Kochen-Specker theorem for von Neumann algebras

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Abstract

The Kochen-Specker theorem has been discussed intensely ever since its original proof in 1967. It is one of the central no-go theorems of quantum theory, showing the non-existence of a certain kind of hidden states models. In this paper, we first offer a new, non-combinatorial proof for quantum systems with a type I_n factor as algebra of observables, including I_∞. Afterwards, we give a proof of the Kochen-Specker theorem for an arbitrary von Neumann algebra $\mathcal{R}$ without summands of types I_1 and I_2, using a known result on two-valued measures on the projection lattice $\mathcal{P}(\mathcal{R})$. Some connections with presheaf formulations as proposed by Isham and Butterfield are made.

1 Introduction

In quantum theory, including quantum mechanics in the von Neumann representation, quantum field theory and quantum information theory, observables are represented by self-adjoint operators A in some von Neumann algebra $\mathcal{R}$, the *algebra of observables*. The algebra $\mathcal{R}$ is contained in $\mathcal{L}(\mathcal{H})$, the set of bounded linear operators on some separable Hilbert space $\mathcal{H}$. The self-adjoint operators $\mathcal{R}_{sa}$ form a real linear space in the algebra $\mathcal{R}$. It is important for the interpretation of quantum theory to see if there is a possibility to assign a *value* to each observable such that (i) an observable $A \in \mathcal{R}_{sa}$ is assigned one element of its spectrum and (ii) if for two observables $A, B \in \mathcal{R}_{sa}$ one has $B = g(A)$ for some (Borel) function g, then the value assigned to B, say b, is given as $g(a)$, where a is the value assigned to A. If this were possible, one could imagine to build some realist model of the quantum world where all observables have definite values, like in classical mechanics.

The first condition, namely that each observable should be assigned one of its spectral values, is quite obvious. The second condition implements the fact that the observables are not all independent. In fact, for every abelian von Neumann algebra $\mathcal{M}$ (think of some abelian subalgebra of $\mathcal{R}$), there is a self-adjoint operator A generating $\mathcal{M}$, i.e. $\mathcal{M} = \{A, I\}''$. This was already proved by von Neumann ([vNeu32]). For a modern reference, see Thm. III.1.21 in [TakI79]. Every operator $B \in \mathcal{M}$ is a Borel function of A, $B = g(A)$. One has $g(\mathrm{sp}\, A) \subseteq \mathrm{sp}\, B$ (see Ch. 5.2 in [KadRinI97]). If g is continuous or if $\mathcal{H}$ is finite-dimensional, equality holds. It is natural to demand that the spectral value b assigned to B is given as $g(a)$, where a is the value assigned to A. This condition is often called the *FUNC principle*.

Kochen and Specker started from a related question in their classical paper [KocSpe67]: is there a space of *hidden states*? A hidden state ψ would be given by a probability measure μ_ψ on a generalized quantum mechanical phase space Ω, such that an observable A is given as a mapping

$$f_A : \Omega \longrightarrow \mathbb{R},$$

a *hidden variable*. When the system is in the hidden state ψ and the observable A is measured, the probability to find a value r lying in the Borel set U is required to be

$$P_{A,\psi}(U) = \mu_\psi(f_A^{-1}(U)). \tag{1}$$

Furthermore, the expectation value $E_\psi(A)$ of A when the system is in the hidden state ψ is required to be

$$E_\psi(A) = \int_\Omega f_a(\omega) d\mu_\psi(\omega).$$

Kochen and Specker demonstrate that it is trivial to construct such a generalized phase space Ω if functional relations between the observables are neglected, but the problem really starts when one takes these relations into account. If $B = g(A)$ for some observables $A, B \in \mathcal{R}_{sa}$ and a Borel function g, one should have

$$f_B = f_{g(A)} = g \circ f_A. \tag{2}$$

This simply translates the functional relation between the operators A and B into the corresponding relation between the hidden variables f_A and f_B. Since $B = g(A)$ can only be if A and B commute and since every abelian von Neumann algebra is generated by a self-adjoint operator, Kochen and Specker go on to introduce *partial algebras*, where algebraic relations are defined exclusively between observables that are *commeasurable*. If one regards a von Neumann algebra $\mathcal{R}$ as the algebra of observables, as we do here, $\mathcal{R}$ is a partial algebra in an obvious way: one just keeps the algebraic relations between commuting operators and neglects the algebraic relations between non-commuting operators, since at first sight (2) is a condition on commuting operators only and those are commeasurable. This point seems important, because a hidden variable no-go theorem by J. von Neumann ([vNeu32]) has been criticized (see e.g. [Bell66]) for the fact that von Neumann required additivity to be preserved even between non-commuting operators, which does not seem adequate in the light of (2). However, in fact (2) does *not* just pose conditions on commuting, but also on non-commuting operators. The reason is that typically an observable B is given as a function $B = g(A) = h(C)$ of non-commuting observables $A, C \in \mathcal{R}_{sa}$. We will see that in this way (2) becomes a very strong condition, ruling out hidden states models of the kind described above.

An abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$ is often called a *context* in the physics literature. The self-adjoint operators $\mathcal{M}_{sa}$ in a maximal context $\mathcal{M}$ form a maximal set of commeasurable observables. Condition (2) seems to be a condition within each context solely, but in fact it is a condition "across contexts", because each observable B typically is contained in many contexts.

The elements of the hypothetical generalized phase space Ω would be generalized pure states. In a slight abuse of language, Ω is also called the space of hidden states. If one assumes that there is some space Ω of hidden states, such that (1) and (2) are satisfied and that there is an embedding $f : \mathcal{R}_{sa} \to \mathbb{R}^{\Omega}$ of the quantum mechanical observables into the mappings from Ω to $\mathbb{R}$, one would have a lot of valuations as described above, assigning a spectral value to each observable and preserving functional relations: every point $\omega \in \Omega$ defines such a valuation v by

$$v(A) := f_A(\omega).$$

Demonstrating that there are no such valuations (Kochen and Specker called them *prediction functions*) thus shows that there is no space Ω of hidden states as described above. More directly, the non-existence of valuation functions means that no realist interpretation of quantum mechanics is possible which assumes that all the observables have definite values at the same time.

It is a funny fact that in spite of many references to it, there seems to be no single result called *the* Kochen-Specker theorem. Above, we tried to lay out (very roughly, admittedly) the train of thought in [KocSpe67], and it seems sensible to spell out the Kochen-Specker theorem as follows:

Kochen-Specker theorem: Let $\mathcal{R} \simeq \mathcal{L}(\mathcal{H}),\ \dim \mathcal{H} \geq 3$ be the algebra of observables of some quantum system ($\mathcal{R}$ is a type I_n factor, $n = \dim \mathcal{H}$). There is no space Ω of hidden states such that (1) and (2) are satisfied, i.e. there is no realist phase space model of quantum theory assigning spectral values to all observables at once, preserving functional relations between them.

It is not obvious at first sight if the Kochen-Specker theorem holds for more general von Neumann algebras $\mathcal{R}$, since each $\mathcal{R}$ that is not a type I_n factor is properly contained in some $\mathcal{L}(\mathcal{H})$, and so there are less conditions (encoded in the FUNC principle) than for $\mathcal{L}(\mathcal{H})$ itself. This might lead to speculation if some hidden states, realist model of quantum systems with an observable algebra $\mathcal{R}$ other than a type I_n factor exists. A necessary condition would be the existence of a valuation function $v : \mathcal{R}_{sa} \rightarrow \mathbb{R}$.

