RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS

A STUDY OF UNIVERSAL ALGEBRAS IN FUZZY SET THEORY

by

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ABSTRACT

This thesis attempts a synthesis of two important and fast developing branches of mathematics, namely universal algebra and fuzzy set theory. Given an abstract algebra [X,F] where X is a non-empty set and F is a set of finitary operations on X, a fuzzy algebra $[I^X,F]$ is constructed by extending operations on X to that on I^X, the set of fuzzy subsets of X (I denotes the unit interval), using Zadeh's extension principle. Homomorphisms between fuzzy algebras are defined and discussed. Fuzzy subalgebras of an algebra are defined to be elements of a fuzzy algebra which respect the extended algebra operations under inclusion of fuzzy subsets. The family of fuzzy subalgebras of an algebra is an algebraic closure system in I^X. Thus the set of fuzzy subalgebras is a complete lattice. A fuzzy equivalence relation on a set is defined and a partition of such a relation into a class of fuzzy subsets is derived. Using these ideas, fuzzy functions between sets, fuzzy congruence relations, and fuzzy homomorphisms are defined. The kernels of fuzzy homomorphisms are proved to be fuzzy congruence relations, paving the way for the fuzzy isomorphism theorem. Finally, we sketch some ideas on free fuzzy subalgebras and polynomial algebras. In a nutshell, we can say that this thesis treats the central ideas of universal algebras, namely subalgebras, homomorphisms, equivalence and congruence relations, isomorphism theorems and free algebra in the fuzzy set theory setting.

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CONTENTS

PREFACE

Universal Algebras started to evolve when mathematics departed from the traditional study of operations on real numbers only. Hamilton's quaternions, Boole's symbolic logic and so forth, brought to light operations on objects other than real numbers and operations which are very different from the traditonal ones. Thus Universal Algebra is the study of finitary operations on a set. On the other hand, during the 60's fuzzy sets were introduced by Zadeh and others (see [43], [44], [17]), as a mathematical basis for multivalued logic. Since its inception the theory of fuzzy subsets has developed in many directions and is finding applications increasingly in a wide variety of fields. For example, in [38] Rosenfeld used this concept to develop the theory of fuzzy groups and subgroups, in [23] Katsaras and Liu discusses fuzzy subspaces of vector spaces and others [29], [22] have considered fuzzy ideals and fuzzy subrings, not to mention the explosion of papers on fuzzy topological spaces.

In this thesis, we attempt a synthesis of the above two important concepts and produce the idea of fuzzy universal algebra to unify certain similar studies made in fuzzy subgroups, fuzzy rings and fuzzy vector spaces etc. from the purely algebraic point of view. As far back as 1975, Negoita and Relescu in their book [33] made a scanty sketch of such ideas and suggested that one could make a serious study of those concepts. The purpose of this thesis is to give a systematic treatment of the most important results in the field of fuzzy universal algebras. Thus we define and discuss several notions central to universal algebras such as subalgebras, homomorphism, quotients, equivalences, congruences, isomorphism theorems and free algebras in the fuzzy universal algebra setting.

Even though the language of category theory would have been more convenient in some ways, we have avoided categorical terms for basically three reasons. Firstly, we did not want to bring another theory into our synthesis. The second reason is to preserve the directness and simplitiy of presentation of universal algebraic concepts. Thirdly, we believe that a full length study (a separate thesis) can profitably be taken up to tie up category theory (theory of monads and triples) and fuzzy universal algebra.

Power algebras have been studied in the 50's by Jonsson and Tarski [21] Foster [10], [11] although in a different context. They tried to unite universal algebra with Boolean algebras. This thesis can also be considered as a study of power algebras. Given a non-empty set X and the unit interval I, I^X, the power set consists of all the fuzzy subsets of X, finitary operations on X are then extended from X to I^X using Zadeh's extension principle [44]. E.G. Manes tackles the problem of fuzzy universal algebra in his paper "A class of fuzzy theories" [30] along a different line.

Since each chapter has its own introduction in some detail we present here only a brief general descripition of each chapter.

In chapter one, we collect the basic concepts of fuzzy set theory and universal algebras. We show how to extend operations from a given set to that of the set of fuzzy subsets of the given set. We consider homomorphisms and their fuzzy extensions. Chapter two deals with the definition of fuzzy subalgebras of an algebra. We prove some simple consequences. Homomorphisms between fuzzy subalgebras and crisp α-level ($0 \leq \alpha \leq 1$) cuts of fuzzy subalgebras are also studied in the same chapter.

Closure systems are just as important in fuzzy set theory as in crisp set theory of universal algebra. In Chapter 3 we define and study closure systems in fuzzy set theory in brief. We apply algebraic closure systems that were defined in Chapter 3 to the lattice of fuzzy subalgebras and prove that the class of fuzzy subalgebras of X is an algebraic closure system in I^X. In that process, we also describe the method to generate the smallest fuzzy subalgebra containing a given fuzzy subset. These are the results of Chapter 4. In Chapter 5, fuzzy equivalence relations are defined and studied. They were first defined in Goguen [17] and later taken up by Kaufmann [24]. The results obtained are analogous to the crisp case and reduce to the usual definitions used in crisp set theory if considered in 2^X. Using the characterisation of a partition of fuzzy equivalence relations into a class of fuzzy subsets, fuzzy functions are defined. They are basically fuzzy relations satisfying certain properties. Composition of fuzzy functions, converse or inverse fuzzy functions, the kernel of a fuzzy function are defined and studied in Chapter 6. The main result is that the kernel of a fuzzy function is a fuzzy equivalence relation. Fuzzy congruence relations are fuzzy equivalence relations on algebras which respect the algebraic operations in a certain sense. In Chapter 7, we define and study fuzzy congruence relations. Chapter 8 deals with isomorphism theorems. Fuzzy homomorphisms between universal algebras are defined and studied in section 8.2. These are different from fuzzy extensions of crisp homomorphisms studied in Chapter 1. The kernel of a fuzzy homomorphism is proved to be a fuzzy congruence relation. Using this, we prove the first isomorphism theorem. Finally, in the appendix we briefly discuss the question of freeness of algebras in fuzzy set theory and polynomial algebras.

In conclusion, I wish to quote P. Freyd [12]. "In a new subject it is often very difficult to decide what is trivial, what is obvious, what is hard, what is worth bragging about. A man learns to think categorically (fuzzy set theoretically), he works out a few definitions, perhaps a theorem, more likely a lemma, and then he publishes it. Very often his exercise,though unpublished, has been in the folklore from the beginning. Very often it has been published faithfully every year...." I hope the work in this thesis proves to be significant and stimulating for further research.

The Theorems, Definitions, Propositions etc. are numbered serially within the chapter and section. For example Proposition 2.4.5. refers to 5th article in section 4 of Chapter 2.

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CHAPTER 1

FUZZY ALGEBRAS

1.1 Introduction

In this thesis, we accept the intuitive concept of a set. In section 1.2, we define what is meant by a fuzzy subset of a set and introduce such concepts as union, intersection, complement, empty fuzzy set etc. In section 1.3, algebras are considered. They are sets together with a set of finitary operations defined on them. A certain amount of familiarity with universal algebras is assumed, see [3], [19]. Using Zadeh's extension principle [44], finitary operations on an algebra are extended to finitary operations defined on the set of fuzzy subsets of the set underlying the given algebra. Such algebras are called fuzzy algebras. In section 1.4, we consider the extensions of homomorphisms between given algebras to fuzzy algebras. Such extensions are proved to be homomorphisms. Finally in section 1.5, the notion of fuzzy points are considered. They were first defined by Zadeh [45] as fuzzy singletons. Using this notion, an embedding theorem is proved where a given algebra is identified as a subalgebra of the fuzzy algebra associated with it.

1.2 Fuzzy sets and mappings

We accept the intuitive concept of a **set** (an ordinary set or crisp set as opposed to fuzzy set) as a collection of objects, called elements or members of the set. The notation $x\in X (x\notin X)$ means that x is (is not) an element of the set X. The **void set** is denoted by $\emptyset$. If X and Y are sets, then $X\subseteq Y$ (**inclusion**), $X=Y$ (**equality**), $X\subset Y$ (**proper inclusion**) are defined in the usual way. Also the set theoretic operations $\cup,\cap,',-$ (they are called **union**, **intersection**, **complement** and **difference** respectively) have their usual meaning. If X is a set, then $\mathbb{P}(X)$ called the **power set** of X, denotes the set of all subsets of X. If 2 denotes the 2-element set $\{0,1\}$ with the **ordering** $0\leq 1$, every subset Y of X can be identified with the **charactersitic function** χ_Y defined by, for any $x\in X$

$$\chi_Y(x) = \begin{cases} 1 & x\in Y \\ 0 & x\notin Y \end{cases} .$$

Thus $\mathbb{P}(X)$ can be identified with 2^X, the set of functions from X to $2=\{0,1\}$.

Now let $I=[0,1]$ be the unit interval of real numbers with the usual ordering. Let X be a set. A **fuzzy subset** of X is characterised by a function μ from X to I. That is $\mu:X\to I$. μ is called the **membership function** and $\mu(x)$ is thought of as the **degree of membership** of the element x to the fuzzy subset of X defined by μ. Thus we identify a fuzzy subset of X with its membership function μ. In this thesis, fuzzy subsets are denoted by lower case Greek letters μ,ν,γ etc. The power set I^X, the set of all functions from X to I is the set of all fuzzy subsets of X and is denoted by $\mathbb{F}(X)$. I^X is a **lattice** (see [1, p50]) with the pointwise ordering induced by the ordering of I. Using this ordering, the notions of inclusion, equality, strict inclusion, union, intersection and complement of fuzzy subsets are defined in the following way.

For $\mu,\nu\in\mathcal{F}(X)$ and for every $x\in X$,

(i) $\mu\leq\nu$ **(inclusion)** if and only if $\mu(x)\leq\nu(x)$

(ii) $\mu=\nu$ **(equality)** if and only if $\mu(x)=\nu(x)$

(iii) $\mu<\nu$ **(strict inclusion)** if and only if $\mu\leq\nu$ and $\mu(x) < \nu(x)$ for at least one x.

(iv) $\mu\vee\nu$ **(union)** is defined as $(\mu\vee\nu)(x)=\sup(\mu(x),\nu(x))$ where sup stands for supremum.

If $(\mu_j)_{j\in J}$ is a collection (finite or infinite) of fuzzy subsets $\bigvee_{j\in J}\mu_j$ **(arbitrary union)** is defined as

$$(\bigvee_{j\in J}\mu_j)(x)=\sup_{j\in J}(\mu_j(x))$$

(v) $\mu\wedge\nu$ **(intersection)** is defined as $(\mu\wedge\nu)(x)=\inf(\mu(x),\nu(x))$ where inf stands for infimum.

If μ_j's are as in (iv) then $\bigwedge_{j\in J}\mu_j$ **(arbitrary intersection)** is defined as $(\bigwedge_{j\in J}\mu_j)(x)=\inf_{j\in J}(\mu_j(x))$

(vi) μ^c **(complement)** is defined as $(\mu^c)(x)=1-\mu(x)$

(vii) The **whole fuzzy set** μ_X of X is defined as $\mu_X(x)=1$ for all $x\in X$ and the **void or empty fuzzy set** $\mu_\emptyset$ of X is defined as $\mu_\emptyset(x)=0$ for all $x\in X$.

Crisp (ordinary) subsets of X are fuzzy subsets of X when identified with characteristic functions of these subsets.

We now turn our attention to defining mappings between fuzzy subsets of two sets. So let X and Y be two non-empty sets and let f be a mapping from X to Y. Then f extends to a mapping from $\mathbb{F}(X)$ to $\mathbb{F}(Y)$ in the following way.

For each $\mu \in I^X$, $f(\mu) \in I^Y$ is defined as

$$f(\mu)(y) = \begin{cases} 0 & \text{if } y \notin f(X) \\ \sup\limits_{y=f(x)} \mu(x) & \text{if } y \in f(X),\ y \in Y \end{cases}$$

$f(\mu)$ is referred to as the **image** of the fuzzy set μ under f. Conversely, given a $\nu \in I^Y$, $f^{-1}(\nu) \in I^X$ is defined by the equation

$$f^{-1}(\nu)(x) = \nu(f(x)) \text{ for } x \in X.$$

$f^{-1}(\nu)$ is the **Inverse image** of ν under f.

1.3 Fuzzy algebras

If X and Y are sets, the **Cartesian** product $X \times Y$ of X and Y is defined as the set of all ordered pairs (x,y) with $x \in X$ and $y \in Y$.

$$X \times Y = \{(x,y) : x \in X, y \in Y\}.$$

More generally, if $X_1, X_2, \ldots, X_n$ are sets then $X_1 \times X_2 \ldots \times X_n$ is defined as $\{(x_1, x_2, \ldots, x_n) : x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n\}$. If $X_1 = X_2 = \ldots = X_n = X$ then we set $X^n = X_1 \times X_2 \times \ldots \times X_n$. That is X^n is the set of all **n-tuples** $(x_1, x_2, \ldots, x_n)$ with $x_i \in X$ for each $i=1,2,\ldots,n$. We define X° to be $\{\emptyset\}$. For a positive integer n and for a set X, we define an **n-ary relation** r on X as a subset of X^n. If r is an n-ary relation on X and $x_1, x_2, \ldots, x_n \in X$, we say that $x_1, x_2, \ldots, x_n$ are r-related , in notation $r(x_1, x_2, \ldots, x_n)$, if and only if $(x_1, x_2, \ldots, x_n) \in r$. An **n-ary operation** on X is defined as a mapping f of X^n into X. In both cases, n is called the **type** of f (of r). Thus an n-ary operation assigns to every n-tuple $(x_1, x_2, \ldots, x_n)$ of elements of X a unique element of X which will be denoted by $f(x_1, x_2, \ldots, x_n)$. Since such an operation is a mapping of X^n into X, we can also say that an n-ary operation is an element of X^{X^n}.

Moreover, we define a 0-ary (nullary) operation as a constant mapping on X, i.e. f assigns to every $x \in X$ a unique element $e \in X$. So a nullary operation on X can be seen as a constant unary operation on X. An n-ary operation f on X can also be described by an (n+1)-ary relation r defined by $r(x_1, x_2, \ldots x_n, x)$ if and only if $f(x_1, x_2, \ldots, x_n) = x$.

We shall also refer to an n-ary operation (for n a non-negative integer) as a finitary operation. In the definition below, $n(\alpha)$ denotes the integer associated with a finitary operation f_α on S.

Definition 1.3.1: An **universal algebra** (or an **abstract algebra**) or briefly an **algebra** A is a pair [S,F] where S is a non-empty set and F is a specified family of finitary operations $\{f_\alpha\}_{\alpha\in\mathcal{a}}$ on S. F is not necessarily finite and it may be void.

Each operation $f_\alpha, \alpha\in\mathcal{a}$ induces a corresponding operation $\mathfrak{f}_\alpha$ on $\mathbb{F}(S)$ as follows:

$$\mathfrak{f}_\alpha : \mathbb{F}(S) \times \mathbb{F}(S) \times \ldots \times \mathbb{F}(S) \to \mathbb{F}(S)$$

$$(\mu_1, \mu_2, \ldots, \mu_{n(\alpha)}) \to \mu$$

is defined as

$$\mu(x) = \sup\{\mu_1(x_1) \wedge \mu_2(x_2) \wedge \ldots \wedge \mu_{n(\alpha)}(x_{n(\alpha)}) :$$

$$f_\alpha(x_1, x_2, \ldots\ldots, x_{n(\alpha)}) = x\},$$

Supremum being taken over all $n(\alpha)$-tuples $(x_1, x_2, \ldots, x_{n(\alpha)})$ for which $f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) = x$. If there exists no such $n(\alpha)$-tuples $\mu(x)$ is defined as 0. As before, we denote $\mu = \mathfrak{f}_\alpha(\mu_1, \mu_2, \ldots, \mu_{n(\alpha)})$. If $n(\alpha) = 0$ then f_α is defined as

$$f_\alpha : S \to S,\ x \to e \text{ for all } x \in S,\ e \text{ is a fixed element of } S.$$

In this case $\mathfrak{f}_\alpha$ is defined as $\mathfrak{f}_\alpha : \mathbb{F}(S) \to \mathbb{F}(S)$, $\mu \to \mu_e$ where the fuzzy subset μ_e is defined as

$$\mu_e(x) = \begin{cases} 0 & \text{if } x \neq e \\ \sup_{x \in S} \mu(x) & x = e \end{cases}$$

If $\mathcal{F}=\{\delta_\alpha : \text{for every } f_\alpha \in F\}$, then $[I^S, \mathcal{F}]$ is an abstract algebra. It is usually denoted by A.

$A = [I^S, \mathcal{F}]$ is called the **fuzzy universal algebra** (**fuzzy abstract algebra**) or briefly **fuzzy algebra** associated with the given algebra $A = [S,F]$.

Remark 1.3.2: Together with the fuzzy universal algebra $A = [I^S, \mathcal{F}]$, one can consider $[2^S, \mathcal{F}]$. Then the extended n-ary operation $\delta_\alpha \in \mathcal{F}$ on 2^S simply becomes

$$\delta_\alpha(X_1, X_2, \ldots, X_{n(\alpha)}) = \{f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) : x_j \in X_j \text{ for } 1 \leq j \leq n(\alpha)\}$$

where $X_1, X_2, \ldots, X_{n(\alpha)}$ are crisp subsets of S. Jonsson and Tarski [21] introduced and studied such objects in the 50's. They called $[2^S, \mathcal{F}]$ Boolean algebra with operators. Brink [2], Henki, Monk and Tarski [20] and Foster [9] also studied such algebras.

Example 1.3.3: A semigroup $[S,.]$ is an algebra with one binary operation . such that $(a.b).c = a.(b.c)$ for all $a,b,c \in S$. The **fuzzy semigroup** associated with $[S,.]$ is defined as $[I^S,.]$ where

$$(\mu_1 . \mu_2)(a) = \sup_{a=a_1 . a_2} (\mu_1(a_1) \wedge \mu_2(a_2)), \ \mu_1, \mu_2 \in I^S \ a, a_1, a_2 \in S.$$

Note that it is easy to check the associative law, namely $(\mu_1.\mu_2).\mu_3 = \mu_1.(\mu_2.\mu_3)$ for $\mu_1, \mu_2, \mu_3 \in I^S$, holds in $[I^S,.]$.

Example 1.3.4: A group is an algebra $G=[S,.,e]$ with one binary operation . and one nullary operation e. The **fuzzy group** associated with $[S,.,e]$ is defined as $[I^S,.,e]$ where $\mu_1.\mu_2$ is defined as in 1.3.3 and

$$\mu_e(x) = \begin{cases} 0 & \text{if } x \neq e \\ \sup\limits_{x \in S} \mu(x) & \text{if } x=e. \end{cases}$$

Such objects were first defined and studied by Rosenfeld [38] and subsequently by D. Foster [11].

Example 1.3.5: Vector spaces may be regarded as algebras. A vector space V over a field F is an abelian group V with scalar multiplications regarded as unary operations. Fuzzy vector spaces were first defined and studied by Katsaras and Liu [23].

Example 1.3.6: Rings are algebras with two binary operations and a nullary operation. Fuzzy rings were studied by Liu [29].

More concretely one can give the following example:

Example 1.3.7: $A = [\mathbb{R},F]$ where $\mathbb{R}$ is the set of real numbers and F is the set operations $(+,X,0)$ where + is the addition, X is the multiplication, 0 is the nullary operation 0. The fuzzy real line from the algebraic point of view is taken up by Rodabaugh [37], Lowen [28] and others [6], [22].

