THE SYMMETRY GROUP OF A MODEL OF HYPERBOLIC PLANE GEOMETRY AND SOME ASSOCIATED INVARIANT OPTIMAL CONTROL PROBLEMS

A thesis submitted in fulfillment of the
requirements for the degree of

MASTERS IN SCIENCE

of

RHODES UNIVERSITY

by

HELEN CLARE HENNINGER

December 2011

Abstract

In this thesis we study left-invariant control affine systems on the symmetry group of a model of hyperbolic plane geometry, the matrix Lie group $\mathsf{SO}(1,2)_0$. We determine that there are 10 distinct classes of such control systems and for typical elements of two of these classes we provide solutions of the left-invariant optimal control problem with quadratic costs. Under the identification of the Lie algebra $\mathfrak{so}(1,2)$ with Minkowski spacetime $\mathbb{R}^{1,2}$, we construct a controllabilility criterion for all left-invariant control affine systems on $\mathsf{SO}(1,2)_0$ which in the inhomogeneous case depends only on the presence or absence of an element in the image of the system's trace in $\mathbb{R}^{1,2}$ which is identifiable using the inner product $\odot$. For the solutions of both the optimal control problems, we provide explicit expressions in terms of Jacobi elliptic functions for the solutions of the reduced extremal equations and determine the nonlinear stability of the equilibrium points.

Key words and phrases: Hyperbolic plane geometry, (reduced) Minkowski spacetime, Lorentz group, symmetry group, matrix Lie groups, left-invariant control affine systems, local detached feedback equivalence, controllability, extremal equation, nonlinear stability, energy-Casimir method.

Acknowledgements

I am fortunate to owe too much to too many:

To my supervisor Dr Claudiu Remsing: for giving me much of my space and your time, your guidance and discussions. Thank you.

To my fellow Master's students Rory Biggs: for enthusiasm and the concept of detached feedback equivalence, and Ross Adams: for your focused clarity in all our discussion groups.

To my mother: the wisdom with which you raised me is always present in my heart and thoughts, and my father: for unstinting support, both financial and otherwise, and Dr Helen Robinson: for my first Maths lessons.

To my twin, Katharine Rose: for going through the Master's process with me, and being my support at every step.

To Robert Rudniak: for all you've done for me.

Contents

Chapter 1

Introduction

1.1 Background

We discuss in a broad sense elements of the fields of geometric control theory and hyperbolic geometry. Precise definitions of the terms and concepts used from these fields are provided in Appendix A.

Control theory

Control theory is a relatively modern field of mathematics which originated from the seminal work carried out in the 1950's by L.S. Pontryagin and his co-workers, who developed the field in response to problems of theoretical engineering. An advantage of control theory is its simultaneously theoretical and practical nature: it has applications in engineering, robotics and biology, but also has the ability to give insight into problems of theoretical physics and the study of ordinary differential equations. It is a fast-growing and active field.

Control theory concerns itself with the study of *control systems*. A control system is a parametrized family of vector fields on an underlying manifold, the *state space*, where the set of parameters are known as *controls*. These controls are elements of the *control space*. The *trajectories* of the control system are the integral curves of the vector field or concatenations of these curves. At any state the control parameter may be changed, and correspondingly a new trajectory chosen in the state space; thus the trajectories of the system are *controlled* by the choice of the control parameter.

If the controls are changed according to a function on either the control space, the state space or both, the system is said to have *feedback* given by this feedback function. If the feedback function does not depend on the state space, then it is said to be *detached*.

Given two states in the underlying manifold, the control system *transfers* one state to the other if the two may be joined by a trajectory of the system. If *any* two states in the state space may be joined by such a trajectory, the control system is *controllable*.

A given control system may produce very similar families of trajectories in the state space to another control system. If these similarities are great enough, we may wish to reduce to studying only one of the two systems with similar trajectories. This leads us to the problem of *classifying control systems*, which can be done according to several criteria. In this thesis we restrict ourselves to a classification under *detached feedback equivalence.*

We may wish to consider properties of the trajectories of these systems; particularly, given two points of the state space, we could consider minimizing or maximizing a function of the control variable along a trajectory which transfers the one state to the other. Such a problem is a problem of *optimal control.* This concept was first developed in the 1950's by Pontryagin, resulting in the celebrated *Pontryagin Maximum Principle* which provides necessary conditions for optimality.

In this thesis we discuss problems of controllability, equivalence classification and optimality for a specific and widely applicable class of control systems, the *control affine systems*, and take as their state space the symmetry group of a model of hyperbolic geometry. Since hyperbolic metrics appear very naturally in theoretical physics, for example in the construction of Minkowski spacetime, we consider that the results of this study may putatively give some insight into physical problems of special relativity.

Hyperbolic geometry

Euclidean plane geometry can be considered axiomatically, where the basic structures of points, lines and angles are assumed and related to each other by the 5 *axioms of Euclid*:

1. For every two points p, q such that $p \neq q$, there exists a unique line ℓ that passes through p and q.
2. For every line segment $\overline{ab}$ and every line segment $\overline{cd}$ there exists a unique point e such that e is between a and b and $\overline{cd}$ is congruent to $\overline{eb}$ (in the sense that their lengths are equal).
3. For all points a and $b \neq a$, there exists a circle centered at a with radius the length ab.
4. All right angles are congruent.
5. For every line ℓ and every point $p \notin \ell$, there exists a unique line ℓ' through p such that ℓ' is parallel to ℓ (where parallelism is defined by $\ell \| \ell' \;\Leftrightarrow\; \ell \cap \ell' = \emptyset$).

For many centuries it was hypothesised that the fifth axiom was dependent on the first four, and as such could be derived by a theorem assuming only statements 1 to 4. However, in the early 1800's the independent work of C. F. Gauss, N. Lobachevskij and J. Bolyai showed that there exist spaces where the first four axioms are assumed and the fifth is negated, to give a completely consistent geometry. This geometry was *hyperbolic geometry*, and it grew out of the negation of Euclid's fifth axiom in this sense: "For every line ℓ and every point $p \notin \ell$, there exists more than one line through p that is parallel to ℓ".

In order to develop the concept of hyperbolic geometry, a *model* of this geometry was required. The first identified hyperbolic model, (inadvertently) discovered by the Dutch physicist

C. Huygens in 1639 was the pseudosphere, for which the most natural parametrization showed a constant *Gaussian curvature* $K = -1$.

When the geodesics of the pseudosphere were calculated, they were found to be extendible infinitely in both directions, a property which was subsequently formalised as *geodesic completeness*. It was later discovered that *all geodesically complete surfaces with constant Gaussian curvature -1 are models of hyperbolic geometry.*
An extention of the concept of surfaces is the notion of an *abstract surface*, or 2-dimensional manifold. In the 1860's the work of B. Riemann led to the concept of a *Riemannian metric*, a variable distance measure on the tangent bundle of an abstract surface which allows for various expressions of the curvature of the surface, and equips it as a *geometric surface*. An *immersion* of an abstract surface into $\mathbb{R}^3$ is a mapping from the surface to $\mathbb{R}^3$ with an injective tangent map. If a Riemannian metric is assigned to an abstract surface, an immersion is *isometric* if the metric is preserved under the immersion. A theorem of D. Hilbert states that *no geodesically complete surface of constant negative curvature can be isometrically immersed in* $\mathbb{R}^3$. In consequence, the only geodesically complete surfaces of constant negative curvature that exist in three dimensions cannot be discussed in a Euclidean environment.

Since geometric surfaces are geometric objects independent of embedding in any ambient space, they are thus essential to the realization of hyperbolic models, which however have not been formalized as abstract surfaces. This thesis begins by establishing independently two hyperbolic models as abstract surfaces with their groups of isometries.

1.2 Control problems on the symmetry group of a model of hyperbolic plane geometry

In this thesis we will approach the problems of geometric control theory in two distinct ways. Firstly, by considering the state space as a symmetry group of a model of hyperbolic plane geometry, we may access control theoretical results by considering the structure of the space on which it acts. Secondly, if the action of the symmetry group is determined to be transitive, then by a result of Myers and Steenrod (1951), it is what is termed a *Lie transformation group* (of dimension ≤ 3), and so, particularly, is a Lie group. Thus on determination of transitivity of action of the symmetry group, we may determine its expression as a Lie group and use the structure of its Lie algebra to access other aspects of control theory on the group.

In each case, the most direct approach is to start by considering a space (set) on which there is a metric structure. Thus, in the geometric approach, we will attempt to establish the symmetry groups of models of hyperbolic plane geometry, beginning with an abstract surface structured with a negative-curvature metric. We then endeavour to use this metric and abstract surface to establish the hyperbolic model as a geometeric surface, in each case contructing its symmetry group independently of maps to or from any other models. This approach differs from the usual approach (see [7], [31], [32], [25]), which relies heavily on maps between models or analogy from one to the other to establish the geodesics and symmetry groups. There are five models of hyperbolic plane geometry in current use; that is, the Poincaré disk, the Klein disk, the upper half-plane, the hemisphere and the hyperboloid. In this thesis we choose to work chiefly with this last model, due to its interelation with the rich structure of Minkowski spacetime. We also consider the upper half-plane model, which has the simplest metric of the five. Since the hyperboloid model is a surface which is not contained within a 2-dimensional plane, we do not work with the hemisphere, which would provide no additional geometric insights, and which we established shares a symmetry group with the upper half-plane. Equally, we do not wish to introduce the symmetry group of the Poincaré disk, which is different to that of the upper half-plane and the hyperboloid model, and the coverage of which would have made this thesis too broad. The metric of the Klein disk also provided unnecessary broadening of topic, due to the natural expression of its symmetries in terms of structures of projective geometry. Establishing the two symmetry groups of the models and aspects of the actions of their symmetry groups completely separately, particularly transitive action, we will then consider group isomorphisms between them so that structural aspects which are more accessible in one symmetry group might be mapped to structures in the other.

On establishing transitivity, we will choose the group with the most direct representation and easily-accesible stuctures, establish it as a matrix Lie group and construct and discuss its Lie algebra. Consideration of its group-theoretical and topological properties will provide more insight when we consider control problems developing on the group as a smooth manifold. We will also construct the adjoint and co-adjoint orbits of the chosen group's Lie algebra. These play a role in the discussion of optimal control problems on the group.

In order to discuss the control problem of optimality, we will consider only full-rank control affine systems, which we wish to group together into classes which share similar properties in terms of optimality and control. In order to do this, we will use the concept of *local detached feedback equivalence* [5]. This equivalence relates two control systems as equivalent if, at least locally, the trajectories of the first system may be mapped smoothly to the trajectories of the second. This mapping is dependent only on the control variable. Thus, if one element of a class produces trajectories optimising the control problem, then for all systems within the class a smooth mapping of their control variables exist, which will map their trajectories to the optimizing trajectory. In this way, optimality results for one representative of the class are shared by all elements of that class.

To determine the equivalence classification will require the use of the group of Lie algebra automophisms for the chosen Lie algebra (cf. [5]).

We next wish to consider the problem of controllability of systems ([29], [2], [30]). In this thesis we restrict to left-invariant control affine systems with piecewise-constant controls. Applying some of the known conditions for controllability of such systems ([29], [30]) on Lie groups, we will attempt to constuct a controllability criterion for these systems on our chosen symmetry group. At all times our approach considers the structure of the symmetry group brought about by the constant-curvature metric of the space on which it acts.

Using the classification results, we will consider the optimal control problem with quadratic costs for representatives of certain classes in this classification. In order to determine the optimal Hamiltonians we will use the Pontryagin Maximum Principle (PMP) as stated in [13]. This principle provides a set of necessary conditions for a trajectory to be optimal, in the form of a system of differential equations. Following [13], we will work in the trivialization of the cotangent bundle of the state space. This trivialization expresses the cotangent bundle as a direct product of the group itself with the dual space of its Lie algebra. In this setting, the differential equations set up by the PMP become a Hamiltonian system with this direct product as its phase space. The solution of this Hamiltonian system is termed an *extremal curve*. We will set up the *reduced extremal equations* (which are the projections onto the dual space of the Lie algebra of the Hamiltonian system generated by the PMP), and attempt to solve these to find the projection of the extremal curve onto the dual space. We consider the extremal curves as developing on the intersection of the level surfaces of their optimal Hamiltonians and a Casimir function, and use this approach to express the development of the projections of these extremal curves on the dual space of the Lie algebra and on the group itself. We investigate the possibilities of expressing the solutions of the reduced extremal equtions in terms of the *Jacobi elliptic functions*.

