX and has a limit in fX. Call it u. Let $v \in f^{-1}u$. Then $fv = u$. The subsequence $\{y_{2n+1}\}$ also converges to u. Now we show that $Sv = fv = u$. By (3.3.2),

$$\begin{aligned}\psi(d(Sv, y_{2n})) &= \psi(d(Sv, Tx_{2n})) \leq \psi(\max\{d(fv, gx_{2n}), d(fv, Sv), d(gx_{2n}, Tx_{2n}),\\ &\quad \frac{1}{2}[d(fv, Tx_{2n}) + d(gx_{2n}, Sv)]\}) - \phi(\max\{d(fv, gx_{2n}),\\ &\quad d(fv, Sv), d(gx_{2n}, Tx_{2n}), \frac{1}{2}[d(fv, Tx_{2n}) + d(gx_{2n}, Sv)]\}).\end{aligned}$$

Making $n \to \infty$,

$$\begin{aligned}\psi(d(Sv, u)) &\leq \psi(\max\{d(fv, u), d(fv, Sv), d(u, u), \frac{1}{2}[d(fv, u) + d(u, Sv)]\})\\ &\quad -\phi(\max\{d(fv, u), d(fv, Sv), d(u, u), \frac{1}{2}[d(fv, u) + d(u, Sv)]\}).\end{aligned}$$

that is

$$\psi(d(Sv, fv)) = \psi(d(Sv, u)) \leq \psi(d(fv, Sv)) - \phi(d(fv, Sv)),$$

a contradiction.

Therefore, $Sv = fv = u$. This proves that $C(S, f)$ is nonempty.

Since $SX \subseteq gX$, $Sv = u$ implies $u \in gX$. Let $w \in g^{-1}u$. Then $gw = u$. Now we show that $Tw = u$. If $Tw \neq u$,

$$\begin{aligned}\psi(d(u, Tw)) &= \psi(d(Sv, Tw)) \leq \psi(\max\{d(fv, gw), d(fv, Sv), d(gw, Tw), \frac{1}{2}[d(fv, Tw) \\ &\quad +d(gw, Sv)]\}) - \phi(\max\{d(fv, gw), d(fv, Sv), d(gw, Tw), \\ &\quad \frac{1}{2}[d(fv, Tw) + d(gw, Sv)]\}).\end{aligned}$$

Therefore, $\psi(d(Sv, Tw)) \leq \psi(d(Sv, Tw)) - \phi(d(Sv, Tw))$, a contradiction.

Hence $Tw = u$. So $Tw = gw = u = fv = Sv$. This proves that $C(T, g)$ is nonempty. If gX is complete then an analogous argument establishes (I). If SX is complete then $u \in SX \subset gX$ and we see that $C(T, g)$ is nonempty. Similarly if TX is complete then $u \in TX \subset fX$, and $C(S, f)$ is nonempty. Thus (I) is completely established.

Now suppose that S and f are commuting at their coincidence point v. Then $SSv = Sfv = fSv = ffv$. So, by (3.3.2), we have

$$\begin{aligned}\psi(d(Sv, SSv)) &= \psi(d(SSv, Tw)) \leq \psi(\max\{d(fSv, gw), d(fSv, SSv), d(gw, Tw), \\ &\quad \frac{1}{2}[d(fSv, Tw) + d(gw, SSv)]\}) - \phi(\max\{d(fSv, gw), d(fSv, SSv), \\ &\quad d(gw, Tw), \frac{1}{2}[d(fSv, Tw) + d(gw, SSv)]\}) \\ &= \psi(d(SSv, Tw)) - \phi(d(SSv, Tw)),\end{aligned}$$

a contradiction. This proves (II). Analogously, if T and g are commuting at their coincidence point w, then w is a common fixed point of T and g. This proves (III). Now, (IV) is immediate.

In order to see the uniqueness part of (IV), assume that there are two distinct common fixed points y and z, that is $fy = gy = Sy = Ty = y$ and $fz = gz = Sz =$

$Tz = z$. Using (3.3.2),

$$\begin{aligned}
\psi(d(y,z)) &= \psi(d(Sy,Tz)) \le \psi(\max\{d(fy,gz), d(fy,Sy), d(gz,Tz), \\
&\quad \frac{1}{2}[d(fy,Tz)+d(gz,Sy)]\}) - \phi(\max\{d(fy,gz), \\
&\quad d(fy,Sy), d(gz,Tz), \frac{1}{2}[d(fy,Tz)+d(gz,Sy)]\}) \\
&\le \psi(d(y,z)) - \phi(d(y,z)) < \psi(d(y,z)),
\end{aligned}$$

a contradiction. This completes the proof.

Self mappings S and f of a metric space X are compatible (see Jungck [61]) if $\lim_{n\to\infty}(Sfx_n, fSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$. For a detailed comparison of various weaker forms of commuting maps, one refer to Singh and Tomar [123] and Stofile [130].

The following result is another variant of Theorem 3.2.3.

Theorem 3.3.2. *Let S, T, f and g be self mappings of a complete metric space (X, d) such that conditions (3.3.1) and (3.3.2) of Theorem 3.3.1 are satisfied. If the pairs (S, f) and (T, g) are compatible and one of the mappings S, T, f and g is continuous then S, T, f and g have a unique common fixed point in X.*

Proof. As in Theorem 3.3.1, the sequence $\{y_n\}$ defined by

$$y_{2n} = Tx_{2n} = fx_{2n+1} \text{ and } y_{2n+1} = Sx_{2n+1} = gx_{2n+2},\ n = 0, 1, 2, ...$$

is a Cauchy sequence. By completeness of X, sequence $\{y_n\}$ converges to a point $u \in X$. Also

$$y_{2n} = Tx_{2n} = fx_{2n+1} \to u\ ,\ y_{2n+1} = Sx_{2n+1} = gx_{2n+2} \to u.$$

Suppose that f is continuous. Then $ffx_{2n} \to fu$, $fSx_{2n} \to fu$, and the compatibility of (S, f) implies $Sfx_{2n} \to fu$. Using (3.3.2), we have

$$\psi(d(Sfx_{2n}, Tx_{2n})) \leq \psi(M(fx_{2n}, x_{2n})) - \phi(M(fx_{2n}, x_{2n})).$$

Making $n \to \infty$,

$$\begin{aligned} \psi(d(fu, u)) &\leq \psi(\max\{d(fu, u), d(fu, fu), d(u, u), \frac{1}{2}[d(fu, u) + d(u, fu)]\}) \\ &\quad -\phi(\max\{d(fu, u), d(fu, fu), d(u, u), \frac{1}{2}[d(fu, u) + d(u, fu)]\}), \end{aligned}$$

that is $\psi(d(fu, u)) \leq \psi(d(fu, u)) - \phi(d(fu, u))$, a contradiction.

So $fu = u$. Further, from (3.3.2),

$$\psi(d(Su, Tx_{2n})) \leq \psi(M(u, x_{2n})) - \phi(M(u, x_{2n})).$$

Making $n \to \infty$,

$$\begin{aligned} \psi(d(Su, u)) &\leq \psi(\max\{d(fu, u), d(fu, Su), d(u, u), \frac{1}{2}[d(fu, u) + d(u, Su)]\}) \\ &\quad -\phi(\max\{d(fu, u), d(fu, Su), d(u, u), \frac{1}{2}[d(fu, u) + d(u, Su)]\}) \\ &= \psi(d(Su, u)) - \phi(d(Su, u), \end{aligned}$$

a contradiction. Therefore, $Su = u$. Since $SX \subseteq gX$, there exists a point $v \in X$ such that $u = fu = Su = gv$. We claim that $u = Tv$. If not, then using (3.3.2),

$$\begin{aligned} \psi(d(Su, Tv)) &\leq \psi(\max\{d(fu, gv), d(fu, Su), d(gv, Tv), \frac{1}{2}[d(fu, Tv) + d(gv, Su)]\}) \\ &\quad -\phi(\max\{d(fu, gv), d(fu, Su), d(gv, Tv), \frac{1}{2}[d(fu, Tv) + d(gv, Su)]\}) \\ &= \psi(d(u, Tv)) - \phi(d(u, Tv)), \end{aligned}$$

a contradiction. Hence $u = Tv = gv$.

Now compatibility of (T, g) implies that $Tu = Tgv = gTv = gu$. From (3.3.2), we get

$$\begin{aligned}\psi(d(Su,Tu)) &\leq \psi(\max\{d(fu,gu), d(fu,Su), d(gu,Tu), \frac{1}{2}[d(fu,Tu)+d(gu,Su)]\}) \\ &\quad -\phi(\max\{d(fu,gu), d(fu,Su), d(gu,Tu), \frac{1}{2}[d(fu,Tu)+d(gu,Su)]\}),\end{aligned}$$

that is $\psi(d(u,Tu)) \leq \psi(d(u,Tu)) - \phi(d(u,Tu))$, a contradiction. So $u = Tu = Su = fu = gu$. This proves that S, T, f and g have a common fixed point. This result holds if g is continuous instead of f. Suppose S is continuous then the compatibility of (S, f) gives $Sf_{2n} = SSx_{2n} = fSx_{2n} \to Su$.