1.4 Fuzzy morphisms:

Two algebras $A = [S,F]$ and $B = [T,F']$ are called <u>similar</u> if $F = \{f_\alpha : \alpha \in \mathcal{C}\}$, $F' = \{f'_\alpha : \alpha \in \mathcal{C}\}$ and for each $\alpha \in \mathcal{C}$ the types of f_α and f'_α are the same.

Definition 1.4.1: Let $A=[S,F]$ and $B=[T,F']$ be two similar algebras. A function $\varphi:S\to T$ is called a **homomorphism** of A into B if and only if for all $f_\alpha \in F$ and $x_i \in S$, $i=1,2,\dots,n(\alpha)$,

$$f'_\alpha(\varphi(x_1),\varphi(x_2),\dots,\varphi(x_{n(\alpha)})) = \varphi(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}))$$

Epimorphism, monomorphism, isomorphism, automorphism and endomorphism are defined in the usual way.

Remark 1.4.2: We retain the same symbol φ for the given mapping between the algebras A and B, and its extension (see 1.2) between the fuzzy algebras $\mathcal{A}=[\mathcal{F}(S),\mathcal{F}]$ and $\mathcal{B}=[\mathcal{F}(T),\mathcal{F}']$.

Proposition 1.4.3: Let A,B be two similar algebras; φ be a homomorphism of A into B. Then the extension φ from $\mathcal{A}=[\mathcal{F}(S),\mathcal{F}]$ to $\mathcal{B}=[\mathcal{F}(T),\mathcal{F}']$ is a homomorphism of the fuzzy algebras $\mathcal{A}$ into $\mathcal{B}$.

Proof: Let $R=\{\varphi(x):x\in S\}$. Then $[R,F]$ is a subalgebra of B since φ is a homomorphism.

Suppose

$$f_\alpha(\mu_1,\mu_2,\dots,\mu_{n(\alpha)})=\mu \text{ where } \mu_1,\mu_2,\dots,\mu_{n(\alpha)},\mu\in I^S$$

and

$$f_\alpha(\varphi(\mu_1),\varphi(\mu_2),\dots,\varphi(\mu_{n(\alpha)}))=\nu \text{ where } \nu\in I^T.$$

It is enough to show that $\varphi(\mu)=\nu$. That is, to show that $\varphi(\mu)(y)=\nu(y)$ for all $y\in T$. We distinguish between two cases.

Case i: $y \notin R$. Then $\varphi(\mu)(y)=0$. So, if $y=f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})$ then at least one of y_i, $i=1,2,\ldots,n(\alpha)$ does not belong to R. Otherwise $y \in R$, since R is a subalgebra. Let for some j with $1 \leq j \leq n(\alpha)$, $y_j \notin R$.

Then $\varphi(\mu_j)(y_j)=0$.

Therefore $\varphi(\mu_1)(y_1) \wedge \varphi(\mu_2)(y_2) \ldots \wedge \varphi(\mu_{n(\alpha)})(y_{n(\alpha)})=0$

$$\nu(y) = (\delta_\alpha(\varphi(\mu_1),\varphi(\mu_2),\ldots,\varphi(\mu_{n(\alpha)})))(y)$$

$$= \sup \{ \varphi(\mu_1)(y_1) \wedge \varphi(\mu_2)(y_2) \wedge \ldots \wedge \varphi(\mu_{n(\alpha)})(y_{n(\alpha)}) : y = f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})\}$$

$$= 0$$

Therefore $\varphi(\mu)(y) = \nu(y)$ for $y \notin R$.

Case ii: Let $y \in R$. Then $(\varphi(\mu))(y)=\sup_{y=\varphi(x)} \mu(x)= a$ say. Without loss of generality, we can assume $a>0$. (If $a=0$, an argument similar to case i will suffice). Given $\epsilon>0$ such that $a-\epsilon>0$, there exists an $x \in S$ with $\varphi(x)=y$ and $\mu(x)>a-\epsilon$.

That is $(\delta_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)}))(x)>a-\epsilon$.

This implies that there exist $x_1,x_2,\ldots,x_{n(\alpha)} \in S$ such that $x=f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})$ and $\mu_1(x_1) \wedge \mu_2(x_2) \wedge \ldots \wedge \mu_{n(\alpha)}(x_{n(\alpha)})>a-\epsilon$.

Since φ is a homomorphism
$$y = \varphi(x) = \varphi(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})) = f_\alpha(\varphi(x_1),\varphi(x_2),\ldots,\varphi(x_{n(\alpha)}))$$

Now we have

$$\nu(y) = (\mathcal{f}_\alpha(\varphi(\mu_1),\varphi(\mu_2),\ldots,\varphi(\mu_{n(\alpha)})))(y)$$

$$\geq \varphi(\mu_1)(\varphi(x_1)) \wedge \varphi(\mu_2)(\varphi(x_2)) \wedge \ldots \wedge \varphi(\mu_{n(\alpha)})(\varphi(x_{n(\alpha)}))$$

$$\geq \mu_1(x_1) \wedge \mu_2(x_2) \wedge \ldots \wedge \mu_{n(\alpha)}(x_{n(\alpha)})$$

$$> a-\epsilon$$

$$= (\varphi(\mu))(y)-\epsilon$$

As $\epsilon>0$ is arbitrary $\nu(y)\geq(\varphi(\mu))(y)$.

On the other hand, given $\delta>0$ there exist $y_1,y_2,\ldots y_{n(\alpha)}\in T$ with $f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})=y$ such that

$$\nu(y)-\delta < (\varphi(\mu_1))(y_1) \wedge (\varphi(\mu_2))(y_2) \wedge \ldots \wedge (\varphi(\mu_{n(\alpha)}))(y_{n(\alpha)})$$

Taking $0<\delta<\nu(y)$ (if $\nu(y) = 0$ then $\varphi(\mu)(y)\geq\nu(y)$), we have $y_1, y_2, \ldots, y_{n(\alpha)}\in \mathbb{R}$.

$$\nu(y)-\delta < \mu_1(x_1) \wedge \mu_2(x_2) \wedge \ldots \wedge \mu_{n(\alpha)}(x_{n(\alpha)}).$$

Since $\varphi(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}))=f_\alpha(\varphi(x_1),\varphi(x_2),\ldots,\varphi(x_{n(\alpha)})$, we get

$$\varphi(\mu)(y)>\nu(y)-\delta$$

$$\varphi(\mu)(y)\geq\nu(y) \quad \text{since } \delta \text{ is arbitrary.}$$

Therefore $\varphi(\mu)(y)=\nu(y)$ for all $y\in R$.

<u>Remark 1.4.4</u>: The above argument is similar to the one in [23].

1.5 Embedding of A in A

A **fuzzy point** in X, denoted by μ_x^r, is defined as the following fuzzy subset in X.

$$\mu_x^r(y) = \begin{cases} r & y=x \\ & \\ 0 & y\neq x \end{cases} \qquad 0<r\leq 1, \quad y\in X$$

If r=1 we write x_x for μ_x^1. In this way the singleton crisp subset {x} is identified with the fuzzy point x_x. The set $\mathbb{FP}(S)$ consists of all fuzzy points of the form $\{x_x : x\in S\}$. It is a subset of $\mathbb{F}(S)$. Each δ_α, corresponding to f_α, on $\mathbb{F}(S)$ defines an operation on $\mathbb{FP}(S)$ by restriction. It is well-defined. For

consider $\delta_\alpha : (\mathbb{FP}(S))^{n(\alpha)} \to \mathbb{FP}(S)$ given by $\delta_\alpha : (\mu_1, \mu_2, \ldots, {}_{n(\alpha)}) \to \mu$

where $\mu_1, \mu_2, \ldots, \mu_{n(\alpha)}$ are elements of $\mathbb{FP}(S)$, $\mu\in\mathbb{F}(S)$.

There exists $x_i, i=1,2,\ldots,n(\alpha)$ such that $\mu_i = x_{x_i}$. Therefore if $\delta_\alpha(x_{x_1}, x_{x_2}, \ldots, x_{x_{n(\alpha)}}) = \mu$ then μ is given by, for $x\in S$,

$$\mu(x) = (\delta_\alpha(x_{x_1}, x_{x_2}, \ldots, x_{x_{n(\alpha)}}))(x)$$

$$= \sup\{x_{x_1}(x_1') \wedge x_{x_2}(x_2') \wedge \cdots \wedge x_{x_{n(\alpha)}}(x'_{n(\alpha)}) : f_\alpha(x_1', x_2', \ldots, x'_{n(\alpha)}) = x\}$$

$$= \begin{cases} 0 & \text{if } f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) \neq x \\ 1 & \text{if } f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) = x \end{cases}$$

That is $\mu = x_{f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})}$ and that $\mu \in \mathbb{FP}(S)$. Thus we have

$$\delta_\alpha(x_{x_1}, x_{x_2}, \ldots, x_{x_{n(\alpha)}}) = x_{f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})}$$

Theorem 1.5.1: The algebra $\mathbb{FP}A = [\mathbb{FP}(S), \mathcal{F}]$ is a subalgebra of A.

Proof: Straightforward.

Theorem 1.5.2: The algebra A=[S,F] is isomorphic to the algebra $\mathbb{FP}A = [\mathbb{FP}(S), \mathcal{F}]$.

Proof: Consider the mapping $S \to \mathbb{FP}(S)$ given by $x \longrightarrow x_x$. Then it is easy to check that this mapping is an isomorphism.

Remark 1.5.3: There are controversial arguments in the way fuzzy points are defined and in dealing with fuzzy points belonging to fuzzy subsets. Fuzzy points were used in the separation axioms of fuzzy topological spaces. See [8], [26], [40], [42].
We need not concern ourselves with the controversies here.

CHAPTER 2

FUZZY SUBALGEBRAS

2.1 Introduction

In Chapter 1, we were concerned with extending the algebraic operations defined on an algebra to fuzzy algebra. Fuzzy subalgebras of an algebra are certain elements of fuzzy algebras which respect the extended algebraic operations under inclusion. This is defined in section 2.2 and some simple consequences are derived. These have been defined and studied in particular cases, for example, by Katsaras and Liu [23], Rosenfeld [38]. In section 2.3, we study union and intersection of fuzzy subalgebras. In the crisp case, if B is a subalgebra of A and if φ is a homomorphism from A to C, then $\varphi(B)$ is a subalgebra of C. Analogues of such results in the fuzzy case are studied in section 2.4. In section 2.5 and 2.6, we study the cuts of fuzzy subalgebras and the effect of a fuzzy subalgebra on the constants of the algebra. It turns out that fuzzy subalgebra takes the maximum value at a constant of an algebra and if there are more than one constant present in an algebra, then the fuzzy subalgebra takes the same value at all the constants of the algebra.

2.2 Fuzzy subalgebras

Definition 2.2.1. A fuzzy subset μ of S is called a **fuzzy subalgebra** of an algebra $A = [S,F]$ if

$$\mathfrak{f}_\alpha \underbrace{(\mu,\mu,\ldots,\mu)}_{n(\alpha)\text{-times}} \leq \mu.$$

for every induced operation $\mathfrak{f}_\alpha, f_\alpha \in F$.

Remark 2.2.2. It is easily seen that the above definition coincides with the definition of subalgebra of an algebra in the crisp case:

For consider $[2^S,F]$ as defined in Remark 1.3.2. By the usual definition, a crisp subset X of S is a subalgebra of S if X is closed under the operations of F. That is $\underbrace{\mathfrak{f}_\alpha(X,X,\ldots,X)}_{n(\alpha)\text{-times}} \subseteq X$.

This is precisely the above definition 2.2.1 applied to $[2^S,F]$.

2.2.3. Every crisp subalgebra is a fuzzy subalgebra.

2.2.4. Every constant fuzzy subset is a fuzzy subalgebra.

Theorem 2.2.5. Let $f_\alpha \in F$ and $\mu_1,\mu_2,\ldots,\mu_{n(\alpha)}$ and μ be $(n(\alpha)+1)$ fuzzy subsets of an algebra $A = [S,F]$. Then

$$\mathfrak{f}_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)}) \leq \mu$$

if and only if for all $n(\alpha)$-tuples $(x_1,x_2,\ldots,x_{n(\alpha)}) \in S^{n(\alpha)}$ the following is true.

$$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} (\mu_i(x_i)).$$

Proof: Let $n(\alpha) \neq 0$. Assume $\mathfrak{f}_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)}) \leq \mu$.

Then $\mathfrak{f}_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)})(x) \leq \mu(x)$ for all $x \in S$.

That is,

$$\sup_{x = f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})} \left(\min_{1\leq i\leq n(\alpha)} \mu_i(x_i)\right) \leq \mu(x),$$

the supremum being taken over all such $n(\alpha)$-tuples $(x_1,x_2,\ldots,x_{n(\alpha)}) \in S^{n(\alpha)}$.

$$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})) \geqq f_\alpha(\mu_1,\mu_2,\ldots\mu_{n(\alpha)})(f_\alpha(x_1,\ldots,x_{n(\alpha)}))$$

$$\geqq \min_{1\leqq i\leqq n(\alpha)} \mu_i(x_i).$$

Conversely, suppose $\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} \mu_i(x_i)$ for all $n(\alpha)$-tuples $(x_1,x_2,\ldots,x_{n(\alpha)})$ in $S^{n(\alpha)}$.

Let $x = f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})$ 2.2.6.

Since $\min_{1\leq i\leq n(\alpha)} (\mu_i(x_i)) \leq \mu(x)$

for all $x = f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})$

$$\sup_x \left(\min_{1\leq i\leq n(\alpha)} (\mu_i(x_i)\right) \leq \mu(x).$$

But the left hand side is precisely $f_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)})$ at x. Since this is true for all x of the form as in 2.2.6, $f_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)})(x) \leq \mu(x)$. On the other hand, if for some x there exists no such $n(\alpha)$-tuples for which 2.2.6. is true, then $f_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)})(x)=0$ which is $\leq \mu(x)$ trivially. Therefore in this case $f_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)}) \leq \mu$.

Now, if $n(\alpha) = 0$, then $f_\alpha(x) = e$ for a fixed element $e \in S$ and for all $x \in S$. We have to show that $f_\alpha(\nu) \leq \mu$ if and only if $\mu(f_\alpha(x)) \geq \nu(x)$ for all $x \in S$. Indeed $f_\alpha(\nu) \leq \mu$ if and only if

$$f_\alpha(\nu)\ (x) \leq \mu(x) \text{ for all } x \in S.$$

But by definition, $f_\alpha(\nu)(x) = \begin{cases} 0 & x \neq e \\ \sup_{x\in S} \nu(x) & x = e \end{cases}$

Therefore $f_\alpha(\nu)(x) \leq \mu(x)$ if and only if $\mu(e) \geq f_\alpha(\nu)\ (e) = \sup_{x\in S} \nu(x)$.

That is, if and only if $\mu(f_\alpha(x)) \geq \nu(x)$ for all $x\in S$.

This completes the proof.

Corollary 2.2.7. A fuzzy subset μ of an algebra A is a fuzzy subalgebra of A if and only if

$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} \mu(x_i)$ for all $f_\alpha \in F$ and for all $n(\alpha)$-tuples $(x_1,x_2,\ldots,x_{n(\alpha)})$ belonging to $S^{n(\alpha)}$.

Proof: Straightforward from the definition and the proposition above.

Example 2.2.8: A **fuzzy subgroup** of a group G is a fuzzy subset $\mu : G\to I$ such that

$$\mu(xy) \geq \mu(x) \wedge \mu(y)$$

$$\mu(x^{-1}) = \mu(x) \quad \text{for all } x, y \in G.$$

See Rosenfeld [38].

Example 2.2.9: A fuzzy subspace of a vector space V is a fuzzy subset $\mu : V\to I$ such that

$$\mu(x+y) \geq \mu(x) \wedge \mu(y)$$

$$\mu(\alpha x) \geq \mu(x) \quad \text{for all } x, y \in X,\ \alpha \in K.$$

See Katsaras and Liu [23].

Remark 2.2.10: The proof in Theorem 2.2.5 is analogous to one in [23].

2.3 Intersection and union of fuzzy subalgebras

Theorem 2.3.1. If $\{\mu_t\}_{t\in\tau}$ is a family of fuzzy subalgebras of an algebra A, then the intersection $\mu = \bigcap_{t\in\tau} \mu_t = \inf_{t\in\tau} \mu_t$ is a fuzzy subalgebra of A.

Proof: Consider for any $f_\alpha \in F$ and for the corresponding $n(\alpha)$, $(x_1,x_2,\ldots x_{n(\alpha)}) \in S^{n(\alpha)}$

$$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})) = \inf_{t\in\tau} (\mu_t(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})))$$

$$\geq \inf_{t\in\tau} (\min_{1\leq i\leq n(\alpha)} \mu_t(x_i)) \text{ since each}$$

μ_t is a fuzzy subalgebra and by Theorem 2.2.5.

$$= \min_{1\leq i\leq n(\alpha)} (\inf_{t\in\tau} \mu_t(x_i))$$

$$= \min_{1\leq i\leq n(\alpha)} (\mu(x_i))$$

Another application of Theorem 2.2.5., shows that μ is a fuzzy subalgebra of A.

Corollary 2.3.2. The fuzzy subalgebras of any algebra A form a complete lattice in which the meet operation is the intersection of fuzzy subalgebras.

Proof: See [3]; also see Proposition 3.2.3.

The join operation is given as follows:

Given a fuzzy subset ν of an algebra A, the set of all fuzzy subalgebras of A which contain ν is not empty. For example χ_S is in the set. Then the intersection ω of all fuzzy subalgebras in this set is a fuzzy subalgebra and ω contains ν. Moreover, ω is the smallest fuzzy subalgebra that contains ν. ω is called the fuzzy subalgebra generated by ν. An explicit construction of a fuzzy subalgebra generated by a fuzzy subset of an algebra will be given later. The join of a class of fuzzy subalgebras is defined as the subalgebra generated by the union of the fuzzy subalgebras belonging to the given class. The fact that the fuzzy subalgebras form a complete lattice follows from general lattice theoretical constructions which are standard in lattice theory (see [3]). We shall prove later on, that this complete lattice is an algebraic closure system in I^S. They are the analogues of corresponding notions in the crisp set theory of universal algebras (see [19]).

2.4 Fuzzy subalgebras and homomorphisms between fuzzy algebras

Let $A = [S,F]$ and $B = [T,F]$ be two given similar algebras, and let $\varphi: S \to T$ be a homomorphism between A and B. Then φ extends to a homomorphism between fuzzy algebra $\mathcal{A} = [I^S, F]$ and $\mathcal{B} = [I^T, F]$. We wish to determine the effect of φ on fuzzy subalgebras. In particular we have

Theorem 2.4.1. If μ is a fuzzy subalgebra of A, then $\varphi(\mu)$ is a fuzzy subalgebra of B. Similarly, $\varphi^{-1}(\nu)$ is a fuzzy subalgebra of A whenever ν is a fuzzy subalgebra of B.

Lemma 2.4.2. If $\mu, \nu \in I^S$ such that $\mu \leq \nu$, then $\varphi(\mu) \leq \varphi(\nu)$ in I^T.

Proof: For all $x \in S, \mu(x) \leq \nu(x)$. For any $y \in B$, $\varphi(\mu)(y) = 0 = \varphi(\nu)(y)$ if $y \notin \varphi(A)$. If $y \in \varphi(A)$, then

$$\varphi(\mu)(y) = \sup_{\varphi(x)=y} \mu(x)$$

$$\leq \sup_{\varphi(x)=y} \nu(x)$$

$$= \varphi(\nu)(y)$$

Therefore for all $y \in B$, $\varphi(\mu)(y) \leq \varphi(\nu)(y)$

That is $\varphi(\mu) \leq \varphi(\nu)$.

Proof of theorem: Since μ is a fuzzy subalgebra of A for each $f_\alpha \in F$, $\mathfrak{f}_\alpha(\mu,\mu,\ldots,\mu) \leq \mu$.