In analogy with the Euclidean case, we will consider a possible alternative to the expression of curves in $\mathsf{SO}(3)$ in terms of Euler angles to determine the projection of the extremal curves onto $\mathsf{SO}(1,2)_0$.

Finally, we will consider the (nonlinear) stability of the equlibrium points of the solutions, which serve as a good indication of the local behaviour of these optimal trajectories (that is, they give an indication of how great a fluctuation from the optimal trajectory may occur for the trajectory to remain at least "approximately" optimal). To determine the nonlinear stability properties, we will use the energy-Casimir method as stated by [17], and if this provides indeterminate cases we will use the extended energy-Casimir method due to Ortega, Ratiu and Planas-Bielsa [23].

1.3 Overview

- **Chapter 2**: **Section 2.1.** $\mathbb{R}^{1,2}$ as an inner product space; planes Γ in $\mathbb{R}^{1,2}$ and the restriction $J|_\Gamma$. **Section 2.2.** Constructs the geometric surface $\mathbb{HL}$ and represents its symmetry group as the matrix group $\mathsf{SO}(1,2)_0$. Trasitivity of $\mathsf{SO}(1,2)_0$. Decomposition $\mathsf{SO}(1,2)_0 = \mathsf{BK}$. **Section 2.3.** Constructs the geometric surface $\mathbb{HP}$ and its symmetry group $\mathsf{Sym}(\mathbb{HP})$. Represents the symmetry group as the matrix group $\mathsf{PGL}(2,\mathbb{R})$. Trasitivity of $\mathsf{Sym}(\mathbb{HP})$. **Section 2.4.** Constructs the group isomorphisms between the symmetry groups of $\mathbb{HL}$ and $\mathbb{HP}$ and their representations as matrix groups.

- **Chapter 3**: **Section 3.1.** $\mathsf{SO}(1,2)_0$ is a connected, non-compact matrix Lie group. **Section 3.2.** Constructs the Lie algebra $\mathfrak{so}(1,2)$ and its commutator relations; $\mathfrak{so}(1,2)$ is simple and semisimple with trivial centre; the Killing form; adjoint operators. **Section 3.3.** The Lorentz cross product $\otimes$; the hat map $\mathfrak{so}(1,2) \to \mathbb{R}^{1,2}_{\otimes}$; interrelation of $\otimes$ with $\odot$; $\mathbb{R}^{1,2}_{\otimes}$ is a Lie algebra. **Section 3.4.** Constructs the Lie algebra automorphisms of $\mathfrak{so}(1,2)$ and $\mathbb{R}^{1,2}_{\otimes}$. **Section 3.5.** Adjoint and co-adjoint orbits of $\mathfrak{so}(1,2)$ expressed in $\mathbb{R}^{1,2}_{\otimes}$. **Section 3.6.** Action of $\mathsf{SO}(1,2)_0$ on the structures of $\mathbb{R}^{1,2}$; $\mathsf{SO}(1,2)_0$ is simple, not simply-connected; Iwasawa decomposition of $\mathsf{SO}(1,2)$.

- **Chapter 4**: **Section 4.1.** The classification of full-rank control affine systems on $\mathsf{SO}(1,2)_0$ under local detached feedback equivalence. This is achieved using the hat map of $\mathfrak{so}(1,2)$ to $\mathbb{R}^{1,2}_{\otimes}$ to send the trace Γ to Γ; properties of the restriction of the metric $J|_\Gamma$. **Section 4.2.** Preliminary results for controllability of left-invariant control affine systems on $\mathsf{SO}(1,2)_0$ and the image hat$\Gamma = \Gamma \subseteq \mathbb{R}^{1,2}$ of the trace; the controllability criterion for all such systems, using the hat map.

- **Chapter 5**: **Section 5.1.** Constructs a Casimir function and expresses the representation $\mathfrak{so}(1,2) = \kappa^\flat(\mathfrak{so}(1,2)^*)$. **Section 5.2.** Reduced extremal equations for $\Sigma_1^{(2,0)}$; explicit solution of extremal curve on $\mathfrak{so}(1,2)$ in terms of Jacobi elliptic functions; projection onto $\mathsf{SO}(1,2)_0$ of the extremal curves as solution to a system of differential equations; equilibrium points of reduced extremal equations, nonlinear stability of equilibrium points. **Section 5.3.** Reduced extremal equations for $\Sigma_2^{(2,0)}$; explicit solution of extremal curve on $\mathfrak{so}(1,2)$ in terms of Jacobi elliptic functions; projection onto $\mathsf{SO}(1,2)_0$ of the extremal curves as solution to a system of differential equations; equilibrium points of reduced extremal equations, nonlinear stability of equilibrium points.

- **Chapter 6**: Discusses and summarizes the results; conclusion.

1.4 Contributions

To our knowledge, we present these original contributions:

- $\mathbb{HL}$ as a geometric surface (PROPOSITIONS 2.2.1-2.2.3; 2.2.24; 2.2.26). $\mathbb{HP}$ as a geometric surface (PROPOSITION 2.3.2; 2.3.18).
- Action of $SO(1,2)_0$ on $\mathbb{R}^{1,2}$ (PROPOSITIONS 3.6.6; 3.6.7); the map hat : $\mathfrak{so}(1,2) \to \mathbb{R}^{1,2}_{\wedge}$ (DEFINITION 3.3.6); $\mathbb{R}^{1,2}_{\wedge}$ is a Lie algebra (PROPOSITION 3.3.5, THEOREM 3.3.7); automorphism group $\mathrm{Aut}(\mathbb{R}^{1,2}_{\wedge})$ (PROPOSITIONS 3.3.9, 3.4.1-3.4.3)
- L.d.f.e. classification of full-rank homogeneous 2-input control affine systems (LEMMA 4.1.6; PROPOSITIONS 4.1.8-4.1.9, THEOREM 4.1.10); l.d.f.e. classification of full-rank 2-input control affine inhomogeneous systems (THEOREMS 4.1.12, 4.1.14, 4.1.19, 4.1.20, 4.1.26- 4.1.27; PROPOSITIONS 4.1.16,4.1.18, 4.1.23- 4.1.25 and COROLLARY 4.1.13); l.d.f.e. classification of single-input control affine sytems (PROPOSITIONS 4.1.30-4.1.34).
- Spacelike, lightlike and timelike elements of $\mathfrak{so}(1,2)$ and their properties (PROPOSITIONS 4.2.2- 4.2.5, LEMMA 4.2.6); controllability criterion for all control affine systems on $SO(1,2)_0$ (PROPOSITION 4.2.1, THEOREMS 4.2.7- 4.2.9).
- Optimal control and optimal Hamiltonian for control problem with quadratic costs on $\Sigma_1^{(2,0)}$ [$\Sigma_2^{(2,0)}$] (THEOREM 5.2.1) [THEOREM 5.3.1]; projection of extremal curves onto $\mathfrak{so}(1,2)^*$ in terms of Jacobi elliptic functions or trigenometric functions (THEOREM 5.2.2) [THEOREMS 5.3.2-5.3.3]; projection onto $SO(1,2)_0$ of extremal curve as a solution of a system of differential equations (THEOREMS 5.2.3-5.2.4) [THEOREM 5.3.4]; equilibrium points of the system of reduced extremal equations classified nonlinear stable/unstable (THEOREM 5.2.5) [THEOREM 5.3.5].

Chapter 2

Two Models of Hyperbolic Plane Geometry

We choose to concentrate our study on the hyperboloid and upper half-plane models and their symmetry groups. In order to discuss the hyperboloid model (which can be considered as a submanifold of reduced Minkowski spacetime $\mathbb{R}^{1,2}$ equipped with the induced metric) we consider properties of $\mathbb{R}^{1,2}$ as an inner product space.

2.1 Reduced Minkowski spacetime $\mathbb{R}^{1,2}$

Minkowski spacetime refers to the 4-dimensional real space $\mathbb{R}^4$ equipped with the the symmetric, nondegenerate bilinear product $\odot$, called the Lorentz product,

$$\odot : \mathbb{R}^4 \to \mathbb{R}^4, \quad \mathbf{p} \odot \mathbf{q} = -p_1q_1 + p_2q_2 + p_3q_3 + p_4q_4$$

This structure $\mathbb{R}^{1,3} = (\mathbb{R}^4, \odot)$ is the geometric setting of special relativity: the points in $\mathbb{R}^{1,3}$ are considered as physical events taking place at a point within a frame of reference $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, where the $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ directions relate the events spatially and the $\mathbf{e}_1$-direction relates them in time. In this thesis we suppress one spatial dimension.

2.1.1 Definition. The **reduced Minkowski spacetime** $\mathbb{R}^{1,2}$ is the 3-dimensional real space $\mathbb{R}^3$ equipped with the Lorentz product,

$$\odot : \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{p} \odot \mathbf{q} = -p_1q_1 + p_2q_2 + p_3q_3.$$

We henceforth identify $\mathbf{p} = (p_1, \ldots, p_n)$ with the column matrix $\begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$.

We may express the Lorentz product in the matrix form

$$\mathbf{p}^\top J\,\mathbf{q} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

Particularly, if we consider the quadratic forms $\mathbf{p}^\top J\,\mathbf{p}$, then $\mathbb{R}^{1,2} = (\mathbb{R}^3, \odot)$ becomes an inner product space as in (A.1.5). The Lorentz product is not positive-definite, and $\mathbf{p} \odot \mathbf{p}$ can take positive, negative or zero values.

2.1.2 REMARK. By a slight abuse of notation, we associate the symmetric bilinear forms

$$ds^2|_{\mathbf{p}} : \mathrm{T}_{\mathbf{p}}\mathbb{R}^3 \times \mathrm{T}_{\mathbf{p}}\mathbb{R}^3 \to \mathbb{R}, \quad ds^2|_{\mathbf{p}} = -dx^2|_{\mathbf{p}} + dy^2|_{\mathbf{p}} + dz^2|_{\mathbf{p}}$$

with $\odot$. Here, $\mathrm{T}_{\mathbf{p}}\mathbb{R}^3$ is an isomorphic copy of $\mathbb{R}^3$. We may express $ds^2|_{\mathbf{p}}$ in matrix form as

$$\mathbf{v}^\top J\,\mathbf{w} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{2.1.1}$$

where $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ associated to the tangent vectors $\mathbf{v} = v_1 \frac{\partial}{\partial u}|_{\mathbf{p}} + v_2 \frac{\partial}{\partial v}|_{\mathbf{p}} + v_3 \frac{\partial}{\partial w}|_{\mathbf{p}}$ and $\mathbf{w} = w_1 \frac{\partial}{\partial u}|_{\mathbf{p}} + w_2 \frac{\partial}{\partial v}|_{\mathbf{p}} + w_3 \frac{\partial}{\partial w}|_{\mathbf{p}}$ in the fibre $\mathrm{T}_{\mathbf{p}}\mathbb{R}^3$ of $\mathrm{T}\mathbb{R}^3$. Since $ds^2|_{\mathbf{p}}$ acts on the tangent vectors $v_1 \frac{\partial}{\partial u}|_{\mathbf{p}} + v_2 \frac{\partial}{\partial v}|_{\mathbf{p}} + v_3 \frac{\partial}{\partial w}|_{\mathbf{p}}$ and $w_1 \frac{\partial}{\partial u}|_{\mathbf{p}} + w_2 \frac{\partial}{\partial v}|_{\mathbf{p}} + w_3 \frac{\partial}{\partial w}|_{\mathbf{p}}$ to give $-v_1 w_1 + v_2 w_2 + v_3 w_3$ (which is exactly the matrix product $\mathbf{v}^\top J\,\mathbf{w}$) this association is justified.