Using (3.3.2), we get

$$\psi(d(SSx_{2n}, Tx_{2n})) \leq \psi(M(Sx_{2n}, x_{2n})) - \phi(M(Sx_{2n}, x_{2n})).$$

Making $n \to \infty$,

$$\begin{aligned}\psi(d(Su,u)) &\leq \psi(\max\{d(Su,u), d(Su,Su), d(u,u), \frac{1}{2}[d(Su,u)+d(u,Su)]\}) \\ &\quad -\phi(\max\{d(Su,u), d(Su,Su), d(u,u), \frac{1}{2}[d(Su,u)+d(u,Su)]\}) \\ &= \psi(d(Su,u)) - \phi(d(Su,u)),\end{aligned}$$

a contradiction. Hence $u = Su$. Since $SX \subseteq gX$, there exists a point v in X such that $u = Su = gv$. We now show that $Tv = gv$. By (3.3.2),

$$\psi(d(SSx_{2n}, Tv)) \leq \psi(M(Sx_{2n}, v)) - \phi(M(Sx_{2n}, v)).$$

Letting $n \to \infty$,

$$\begin{aligned}\psi(d(Su,Tv)) &\leq \psi(\max\{d(Su,gv), d(Su,Su), d(Su,Tv), \frac{1}{2}[d(Su,Tv)+d(Su,Su)]\}) \\ &\quad -\phi(\max\{d(Su,gv), d(Su,Su), d(Su,Tv), \frac{1}{2}[d(Su,Tv)+d(Su,Su)]\}),\end{aligned}$$

that is $\psi(d(u, Tv)) = \psi(d(u, Tv)) - \phi(d(u, Tv))$,

a contradiction. Therefore, $u = Tv = gv = Su$, and the compatibility of (T, g) yields $Tu = Tgv = gTv = gu$. Now using (3.3.2), we get

$$\psi(d(Sx_{2n}, Tu)) \leq \psi(M(x_{2n}, u)) - \phi(M(x_{2n}, u)).$$

Making $n \to \infty$,

$$\psi(d(u, Tu)) = \psi(d(u, Tu)) - \phi(d(u, Tu)). \text{ So } u = Tu = gu.$$

As $TX \subseteq fX$, there exists a point w in X such that $u = Tu = gu = fw$. We now show that $u = Sw$. Using (3.3.2) with $x = w$ and $y = u$, we have

$$\psi(d(Sw, Tu)) = \psi(M(w, u)) - \phi(M(w, u)) = \psi(d(u, Sw)) - \phi(M(u, Sw)$$

a contradiction. Therefore, $u = Sw = fw = Tu = gu$. Compatibility of the pair (S, f) implies $u = Sw = Sfw = fSw = fu$. Hence $u = Su = Tu = gu = fu$. Thus u is a common fixed point of S, T, f and g. If T is continuous then analogous argument establishes the existence of common fixed point of S, T, f and g. Uniqueness of the fixed point follows.

In case $f = g$ in Theorem 3.3.1, we obtain a slightly improved version which we state below.

Theorem 3.3.3. *Let S, T and f be self mappings of a metric space X such that $SX \cup TX \subseteq fX$, and the condition (3.3.2) with $f = g$ holds. If one of SX, TX or fX is a complete subspace of X, then S, T and f have a common coincidence point. Further, if f commutes with each of S and T at one of their coincidence points, then S, T and f have a unique common fixed point.*

We remark that Theorem 3.3.3 with f as the identity map on X is a common fixed point theorem for two self mappings S and T of a complete metric space due to Doric [39]. Further, Theorem 3.3.3 with f as the identity mapping on X and $\psi(t) = t$ for all $t \in [0, \infty)$ is a common fixed point theorem for two self mappings S and T of a complete metric space due to Zhang and Song [149]. The following example demonstrates the generality of Theorem 3.3.3 over results of Azam and Shakeel[18], Doric [39] and Zhang and Song [149].

Example 3.3.1. *Let $X = \{1, 2, 3, 4, 5\}$ be endowed with the usual metric. Let $f = g$ and $S1 = S3 = S4 = 1$, $S2 = S5 = 2$, $T1 = T3 = 1$, $T2 = T4 = T5 = 2$ and $f1 = 3$, $f2 = 2$, $f3 = 5$, $f4 = 4$, $f5 = 1$. Notice that SX and TX are contained in fX, and the condition (3.3.2) of Theorem 3.3.3 is satisfied, when $\psi : [0, \infty) \to [0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$ are respectively defined as $\psi(t) = 2t$ and $\phi(t) = \frac{1}{5}t$. So all the hypothesis of Theorem 3.3.1(or Theorem 3.3.1 with $f = g$) are true.*

Evidently, S, T and f have a unique common fixed point at $x = 2$. Notice that the condition (3.2.5) used by Azam and Shakeel [18] is not satisfied for $x = 2$ and $y = 1$. Further, it can be easily verified that the condition (3.2.8) of Zhang and Song [149] and condition used by Doric [39] are not satisfied for $x = 2$ and $y = 1$ with f as identity map.

We remark that under the hypotheses of Theorem 3.3.1, the pairs (S, f) and (T, g) may have different coincidence points. Further, if $f = g$ in Theorem 3.3.1, then maps S, T and f have a common coincidence point(cf. Theorem 3.3.3). However, it is interesting to note that if f and g are distinct and $S = T$ in Theorem 3.3.1, then maps f, g and S need not have a common coincidence point. The following examples authenticate these remarks.

Example 3.3.2. *Let $X = [0, \infty)$ be endowed with the usual metric and define*

$$\psi, \phi : [0, \infty) \to [0, \infty) \text{ as } \psi(t) = t \text{ and } \phi(t) = \frac{1}{6}t.$$

Let $SX = x^2 + \frac{5}{9}$, $TX = x^3 + \frac{5}{9}$, $fX = 6x^2$, $gX = 6x^3$, $x \in X$. Then:

$$\begin{aligned} \psi(d(Sx, Ty)) &= \psi(|x^2 - y^3|) = |x^2 - y^3| < \psi(6|x^2 - y^3|) \\ &- \phi(6|x^2 - y^3|) = \psi(d(fx, gy) - \phi(fx, gy), \end{aligned}$$

for all $x, y \in X$. So, (3.3.2) and other hypotheses of Theorem 3.3.1 are satisfied. We see that $S(\frac{1}{3}) = f(\frac{1}{3}) = \frac{6}{9}$ and $T(\frac{1}{9})^{\frac{1}{3}} = g(\frac{1}{9})^{\frac{1}{3}} = \frac{6}{9}$, that is, S and f have a coincidence at $x = \frac{1}{3}$, while T and g have a (different) coincidence at $x = (\frac{1}{9})^{\frac{1}{3}}$.

The following example shows that if $f = g$ in Theorem 3.3.1, then mappings S, T and f have a common coincidence point as guaranteed by Theorem 3.3.3. The example also shows that the requirement of commutativity in Theorem 3.3.3 is imperative and cannot be relaxed.

Example 3.3.3. *Let $X = \{1, 2, 3, 4\}$ be endowed with the usual metric. Let $f = g$ and $S1 = S2 = S3 = S4 = 1$, $T1 = T2 = T3 = 1$, $T4 = 2$ and $f1 = 3$, $f2 = 2$, $f3 = 1$, $f4 = 4$. Then S, T and f satisfy all the hypotheses of Theorem 3.3.3 with $\psi(t) = 2t$ and $\phi(t) = \frac{1}{3}t$. Evidently, S, T and f have a common coincidence at $x = 3$, while f does not commute with either S or T. So the only common coincidence point 3 of S, T and f is not their common fixed point.*

The following example shows that if $S = T$ in Theorem 3.3.1, then mappings f, g and S need not have a common coincidence point.

Example 3.3.4. *Let $X = \{1, 2, 3\}$ be endowed with the usual metric and $S1 = S2 = S3 = 1$, $f1 = 2$, $f2 = 3$, $f3 = 1$ and $g1 = 3$, $g2 = 1$, $g3 = 2$. It is easy to see* *that S, f and g satisfy all the hypotheses of Theorem 3.3.1 with $S = T$, $\psi(t) = t$ and $\phi(t) = \frac{1}{2}t$. Notice that $S3 = f3 = 1 = S2 = g2$, that is, S and f have a coincidence at $x = 3$ and S and g have a (different) coincidence at $x = 2$.*

From Theorem 3.3.1 we derive the following result which generalizes fixed point theorems of Chang [33], Jachymiski [57], Jungck and Pathak [62], Kang and Rhoades [69], Song [128], Tan [137] and others.

Corollary 3.3.1. *Let S, T, f and g be self mappings of a metric space X such that the condition (3.3.1) of Theorem 3.3.1 and the following is satisfied:*

$$d(Sx, Ty) \leq \varphi(M(x, y)), \tag{3.3.3}$$

for all $x, y \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$. If one of fX, gX, SX or TX is a complete subspace of X then the conclusions of Theorem 3.3.1 hold.

Proof. Let $\phi(t) = t - \varphi(t)$. Then from (3.3.3), we have

$$d(Sx, Ty) \leq M(x, y) - \phi(M(x, y)), \text{ for all } x, y \in X.$$

Then $\phi : [0, \infty) \to [0, \infty)$ is lower semi-continuous from the right such that $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$. The result follows from the proof of Theorem 3.3.1 with $\psi(t) = t$.