Therefore $\varphi(\mathfrak{f}_\alpha(\mu,\mu,\ldots,\mu)) \leq \varphi(\mu)$ by the Lemma.

The LHS is $\mathfrak{f}_\alpha(\varphi(\mu),\varphi(\mu),\ldots,\varphi(\mu))$.

That is $\varphi(\mu)$ is a subalgebra of A.

For the second part, consider for $f_\alpha \in F$ and for every $n(\alpha)$-tuples $(x_1, x_2, \ldots, x_{n(\alpha)}) \in S^{n(\alpha)}$

$$\varphi^{-1}(\nu)(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)}))$$

$$= \nu(\varphi(f_\alpha(x_1,x_2,\dots,x_{n(\alpha)})))$$

$$= \nu(f_\alpha(\varphi(x_1),\varphi(x_2),\dots,\varphi(x_{n(\alpha)})))$$

$$\geq \nu(\varphi(x_1)) \wedge \nu(\varphi(x_2)) \wedge \dots \wedge \nu(\varphi(x_{n(\alpha)}))$$

$$= (\varphi^{-1}(\nu))(x_1) \wedge (\varphi^{-1}(\nu))(x_2) \wedge \dots \wedge (\varphi^{-1}(\nu))(x_{n(\alpha)})$$

Now from Theorem 2.2.5.,

$$\delta_\alpha(\varphi^{-1}(\nu),\varphi^{-1}(\nu),\dots,\varphi^{-1}(\nu)) \leq \varphi^{-1}(\nu)$$

That is $\varphi^{-1}(\nu)$ is a fuzzy subalgebra of B.

2.5 Cuts of fuzzy subalgebras

For any crisp subalgebra B = [T,F] of an algebra A = [S,F], define a fuzzy subset $\mu_{r,T}$ of S as follows: For $0 \leq r \leq 1$,

$$\mu_{r,T}(x) = \begin{cases} r & \text{if } x \in T \\ 0 & \text{if } x \notin T \end{cases}$$

It is easily seen that $\mu_{r,T}$ is a fuzzy subalgebra of A. Further, we have the following:

Proposition 2.5.1. Suppose μ is a fuzzy subalgebra of an algebra A = [S,F]. Let, for any $0 \leq r \leq 1$, $T_{\mu,r}$ be defined as

$T_{\mu,r} = \{x \in S : \mu(x) \geq r\}$. Then $T_{\mu,r}$ is a crisp subalgebra of A.

Moreover μ can be represented as the union of the fuzzy subsets $r\chi_{T_{\mu,r}}$.

That is $\mu = \bigvee_{r=0}^{1} r\chi_{T_{\mu,r}}$.

Proof: Let $f_\alpha \in F$. Suppose $x_1, x_2, \ldots, x_{n(\alpha)} \in T_r$.

Then $\mu(x_i) \geq r$ for each $i = 1,2,\ldots,n(\alpha)$.

Therefore $\min_{1 \leq i \leq n(\alpha)} \mu(x_i) \geq r$.

By Theorem 2.2.5.,

$$\mu(f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)})) \geq \min_{1 \leq i \leq n(\alpha)} (\mu(x_i)) \geq r.$$

Therefore $f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) \in T_r$.

That is, T_r is a crisp subalgebra of A. The second part follows from the remarks from section 1.1.

Corollary 2.5.2. $T_{\mu,0} = \mu_1 = \chi_S = A$ and $T_{\mu,1}$ is the smallest subalgebra contained in $T_{\mu,r}$ for every $0 \leq r < 1$. $T_{\mu,1} \subset T_{\mu,r}$ for all $0 \leq r < 1$. ($T_{\mu,1}$ can be void).

From the above it is easy to see that $T_{\mu,r_1} \subseteq T_{\mu,r_2}$ for $1 \geq r_1 \geq r_2 \geq 0$. Thus $\{T_{\mu,r}\}_{r=0}^{r=1}$ from a complete chain of crisp subalgebras in the complete lattice of all subalgebras of A.

Remark 2.5.3: In the finitely generated algebras, especially finite groups and finite dimensional vector spaces, there are useful forms of representations of the complain chain $\{T_{\mu,r}\}$. See [4], [28].

Remark 2.5.4: Representing a fuzzy subset μ in the form $\bigvee_{r=0}^{1} r\chi_{T_{\mu,r}}$ is sometimes very useful. It was exploited in the study of compactness in fuzzy topological spaces [13],[14].

2.6 Fuzzy subalgebras and constants of an algebra

Proposition 2.6.1. Let A = [S,F] be an algebra. Suppose $\{f_j\}_{j\in J}$ is a subset of F consisting of all nullary operations of A and $\{e_j\}_{j\in J}$ are the corresponding constant elements of S. That is, for each $j \in J$, there is a fixed element $e_j \in S$ such that $f_j(x) = e_j$ for all $x \in S$. If μ is a fuzzy subalgebra of A, then

(i) there is a constant c such that $0 \leq c \leq 1$ and $\mu(e_j) = c$ for all $j \in J$.

(ii) $\mu(x) \leq c$ for all $x \in S$.

(iii) The ordinary subset $T = \{x \in S : \mu(x) = c\}$ is a crisp subalgebra of A.

Proof: (i) Since μ is a fuzzy subalgebra of A, by Theorem 2.2.5., for each $k \in J$, $\mu(f_k(x)) \geq \mu(x)$ for all $x \in S$. This implies $\mu(e_k) \geq \mu(x)$ for all $x \in S$. $\mu(e_j) \leq \mu(e_k)$ for all $j \in J$. By interchanging the rolls j and k, we have $\mu(e_j) = \mu(e_k)$ for every $j \in J$. If we set $\mu(e_k) = c$, then we have $\mu(e_j) = c$ for all $j \in J$.

(ii) Fix a $j \in J$; then $f_j(x) = e_j$ for all $x \in S$.

Then $\mu(f_j(x)) \geq \mu(x)$ for all $x \in X$ as in (i).

Therefore $\mu(x) \leq \mu(e_j) = c$.

(iii) Again by Theorem 2.2.5., for each $f_\alpha \in F$ and for all $n(\alpha)$-tuples $(x_1,x_2,\ldots,x_{n(\alpha)}) \in S^{n(\alpha)}$,

$$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} (\mu(x_i)).$$

Therefore if $x_1, x_2, \ldots, x_{n(\alpha)} \in T$, then

$$\mu(x_i) = c \text{ for } i = 1,2,..,n(\alpha)$$

and so $$\min_{1 \leq i \leq n(\alpha)} \mu(x_i) = c.$$

Therefore $c \leq \mu(f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)})) \leq c$

Therefore, $\mu(f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)})) = c$ which implies

$$f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) \in T.$$

Thus T is a subalgebra.

Remark 2.6.2: Gerla & Tortora [15] have proposed concepts of fuzzy algebras. They assume fuzzy subalgebras take highest value at the constants of the given algebra.

CHAPTER 3

CLOSURE SYSTEMS IN FUZZY SET THEORY

3.1 Introduction

Closure systems in lattice theory are well-known and have been studied in depth by several authors [3], [1], [41]. In this chapter we define and study closure systems in the lattice of fuzzy subsets $\mathbb{F}(X)$ of a set X. Most of the results are standard and can be found with some modification in [3]. Closure systems arise in two different contexts, viz. topology and algebra. Our interests lie in algebraic closure systems. These are defined in section 3.4. They are analogues of algebraic closure systems in crisp set theory. The connection between closure systems and closure operators is studied in section 3.3. We give some examples of closure systems in fuzzy set theory.

3.2 Closure systems in Fuzzy set theory

Let $\mathcal{F}(X) = I^X$ be the set of all fuzzy subsets of a non-empty crisp set X. Let $\mathcal{C}$ be a non-empty collection of fuzzy subsets of X. That is $\mathcal{C} \subseteq \mathcal{F}(X)$.

Definition 3.2.1: $\mathcal{C}$ is said to be a **closure system** in $\mathcal{F}(X)$ if for any subcollection $\mathcal{D}$ of fuzzy subsets in $\mathcal{C}$, $\inf_{\mu \in \mathcal{D}} \mu \left(= \bigwedge_{\mu \in \mathcal{D}} \mu\right) \in \mathcal{C}$.

Remark 3.2.2: (i) A closure system is also known as "Moore family" of fuzzy subsets.

(ii) If $\mathcal{D} = \varphi$, the empty collection in $\mathcal{C}$, then by definition, $\inf_{\mu \in \mathcal{D}} \mu = \chi_X$ and thus $\chi_X \in \mathcal{C}$.

Proposition 3.2.3: If $\mathcal{C}$ is a closure system in $\mathcal{F}(X)$, then $\mathcal{C}$ is a complete lattice.

Proof: Let $\mathcal{D}$ be a subcollection of fuzzy subsets in $\mathcal{C}$. Then $\bigwedge_{\mu \in \mathcal{D}} \mu \in \mathcal{C}$. Also, let $\mathcal{E}$ be the collection of fuzzy subsets $\{\nu\}$ in $\mathcal{C}$ such that $\nu \geq \mu$ for every μ in $\mathcal{D}$. Let $\inf_{\nu \in \mathcal{E}} \nu = \bigwedge_{\nu \in \mathcal{E}} \nu = \nu_1$. Then any element μ of $\mathcal{D}$ is a lower bound of $\mathcal{E}$. Therefore $\mu \leq \nu_1$ for all $\mu \in \mathcal{D}$. Moreover, if $\mu \leq \nu_2$ for every $\mu \in \mathcal{D}$, then $\nu_2 \in \mathcal{E}$ and hence $\nu_1 \leq \nu_2$. Therefore $\nu_1 = \sup_{\mu \in \mathcal{D}} \mu = \bigvee_{\mu \in \mathcal{D}} \mu$ exists in $\mathcal{C}$. Thus $\mathcal{C}$ is a complete lattice. This completes the proof.

We note that the ordering in $\mathcal{C}$ is the same as in $\mathcal{F}(X)$. But $\mathcal{C}$ need not be a sublattice of $\mathcal{F}(X)$. This is so because, in general,

$$\sup_{\mu \in \mathcal{D}}{}^{\mathcal{F}} \mu \leq \sup_{\mu \in \mathcal{D}}{}^{\mathcal{C}} \mu.$$

Example 3.2.4: Let $\mathbb{R}^2$ be the two dimensional plane. Define $\mu_X(x)=\frac{1}{2}$ for all $x \in X$, the X-axis and $\mu_X(x)=0$ for $x \notin X$. Similarly $\mu_Y(x)=\frac{3}{4}$ for all $x \in Y$, the Y-axis and $\mu_Y(x)=0$ for $x \notin Y$. Then μ_X and μ_Y are fuzzy subspaces of $\mathbb{R}^2$. $\mu_X \vee \mu_Y$ takes $\frac{1}{2}$ on the X-axis except the origin and takes $\frac{3}{4}$ on the Y-axis and 0 everywhere else. On the other hand the fuzzy subspace generated by μ_X and μ_Y takes $\frac{3}{4}$ on the Y-axis and $\frac{1}{2}$ everywhere else. Therefore $\mu_X \vee \mu_Y \leq$ the fuzzy subspace generated by μ_X and μ_Y. That is $\sup_{\mathbb{F}}(\mu_X,\mu_Y) \leq \sup_{\mathbb{C}}(\mu_X,\mu_Y)$ where $\mathbb{C}$ denotes the class of fuzzy subspaces of $\mathbb{R}^2$.

If $\mathbb{P}(X)$ denotes the set of all crisp subsets of X, then $\mathbb{P}(X)$ is a closure system in $\mathbb{F}(X)$, with every element of $\mathbb{P}(X)$ identified with characteristic function of that element. Moreover, in the same way, every closure system $\mathbb{C}$ of crisp subsets is a closure system of fuzzy subsets in $\mathbb{F}(X)$. Some more examples are the following.

(i) Let (X,τ) be a fuzzy topological space; $\mathbb{T}$ be the collection of τ-closed fuzzy subsets of X. Then $\mathbb{T}$ is a closure system.

(ii) The collection of all fuzzy subalgebras of an algebra is a closure system. Thus, in particular, the collection of all fuzzy subspaces (Katsaras and Liu [23]),, fuzzy subgroups (D.H. Foster [11]), fuzzy ideals (Liu [29]) are closure systems. These are algebraic closure systems (see definition 3.4.4).

Definition 3.2.5: A closure system $\mathbb{T}$ in $\mathbb{F}(X)$ is said to be a **topological closure system** if for any $\mu,\nu \in \mathbb{T}$, $\mu \vee \nu \in \mathbb{T}$. $\mathbb{T}$ in example (i) is a topological closure system. Conversely, with every topological closure system, we can associate a fuzzy topology in a natural way, namely, take every element of the topological closure system to be the closed fuzzy subsets of X. See [27].

We shall define algebraic closure system in $\mathbb{F}(X)$ later.

3.3 Closure operators in fuzzy sets

A **closure operator** J on $\mathbb{F}(X)$ is a mapping of $\mathbb{F}(X)$ into $\mathbb{F}(X)$ with the following properties.

(i) If $\mu \leq \nu$, then $J(\mu) \leq J(\nu)$ for all $\mu, \nu \in \mathbb{F}(X)$,

(ii) $\mu \leq J(\mu)$ for all $\mu \in \mathbb{F}(X)$,

(iii) $J(J(\mu)) = J(\mu)$ for all $\mu \in \mathbb{F}(X)$.

An element $\mu \in \mathbb{F}(X)$ is said to be **closed** if and only if $J(\mu) = \mu$.

There is a close connection between closure systems in and closure operators on $\mathbb{F}(X)$. It is given in the following

Theorem 3.3.1: Every closure system $\mathbb{C}$ in $\mathbb{F}(X)$ defines a closure operator J on $\mathbb{F}(X)$ by the rule

$$J(\nu) = \bigwedge \{\mu \in \mathbb{C} : \nu \leq \mu\} \text{ for } \nu \in \mathbb{F}(X).$$

Conversely every closure operator J on $\mathbb{F}(X)$ defines a closure system defined by

$$\mathbb{C} = \{\mu \in \mathbb{F}(X) : J(\mu) = \mu\}$$

and the correspondence $\mathbb{C} \longleftrightarrow J$ between the closure systems and closure operators thus defined is bijective.

Proof: Let $\mathbb{C}$ be a closure system in $\mathbb{F}(X)$. Let J be as defined in the theorem. Suppose $\nu_1 \leq \nu_2$ in $\mathbb{F}(X)$. Let $\mathbb{D}_{\nu_1}$ and $\mathbb{D}_{\nu_2}$ be the subcollections of $\mathbb{C}$ defined by the set of all $\mu \in \mathbb{C}$ such that $\nu_1 \leq \mu$ and $\nu_2 \leq \mu$ respectively. Then $\mathbb{D}_{\nu_2}$ is contained in $\mathbb{D}_{\nu_1}$.

As $J(\nu_i) = \bigwedge_{\mathcal{D}_{\nu_i}} \mu$ $i = 1,2$, we see that $J(\nu_1) \leq J(\nu_2)$. Thus J satisfies the condition (i) of closure operator. By definition of the closure system $J(\mu) \in \mathcal{C}$ and clearly, $\mu \leq J(\mu)$, thus satisfying the condition (ii). Further, we note that $J(\mu) = \mu$ if and only if $\mu \in \mathcal{C}$, since $\mathcal{C}$ is a closure system. Hence, $J(J(\mu)) = J(\mu)$ for all $\mu \in \mathcal{F}(X)$ proving condition (iii) of closure operator.

Conversely, let J be a closure operator on $\mathcal{F}(X)$ and $\mathcal{C}$ be the collection of fuzzy subsets as defined in the theorem.
Let $\mathcal{D} = \{\mu_i : i \in I\}$ be a subcollection of $\mathcal{C}$.
Let $\mu = \bigwedge_{i \in I} \mu_i = \inf_{i \in I} \mu_i$; then $\mu \leq \mu_i$ for each $i \in I$. By property (i) of closure operator, $J(\mu) \leq J(\mu_i) = \mu_i$. Therefore $J(\mu) \leq \bigwedge_{i \in I} \mu_i = \mu$.
By property (ii) of closure operator $\mu \leq J(\mu)$. Therefore $\mu = J(\mu)$ implying $\mu \in \mathcal{C}$. Finally, let $\mathcal{C}$ be any closure system and J be the operator associated with $\mathcal{C}$. Let $\mathcal{C}'$ be the closure system associated with J. Since $J(J(\mu)) = J(\mu)$, $\mathcal{C} = \mathcal{C}'$.

Finally, let J be a closure operator on $\mathcal{F}(X)$ and let $\mathcal{C}$ be the closure system associated with J and J' be the closure operator associated with $\mathcal{C}$. By the above $\mathcal{C}$ is also the closure system associated with J'. Hence $J(\mu) = \mu$ if and only if $J'(\mu) = \mu$. But $J(J(\mu)) = J(\mu)$ by property (iii) of closure operator. Hence $J'(J(\mu)) = J(\mu)$. But $\mu \leq J(\mu)$ and applying J' we obtain $J'(\mu) \leq J'(J(\mu)) = J(\mu)$. By symmetry we can show that $J(\mu) \leq J'(\mu)$. Therefore we have $J(\mu) = J'(\mu)$ for any $\mu \in \mathcal{F}(X)$. This shows that $\mathcal{C} \longleftrightarrow J$ is a bijection from the set of all closure systems in $\mathcal{F}(X)$ and onto the set of all closure operators on $\mathcal{F}(X)$. This completes the proof.

3.4 Algebraic closure systems in $\mathbb{F}(X)$

In this section, we define and study algebraic closure system in $\mathbb{F}(X)$. First we have the following well-known proposition on an ordered set. We leave out the proof, see [3] .

Proposition 3.4.1: Let A be a partially ordered set; then the following three conditions on A are equivalent.

(i) Every non-empty directed subset of A has a supremum.

(ii) Every non-empty chain in A has a supremum.

(iii) Every non-empty well-ordered chain of A has supremum.

Definition 3.4.2: A non-empty collection $\mathcal{C}$ of fuzzy subsets of X is called **inductive** if every non-empty chain in $\mathcal{C}$ has a supremum in $\mathcal{C}$.

Remark 3.4.3: We could replace the condition "Every non-empty chain in $\mathcal{C}$ has a supremum in $\mathcal{C}$ " by the equivalent condition "Every directed set in $\mathcal{C}$ has a supremum in $\mathcal{C}$ " by the above proposition.

Definition 3.4.4: A closure operator J on $\mathbb{F}(X)$ is said to be an **algebraic operator** if for any $\mu \in \mathbb{F}(X)$, we have

$$J(\mu) = \bigvee_{f} J(\mu_f)$$

the union being taken over all $\mu_f \in \mathbb{F}(X)$ where $\mu_f \leq \mu$ has finite support and $\mu_f(x) < \mu(x)$ if $\mu_f(x) > 0$.

A closure system in $\mathbb{F}(X)$ is said to be an **algebraic closure system** if the associated closure operator is algebraic.

Theorem 3.4.5: A closure system in $\mathbb{F}(X)$ is algebraic if and only if it is inductive.

Proof: Let $\mathbb{C}$ be an algebraic closure system in $\mathbb{F}(X)$ and $\mathbb{K}$ be a non-empty chain in $\mathbb{C}$. Let $\mu = \sup \mathbb{K}$ in $\mathbb{F}(X)$. It is enough to show that $\mu \in \mathbb{C}$ in order to show that $\mathbb{C}$ is inductive. Let $\mu_f \in \mathbb{F}(X)$ be a fuzzy subset with a finite crisp subset of X as support and let $\mu_f \leq \mu$. Since $\mu = \sup \mathbb{K}$ and $\mathbb{K}$ is a chain, there exists a $\nu \in \mathbb{K}$ such that $\mu_f \leq \nu \leq \mu$. Therefore, $J(\mu_f) \leq J(\nu) = \nu \leq \mu$, since J is a closure operator. Hence $\bigvee_f J(\mu_f) \leq \mu$, and since J is algebraic, $J(\mu) \leq \mu$. But for any $\mu \in \mathbb{F}(X)$, $\mu \leq J(\mu)$. Therefore $J(\mu) = \mu$ implying $\mu \in \mathbb{C}$.