2.1.3 DEFINITION. An element $\mathbf{p} = (p_1, p_2, p_3)$ of $\mathbb{R}^{1,2}$ is **timelike** if $\mathbf{p} \odot \mathbf{p} < 0$, **lightlike** if $\mathbf{p} \odot \mathbf{p} = 0$ and **spacelike** if $\mathbf{p} \odot \mathbf{p} > 0$.

2.1.4 DEFINITION. The **Minkowski length** is the mapping $\|\cdot\| : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $\|\mathbf{p}\| = \mathrm{sgn}(\mathbf{p})\sqrt{|\mathbf{p} \odot \mathbf{p}|}$ where $\mathrm{sgn}(\mathbf{p}) = 1$ if $\mathbf{p} \odot \mathbf{p} > 0$, $\mathrm{sgn}(\mathbf{p}) = -1$ if $\mathbf{p} \odot \mathbf{p} < 0$, and $\mathrm{sgn}(\mathbf{p}) = 0$ if $\mathbf{p} \odot \mathbf{p} = 0$.

Note that this mapping is not positive-definite.

2.1.5 DEFINITION. A timelike vector $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^{1,2}$ is **positively-oriented** if $p_1 > 0$ and **negatively-oriented** if $p_1 < 0$.

2.1.6 PROPOSITION. *Orientation of timelike vectors is an orientation relation on this subset of* $\mathbb{R}^{1,2}$.

PROOF. We show that this relation is an equivalence relation. Note that if $\mathbf{p}$ is timelike, then $p_1 \neq 0$, so p_1 is either positive or negative. Firstly, if $p_1 > 0[p_1 < 0]$, then $\mathbf{p}$ is positively-oriented [negatively-oriented] and in each case $\mathbf{p} \sim \mathbf{p}$. Secondly, if $\mathbf{p} \sim \mathbf{q}$, then $p_1 > 0, q_1 > 0[p_1 < 0, q_1 < 0]$, and in each case $\mathbf{q} \sim \mathbf{p}$. Finally, if $\mathbf{p} \sim \mathbf{q}$ and $\mathbf{q} \sim \mathbf{s}$, then $p_1 > 0, q_1 > 0$ and $q_1 > 0, s_1 > 0$ $[p_1 > 0, q_1 > 0$ and $q_1 > 0, s_1 > 0]$ and in both cases it follows that $\mathbf{p} \sim \mathbf{s}$. Since either $p_1 > 0$ or $p_1 < 0$, there are exactly two equivalence classes. The result follows. □

2.1.7 PROPOSITION. *[22] Given a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for an n-dimensional real vector space V on which is defined a nondegenerate symmetric bilinear form $\chi^s : V \times V \to \mathbb{R}$, $\chi^s(\mathbf{e}_i, \mathbf{e}_j) = 0$ for $i \neq j$, then the number of basis vectors $\mathbf{e}_i$ for which $\chi^s(\mathbf{e}_i, \mathbf{e}_i) = -1$ is the same for any such basis.*

2.1.8 COROLLARY. *If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis for $\mathbb{R}^{1,2}$, then exactly one of $\mathbf{e}_1, \mathbf{e}_2$ or $\mathbf{e}_3$ is timelike.*

2.1.9 REMARK. This result is not generally true of bases which are not orthonormal; indeed, we may construct a timelike basis

$$\{(1,0,0),(2,1,0),(3,2,1)\} \tag{2.1.2}$$

for $\mathbb{R}^{1,2}$.

2.1.10 PROPOSITION. *A nonzero vector of $\mathbb{R}^{1,2}$ orthogonal to a timelike vector must be spacelike.*

PROOF. We will show that a timelike vector cannot be orthogonal to a timelike vector or a (nonzero) null vector. Given a timelike vector $\mathbf{p} = (p_1, p_2, p_3)$ and a (nonzero) null or timelike vector $\mathbf{q} = (q_1, q_2, q_3)$, then $\mathbf{p} \odot \mathbf{p} = -p_1^2 + p_2^2 + p_3^2 < 0$ and also the product $\mathbf{q} \odot \mathbf{q} = -q_1^2 + q_2^2 + q_3^2 \leq 0$. Thus $p_1^2 q_1^2 > (p_2^2 + p_3^2) \cdot (q_2^2 + q_3^2) \geq (p_2 q_2 + p_3 q_3)^2$, where the second inequality follows from the Cauchy-Schwartz inequality on $(\mathbb{R}^2, \bullet)$, where $\bullet$ is the usnal dot product. Thus $|p_1 q_1| > |p_2 q_2 + p_3 q_3|$, and particularly $p_1 q_1 \neq 0$. Thus $\mathbf{p} \odot \mathbf{q} \neq 0$. □

We now consider as in (A.1.14) the restriction to a hyperplane Γ in $\mathbb{R}^{1,2}$ of the inner product $\odot$, which we denote by $\mathbf{p}^\top J\,\mathbf{p}$. In the next 5 statements we refer to (A.1.1)-(A.1.16) for the necessary resnlts from the theory of quadratic forms on vector spaces.

2.1.11 PROPOSITION. *The restriction $J|_\Gamma$ of the Lorentz product to a hyperplane Γ in $\mathbb{R}^{1,2}$ acting as a quadratic form has signature* $(0,1,1),(0,2,0)$ *or* $(1,1,0)$ *only.*

PROOF. Let a, b and c be real. Given a hyperplane $\Gamma = \{ax + by + cz = h | x, y, z \in \mathbb{R}\}$ of $\mathbb{R}^{1,2}$, then firstly taking $a \neq 0$, we may write $x = -\frac{b}{a}y - \frac{c}{az}z + h$. Thus $dx = -\frac{b}{a}dy - \frac{c}{a}dz$, and ds^2 restricted to Γ can be expressed as

$$ds^2|_\Gamma = -\left(-\frac{b}{a}dy - \frac{c}{a}dz\right)^2 + dy^2 + dz^2 = \left(1 - \frac{b^2}{a^2}\right)dy^2 + \left(1 - \frac{c^2}{a^2}\right)dz^2 - \frac{2bc}{a^2}dydz.$$

As a quadratic form, making the association of $ds^2|_\Gamma$ with J_Γ, then

$$\mathbf{p}^\top J|_\Gamma\,\mathbf{p} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} \left(1 - \frac{b^2}{a^2}\right) & \frac{-bc}{a^2} \\ \frac{-bc}{a^2} & \left(1 - \frac{c^2}{a^2}\right) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

Using Mathematica (C.1), $J|_\Gamma$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{(a^2 - b^2 - c^2)}{a^2}$. The eigenvalue λ_1 is clearly always positive, while λ_2 may take both positive and negative real values. Taking $b \neq 0$, we may write $y = -\frac{a}{b}x - \frac{c}{b}z + h$, and so $dy = -\frac{a}{b}dx - \frac{c}{b}dz$, and

$$ds^2|_\Gamma = -dx^2 + \left(-\frac{a}{b}dx - \frac{c}{b}dz\right)^2 + dz^2 = \left(-1 + \frac{a^2}{b^2}\right)dx^2 + \left(1 + \frac{c^2}{b^2}\right)dz^2 + \frac{2ac}{b^2}dxdz.$$

As a quadratic form,

$$\mathbf{p}^\top J|_\Gamma\,\mathbf{p} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} \left(\frac{a^2}{b^2} - 1\right) & \frac{ac}{b^2} \\ \frac{ac}{b^2} & \left(\frac{c^2}{b^2} + 1\right) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

$J|_\Gamma$ has eigenvalues

$$\lambda_1 = \frac{a^2 + c^2 - \sqrt{a^4 + (2b^2 + c^2)^2 + a^2(-4b^2 + 2c^2)}}{2b^2}, \quad \lambda_2 = \frac{a^2 + c^2 + \sqrt{a^4 + (2b^2 + c^2)^2 + a^2(-4b^2 + 2c^2)}}{2b^2}$$

from (C.1). The eigenvalue λ_2 clearly always has a positive real part. Finally, taking $c \neq 0$, then we may express $z = -\frac{a}{c}x - \frac{b}{c}y + h$, and so $dz = -\frac{a}{c}x - \frac{b}{c}y$, and

$$-dx^2 + dy^2 + \left(-\frac{a}{c}x - \frac{b}{c}y\right)^2 = \left(\frac{a^2}{c^2} - 1\right)dx^2 + \left(1 + \frac{b^2}{c^2}\right)dy^2 + \frac{2ab}{c^2}dydx.$$

As a quadratic form,

$$\mathbf{p}^\top J|_\Gamma \mathbf{p} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} \left(\frac{a^2}{c^2} - 1\right) & \frac{ab}{c^2} \\ \frac{ab}{c^2} & \left(\frac{b^2}{c^2} + 1\right) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

$J|_\Gamma$ has eigenvalues

$$\lambda_1 = \frac{a^2 + b^2 - \sqrt{a^4 + 2a^2(b^2 - 2c^2) + (b^2 + 2c^2)^2}}{2c^2}, \quad \lambda_2 = \frac{a^2 + b^2 + \sqrt{a^4 + 2a^2(b^2 - 2c^2) + (b^2 + 2c^2)^2}}{2c^2}$$

from (C.1). The eigenvalue λ_2 clearly always has a positive real part. Thus in each case by (A.1.9), these quadratic forms have signature $(0, 1, 1)$, $(0, 2, 0)$ or $(1, 1, 0)$ only. □

We make the definition

2.1.12 DEFINITION. A hyperplane $\mathbf{\Gamma}$ in $\mathbb{R}^{1,2}$ is said to be **elliptic** if the restriction of the scalar product $\odot$ to $\mathbf{\Gamma}$ is a quadratic form with signature $(0, 2, 0)$. $\mathbf{\Gamma}$ is **parabolic** if the restriction of $\odot$ to $\mathbf{\Gamma}$ is a quadratic form with signature $(1, 1, 0)$. $\mathbf{\Gamma}$ is **hyperbolic** if the restriction of $\odot$ to $\mathbf{\Gamma}$ is a quadratic form with signature $(0, 1, 1)$.

By PROPOSITION 2.1.11, these classes partition the family of all hyperplanes of $\mathbb{R}^3$ into three distinct classes: the elliptic, hyperbolic and parabolic classes. Particularly, the 2-dimensional linear subspaces are partitioned into classes of elliptic, hyperbolic and parabolic subspaces.

2.1.13 DEFINITION. An element $\mathbf{p} = \mathbf{a} + u_1^0\mathbf{b}_1 + \ldots u_i^0\mathbf{b}_i$ of a hyperplane $\mathbf{\Gamma} = \mathbf{a} + \langle \mathbf{b}_1, \ldots, \mathbf{b}_\ell \rangle$ is a **timelike [spacelike, lightlike] vector** if $\mathbf{p} \odot \mathbf{p} < 0$ [$\mathbf{p} \odot \mathbf{p} > 0$, $\mathbf{p} \odot \mathbf{p} = 0$].

2.1.14 PROPOSITION. *Given any two 2-dimensional linear subspaces $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ of $\mathbb{R}^{1,2}$ of the same type (elliptic, parabolic or hyperbolic), there exists an element of the group of inner product space isometries of $\mathbb{R}^{1,2}$ which maps $\mathbf{\Gamma}_1$ to $\mathbf{\Gamma}_2$ bijectively.*

PROOF. Since $J|_{\Gamma_1} = J|_{\Gamma_2}$, then the subspaces $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are (inner product space) isometric, and so by WITT'S THEOREM (A.1.16) they lie within the same orbit (A.1.13) in $\mathbb{R}^{1,2}$. Thus there exists an inner product space isometry of $\mathbb{R}^{1,2}$ which maps the one to the other bijectively. □

We now state and prove a simple result concerning the visualization of the subsets of spacelike, timelike and lightlike vectors in $\mathbb{R}^{1,2}$.