For a pair of self mappings T and f of X, we obtain the following result which generalizes certain results of Beg and Abbas [24], Dutta and Choudhury [41] and Song [128].

Corollary 3.3.2. *Let T and f be self mappings of a metric space X such that*

$$TX \subseteq fX, \tag{3.3.4}$$

if for each $x, y \in X$,

$$\psi(d(Tx,Ty)) \leq \psi(M(f,T)) - \phi(M(f,T)), \tag{3.3.5}$$

where $\phi \in \Phi$ and $\psi \in \Psi$. If one of TX or fX is a complete subspace of X then $C(T,f)$ is nonempty. Further, T and f have a unique common fixed point provided T and f commute at a coincidence point.

Proof. It comes from Theorem 3.3.3 when $S = T$. Indeed, if $v \in C(T,f)$, then $Tv = fv = u$ (say). If T and f commute at v then u is their unique common fixed point.

We remark that if f is the identity map on X and $\psi(t) = t$ in Corollary 3.3.2 then we obtain [149, Cor. 2.2] as a special case. From Corollary 3.3.2 we immediately obtain the following result.

Corollary 3.3.3. *Let T and f be self mappings of a metric space X such that the condition (3.3.4) of Corollary 3.3.2 and the following are satisfied:*

$$\psi(d(Tx,Ty)) \leq \psi(m(f,T)) - \phi(m(f,T)) \tag{3.3.6}$$

for all $x, y \in X$, where $\phi \in \Phi$ and $\psi \in \Psi$. If one of TX or fX is a complete subspace of X, then conclusions of Corollary 3.3.2 are true.

The following example shows that maps T and f satisfy all the conditions of Corollary 3.3.2 but the condition (3.2.2) due to Beg and Abbas [24] and Song [128] is not satisfied.

Example 3.3.5. *Let $X = \{1, 2, 3, 4, 5\}$ with the usual metric and $T1 = T3 = T4 = 1$, $T2 = T5 = 2$ and $f1 = 3$, $f2 = 2$, $f3 = 5$ $f4 = 4$, $f5 = 1$. Notice that the condition*

(3.3.5) of Corollary 3.3.2 is satisfied if $\psi, \phi : [0, \infty) \to [0, \infty)$ are defined as $\psi(t) = t$ and $\phi(t) = \frac{1}{5}t$. It can be verified that T and f satisfy all the conditions of Corollary 3.3.2. Further, it is easy to see that condition (3.2.2) of Theorem 3.2.2 is not satisfied for $x = 2$ and $y = 1$.

3.4 Convergence Theorems

In this section we obtain certain convergence theorems for modified Mann iterations and modified Ishikawa iterations in a convex metric space. Takahashi [136] introduced a varient of the notion of convexity in metric spaces. He obtained some interesting generalizations of fixed point theorems in Banach spaces (see, for instance [23], [25], [37], [85] and [100]). The following definition is essentially due to Takahashi [136] (see also Agarwal et al. [7], Mishra [85] and Rafiq and Zafar [100]).

Definition 3.4.1. Let X be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is a convex structure on X, if for all $x, y \in X$ and $\lambda \in [0, 1]$, the condition

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

is satisfied for all $u \in X$.

A metric space with a convex structure is called a convex metric space. A nonempty subset C of a convex metric space X is convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. Notice that all normed spaces and each of their convex subsets are convex metric spaces. However, there are examples of convex metric spaces which are not embedded in a normed space. For details, one may refer to Takahashi [136] and Rafiq and Zafar [100].

Definition 3.4.2. Let X be a convex metric space and T, f self mappings of X such that $TX \subseteq fX$. Let $f(X)$ be a convex subset of X. Define a sequence $\{y_n\}$ in $f(X)$ as follows:

$$y_n = fx_{n+1} = W(Tx_n, fx_n, \alpha_n) = (1-\alpha_n)fx_n + \alpha_n Tx_n, x_0 \in X, n \geq 0,$$

where $0 \leq \alpha_n \leq 1$ for each $n \geq 0$. The sequence $\{y_n\}$ so obtained is called modified Mann iterative scheme.

Definition 3.4.3. If we take f the identity mapping on X in Definition 3.4.2, then the sequence $\{x_n\}$ is called the Mann sequence of iterates.

Definition 3.4.4. Let X be a convex metric space and T, f self mappings of X such that $TX \subseteq fX$. Let $f(X)$ be a convex subset of X. Define two sequences $\{z_n\}$ and $\{y_n\}$ in $f(X)$ as follows:

$$z_n = fx_{n+1} = W(Tp_n, fx_n, \alpha_n) = (1-\alpha_n)fx_n + \alpha_n Tp_n,$$
$$y_n = fp_n = W(Tx_n, fx_n, \beta_n) = (1-\beta_n)fx_n + \beta_n Tx_n, x_0 \in X, n \geq 0,$$

where $0 \leq \alpha_n, \beta_n \leq 1$ for each $n \geq 0$. The sequence $\{z_n\}$ so obtained is called the modified Ishikawa iterative scheme.

Now we present a convergence theorem for modified Mann iterations in a convex metric space.

Theorem 3.4.1. *Let T and f be self mappings of a convex metric space X such that conditions of Corollary 3.3.3 are satisfied. If TX or fX is a complete convex subspace of X, then the modified Mann iterative scheme converges to a common fixed point of T and f provided that T and f commute at their coincidences.*

Proof. From Corollary 3.3.3 (see also the proof of Corollary 3.3.2), we suppose that $v \in C(T, f)$ and $Tv = fv = u$. Suppose T and f commute at their coincidence point v. Then u is their unique common fixed point. Now consider:

$$\begin{aligned}
\psi(d(y_n, u)) &= \psi(d(fx_{n+1}, fv)) = \psi(d(W(Tx_n, fx_n, \alpha_n), fv)) \\
&\leq \psi[(1-\alpha_n)d(fx_n, fv) + \alpha_n d(Tx_n, fv)] \\
&= (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n \psi(d(Tx_n, Tv)) \\
&\leq (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n[\psi(\max\{d(fx_n, fv), d(fx_n, Tx_n), d(fv, Tv)\}) \\
&\quad -\phi(\max\{d(fx_n, fv), d(fx_n, Tx_n), d(fv, Tv)\})] \\
&= (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n[\psi(m(f, T)) - \phi(m(f, T))]. \qquad (3.4.1)
\end{aligned}$$

Now we consider the following two cases.

Case 1: $m(f, T) = d(fx_n, fv)$. Then from (3.4.1),

$$\begin{aligned}
\psi(d(y_n, u)) &\leq (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n[\psi(d(fx_n, fv)) - \phi(d(fx_n, fv))] \\
&= \psi(d(fx_n, fv)) - \alpha_n\psi(d(fx_n, fv)) + \alpha_n\psi(d(fx_n, fv)) - \alpha_n(\phi(d(fx_n, fv))) \\
&= \psi(d(fx_n, fv)) - \alpha_n(\phi(d(fx_n, fv))) \qquad (3.4.2) \\
&\leq \psi(d(y_{n-1}, u)).
\end{aligned}$$

Case 2: $m(f, T) = d(fx_n, Tx_n)$. Then from (3.4.1),

$$\begin{aligned}
\psi(d(y_n, u)) &\leq (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n[\psi(d(fx_n, Tx_n)) - \phi(d(fx_n, Tx_n))] \\
&= \psi(d(fx_n, fv)) - \alpha_n\psi(d(fx_n, fv)) + \alpha_n(\psi(d(fx_n, Tx_n))) \\
&\quad -\alpha_n(\phi(d(fx_n, Tx_n))) \\
&\leq \psi(d(fx_n, fv)) - \alpha_n\psi[d(fx_n, Tx_n) + d(Tx_n, fv)] + \alpha_n\psi(d(fx_n, Tx_n)) \\
&\quad -\alpha_n(\phi(d(fx_n, Tx_n)))
\end{aligned}$$

$$\begin{aligned}
&= \psi(d(fx_n, fv)) - \alpha_n(\psi(d(Tx_n, fv))) - \alpha_n(\phi(d(fx_n, Tx_n))) \\
&\leq \psi(d(fx_n, fv)) - \alpha_n(\phi(d(fx_n, Tx_n))) \\
&\leq \psi(d(fx_n, fv)) - \alpha_n(\phi[d(fx_n, fv) + d(fv, Tx_n)]) \\
&= \psi(d(fx_n, fv)) - \alpha_n(\phi(d(fx_n, fv))) \qquad (3.4.3) \\
&\leq \psi(d(y_{n-1}, u)).
\end{aligned}$$

Therefore, in both cases, by monotonic nature of ψ, $d(y_n, u) \leq d(y_{n-1}, u)$. This shows that $\{d(y_n, u)\}$ is a monotone non-increasing sequence of real numbers and tends to a limit $r \geq 0$. So, $\lim_{n\to\infty} d(y_n, u) = r \geq 0$. Now if $r > 0$, then from (3.4.2) and (3.4.3), using the lower semi-continuity of ϕ, we have

$\psi(r) \leq \psi(r) - \lim_{n\to\infty} \inf \phi(d(fx_{2n}, fv)) \leq \psi(r) - \phi(r)$, i.e., $\phi(r) \leq 0$,

a contradiction. Therefore, the modified Mann iterative scheme converges to a common fixed point of the mappings T and f.