To prove the non-existence of valuation functions, Kochen and Specker ([KocSpe67]) concentrate on the projections $\mathcal{P}(\mathcal{R})$ of $\mathcal{R}$. These form a partial Boolean algebra. A valuation function v can assign 0 or 1 to a projection E, since $\operatorname{sp} E = \{0, 1\}$. Kochen and Specker examine if the partial Boolean algebra $\mathcal{P}(\mathcal{R})$ can be embedded into a Boolean algebra. The existence of such an embedding is a necessary condition for the existence of a valuation function. For the case $\mathcal{H} = \mathbb{R}^3, \mathcal{R} = \mathbb{M}_n(3)$, they construct a finitely generated subalgebra $D \subset \mathcal{R}$ that cannot be embedded into a Boolean algebra, thus showing that there is no valuation function in this case. Kochen and Specker use 117 vectors in their construction, corresponding to 117 projections onto one-dimensional subspaces. Later on, this number could be reduced to 33 by A. Peres and 31 by Conway and Kochen, see [Per93] and references therein. The proofs are combinatorial in nature, giving a counterexample.

In this paper, we will use another approach. Let $\mathcal{R}$ be a von Neumann algebra. Assuming the existence of a valuation function $v : \mathcal{R}_{sa} \rightarrow \mathbb{R}$ (see Def. 3 below), we show that v induces a so-called *quasi-state* $v' : \mathcal{R} \rightarrow \mathbb{C}$ (see Def. 7) such that $v'|_{\mathcal{R}_{sa}} = v$. This quasi-state is a pure state of every abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$. Restricting v' to the projections $\mathcal{P}(\mathcal{R})$, we obtain a finitely additive probability measure on $\mathcal{P}(\mathcal{R})$.

If $\mathcal{R}$ is a type I_n factor ($n \in \{3, 4, ...\}$), Gleason's theorem ([Gle57]) shows that v' is a state of $\mathcal{R}$ of the form $v'(\_) = tr(\rho\_)$. But such a state does *not* assign 0 or 1 to every projection, hence we have a contradiction of one of the defining conditions of the valuation function v. The case of a type I_∞ factor can be treated easily.

For more general von Neumann algebras, another proof is presented. Using the Gleason-Christensen-Yeadon theorem (Thm. 15), a generalization of Gleason's theorem, we again show that v' is a state if $\mathcal{R}$ has no summand of type I_2. Hamhalter showed in [Ham93] that every finitely additive two-valued probability measure $\mu : \mathcal{P}(\mathcal{R}) \to \{0, 1\}$ gives rise to a *multiplicative* state of $\mathcal{R}$. Since $v'|_{\mathcal{P}(\mathcal{R})}$ is of this kind, a valuation function v induces a multiplicative state v'. If a von Neumann algebra $\mathcal{R}$ contains no summand of type I_1, then there are no multiplicative states of $\mathcal{R}$, so there is no valuation function for a von Neumann algebra $\mathcal{R}$ without summands of types I_1 and I_2. Thus, the generalized Kochen-Specker theorem holds for all von Neumann algebras $\mathcal{R}$ without summands of types I_1 and I_2.

In section 3, we give two different reformulations of the generalized Kochen-Specker theorem in the language of presheafs. For $\mathcal{R} = \mathcal{L}(\mathcal{H})$, this has been proposed by Isham, Butterfield and Hamilton ([IshBut98, IshBut99, HIB00, IshBut02]). They observed that the FUNC principle means that certain presheafs on small categories have global sections.

Our first presheaf formulation of the Kochen-Specker theorem, generalizing the category and presheaf chosen in [HIB00], is closely related to our first proof (subsections 2.1, 2.2). The second formulation uses another presheaf which does have global sections: each state of the von Neumann algebra $\mathcal{R}$ induces one. However, the Kochen-Specker theorem means that there are no global sections of the kind a valuation function would induce, giving a pure state of every abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$.

2 The new proofs

2.1 Valuation functions and quasi-states

All von Neumann and C^*-algebras treated here are unital subalgebras of some $\mathcal{L}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space $\mathcal{H}$. $\mathcal{R}_{sa}$ denotes the real linear space of self-adjoint elements of a C^*- or von Neumann algebra $\mathcal{R}$, $\mathcal{P}(\mathcal{R})$ is the lattice of projections of $\mathcal{R}$.

Definition 1 *Let $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. A* ***finitely additive probability measure*** *μ is a mapping from $\mathcal{P}(\mathcal{R})$ to $\mathbb{R}$ such that*

(M1) $\forall E \in \mathcal{P}(\mathcal{R}) : 0 \leq \mu(E) \leq 1$ and $\mu(I) = 1$,

(M2) If $E, F \in \mathcal{P}(\mathcal{R})$ such that $EF = 0$, then $\mu(E \vee F) = \mu(E) + \mu(F)$.

If in addition to (M2) one of the stronger conditions

(M2σ) $\mu(\bigvee_{n \in I} P_n) = \sum_{n \in I} \mu(P_n)$ for every countable family $\{P_n\}_{n \in I}$ of orthogonal projections in $\mathcal{P}(\mathcal{R})$,

(M2c) $\mu(\bigvee_{j\in J} P_j) = \sum_{j\in J} \mu(P_j)$ *for every family* $\{P_j\}_{j\in J}$
of orthogonal projections in $\mathcal{P}(\mathcal{R})$

holds, then μ *is called a* ***σ-additive (countably additive)*** *or a* ***completely additive probability measure****, respectively.*

Every *normal* state $\phi : \mathcal{R} \to \mathbb{C}$ is of the form $\phi(\_) = tr(\rho\_)$ for some positive trace class operator of trace 1, see Thm. 7.1.12 in [KadRinII97]. Such a normal state induces a completely additive probability measure by restriction to $\mathcal{P}(\mathcal{R})$. For type I factors, the converse is also true, as Gleason showed in his classical paper [Gle57]. For ease of reference, we cite Gleason's theorem:

Theorem 2 *(Gleason 1957) Let* $\mathcal{R}$ *be a type* I_n *factor,* $n \in \{3, 4, ...\}$, $\mathcal{R} \simeq \mathcal{L}(\mathcal{H})$, $\dim \mathcal{H} = n$*, and let* μ *be a finitely additive probability measure on* $\mathcal{P}(\mathcal{R})$*. There is some positive trace class operator* ρ *of trace* 1 *such that*

$$\forall E \in \mathcal{P}(\mathcal{R}) : \mu(E) = tr(\rho E). \tag{1}$$

If $\mathcal{H}$ *is infinite-dimensional and separable,* μ *is* σ*-additive and* $\mathcal{R}$ *is isomorphic to the type* I_∞ *factor* $\mathcal{L}(\mathcal{H})$*, then there is some positive trace class operator* ρ *of trace* 1 *such that (1) holds.*

If $\mathcal{H}$ *is an arbitrary infinite-dimensional Hilbert space (possibly non-separable),* μ *is completely additive and* $\mathcal{R}$ *is isomorphic to the type* I_∞ *factor* $\mathcal{L}(\mathcal{H})$*, then there is some positive trace class operator* ρ *of trace* 1 *such that (1) holds. (For this partial result, see Thm. 2.3 in [Mae89].)*

This classifies the probability measures on the projection lattices of type I factors. In particular, they all come from normal states of the form $tr(\rho\_)$.

From now on, we will assume that $\mathcal{H}$ is separable.

We now give the precise definition of a valuation function, which is the starting point for the proof of the Kochen-Specker theorem.

Definition 3 *Let* $\mathcal{H}$ *be a Hilbert space,* $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ *a von Neumann algebra. A* ***valuation function*** *is a mapping* $v : \mathcal{R}_{sa} \to \mathbb{R}$ *such that*
(a) $v(A) \in \operatorname{sp} A$ *and*
(b) for all Borel functions $f : \mathbb{R} \to \mathbb{R}$*, one has* $v(f(A)) = f(v(A))$*.*

Kochen and Specker call this a *prediction function*, see [KocSpe67]. $v(I) = 1$ and $v(0) = 0$ follow. Condition (a) is often called the **Spectrum rule**, condition (b) is the **FUNC principle**.