Conversely, suppose $\mathbb{C}$ is an inductive closure system in $\mathbb{F}(X)$ and J is the associated closure operator corresponding to $\mathbb{C}$. We have to show that

$$J(\mu) = \bigvee_f J(\mu_f) \text{ for any } \mu \in \mathbb{F}(X)$$

where $\mu_f \leq \mu$ and μ_f is a fuzzy subset with a finite support.
Let $\mu \in \mathbb{F}(X)$. Denote by $\mathbb{K}$ the set of all such $J(\mu_f)$ where $\mu_f \leq \mu$.
If μ_f and ν_f are two fuzzy subsets with finite support and $\mu_f \leq \mu$ and $\nu_f \leq \mu$, then $\mu_f \vee \nu_f$ is again a fuzzy subset with finite support and $\mu_f \vee \nu_f \leq \mu$. Moreoever,

$$\mu_f \leq \mu_f \vee \nu_f$$

and

$$\nu_f \leq \mu_f \vee \nu_f$$

imply

$$J(\mu_f) \leq J(\mu_f \vee \nu_f)$$

$$J(\nu_f) \leq J(\mu_f \vee \nu_f)$$

which in turn imply $J(\mu_f) \vee J(\nu_f) \leq J(\mu_f \vee \nu_f)$. Therefore $\mathbb{K}$ is a directed system and hence has a supremum in $\mathcal{C}$ say τ. Since $\mu_f \leq \mu$, $J(\mu_f) \leq J(\mu)$. As this is true for every $J(\mu_f)$ in $\mathbb{K}$, $\tau \leq J(\mu)$. Also $\mu_f \leq J(\mu_f)$ for each $\mu_f \in \mathbb{K}$ and $\mu = \bigvee_f \mu_f$. Therefore $\mu \leq \tau$. This implies that $J(\mu) \leq J(\tau) = \tau$. Hence $\tau = J(\mu)$. Thus $J(\mu) = \tau = \bigvee_f J(\mu_f)$. Therefore J is algebraic implying $\mathcal{C}$ is an algebraic closure system.

Collollary: If $\mathcal{C}$ is an algebraic closure system in $\mathcal{F}(X)$ and $\mathbb{K}$ is a directed subsystem in $\mathcal{C}$, then sup $\mathbb{K}$ is in $\mathcal{C}$.

CHAPTER 4

LATTICE OF FUZZY SUBALGEBRAS

4.1 Introduction

We apply the results of the last chapter on algebraic closure systems to prove that the lattice of fuzzy subalgebras of an algebra is an algebraic closure system of fuzzy subsets of the algebra. In section 4.2, we give a method by which to generate a fuzzy subalgebra from a given fuzzy subset of an algebra. This is an extension of the method used to generate a subalgebra from a given subset of an algebra in the crisp case.

4.2 Generation of fuzzy subalgebras

Let $A = [S,F]$ be an algebra. For each $f_\alpha \in F$ and $x \in S$, define $C_\alpha^x = \{c = (x_1,x_2,\ldots,x_{n(\alpha)}) \in S^{n(\alpha)} : f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}) = x\}$. Given a fuzzy subset μ of S, define, for each non-negative integer k, a fuzzy subset μ_k of S as follows: Let

$$\mu_0 = \mu \text{ on } S \text{ and for each } x \in S$$

$$\mu_{k+1}(x) = \max\left\{\mu_k(x), \sup_{c \in C_\alpha^x} \min_{1 \leqq i \leqq n(\alpha)} \mu_k(x_i)\right\}.$$

Theorem 4.2.1: $\tilde{\mu} = \bigvee_{k=0}^{\infty} \mu_k = \sup_k \mu_k$ is the smallest fuzzy subalgebra of A containing μ.

Proof: By definition $\mu = \mu_0 \leq \bigvee_{k=0}^{\infty} \mu_k = \tilde{\mu}$. Therefore $\tilde{\mu}$ contains μ. Next we show that $\tilde{\mu}$ is a fuzzy subalgebra of A. For this, given any $f_\alpha \in F$, consider an $n(\alpha)$-tuple $c = (x_1,x_2,\ldots x_{n(\alpha)}) \in S^{n(\alpha)}$. We have to show

$$\tilde{\mu}(f_\alpha(x_1,\ldots,x_{n(\alpha)})) \geqq \min_{1 \leqq i \leqq n(\alpha)} \tilde{\mu}(x_i) .$$

Let $\theta < \tilde{\mu}(x_i)$ for all i. Since $\mu_{k+1} \geqq \mu_k$, there exists k such that $\mu_k(x_i) > \theta$ for $i = 1, \ldots, n(\alpha)$. Now

$$\tilde{\mu}(f_\alpha(x_1,\ldots, x_{n(\alpha)})) \geqq \mu_{k+1}(f_\alpha(x_1,\ldots,x_{n(\alpha)}))$$

$$\geqq \min_i \mu_k(x_i) > \theta$$

That is $\tilde{\mu}(x) \geq \min_{1 \leq i \leq n(\alpha)} (\tilde{\mu}(x_i))$

where $x = f_\alpha(c) = f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)})$

By Theorem 2.2.5, $\tilde{\mu}$ is a fuzzy subalgebra. Lastly, we show that $\tilde{\mu}$ is the smallest fuzzy subalgebra of A containing μ. So, let ν be a fuzzy subalgebra of A containing μ. Since by definition $\mu_0 = \mu$, $\mu_0 \leq \nu$.

Suppose $\mu_k \leq \nu$ for some positive integer k, then for any

$x = f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}) = f_\alpha(c)$, we have

$$\min_{1 \leq i \leq n(\alpha)} \mu_k(x_i) \leq \min_{1 \leq i \leq n(\alpha)} \nu(x_i) \leq \nu(f_\alpha(x_1, x_2, \ldots . x_{n(\alpha)})) = \nu(x)$$

since ν is a fuzzy subalgebra of A.

Now
$$\mu_{k+1}(x) = \sup_{\substack{c \in C_\alpha^X \\ f_\alpha \in F}} \left(\min_{1 \leq i \leq n(\alpha)} (\mu_k(x_i)) \right)$$
$$\leq \nu(x)$$

Therefore, $\mu_{k+1} \leq \nu$.

By induction on k, $\mu_k \leq \nu$ for k=0,1,2,...

and so $\sup_{k=0}^{\infty} \mu_k = \bigvee_{k=0}^{\infty} \mu_k = \tilde{\mu} \leq \nu$.

Thus $\tilde{\mu}$ is the smallest fuzzy subalgebra of A containing μ.

4.3 Lattice of fuzzy subalgebras

In this section we prove that the lattice of fuzzy subalgebras form an algebraic closure system in the lattice of all fuzzy subsets of an algebra.

Let $A = [S,F]$ be an algebra. Consider the set of all fuzzy subalgebras of A denoted by $\mathbb{FA}(S)$. By Theorem 2.3.1, we know that $\mathbb{FA}(S)$ is a closure system and hence a complete lattice. Let $J_{\mathbb{FA}}$ be the closure operator on I^S associated with the closure system $\mathbb{FA}(S)$. Thus

$$J_{\mathbb{FA}} : \mathbb{F}(S) \to \mathbb{F}(S)$$
$$\mu \mapsto \wedge\{\nu \in \mathbb{FA}(S) : \mu \leq \nu\}$$

By Theorem 4.2.1, $J_{\mathbb{FA}}(\mu) = \tilde{\mu}$.

<u>Theorem 4.3.1</u>: $\mathbb{F}\mathbb{A}(S)$ is an algebraic closure system.

<u>Proof</u>:

By Theorem 3.4.5 it suffices to show that $\mathbb{F}\mathbb{A}(S)$ is inductive. So, let $\mathbb{K}$ be a chain in $\mathbb{F}\mathbb{A}(S)$ and let $\mu = \sup \mathbb{K}$ (union in $\mathbb{F}(S)$). Then μ is a fuzzy subalgebra. In fact, let $f_\alpha \in F$, $x_i \in S$, $i = 1, \ldots n(\alpha)$, $x = f(x_1, \ldots, x_{n(\alpha)})$. We must show that $\mu(x) \geq \beta = \min_i \mu(x_i)$. This is clearly true if $\beta = 0$.

If $\beta > \theta > 0$, there are $\mu_1, \mu_2, \ldots, \mu_{n(\alpha)} \in \mathbb{K}$, $\mu_i(x_i) > \theta$. Let $\nu \in \mathbb{K}$, $\nu \geq \mu_1, \mu_2, \ldots \mu_{n(\alpha)}$

Then $\mu(x) > \nu(x) \geq \min_i \nu(x_i) > \theta'$. This clearly completes the proof.

Remark 4.3.2: The notion of **algebraic lattice** was discovered by Birkhoff and Frink [1], to describe the lattice of subalgebras of an algebra in the crisp set theory. Let $\mathbb{L} = [L, \leq]$ be a complete lattice. Let $a \in L$. The element a is called **compact** if the following condition is satisfied.

If $a \leq \vee(x_i : i \in I)$ where $x_i \in L$ for each $i \in I$, an indexing set, then there exists $I_1 \subseteq I$ such that I_1 is finite and $a \leq \vee(x_i : i \in I_1)$.

The adjective compact is used in analogy with the concept of compact subspaces in topology. A complete lattice is called **algebraic** if every a in the lattice can be written as a join of compact elements [1]. Now it is a classical result that given a lattice $\mathbb{L}$, the following conditions are equivalent, see [19, p25].

(i) $\mathbb{L}$ is an algebraic lattice.

(ii) $\mathbb{L}$ is isomorphic to some ideal lattice $\mathcal{I}(\mathbb{L}')$ where $\mathbb{L}'$ is a semilattice with 0.

(iii) There exists an algebraic closure system $\mathbb{A}$ in 2^X such that $\mathbb{L}$ is isomorphic to $(\mathbb{A}, \subseteq)$.

Continuous lattices are generalisation of algebraic lattices in the context of the power lattices of the form I^X, see [16]. It will be interesting to study whether the lattice of fuzzy subalgebras of an algebra is a continuous lattice and whether a proposition similar to the above classical result holds in the fuzzy set theory.

Remark 4.3.3: We conjecture that if A is a finitely generated algebra and μ is a fuzzy subalgebra of A, then μ takes only a finite number of distinct values in I on A. We are not able to prove it. The basis for this conjecture is a representation theorem proved by Lowen [28] for finite dimensional vector spaces.

CHAPTER 5

FUZZY EQUIVALENCE RELATIONS

5.1 Introduction

L.A. Zadeh in his paper [43] first proposed a fuzzy relation between two sets X and Y to be a fuzzy subset of the product set X x Y. Subsequently J.A. Goguen [17], A. Kaufmann [34], E. Sanchez [39] and others have studied fuzzy relations in various contexts. In this Chapter, we define and study fuzzy equivalence relations. Kaufmann [24] calls it a relation of similitude. In section 5.2, we define a fuzzy equivalence relation and derive some simple consequences. Cuts of fuzzy equivalence relations are crisp equivalence relations. Section 5.4 describes these results. In section 5.5 a partition of a fuzzy equivalence relation is given. Conversely, starting with a class of fuzzy subsets satisfying conditions similar to those satisfied by the class given in section 5.5 associated with a fuzzy equivalence relation, can one construct a fuzzy equivalence relation such that its partition coincides with the given class? The answer is yes and these are discussed in section 5.6. Section 5.3 is devoted to the lattice theoretical properties of fuzzy equivalence relations.

5.2 Fuzzy relations

Let $X_1, X_2, \ldots, X_n$ be non-empty sets. A fuzzy n-ary relation on $X_1, X_2, \ldots, X_n$ was defined as a fuzzy subset of the product set $X_1 x X_2 x \ldots x X_n$. In particular, a fuzzy **binary relation** on X and Y is a fuzzy subset μ on XxY. We are only interested in the case in which Y=X in this section. Accordingly, unless otherwise stated, by a fuzzy relation, we mean a fuzzy binary relation given by $\mu : XxX \to I$.

Definition 5.2.1: A fuzzy relation μ on X is said to **reflexive** if $\mu(x,x) = 1$ for all $x \in X$ and said to be **symmetric** if $\mu(x,y) = \mu(y,x)$ for all $x,y \in X$.

Definition 5.2.2: Suppose μ_1 and μ_2 are two fuzzy relations on X. Then their **composition**, denoted by $\mu_1 \circ \mu_2$, is defined as $(\mu_1 \circ \mu_2)(x,y) = \sup_{z \in X} (\mu_2(x,z) \wedge \mu_1(z,y))$. If $\mu_1 = \mu_2 = \mu$ say, and $\mu \circ \mu \leq \mu$, then the fuzzy relation μ is called **transitive.**

A fuzzy relation μ on X is said to be a **fuzzy equivalence relation** if μ is reflexive, symmetric and transitive.

Proposition 5.2.3: Composition of fuzzy relations on a set X is associative.

Proof: Suppose μ_1, μ_2, μ_3 are three fuzzy relations on X. Consider for any $x,y \in X$,

$$((\mu_1 \circ \mu_2) \circ \mu_3)(x,y) = \sup_{z \in X} ((\mu_1 \circ \mu_2)(x,z) \wedge \mu_3(z,y))$$

$$= \sup_{z \in X} (\sup_{t \in X} (\mu_1(x,t) \wedge \mu_2(t,z)) \wedge \mu_3(z,y))$$

$$= \sup_{z \in X} \sup_{t \in X} ((\mu_1(x,t) \wedge \mu_2(t,z)) \wedge \mu_3(z,y))$$

$$= \sup_{t \in X} \sup_{z \in X} (\mu_1(x,t) \wedge (\mu_2(t,z) \wedge \mu_3(z,y)))$$

$$= \sup_{t \in X} (\mu_1(x,t) \wedge \sup_{z \in X} (\mu_2(t,z) \wedge \mu_3(z,y)))$$

$$= \sup_{t \in X} (\mu_1(x,t) \wedge (\mu_2 \circ \mu_3)(t,y))$$

$$= (\mu_1 \circ (\mu_2 \circ \mu_3))(x,y)$$

Therefore $(\mu_1 \circ \mu_2) \circ \mu_3 = \mu_1 \circ (\mu_2 \circ \mu_3)$.

Definition 5.2.4: The **identity relation** Id_X on X is defined as for any $x,y \in X$, $Id_X(x,y) = 0$ if $x \neq y$ and $Id_X(x,y) = 1$ for $x=y$.

The **zero relation** and the **equality relation** are defined as

$$O(x,y) = 0$$
$$I(x,y) = 1, \quad \text{for all } x,y \in X, \text{ respectively.}$$

A **partial ordering** $\leq$ in the set of all fuzzy relations on X is given by $\mu_1 \leq \mu_2$ if and only if $\mu_1(x,y) \leq \mu_2(x,y)$ for all $x,y \in X$.

Proposition 5.2.5: The set of all fuzzy relations on a set X denoted by $\mathcal{R}(X)$ forms a complete, completely distributive lattice under the ordering $\leq$, with the universal bounds given by O and I. Moreover, under the composition $\circ$ defined above, $\mathcal{R}(X)$ is a monoid with identity Id_X.

Proof: Since $R(X) \cong I^{X \times X}$ with the pointwise ordering $\leq$ on $I^{X \times X}$ induced by the usual ordering of I, $R(X)$ is a complete, completely distributive lattice under $\leq$. The other assertions are straightforward.

Remark 5.2.6: M. Erceg in [8] also made a study of equivalence relations and functions in fuzzy set theory. They are different from the results obtained in this chapter. We do not know whether there are any connections between the two studies.

5.3 Lattice of fuzzy equivalence relations

Let $E(X)$ denote the subset of $R(X)$ consisting of all fuzzy equivalence relations on a nonempty set X. We introduce a partial ordering $\leq$ on $E(X)$ as the partial ordering on $E(X)$ induced by the ordering on $R(X)$.

Proposition 5.3.1: $(E(X),\leq)$ is a complete lattice.

Proof: Firstly we note that both relations I and Id_X are in $E(X)$ and that Id_X is the least element of $E(X)$ and I is the greatest element of $E(X)$ with respect to the ordering $\leq$. Let $\{\mu_j\}_{j\in J}$ be a non-empty family of fuzzy equivalence relations in $E(X)$. Then the relation μ defined by

$$\mu(x,y) = \inf_{j\in J} \mu_j(x,y) \quad \text{for all } x,y\in X$$

is a fuzzy equivalence relation on X. For

(i) $\mu(x,x) = \inf_{j\in J} \mu_j(x,x) = 1$ for $x\in X$

(ii) $\mu(x,y) = \inf \mu_j(x,y) = \inf \mu_j(y,x) = \mu(y,x)$ for $x,y\in X$

(iii)
$$\begin{aligned}
\mu\circ\mu(x,y) &= \sup_{z\in X} (\mu(x,z)\wedge\mu(z,y)) \\
&= \sup_{z\in X} (\inf_{j\in J} \mu_j(x,z)\wedge \inf_{j\in J} \mu_j(z,y)) \\
&= \sup_{z\in X} (\inf_{j\in J} (\inf_{k\in J} (\mu_j(x,z)\wedge\mu_k(z,y)))) \\
&\leq \sup_{z\in X} (\inf_{j\in J} (\mu_j(x,z)\wedge\mu_j(z,y))) \\
&\leq \inf_{j\in J} (\sup_{z\in X} (\mu_j(x,z)\wedge\mu_j(z,y)))
\end{aligned}$$

$$= \inf_{j\in J} \mu_j(x,y)$$

$$= \mu(x,y)$$

Also $\mu \leq \mu_j$ for every $j \in J$ by definition. Moreover if $\mu_j \leq \nu$, again by the definition of infimum, $\nu \leq \mu$. This shows that $\mu = \bigwedge_{j\in J} \mu_j$. Hence by 3.2.2, $(E(X), \leq)$ is a complete lattice.

We shall now describe the join V in $(E(X), \leq)$.
Let $\mu, \nu \in E(X)$. Then $\mu \vee \nu$ is the smallest fuzzy equivalence relation containing both μ and ν in the ordering $\leq$. That is

$\mu \vee \nu = \bigwedge\{\gamma \in E(X) : \mu \leq \gamma$ and $\nu \leq \gamma\}$; at least one such γ exists, for example, Id_X. Consider

$$\mu_0 = \mu$$

$$\mu_1 = \mu\circ\nu$$

$$\mu_2 = \mu\circ\nu\circ\mu$$

$$\mu_3 = \mu\circ\nu\circ\mu\circ\nu$$

$$\vdots$$

It is immediate that

$$\mu_0 \leq \mu_1 \leq \mu_2 \leq \dots$$

and also, it is easy to check that

$$\mu_n \leq \mu \vee \nu \quad \text{for all } n=0,1,2,\dots$$

We claim that $\mu \vee \nu = \bigvee_{n=0}^{\infty} \mu_n$. For

(i) $\bigvee_{n=0}^{\infty} \mu_n(x,x) = \sup_n \mu_n(x,x) = 1$ for $x \in X$.

(ii) $\bigvee_{n=0}^{\infty} \mu_n(x,y) = \bigvee_{n=0}^{\infty} \mu_n(y,x)$ is straightforward.