We remark that Theorem 3.4.1 (under the weak contractive condition (3.3.6)) substantially improves certain results of Azam and Shakeel [18], Beg and Abbas [24], Bose and Roychowdhury [29] and Rhoades [109]. Notice that the requirement $\sum \alpha_n = \infty$ on the sequence $\{\alpha_n\}$ in Theorem 3.1 is not needed.

The following is the convergence result for modified Ishikawa iterations in a convex metric space.

Theorem 3.4.2. *Let T and f be self mappings of a convex metric space such that condition (3.3.4) of Corollary 3.3.2 and the following condition is satisfied:*

$$\psi(Tx, Ty)) \leq \psi(d(fx, fy)) - \phi(d(fx, f)) \qquad (3.4.4)$$

for all $x, y \in X$, where $\phi \in \Phi$ and $\psi \in \Psi$. If TX or fX is a convex and complete *subspace of X, then the modified Ishikawa iterative scheme with $\sum \alpha_n\beta_n = \infty$ converges to a common fixed point of T and f provided that T and f commute at their coincidences.*

Proof. From Corollary 3.3.2, as in the proof of Theorem 3.4.1, T and f have a coincidence point v and a unique common fixed point u in X such that $Tv = fv = u$. We consider:

$$\begin{aligned}
\psi(d(z_n, u)) &= \psi(d(W(Tp_n, fx_n, \alpha_n), u)) \leq \alpha_n(\psi[d(Tp_n, u) + (1-\alpha_n)d(fx_n, u)]) \\
&\leq \alpha_n\psi(d(Tp_n, Tv)) + (1-\alpha_n)\psi(d(fx_n, u)) \\
&\leq \alpha_n[\psi(d(fp_n, fv)) - \phi(d(fp_n, fv))] + (1-\alpha_n)\psi(d(fx_n, u)) \\
&= \alpha_n\psi(d(fp_n, u)) - \alpha_n\phi(d(fp_n, u)) + (1-\alpha_n)\psi(d(fx_n, u)) \\
&= \alpha_n\psi(d(W(Tx_n, fx_n, \beta_n), u)) - \alpha_n\phi(d(fp_n, u)) + (1-\alpha_n)\psi(d(fx_n, u)) \\
&\leq \alpha_n(\psi[\beta_n d(Tx_n, u) + (1-\beta_n)d(fx_n, u)]) - \alpha_n\phi(d(fp_n, u)) + \\
&\quad (1-\alpha_n)\psi(d(fx_n, u)) \\
&= \alpha_n\beta_n(\psi(d(Tx_n, Tu))) + \alpha_n(1-\beta_n)\psi(d(fx_n, u)) - \alpha_n\phi(d(fp_n, u)) \\
&\quad +(1-\alpha_n)\psi(d(fx_n, u)) \\
&\leq \alpha_n\beta_n[\psi(d(fx_n, u)) - \phi(d(fx_n, u))] + \alpha_n(1-\beta_n)\psi(d(fx_n, u)) \\
&\quad -\alpha_n\phi(d(fp_n, u)) + (1-\alpha_n)\psi(d(fx_n, u)) \\
&= \alpha_n\beta_n\psi(d(fx_n, u)) - \alpha_n\beta_n\phi(d(fx_n, u)) + \alpha_n\psi(d(fx_n, u)) - \alpha_n\beta_n\psi(d(fx_n, u)) \\
&\quad -\alpha_n\phi(d(fp_n, u)) + \psi(d(fx_n, u)) - \alpha_n\psi(d(fx_n, u)) \\
&\leq \psi(d(fx_n, u)) - \alpha_n\beta_n\phi(d(fx_n, u)) - \alpha_n\phi(d(fp_n, u)) \leq \psi(d(fx_n, u)).
\end{aligned}$$

Therefore, the sequence $\{d(z_n, u)\}$ is nonincreasing nonnegative sequence which converges to the limit $r \geq 0$. Suppose that $r > 0$. Then for a fixed integer N, we have

$$\sum_{n=N}^{\infty} \alpha_n \beta_n \phi(r) \leq \sum_{n=N}^{\infty} \alpha_n \beta_n \phi(d(z_n, u)) \leq \sum_{n=N}^{\infty} (d(z_n, u) - d(z_{n+1}, u)) \leq d(z_N, u).$$

This is a contradiction to $\sum \alpha_n \beta_n = \infty$. Hence, the iterative scheme $\{z_n\}$ converges to the common fixed point u. This completes the proof.

Chapter 4

Fixed and Stationary Points of Generalized Weak Contractions

4.1 Introduction

In this Chapter, we introduce the notions of quasi weak contraction and multi valued quasi weak contraction. Under the assumption of these notions we obtain some results on stationary points for a multi valued mapping on a metric space satisfying the property (E.A). We further, extend and generalize the results of Amini-Harandi [12] and Moradi and Khojasteh [88] to a hybrid pair of single-valued and multi-valued mappings.

4.2 Preliminaries

In this section we recall definitions of fixed and stationary point for multi-valued mappings and for a hybrid pair of mappings, approximate endpoint property, compatible mappings, property (E.A) and quasi-contraction . The relationship between compatibility and property (E.A) is discussed and supported with examples. We further introduce the notion of quasi weak and multi valued quasi weak contraction.

Definition 4.2.1. Given a metric space (X, d), let

$$CL(X) = \{A \subseteq X : A \neq \emptyset \text{ and closed}\},$$

$$CB(X) = \{A \subseteq X : A \neq \emptyset, \text{ closed and bounded}\},$$

for $A, B \in CL(X)$, define

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B),\ \ \sup_{b \in B} d(b, A)\right\}.$$

Then the map $(A, B) \to H(A, B)$ is a metric except that it can possibly take infinite values. This situation could be rectified by restricting attention to the subcollection of nonempty closed and bounded subsets or by considering an equivalent bounded metric. This metric H is called the Hausdorff metric induced by d on $CL(X)$ and the space $(CL(X), H)$ is called hyper space of (X, d).

Definition 4.2.2. Let $T : X \to CL(X)$. A point $z \in X$ is called a fixed point of T if $z \in Tz$ and is called a stationary point (or endpoint) of T if $Tz = \{z\}$.

The above definition can be extended to a hybrid pair of mappings as follows.

Definition 4.2.3. Let $S : X \to X$ and $T : X \to CL(X)$. A point $z \in X$ is called a coincidence point of the mappings S and T if $Sz \in Tz$ and a fixed point if $z = Sz \in Tz$. Further, $z \in X$ is called a stationary point (or endpoint) of S and T if $Tz = \{Sz\} = \{z\}$.

Notice that every stationary point of T is a fixed point of T, but not conversely.

Example 4.2.1. *Let* $X = [0, \infty)$ *with the usual metric and* $T : X \to CL(X)$ *be defined by*

$$Tx = \begin{cases} \{1/4\} & \text{if } x = 0, \\ [0, x] & \text{if } x \neq 0. \end{cases}$$

Then every point of $(0, x]$ *is a fixed point of* T *but* T *has no stationary point.*

Definition 4.2.4. Let $S : X \to X$ and $T : X \to CL(X)$. We say that S and T have approximate endpoint property if

$$\inf_{z \in X} \sup_{y \in Tz} d(Sz, y) = 0.$$

Furthermore, T has an approximate endpoint property if

$$\inf_{z \in X} \sup_{y \in Tz} d(z, y) = 0.$$

Example 4.2.2. *Let* $X = [0, 1]$ *endowed with the usual metric. Let* $S : X \to X$ *and* $T : X \to CL(X)$ *be defined by*

$$Sx = x \text{ for all } x \in X \text{ and } Tx = \begin{cases} \{1/2\} & \text{if } x = 0, \\ [0, x] & \text{if } x \neq 0. \end{cases}$$

Then

$$\inf_{z \in X} \sup_{y \in Tz} d(Sz, y) = 0.$$

So S *and* T *have approximate endpoint property. Again, the mappings* S *and* T *have no stationary points.*

The well known notion of compatible mappings due to Gerald Jungck [61] was introduced as a weaker form of commutativity. First, we recall the above notion of compatibility.

Definition 4.2.5. Self mappings S and T of a metric space (X, d) are said to be compatible if

$$\lim_n d(TSx_n, STx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_n Sx_n = \lim_n Tx_n = z, \quad \text{for some } z \in X.$$

Example 4.2.3. *[130] Let $X = \mathbb{R}$ with the usual metric. Define $S, T : X \to X$ by*

$$Sx = e^x - 1 \textit{ and } Tx = x^2$$

then

$$d(Sx_n, Tx_n) = |e^{x_n} - 1 - x_n^2| \to 0 \textit{ iff } x_n \to 0$$

$$d(TSx_n, STx_n) = |(e^{x_n} - 1)^2 - (e^{x_n^2} - 1)| = |e^{2x_n} - 2e^{x_n} + 2 - e^{x_n^2}| \to 0 \textit{ iff } x_n \to 0$$

so that S and T are compatible. But they are not commuting since

$$d(TSx, STx) = |e^{2x} - 2e^x + 2 - e^{x^2}| = |e^2 - 3e + 2| \neq 0 \textit{ for } x = 1 \in X.$$

Subsequently, the notion of $(E.A)$ property has been introduced by Aamri and Moutawakil [1] as a generalization of noncompatible mappings as follows.