Definition 4 *Let* $v : \mathcal{R}_{sa} \to \mathbb{R}$ *be a valuation function. We extend* v *in a canonical manner to a function*

$$v' : \mathcal{R} \longrightarrow \mathbb{C},$$
$$B = A_1 + iA_2 \longmapsto v(A_1) + iv(A_2),$$

where $B = A_1 + iA_2$ is the unique decomposition of B into self-adjoint operators $A_1, A_2 \in \mathcal{R}_{sa}$.

Obviously, $v'(A) = v(A)$ for a self-adjoint operator $A \in \mathcal{R}_{sa}$. This will be used throughout.

Lemma 5 *Let $g : \mathbb{R} \to \mathbb{C}$ be a Borel function, $g_r : \mathbb{R} \to \mathbb{R}$ its real part, $g_i : \mathbb{R} \to \mathbb{R}$ its imaginary part, $g = g_r + ig_i$. Thus g acts on $a \in \mathbb{R}$ as*

$$g(a) = g_r(a) + ig_i(a)$$

and on self-adjoint operators A as

$$g(A) = g_r(A) + ig_i(A).$$

Let $v : \mathcal{R}_{sa} \to \mathbb{R}$ be a valuation function, $v' : \mathcal{R} \to \mathbb{C}$ its extension. Then $v'(g(A)) = g(v'(A))$ holds for all self-adjoint operators $A \in \mathcal{R}_{sa}$.

Proof. One has

$$\begin{aligned} v'(g(A)) &= v'(g_r(A) + ig_i(A)) \\ &= v(g_r(A)) + iv(g_i(A)) \\ &= g_r(v(A)) + ig_i(v(A)) \\ &= g(v(A)) \\ &= g(v'(A)). \end{aligned}$$

■

Lemma 6 *If $v : \mathcal{R}_{sa} \to \mathbb{R}$ is a valuation function and $\mathcal{M} \subseteq \mathcal{R}$ is an abelian von Neumann subalgebra, then $v'|_{\mathcal{M}}$ is a character of $\mathcal{M}$. $v|_{\mathcal{M}_{sa}}$ is a real-valued, $\mathbb{R}$-linear, linear functional .*

Proof. Let $A \in \mathcal{M}_{sa}$ be a self-adjoint operator that generates $\mathcal{M}$, i.e. $\mathcal{M} = \{A, I\}''$ (see [TakI79, Prop. III.1.21, p. 112]). All operators $B, C \in \mathcal{M}$ are Borel functions of A:

$$B = f(A), \quad C = g(A),$$

where $f, g : \mathbb{R} \to \mathbb{C}$ are Borel functions on $\operatorname{sp} A \subseteq \mathbb{R}$. Since $B + C \in \mathcal{M}$, there also is a Borel function $h : \mathbb{R} \to \mathbb{C}$ such that $B + C = f(A) + g(A) =: h(A)$ and hence

$$\begin{aligned} v'(B + C) &= v'(f(A) + g(A)) \\ &= v'(h(A)) \\ &= h(v'(A)) \\ &= f(v'(A)) + g(v'(A)) \\ &= v'(f(A)) + v'(g(A)) \\ &= v'(B) + v'(C). \end{aligned}$$

Analogously for $BC = CB$: there is a Borel function $j : \mathbb{R} \to \mathbb{C}$ such that $BC = f(A)g(A) =: j(A)$ and hence

$$\begin{aligned} v'(BC) &= v'(f(A)g(A)) \\ &= v'(j(A)) \\ &= j(v'(A)) \\ &= f(v'(A))g(v'(A)) \\ &= v'(f(A))v'(g(A)) \\ &= v'(B)v'(C). \end{aligned}$$

The $\mathbb{C}$-linearity of $v'|_{\mathcal{M}}$ is obvious. Let $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} v'(\alpha B) &= v'(\alpha f(A)) \\ &= v'(k(A)) \\ &= k(v'(A)) \\ &= \alpha f(v'(A)) \\ &= \alpha v'(f(A)) \\ &= \alpha v'(B), \end{aligned}$$

where $k := M_\alpha \circ f$. This shows that $v'|_{\mathcal{M}}$ is a character of $\mathcal{M}$. Restricting to $\mathcal{M}_{sa}$, one obtains a real-valued, $\mathbb{R}$-linear, linear functional. ∎

A character (multiplicative linear functional) of an abelian C^*-algebra $\mathcal{M}$ is a pure state of $\mathcal{M}$. So the above lemma shows that a valuation function induces a pure state on every abelian subalgebra $\mathcal{M} \subseteq \mathcal{R}$, which is exactly what one would expect from a physical point of view. Lemma 6 is closely related to the *sum rule* and the *product rule* first described in [Fin74], see also [Red87].

J. F. Aarnes has introduced the notion of a quasi-state on a C^*-algebra in his paper [Aar69]:

Definition 7 *Let $\mathcal{A}$ be a unital C^*-algebra. A* ***quasi-state*** *of $\mathcal{A}$ is a functional ρ satisfying the following three conditions:*

(1) For each $B \in \mathcal{A}_{sa}$, ρ is linear and positive on the abelian C^*-subalgebra $\mathcal{A}_B \subseteq \mathcal{A}$ generated by B and I.

(2) If $C = A_1 + iA_2$ for self-adjoint $A_1, A_2 \in \mathcal{A}_{sa}$, then $\rho(C) = \rho(A_1) + i\rho(A_2)$.

(3) $\rho(I) = 1$.

Lemma 8 *v' is a quasi-state.*

Proof. (1) v' is linear on every abelian subalgebra $\mathcal{M} \subseteq \mathcal{R}$. v' is positive on each such $\mathcal{M}$ (and on the whole of $\mathcal{R}$), since a positive operator B^*B is assigned some element of its spectrum, $v'(B^*B) = v(B^*B) \in sp(B^*B)$.

(2) For $B = A_1 + iA_2$ ($A_1, A_2 \in \mathcal{R}_{sa}$) one has

$$v'(B) = v(A_1) + iv(A_2) = v'(A_1) + iv'(A_2),$$

the former because of the definition of v', the latter because $v'(A) = v(A)$ for $A \in \mathcal{R}_{sa}$.

(3) $v'(I) = 1$ holds.

Thus v' is a quasi-state. ■

A quasi-state of an abelian von Neumann algebra is a state. This follows from the fact that an abelian von Neumann algebra on a separable Hilbert space is generated by an operator A ([TakI79, Prop. III.1.21]), a result we already used in Lemma 6.

It is easy to see that a quasi-state on an arbitrary von Neumann algebra $\mathcal{R}$, when restricted to the lattice of projections $\mathcal{P}(\mathcal{R})$, gives a finitely additive probability measure. We follow the proof given in [Mae89, Cor. 7.9, p. 264]:

Lemma 9 *If ρ is a quasi-state of a von Neumann algebra $\mathcal{R}$, then $\rho|_{\mathcal{P}(\mathcal{R})}$ is a finitely additive probability measure $\mathcal{P}(\mathcal{R}) \to [0,1]$.*

Proof. For $E \in \mathcal{P}(\mathcal{R})$, we have $0 = \rho(0) \leq \rho(E) \leq \rho(I) = 1$, since ρ is positive on $\{E, I\}''$. If $EF = 0$ for $E, F \in \mathcal{P}(\mathcal{R})$, then $\mathcal{M} := \{E, F, I\}'' \subseteq \mathcal{R}$ is abelian and $\rho|_{\mathcal{M}}$ is a state, in particular, it is additive. Hence,

$$\rho(E \vee F) = \rho(E + F) = \rho(E) + \rho(F).$$

■

To clarify the relation between normal states and valuation functions, we will need the following fact (Lemma 6.5.6 in [KadRinII97]):

Lemma 10 *Let $\mathcal{R}$ be a von Neumann algebra with no central portion of type I (equivalently, with no non-zero abelian projections), and let $E \in \mathcal{P}(\mathcal{R})$. For each positive integer n, there are n equivalent orthogonal projections with sum E.*

Lemma 11 *Let $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ be a von Neumann algebra of type I_n, $n \in \{2, 3, ...\} \cup \{\infty\}$, or of type II or III, and let ϕ be a normal state of $\mathcal{R}$. There is a projection $E \in \mathcal{P}(\mathcal{R})$ such that $\phi(E) \notin \{0, 1\} = \operatorname{sp} E$.*

Proof. Since ϕ is a normal state, it is weakly continuous and of the form

$$\phi(\_) = tr(\rho\_)$$

for some positive trace class operator ρ of trace 1, see [KadRinII97, Thm. 7.1.12].