2.1.15 PROPOSITION. *The lightlike elements of* $\mathbb{R}^{1,2}$ *lie on* $\mathcal{K}_L : \{(x, y, z) \in \mathbb{R}^3 \ : \ -x^2 + y^2 + z^2 = 0\}$, *the cone with axis of rotation the* $\mathbf{e}_1$*-axis. The spacelike elements lie on the hyperboloids of one sheet* $\mathcal{H}^1_r : \{(x, y, z) \in \mathbb{R}^3 \ : \ -x^2 + y^2 + z^2 = r^2\}$. *The timelike elements lie on the hyperboloids of two sheets* $\mathcal{H}^2_r : \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 - y^2 - z^2 = r^2\}$.

PROOF. Consider the spacelike element $\mathbf{p} \in \mathsf{SO}(1,2)_0$; then $\mathbf{p} \odot \mathbf{p} = -p_1^2 + p_2^2 + p_3^2 = 0$ and $(p_1, p_2, p_3) \in \mathcal{K}_L$. Taking $\mathbf{p}$ spacelike, then $\mathbf{p} \odot \mathbf{p} = -p_1^2 + p_2^2 + p_3^2 = \|\mathbf{p}\|^2$ and thus $\mathbf{p} \in \mathcal{H}^1_{\|\mathbf{p}\|}$. Similarly, taking $\mathbf{q}$ timelike, then $\mathbf{q} \odot \mathbf{q} = -q_1^2 + q_2^2 + q_3^2 = \|\mathbf{q}\|^2 < 0$. Consider the vector $\mathbf{p}$ such that $\|\mathbf{p}\|^2 > 0, \|\mathbf{p}\|^2 = -\|\mathbf{q}\|^2$. Then $-q_1^2 + q_2^2 + q_3^2 = -\|\mathbf{p}\|^2$, and $\mathbf{q} \in \mathcal{H}^2_{\|\mathbf{p}\|}$. □

2.2 The hyperboloid model

2.2.1 The hyperboloid as a geometric surface

Given the open subsets of $\mathbb{R}^2$, $U_1 = (-\infty, \infty) \times (0, 2\pi)$ and $U_2 = (-\infty, \infty) \times (-\pi, \pi)$, then the mappings

$$\epsilon^1 : U_1 \to \mathbb{R}^3, \quad \epsilon^1(u, v) = (\cosh u, \sinh u \cos v, \sinh u \sin v)$$

and

$$\epsilon^2 : U_2 \to \mathbb{R}^3, \quad \epsilon^2(u, v) = (\cosh u, \sinh u \cos v, \sinh u \sin v)$$

are continuous and injective since each of their component functions are continuous and injective. We consider the set $\mathcal{HL} = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 - y^2 - z^2 = 1, x > 0\}$ equipped with the symmetric bilinear form (A.2.15) $ds^2 = -dx^2 + dy^2 + dz^2$ acting on the tangent bundle $\mathrm{T}\mathcal{HL}$.

2.2.1 PROPOSITION. $\mathcal{HL}$ *is an abstract surface: that is, the union* $\bigcup_{i=1,2}(\epsilon_i, U_i)$ *covers* $\mathcal{HL}$, *and the transition maps between the two patches are smooth.*

PROOF. Given $(x, y, z) \in \bigcup_{i=1,2}(\epsilon^i, U_i)$, then $(x, y, z) = \epsilon_i(u, v) = (\cosh u, \sinh u \cos v, \sin u \sin v)$ and $-x^2+y^2+z^2 = -1$. Thus $\bigcup_{i=1,2}(\epsilon_i, U_i) \subseteq \mathcal{HL}$. We show that the condition $-p_1^2+p_2^2+p_3^2 = -1$ on $\mathbf{p} = (p_1, p_2, p_3) \in \mathcal{HL}$ defines a pair (u_0, v_0) such that $\mathbf{p}$ lies either in the patch (ϵ_1, U_1) or (ϵ_2, U_2). Consider $(p_1, p_2, p_3) \in \mathbb{R}^{1,2}$ such that $-p_1^2 + p_2^2 + p_3^2 = -1$. Clearly, there exists some $u_0 = \cosh^{-1}(p_1)$ such that $p_1 = \cosh u_0$. Then $p_1^2 = \cosh^2 u_0$ and $p_1^2 - p_2^2 - p_3^2 = 1$, so thus $p_2^2 + p_3^2 = \sinh^2 u_0$. Further, since $\frac{p_2^2+p_3^2}{\sinh(u_0)} = \frac{p_2^2+p_3^2}{-1+p_1^2} = 1$, then

$$\left|\frac{p_2}{\sqrt{-1+p_1^2}}\right| = \left|\frac{p_2}{\sqrt{p_2^2+p_3^2}}\right| \leq 1 \quad \text{and} \quad \left|\frac{p_3}{\sqrt{-1+p_1^2}}\right| = \left|\frac{p_3}{\sqrt{p_2^2+p_3^2}}\right| \leq 1$$

and so we can find a preimage v_1 in $(0, 2\pi)$ or $(-\pi, \pi)$ of $\frac{p_2}{\sinh(u_0)}$ under cos and a preimage v_2 in $(0, 2\pi)$ or $(-\pi, \pi)$ of $\frac{p_3}{\sinh(u_0)}$ under sin. Since

$$\left(\frac{p_2}{\sqrt{p_1^2+p_3^2}}\right)^2 + \left(\frac{p_3}{\sqrt{p_1^2+p_3^2}}\right)^2 = 1$$

then $v_1 = v_2 = v_0$ in $(0, 2\pi)$ or $(-\pi, \pi)$. Thus $p_2 = \sinh u_0 \cos v_0$ and $p_3 = \sinh u_0 \sin v_0$ for some (u_0, v_0) iu U_1 or U_2 such that $(p_1, p_2, p_3) = (\cosh u_0, \sinh u_0 \cos v_0, \sinh u_0 \sin v_0)$. Then

(p_1, p_2, p_3) lies in either the patch (ϵ_1, U_1) or the patch (ϵ_2, U_2). Thus $\mathcal{HL} \subseteq \bigcup_{i=1,2}(\epsilon_i, U_i)$. Since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{bmatrix} \begin{bmatrix} \cosh u \\ \sinh u \cos v \\ \sinh u \sin v \end{bmatrix} = \begin{bmatrix} \cosh u \\ \sinh u \cos(v+\pi) \\ \sinh u \sin(v+\pi) \end{bmatrix}$$

then the transition maps $(\epsilon^1)^{-1} \circ \epsilon^2$ and $(\epsilon^2)^{-1} \circ \epsilon^1$ are smooth rotations about the origin. Thus by (A.2.1), the patches $(\epsilon^1, U_1), (\epsilon^2, U_2)$ express $\mathcal{HL}$ as an abstract surface. □

We have stated that we equip $\mathcal{HL}$ with the bilinear form $ds^2 = -dx^2 + dy^2 + dz^2$. As in the previous section, we express ds^2 as

$$\mathbf{p}^\top J\,\mathbf{q} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

2.2.2 PROPOSITION. *Each element of each tangent plane* $\mathrm{T}_\mathbf{p}\mathcal{HL}$ *is spacelike.*

PROOF. Consider an element $\mathbf{v} \in \mathrm{T}_\mathbf{p}\mathcal{HL}$ where $\mathbf{p} = \epsilon^i(u_0, v_0)$. Then by definition $\mathbf{v}$ is an element of $\langle \epsilon^i_u(u_0, v_0), \epsilon^i_v(u_0, v_0)\rangle$. But $\epsilon^i_u(u_0, v_0) = (\sinh u_0, \cosh u_0 \cos v_0, \cosh u_0 \sin v_0)$ and $\epsilon^i_v(u_0, v_0) = (0, -\sinh u_0 \sin v_0, \sinh u_0 \cos v_0)$. Since $\mathbf{p} = (\cosh u_0, \sinh u_0 \cos v_0, \sinh u_0 \sin v_0)$, then

$$\mathbf{p} \odot \epsilon^i_u(u_0, v_0) = (\cosh u_0, \sinh u_0 \cos v_0, \sinh u_0 \sin v_0) \odot (\sinh u_0, \cosh u_0 \cos v_0, \cosh u_0 \sin v_0) = 0$$

and

$$\mathbf{p} \odot \epsilon^i_v(u_0, v_0) = (\cosh u_0, \sinh u_0 \cos v_0, \sinh u_0 \sin v_0) \odot (0, -\sinh u_0 \sin v_0, \sinh u_0 \cos v_0) = 0$$

and $v = a\epsilon^i_u(u_0, v_0) + b\epsilon^i_v(u_0, v_0)$ is perpendicular to the timelike vector $\mathbf{p} \in \mathcal{HL}$ by the bilinearity of $\odot$. But from PROPOSITION 2.1.10, any vector perpendicular to a timelike vector must be spacelike. The result follows. □

2.2.3 PROPOSITION. $\mathcal{HL}$ *is a geometric surface.*

PROOF. Since we showed in PROPOSITION 2.2.1 that $\mathcal{HL}$ is an abstract surface, we require only to show that $ds^2 = -dx^2 + dy^2 + dz^2$ acts on the tangent spaces of $\mathcal{HL}$ as a pseudo-Riemannian metric. Identifying the tangent vectors v, w with triplets $\mathbf{v}$ and $\mathbf{w}$, we express ds^2 in matrix form

$$\mathbf{v}^\top J\,\mathbf{w} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

1. Bilinearity follows immediately from the linearity of the matrix product.

2. By the symmetry of the matrix J, the bilinear form $\mathbf{v}^\top J\mathbf{w}$ is symmetric.

3. By PROPOSITION 2.2.2, the vectors of each tangent plane to $\mathcal{HL}$ are spacelike. Thus the restriction of ds^2 to each tangent space of $\mathcal{HL}$ is positive-definite.

Then ds^2 fulfils the requirements of a Riemannian metric (A.5.28) on $\mathcal{HL}$. □

2.2.4 DEFINITION. The geometric surface $\mathcal{HL}$ equipped with the Riemannian metric ds^2 is the **hyperboloid model** $\mathbb{HL} = (\mathcal{HL}, ds^2)$.

2.2.5 THEOREM. *The hyperboloid model has a constant Gaussian curvature* $\mathrm{K} = -1$.

PROOF. From (A.2.52), K is a function of $E = \langle \epsilon^i_u, \epsilon^i_u \rangle$, $F = \langle \epsilon^i_u, \epsilon^i_v \rangle$ and $G = \langle \epsilon^i_v, \epsilon^i_v \rangle$, where in this case $\langle \cdot, \cdot \rangle = -dx^2 + dy^2 + dz^2$. Since the derivatives $\epsilon^i_u = (\sinh u, \cos v \cosh u, \cosh u \sin v)$ and $\epsilon^i_v = (0, -\sin v \sinh u, \cos v \sinh u)$ for $i = 1, 2$, then $E = 1, F = 0$ and $G = \sinh v$ in each of the two patches. Since $F = 0$ in both the patches ϵ^1 and ϵ^2, then in each case the Gaussian curvature is given by (A.2.53)

$$\mathrm{K} = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) \right).$$

In both patches $E_u = 0, G_v = \sqrt{\sinh v^2}$; thus substitution into (A.2.53) of E, F and G using Mathematica (6) shows that in each patch

$$\mathrm{K} = \frac{\frac{2\cosh v^2 \sinh[v]^2}{(\sinh v^2)^{3/2}} - \frac{2\cosh v^2}{\sqrt{\sinh v^2}} - \frac{2\sinh v^2}{\sqrt{\sinh v^2}}}{2\sqrt{\sinh v^2}} = \frac{-1}{2\sinh v} \cdot (2\sinh v) = -1.$$ □

It is well known (see [25]) that the family $\mathcal{G}$ of geodesics on a surface contains the family of all distance-minimising curves on that surface: that is, there exists a subfamily of $\mathcal{G}$ of curves which solve the Riemannian problem (A.2.47). As described in (A.2.47), we use the Euler-Lagrange equations (A.2.46) to solve the Riemannian problem on $\mathbb{HL}$, and so establish a subfamily of geodesics for $\mathbb{HL}$.