Definition 4.2.6. Let X be a metric space. Two mappings $S, T : X \to X$ satisfy the property $(E.A)$ if there exits a sequence $\{x_n\}$ in X such that

$$\lim_n Sx_n = \lim_n Tx_n = z, \quad \text{for some } z \in X.$$

It is clear from the above definition that compatible mappings satisfy the property (E.A). The following example shows that the converse is not necessarily true in general.

Example 4.2.4. *Consider $X = [0, 1]$ equipped with the usual metric. Define $S, T : X \to X$ by*

$$Sx = \begin{cases} 1 - x & \text{if } x \in [0, \frac{1}{2}], \\ 0 & \text{if } x \in (\frac{1}{2}, 1]. \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}], \\ \frac{3}{4} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then, for the sequence $\{x_n\} = \{\frac{1}{2} - \frac{1}{n}\}, n \geq 2$, we have $\lim_n Sx_n = \lim_n Tx_n = \frac{1}{2}$. So that S and T satisfy the property (E.A) but are not compatible since $\lim_n d(STx_n, TSx_n) = d(\frac{1}{2}, \frac{3}{4}) \neq 0$.

Remark 4.2.1. We notice from the above example and Definition 4.2.5 that two self mappings S and T of a metric space X are noncompatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_n Sx_n = \lim_n Tx_n = z, \quad \text{for some } z \in X,$$

but $\lim_n d(TSx_n, STx_n)$ is either zero or does not exist. Therefore, two noncompatible self mappings of a metric space satisfy the property (E.A) which shows that the property (E.A) is a generalization of noncompatible mappings.

The following example shows that there are mappings which satisfy the property (E.A) and also compatible.

Example 4.2.5. *[1]. Let $X = [0, \infty)$ endowed with the usual metric. Define $S, T : X \to X$ by $Sx = \frac{x}{4}$ and $Tx = \frac{3x}{4}$ for all $x \in X$. Consider the sequence $x_n = \frac{1}{n}$, $n \in \mathbb{N}$. Clearly $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$, and S and T satisfy the property $(E.A)$ and are clearly compatible since $\lim_n d(STx_n, TSx_n) = 0$*

There are mappings which do not satisfy the property $(E.A)$.

Example 4.2.6. *[1]. Let $X = [2, \infty)$ endowed with the usual metric. Define $S, T : X \to X$ by $Sx = x + 1$ and $Tx = 2x + 1$, for all $x \in X$. Suppose S and T satisfy the property $(E.A)$, then there exists a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$. Therefore $\lim_{n\to\infty} x_n = t - 1$ and $\lim_{n\to\infty} x_n = \frac{t-1}{2}$. Then $t = 1$, which is a contradiction since $1 \notin X$. Hence S and T do not satisfy the property $(E.A)$.*

Commutativity $\Rightarrow$ Compatibility $\Rightarrow$ Property(E.A).

Definition 4.2.6 can be extended to a hybrid pair of mappings as follows:

Definition 4.2.7. [63] Let X be a metric space. Two mappings $S : X \to X$ and $T : X \to CL(X)$ satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = z \in A = \lim_{n\to\infty} Tx_n$$

for some $z \in X$ and $A \in CL(X)$.

When S is an identity mapping on X, we obtain the corresponding definition for a (single) mapping satisfying the property $(E.A)$ (see [98]).

Example 4.2.7. *Let $X = [0, 1]$ endowed with the usual metric. Define $S : X \to X$ and $T : X \to CL(X)$ by $Sx = x/2$ and $Tx = [0, x]$. Consider the sequence $x_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then,*

$$\lim_{n\to\infty} Sx_n = 0 \in \{0\} = \lim_{n\to\infty} Tx_n.$$

Therefore, S and T satisfy the property $(E.A)$.

We shall use the following notations. Let

1. Φ denote the class of functions: $\varphi : [0, \infty) \to [0, \infty)$ satisfying:

(a) φ is continuous and monotone nondecreasing,

(b) $\varphi(t) = 0 \Leftrightarrow t = 0$.

2. Ψ denote the class of functions: $\phi : [0, \infty) \to [0, \infty)$ is upper semicontinuous, $\phi(t) < t$ for each $t > 0$ and

$$\lim_{n\to\infty} \inf_n (t - \phi(t)) > 0.$$

3. $M_T(x, y) := \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

4. $m(x, y) := \max\left\{d(x, y), d(x, Tx), d(y, Ty), \dfrac{d(x, Ty) + d(y, Tx)}{2}\right\}$.

5. $M_{S,T}(x, y) := \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \dfrac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}$.

The following notion of quasi-contraction is due to Ćirić [35]. We note that it is one of the most general contractive conditions used in fixed point theory (cf. Rhoades [104])

Definition 4.2.8. A self mapping T of a metric space X is said to be a quasi-contraction if there exists k, $0 \le k < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \le k \max\{d(x, y), d(x, Tx), d(y, Ty)d(x, Ty), d(y, Tx)\}. \tag{4.2.1}$$

Definition 4.2.9. [41] Let X be a metric space and $T : X \to X$. The mapping T is said to be (ψ, φ)-weak contraction if

$$\psi(d(Tx, Ty)) \le \psi(d(x, y)) - \varphi(d(x, y)) \tag{4.2.2}$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$.

We extend Definition 4.2.9 and introduce the notion quasi weak contraction as follows:

Definition 4.2.10. Let X be a metric space and $T : X \to X$. The mapping T will be called a quasi weak contraction if

$$\psi(d(Tx, Ty)) \leq \psi(M_T(x, y)) - \varphi(M_T(x, y)) \tag{4.2.3}$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$.

Remark 4.2.2. When $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ with $k \in (0, 1)$, in Definition 4.2.10, we recover Definition 4.2.8.

In case T is a multi-valued mapping, we have the following definition.

Definition 4.2.11. Let X be a metric space and $T : X \to CL(X)$. The mapping T will be called multi-valued quasi weak contraction if

$$\psi(H(Tx, Ty)) \leq \psi(M_T(x, y)) - \varphi(M_T(x, y)) \tag{4.2.4}$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$.

4.3 Results concerning stationary points and approximate endpoint property

This section is about the existence theorem for stationary points of a multi-valued quasi weak contraction satisfying the property $(E.A)$ and the relationship between the existence of a unique endpoint and approximate endpoint property for a hybrid pair of mappings satisfying certain conditions.

Theorem 4.3.1. *Let X be a metric space and $T : X \to CL(X)$ a multi-valued quasi weak contraction satisfying the property $(E.A)$. Then T has a unique stationary point in X.*

Proof. Since T satisfies the property $(E.A)$, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} x_n = z \in A = \lim_{n\to\infty} Tx_n$$

for some $z \in X$ and $A \in CL(X)$. By (4.2.4), we get

$$\psi(H(Tz, Tx_n)) \leq \psi(M_T(z, x_n)) - \varphi(M_T(z, x_n)). \tag{4.3.1}$$

Notice that

$$\begin{aligned}
\lim_{n\to\infty} M_T(z, x_n) &= \lim_{n\to\infty} \max\{d(z, x_n), d(z, Tz), d(x_n, Tx_n), d(z, Tx_n), d(x_n, Tz)\} \\
&= \max\{d(z, z), d(z, Tz), d(z, A), d(z, A), d(z, Tz)\} \qquad (4.3.2) \\
&= \max\{0, d(z, Tz), 0, 0, d(z, Tz)\} \\
&= d(z, Tz).
\end{aligned}$$

Since $\psi,\ \varphi \in \Phi$, (4.3.1) and (4.3.2) implies

$$\lim_{n\to\infty} \psi(H(Tz, Tx_n)) = \psi(H(Tz, A)) \leq \psi(d(z, Tz)) - \varphi(d(z, Tz)),$$

a contradiction, unless $H(Tz, A) = 0$. Thus $Tz = A = \{z\}$.

To prove the uniqueness, we suppose that T has two distinct stationary points u and v in X.

Again notice that

$$\begin{aligned}
M_{S,T}(u, v) &= \max\{d(u, v), d(u, Tu), d(v, Tv), d(u, Tv), d(v, Tu)\} \\
&= \max\{d(u, v), d(u, u), d(v, v), d(u, v), d(v, u)\} \\
&= d(u, v). \qquad (4.3.3)
\end{aligned}$$

Since $\psi,\ \varphi \in \Phi$, by (4.2.4) and (4.3.3)

$$\begin{aligned}
\psi(d(u, v)) &= \psi(H(Tu, Tv)) \leq \psi(M_T(u, v)) - \varphi(M_T(u, v)) \\
&= \psi(d(u, v)) - \varphi(d(u, v)),
\end{aligned}$$

a contradiction, unless $d(u, v) = 0$.

If T is a single valued mapping on X, we have the following result.

Corollary 4.3.1. *Let X be a metric space and $T : X \to X$ a quasi weak contraction satisfying the property $(E.A)$. Then T has a unique fixed point in X.*

When $\psi(t) = t$ in Theorem 4.3.1, we have the following corollary.