(iii) Let $\gamma = \bigvee_{n=0}^{\infty} \mu_n$. For transitivity of γ, we have to show that $\gamma \circ \gamma \leq \gamma$. That is, for any $x,y \in X$, we have to show that $(\gamma \circ \gamma)(x,y) \leq \gamma(x,y)$. Now

$$\gamma \circ \gamma(x,y) = \sup_{z \in X} (\gamma(x,z) \wedge \gamma(z,y))$$

$$= \sup_{z \in X} \left(\bigvee_{n=0}^{\infty} \mu_n(x,z) \wedge \bigvee_{n=0}^{\infty} \mu_n(z,y)\right)$$

$$\leq \bigvee_{n=0}^{\infty} \mu_n(x,y) \text{ since for any } m,n$$

$$\sup_{z} \mu_n(x,z) \wedge \mu_m(z,y) \leq \mu_{n+m}(x,y).$$

Therefore $\gamma \circ \gamma(x,y) \leq \gamma(x,y)$, implying the transitivity of γ.

More generally, suppose $S = \{\nu_j\}_{j \in J}$ is a class of fuzzy equivalence relations on X, then $\bigvee_{j \in J} \nu_j = \gamma$ is defined as $\gamma = \sup_{n=0}^{\infty} \mu_n$ where

$$\mu_0(x,y) = \sup_{j \in J} \nu_j(x,y)$$

$$\mu_n(x,y) = \sup_{(\nu_1,\nu_2,\ldots,\nu_n) \in S^n} \left(\sup_{(z_0=x,z_1,\ldots,z_n=y) \in X^{n+1}} \left(\min_{i=1 \text{ to } n} (\nu_i(z_{i-1},z_i))\right)\right)$$

It is straightforward, though tedious, to check that

1. $\mu_0 \leq \mu_1 \leq \mu_2 \leq \ldots.$
2. $\nu_j \leq \gamma$
3. γ is a fuzzy equivalence relation on X.
4. γ is the smallest fuzzy equivalence relation on X such that (2) is true.

Hence $\gamma = \bigvee_{j \in J} \nu_j$ in $E(X)$.

Finally, we require the following proposition later.

Proposition 5.3.2: If μ is a fuzzy equivalence relation on X, then μ is an idempotent element of $E(X)$. That is, $\mu \circ \mu = \mu$.

Proof: For any $x, y \in X$, $(\mu \circ \mu)(x,y) \leq \mu(x,y)$. On the other hand,

$$(\mu \circ \mu)\,(x,y) = \sup_{z \in X} (\mu(x,z) \wedge \mu(z,y))$$

$$\geq \mu(x,x) \wedge \mu(x,y)$$

$$= \mu(x,y) \text{ since } \mu(x,x) = 1.$$

Therefore $(\mu \circ \mu)(x,y) = \mu(x,y)$ implying $\mu \circ \mu = \mu$.

5.4 Cuts of fuzzy equivalence relations

In this section, we study cuts of fuzzy equivalence relations on a non-empty set X. The cuts are proved to be crisp equivalence relations on X. In particular, the 1-cut equivalence relation proves to be an important one from the point of view of the partition associated with a fuzzy equivalence relation. See section 5.5.

Given a fuzzy equivalence relation μ on X, two crisp relations on X are defined for each $\alpha\in[0,1]$ as follows:

Definition 5.4.1: A **weak α-relation** denoted by $\underset{\sim}{\omega}_{\alpha}$ is defined on X as $x \,\underset{\sim}{\omega}_{\alpha}\, y$ if and only if $\mu(x,y) \geq \alpha$ and a **strong α-relation** denoted by $\underset{\sim}{\sigma}_{\alpha}$ is defined on X as $x\underset{\sim}{\sigma}_{\alpha}y$ if and only if $\mu(x,y)>\alpha$, $x,y\in X$.

Proposition 5.4.2: If μ is a fuzzy equivalence relation on X, then the following are crisp equivalence relations on X.

(i) $\underset{\sim}{\omega}_{\alpha}$ for each $\alpha\in[0,1]$.

(ii) $\underset{\sim}{\sigma}_{\alpha}$ for each $\alpha\in[0,1)$.

Proof: (i) Let $0\leq\alpha\leq1$. For each $x\in X, \mu(x,x) = 1\geq\alpha$; hence $x \,\underset{\sim}{\omega}_{\alpha} x$. Also $x \,\underset{\sim}{\omega}_{\alpha}\, y$ implies $\mu(x,y)\geq\alpha$, and so $\mu(y,x) = \mu(x,y)\geq\alpha$, implying $y \,\underset{\sim}{\omega}_{\alpha}\, x$. Finally suppose $x \,\omega_{\alpha}\, y$ and $y \,\underset{\sim}{\omega}_{\alpha}\, z$, then

$$\begin{aligned}\mu(x,z) = \mu\circ\mu(x,z) &= \sup_{t\in X} (\mu(x,t) \wedge \mu(t,z)) \\ &\geq \mu(x,y) \wedge \mu(y,z) \\ &\geq \alpha \wedge \alpha = \alpha.\end{aligned}$$

Therefore $x \underset{\sim}{\omega}_\alpha z$.

That is $\underset{\sim}{\omega}_\alpha$ is indeed a crisp equivalence relation on X.

(ii) is proved similarly.

For each $x \in X$, let $[x]_{\underset{\sim}{\omega}_\alpha}$ denote the crisp equivalence class containing x with respect to the weak α-relation $\underset{\sim}{\omega}_\alpha$.

Proposition 5.4.3: (i) $[x]_{\underset{\sim}{\omega}_\alpha} \subseteq [x]_{\underset{\sim}{\omega}_\beta}$ for $1 \geq \alpha \geq \beta \geq 0$

(ii) $\bigcap_{0 \leq \alpha \leq 1} [x]_{\underset{\sim}{\omega}_\alpha} = [x]_{\underset{\sim}{\omega}_1}$

(iii) $\bigcup_{0 \leq \alpha \leq 1} [x]_{\underset{\sim}{\omega}_\alpha} = [x]_{\underset{\sim}{\omega}_0}$

Proof: (i) Let $1 \geq \alpha \geq \beta \geq 0$, $y \in [x]_{\underset{\sim}{\omega}_\alpha}$. Then $\mu(x,y) \geq \alpha \geq \beta$.

This implies $\mu(x,y) \geq \beta$ and in turn implies $y \in [x]_{\underset{\sim}{\omega}_\beta}$.

(ii) Let $y \in \bigcap_{0 \leq \alpha \leq 1} [x]_{\underset{\sim}{\omega}_\alpha}$. Then $\mu(x,y) \geq \alpha$ for every $\alpha \in [0,1]$.

Therefore $\mu(x,y) \geq 1$ implying $\mu(x,y) = 1$. That is $y \in [x]_{\underset{\sim}{\omega}_1}$.

Conversely, let $y \in [x]_{\underset{\sim}{\omega}_1}$. Then $\mu(x,y) = 1 \geq \alpha$ for every $\alpha \in [0,1]$.

Therefore $y \in [x]_{\underset{\sim}{\omega}_\alpha}$ for $\alpha \in [0,1]$. So $y \in \bigcap_{0 \leq \alpha \leq 1} [x]_{\underset{\sim}{\omega}_\alpha}$.

(iii) $[x]_{\underset{\sim}{\omega}_0} \subseteq \bigcup_{0 \leq \alpha < 1} [x]_{\underset{\sim}{\omega}_\alpha}$. Conversely if $y \in [x]_{\underset{\sim}{\omega}_\alpha}$ for some $0 \leq \alpha \leq 1$, then $\mu(x,y) \geq \alpha \geq 0$. Therefore $y \in [x]_{\underset{\sim}{\omega}_0}$.

A similar proposition holds for the equivalance classes in X with respect to α-strong relation. Moreover, for every $0 < \alpha < 1$, $[x]_{\underset{\sim}{\sigma}_\alpha} \subseteq [x]_{\underset{\sim}{\omega}_\alpha}$.

Theorem 5.4.4: Let $x_1, x_2 \in [x]_{\underset{\sim}{\omega}_\alpha}$ and y is any element of X such that $\mu(x_1,y) \leq \alpha$ and $\mu(x_2,y) \leq \alpha$. Then $\mu(x_1,y) = \mu(x_2,y)$.

Proof: $\mu(x_1,y) = \mu \circ \mu(x_1,y) = \sup_{z \in X} (\mu(x_1,z) \wedge \mu(z,y))$

$$\geq \mu(x_1,x_2) \wedge \mu(x_2,y)$$

$$= \mu(x_2,y)$$

Interchanging x_1 amd x_2, we get similarly $\mu(x_2,y) \geq \mu(x_1,y)$.

Hence they are equal.

Corollary 5.4.5: Let $[x]_{\underset{\sim}{\omega}_\alpha}$ and $[y]_{\underset{\sim}{\omega}_\alpha}$ be two distinct crisp α-weak equivalence classes in X to which x and y belong respectively, with $0<\alpha \leq 1$. If there exists a β such that $0<\beta<\alpha$ with $\mu(x,y) \geq \beta$, then for all $x' \in [x]_{\underset{\sim}{\omega}_\alpha}$ and $y' \in [y]_{\underset{\sim}{\omega}_\alpha}$, $\mu(x',y') \geq \beta$.

Proof: By the theorem, $\mu(x,y) = \mu(x',y) = \mu(x,y') = \mu(x',y')$. Therefore $\mu(x',y') \geq \beta$.

This corollary shows that every $[x]_{\underset{\sim}{\omega}_\beta}$, $0 \leq \beta < 1$ consists of union of some crisp α-weak equivalence classes for each $\beta < \alpha \leq 1$. That is each $[x]_{\underset{\sim}{\omega}_\alpha}$ is a refinement of $[x]_{\underset{\sim}{\omega}_\beta}$.

5.5 Partition associated with a fuzzy equivalence relation

Given a fuzzy equivalence relation on X, we generate fuzzy subsets of X which form a "partition" of X in the sense that the union of these subsets is π_X. This is analogous to representing a crisp equivalence relation on X as a partition of X. Let μ be a fuzzy equivalence relation on X.

Consider the crisp equivalence classes $[x]_{\underset{\sim}{\omega_1}}$ for $x \epsilon X$. Let us denote $[x]_{\underset{\sim}{\omega_1}} = \mathcal{x}$. Define a fuzzy subset $\mu_{\mathcal{x}}$ on X corresponding to $\mathcal{x} = [x]_{\underset{\sim}{\omega_1}}$ as follows:

$$\mu_{\mathcal{x}}(z) = \mu(x,z) \quad \text{for all } z \epsilon X.$$

<u>Proposition 5.5.1:</u> $\mu_{\mathcal{x}}: X \to I$ is well-defined.

<u>Proof:</u> Since $\mu_{\mathcal{x}}$ is defined in terms of μ, $0 \leq \mu_{\mathcal{x}}(z) \leq 1$ for all $z \epsilon X$. Suppose $y \epsilon \mathcal{x}$. Then $\mu(x,y) = 1$, $\mu_{\mathcal{x}}(z) = \mu(y,z) \leq 1$ and $\mu_{\mathcal{x}}(z) = \mu(x,z) \leq 1$. Therefore by Theorem 5.4.5, $\mu(x,z) = \mu(y,z)$ implying $\mu_{\mathcal{x}}$ is well-defined.

For a fuzzy subset $\mu: X \to I$, let the crisp subsets of X denoted by $W(\mu,\alpha)$ and $S(\mu,\alpha)$, $0 \leq \alpha \leq 1$, called the weak α-cuts and strong α-cuts respectively of μ be defined as

$$W(\mu,\alpha) = \{x \epsilon X : \mu(x) \geq \alpha\}$$

$$\text{and} \quad S(\mu,\alpha) = \{x \epsilon X : \mu(x) > \alpha\}.$$

<u>Proposition 5.5.2:</u> For each fixed α such that $0 \leq \alpha \leq 1$, the collection of subsets $\{W(\mu_{\mathcal{x}},\alpha) : x \epsilon X\}$ form a crisp partition of X.

Proof: Every $x \in X$ belongs to $W(\mu_x,\alpha)$. Therefore $\bigcup_{x\in X} W(\mu_x,\alpha) = X$.

Let $x,y \in X$, $x \in x$ and $y \in y$. Suppose $x \neq y$. Then we claim that either $W(\mu_x,\alpha) \cap W(\mu_y,\alpha) = \emptyset$ or if there exists a $z \in W(\mu_x,\alpha) \cap W(\mu_y,\alpha)$ then $W(\mu_x,\alpha) = W(\mu_y,\alpha)$. For if $z \in W(\mu_x,\alpha) \cap W(\mu_y,\alpha)$ then $\mu_x(z) \geq \alpha$ and $\mu_y(z) \geq \alpha$. These imply that $\mu(x,z) \geq \alpha$ and $\mu(y,z) \geq \alpha$. By transitivity

$$\mu(x,y) \geq \alpha \tag{1}$$

If $t \in W(\mu_x,\alpha)$, then $\mu(x,t) \geq \alpha$. By symmetry

$$\mu(t,x) \geq \alpha \tag{2}$$

Combining (1) and (2) we get $\mu(t,y) \geq \alpha$. That is $\mu(y,t) \geq \alpha$ implying $\mu_y(t) \geq \alpha$. Therefore $t \in W(\mu_y,\alpha)$. Hence $W(\mu_x,\alpha) \subseteq W(\mu_y,\alpha)$. Interchanging the roles of x and y, we get $W(\mu_y,\alpha) \subseteq W(\mu_x,\alpha)$. That is $W(\mu_x,\alpha) = W(\mu_y,\alpha)$. This completes the proof.

The next proposition shows that a fuzzy equivalence relation μ on X is such that $\mu(x,y) = \alpha$ for some $x,y \in X$, then μ_x and μ_y are α-disjoint. That is $\mu_x \wedge \mu_y \leq \alpha$. The corollary is if $\mu(x,y)=0$ for some $x,y \in X$, then μ_x and μ_y are totally disjoint. That is, the supports of μ_x and μ_y are disjoint as crisp subsets.

Proposition 5.5.3: If for some $(x,y) \in X \times X$, $\mu(x,y)=\alpha$, then $\mu_x \wedge \mu_y \leq \alpha$.

Proof: Suppose not. Then for some $t \in X$,

$$(\mu_x \wedge \mu_y)(t) > \alpha.$$

That is $\inf(\mu_x(t), \mu_y(t)) > \alpha$.

$\mu_x(t) > \alpha$ and $\mu_y(t) > \alpha$. By transitivity, $\mu(x,y) > \alpha$. This is a contradiction to the fact that $\mu(x,y) = \alpha$.

This completes the proof.

Corollary 5.5.4: If $\mu(x,y) = 0$ then $\mu_x \wedge \mu_y = 0$.

In the above we constructed the class of fuzzy subsets $\{\mu_x\}, x \in X$, on X associated with a fuzzy equivalence relation μ. We call this collection the **fuzzy partition** of X with respect to μ. It is uniquely determined by μ.

5.6 Fuzzy partition and fuzzy equivalence relations

In this section, we construct a fuzzy equivalence relation associated with a class of fuzzy subsets on X. Let $\{\mu_j\}_{j\in J}$, J an indexing set, be a class of fuzzy subsets on X. It is called a **fuzzy partition** of X_X, if the following are true.

(i) For each $x \in X$, there is an unique $j \in J$ such that $\mu_j(x)=1$.

(ii) For each $\alpha \in [0,1]$, $\{W(\mu_j,\alpha)\}(j \in J)$ form a crisp partition of X.

Proposition 5.6.1 For each α such that $0<\alpha\leq 1$, $\{W(\mu_j, \alpha), j \in J\}$ is a refinement of $\{W(\mu_j, \beta), j \in J\}$ with $0\leq\beta\leq\alpha\leq 1$.

Proof: Let $x \in W[\mu_j, \beta]$. There exists $j' \in J$ such that $x \in W[\mu_{j'},\alpha]$(since there exists j_0 with $\mu_{j_0}(x) = 1$). For any such j' we have $x \in W[\mu_j,\beta] \cap W[\mu_{j'},\beta]$ and so $W[\mu_j,\beta] = W[\mu_{j'},\beta]$. Now

$$x \in W[\mu_{j'},\alpha] \subset W[\mu_{j'},\beta] = W[\mu_j,\beta].$$

In this case, we can associate an unique fuzzy equivalence relation μ on X as follows:

$$\mu(x,y) = \sup_{j\in J} (\mu_j(x) \wedge \mu_j(y)) \text{ for all } x,y \in X.$$

Theorem 5.6.1: μ is a fuzzy equivalence relation on X.

Proof: By condition (i) of fuzzy partition, for each $x \in X$, $\mu(x,x)=1$. By definition of μ, $\mu(x,y)=\mu(y,x)$ for all $x,y \in X$. That is μ is symmetric.

Now we shall prove that μ is transitive. For this consider,

$$\begin{aligned}(\mu \circ \mu)(x,y) &= \sup_{z \in X} (\mu(x,z) \wedge \mu(z,y)) \\ &= \sup_{z} (\sup_{j}((\mu_j(x) \wedge \mu_j(z)) \wedge \sup_{j}(\mu_j(z) \wedge \mu_j(y))) \\ &\leq \sup_{z} (\sup_{j} (\mu_j(x) \wedge \mu_j(z) \wedge \mu_j(y))) \\ &\leq \sup_{j} (\mu_j(x) \wedge \mu_j(y)) \\ &= \mu(x,y) \quad \text{for all } x,y \in X.\end{aligned}$$

Therefore μ is transitive. This completes the proof. By condition (i) of fuzzy partition, each $x \in X$ determines a unique $j_x \in J$ such that $\mu_{j_x}(x)=1$. Now consider $\mathscr{x}=\{y \in X : \mu_{j_x}(x)=1\}$. By condition (ii) on μ_j's, the class of subsets $(\mathscr{x})_{x \in X}$ of X form a crisp partition of X.

Proposition 5.6.2: For each $x \in X$, $\mathscr{x}$ as defined above is equal to the set $[x]_{\underset{\sim}{\omega_1}}$, the crisp equivalence class containing x with respect to the weak 1-relation $\underset{\sim}{\omega_1}$ of the fuzzy equivalence relation μ.

Proof: $[x]_{\underset{\sim}{\omega_1}} = \{y \in X : \mu(x,y)=1\}$.

Suppose $\mu_{j_x}(y) \neq 1$. Then $\mu_{j_x}(y)=\gamma$ say, which is such that $\gamma<1$. Consider an α such that $\gamma<\alpha<1$. Since $\{W(\mu_j,\alpha), j \in J\}$ form a crisp partition of X, y does not belong to $[x]_{\underset{\sim}{\omega_1}}$ the class containing x. Conversely, it is easy to see that $\mathscr{x} \subseteq [x]_{\underset{\sim}{\omega_1}}$. Therefore $\mathscr{x}=[x]_{\underset{\sim}{\omega_1}}$.

We next prove that the class of fuzzy subsets $\{\mu_j\}$ on X is the same as the fuzzy partition associated with μ.

Proposition 5.6.3: For each $x \in X$, the fuzzy subset μ_x defined by $\mu_x(y)=\mu(x,y)$ for all $y \in X$ is equal to the fuzzy subset μ_{j_x}.

Proof: We have to show that $\mu_x(y)=\mu_{j_x}(y)$ for all $y \in X$. For any $y \in X$, $\mu_x(y)=\mu(x,y)=\sup_j (\mu_j(x) \wedge \mu_j(y)) \geq \mu_{j_x}(x) \wedge \mu_{j_x}(y)=\mu_{j_x}(y)$. (1)

On the other hand, given $\epsilon>0$, there exists a $j' \in J$ such that

$\mu_j(x) \wedge \mu_{j'}(y)>\mu(x,y)-\epsilon=\mu_x(y)-\epsilon$. (If $\mu(x,y)=0$ then there is nothing to prove).