2.2.6 PROPOSITION. *The paths* $\gamma = \{(\cosh t, \sinh t \cos\theta, \sinh t \sin\theta) : t \in \mathbb{R}\}$ *are geodesics of* $\mathbb{HL}$.

PROOF. From the definition in (A.2.25), the pullback of $ds^2 = -dx^2 + dy^2 + dz^2$ by the patches ϵ^1 and ϵ^2 is given by $Edu^2 + 2Fdudv + Gdv^2$. From the calculation of E, F and G in PROPOSITION 2.2.5, this pullback is given by $-du^2 + \sinh^2 u dv^2$ in both cases. Thus the Lagrangian for the Riemannian problem is $\mathcal{L}(u, v) = -\dot{u}^2 + \sinh^2 u \dot{v}^2$, and using the partial derivatives

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial u} & = 2\cosh u \sinh u (\dot{v}^2) \\ \frac{\partial \mathcal{L}}{\partial \dot{u}} & = -2\dot{u} \end{cases} \qquad \begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{v}} & = 2(\sinh u)^2 \dot{v} \\ \frac{\partial \mathcal{L}}{\partial v} & = 0 \end{cases}$$

we determine the Euler-Lagrange equations (A.2.50)

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) = 2(\sinh u \cosh u)\dot{v} + 2\ddot{u} = 0 \qquad (2.2.1)$$

$$\frac{\partial \mathcal{L}}{\partial v} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{v}}\right) = 2(\sinh u)^2 \dot{v} = k. \qquad (2.2.2)$$

which we solve in the particular case $v = \frac{\pi}{2}$. From (2.2.1), taking $v = \frac{\pi}{2}$, then $-2\ddot{u} = 0$ and so $u = at + b$ for some $a, b \in \mathbb{R}$. Thus the paths $\gamma_0 = \{(\cosh u, \sinh u, 0) \, : \, u \in \mathbb{R}\}$ are geodesics of $\mathbb{HL}$, since they are images of the maps which solve the Riemannian problem and thus are distance-minimising. Since from PROPOSITION 2.2.9 the Euclidean rotations about the $\mathbf{e}_1$-axis are isometries of $\mathbb{HL}$, it follows from (A.2.42) that for each θ, the image

$$\gamma = \{(\cosh u, \sinh u \cos\theta, \sinh u \sin\theta) \, : \, u \in \mathbb{R}\}$$

of this curve under rotation is similarly a geodesic of $\mathbb{HL}$. □

2.2.7 PROPOSITION. *Given any point* $\mathbf{p}$ *on* $\mathcal{HL}$*, there exists a geodesic of the form* γ *passing through* $(1, 0, 0)$ *and* $\mathbf{p}$.

PROOF. The path γ_0 in $\mathcal{HL}$ has a direction vector $\mathbf{v}$ in the tangent plane to $\mathcal{HL}$ at $(1, 0, 0)$. But using the parametrization of $\mathcal{HL}$ in terms of patches ϵ^1 and ϵ^2, the tangent plane at $\epsilon^1(u, v) = (1, 0, 0)$, which occurs at $(u, v) = (0, 0)$ is the span of orthogonal vectors

$$\epsilon^1_u(0,0) \quad \text{and} \quad \epsilon^1_v(0,0), \quad \text{where } \epsilon^1_u(0,0) = (0,1,0) \quad \text{and} \quad \epsilon^1_v(0,0) = (0,0,1).$$

Thus the plane $\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ is tangent to $\mathcal{HL}$ at the point $(1, 0, 0)$. Then under the Euclidean rotations about the $\mathbf{e}_1$-axis, the direction vector $\mathbf{v}$ of γ_0 is mapped to the direction vector $R_\theta \mathbf{v}$ of γ. Thus there exists a geodesic of the form γ passing through $(1, 0, 0)$ in the direction of every tangent vector to $\mathcal{HL}$ at $(1, 0, 0)$. Stated differently, given any $\mathbf{p}$ in $\mathcal{HL}$, there is a geodesic of the form γ passing through $(1, 0, 0)$ and $\mathbf{p}$. □

2.2.2 The symmetry group of $\mathbb{HL}$

In order to find the isometries of $\mathbb{HL}$, we consider first the inner product space isometries of $\mathbb{R}^{1,2}$ and then restrict these isometries to those preserving the abstract surface $\mathcal{HL}$. We then show that all isometries of $\mathbb{HL}$ are linear transformations which preserve the scalar product $\odot$. For the initial propositions 2.2.9-2.2.11 we follow a similar approach to [10].
For any bilinear bijections $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, we use the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to make the identification of ϕ with the 3×3 matrix $[g_{ij}]$, where the j-th column $\begin{bmatrix} g_{1j} & g_{2j} & g_{3j} \end{bmatrix} = \phi(\mathbf{e}_j)$.

2.2.8 REMARK. Given an inner product space isometry of $\mathbb{R}^{1,2}$ identified with the matrix g, then for each standard basis element $\mathbf{e}_i, \mathbf{e}_j$ of $\mathbb{R}^3$, $(g\mathbf{e}_i)^\top \odot g\mathbf{e}_j = \mathbf{e}_i^\top g^\top J g \mathbf{e}_j = \mathbf{e}_i^\top J \mathbf{e}_j = J_{ij}$. Then $(g\mathbf{e}_i)^\top \odot g\mathbf{e}_j = \sum_{k=1}^3 g_{ki} g_{kj} = \mathbf{e}_i^\top J \mathbf{e}_j = J_{ij}$, and it follows that

$$J = \begin{bmatrix} \sum_{k=1}^3 g_{k1}g_{k1} & \sum_{k=1}^3 g_{k1}g_{k2} & \sum_{k=1}^3 g_{k1}g_{k3} \\ \sum_{k=1}^3 g_{k2}g_{k1} & \sum_{k=1}^3 g_{k2}g_{k2} & \sum_{k=1}^3 g_{k2}g_{k3} \\ \sum_{k=1}^3 g_{k3}g_{k1} & \sum_{k=1}^3 g_{k3}a_{k2} & \sum_{k=1}^3 g_{k3}g_{k3} \end{bmatrix} = gJg^\top.$$

Thus all inner product space isometries of $\mathbb{R}^{1,2}$ may be expressed as matrices in $\mathbb{R}^{3\times 3}$ satisfying the property $g \in \mathbb{R}^{3\times 3}, g^\top J \, g = J$.

2.2.9 THEOREM. $gJg^\top = J \Rightarrow g = \begin{bmatrix} c & \mathbf{q}^\top \\ \mathbf{q} & \sqrt{\mathbf{1}+\mathbf{q}\mathbf{q}^\top} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_\theta \end{bmatrix}$ *for* $R_\theta \in \mathrm{O}(2), c \in \mathbb{R}^+, \mathbf{q} \in \mathbb{R}^2$.

PROOF. The condition $gJg^\top = J$ can be expressed equivalently as $g^\top J g = J$, since the diagonal matrix J is symmetric. For $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2, \mathbf{p} = (p_1, p_2), \mathbf{q} = (q_1, q_2)$ and $c \in \mathbb{R}$, we may express these conditions as

$$\begin{bmatrix} c & \mathbf{q}^\top \\ \mathbf{p} & m^\top \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & \mathbf{p}^\top \\ \mathbf{q} & m \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c & \mathbf{p}^\top \\ \mathbf{q} & m \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & \mathbf{q}^\top \\ \mathbf{p} & m^\top \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

respectively. Multiplying out, then

$$\begin{bmatrix} -c^2 + \mathbf{q}^\top\mathbf{q} & -c\mathbf{p}^\top + \mathbf{q}^\top m \\ c\mathbf{p} + m^\top \mathbf{q} & -\mathbf{p}\mathbf{p}^\top + m^\top m \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -c^2 + \mathbf{p}^\top\mathbf{p} & -c\mathbf{q}^\top + \mathbf{p}^\top m^\top \\ -c\mathbf{q} + \mathbf{p}m & -\mathbf{q}\mathbf{q}^\top + mm^\top \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and thus

$$-c^2 + \mathbf{q}^\top\mathbf{q} \;=\; 1 = -c^2 + \mathbf{p}^\top\mathbf{p} \tag{2.2.3}$$

$$c\mathbf{p} \;=\; \mathbf{q}m^\top \tag{2.2.4}$$

$$c\mathbf{q} \;=\; \mathbf{p}m \tag{2.2.5}$$

$$m^\top m - \mathbf{p}\mathbf{p}^\top \;=\; \mathbf{1} = \; mm^\top - \mathbf{q}\mathbf{q}^\top \tag{2.2.6}$$

where 2.2.6 implies that m is symmetric and positive-definite and 2.2.3 implies that $|c| \geq 1$. Then m has a polar decomposition (A.5.7) $m = sR_\theta$ where R_θ is in $\mathrm{O}(2)$, and since m is positive-definite, then s is symmetric positive-definite. Thus $s^2 = sh(sh)^\top = mm^\top = \mathbf{q}\mathbf{q}^\top + \mathbf{1}$. But

$$\mathbf{q}\mathbf{q}^\top = \begin{bmatrix} q_1^2 & q_1q_2 \\ q_1q_2 & q_2^2 \end{bmatrix}$$

where by (A.5.10), the characteristic polynomial of $\mathbf{q}\mathbf{q}^\top$ is $\lambda^2 - \lambda \mathrm{tr}(\mathbf{q}\mathbf{q}^\top) + \det(\mathbf{q}\mathbf{q}^\top)$. But clearly $\det(\mathbf{q}\mathbf{q}^\top) = q_1^2q_2^2 - (q_1q_2)^2 = 0$, and, since

$$\begin{bmatrix} q_1^2 & q_1q_2 \\ q_1q_2 & q_2^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \|\mathbf{q}\| \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

then $\mathbf{q}\mathbf{q}^\top$ has an eigenvalue 0 and an eigenvalue $\|\mathbf{q}\| = c^2 - 1$ which has a corresponding eigenvector $\mathbf{q}$. Thus the matrix $s^2 = \mathbf{1} + \mathbf{q}\mathbf{q}^\top$ has the eigenvalues 1 and $c^2 = \|\mathbf{q}\| + 1$. Thus we have the cases

Case 1: $c = \sqrt{1 + \|\mathbf{q}\|^2}$

We have shown that s has an eigenvalue c and $\mathbf{q}^\top s = cs$. Then $\mathbf{q}^\top m = \mathbf{q}^\top sR_\theta = c\mathbf{q}^\top R_\theta$. But by (2.2.4), $c\mathbf{p}^\top = \mathbf{q}^\top m$. Thus $\mathbf{q}^\top R_\theta = \mathbf{p}^\top$, and so

$$g = \begin{bmatrix} c & \mathbf{p}^\top \\ \mathbf{q} & m \end{bmatrix} = \begin{bmatrix} c & \mathbf{q}^\top R_\theta \\ \mathbf{q} & sR_\theta \end{bmatrix} \quad \Rightarrow \quad g = \begin{bmatrix} c & \mathbf{q} \\ \mathbf{q}^\top & \sqrt{\mathbf{1}+\mathbf{q}\mathbf{q}^\top} \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ 0 & R_\theta \end{bmatrix}.$$

Case 2: $c = -\sqrt{\mathbf{1} + \|\mathbf{q}\|^2}$

We have shown that $\mathbf{q}^\top s = \sqrt{\mathbf{1} + \|\mathbf{q}\|^2}\mathbf{q}$. Then $-\mathbf{q}^\top s = -\sqrt{1 + \|\mathbf{q}\|^2}\mathbf{q}$ and so in this case $\mathbf{q}^\top s =$