Corollary 4.3.2. *Let X be a metric space and $T : X \to CL(X)$ a multi-valued mapping satisfying the property $(E.A)$ such that*

$$H(Tx, Ty) \leq M_T(x, y) - \varphi(M_T(x, y)) \tag{4.3.4}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then T has a unique stationary point.

Corollary 4.3.3. *Let X be a metric space and $T : X \to CL(X)$ a multi-valued (ψ, φ)-weak contraction mapping satisfying the property $(E.A)$. Then T has a unique stationary point.*

Proof. It comes from Theorem 4.3.1, when $M_T(x, y) = d(x, y)$.

When $M_{S,T}(x, y) = d(x, y)$, $\psi(t) = t$ and $\varphi(t) = 0$, then we have the following result for nonexpansive mappings.

Corollary 4.3.4. *Let X be a metric space and $T : X \to CL(X)$ a multi-valued mapping satisfying the property $(E.A)$ such that*

$$H(Tx, Ty) \leq d(x, y) \tag{4.3.5}$$

for all $x, y \in X$. Then T has a unique stationary point.

Now we present an example to illustrate our results.

Example 4.3.1. *Let* $X = [0, 1)$ *endowed with the usual metric. Define* $T : X \to CL(X)$ *by* $Tx = [0, x/2]$. *Let* $\psi(t) = 2t$ *and* $\varphi(t) = \frac{t}{2}$.

Consider the sequence $\{x_n\}$ *defined by* $x_n = \frac{1}{n}, \quad n \in \mathbb{N}$. *Clearly,*

$$\lim_{n\to\infty} x_n = 0 \in \{0\} = \lim_{n\to\infty} Tx_n,$$

and T *satisfies the property* $(E.A)$. *For all* $x, y \in X$

$$\psi(H(Tx, Ty)) = |x - y| \leq 2|x - y| - \frac{|x - y|}{2}.$$

Therefore T *satisfies all the hypotheses of Corollary 4.3.3 and* $T0 = \{0\}$. *It is interesting to note that* X *is not complete.*

Recently, extending nonlinear contraction of Boyd and Wong [30] for a multi-valued mapping, Amini-Harandi [12] obtained the following stationary point theorems.

Theorem 4.3.2. *Let* X *be a complete metric space and* $T : X \to CB(X)$ *such that*

$$H(Tx, Ty) \leq \phi(d(x, y)) \text{ for all } x, y \in X. \tag{4.3.6}$$

Then T *has a unique endpoint if and only if* T *has approximate endpoint property.*

Moradi and Khojasteh [88] extended the above theorem to a multi-valued generalized weak contraction as follows:

Theorem 4.3.3. *Let* X *be a complete metric space and* $T : X \to CB(X)$ *such that*

$$H(Tx, Ty) \leq \phi(m(x, y)) \text{ for all } x, y \in X. \tag{4.3.7}$$

Then T *has a unique endpoint if and only if* T *has approximate endpoint property.*

The following result may be considered as an extension of Theorems 4.3.2 and 4.3.1.

Theorem 4.3.4. *Let X be a metric space, $S : X \to X$ and $T : X \to CB(X)$ such that*

(A1) $TX \subseteq SX$;

(A2) $H(Tx, Ty) \leq \phi(M_{S,T}(x, y))$ *for all* $x, y \in X$;

(A3) SX *or* TX *is a complete subspace of* X.

Then S and T have a unique endpoint if and only if S and T have approximate endpoint property.

Proof. It is obvious that if S and T have an endpoint then they have approximate endpoint property. Conversely, suppose that S and T have approximate endpoint property. Then there exists a sequence $\{x_n\} \in X$ such that $\lim\limits_{n\to\infty} H(Sx_n, Tx_n) = 0$. We shall show that the sequence $\{Sx_n\}$ is Cauchy. For all $m, n \in \mathbb{N}$

$$\begin{aligned}
M_{S,T}(x_n, x_m) &= \max\Big\{ d(Sx_n, Sx_m), d(Sx_n, Tx_n), d(Sx_m, Tx_m), \\
&\qquad \frac{d(Sx_n, Tx_m) + d(Sx_m, Tx_n)}{2} \Big\} \\
&\leq \max\Big\{ d(Sx_n, Sx_m), H(Sx_n, Tx_n), H(Sx_m, Tx_m), \\
&\qquad \frac{H(Sx_n, Tx_m) + H(Sx_m, Tx_n)}{2} \Big\} \\
&\leq d(Sx_n, Sx_m) + H(Sx_n, Tx_n) + H(Sx_m, Tx_m)
\end{aligned}$$

$$
\begin{aligned}
&= d(Sx_n, Sx_m) - H(Sx_n, Tx_n) - H(Sx_m, Tx_m) + \\
&\quad 2H(Sx_n y, Tx_n) + 2H(Sx_m, Tx_m) \\
&\leq H(Tx_n, Tx_m) + 2H(Sx_n, Tx_n) + 2H(Sx_m, Tx_m) \\
&\leq \phi(M_{S,T}(x_n, x_m)) + 2H(Sx_n, Tx_n) + 2H(Sx_m, Tx_m).
\end{aligned}
$$

Therefore for all $m, n \in \mathbb{N}$, we have

$$M_{S,T}(x_n, x_m) - \phi(M_{S,T}(x_n, x_m)) \leq 2[H(Sx_n, Tx_n) + H(Sx_m, Tx_m)]. \tag{4.3.8}$$

Since ϕ is upper semicontinuous, $\phi(t) < t$ for each $t > 0$ and $\lim_{n\to\infty} \inf(t - \phi(t)) > 0$, by (4.3.8) we conclude that

$$\limsup_{m,n} M_{S,T}(x_n, x_m) = 0,$$

and $\{Sx_n\}$ is Cauchy sequence.

Suppose SX is complete subspace of X then $\{Sx_n\}$ being contained in SX has a limit in SX. Call it z. Let $Su = z$ for some $u \in X$. Now we have

$$
\begin{aligned}
M_{S,T}(x_n, u) &= \max\left\{d(Sx_n, Su), d(Sx_n, Tx_n), d(Su, Tu), \frac{d(Sx_n, Tu) + d(Su, Tx_n)}{2}\right\} \\
&\leq \max\left\{d(Sx_n, u), H(Sx_n, Tx_n), H(Su, Tu), \frac{H(Sx_n, Tu) + H(Su, Tx_n)}{2}\right\} \\
&\leq d(Sx_n, Su) + H(Sx_n, Tx_n) + H(Su, Tu),
\end{aligned}
$$

and

$$\lim_{n\to\infty} M_{S,T}(x_n, u) \leq H(Su, Tu).$$

Since ϕ is upper semicontinuous

$$\lim_{n\to\infty} \sup \phi(M_{S,T}(x_n, u)) \leq \phi(H(Su, Tu)). \tag{4.3.9}$$

By the triangle inequality and using (A2), we have

$$\begin{aligned} H(Sx_n, Tu) &\leq H(Sx_n, Tx_n) + H(Tx_n, Tu) \\ &\leq H(Sx_n, Tx_n) + \phi(M_{S,T}(x_n, u)). \end{aligned}$$

Making $n \to \infty$ and using (4.3.9), we get

$$H(Su, Tu) \leq \phi(H(Su, Tu)),$$

and $\phi(t) < t$ for all $t > 0$, implies that $H(Su, Tu) = 0$. It follows that $Tu = \{Su\}$ and u is an endpoint of S and T.

In case TX is a complete subspace of X, the condition $TX \subseteq SX$ implies that the sequence $\{Sx_n\}$ converges in SX and the previous argument works. The uniqueness of endpoint follows easily.

Corollary 4.3.5. *Theorem* 4.3.1.

Proof. It comes from Theorem 4.3.4 when S is an identity mapping on X.

Corollary 4.3.6. *Let X be a metric space, $S : X \to X$ and $T : X \to CB(X)$ such that*

(C1) $TX \subseteq SX$*;*

(C2) $H(Tx, Ty) \leq q\ M_{S,T}(x, y)$ *for all* $x, y \in X$*, where* $0 < q < 1$*;*

(C3) SX *or* TX *is a complete subspace of* X*.*

Then S and T have a unique endpoint if and only if S and T have approximate endpoint property.

Proof. It comes from Theorem 4.3.4 when $\phi(t) = qt$.

Corollary 4.3.7. *Theorem 4.3.2.*

Proof. It comes from Corollary 4.3.5 when $m(x, y) = d(x, y)$.

The following example shows the generality of our Theorem 4.3.4.

Example 4.3.2. *Let $X = [0, \infty)$ endowed with the usual metric. Let $S : X \to X$ and $T : X \to CB(X)$ be defined by*

$$Sx = 2x \text{ for all } x \in X \text{ and } \quad Tx = \begin{cases} \{0\} & \text{if } x = 0, \\ [0, x] & \text{if } x \neq 0. \end{cases}$$

Let $\phi(t) = \frac{3}{4}t$ for $t > 0$.