We assume that $\phi(E) = tr(\rho E) \in \{0,1\}$ holds for all $E \in \mathcal{P}(\mathcal{R})$. Let $\{e_k\}_{k\in K}$ be an orthonormal basis of $\mathcal{H}$ that is adapted to E, i.e. for all k, $e_k \in \operatorname{im} E \cup \operatorname{im}(I-E)$. Let $K_E := \{k \in K \mid e_k \in \operatorname{im} E\}$. If $tr(\rho E) = 1$, we have

$$\begin{aligned} 1 &= \sum_k \langle \rho e_k, E e_k \rangle \\ &= \sum_{k \in K_E} \langle \rho e_k, e_k \rangle . \end{aligned}$$

Since $\langle \rho e_k, e_k \rangle \geq 0$ for all $k \in K$ and $tr\rho = 1$, we see that $\langle \rho e_k, e_k \rangle = 0$ for all $k \in K \backslash K_E$, and hence $\rho e_k = 0$ for all $k \in K \backslash K_E$. Therefore, $\rho(I-E) = 0$, i.e. $\rho = \rho E$ and

$$\rho E = \rho = \rho^* = E\rho.$$

If $tr(\rho E) = 0$, we have $tr(\rho(I-E)) = 1$, since $tr(\rho I) = 1 = tr(\rho E) + tr(\rho(I-E))$. It follows that $\rho(I-E) = (I-E)\rho$ and thus $\rho E = E\rho$ in this case, too. Since a von Neumann algebra is generated by its projections, we obtain

$$(\forall E \in \mathcal{P}(\mathcal{R}) : tr(\rho E) \in \{0,1\}) \Longrightarrow \rho \in \mathcal{R}',$$

where $\mathcal{R}'$ is the commutant of $\mathcal{R}$. Now let $\theta \in \mathcal{R}$ be a partial isometry such that $\theta^*\theta = E$ and $F := \theta\theta^*$. One has

$$tr(\rho E) = tr(\rho\theta^*\theta) = tr(\theta\rho\theta^*) = tr(\theta\theta^*\rho) = tr(F\rho) = tr(\rho F),$$

so from $E \sim F$ it follows that $\phi(E) = tr(\rho E) = tr(\rho F) = \phi(F)$.

If $\mathcal{R}$ is of type I_n, $n \geq 2$, then the identity I is the sum of n equivalent abelian (orthogonal) projections E_j $(j = 1, ..., n)$. We have

$$1 = \phi(I) = \phi(\sum_{j=1}^{n} E_j) = \sum_{j=1}^{n} \phi(E_j),$$

so $\phi(E_j) = \frac{1}{n}$ $(j = 1, ..., n)$, which contradicts our assumption $\phi(E) \in \{0,1\}$ for all $E \in \mathcal{P}(\mathcal{R})$.

If $\mathcal{R}$ is of type I_∞, we use the halving lemma (6.3.3 in [KadRinII97]) to show that there is a projection $F \in \mathcal{P}(\mathcal{R})$ such that $F \sim F^\perp := I - F$. If $\mathcal{R}$ is of type II or III, then we employ Lemma 10 for $E = I$ and $n = 2$ to obtain the same. We have

$$1 = \phi(I) = \phi(F + I - F) = \phi(F) + \phi(F^\perp),$$

so $\phi(F) = \phi(F^\perp) = \frac{1}{2}$, which contradicts our assumption. ■

This lemma means that restricting a normal state ϕ of a von Neumann algebra $\mathcal{R}$ that is not of type I_1 (i.e. abelian) to $\mathcal{R}_{sa}$ can never give a valuation function. The proof of this lemma is based on the proof of Thm. 6.4 in [deG01] (Thm. 13 in our paper).

2.2 Type I_n factors

In this subsection, let $\mathcal{R}$ be a type I factor.

Definition 12 *Using some notions from [deG01], we call a maximal distributive sublattice of $\mathcal{P}(\mathcal{R})$ a* ***Boolean sector*** *$\mathbb{B}$. The abelian von Neumann algebra $\mathcal{M}(\mathbb{B}) \subseteq \mathcal{R}$ generated by $\mathbb{B}$ has Gelfand spectrum $\Omega(\mathcal{M}(\mathbb{B}))$. One can define the* ***Stone spectrum*** *$\mathcal{Q}(\mathbb{B})$ of $\mathbb{B}$ as the set of maximal dual ideals in $\mathbb{B}$, equipped with the topology induced by the sets*

$$\mathcal{Q}_E(\mathbb{B}) := \{\beta \in \mathcal{Q}(\mathbb{B}) \mid E \in \beta\},$$

see Ch. 4 in [deG01] (the Stone spectrum of $\mathbb{B}$ is called the Stone space of $\mathbb{B}$ there). The elements β of the Stone spectrum $\mathcal{Q}(\mathbb{B})$ are called ***quasipoints****. To every quasipoint β of $\mathcal{Q}(\mathbb{B})$, there corresponds an element ω of $\Omega(\mathcal{M}(\mathbb{B}))$, since the Stone spectrum and the Gelfand spectrum are homeomorphic ([deG01, Thm. 5.2]). More generally, the Stone spectrum $\mathcal{Q}(\mathcal{M}) := \mathcal{Q}(\mathcal{P}(\mathcal{M}))$ of an abelian von Neumann algebra $\mathcal{M}$ is defined as the set of maximal dual ideals β in $\mathcal{P}(\mathcal{M})$, equipped with the topology induced by the sets*

$$\mathcal{Q}_E(\mathcal{M}) := \{\beta \in \mathcal{Q}(\mathcal{M}) \mid E \in \beta\}.$$

$\mathcal{Q}(\mathcal{M})$ is homeomorphic to the Gelfand spectrum $\Omega(\mathcal{M})$.

Every state ρ of $\mathcal{R}$ induces a bounded positive Radon measure $\mu_\rho^{\mathbb{B}}$ of norm 1 on $\mathcal{Q}(\mathbb{B})$ by

$$\begin{aligned} \mu_\rho^{\mathbb{B}} : C(\mathcal{Q}(\mathbb{B})) &\longrightarrow \mathbb{C} \\ A &\longmapsto tr(\rho A). \end{aligned}$$

We will use the following result (Thm. 6.4 in [deG01]):

Theorem 13 *Let ρ be a state of $\mathcal{R}$, and let $\mathbb{B} \subseteq \mathcal{P}(\mathcal{R})$ be a Boolean sector. Then the Radon measure $\mu_\rho^{\mathbb{B}}$ on the Stone spectrum $\mathcal{Q}(\mathbb{B})$ is the point measure ε_{β_0} for some $\beta_0 \in \mathcal{Q}(\mathbb{B})$, if and only if there is an $x \in S^1(\mathcal{H})$ such that $\mathbb{C}x \in \mathbb{B}$, $\beta_0 = \beta_{\mathbb{C}x}$ and $\rho = P_{\mathbb{C}x}$. Here $\beta_{\mathbb{C}x}$ is the unique quasipoint containing $P_{\mathbb{C}x}$.*

Proposition 14 *Let $\mathcal{R}$ be a factor of type I_n, $n \in \{3, 4, ...\} \cup \{\infty\}$. There is no valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$.*

Proof. We first treat the case of finite n. As shown above, assuming the existence of a valuation function v, one has a quasi-state v' of $\mathcal{R}$, which is a pure state (i.e. a character) of every abelian von Neumann subalgebra $\mathcal{M} \subseteq \mathcal{R}$. Lemma 9 shows that $v'|_{\mathcal{P}(\mathcal{R})}$ is a finitely additive probability measure. Gleason's theorem (Thm. 2) shows that $v'|_{\mathcal{P}(\mathcal{R})}$ comes from a state of $\mathcal{R} \simeq \mathcal{L}(\mathcal{H}_n)$, where $\mathcal{H}_n$ is an n-dimensional Hilbert space.