That is $\mu_{j'}(x)>\mu(x,y)-\epsilon=\mu_x(y)-\epsilon=\alpha$ say

$\mu_{j'}(y)>\mu(x,y)-\epsilon=\mu_x(y)-\epsilon=\alpha$.

This implies that $x,y \in W(\mu_{j'},\alpha)$. But $\{W(\mu_j,\alpha), j \in J\}$ form a crisp partition of X. Therefore, either $W(\mu_{j'},\alpha) \cap W(\mu_{j_x},\alpha)=\emptyset$ or $W(\mu_j,\alpha)=W(\mu_{j_x},\alpha)$. Certainly $W(\mu_{j'},\alpha) \cap W(\mu_{j_x},\alpha) \neq \emptyset$ since x is a common element.

Therefore $W(\mu_{j'},\alpha)=W(\mu_{j_x},\alpha)$

That is $\mu_{j_x}(y) \geq \alpha=\mu_x(y)-\epsilon$.

As $\epsilon>0$ is arbitrary, $\mu_{j_x}(y) \geq \mu_x(y)$. (2)

From (1) and (2) we get $\mu_{j_x}(y)= \mu_x(y)$.

As $y \in X$ is arbitrary, $\mu_{j_x}=\mu_x$ as fuzzy subsets on X. This completes the proof.

CHAPTER 6

6.1 Introduction

In Chapter 5 we defined fuzzy relations and fuzzy equivalence relations. A fuzzy relation on X and Y, satisfying certain properties is called a fuzzy function from X to Y. This is defined in section 6.2 In section 6.3 the relationship between fuzzy functions and fuzzy equivalence relations is explored. We prove that the composition of a fuzzy function and its converse is a fuzzy equivalence relation. Similar results were obtained by W.C. Nemitz [34] recently, although in a different context.

6.2 Fuzzy functions

Let X and Y be two crisp non-empty sets. Let f be a fuzzy relation from X to Y . That is $f \in \mathcal{R}(X,Y)$ such that $f: X\times Y \to I$.

Each $y \in Y$ determines a fuzzy subset of X as follows:

$$\mu_y : X \to I$$

$$x \to f(x,y) \quad \text{for all } x \in X, \ y \text{ is a fixed element in } Y.$$

The fuzzy relation is said to be a **fuzzy function** from X to Y if

(i) For each x in X, there exists a unique $y \in Y$ such that $f(x,y) = 1$.

(ii) For each $\alpha \in [0,1]$, the weak α-cuts $\{W(\mu_y, \alpha): y \in Y\}$ form a crisp partition of X.

It follows (see Proposition 5.6.1) that for each $\alpha \in [0,1]$, $\{W(\mu_y, \alpha): y \in Y\}$ is a refinement of

$$\{W(\mu_y, \beta): y \in Y\} \quad \text{for } 0 \leq \beta \leq \alpha \leq 1.$$

If for each y, there exists an $x \in X$ such that $f(x,y) = 1$, then f is said to be an **onto function** from X onto Y. If each pair $x_1, x_2 \in X$ such that $f(x_1,y) = f(x_2,y) = 1$ implies that $x_1 = x_2$, then f is said to be a **one-to-one function** from X to Y.

It is easy to check that the above definitions reduce to the usual definitions in the case of crisp set theory.

Given a fuzzy function $f: X\times Y \to I$, its **converse (transpose)** denoted by $\breve{f}$ is defined as $\breve{f}: Y\times X \to I$, $\breve{f}(y,x) = f(x,y)$, $x \in X$ and $y \in Y$.

In the following , the **image** of a fuzzy subset of X under a fuzzy function f from X to Y is defined. (see J.A. Goguen [17]). Let $\gamma:X\to I$ be a given fuzzy subset. Then $f(\gamma):Y\to I$ is defined as

$$f(\gamma)(y) = \sup_{x\in X} (\gamma(x) \wedge f(x,y)) \text{ for } y\in Y.$$

If $\gamma:Y\to I$ is a fuzzy subset of Y, then its preimage under f denoted by $f^{-1}(\gamma)$ is defined as a fuzzy subset of X as follows

$$f^{-1}(\gamma)(x) = \sup_{y\in Y} (\gamma(y) \wedge \breve{f}(y,x)) \text{ for } x\in X.$$

A fuzzy function f: $X\to Y$ is said to be of finite type if $f(X\times Y)$ is a finite subset of I.

For example all crisp functions between X and Y, and all fuzzy functions between finite crisp sets are of finite type.

If $f_1:X\to Y$ and $f_2:Y\to Z$ are two fuzzy functions, then their **composition** denoted by $f_2 \circ f_1$ is defined as $f_2 \circ f_1:X\to Z$

$$f_2 \circ f_1 : X\times Z \to I$$

$$(f_2 \circ f_1)(x,z) = \sup_{y\in Y} (f_1(x,y) \wedge f_2(y,z)) \text{ for all } x\in X,\ z\in Z.$$

This is the usual composition of fuzzy relation.

6.3 Fuzzy function and fuzzy equivalence relations

Given a fuzzy function $f: X \times Y \to I$, consider the following composition of $f: X \to Y$ and $\breve{f}: Y \to X$, given by

$$\breve{f} \circ f(x_1, x_2) = \sup (f(x_1, y) \wedge \breve{f}(y, x_2)) \text{ for all } x_1, x_2 \in X$$

Let μ be the relation $\breve{f} \circ f$. That is $\mu(x_1, x_2) = f \circ \breve{f}(x_1, x_2)$.

Proposition 6.3.1: The fuzzy relation μ on X as defined above is a fuzzy equivalence relation.

Proof: By condition (i) of fuzzy function, $\mu(x,x)=1$ since f is a fuzzy function. That is μ is fuzzy reflexive on X. That μ is fuzzy symmetric is straightforward. Finally μ is fuzzy transitive is proved as follows: For $x, x' \in X$

Let $\alpha = (\mu \circ \mu)(x, x')$, $\varepsilon > 0$. There exists $z \in X$ such that $\mu(x,z), \mu(z,x') > \alpha - \varepsilon$.

Thus, there are $y, w \in Y$ with $f(x,y) \wedge f(z,y) > \alpha - \varepsilon$ and $f(z,w) \wedge f(x',w) > \alpha - \varepsilon$.

Since $z \in W[\mu_y, \alpha - \varepsilon] \cap W[\mu_w, \alpha - \varepsilon]$, we have $W[\mu_y, \alpha - \varepsilon] = W[\mu_w, \alpha - \varepsilon]$.

Thus $x' \in W[\mu_w, \alpha - \varepsilon] = W[\mu_y, \alpha - \varepsilon]$ and so $f(x', y) > \alpha - \varepsilon$.

Thus $\mu(x, x') \geq f(x,y) \wedge f(x',y) > \alpha - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get

$\mu(x, x') \geq \alpha$.

This completes the proof.

The fuzzy equivalence relation μ associated with the fuzzy function f as above, is called the **kernel** of f.

We have the following analogue of Theorem 3.1 [3].

Theorem 6.3.2: Suppose $f:X \to Y$ is a fuzzy function from X to Y. Let μ be the kernel of f. Then there is a decomposition of f given by the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{\ g\ } & Y \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle i} \\ X/\mu & \xrightarrow[\ f'\ ]{} & Xf \end{array}$$

where X/μ denotes the class of fuzzy subsets $(\mu_x)_{x \in X}$ associated with μ, Xf is the crisp subset of Y given by $\{y \in Y$: there exists a $x \in X$ with $f(x,y)=1\}$ and ε is the natural mapping given by $\varepsilon(x,\mu_x)=\mu_x(x)$. f' and i are crisp mappings given by $f'(\mu_x)=y$ where x is the class given by x such that $f(x,y)=1$ and i is given by the inclusion , $i:X_f \hookrightarrow Y$, $i:y \to y$. and where $g(x) = y$ is defined by $f(x,y) = 1$.

Proof: First we show that f' is well-defined. Suppose $\mu(x,x') = 1$ and $y,y' \in T$ are such that $f(x,y) = f(x',y') = 1$.

We have to show that $y = y'$.

Let $x = [x]_{\omega_1}$ where $f(x,y) = 1$

and $x' = [x']_{\omega_1}$ where $f(x',y') = 1$.

Since $\mu(x,x') = 1$, we have $x = x'$.

That is $f(x,y) = f(x,y') = 1$ which implies $y = y'$.

Everything else is clear.

CHAPTER 7

FUZZY CONGRUENCE RELATIONS

7.1 Introduction

If A = [S,F] is an algebra and if $\mathcal{R}(S)$ denotes the set of all fuzzy relations on A, then the algebraic operations $f_\alpha \in F$ on A can be extended to algebraic operation f on $\mathcal{R}(S)$. Thus $(\mathcal{R}(S),\mathbb{F})$ is an algebra. It is called the relation algebra of A. This algebra is studied in section 7.2. The subset $\mathcal{E}(S)$ of $\mathcal{R}(S)$ consisting of all fuzzy equivalence relations is a subalgebra of $(\mathcal{R}(S),\mathbb{F})$. In section 7.3, a fuzzy congruence relation is defined as a fuzzy equivalence relation which is at the same time a fuzzy subalgebra of $\mathcal{R}(S)$. Fuzzy congruence relation is studied in section 7.3. The behaviour of a fuzzy congruence relation under a homomorphism between algebras is studied in section 7.4. In the last section, the set of all fuzzy congruence relations on an algebra is proved to be a complete lattice and an algebraic closure system.

7.2 Algebraic operations on relations and equivalence relations

In this section we define what is meant by a fuzzy congruence relation on an algebra A=[S,F] and derive some simple consequences. We already noted in section 5.1, that $\mathcal{R}(S)$ and $\mathcal{E}(S)$ are complete lattices. Here we extend the algebraic operations F on A to $\mathbb{F}$ on $\mathcal{R}(S)$ as follows:

Definition 7.2.1: Let $f_\alpha \in F$ be given. For any $n(\alpha)$-tuples $(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)}) \in \mathcal{R}(S)^{n(\alpha)}$, we define, for any $x,y \in S$, $\mathcal{f}_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)})$ to be an element of $\mathcal{R}(S)$ given by $\mathcal{f}_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)})(x,y)$

$$= \sup_{\substack{x=f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}) \\ y=f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})}} \left(\min_{1\le i\le n(\alpha)} \mu_i(x_i,y_i)\right)$$

where the supremum is taken over all representations of x and y of the form indicated above.

Thus $[\mathcal{R}(S),\mathbb{F}]$ is an algebra. It is called the **relational algebra.**

7.3 Fuzzy congruence relations

In the previous section we extended algebraic operations from a given algebra to the relation algebra, namely, the set of all relations defined on the algebra. Here we define fuzzy congruence relation to be a fuzzy subalgebra of the relational algebra. Precisely,

Definition 7.3.1: Let A=[S,F] be an algebra. An element $\mu\in(E(S),\mathcal{F})$ is said to be a **fuzzy congruence relation** on A if and only if, for each $f_\alpha\in\mathcal{F}$,

$$f_\alpha(\mu,\mu,\ldots,\mu)\leq\mu.$$

In other words, if the algebraic structure of A is preserved by the fuzzy equivalence relation in A, then we call it a fuzzy congruence relation. This is a generalisation of ordinary congruence relation, in the sense that if a $\mu(x,y)$ on S takes only the values 0 and 1, then the above definition reduces to μ being an ordinary congruence relation. Definition 7.3.1 can be thought of as the "substitution property" normally satisfied by the crisp congruence relation.

Proposition 7.3.2: If μ is a fuzzy congruence relation on an algebra A=[S,F], then $\underset{\sim}{\omega}_r$, the weak r-cut relation is a crisp congruence relation on A for each $0\leq r\leq 1$.

Proof: Since μ is a fuzzy equivalence relation $\underset{\sim}{\omega}_r$ is a crisp equivalence relation on A. So we need to check only that $\underset{\sim}{\omega}_\alpha$ satisfies the substitution property. Let $f_\alpha\in F$ and let $x=f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})$ and $y=(y_1,y_2,\ldots,y_{n(\alpha)})$for x,y∈S and for two $n(\alpha)$-tuples $(x_1,x_2,\ldots,x_{n(\alpha)})$ and $(y_1,y_2,\ldots,y_{n(\alpha)})$ belonging to $S^{n(\alpha)}$. Suppose now that $x_i\underset{\sim}{\omega}_r y_i$ for

$i=1,2,\ldots,n(\alpha)$. Then $\mu(x_i,y_i)\geq r$ for $0\leq r\leq 1$.

Since $f_\alpha(\mu,\mu,\ldots,\mu)\leq\mu$,

$$\mu(x,y) \geq \sup_{\substack{x=f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})\\ y=f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})}} \left(\min_{1\leq i\leq n(\alpha)} \mu(x_i,y_i)\right)$$

$$\geq r$$

Therefore $\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),\ f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}))\geq r$.

That is $f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})\ \underset{\sim}{\omega}_r\ f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})$

This completes the proof.

In particular, $\underset{\sim}{\omega}_1$ is a crisp congruence relation on A. We shall make use of this fact in isomorphism theorems. The following proposition is similar to Theorem 2.5.1.

<u>Proposition 7.3.3</u>: A fuzzy equivalence relation μ on an algebra A is a fuzzy congruence relation if and only if

$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}))$ is greater than or equal to

$\min_{1\leq i\leq n(\alpha)} \mu(x_i,y_i)$ for all $f_\alpha\in F$ and for all $n(\alpha)$-tuples $(x_1,x_2,\ldots,x_{n(\alpha)})$,

$(y_1,y_2,\ldots,y_{n(\alpha)})$ belonging to $S^{n(\alpha)}$.

<u>Proof</u>: Similar to 2.5.1.

For the following proposition we recall from 6.1 that μ_x is defined as $\mu_x(y)=\mu(x,y)$ where μ is a fuzzy equivalence relation and μ_x is an associated fuzzy subset. If θ is a crisp congruent relation on an algebra and if there exists a single element subalgebra {e} in A, then the subset $\{x \in A : x\theta e\}$ is a subalgebra of A. For instance, normal subgroups in groups, ideals in rings and ideals in lattices etc. We prove a similar result in the following:

Proposition 7.3.4: Let A=[S,F] be an algebra with a single element subalgebra {e}. Let μ be a fuzzy congruence relation on A. Then, if $e=[e]_{\underset{\sim}{\omega}_1}$, μ_e is a fuzzy subalgebra of A.

Proof: First we observe that $f_\alpha(e,e,\ldots,e)=e$ for every $f_\alpha \in F$. Also $e=[e]_{\underset{\sim}{\omega}_1}=\{x \in S : \mu(x,e)=1\}$. Now by definition $\mu_e(x)=\mu(x,e)$ for all $x \in S$. For any $f_\alpha \in F$ and $n(\alpha)$-tuples $(x_1,x_2,\ldots,x_{n(\alpha)}) \in S^{n(\alpha)}$ we have

$$\begin{aligned}
\mu\,(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}) &= \mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),e) \\
&= \mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(e,e,\ldots,e)) \\
&\geq \min_{1>i\geq n(\alpha)} \mu(x_i,e) \quad \text{by Proposition 7.3.3} \\
&= \min_{1\leq i\leq n(\alpha)} \mu_e(x_i)
\end{aligned}$$

Therefore by Theorem 2.5.1, μ_e is a fuzzy subalgebra of A.

7.4 Fuzzy congruence relations under a homomorphism

In this section we examine fuzzy congruence relations under a homomorphism between two similar algebras.

Let $A=[S,F]$ and $B=[T,F]$ be two similar algebras and $\varphi:A\to B$ be a homomorphism. Let μ and ν be relations on the algebras A and B respectively. We define the **inverse image** of ν, denoted by $\varphi^{-1}(\nu)$ to be a relation on A given by $\varphi^{-1}(\nu)(x,y)=\nu(\varphi(x),\varphi(y))$ for all $x,y\in S$, and the image of μ, denoted by $\varphi(\mu)$ to be a relation on B given by

$$\varphi(\mu)(x',y') = \begin{cases} \sup\limits_{\substack{x\in\varphi^{-1}(x') \\ y\in\varphi^{-1}(y')}} \mu(x,y) & \text{if } \varphi^{-1}(x')\neq\emptyset \text{ and } \varphi^{-1}(y')\neq\emptyset . \\ 0 & \text{if } \varphi^{-1}(x)=\varphi \text{ or } \varphi^{-1}(y)=\emptyset \end{cases}$$

for all $x',y'\in T$.

Since $\varphi(S)\subseteq T$ and $B'=[\varphi(S),F]$ is a subalgebra of B, we can restrict ourselves to φ being a homomorphism of A onto B. Thus

$$\varphi(\mu)(x',y') = \sup_{\substack{x\in\varphi^{-1}(x') \\ y\in\varphi^{-1}(y')}} \mu(x,y) \text{ with } x',y'\in\varphi(S)$$

In general we note that $\nu\leq\varphi(\varphi^{-1}(\nu))$ and also $\mu\leq\varphi^{-1}(\varphi(\mu))$, if φ is onto. If φ is one-to one and onto, i.e. if φ is an isomorphism of A onto B, then $\mu = \varphi^{-1}(\varphi(\mu))$ for every $\mu\in R(A)$ and $\nu=\varphi(\varphi^{-1}(\nu))$ for every $\nu\in R(B)$.

Proposition 7.4.1: Let φ be a homomorphism between two similar algebras $A=[S,F]$ and $B[T,F]$. If ν is a fuzzy congruence relation on B, then $\varphi^{-1}(\nu)$ is a fuzzy congruence relation on A.

Proof: $\varphi^{-1}(\nu)(x,x)=\nu(\varphi(x),\varphi(x))=1$ for all $x\in S$. Also

$\varphi^{-1}(\nu)(x,y)=\nu(\varphi(x),\varphi(y))=\nu(\varphi(y),\varphi(x))=\varphi^{-1}(\nu)(y,x)$ for all $x,y\in S$. Thus $\varphi^{-1}(\nu)$ is reflexive and symmetric. We shall next show that $\varphi^{-1}(\nu)$ is transitive. Consider for any $x,y\in S$,

$$\begin{aligned}(\varphi^{-1}(\nu)\circ\varphi^{-1}(\nu))(x,y) &= \sup_{z\in S} (\varphi^{-1}(\nu)(x,z)\wedge\varphi^{-1}(\nu)(z,y)) \\ &= \sup_{z\in S} (\nu(\varphi(x),\varphi(z))\wedge\nu(\varphi(z),\varphi(y)) \\ &\leq \sup_{z'\in T} (\nu(\varphi(x),z')\wedge\nu(z',\varphi(y))) \\ &= (\nu\circ\nu)(\varphi(x),\varphi(y)) \\ &= \nu(\varphi(x),\varphi(y)) \\ &= \varphi^{-1}(\nu)(x,y)\end{aligned}$$

Therefore, $\varphi^{-1}(\nu)$ is transitive.

Thus $\varphi^{-1}(\nu)$ is a fuzzy equivalence relation. Lastly we show that $\varphi^{-1}(\nu)$ has the "substitution property".