Then for $x, y > 0$

$$H(Tx, Ty) = |x - y| > \frac{3}{4}|x - y|,$$

and the condition (4.3.6) (of Theorem 4.3.2) and condition (4.3.7) (of Theorem 4.3.1) are not satisfied. However, the mappings S and T satisfy all the conditions of Theorem 4.3.4 and $S0 \in T0 = \{0\}$, showing that 0 is the unique endpoint of S and T.

Chapter 5

Fixed Point Theorems for Set-valued Generalized Asymptotic Contractions

5.1 Introduction

In this chapter we introduce the notion of *set-valued generalized asymptotic contraction of Meir-Keeler type*, which includes the known notions of asymptotic contractions due to Kirk [75], Suzuki [131] and Fakhar [43]. Subsequently, this notion is utilized to obtain coincidence and fixed point theorems for such contractions which generalize, and unify a number of known results due to [43], [142] and others.

5.2 Preliminaries

One of the generalizations of the well-known contraction principle of Banach [20] is due to Meir-Keeler [80]. We first recall the above notion as follows:

Definition 5.2.1. Let (X, d) be a metric space. A mapping T on X is said to be a

Meir-Keeler contraction if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \text{ implies } d(Tx, Ty) < \epsilon \text{ for all } x, y \in X.$$

Meir-Keeler [80] proved the following fixed point theorem which is a generalization of the Banach contraction principle [20].

Theorem 5.2.1. *Let (X, d) be a complete metric space and let T be a Meir-Keeler contraction on X. Then T has a unique fixed point.*

Kirk [75] introduced a new class of mappings known as *asymptotic contractions* on a metric space and obtained a fixed point theorem (see Definition 5.2.2 and Theorem 5.2.2) below.

Definition 5.2.2. Let (X, d) be a metric space. A self mapping T of X is an *asymptotic contraction* on X if

$$d(T^n x, T^n y) \leq \varphi_n(d(x, y)) \text{ for } x,\ y \in X,$$

where φ is a continuous function, from $[0, \infty)$ into itself, $\varphi(t) < t$ for all $t > 0$ and $\{\varphi_n\}$ is a sequence of functions from $[0, \infty)$ into itself such that $\varphi_n \to \varphi$ uniformly on the range of d.

Theorem 5.2.2. *Let (X, d) be a complete metric space and T an asymptotic contraction on X with $\{\varphi_n\}$ and φ as in Definition 5.2.2. Assume that there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ of x is bounded, and that φ_n is continuous for $n \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover $\lim_n T^n x = z$ for all $x \in X$.*

Remark 5.2.1. We remark that:

1. Theorem 5.2.2 is an asymptotic version of Boyd and Wong contraction [30] (see [58]).

2. Jachymski and Jóźwic [58] showed that the continuity of the map T is essential for the conclusion of Theorem 5.2.2 to hold.

3. In respect of Definition 5.2.2, it has been observed that $\varphi(0) = 0$ (cf. [15, 58, 131, 132, 133]).

Recently Suzuki [131] combined the ideas of Meir-Keeler contraction and Kirk's asymptotic contraction and introduced the following notion of *asymptotic contraction of Meir-Keeler type*.

Definition 5.2.3. Let (X, d) be a metric space. A self-map T of X is called an *asymptotic contraction of Meir-Keeler type* if there exists a sequence φ_n of functions from $[0, \infty)$ into itself satisfying the following conditions:

(S1) $\limsup \varphi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$;

(S2) for each $\varepsilon > 0$, there exists $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\varphi_\nu(t) \leq \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$;

(S3) $d(T^n x, T^n y) < \varphi_n(d(x, y))$, for all $n \in \mathbb{N}$ and x, $y \in X$ with $x \neq y$.

Definition 5.2.4. Let (X, d) be a metric space. We say that the set-valued dynamic system $T : X \to 2^X$ satisfies condition (C), if one of the following conditions hold:

(I) For each $0 < \alpha < \beta < \infty$, there exist maps $\varphi_{\alpha,\beta;m},\ \varphi_{\alpha,\beta} : [\alpha, \beta] \to [0, \infty), m \in \mathbb{N}$, such that $\varphi_{\alpha,\beta;m}$, $m \in \mathbb{N}$, are continuous on $[\alpha, \beta]$, $\varphi_{\alpha,\beta}(r) < r$ for any $r \in [\alpha, \beta]$, $\varphi_{\alpha,\beta;m} \to \varphi_{\alpha,\beta}$ uniformly on $[\alpha, \beta]$ and, for any $A \in B(X)$ and $m \in \mathbb{N}$, if $\alpha \leq \delta(A) \leq \beta$, then $\delta(T^{[m]}(A)) \leq \varphi_{\alpha,\beta;m}(\delta(A))$.

(II) For each $\alpha > 0$, there exist maps $\varphi_{\alpha;m}$, $\varphi_\alpha : [\alpha, \infty) \to [0, \infty)$, $m \in \mathbb{N}$, such that $\varphi_{\alpha;m}$, $m \in \mathbb{N}$, are continuous on $[\alpha, \infty)$, $\varphi_\alpha(r) < r$ for any $r \in [\alpha, \infty)$, $\varphi_{\alpha;m} \to \varphi_\alpha$ uniformly on $[\alpha, \infty)$ and, for any $A \in B(X)$ and $m \in \mathbb{N}$, if $\delta(A) \geq \alpha$, then $\delta(T^{[m]}(A)) \leq \varphi_{\alpha;m}(\delta(A))$.

(III) There exist maps $\varphi_m, \varphi : [0, \infty) \to [0, \infty), m \in \mathbb{N}$, such that $\varphi_m, m \in \mathbb{N}$, are continuous on $[0, \infty)$, $\varphi(0) = 0$, $\varphi(r) < r$ for any $r \in [0, \infty)$, $\varphi_m \to \varphi$ uniformly on $[0, \infty)$ and, for any $A \in B(X)$ and $m \in \mathbb{N}$, $\delta(T^{[m]}(A)) \leq \varphi_m(\delta(A))$.

We call maps which satisfy condition (C) set-valued asymptotic contractions.

5.3 Generalized Asymptotic Contractions

Motivated by Suzuki [131], Fakhar [43] and Wlodarczyk et al. [142], we now introduce the notion of generalized asymptotic contraction for a hybrid pair of mappings, but first we recall certain notions that will be used in the sequel.

Throughout this section, Y denotes an arbitrary nonempty set, (X, d) a metric space, $CB(X)$ the collection of all nonempty closed bounded subsets of X, φ_n as in Definition 5.2.3 and H the Hausdorff metric induced by d.

Further, let

$$\begin{aligned} m(x, y) : &= \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}; \\ M(x, y) : &= \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right\}. \end{aligned}$$

Now, we have the following:

Definition 5.3.1. Let (X,d) be a metric space $f : Y \to X$ and $T : Y \to CB(X)$. The map T will be called *a generalized asymptotic contraction of Meir-Keeler type* with respect to f if the following hold:

(G1) $\limsup_n \varphi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$;

(G2) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi_k(t) < \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$ and $k \in \mathbb{N}$;

(G3) $H(T^n x, T^n y) < \varphi_n(M(x,y))$ for all $n \in \mathbb{N}$ and $x, y \in Y$ with $M(x,y) > 0$.

As a special case of the above definition, we have the following:

Definition 5.3.2. Let (X,d) be a metric space and $T : X \to CB(X)$. The map T will be called *a generalized asymptotic contraction of Meir-Keeler type* if the following hold:

- $\limsup_n \varphi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$;
- for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi_k(t) < \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$ and $k \in \mathbb{N}$;
- $H(T^n x, T^n y) < \varphi_n(m(x,y))$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $m(x,y) > 0$.

Remark 5.3.1. We remark that a set-valued *asymptotic contraction of Meir-Keeler type* is the set-valued *generalized contraction of Meir-Keeler type* when $m(x,y) = d(x,y)$. Further it includes the set-valued *asymptotic contraction* given in Definition 5.2.4.

Now, we recall the following results:

Theorem 5.3.1. *[43] Assume that T is a uniformly continuous multi-valued asymptotic ϕ-contraction. Then T has a unique endpoint if and only if T has the approximate endpoint property. Furthermore, the fixed point problem is well posed for T with respect to H.*

Theorem 5.3.2. *[142] Let (X, d) be a metric space and let $T : X \to 2^X$. Suppose that:*

(a) *X is complete;*

(b) *T is closed;*

(c) *there exists $u^1 \in X$ and $u^{m+1} \in T^{[m]}(u^1)$ for $m \in \mathbb{N}$ such that the sequence $\{u^m\}$ is bounded;*

(d) *T satisfies condition (C).*

Then the following hold:

(i) *T has a unique endpoint u in X and*

(ii) *each sequence $\{w^m\}$, where $w^1 \in X$ and $w^{m+1} \in T^{[m]}(w^1)$ for $m \in \mathbb{N}$, converges to v.*

The following theorem is our main result.

Theorem 5.3.3. *Let (X, d) be a metric space, $f : Y \to X$ and $T : Y \to CB(X)$ such that $TY \subseteq fY$. Let T be a generalized asymptotic contraction of Meir-Keeler type with respect to f.*

If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence point.

Further, if $Y = X$, then T and f have a common fixed point provided that $ffu = fu$ and T and f commute at a coincidence point.