The state given by Gleason's theorem is of the form $tr(\rho\_)$, where ρ is a positive trace class operator of trace 1. According to the spectral theorem, every operator $A \in \mathcal{R}$ is the norm limit of complex linear combinations of projections, hence there is a unique possibility to extend the probability measure $v'|_{\mathcal{P}(\mathcal{R})}$ to a state (given by linearly extending $v'|_{\mathcal{P}(\mathcal{R})}$). Of course, this extension simply is v', so

$$v'(\_) = tr(\rho\_).$$

Let $\mathbb{B}$ be a *Boolean sector* of $\mathcal{P}(\mathcal{R})$, and let $\mathcal{M}(\mathbb{B})$ be the maximal abelian subalgebra of $\mathcal{R}$ generated by $\mathbb{B}$. Since $v'|_{\mathcal{M}(\mathbb{B})}$ is a pure state, it corresponds to exactly one element $\beta_0 \in \mathcal{Q}(\mathbb{B})$. v' induces a point measure on $\mathcal{M}(\mathbb{B}) = C(\mathcal{Q}(\mathbb{B}))$ in this way.

According to Thm. 13, ρ and β are of the form

$$\rho = P_{\mathbb{C}x},$$
$$\beta = \beta_{\mathbb{C}x},$$

and thus

$$v'(\_) = tr(P_{\mathbb{C}x}\_).$$

This form does not depend on the chosen Boolean sector $\mathbb{B}$.

Now let $\mathbb{B}'$ be a different Boolean sector that does not contain the projections $P_{\mathbb{C}x}, I - P_{\mathbb{C}x}$. There is a projection $F' \in \mathbb{B}'$ such that

$$v'(F') = tr(P_{\mathbb{C}x}F') \notin \{0,1\},$$

contradicting the defining condition $v'(E) = v(E) \in \operatorname{sp} E = \{0,1\}$ ($E \in \mathcal{P}(\mathcal{R})$) of a valuation function. This shows that there is no valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ for factors $\mathcal{R}$ of type $I_n,\ n \in \{3,4,...\}$, from which the Kochen-Specker theorem follows. Instead of referring to [deG01], we could have used Lemma 11.

Now let $\mathcal{R}$ be a type I_∞ factor, $\mathcal{R} \simeq \mathcal{L}(\mathcal{H})$ for an infinite-dimensional separable Hilbert space $\mathcal{H}$. $\mathcal{R}$ contains a subfactor of type I_n for every $n \in \{3,4,...\}$: Let $\mathcal{S}$ be a type I_n factor, $\mathcal{S} \simeq \mathcal{L}(\mathcal{H}_n)$. The separable Hilbert spaces $\mathcal{H}_n \otimes \mathcal{H}$ and $\mathcal{H}$ are isomorphic and will be identified. Embed $\mathcal{L}(\mathcal{H}_n)$ into $\mathcal{L}(\mathcal{H})$ via the mapping

$$\mathcal{L}(\mathcal{H}_n) \longrightarrow \mathcal{L}(\mathcal{H}_n \otimes \mathcal{H}) \simeq \mathcal{L}(\mathcal{H})$$
$$A \longmapsto A \otimes I.$$

This guarantees that the identity I_n of $\mathcal{L}(\mathcal{H}_n)$ is mapped to the identity I of $\mathcal{L}(\mathcal{H})$.

We assume that there is a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$. Restricting v to the self-adjoint part of a type I_n subfactor $\mathcal{S}$ of $\mathcal{R}$ gives a valuation function for $\mathcal{S}$. Since we saw that there is no such valuation function, there can be none for $\mathcal{R}_{sa}$. $\blacksquare$

2.3 Von Neumann algebras without type I_2 summand

The proof for the type I_n case proceeded in two steps: first, assuming that there is a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ for $\mathcal{R}$ a type I_n algebra, we showed that it induces a quasi-state $v' : \mathcal{R} \to \mathbb{C}$ in a canonical manner and thus a finitely additive probability measure $v'|_{\mathcal{P}(\mathcal{R})}$. In a second step, we used Gleason's theorem to see that v' is a *state* of $\mathcal{R}$ of the form $v'(\_) = tr(\rho\_)$, which cannot satisfy the defining conditions for a valuation function, because there are projections $F' \in \mathcal{P}(\mathcal{R})$ such that $v'(F') \notin \operatorname{sp} F' = \{0, 1\}$.

If we want to treat more general von Neumann algebras $\mathcal{R}$, we first must assure that the quasi-state v' is a state of $\mathcal{R}$. Since we know that $v'|_{\mathcal{P}(\mathcal{R})}$ is a finitely additive probability measure for an arbitrary von Neumann algebra $\mathcal{R}$ (Lemma 9), a generalization of Gleason's theorem is needed, showing that this probability measure comes from a state. There is a beautiful and detailed paper by S. Maeda [Mae89] on the generalizations of Gleason's theorem. Maeda is drawing on results by J. F. Aarnes [Aar69, Aar70], J. Gunson [Gun72], E. Christensen [Chr82, Chr85], F. J. Yeadon [Yea83, Yea84] and K. Saito [Sai85]. The proofs given in Maeda's paper are by no means trivial. The central point of course is to show that a quasi-state is linear on $\mathcal{R}_{sa}$ for *non-commuting* self-adjoint operators A, B. Maeda uses Gleason's theorem [Gle57] for the type I_n algebras. Types II and III require a lot more work. We cite the main result (Thm. 12.1 in [Mae89]):

Theorem 15 *(Christensen, Yeadon, Maeda et. al.) Let $\mathcal{R}$ be a von Neumann algebra without direct summand of type I_2, and let μ be a finitely additive probability measure on the complete orthomodular lattice $\mathcal{P}(\mathcal{R})$. μ can be extended to a state $\widehat{\mu}$ of $\mathcal{R}$, and moreover*

$$\forall E, F \in \mathcal{P}(\mathcal{R}) : |\mu(E) - \mu(F)| \leq ||E - F||.$$

It follows that the quasi-state v' is a state of $\mathcal{R}$ if the von Neumann algebra $\mathcal{R}$ has no summand of type I_2. However, the state v' is not normal necessarily, i.e. it need not be of the form $v'(\_) = tr(\rho\_)$, so we cannot use the same argument as before. Instead, we will show that v' is a *multiplicative* state, using a result by J. Hamhalter ([Ham93]), and give a second proof of the Kochen-Specker theorem, valid for all von Neumann algebras without summands of types I_1 and I_2.

We will exploit the fact that $v'|_{\mathcal{P}(\mathcal{R})}$ is a *two-valued* measure, i.e. $v'(E) \in \{0, 1\}$ for all $E \in \mathcal{P}(\mathcal{R})$. From now on, measure will always mean finitely additive probability measure. We cite Lemma 5.1 of [Ham93] with proof:

Lemma 16 *Let $\mathcal{R}$ be a von Neumann algebra without type I_2 summand. Every two-valued measure on $\mathcal{P}(\mathcal{R})$ can be extended to a multiplicative state of $\mathcal{R}$.*

Proof. Let μ be a two-valued measure on $\mathcal{P}(\mathcal{R})$. Using the Gleason-Christensen-Yeadon theorem (Thm. 15), we can extend μ to a state ϕ of $\mathcal{R}$.