Let $f_\alpha\in F$. For any $x,y\in S$, consider $f_\alpha(\varphi^{-1}(\nu),\varphi^{-1}(\nu),\ldots,\varphi^{-1}(\nu)))(x,y)$

$$= \sup_{\substack{x=f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}) \\ y=f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})}} (\min_{1\leq i\leq n(\alpha)} (\varphi^{-1}(\nu)(x_i,y_i)))$$

$$= \sup_{\substack{x=f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}) \\ y=f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})}} (\min_{1\leq i\leq n(\alpha)} (\nu(\varphi(x_i),\varphi(y_i))) \tag{1}$$

Since $\varphi(x) = \varphi(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})$

$$= f_\alpha(\varphi(x_1),\varphi(x_2),\ldots,\varphi(x_{n(\alpha)}))$$

and similarly $\varphi(y) = f_\alpha(\varphi(y_1),\varphi(y_2),\ldots,\varphi(y_{n(\alpha)}))$ because φ is a homomorphism, we have

$$\text{L.H.S. of } (1) \leq \sup_{\substack{\varphi(x)=f_\alpha(\varphi(x_1),\varphi(x_1),\dots,\varphi(x_{n(\alpha)})) \\ \varphi(y)=f_\alpha(\varphi(y_1),\varphi(y_2),\dots,\varphi(y_{n(\alpha)}))}} (\min_{1\leq i\leq n(\alpha)} \nu(\varphi(x_i),\varphi(y_i)))$$

$$= (f_\alpha(\nu,\nu,\dots,\nu))(\varphi(x),\varphi(y))$$

$$\leq \nu(\varphi(x),\varphi(y))$$

$$= \varphi^{-1}(\nu)(x,y)$$

implying the "substitution property" for $\varphi^{-1}(\nu)$. Thus $\varphi^{-1}(\nu)$ is a fuzzy congruence relation on A.

This completes the propf.

The next proposition deals with the image of a fuzzy congruence relation under a homomorphism.

Proposition 7.4.2: Let φ be a one-one epimorphism of an algebra A onto an algebra B. If μ is a fuzzy congruence relation on A, then $\varphi(\mu)$ is also so on B.

Proof: For any $x'\epsilon B$, $\varphi(\mu)(x',x') = \sup_{x,y\epsilon\varphi^{-1}(x')} \mu(x,y)=1$ since μ is reflexive. So $\varphi(\mu)$ is reflexive on B. It is easy to check that $\varphi(\mu)$ is symmetric. Next we prove that $\varphi(\mu)$ is transitive. For any $x',y'\epsilon B$,

$$(\varphi(\mu)\circ\varphi(\mu))(x',y') = \sup_{z'\epsilon B} (\varphi(\mu)(x',z') \wedge \varphi(\mu)(z',y'))$$

$$= \sup_{z'\in B} \Big(\sup_{\substack{x\in\varphi^{-1}(x')\\ z\in\varphi^{-1}(z')}} (\mu(x,z)) \wedge \sup_{\substack{z\in\varphi^{-1}(z')\\ y\in\varphi^{-1}(y')}} (\mu(z,y))\Big)$$

$$\leq \sup_{z'\in B} \Big(\sup_{\substack{x\in\varphi^{-1}(x')\\ z\in\varphi^{-1}(z')\\ y\in\varphi^{-1}(y')}} (\mu(x,z) \wedge \mu(z,y))\Big) \quad \text{Since } \varphi \text{ is one-one}$$

$$\leq \sup_{\substack{x\in\varphi^{-1}(x')\\ y\in\varphi^{-1}(y')}} \Big(\sup_{z\in A} (\mu(x,z) \wedge \mu(z,y))\Big)$$

$$= \sup_{\substack{x\in\varphi^{-1}(x')\\ y\in\varphi^{-1}(y')}} (\mu(x,y)) = \varphi(\mu)(x,y)$$

This implies that $\varphi(\mu)\circ\varphi(\mu)\leq\varphi(\mu)$.

Finally, we show that $\varphi(\mu)$ has substitution property.

Let $f_\alpha \in F$ and $\mathcal{f}_\alpha$ be the corresponding operation on $\mathcal{R}(B)$.

Consider for any $x',y'\in B$,

$\mathcal{f}_\alpha(\varphi(\mu),\varphi(\mu),\ldots,\varphi(\mu)(x',y')$

$$= \sup_{\substack{x'=f_\alpha(x_1',x_2',\ldots,x_{n(\alpha)}')\\ y'=f_\alpha(y_1',y_2',\ldots,y_{n(\alpha)}')}} \Big(\min_{1\leq i\leq n(\alpha)} (\varphi(\mu)(x_i',y_i'))\Big)$$

$$= \sup_{\substack{x'=f_\alpha(x_1',x_2',\ldots,x_{n(\alpha)}') \\ y'=f_\alpha(y_1',y_2',\ldots,y_{n(\alpha)}')}} (\min_{1\leq i\leq n(\alpha)} (\sup_{\substack{x_i\in\varphi^{-1}(x_i') \\ y_i\in\varphi^{-1}(y_i')}} (\mu(x_i,y_i))))$$

$$\leq \sup_{\substack{x'=f_\alpha(\varphi(x_1),\varphi(x_2),\ldots,\varphi(x_{n(\alpha)})) \\ y'=f_\alpha(\varphi(y_1),\varphi(y_2),\ldots,\varphi(y_{n(\alpha)}))}} (\min_{1\leq i\leq n(\alpha)} (\mu(x_i,y_i)))$$

$$= \sup_{\substack{x'=\varphi(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})) \\ y'=\varphi(f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}))}} (\min_{1\leq i\leq n(\alpha)} (\mu(x_i,y_i)))$$

$$= \mathcal{f}_\alpha(\mu,\mu,\ldots,\mu)(\varphi^{-1}(x'),\varphi^{-1}(y'))$$

$\leq \mu(\varphi^{-1}(x'),\varphi^{-1}(y'))$ since μ is a fuzzy congruence relation

$\leq \varphi(\mu)(x',y')$.

Thus $\varphi(\mu)$ is a fuzzy congruence relation on B.

7.5 Lattice of fuzzy congruence relations

In this section we consider the set of all fuzzy congruence relations on an algebra A. We prove that this set is a complete lattice. Moreover, they form an algebraic closure system in I^{AXA}.

Proposition 7.5.1: Let A = [S,F] be an algebra and let $\mu_j (j \in J)$ be a family of fuzzy congruence relations on A. Then $\bigwedge_{j \in J} \mu_j = \inf_j \mu_j$ is a fuzzy congruence relation on A.

Proof: Recall that from Proposition 5.3.1, if μ_j are fuzzy equivalence relations on a set, then $\wedge \mu_j$ is also so. Since fuzzy congruence relations are fuzzy equivalence relations, we have already $\bigwedge_{j \in J} \mu_j$ is a fuzzy equivalence relation. We need only check that $\bigwedge_{j \in J} \mu_j$ satisfies the substitution property. Let $\bigwedge_{j \in J} \mu_j = \mu$. Consider for any $f_\alpha \in F$, and any two $n(\alpha)$-tuples $(x_1, x_2, \ldots, x_{n(\alpha)})$ and $(y_1, y_2, \ldots, y_{n(\alpha)}) \in S^{n(\alpha)}$,

$$\mu(f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}), f_\alpha(y_1, y_2, \ldots, y_{n(\alpha)}))$$

$$= (\inf \mu_j)(f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}), f_\alpha(y_1, y_2, \ldots, y_{n(\alpha)}))$$

$$= \inf(\mu_j \; (f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)}), f_\alpha(y_1, y_2, \ldots, y_{n(\alpha)}))$$

$$\geq \inf_j (\min_{1 \leq i \leq n(\alpha)} \mu_j(x_i, y_i)) \quad \text{since each } \mu_j \text{ is a fuzzy congruence relation.}$$

$$= \min_{1 \leq i \leq n(\alpha)} (\inf_j \mu_j(x_i, y_i))$$

$$= \min_{1 \leq i \leq n(\alpha)} (\mu(x_i, y_i))$$

Now, by Proposition 7.3.3, it follows that μ satisfies the substitution property.

Let $C(A)$ denote the set of all fuzzy congruence relations on A. By Proposition 3.2.2 and the above Proposition 7.3.3, it follows that $C(A)$ is a complete lattice in which the meet operation is given by the infinimum. We also defined the join operation, that is the union, in the lattice of fuzzy equivalence relation in section 5.3. We shall now show that the join operation in $C(A)$ is the same as the one defined in section 5.3. Let $\Gamma=(\nu_j)_{j\in J}$ be a set of fuzzy congruence relations on A. $\gamma=\bigvee_{j\in J}\nu_j$ was defined as

$$\gamma = \sup_{n=0}^{\infty} \mu_n \quad \text{where}$$

$$\mu_n(x,y) = \sup_{(\nu_1,\nu_2,\ldots,\nu_n)\in\Gamma^n} \left(\sup_{(z_0,z_1,\ldots z_n)\in S^{n+1}} \left(\min_{i=1 \text{ to } n} (\nu_i(z_{i-1},z_i))\right)\right)$$

$$\mu_0(x,y) = \sup_{j\in J} \nu_j$$

where $z_0 = x$, $z_n = y$.

Now if $f_\alpha \in F$ and $x = f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})$

$$y = f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}).$$

<u>Lemma 7.5.2</u>: If μ and ν are two relations on an algebra $A=[S,F]$ satisfying the substitution property, when $\mu\circ\nu$ also satisfies the substitution property on A.

<u>Proof</u>: Let $f_\alpha \in F$ and $x = f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})$,

$$y = f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}).$$

By Proposition 7.3.3, we have to show that

$$\mu\circ\nu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} \mu\circ\nu(x_i,y_i).$$

That is

$$\sup_{z\in S}(\mu(x,z)\wedge\nu(z,y)) \geq \min_{1\leq i\leq n(\alpha)} (\sup_{z\in S}(\mu(x_i,z)\wedge\nu(z,y_i))).$$

Suppose not. Then there exists an $\alpha > 0$ such that

$$\min_{1\leq i\leq n(\alpha)} (\sup_{z\in S}(\mu(x_i,z)\wedge\nu(z,y_i))) > \alpha > \sup_{z\in S}(\mu(x,z)\wedge\nu(z,y)).$$

This implies that for each $1\leq i\leq n(\alpha)$,

$$\sup_{z\in S} (\mu(x_i,z)\wedge\nu(z,y_i)) > \alpha.$$

That is, for each $1\leq i\leq n(\alpha)$ there exists an z_i such that

$\mu(x_i,z_i)\wedge\nu(z_i,y_i) > \alpha$.

Consider $z = f_\alpha(z_1,z_2,\ldots,z_{n(\alpha)})$.

Since μ and ν satisfy the substitution property,

$$\mu(x,z) \geq \min_{1\leq i\leq n(\alpha)} \mu(x_i,z_i) > \alpha$$

and

$$\nu(z,y) \geq \min_{1\leq i\leq n(\alpha)} \nu(z_i,y_i) > \alpha.$$

Hence $\mu(x,z) \wedge \nu(z,y) > \alpha$.

Therefore $\sup_{z\in S} (\mu(x,z)\wedge\nu(z,y)) > \alpha$.

This is a contradiction, and the proof of the Lemma is complete.

Now, we note that μ_1 is built up from a composition of ν_j's taken two at a time. μ_n is built up from a composition of ν_j's taken (n+1) at a time. Since ν_j's are congruence relations, we have

$$\mu_0(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} (\mu_0(x_i,y_i))$$

Using the above Lemma 7.5.2, and an induction on n, gives

$$\mu_n(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} \mu_n(x_i,y_i)$$

for all n=1,2,... .

Now $\gamma = \sup_{n=0}^{\infty} \mu_n$

Therefore $\gamma(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} (x_i,y_i)$.

Thus γ is a fuzzy congruence relation.

<u>Corollary 7.5.3</u>: $C(A)$ is a complete sublattice of $E(A)$, the lattice of all fuzzy equivalence relations.

<u>Lemma 7.5.4</u>: Let $\{\nu_j\}_{j\in J}$ be a directed family of fuzzy congruence relations on A. Then $\sup_{j\in J} \nu_j = \bigvee_{j\in J} \nu_j$ where $\bigvee_{j\in J}$ denotes the union taken in the lattice $E(A)$.

<u>Proof</u>: Since $\mu_0 = \sup_{j\in J} \nu_j$ in the definition of $\bigvee_{j\in J} \nu_j$, $\sup_{j\in J} \nu_j \leq \bigvee_{j\in J} \nu_j$.

For the reverse inequality, we observe that μ_n is built up from a composition of ν_j's taken (n+1) at a time. Since ν_j's are directed, each composition of (n+1) congruence relations is majorised by one of ν_j. That is

$$\mu_n \leq \sup_{j\in J} \nu_j \quad \text{for } n=1,2,\ldots$$

and trivially

$$\mu_0 \leq \sup_{j\in J} \nu_j .$$

That is, $\sup_{n=0}^{\infty} \mu_n \leq \sup_{j\in J} \nu_j$

implying $\bigvee_{j \in J} \nu_j \leq \sup_{j \in J} \nu_j$.

This completes the proof.

Finally, Theorem 3.4.5, together with the above Lemma 7.5.4, implies the following important result.

<u>Theorem 7.5.5:</u> $C(A)$ is an algebraic closure system in $I^{A \times A}$.

CHAPTER 8

ISOMORPHISM THEOREMS OF FUZZY ALGEBRAS

8.1 Introduction

The three isomorphism theorems of groups, rings and other algebraic structures are well-known. They have been generalised and proved in the setting of universal algebras. In this chapter, we examine the relevance of these three theorems in fuzzy algebra. We are able to properly state these three theorems in the context of fuzzy algebras and prove these theorems. In section 8.2, we define fuzzy homomorphisms between algebras. These are different from the extensions of ordinary homomorphisms discussed in Chapter 1. In section 8.3 we discuss the first isomorphism theorem, and in section 8.4, the second and third isomorphism theorems.

8.2 Fuzzy homomorphisms and fuzzy subalgebras

Let A=[S,F] and B=[T,F] be two similar algebras. In section 7.1, we definedthe relational algebra $\mathcal{R}(A,B)$. In this section, we study certain fuzzy subalgebras of $\mathcal{R}(A,B)$. Such objects are called fuzzy homomorphism from A to B. A fuzzy function μ from A to B is said to be a **fuzzy homomorphism** from A to B if and only if for each $f_\alpha \in F$, $f_\alpha(\mu,\mu,\ldots,\mu) \leq \mu$ That is, μ is a fuzzy subalgebra of AxB in the induced operation f_α. If A=B, then a fuzzy homomorphism from A to A is also called a **fuzzy endomorphism**.

For the next proposition we recall the definition in 6.2 of an image of fuzzy subset γ under a fuzzy function μ as $\mu(\gamma)=\sup_{x\in S}(\gamma(x) \wedge \mu(x,y))$.

Proposition 8.2.1: If γ is a fuzzy subalgebra of A and μ is a fuzzy homomorphism from A to B, then the image $\mu(\gamma)$ is a fuzzy subalgebra of B.

Proof: Let $f_\alpha \in F$. We have to show that

$$f_\alpha(\mu(\gamma),\mu(\gamma),\ldots,\mu(\gamma)) \leq \mu(\gamma)$$

Put $\mu(\gamma)=\nu$. Let $x=f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})$ and $y=f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})$

Since μ is a fuzzy homomorphism,

$$\mu(x,y) \geq \min_{1 \leq i \leq n(\alpha)} (\mu(x_i, y_i))$$

and since γ is a fuzzy subalgebra,

$$\gamma(x) \geq \min_{1 \geq i \geq n(\alpha)} \gamma(x_i).$$

Let $\theta < \min_i \nu(y_i)$.

There are $x_i \in S$ with $\gamma(x_i) \wedge \mu(x_i, y_i) > \theta$. If $x = f_\alpha(x_1), \ldots x_{n(\alpha)})$, then

$$\nu(y) \geqq \gamma(x) \wedge \mu(x,y) \geqq [\min_i \gamma(x_i)] \wedge [\min_i \mu(x_i, y_i)] > \theta.$$

This clearly proves that $\nu(y) \geqq \min_i \nu(y_i)$.

By Theorem 2.2.5, we have $\qquad \delta_\alpha(\nu, \nu, \ldots, \nu) \leq \nu$.

That is, ν is a fuzzy subalgebra of B.

Proposition 8.2.2: If ν is a fuzzy subalgebra of B and μ is a fuzzy homomorphism from A to B, then the inverse image $\mu^{-1}(\nu)$ is a fuzzy subalgebra of A.

Proof: Similar to proposition 8.2.1.

8.3 Fuzzy homomorphism theorem and First isomorphism theorem

Given an algebra $A = [S,F]$ and μ a fuzzy congruence relation, S/μ is made into an algebra of fuzzy subsets on A similar to A by defining suitable algebraic operations on S/μ. We also prove that the kernel of a fuzzy homomorphism between two similar algebras is a fuzzy congruence relation on A. Finally, using the above results we establish the first isomorphism theorem of fuzzy algebras.

Let $A = [S,F]$ be an algebra and μ be a fuzzy congruence relation on A. Then S/μ consisting of all fuzzy subsets of S of the form μ_x where $x = [x]_{\underset{\sim}{\omega}_1}$ $x \in S$, can be made into an algebra similar to A as follows. It is denoted by A/μ. Let $f_\alpha \in F$. Consider $\mu_{x_1}, \mu_{x_2}, \ldots \mu_{x_{n(\alpha)}} \in S/\mu$.

Then the algebraic operation f'_α on S/μ is defined as

$$f'_\alpha (\mu_{x_1}, \mu_{x_2}, \ldots \mu_{x_{n(\alpha)}}) = \mu_z \text{ with}$$

$\mu_z(t) = \mu(z,t)$ and z is the class given by $z = [z]_{\underset{\sim}{\omega}_1}$ where

$z = f_\alpha(x_1, x_2, \ldots x_{n(\alpha)})$ with $x_i \in x_i$ for each $i = 1,2,\ldots n(\alpha)$.

We have to verify that this is well-defined.

Suppose $x'_i \in x_i$ for $i = 1,2,\ldots, n(\alpha)$. Then $x_i \overset{\underset{\sim}{\omega}_1}{} x'_i$ for $i = 1,2,\ldots,n(\alpha)$.

As the weak 1-cut of μ is a crisp congruence relation by Proposition 7.3.2, we have $f_\alpha(x_1, x_2, \ldots x_{n(\alpha)}) \underset{\sim}{\omega}_1 f_\alpha (x'_1, x'_2, \ldots x'_{n(\alpha)})$.

That is $z \underset{\sim}{\omega}_1 z'$ where

$z' = f_\alpha(x'_1, x'_2, \ldots x'_{n(\alpha)})$. Hence $z' \in [z]_{\underset{\sim}{\omega}_1}$ and

$\mu_{z'}(t) = \mu(z',t) = \mu(z,t) = \mu_z(t)$.

Thus $A/\mu = [S/\mu, F']$ is an algebra similar to $A = [S,F]$.

Theorem 8.3.1: Let ε be a mapping from A to A/μ defined by $\varepsilon(x) = \mu_{x}$.

Then ε is a homomorphism.

Proof: By definition $\varepsilon(f_\alpha(x_1, x_2, \ldots x_{n(\alpha)})$

$$= \mu_z \text{ where } z = [f_\alpha (x_1, x_2, \ldots, x_{n(\alpha)}]_{\omega}.$$

$$= f_\alpha (\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_{n(\alpha)}})$$

$$= f_\alpha (\varepsilon (x_1), \varepsilon (x_2) \ldots, \varepsilon (x_{n(\alpha)}))$$

Therefore ε is a homomorphism.

Given a fuzzy homomorphism f from an algebra A=[S,F] to B=[T,F], we associate with f the fuzzy equivalence relation μ on A as in section 6.5, known as the kernel of f; that is $\mu = \breve{f} \circ f$.

Proposition 8.3.2: If $f: A \to B$ is a fuzzy homomorphism, then the kernel μ of f is a fuzzy congruence relation on A.

Proof: We know that μ is a fuzzy equivalence relation on A. Therefore it is enough to check that μ satisfies the substitution property. So consider for any $f_\alpha \in F$, two $n(\alpha)$-tuples $(x_1,x_2,\ldots,x_{n(\alpha)})$ and $(y_1,y_2,\ldots,y_{n(\alpha)}) \in S^{n(\alpha)}$.