Proof. Pick $x_0 \in Y$. We construct a sequence $\{x_n\}$ in the following manner. Since $TY \subseteq fY$, we may choose a point $x_1 \in Y$ such that $fx_1 \in Tx_0$. If $Tx_0 = Tx_1$ then $x_1 = x_0$ is a coincidence point of T and f and we are done. So assume that $Tx_0 \neq Tx_1$ and choose $x_2 \in Y$ such that $fx_2 \in Tx_1$ and

$$d(fx_1, fx_2) \leq H(Tx_0, Tx_1).$$

If $Tx_1 = Tx_2$, i.e., x_2 is a coincidence point of T and f, we are done. If not continuing in the same manner we have

$$d(fx_{n+1}, fx_{n+2}) \leq H(Tx_n, Tx_{n+1}).$$

By (G3),

$$d(fx_n, fx_{n+1}) \leq H(Tx_{n-1}, Tx_n) < \varphi_n(M(x_0, x_1)).$$

First we show that

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0. \tag{5.3.1}$$

It initially holds if $x_1 = x_2$. In the other case of $x_1 \neq x_2$, we assume that

$$\alpha := \limsup_n d(fx_{n+1}, fx_{n+2}) > 0.$$

From the condition (G2), we can choose $k \in \mathbb{N}$ satisfying $\varphi_k(d(fx_1, fx_2)) < d(fx_1, fx_2)$. By (G3) and (G1),

$$d(fx_{k+1}, fx_{k+2}) \leq H(Tx_k, Tx_{k+1}) < \varphi_k(M(x_0, x_1)) < M(x_1, x_2). \tag{5.3.2}$$

Now, we have

$$\begin{aligned}
\alpha : &= \limsup_{n\to\infty} d(fx_{k+n+1}, fx_{k+n+2}) \le \limsup_{n\to\infty} H(Tx_{k+n}, Tx_{k+n+1}) \\
&\le \limsup_{n\to\infty} \varphi_n(M(x_k, x_{k+1})) \le M(x_k, x_{k+1}) \\
&= \max\{d(fx_k, fx_{k+1}), d(fx_k, Tx_k), d(fx_{k+1}, Tx_{k+1}), \\
&\quad \frac{1}{2}[d(fx_k, Tx_{k+1}) + d(fx_{k+1}, Tx_k)\} \\
&= \max\{d(fx_k, fx_{k+1}), d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2}), \\
&\quad \frac{1}{2}[d(Tx_k, Tx_{k+2}) + 0)\} \\
&= \max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2}), \\
&\quad \frac{1}{2}[d(fx_k, fx_{k+1}) + d(fx_{k+1}, fx_{k+2})\} \\
&= \max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2})\}.
\end{aligned}$$

If

$$\max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2})\} = d(fx_{k+1}, fx_{k+2})$$

then

$$\begin{aligned}
d(fx_{k+1}, fx_{k+2}) &\le H(Tx_k, Tx_{k+1}) \\
&< \varphi_1(M(x_k, x_{k+1})) < M(x_k, x_{k+1}) \\
&= \max\{d(fx_k, fx_{k+1}), d(fx_k, Tx_{k+1}), d(fx_{k+1}, Tx_{k+1}), \\
&\quad \frac{1}{2}[d(fx_k, Tx_{k+1}) + d(fx_{k+1}, Tx_k)]\} \\
&= \max\{d(fx_k, fx_{k+1}), d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2}), \\
&\quad \frac{1}{2}[d(fx_k, fx_{k+1}) + 0]\} \\
&= \max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2})\} \\
&= d(fx_{k+1}, fx_{k+2}),
\end{aligned}$$

a contradiction. Therefore

$$\max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2})\} = d(fx_k, fx_{k+1})$$

and we conclude that $M(x_k, x_{k+1}) = d(fx_k, fx_{k+1})$.

By (5.3.2),

$$\begin{aligned}
d(fx_{k+2}, fx_{k+3}) &\leq H(Tx_{k+1}, Tx_{k+2}) \\
&< \varphi_k(M(x_1, x_2)) < M(x_1, x_2) \\
&= \max\{d(fx_1, fx_2), d(fx_1, Tx_2), d(fx_1, Tx_2), \\
&\quad \frac{1}{2}[d(fx_1, Tx_2) + d(fx_1, Tx_2)]\} \\
&= d(fx_1, fx_2).
\end{aligned}$$

So $\alpha < d(fx_1, fx_2)$. By a similar argument, we obtain $\alpha < d(fx_{k+1}, fx_{k+2})$ for all $k \in \mathbb{N}$. Hence $\{d(fx_n, fx_{n+1}\}$ converges to α.

Since $0 < \alpha < d(fx_1, fx_2) < \infty$, there exists $\delta_2 > 0$ and $l \in \mathbb{N}$ such that

$$\varphi_l(t) \leq \alpha \text{ for all } t \in [\alpha, \alpha + \delta_2].$$

We choose $p \in \mathbb{N}$ with $d(fx_{p+1}, fx_{p+2}) < \alpha + \delta_2$. Then we have

$$d(fx_{l+p+1}, fx_{l+p+2}) \leq H(Tx_{l+p}, Tx_{l+p+1}) < \varphi_l d(fx_p, fx_{p+1}) \leq \alpha,$$

a contradiction. This proves that $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0$. Now following the proof of Theorem 3.1 [121], it can be easily shown that $\{fx_n\}$ is a Cauchy sequence.

Suppose $f(Y)$ is complete. Then $\{fx_n\}$ being contained in $f(Y)$ has a limit in $f(Y)$. Call it z. Let $u \in f^{-1}z$. Then $fu = z$. Using (G2),

$$\begin{aligned}
d(fu, Tu) &\leq H(Tx_n, Tu) < \varphi_1(M(u, x_n)) \\
&= \varphi_1(\max\{d(fu, fx_n), d(fu, Tu), d(fx_n, Tx_n), \\
&\quad \frac{1}{2}[d(fu, Tx_n) + d(fx_n, Tu)]\}).
\end{aligned}$$

Making $n \to \infty$, $d(fu, Tu) \leq \varphi_1(d(fu, Tu)) < d(fu, Tu)$. This yields $fu \in Tu$.

Further, if $Y = X$, $ffu = fu$, and the maps f and T commute at their coincidence point u then $fu \in fTu \subseteq Tfu$ and fu is a common fixed point of f and T.

In case TY is a complete subspace of X, the condition $TY \subseteq fY$ implies that the sequence $\{fx_n\}$ converges in fY and the previous argument works.

Now in the view of Definition 5.3.2 and Remark 5.3.1 we have the following remark.

Remark 5.3.2.

1. Let (X, d) be a complete metric space and $T : X \to CB(X)$ a generalized asymptotic contraction of Meir-Keeler type. Then T has a fixed point in X.

2. Let (X, d) be a complete metric space and $T : X \to CB(X)$ an asymptotic contraction of Meir-Keeler type. Then T has a fixed point in X.

The following example shows the generality of Theorem 5.3.3 over Theorem 5.3.2 and Theorem 5.3.1.

Example 5.3.1. *Let $Y = (-\infty, \infty)$ and $X = [0, \infty)$ endowed with the usual metric d. Let $f : y \to X$ and $T : Y \to CB(X)$ be defined by*

$$fx = \begin{cases} -2x & \text{if } x < 0, \\ 2x & \text{if } x \geq 0 \end{cases} \quad \text{and } Tx = \begin{cases} \{-x\} & \text{if } x < 0, \\ [0, x] & \text{if } 0 \leq x \leq 1, \\ \{x\} & \text{if } x > 1 \end{cases}$$

for all $x \in Y$. Let $\varphi_n(t) = \frac{3}{4}t$ for $t > 0$.

Then for $x > 1$ and $y > 1$,

$$H(T^n x, T^n y) = \mid x - y \mid > \frac{3}{4} \mid x - y \mid = \varphi_n(d(x, y)),$$

and the contractive condition of Theorem 5.3.1 is not satisfied.

Further, $\delta(T^n([0,1])) = \delta([0,1])$ *and condition (d) of Theorem 5.3.2 is not satisfied. It can be verified that the maps* f *and* T *satisfy all the hypotheses of Theorem 5.3.3. Notice that* $TY \subseteq fY$ *and* f *and* T *commute at* 0*. Hence* $f0 \in T0$ *is a common fixed point of* f *and* T*.*

Appendix

List of Published papers

1. *Stability of common fixed points in uniform spaces*, Fixed point theory and Applications, 2011 : 36 (with S. N Mishra and S. L Singh).

2. *Suzuki theorem on a 2-metric space*, Journal of Advanced Mathematical Studies, 5(1) (2012) 71-76 (with S. L. Singh and S. N. Mishra).

3. *Some new notions of convergence and stability of common fixed points in 2-metric spaces*, Advances in Fixed Point Theory, 2(1) (2012), 64-78 (with S. N Mishra, S. L Singh and Rajendra Pant).

4. *Fixed and stationary points of generalized weak contractions*, Journal of Advanced Mathematical Studies, 5(2) (2012), 46-53 (with S. N. Mishra and Rajendra Pant).

5. *Coincidence and fixed points of weakly contractive maps*, Journal of Advanced Mathematical Studies, 5(2) (2012), 68-81 (with S. L. Singh and Ashish Kumar).

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