Let $\pi_\phi : \mathcal{R} \to \mathcal{H}_\phi$ be the GNS representation engendered by ϕ. Let x_ϕ be a unit cyclic vector of π_ϕ such that $\phi = \omega_{x_\phi} \circ \pi_\phi$, where ω_{x_ϕ} is the vector state given by x_ϕ. For every $E \in \mathcal{P}(\mathcal{R})$ we have $\mu(E) = \langle \pi_\phi(E)x_\phi, x_\phi \rangle$. We see that $\mu(E)$ is either 0 or 1. It follows that either $\pi_\phi(E)x_\phi = x_\phi$ or $\pi_\phi(E)x_\phi = 0$. Hence, $\mathcal{H}_\phi = \overline{\text{lin}}\{\pi_\phi(A)x_\phi \mid A \in \mathcal{R}\} = \overline{\text{lin}}\{\pi_\phi(E) \mid E \in \mathcal{P}(\mathcal{R})\} = \text{lin}\{x_\phi\}$, where lin means the linear span and $\overline{\text{lin}}$ its closure. Therefore, for every $A \in \mathcal{R}$ there is a complex number λ_A such that $\pi_\phi(A)x_\phi = \lambda_A x_\phi$. Obviously, $\lambda_{AB} = \lambda_A \lambda_B$ for $A, B \in \mathcal{R}$, and therefore

$$\begin{aligned} \phi(AB) &= \langle \pi_\phi(AB)x_\phi, x_\phi \rangle = \lambda_{AB} \\ &= \lambda_A \lambda_B = \langle \pi_\phi(A)x_\phi, x_\phi \rangle \langle \pi_\phi(B)x_\phi, x_\phi \rangle \\ &= \phi(A)\phi(B) \end{aligned}$$

for all $A, B \in \mathcal{R}$. ■

Corollary 17 *Let $\mathcal{R}$ be a von Neumann algebra without type I_2 summand. The state v' induced by a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ is multiplicative.*

We now give a short proof (in two lemmata) for the well-known fact that a von Neumann algebra $\mathcal{R}$ of fixed type has no multiplicative states unless $\mathcal{R}$ is of type I_1, i.e. abelian. See also Thm. 5.3 in [Ham93].

Lemma 18 *Let $\mathcal{R}$ be a von Neumann algebra of type I_n, $n \geq 2$. There are no multiplicative states of $\mathcal{R}$.*

Proof. If $\mathcal{R}$ is of type I_n, then I is the sum of n equivalent abelian orthogonal projections E_j $(j = 1, ..., n)$. $E_1 \sim E_2$ means that there is a partial isometry $\theta \in \mathcal{R}$ such that $E_1 = \theta^*\theta$ and $E_2 = \theta\theta^*$. Let ϕ be a multiplicative state of $\mathcal{R}$. In particular, ϕ is a tracial state, i.e. $\phi(AB) = \phi(BA)$ for all $A, B \in \mathcal{R}$, hence

$$\phi(E_1) = \phi(\theta^*\theta) = \phi(\theta\theta^*) = \phi(E_2).$$

In the same manner, one obtains $\phi(E_1) = \phi(E_2) = \phi(E_3) = ... = \phi(E_n)$. But $\phi(E_1) \in \{0, 1\}$, since ϕ is multiplicative, so

$$\phi(I) = \phi(\sum_{j=1}^{n} E_j) = \sum_{j=1}^{n} \phi(E_n) \in \{0, n\},$$

which is a contradiction. ■

Lemma 19 *Let $\mathcal{R}$ be a von Neumann algebra of type I_∞, II or III. There are no multiplicative states of $\mathcal{R}$.*

Proof. First regard the case that $\mathcal{R}$ is of type I_∞. Since $\mathcal{R}$ is properly infinite, we can use the halving lemma (Lemma 6.3.3 in [KadRinII97]) to show that there is a projection $F \in \mathcal{P}(\mathcal{R})$ such that $F \sim F^\perp := I - F$. For $\mathcal{R}$ a type II or III algebra, we use lemma 10 (choose $E = I$ and $n = 2$) to the same effect. $F \sim F^{\perp}$ means that there is a partial isometry $\theta \in \mathcal{R}$ such that $F = \theta^*\theta$ and $F^{\perp} = \theta\theta^*$. Let ϕ be a multiplicative state of $\mathcal{R}$, so

$$\phi(F) = \phi(\theta^*\theta) = \phi(\theta\theta^*) = \phi(F^{\perp}).$$

Since $\phi(F) \in \{0, 1\}$, we have

$$\phi(I) = \phi(I - F + F) = \phi(F^{\perp}) + \phi(F) \in \{0, 2\},$$

which is a contradiction. ■

Now let $\mathcal{R}$ be an arbitrary von Neumann algebra without summand of type I_2. Let $P_{I_1} \in \mathcal{P}(\mathcal{R})$ be the maximal abelian central projection, P_I the maximal central projection such that $\mathcal{R}P_I$ is of type I, but has no central abelian portion, P_{II} the maximal central projection such that $\mathcal{R}P_{II}$ is of type II and P_{III} the maximal central projection such that $\mathcal{R}P_{III}$ is of type III. We have $I = P_{I_1} + P_I + P_{II} + P_{III}$ (see Thm. 6.5.2 in [KadRinII97]).

Every projection $E \in \mathcal{P}(\mathcal{R})$ can be written as $E = E_{I_1} + E_I + E_{II} + E_{III}$ for orthogonal projections $E_{I_1} \in \mathcal{R}P_{I_1}$, $E_I \in \mathcal{R}P_1$, $E_{II} \in \mathcal{R}P_{II}$ and $E_{III} \in \mathcal{R}P_{III}$. Let $v : \mathcal{R}_{sa} \to \mathbb{R}$ be a valuation function and let v' be the induced state of $\mathcal{R}$. Since $v'|_{\mathcal{P}(\mathcal{R})}$ is finitely additive, $v'|_{\mathcal{R}_{sa}} = v$ and $v(I) = 1 = v(P_{I_1}) + v(P_I) + v(P_{II}) + v(P_{III})$, exactly one term on the right hand side equals 1, the others are zero. Let P_x ($x \in \{I_1, I, II, III\}$) denote the central projection such that $v(P_x) = 1$. It follows that $v(E) = 0$ for all $E \leq P_y$ ($y \in \{I_1, I, II, III\}$) for all $y \neq x$, since $v'|_{\{E, P_y\}''}$ is positive. This means that the valuation function is concentrated at $\mathcal{R}P_x$ in the sense that $v(E) = 0$ for all projections E orthogonal to P_x.

$v|_{\mathcal{R}P_I}$ cannot be a valuation function for $(\mathcal{R}P_I)_{sa}$, since the induced state $(v|_{\mathcal{R}P_I})'$ on $\mathcal{R}P_I$ would be multiplicative, but $\mathcal{R}P_I$ is a sum of type I_n algebras, $n \in \{3, 4, ..., \infty\}$, and none of these algebras has a multiplicative state. Similarly, $v|_{\mathcal{R}P_{II}}$ cannot be a valuation function for $\mathcal{R}P_{II}$ and $v|_{\mathcal{R}P_{III}}$ cannot be a valuation function for $\mathcal{R}P_{III}$, because $\mathcal{R}P_{II}$ and $\mathcal{R}P_{III}$ have no multiplicative states. It follows that $v|_{\mathcal{R}P_{I_1}}$ is a valuation function for $(\mathcal{R}P_{I_1})_{sa}$ and the induced multiplicative state $(v|_{\mathcal{R}P_{I_1}})'$ equals v'. For the abelian part $\mathcal{R}P_{I_1}$, a "hidden state space" is given by the Gelfand spectrum $\Omega(\mathcal{R}P_{I_1})$, each element $\omega \in \Omega(\mathcal{R}P_{I_1})$ is a hidden pure state and induces a valuation function, assigning a spectral value to each $A \in \mathcal{R}P_{I_1}$ by evaluating $\omega(A)$, preserving functional relations. We have shown that only in this trivial situation one can have a valuation function. We obtain:

Lemma 20 *Let $\mathcal{R}$ be a von Neumann algebra without type I_2 summand, and let $P_{I_1} \in \mathcal{P}(\mathcal{R})$ be the maximal abelian central projection. There exists a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ if and only if $\mathcal{R}$ has a summand of type I_1, i.e. $P_{I_1} \neq 0$. In this case, $v = v|_{\mathcal{R}P_{I_1}}$, and the valuation function v is completely trivial on the non-abelian part $\mathcal{R}(I - P_{I_1})$ of $\mathcal{R}$, $v|_{(\mathcal{R}(I-P_{I_1}))_{sa}} = 0$.*

Summing up, we have a generalized Kochen-Specker theorem:

Theorem 21 *Let $\mathcal{R}$ be a von Neumann algebra without type I_2 summand. If $\mathcal{R}$ has no type I_1 summand, then the Kochen-Specker theorem holds. If $\mathcal{R}$ has a type I_1 summand, then there is a hidden state space in the sense described in the introduction, but only for the trivial, abelian part $\mathcal{R}P_{I_1}$ of $\mathcal{R}$.*

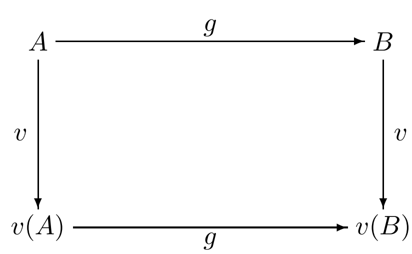
3 The presheaf perspective

In a remarkable series of papers, C. J. Isham and J. Butterfield (with J. Hamilton as co-author of the third paper) have given several reformulations of the Kochen-Specker theorem ([IshBut98, IshBut99, HIB00, IshBut02]). They use the language of *presheafs on a category*:

Definition 22 *Let $\mathcal{C}$ be a small category. A* ***presheaf on*** $\mathcal{C}$ *is a covariant functor*

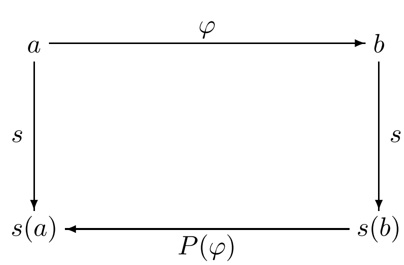
$$P : \mathcal{C}^{op} \longrightarrow \mathrm{Set}\,.$$

The observation is that the FUNC principle, condition (b) in Def. 3, means that a certain square diagram commutes:



This diagram captures the situation $B = g(A)$ and $v(B) = v(g(A)) = g(v(A))$. Isham and Butterfield observe that such a diagram can be read as expressing that there is a *section* of a presheaf on a category:

Definition 23 *Let P be a presheaf on a small category $\mathcal{C}$. A* ***global section*** *s of P is a mapping $\mathcal{C} \to \mathrm{Set}$ such that $s(a) \in P(a)$ for all $a \in \mathcal{C}$ and, whenever there is a morphism $\varphi : a \to b$ $(a, b \in \mathcal{C})$, the following diagram commutes:*

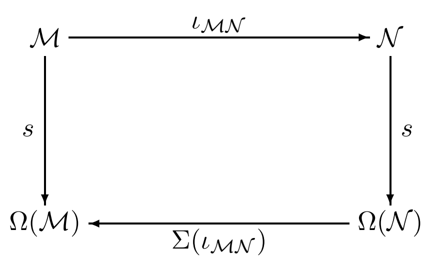


Please notice that the the horizontal arrow at the bottom is reversed, because we are dealing with presheafs, i.e. contravariant functors $\mathcal{C} \to \mathrm{Set}$.

There are several choices for the category and the presheaf that can be used to reformulate the Kochen-Specker theorem. We will generalize the proposal made in [HIB00]: Let $\mathfrak{A}(\mathcal{R})$ denote the category of unital abelian subalgebras of $\mathcal{R}$ (the unit of $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$ is the unit of $\mathcal{R}$). A morphism $\iota_{\mathcal{M}\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ exists whenever $\mathcal{M} \subseteq \mathcal{N}$. Hamilton, Isham and Butterfield only regard the case $\mathcal{R} = \mathcal{L}(\mathcal{H})$ and denote this category by $\mathcal{V}$.

Definition 24 *(compare Def 2.3 in [HIB00]) The* ***spectral presheaf*** *over* $\mathfrak{A}(\mathcal{R})$ *is the contravariant functor* $\Sigma : \mathfrak{A}(\mathcal{R}) \to \mathrm{Set}$ *defined as follows:*
(i) On objects: $\Sigma(\mathcal{M}) := \Omega(\mathcal{M})$*, the Gelfand spectrum of* $\mathcal{M}$*.*
(ii) On morphisms: If $\iota_{\mathcal{M}\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ *is the inclusion, then* $\Sigma(\iota_{\mathcal{M}\mathcal{N}}) : \Omega(\mathcal{N}) \to \Omega(\mathcal{M})$ *is defined by* $\Sigma(\iota_{\mathcal{M}\mathcal{N}})(\omega) := \omega|_{\mathcal{M}}$*.*

If there was a global section s of Σ, the following diagram would commute:



For $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$, $s(\mathcal{M}) \in \Omega(\mathcal{M})$ and $s(\mathcal{M}) = \Sigma(\iota_{\mathcal{M}\mathcal{N}})(s(\iota_{\mathcal{M}\mathcal{N}}(\mathcal{M}))$, where $\iota_{\mathcal{M}\mathcal{N}}(\mathcal{M})$ is the algebra $\mathcal{M}$ seen as part of $\mathcal{N}$ and $s(\iota_{\mathcal{M}\mathcal{N}}(\mathcal{M})) \in \Omega(\mathcal{N})$. The commutativity of the diagram means that $s(\mathcal{M})$ is given as the restriction of $s(\iota_{\mathcal{M}\mathcal{N}}(\mathcal{M}))$ to $\Omega(\mathcal{M}) \subseteq \Omega(\mathcal{N})$.

Such a choice of one element $s(\mathcal{M})$ of the Gelfand spectrum $\Omega(\mathcal{M})$ per abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$, compatible with the spectral presheaf mappings, i.e. with restrictions $\Omega(\mathcal{N}) \to \Omega(\mathcal{M})$, would give a valuation function when restricted to the self-adjoint elements: for all $A \in \mathcal{M}_{sa}$, $s(\mathcal{M})(A) \in \mathrm{sp}\, A$ and $s(\mathcal{M})(f(A)) = f(s(\mathcal{M})(A))$. The generalized Kochen-Specker theorem (Thm. 21) hence shows that for von Neumann algebras $\mathcal{R}$ without summands of types I_1 and I_2, there is no global section of Σ.

In Lemma 6, we saw that having a valuation function $v : \mathcal{R}_{sa} \to \mathbb{R}$ would mean having a character $v'|_{\mathcal{M}}$ (an element of the Gelfand spectrum) for each abelian subalgebra $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$. It follows from the FUNC principle that these characters are subject to the same conditions as above: if $\mathcal{M} \subseteq \mathcal{N}$, then restricting $v'|_{\mathcal{N}}$ to $\Omega(\mathcal{M})$ must give $v'|_{\mathcal{M}}$ (which is not possible globally). This choice of a category and a presheaf thus brings the presheaf formulation of the Kochen-Specker theorem very close to our first proof.

There is a closely related formulation, using Stone spectra instead of Gelfand spectra (see Def. 12):

Definition 25 *The* ***state presheaf*** $\mathcal{M}^1$ *on* $\mathfrak{A}(\mathcal{R})$ *is defined as follows:*

(i) On objects: $\mathcal{M}^1(\mathcal{M}) := \mathcal{M}^1(\mathcal{Q}(\mathcal{M}))$, *the set of positive Radon measures of norm* 1 *on* $\mathcal{Q}(\mathcal{M})$.