We have to show that

$$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})) \geq \min_{1 \leq i \leq n(\alpha)} \mu(x_i,y_i).$$

Now

$$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}))$$

$$= \breve{f} \circ f(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}))$$

$$= \sup_{z\in T} (f(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),z)\wedge \breve{f}(z,f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})))$$

$$= \sup_{z\in T} (f(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),z) \wedge f(f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}),z)).$$

Also $\mu(x_i,y_i) = f\circ\breve{f}(x_i,y_i)$

$$= \sup_{z\in T}(f(x_i,z)\wedge \breve{f}(z,y_i))$$

$$\mu(x_i,y_i) = \sup_{z\in T} (f(x_i,z)\wedge f(y_i,z)).$$

Suppose

$$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}) \not\geq \min_{1\leq i\leq n(\alpha)} \mu(x_i,y_i)$$

then there exists a β such that

$$\mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}), f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})) < \beta < \mu(x_i,y_i)$$

for $i=1,2,\ldots,n(\alpha)$.

That is

$$\sup_{z\in T} (f(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)},z)\wedge f(f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}),z)) < \beta$$

and for each $i=1,2,\ldots,n(\alpha)$, $\sup_{z\in T} (f(x_i,z)\wedge f(y_i,z)) > \beta$.

Now there exists a z_i for each $i=1,2,\ldots,n(\alpha)$ such that $f(x_i,z_i) > \beta$ and $f(y_i,z_i) > \beta$.

Let $z=f_\alpha(z_1,z_2,\ldots,z_{n(\alpha)})$. Since f is a fuzzy homomorphism we have

$$f(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),f_\alpha(z_1,z_2,\ldots,z_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} f(x_i,z_i) > \beta.$$

Similarly $f(f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}),f_\alpha(z_1,z_2,\ldots,z_{n(\alpha)})) \geq \min_{1\leq i\leq n(\alpha)} f(y_i,z_i)$.

Therefore $f(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),z) \wedge f(f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}),z) > \beta$

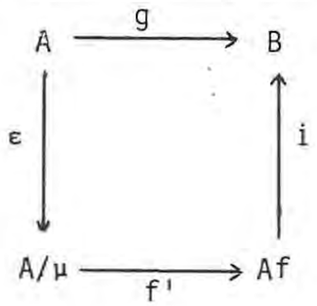
hence

$$\sup_{z\in T} (f(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),z) \wedge f(f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)}),z)) > \beta .$$

This is a contradiction, and the result follows.

Theorem 8.3.3. (First isomorphism theorem):

Suppose $f:A\to B$ is a fuzzy homomorphism from an algebra $A=[S,F]$ to $B=[T,F]$. Let μ be the kernel of f. Then there is a decomposition of f given by the following diagram



where all the symbols have the same meaning as in Theorem 6.5.2.

Proof: Everything is clear from Theorem 6.5.2 and Proposition 8.3.2. So we have only to check that f' is an isomorphism.

$f'(\mu_x) = y$ where

$x = \{x \in S: f(x,y) = 1\}$. f' is clearly bijective and is a homomorphism since $f'(f_\alpha(\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_{n(\alpha)}})) = f_\alpha(f'(\mu_{x_1}), f'(\mu_{x_2}), \ldots, f'(\mu_{x_{n(\alpha)}}))$.

8.4 The Second and Third isomorphism theorems

Remark 8.4.1 The second isomorphism theorem of universal algebra in crisp case states that if $A = [S,F]$ is an algebra, B a subalgebra of A and θ is a congruence relation on A, then $A/\theta \cong B/\theta_B$ provided $[B]\theta = A$ where θ_B is the restriction of θ to B. In fuzzy case, we tried the following.

Let $A = [S,F]$; μ a fuzzy congruence relation and ν a fuzzy subalgebra of A. We need to define the restriction of μ to ν . A natural candidate for such a fuzzy congruence relation is

$$\nu(x,y) = \nu(x) \wedge \nu(y) \wedge \mu(x,y) \quad x, y \in S.$$

Then ν satisfies properties of symmetry, transitivity and substitution but not reflexivity as we defined it. If we define artificially $\nu(x,y) = 1$ if $x = y$ and $\nu(x,y) = \nu(x) \wedge \nu(y) \wedge \mu(x,y)$ if $x \neq y$ then we do have reflexive, symmetric and transitive properties, but the substitution property breaks down. So there seems to be no natural way of defining the restirction of a fuzzy congruence relation to a fuzzy subalgebra. Thus at present we do not have a second isomorphism theorem in the fuzzy case.

Now we turn to the third isomorphism theorem. We recall from Definition 5.2, that a partial ordering "$\leq$" of two fuzzy relations μ and ν on X is given by $\nu \leq \mu$ if and only if $\nu(x,y) \leq \mu(x,y)$ for all $x,y \in X$. Let $A = [S,F]$ be an algebra and let μ,ν be two fuzzy congruence relations on A, with $\nu \leq \mu$. If $\underset{\sim}{\omega}_1(\nu)$ and $\underset{\sim}{\omega}_1(\mu)$ denote the weak 1-equivalence classes of ν and μ respectively, then we observe that $\underset{\sim}{\omega}_1(\nu)$ is a refinement of $\underset{\sim}{\omega}_1(\mu)$. We define below a congruence relation on

$A/\nu = \{\nu_x : x \in \underset{\sim}{\omega}_1(\nu), x \in S\}$.

Definition 8.4.2: $\mu/\nu_x : A/\nu \times A/\nu \longrightarrow I$

$$\mu/\nu(\nu_x,\nu_y) = \mu(x,y) \text{ for } x\in x, y\in y.$$

μ/ν as defined above is called the **quotient fuzzy congruence relation.**

Theorem 8.4.3: μ/ν as defined above is indeed a fuzzy congruence relation on A/ν.

Proof: Firstly, μ/ν is well-defined. For suppose $x,x' \in x$ and $y,y' \in y$. Since $\underset{\sim}{\omega}_1(\nu)$ is a refinement of $\underset{\sim}{\omega}_1(\mu)$, x,x' belong to the same class of the 1-partition associated with μ. Similarly y,y' belong to the same class. By Corollary 5.4.6, $\mu(x,y) = \mu(x',y')$. Thus $\mu/\nu(\nu_x,\nu_y) = \mu(x,y) = \mu(x',y')$.

That μ/ν is reflexive and symmetric, is straightforward to check. We now prove that μ/ν is transitive. Put $\mu/\nu = \gamma$. We have to show that $\gamma\circ\gamma \leq \gamma$.

Consider
$$\begin{aligned}\gamma\circ\gamma(\nu_x,\nu_y) &= \sup_{z\in\underset{\sim}{\omega}_1(\nu)} (\gamma(\nu_x,\nu_z)\wedge\gamma(\nu_z,\nu_y)) \\ &= \sup_{z\in\underset{\sim}{\omega}_1(\nu)} \{(\mu(x,z)\wedge\mu(z,y)) \quad z\in z,\ x\in x, y\in y\} \\ &\leq \sup_{z\in S}(\mu(x,z)\wedge\mu(z,y)) \\ &= \mu\circ\mu(x,y) \\ &\leq \mu(x,y) \\ &= \gamma(\nu_x,\nu_y)\end{aligned}$$

Finally, μ/ν satisfies the substitution property as follows:

Let $f_\alpha \in F$. Then we have to show that for any two $n(\alpha)$-tuples $(\nu_{x_1},\nu_{x_2},\ldots,\nu_{x_{n(\alpha)}})$ and $(\nu_{y_1},\nu_{y_2},\ldots,\nu_{y_{n(\alpha)}})$ belonging to $(A/\nu)^{n(\alpha)}$,

$$\gamma(f_\alpha(\nu_{x_1},\nu_{x_2},\ldots,\nu_{x_{n(\alpha)}}),\ f_\alpha(\nu_{y_1},\nu_{y_2},\ldots,\nu_{y_{n(\alpha)}})) \geq \min_{i=1 \text{ to } n} \gamma(\nu_{x_i},\nu_{y_i}).$$

Suppose not. Then there exists a $\beta > 0$ such that

$\gamma(f_\alpha(\nu_{x_1},\nu_{x_2},\ldots,\nu_{x_{n(\alpha)}}),\ f_\alpha(\nu_{y_1},\nu_{y_2},\ldots,\nu_{y_{n(\alpha)}})) < \beta$ and

$\beta < \gamma(\nu_{x_i},\nu_{y_i})$ for each $i = 1,2,\ldots,n(\alpha)$.

First, we notice that from the definition preceeding Theorem 8.3.1

$$f_\alpha(\nu_{x_1},\nu_{x_2},\ldots,\nu_{x_{n(\alpha)}}) = \nu_x \quad \text{and}$$

$$f_\alpha(\nu_{y_1},\nu_{y_2},\ldots,\nu_{y_{n(\alpha)}}) = \nu_y$$

where $x = [x]_{\underset{\sim}{\omega}_1(\nu)}$ and $y = [y]_{\underset{\sim}{\omega}_1(\nu)}$ with

$x = f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)})$ $\quad x_i \in x_i \quad i=1,2,\ldots,n(\alpha)$

and $y = f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})$ $\quad y_i \in y_i \quad i=1,2,\ldots,n(\alpha)$.

Since $\gamma(\nu_{x_i},\nu_{y_i}) = \mu(x_i,y_i) > \beta$ for each $i=1,2,\ldots,n(\alpha)$,

$$\mu(x,y) = \mu(f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}),\ f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})) \geq \min_{i=1 \text{ to } n} \mu(x_i,y_i)$$

since μ is a fuzzy congruence relation.

That is, $\mu(x,y) > \beta$.

But $\gamma(\nu_x,\nu_y)$ is, by definition $\mu(x,y)$.

Therefore $\gamma(f_\alpha(\nu_{x_1},\nu_{x_2},\ldots,\nu_{x_{n(\alpha)}}),\ f_\alpha(\nu_{y_1},\nu_{y_2},\ldots,\nu_{y_{n(\alpha)}})) > \beta$.

This is a contradiction, and the proof is complete.

<u>Corollary 8.4.4</u>: (Third isomorphism theorem).

With μ and ν as in the Theorem, we have

$$A/\nu_{/\mu/\nu} \cong A/\mu.$$

<u>Proof</u>: Consider $[\nu_x]_{\underset{\sim}{\omega}_1(\mu/\nu)} = \{\nu_y \in A/\nu, \mu/\nu(\nu_x, \nu_y) = 1\}$.

Then by definition of μ/ν in terms of μ, we see that $[\nu_x]_{\underset{\sim}{\omega}_1(\mu/\nu)} \cong x$

as crisp sets. Hence $(\mu/\nu)_{[\nu_x]_{\underset{\sim}{\omega}_1(\mu/\nu)}} \equiv \mu_x$ as fuzzy subsets.

That is $A/\nu_{/\mu/\nu} \cong A/\mu$.

APPENDIX

FUZZY POLYNOMIAL ALGEBRAS AND FREE FUZZY SUBALGEBRAS

These are additional thoughts on free subalgebras and polynomial algebras. The discussion is sketchy, but hopefully leaves ideas for further research.

Let A = [S,F] be an algebra. Let n be a non-negative integer. Fuzzy n-ary polynomials over A are certain mappings from $\underbrace{I^S x I^S x \ldots x I^S}_{n\text{-times}}$ to I^S defined as follows:

Definition A.1: (i) **Elementary fuzzy n-ary polynomials** over A are defined as

$$\varepsilon_i^n : I^S x I^S x \ldots x I^S \rightarrow I^S$$

$$\varepsilon_i^n(\mu_1,\mu_2,\ldots,\mu_n) = \mu \quad \text{for each } i = 1,2,\ldots,n(\alpha)$$

where $\mu(x) = \sup\limits_{\substack{(x_1,x_2,\ldots,x_n) \\ x_i=x}} (\min\limits_{i=1 \text{ to } n} (\mu_i(x_i))$, the supremum being taken over all n-tuples $(x_1,x_2,\ldots,x_n) \in S^n$ with x_i=x at the i^{th} place.

(ii) If $\pi_1,\pi_2,\ldots,\pi_{n(\alpha)}$ are fuzzy n-ary polynomials over A, then so is $f_\alpha(\pi_1,\pi_2,\ldots,\pi_{n(\alpha)})$ defined by

$$(f_\alpha(\pi_1,\pi_2,\ldots,\pi_{n(\alpha)}))(\mu_1,\mu_2,\ldots\mu_n) = f_\alpha(\pi\ (\mu_1,\mu_2,\ldots,\mu_n),\ldots,\pi_{n(\alpha)}(\mu_1,\mu_2,\ldots,\mu_n))$$

(iii) Fuzzy n-ary polynomials are those and only those which we get from (i) and (ii) in a finite number of steps.

Remark A.2: Every fuzzy nullary polynomial is a fuzzy n-ary polynomial for every n. If $f : S \to S$ is a nullary operation the fuzzy nullary polynomial is defined as $f : I^S \to I^S$, with $\mu \to f(\mu)$ where $f(\mu)(x) = \sup_{x \in S} \mu(x)$.

Example A.3: Let (G,.) be a group. An example of an elementary fuzzy binary polynomial is, for $\mu_1, \mu_2 \in I^G$

$$\varepsilon_1^2(\mu_1, \mu_2) = \mu$$

where

$$\mu(x) = \sup_{\substack{(x_1,x_2) \\ x_1 = x}} (\mu_1(x) \wedge \mu_2(x_2))$$

$$= \sup_{x_2} (\mu_1(x) \wedge \mu_2(x_2))$$

$$= \sup_{y \in G} (\mu_1(x) \wedge \mu_2(y))$$

Example A.4: Let $(L, \wedge, V)$ be a lattice.

$$\varepsilon_1^1(\mu_1) = \mu \text{ where } \mu(x) = \sup_{x_1 = x} \mu_1(x_1) = \mu_1(x)$$

$$(\varepsilon_1^1 V \varepsilon_1^1)(\mu_1)(x) = \sup_{x = x_1 V x_1} \mu_1(x_1) = \sup_{x_1 = x} \mu_1(x_1) = \mu_1(x)$$

and other fuzzy binary polynomials are defined similarly.

The collection of all fuzzy n-ary polynomials over A is denoted by $P^{(n)}(A)$. The n-ary operations on $P^{(n)}(A)$ are defined as in I-1(ii). Then $[P^{(n)}(A), \mathbb{F}]$ is an algebra. It is denoted by $\Pi^{(n)}(A)$ and is called the algebra of fuzzy n-ary polynomials over A.

Let A = [S,F] and B = [T,F] be two similar algebras. Since the fuzzy n-ary polynomials over B are built up in the same way as the fuzzy n-ary polynomials over A, we can use certain symbols for n-ary polynomial symbols and use them to generate the fuzzy n-ary polynomials over A or B or any other similar algebra. Thus we define polynomial symbols of certain types as follows.

Definition A.5: (i) $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are n-ary polynomial symbols.

(ii) $\pi_1, \pi_2, \ldots, \pi_{n(\alpha)}$ are n-ary polynomial symbols and $f_\alpha \in F$, then $f_\alpha(\pi_1, \pi_2, \ldots \pi_{n(\alpha)})$ is also a n-ary polynomial symbol.

(iii) n-ary polynomial symbols are those and only those which we get from (i) and (ii) in a finite number of steps.

Now, we connect the notion of n-ary polynomial symbols as defined above with the earlier notion of fuzzy n-ary polynomials over an algebra A by the following:

Definition A.6: The fuzzy n-ary polynomial π over an algebra A associated with or induced by the n-ary polynomial symbol π is defined as follows:

(i) ε_i induces ε_i^n, i=1,2,...,n on A.

(ii) If $\pi = f_\alpha(\pi_1, \pi_2, \ldots, \pi_{n(\alpha)})$ and π_i induces π_i for i=1,2,...,n

then π induces $f_\alpha(\pi_1, \pi_2, \ldots, \pi_{n(\alpha)})$.

The following Proposition describes the action of fuzzy n-ary polynomials on fuzzy points of A.

Proposition A.7: Let $x_1^{r_1}, x_1^{r_2}, \ldots, x_n^{r_n} \in \mathbb{F}\mathbb{P}(S)$ then

$$\varepsilon_i^n(x_1^{r_1}, x_2^{r_2}, \ldots, x_n^{r_n}) = x_i^r \quad \text{for } i=1,2,\ldots,n$$

where $r = \bigwedge_{i=1}^{n} r_i$.

Also $\pi(x_1^{r_1}, x_2^{r_2}, \ldots, x_n^{r_n}) = x_\pi^r$

where $x_\pi = \pi(x_1, x_2, \ldots, x_n)$

$$r = \bigwedge_{i=1}^{n} r_i$$

Proof: Since $\varepsilon_1^n(\mu_1, \mu_2, \ldots, \mu_n) = \mu$ where

$$\mu(x) = \sup_{\substack{(x_1,x_2,\ldots,x_n) \\ x_i = x}} \left(\min_{i=1 \text{ to } n} (\mu_i(x_i))\right)$$

$$= \begin{cases} 0 & \text{if } x_i \neq x \\ \min_{i=1 \text{ to } n} r_i & x_i = x \end{cases}$$

Therefore $\mu(x)$ is the fuzzy point x_i^r.

That is $\varepsilon_i^n(x_1^{r_1}, x_2^{r_2}, \ldots, x_n^{r_n}) = x_i^r$.

The other assertion follows similarly.

More generally if $\pi_1, \pi_2, \ldots, \pi_{n(\alpha)}$ are fuzzy n-ary polynomials over A, and $f_\alpha \in F$, then the action of $\mathcal{f}_\alpha(\pi_1, \pi_2, \ldots, \pi_{n(\alpha)})$ on the n-tuple of the fuzzy point $(x_1^{r_1}, x_2^{r_2}, \ldots, x_n^{r_n})$ is given by

$$\mathcal{f}_\alpha(\pi_1, \pi_2, \ldots, \pi_{n(\alpha)})(x_1^{r_1}, x_2^{r_2}, \ldots, x_n^{r_n}) = x^r$$

where $x = f_\alpha(\pi_1(x_1,x_2,\ldots,x_n),\ \pi_2(x_1,x_2,\ldots,x_n),\ldots,\pi_n(x_1,x_2,\ldots,x_n))$

$$r = \bigwedge_{i=1}^{n} r_i .$$

Since a fuzzy subset μ can be thought of as $\bigvee_{x\in X} \mu_x^{\mu(x)}$, we can restrict ourselves to fuzzy n-ary polynomials' actions on fuzzy points only.

If $K(\tau)$ is a species of similar algebras of certain type τ, then the n-ary polynomial algebra of type τ is $\Pi^{(n)}(\tau) = [P^{(n)}(\tau), F]$ where $P^{(n)}(\tau)$ consists of all n-ary polynomial symbols and F consists of operations f_α on $P^{(n)}(\tau)$ as defined in A.5(ii).

Definition A.8: Let $K(\tau)$ be a species of algebra. Let $A = [S,F] \in K$. Let $X \subseteq S$ be a free generating set of A. That is, A is a free algebra over X in K. Let $\mu : X \to I$ be a fuzzy subset of X. The fuzzy subalgebra of A generated by μ is called a **free fuzzy subalgebra** in K.

Remark A.9: We have proposed the above definition with the idea that a fuzzy subalgebra of a free algebra represents in some sense the most general fuzzy subalgebra one can generate from a given fuzzy subset. We do not know whether such a definition proves to be the right one from the point of view of universal property of category theory. Also we do not know at present whether there are other suitable definitions of "free algebra" concept in fuzzy algebras. Also, we conjecture that a suitable fuzzy congruence relation $\mu(\tau)$ can be defined on $\Pi^{(n)}(\tau)$ such that if a free fuzzy subalgebra exists in K, then it is isomorphic to the quotient $\Pi^{(n)}(\tau)/\mu(\tau)$. Further research is necessary to study thoroughly the actions which polynomial symbols have on a fuzzy algebra and to study fuzzy algebras of varieties or equational classes of algebras.

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