$-c\mathbf{q}^\top$. Then $\mathbf{q}^\top m = \mathbf{q}^\top sR_\theta = (-c\mathbf{q}^\top)R_\theta$. But by (2.2.4), $\mathbf{q}^\top m = c\mathbf{p}^\top$. Thus $(-c\mathbf{q}^\top)R_\theta = c\mathbf{p}^\top \Rightarrow (-\mathbf{q}^\top)R_\theta = \mathbf{p}^\top$, and so

$$g = \begin{bmatrix} c & \mathbf{p}^\top \\ \mathbf{q} & m \end{bmatrix} = \begin{bmatrix} c & (-\mathbf{q}^\top)h \\ \mathbf{q} & sR_\theta \end{bmatrix} \quad \Rightarrow \quad g = \begin{bmatrix} -c & -\mathbf{q}^\top \\ -\mathbf{q} & \sqrt{\mathbf{1}+\mathbf{q}\mathbf{q}^\top} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & R_\theta \end{bmatrix}.$$

Then taking $c' = \sqrt{\|\mathbf{q}\|^2 + 1}$,

$$g = \begin{bmatrix} -c' & (\mathbf{q}^\top)h \\ -\mathbf{q} & sR_\theta \end{bmatrix} \Rightarrow g = \begin{bmatrix} c' & \mathbf{q}^\top \\ \mathbf{q} & \sqrt{\mathbf{1}+\mathbf{q}\mathbf{q}^\top} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & R_\theta \end{bmatrix}$$

where $c' > 0$. □

2.2.10 REMARK. In THEOREM 2.2.9 we showed that the isometries g of $\mathbb{R}^{1,2}$ can be expressed as one of four matrix products of the form

$$\begin{bmatrix} c & \mathbf{q}^\top \\ \mathbf{q} & \sqrt{\mathbf{1}+\mathbf{q}\mathbf{q}^\top} \end{bmatrix} \begin{bmatrix} \pm 1 & 0 \\ 0 & R_\theta \end{bmatrix} \qquad R_\theta \in \mathsf{O}(2),\ c \in \mathbb{R},\ c > 0$$

which indicates that these isometries fall into four disjoint subsets:

$$\{R_\theta \in \mathsf{O}(2) : \det R_\theta < 0,\ 1 \text{ as upper-left entry in orthogonal matrix}\} \tag{2.2.7}$$

$$\{R_\theta \in \mathsf{O}(2) : \det R_\theta < 0,\ -1 \text{ as upper-left entry in orthogonal matrix}\} \tag{2.2.8}$$

$$\{R_\theta \in \mathsf{SO}(2) : \det R_\theta > 0,\ 1 \text{ as upper-left entry in orthogonal matrix}\} \tag{2.2.9}$$

$$\{R_\theta \in \mathsf{SO}(2) : \det R_\theta > 0,\ -1 \text{ as upper-left entry in orthogonal matrix}\} \tag{2.2.10}$$

We denote (2.2.8) by $\mathsf{SO}(1,2)^-$ and (2.2.9) by $\mathsf{SO}(1,2)_0$.

In DEFINITION 2.1.5, we expressed the orientation of a vector in $\mathbb{R}^{1,2}$, which in PROPOSITION 2.1.6 we showed is indeed an orientation on $\mathbb{R}^{1,2}$ as defined in (A.2.31). We now prove

2.2.11 PROPOSITION. *Given an isometry g in one of (2.2.7) to (2.2.10) such that $\mathbf{e}_1^\top g\mathbf{e}_1 = g_{11} > 0$, then g preserves the orientation of all timelike vectors in $\mathbb{R}^{1,2}$. Conversely, if g preserves the orientation of all timelike vectors in $\mathbb{R}^{1,2}$, then $\mathbf{e}_1^\top g\mathbf{e}_1 = g_{11} > 0$.*

PROOF. Express $g \in \mathsf{G}$ as the matrix of row vectors $g = (g_1, g_2, g_3)$. Then particularly, taking $g_1 = (g_{11}, g_{12}, g_{13})$, then it follows that $-g_{11}^2 + (g_{12}^2 + g_{13}^2) = -1$, since $g_1 J g_1^\top = -1$ by the property $gJg^\top = J$. Acting with g on some timelike vector $\mathbf{a}$, then the first component of $g\mathbf{a}$ is given by $(a_2, a_3) \bullet (g_{12}, g_{13}) + (g_{11}a_1)$, where by the Cauchy-Schwartz inequality on $(\mathbb{R}^2, \bullet)$,

$$((a_2, a_3) \bullet (g_{12}, g_{13}))^2 \le \|(a_2, a_3)\|^2 \|(g_{12}, g_{13})\|^2 \le a_1^2(g_{11}^2 - 1) < t^2 g_{11}^2$$

where $a_1^2 \neq 0$ since $\mathbf{a}$ is timelike. Thus $(a_2, a_3) \bullet (g_{12}, g_{13})$ has the same sign as a_1, since $g_{11} > 0$: that is, $a_1 > 0$. Thus if $g_{11} > 0$, then g maps positive timelike vectors to positive timelike vectors and similarly negative timelike vectors to negative timelike vectors.

Conversely, if g preserves the orientation of all timelike vectors, then particularly g preserves the orientation of $\mathbf{e}_1$ and thus $g\mathbf{e}_1 = (g_{11}, g_{21}, g_{31})$ is timelike and positively-oriented: that is, $g_{11} > 0$. □

2.2.12 PROPOSITION. *The elements of (2.2.7) and (2.2.9) preserve the orientation of the* e_1*-axis; the elements of (2.2.8) and (2.2.10) reverse the orientation of the* e_1*-axis.*

PROOF. Given an element g in (2.2.7) or (2.2.9), then

$$e_1^\top g e_1 = e_1^\top \begin{bmatrix} c & q^\top R_\theta \\ -q & \sqrt{1+qq^\top} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_\theta \end{bmatrix} e_1 = c > 0.$$

Thus by PROPOSITION 2.2.11, g preserves the orientation of all timelike vectors and thus specifically preserves the orientation of the e_1-axis. Given an element g in (2.2.8) or (2.2.10), then

$$e_1^\top g e_1 = e_1^\top \begin{bmatrix} -c & q^\top R_\theta \\ -q & \sqrt{1+qq^\top} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_\theta \end{bmatrix} e_1 = -c < 0.$$

Thus by PROPOSITION 2.2.11, g reverses the orientation of all timelike vectors and thus specifically reverses the orientation of the e_1-axis. □

2.2.13 REMARK. Reversing the orientation of the e_1-axis will send the subset $\{(x,y,z) \in \mathbb{R}^3 \ : \ x > 0\}$ to the subset $\{(x,y,z) \in \mathbb{R}^3 \ : \ x < 0\}$. Thus the elements of the sets (2.2.8) and (2.2.10) send $\mathcal{HL} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 - y^2 - z^2 = 1, x > 0\}$ to $\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 - y^2 - z^2 = 1, x < 0\}$, while the elements (2.2.7) and (2.2.9) preserve $\mathcal{HL}$.

2.2.14 PROPOSITION. *The union* $\mathsf{SO}(1,2)_0 \cup \mathsf{SO}^-(1,2)$ *is the set*

$$\mathsf{SO}(1,2) = \left\{ g \in \mathbb{R}^{3\times3} \ : \ gJg^\top = J, \ \det g = 1 \right\}.$$

PROOF. We proved in THEOREM 2.2.9 that all matrics g such that $gJg^\top = J$ are of the form

$$\begin{bmatrix} c & q^\top \\ q & \sqrt{1+qq^\top} \end{bmatrix} \begin{bmatrix} \pm1 & 0 \\ 0 & R_\theta \end{bmatrix} \qquad R_\theta \in \mathsf{O}(2), c \in \mathbb{R}^+$$

and so are elements of (2.2.7) to (2.2.10). Further, the matrices in (2.2.7) and (2.2.10) have determinant -1, since from the proof of THEOREM 2.2.9 $c\sqrt{1+qq^\top} = 1 + \|q\|$ and so

$$\det g = \begin{vmatrix} c & q^\top \\ q & \sqrt{1+qq^\top} \end{vmatrix} \begin{vmatrix} -1 & 0 \\ 0 & R_\theta \end{vmatrix} = (1 + \|q\|^2 - \|q\|^2)(-1|R_\theta|) = -1$$

while those in (2.2.8) ($\mathsf{SO}^-(1,2)$) and 3 ($\mathsf{SO}(1,2)_0$) have determinant +1 , since

$$\det g = \begin{vmatrix} c & q^\top \\ q & \sqrt{1+qq^\top} \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & R_\theta \end{vmatrix} = (1 + \|q\|^2 - \|q\|^2)(1|R_\theta|) = 1.$$

Thus $\mathsf{SO}(1,2) \subseteq \mathsf{SO}(1,2)_0 \cup \mathsf{SO}^-(1,2)$. Since any matrix g in $\mathsf{SO}^-(1,2)$ or $\mathsf{SO}(1,2)_0$ satsifies $gJg^\top = J$, then $\mathsf{SO}(1,2)_0 \cup \mathsf{SO}^-(1,2) \subseteq \mathsf{SO}(1,2)$, and the result follows. □

2.2.15 PROPOSITION. *The set* $\mathsf{SO}(1,2)_0$ *may be written as*

$$\mathsf{SO}(1,2)_0 = \left\{ g \in \mathbb{R}^{3\times3} \ : \ gJg^\top = J, \ e_1^\top g e_1 > 0, \ \det g = 1 \right\}.$$

PROOF. Since $\mathsf{SO}(1,2)_0$ is subset (2.2.9), then if $g \in \mathsf{SO}(1,2)_0$, $gJg^\top = J$. From PROPOSITION 2.2.12, $\mathbf{e}_1^\top g\mathbf{e}_1 > 0$. From the proof of PROPOSITION 2.2.14, it follows that for the element g in $\mathsf{SO}(1,2)_0$, then det $g = 1$. Thus $\mathsf{SO}(1,2)_0 \subseteq \{g \in \mathbb{R}^{3\times 3} : gJg^\top = J, \mathbf{e}_1^\top g\mathbf{e}_1 > 0, \det g = 1\}$. But any element g of the group $\{g \in \mathbb{R}^{2\times 2} : gJg^\top = J, \det g = 1, \mathbf{e}_1^\top g\mathbf{e}_1 > 0\}$ is an isometry of $\mathbb{R}^{1,2}$ which by THEOREM 2.2.9 belongs to subset (2.2.9). Thus g is an element of $\mathsf{SO}(1,2)_0$ and $\{g \in \mathbb{R}^{2\times 2} : gJg^\top = J, \mathbf{e}_1 g\mathbf{e}_1^\top > 0, \det g = 1\} \subseteq \mathsf{SO}(1,2)_0$. The result follows. □

2.2.16 PROPOSITION. $\mathsf{SO}(1,2)_0$ *is a group under matrix multiplication.*

PROOF. Since for all $g \in \mathsf{SO}(1,2)_0$, det $g = 1$, then $\mathsf{SO}(1,2)_0$ is clearly a subset of $\mathsf{GL}(3,\mathbb{R})$. Thus in order to check that $\mathsf{SO}(1,2)_0$ is a group under matrix multiplication, we require only to show that it is a subgroup of $\mathsf{GL}(3,\mathbb{R})$. We use the defining properties $gJg^\top = J$, det $g = 1$ and $\mathbf{e}_1^\top g\mathbf{e}_1 > 0$ of PROPOSITION 2.2.15. Given any two elements $g, g' \in \mathsf{SO}(1,2)_0$, then

$$(gg')^\top J\,(gg') = (g')^\top (gJg^\top)g' = (g')^\top J\,g' = J.$$

Since $\mathbf{e}_1^\top g\mathbf{e}_1 > 0, \mathbf{e}_1^\top g'\mathbf{e}_1 > 0$, then by PROPOSITION 2.2.11, g and g' preserve the orientation of all timelike vectors and so the orientation of the $\mathbf{e}_1$ axis. But then the product gg' preserves the orientation of the $\mathbf{e}_1$-axis, and it follows from 2.2.11 that $\mathbf{e}_1^\top gg'\mathbf{e}_1 > 0$. Thus $gg' \in \mathsf{SO}(1,2)_0$ and it is closed under matrix multiplication.