(ii) On morphisms: for $\mathcal{M}, \mathcal{N} \in \mathfrak{A}(\mathcal{R})$ *such that* $\mathcal{M} \subseteq \mathcal{N}$ *let*

$$\begin{aligned} p^{\mathcal{N}}_{\mathcal{M}} : \mathcal{M}^1(\mathcal{N}) &\longrightarrow \mathcal{M}^1(\mathcal{M}) \\ \mu_{\mathcal{N}} &\longmapsto p^{\mathcal{N}}_{\mathcal{M}}.\mu_{\mathcal{N}}, \end{aligned}$$

where $p^{\mathcal{N}}_{\mathcal{M}}.\mu_{\mathcal{N}}$ *is the image measure defined by*

$$(p^{\mathcal{N}}_{\mathcal{M}}.\mu_{\mathcal{N}})(U) := \mu_{\mathcal{N}}((p^{\mathcal{N}}_{\mathcal{M}})^{-1}(U)),$$

where $U \subseteq \mathcal{Q}(\mathcal{M})$ *is a Borel set and* $p^{\mathcal{N}}_{\mathcal{M}} : \mathcal{Q}(\mathcal{N}) \to \mathcal{Q}(\mathcal{M}),\ \beta \mapsto \beta \cap \mathcal{M}$ *is the restriction map between the Stone spectra.*

$\mathcal{M}^1$ really is a presheaf on $\mathfrak{A}(\mathcal{R})$, since obviously $p^{\mathcal{M}}_{\mathcal{M}} = id_{\mathcal{M}^1(\mathcal{M})}$ and since $p^{\mathcal{C}}_{\mathcal{M}} = p^{\mathcal{N}}_{\mathcal{M}} \circ p^{\mathcal{C}}_{\mathcal{N}}$ as mappings $\mathcal{Q}(\mathcal{P}) \to \mathcal{Q}(\mathcal{N}) \to \mathcal{Q}(\mathcal{M})$, the same holds for the mappings $\mathcal{M}^1(\mathcal{P}) \to \mathcal{M}^1(\mathcal{N}) \to \mathcal{M}^1(\mathcal{M})$. Let $\mathcal{R}$ have no type I_1 and I_2 summands. From the generalized Kochen-Specker theorem (Thm. 21) it follows that $\mathcal{M}^1$ has no global sections consisting entirely of point measures. The fact that $\mathcal{M}^1$ has no such global sections is equivalent to the generalized Kochen-Specker theorem, since a valuation function would induce a quasi-state v' (Lemma 8) such that $v'|_{\mathcal{M}}$ is a pure state for every $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$ (and hence gives a point measure on $\mathcal{Q}(\mathcal{M})$).

This presheaf formulation emphasizes the fact that a valuation function would give a *pure* state of every $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$. The presheaf $\mathcal{M}^1$ does have global sections (every state of $\mathcal{R}$ induces one, obviously), but it has no global sections consisting entirely of point measures.

4 Discussion

We have presented two functional analytic proofs for the fact that there are no valuation functions $v : \mathcal{R}_{sa} \to \mathbb{R}$ for $\mathcal{R}$ a von Neumann algebra. The first proof only uses Gleason's classical theorem (Thm. 2) and holds for $\mathcal{R}$ a type I_n factor, $n \geq 3$. The second proof depends on the Gleason-Christensen-Yeadon theorem (Thm. 15) and holds for von Neumann algebras $\mathcal{R}$ without summands of types I_1 and I_2. To the best of our knowledge, for the first time von Neumann algebras other than the type I_n factors $\mathcal{L}(\mathcal{H})$ have been treated. The generalized Kochen-Specker theorem follows: there is no hidden states model of quantum theory in the sense described in the introduction.

Both proofs are based on the fact that having a valuation function v would mean having a *state* v' of $\mathcal{R}$, which follows from Gleason's theorem, and this state has properties that lead to a contradiction. In the first proof, for type I_n factors ($n \geq 3$), the state is of the form $v'(\_) = tr(\rho\_)$, so there are projections $E \in \mathcal{P}(\mathcal{R})$ such that $v'(E) \notin \{0,1\}$. The second proof, which is much more general, uses the fact that v' is a multiplicative state. Since there are no multiplicative states except on type I_1, i.e. abelian, algebras, the Kochen-Specker theorem holds for all von Neumann algebras without summands of types I_1 and I_2. Type I_2 must be excluded since Gleason's theorem and the Gleason-Christensen-Yeadon theorem only hold if $\mathcal{R}$ has no type I_2 summand. It is known that every type I_2 algebra admits a two-valued measure and hence a valuation function, see Rem. 5.4 in [Ham93]. If $\mathcal{R}$ has a type I_1 summand, then there are valuation functions, but they are concentrated at the trivial, abelian part $\mathcal{R}P_{I_1}$ of $\mathcal{R}$, where $P_{I_1} \in \mathcal{P}(\mathcal{R})$ is the maximal abelian central projection.

The fact that the defining conditions of a valuation function v inevitably lead to a multiplicative state shows that these conditions are very strong. Indeed, although the FUNC principle only seems to pose conditions on commuting operators, this is not the case: v' is a state, i.e. additive on non-commuting operators also. In the physics literature, an abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$ is called a *context*. Of course, the contexts give nothing like a partition of $\mathcal{R}$ into abelian, "classical" parts, but are interwoven in an intricate manner, since an observable $A \in \mathcal{R}_{sa}$ typically is contained in many abelian subalgebras. The FUNC principle poses conditions within each context, but since typically $A = f(B) = g(C)$ for non-commuting observables B, C, it also poses conditions on non-commuting observables, across contexts. The presheaf formulations presented in section 3 clearly show that the Kochen-Specker theorem means that there is no state ϕ of $\mathcal{R}$ such that for all contexts $\mathcal{M} \in \mathfrak{A}(\mathcal{R})$, the restriction $\phi|_{\mathcal{M}}$ is a pure state.

The Kochen-Specker theorem is little more than a corollary to Gleason's theorem, in a more general sense than worked out by Bell ([Bell66]). The fact that v' is a state comes from Gleason's theorem (or its generalization): a valuation function v defines a quasi-state v' in a canonical manner, and restricting the quasi-state v' to the projection lattice $\mathcal{P}(\mathcal{R})$ gives a finitely additive probability measure. Gleason's theorem shows that v' must be a state. The deep meaning of Gleason's theorem is that the simple, *lattice-theoretic* condition of finite additivity on each distributive sublattice, which is a condition on finite joins actually ($E + F = E \vee F$ for orthogonal projections E, F), suffices to guarantee additivity of the functional v' defined by linear extension of the probability measure (and taking the appropriate limit, see e.g. Ch. III.7 of [Mae89]). Of course, finite additivity on each distributive sublattice is a condition across distributive sublattices, since each projection E is contained in many distributive sublattices.

But the defining conditions of a valuation function v are even stronger: using the fact that $v(E) \in \operatorname{sp} E = \{0,1\}$ ($E \in \mathcal{P}(\mathcal{R})$), we saw that the state v' is multiplicative, which is only possible if v is concentrated at the abelian part

$\mathcal{R}P_{I_1}$ of $\mathcal{R}$. Thus, a valuation function and a hidden states model can only exist for the trivial, abelian situation. This generalizes to arbitrary von Neumann algebras a result found by J. D. Malley ([Mal04]). It also rebuts the critique of von Neumann's proof from 1932 ([vNeu32]). Von Neumann posed additivity conditions on non-commuting observables, which was strongly criticized by Bell ([Bell66]) as unphysical. Of course, the Gelfand representation of an abelian von Neumann algebra is a hidden states model, the Gelfand spectrum $\Omega(\mathcal{R})$ taking the rôle of the "hidden" state space.

We have shown more than the fact that there are no non-trivial hidden states models: Each element ω of a hidden state space Ω would give a valuation function, as described in the introduction, but having a valuation function would not necessarily mean having a hidden states model. A valuation function would simply assign values to all observables in a manner consistent with the FUNC principle, which would be an important piece of a realist quantum theory. Since we have ruled out this possibility, there are no such naïve realist models of quantum theory.

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