Further, given any element $g \in \mathsf{SO}(1,2)_0$, then $gg^{-1} = g^{-1}g = \mathbf{1}$ and so immediately $(gg^{-1})J(gg^{-1})^\top = J$. But $(gg^{-1})J(gg^{-1})^\top = (g^{-1})gJg^\top(g^{-1})^\top = J$, and so $(g^{-1})^\top J\,g^{-1} = J$, since $gJg^\top = J$. Similarly, $\det(gg^{-1}) = 1 = 1 \cdot \det g^{-1}$ and thus det $g^{-1} = 1$. Assume that $\mathbf{e}_1^\top g^{-1}\mathbf{e}_1 < 0$; that is, g^{-1} reverses the orientation of the $\mathbf{e}_1$ axis. But g preserves the orientation of the $\mathbf{e}_1$-axis, and thus $g^{-1}g$ must reverse this orientation. But $\mathbf{e}_1^\top g^{-1}g\mathbf{e}_1 = 1 > 0$, and so by PROPOSITION 2.2.11, $g^{-1}g$ is orientation-preserving, a contradiction. Thus $\mathbf{e}_1^\top g^{-1}\mathbf{e}_1 > 0$ and $g^{-1} \in \mathsf{SO}(1,2)_0$. Thus $\mathsf{SO}(1,2)_0$ is closed under the taking of inverses. □

2.2.17 COROLLARY. $\mathsf{SO}(1,2)$ *is a group under matrix multiplication.*

PROOF. By definition, $g \in \mathsf{SO}(1,2) \Rightarrow gJg^\top = J$ and det $g = 1$. Thus clearly $\mathsf{SO}(1,2)$ is a subset of $\mathsf{GL}(3,\mathbb{R})$. But in the proof of THEOREM 2.2.16, we showed that given g such that $gJg^\top = J$ and det $g = 1$, then $(g^{-1})J(g^{-1})\top = J$ and det $g^{-1} = 1$. Similarly, given g' such that $g'J(g')^\top = J$ and det $g' = 1$, then $(g'g)J(g'g)^\top = J$ and $\det(g'g) = 1$. Thus $\mathsf{SO}(1,2)$ is a subset of $\mathsf{GL}(3,\mathbb{R})$ which is closed under the matrix product and the taking of inverses, and the result follows. □

We shall now show that all symmetries of $\mathbb{HL}$ are linear. In order to do this, we require a (topological) metric (A.2.44) on $\mathcal{HL}$.

2.2.18 PROPOSITION. *The mapping* $d(\cdot,\cdot) : \mathbb{HL} \times \mathbb{HL} \to \mathbb{R}$, $d(\mathbf{p},\mathbf{q}) = \cosh^{-1}(\mathbf{p} \odot \mathbf{q})$, *is a (topological) metric on* $\mathbb{HL}$.

PROOF. Consider the curve $\gamma_0(\cdot) = (\cosh(\cdot), \sinh(\cdot), 0)$ parametrizing the geodesic γ_0 of PROPOSITION 2.2.6. For each $u \in \mathbb{R}$, $\dot{\gamma}_0(u) \odot \dot{\gamma}(u) = 1$. Thus the parametrization of $\mathcal{HL}$ in PROPOSITION 2.2.1 is in terms of arc length along the geodesics γ_0, and so (A.2.45) $d(\gamma_0(0), \gamma_0(u_0)) = u_0$. From THEOREM 2.2.27 (which does not depend on this result or any which follow from it) the isometries $SO(1,2)_0$ act transitively on $\mathcal{HL}$. Thus given any $\mathbf{p}, \mathbf{q} \in \mathcal{HL}$, not necessarily lying on the geodesic γ_0, there exists a $g \in SO(1,2)_0$ such that $g(\mathbf{p}) = (1,0,0)$. Then $g(\mathbf{q}) = \mathbf{q}'$, where by PROPOSITION 2.2.7 there exists a geodesic of the form $\gamma(u)$ passing through $(1,0,0)$ and $\mathbf{q}'$. Thus for some $\theta \in \mathbb{R}$,

$$g(\mathbf{p}) = \gamma_0(0) = (1,0,0) \quad \text{and} \quad g(\mathbf{q}) = \gamma_0(u_0) = (\cosh u_0, \sinh u_0 \cos\theta, \sinh u_0 \sin\theta)$$

and

$$\begin{aligned} \mathbf{p} \odot \mathbf{q} &= g^{-1}\gamma_0(0) \odot g^{-1}\gamma_0(u_0) \\ &= \gamma_0(0) \odot \gamma_0(u_0) \\ &= (1,0,0) \odot (\cosh u_0, \sinh u_0 \cos\theta, \sinh u_0 \sin\theta) \\ &= \cosh u_0 \\ &= \cosh(d(\gamma_0(0), \gamma_0(u_0))) \\ &= \cosh(d(g(\mathbf{p}), g(\mathbf{q}))) \qquad (2.2.11) \\ &= \cosh(d(\mathbf{p}, \mathbf{q})), \end{aligned}$$

where (2.2.12) follows from THEOREM 2.2.9: $g^*(ds^2) = ds^2$, and so $\int_0^{u_0} g^*(ds) = \int_0^{u_0} ds$. Thus $d(\mathbf{p}, \mathbf{q}) = \cosh^{-1}(\mathbf{p} \odot \mathbf{q})$ for each $\mathbf{p}, \mathbf{q} \in \mathcal{HL}$, and we have established a (topological) metric on $\mathcal{HL}$. □

2.2.19 THEOREM. *All isometries of* $\mathbb{HL}$ *are linear transformations which preserve the Lorentz product.*

PROOF. In PROPOSITION 2.2.18, we derived $d(\cdot,\cdot) : \mathcal{HL} \times \mathcal{HL} \to \mathbb{R}$. For each isometry ρ of $\mathbb{HL}$, $d(\rho(\mathbf{p}), \rho(\mathbf{q})) = d(\mathbf{p}, \mathbf{q})$, since $g^*(ds^2) = ds^2$, and so $\int_0^{u_0} g^*(ds) = \int_0^{u_0} ds$. In (2.1.2) we constructed a basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ of $\mathbb{R}^{1,2}$ consisting entirely of timelike vectors, and so by PROPOSITION 2.1.15 elements of $\mathcal{HL}$. For some isometry ρ of $\mathbb{HL}$, $i = 1,2,3$, define the linear map g taking $\tilde{\mathbf{e}}_i$ to $\rho(\tilde{\mathbf{e}}_i)$. Since $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ are spanning for $\mathbb{R}^{1,2}$, then g is uniquely defined. We express each standard basis element $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0)$ and $\mathbf{e}_3 = (0,0,1)$ in terms of the basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$: $\mathbf{e}_i = \Sigma_{j=1}^3 a_{ij}\tilde{\mathbf{e}}_j$ for $a_{ij} \in \mathbb{R}, i, j = 1,2,3$. Then

$$\begin{aligned} g(\mathbf{e}_i) \odot g(\mathbf{e}_j) &= g(\Sigma_{k=1}^3 a_{ik}\tilde{\mathbf{e}}_k) \odot g(\Sigma_{l=1}^3 a_{jl}\tilde{\mathbf{e}}_l) \\ &= \Sigma_{k=1}^3 a_{ik}\rho(\tilde{\mathbf{e}}_k) \odot \Sigma_{l=1}^3 a_{jl}\rho(\tilde{\mathbf{e}}_l) \\ &= \Sigma_{k,l=1}^3 a_{ik}a_{jl}\rho(\tilde{\mathbf{e}}_k) \odot \rho(\tilde{\mathbf{e}}_l) \qquad (2.2.12) \\ &= \Sigma_{k,l=1}^3 a_{ik}a_{jl}\cosh(d(\rho(\tilde{\mathbf{e}}_k), \rho(\tilde{\mathbf{e}}_l))) \qquad (2.2.13) \\ &= \Sigma_{k,l=1}^3 a_{ik}a_{jl}\cosh(d((\tilde{\mathbf{e}}_k), (\tilde{\mathbf{e}}_l))) \\ &= \Sigma_{k,l=1}^3 a_{ik}a_{jl}\tilde{\mathbf{e}}_k \odot \tilde{\mathbf{e}}_l \\ &= \Sigma_{k=1}^3 a_{ik}\tilde{\mathbf{e}}_k \odot \Sigma_{l=1}^3 a_{il}\tilde{\mathbf{e}}_l \\ &= \mathbf{e}_i \odot \mathbf{e}_j. \end{aligned}$$

where (2.2.12) follows from the linearity of $\odot$, and (2.2.13) follows from the definition (A.2.44) of a (topological) metric. Thus g is an isometry of $\mathbb{R}^{1,2}$, and thus the composition $g^{-1} \circ \rho$ is an isometry of $\mathbb{R}^{1,2}$, and for any $\mathbf{p} \in \mathbb{R}^{1,2}$, then

$$\begin{aligned} g^{-1} \circ \rho(\mathbf{p}) \odot \mathbf{e}_i &= \rho(\mathbf{p}) \odot \Sigma_{j=1}^{3} a_{ij} \tilde{\mathbf{e}}_j \\ &= \Sigma_{j=1}^{3} a_{ij} \rho(\mathbf{p}) \odot \rho(\tilde{\mathbf{e}}_j) \qquad (2.2.14) \\ &= \Sigma_{j=1}^{3} a_{ij} \cosh(d(\rho(\mathbf{p}), \rho(\tilde{\mathbf{e}}_j))) \qquad (2.2.15) \\ &= \Sigma_{j=1}^{3} a_{ij} \cosh(d(\mathbf{p}, \tilde{\mathbf{e}}_j)) \\ &= \mathbf{p} \odot \Sigma_{j=1}^{3} a_{ij} \tilde{\mathbf{e}}_j \\ &= \mathbf{p} \odot \mathbf{e}_i. \end{aligned}$$

where (2.2.14) follows from the fact that the isometry g is linear, and (2.2.15) follows from the fact that ρ is an isometry. Since this is true for $i = 1$ to 3, then it must be that $g^{-1} \circ \rho(\mathbf{p}) = \mathbf{p}$ for each $\mathbf{p} \in \mathbb{R}^{1,2}$. Then $g^{-1} \circ \rho = \iota$, the identity transformation, and it follows that $\rho = g$: that is, ρ is a linear isometry. □

We require the next three lemmas to prove the major result of this section.

2.2.20 Lemma. *The matrix* $\begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}$ *has an eigenspace*

$$\left\langle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle$$

and eigenvalues e^{λ} *and* $e^{-\lambda}$.

Proof. Direct computation using the the characteristic polynomial $1 - 2x\cosh(r) + x^2$, which has roots $x_1 = e^{\lambda}$ and $x_2 = e^{-\lambda}$, where the eigenvector $(1, 1)$ corresponds to x_1 and the eigenvector $(1, -1)$ corresponds to x_2. Thus particularly the eigenspace can be expressed as $\left\langle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle$. □

2.2.21 Lemma. *The matrix* $n = \begin{bmatrix} c & \mathbf{q}^{\top} \\ \mathbf{q} & \sqrt{1 + \mathbf{q}\mathbf{q}^{\top}} \end{bmatrix}$ *has an eigenspace*

$$\left\langle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \right\rangle$$

and eigenvalues $e^{\lambda}, e^{-\lambda}, 1$, *where* $\mathbf{p} \in \mathbb{R}^2$ *such that* $\sqrt{\mathbf{p} \bullet \mathbf{p}} = 1$ *and* $\mathbf{p}$ *is orthogonal to* $\mathbf{q}$, *where the eigenvectors form an orthonormal basis for* $\mathbb{R}^3$.

Proof. In the proof of Theorem 2.2.9, we showed that $\mathbf{q}$ is an eigenvector of $(1 + \mathbf{q}\mathbf{q}^{\top})$ corresponding to the eigenvalue c^2. Consider a value $d \in \mathbb{R}$ such that

$$\begin{bmatrix} c & \mathbf{q}^{\top} \\ \mathbf{q} & \sqrt{1 + \mathbf{q}\mathbf{q}^{\top}} \end{bmatrix} \begin{bmatrix} d \\ \mathbf{q} \end{bmatrix} = \lambda \begin{bmatrix} d \\ \mathbf{q} \end{bmatrix}. \qquad (2.2.16)$$

In order to solve for d, we set up the system of 2 equations

$$\sqrt{1+\mathbf{q}\mathbf{q}^{\top}}\mathbf{q}+d\mathbf{q} = \lambda \tag{2.2.17}$$

$$\mathbf{q}^{\top}\mathbf{q}+cd = \lambda d \tag{2.2.18}$$

But in the proof of THEOREM 2.2.9, we showed that $\sqrt{1+\mathbf{q}\mathbf{q}^{\top}}\mathbf{q} = c\mathbf{q}$ and $c = \sqrt{\|\mathbf{q}\|^2+1}$. Thus substituting $\sqrt{1+\mathbf{q}\mathbf{q}^{\top}}\cdot\mathbf{q} = c\mathbf{q}$ into (2.2.17) and $c = \sqrt{\|\mathbf{q}\|^2+1}$ into (2.2.18), this system of equations simplifies to

$$(c+d)\mathbf{q} = \lambda\mathbf{q} \tag{2.2.19}$$

$$(c^2-1)+cd = \lambda d. \tag{2.2.20}$$

Since $\mathbf{q} \neq 0$, from (2.2.19) we have $\lambda = c+d$. Substituting this into (2.2.20), then

$$c^2-1+cd = (c+d)\cdot d \quad \Rightarrow \quad d^2 = c^2-1.$$

Thus, since $\lambda = c+d$, then

$$\begin{cases} \lambda_1 & = c+\sqrt{c^2-1} \quad \text{for} \quad d = \sqrt{c^2-1} \\ \lambda_2 & = c-\sqrt{c^2-1} \quad \text{for} \quad d = -\sqrt{c^2-1}. \end{cases}$$

Since $c^2 \geq 1$ and $\lambda_1\lambda_2 = (c+\sqrt{c^2-1})(c-\sqrt{c^2-1}) = 1$, we set $\lambda = \ln\lambda_1 = \ln(c+\sqrt{c^2-1}) \geq 0$, which then implies that $-\lambda = \ln\lambda_2$. Then $\lambda_1 = e^{\lambda}$ and $\lambda_2 = e^{-\lambda}$. Thus the corresponding eigenvector of (2.2.16) is given by

$$\begin{bmatrix} d \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \sqrt{c^2-1} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \sqrt{\|\mathbf{q}\|^2+1-1} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \|\mathbf{q}\| \\ \mathbf{q} \end{bmatrix}$$

and so

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} \end{bmatrix}$$

are eigenvectors. If we take $\mathbf{p} \in \mathbb{R}^2$ normal and (Euclidean) orthogonal to $\mathbf{q}$, then

$$\begin{bmatrix} c & \mathbf{q} \\ \mathbf{q} & \sqrt{1+\mathbf{q}\mathbf{q}^{\top}} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix}$$

and we may associate to the eigenvalue 1 the eigenvector $(0,\mathbf{p})$. Then indeed the matrix n has an eigenspace $\left\langle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \right\rangle$ consisting of orthonormal vectors. □

2.2.22 LEMMA. *The product* $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} & \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} & \mathbf{p} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is a matrix* $\begin{bmatrix} 1 & 0 \\ 0 & R_\theta \end{bmatrix}$, *where* $R_\theta \in \mathsf{SO}(2)$.

PROOF. By matrix multiplication,

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} & \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} & \mathbf{p} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\mathbf{q}}{\|\mathbf{q}\|} & \mathbf{p} \end{bmatrix}$$

where the matrix $\begin{bmatrix} \frac{\mathbf{q}}{\|\mathbf{q}\|} & \mathbf{p} \end{bmatrix}$ has two columns that are orthonormal vectors, since by definition $\sqrt{\mathbf{p} \bullet \mathbf{p}} = 1$ and $\mathbf{p}$ is orthogonal to $\mathbf{q}$. But in (A.5.5), we noted that this was a defining property of elements of $\mathsf{O}(2)$: thus $\begin{bmatrix} \frac{\mathbf{q}}{\|\mathbf{q}\|} & \mathbf{p} \end{bmatrix}$ is in $\mathsf{O}(2)$. Since $\begin{vmatrix} \frac{\mathbf{q}}{\|\mathbf{q}\|} & \mathbf{p} \end{vmatrix} = 1$ by the fact that the vectors $\frac{\mathbf{q}}{\|\mathbf{q}\|}$ and $\mathbf{p}$ are orthogonal and of norm 1, then it follows that $\begin{bmatrix} \frac{\mathbf{q}}{\|\mathbf{q}\|} & \mathbf{p} \end{bmatrix} \in \mathsf{SO}(2)$. □

2.2.23 THEOREM. *Each element of* $\mathsf{SO}(1,2)_0$ *can be expressed as the product bk where*

$$b \in \mathsf{B} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cosh\theta_2 & \sinh\theta_2 & 0 \\ \sinh\theta_2 & \cosh\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}^{\mathsf{T}} \middle| \theta_1, \theta_2 \in \mathbb{R} \right\}$$

and

$$k \in \mathsf{K} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{bmatrix} \middle| \theta_3 \in \mathbb{R} \right\}$$

that is, $\mathsf{SO}(1,2)_0 = \mathsf{BK}$.

PROOF. Since from LEMMA 2.2.20,

$$\begin{bmatrix} e^{-\lambda} & 0 \\ 0 & e^{\lambda} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{\mathsf{T}}$$

then for $\theta_1 \in \mathbb{R}$,

$$\begin{bmatrix} e^{-\theta_1} & 0 & 0 \\ 0 & e^{\theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}}.$$

Thus by LEMMA 2.2.21, the matrix n can be expressed as the matrix product

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} & \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} & \mathbf{p} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} & \frac{\mathbf{q}}{\sqrt{2}\|\mathbf{q}\|} & \mathbf{p} \end{bmatrix}^{-1}$$

which by LEMMA 2.2.22 simplifies to

$$\begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2} \end{bmatrix} \begin{bmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2}^{\mathsf{T}} \end{bmatrix}.$$

for $R_{\theta_2} \in \mathsf{SO}(2)$. But by THEOREM 2.2.9, $g = n \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_3} \end{bmatrix}$ for $g \in \mathsf{SO}(1,2)_0$, where $R_{\theta_3} \in \mathsf{SO}(2)$. Thus

$$g = \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2} \end{bmatrix} \begin{bmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2}^{\top} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_3} \end{bmatrix}$$

for $R_{\theta_2}, R_{\theta_3} \in \mathsf{SO}(2)$. □

The transformations $k(\theta_3) \in \mathsf{K}$ are the **Euclidean rotations** of $\mathcal{HL}$. The transformations $b(\theta_1) \in \mathsf{B}$ are generally referred to as the **Lorentz boosts**.

In the subset of Lorentz boosts of $\mathsf{SO}(1,2)_0$ we denote particularly

$$\begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix} = b_1. \tag{2.2.21}$$

We will denote the matrix $\operatorname{diag}(-1,-1,1)$ by g_2.

2.2.24 COROLLARY. *Any element of* $\mathsf{SO}(1,2)$ *may be written as a product*

$$\begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2} \end{bmatrix} \begin{bmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2}^{\top} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_1} \end{bmatrix} g_2$$

where $R_{\theta_1}, R_{\theta_2} \in \mathsf{SO}(2)$.

PROOF. In PROPOSITION 2.2.14, we expressed $\mathsf{SO}(1,2)$ as the union $\mathsf{SO}(1,2)_0 \cup \mathsf{SO}^-(1,2)$. But $\mathsf{SO}^-(1,2)$ is the set of all matrices

$$\begin{bmatrix} c & \mathrm{q}^{\top} \\ \mathrm{q} & \sqrt{1+\mathrm{q}\mathrm{q}^{\top}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & R_{\theta_3} \end{bmatrix}$$

where $R_{\theta_3} \in \mathsf{O}(2), \det R_{\theta_3} = -1$. We show firstly that each element in $\mathsf{O}(2)\backslash\mathsf{SO}(2)$ is of the form $R_{\theta_1} g_2$ where $\det R_{\theta_1} = 1$. Given $R_{\theta_1} \in \mathsf{SO}(2)$, then $\det R_{\theta_1} = 1$ and so $\det(R_{\theta_1} g_2) = 1 \cdot -1 = -1$, and $\{R_{\theta_1} g_2 | \theta_1 \in \mathbb{R}\} \subseteq \mathsf{O}(2)\backslash\mathsf{SO}(2)$. Further, consider some element $R_{\tilde{\theta_1}} \in \mathsf{O}(2)\backslash\mathsf{SO}(2)$. Then $\det R_{\tilde{\theta_1}} = -1$ and thus $\det(R_{\tilde{\theta_1}} g_2) = 1$ and so $R_{\tilde{\theta_1}} g_2 = R_{\theta_1}$ where $R_{\theta_1} \in \mathsf{SO}(2)$. But then $R_{\tilde{\theta_1}} = R_{\theta_1} g_2$ and it follows that $R_{\tilde{\theta_1}} \in \{R_{\theta_1} g_2 \ : \ \theta_1 \in \mathbb{R}\}$. Thus $\mathsf{O}(2)\backslash\mathsf{SO}(2) \subseteq \{R_{\theta_1} g_2 \ : \ \theta_1 \in \mathbb{R}\}$. Since we have both containments, it follows that $\mathsf{O}(2)\backslash\mathsf{SO}(2) = {}_{\theta_1} g_2 \ : \ \theta_1 \in \mathbb{R}\}$ and thus each element of $\mathsf{SO}^-(1,2)$ is a product

$$\begin{bmatrix} c & \mathrm{q}^{\top} \\ \mathrm{q} & \sqrt{1+\mathrm{q}\mathrm{q}^{\top}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_1} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $R_{\theta_1} \in \mathsf{SO}(2)$. But in THEOREM 2.2.23 we expressed

$$\begin{bmatrix} c & \mathrm{q}^{\top} \\ \mathrm{q} & \sqrt{1+\mathrm{q}\mathrm{q}^{\top}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2} \end{bmatrix} \begin{bmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2}^{\top} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_3} \end{bmatrix}.$$

where $R_{\theta_3} \in \mathsf{SO}(2)$. Thus

$$\begin{bmatrix} c & \mathbf{q} \\ \mathbf{q}^\top & \sqrt{1+\mathbf{q}\mathbf{q}^\top} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_3} \end{bmatrix} g_2 = \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2} \end{bmatrix} \begin{bmatrix} \cosh\theta_1 & \sinh\theta_1 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_2}^\top \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_3} \end{bmatrix} g_2$$

and the result follows. □

2.2.25 Corollary. *Given any two orthonormal bases* $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ *and* $\{\mathbf{e}_1'', \mathbf{e}_2'', \mathbf{e}_3''\}$ *of* $\mathbb{R}^{1,2}$ *positively-oriented in time, there exists an element* $g \in \mathsf{SO}(1,2)_0$ *such that* $(\mathbf{e}_1), g(\mathbf{e}_2), g(\mathbf{e}_3)\} = \{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$.

Proof. Consider the standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^{1,2}$. By the conditions $\mathbf{e}_1^\top g \mathbf{e}_1 > 0, gJg^\top = J, \det g = 1$ on $g \in \mathsf{SO}(1,2)_0$, as in Proposition 2.2.15, each element of $\mathsf{SO}(1,2)_0$ can be considered as a matrix $g = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$ where g_1, g_2, g_3 are Minkowski-orthonormal column vectors, and the basis is positively-oriented in time since $g_{11} > 0$. Further, from Proposition 2.2.15, we see that any matrix $g = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$ where g_1, g_2, g_3 are Minkowski-orthonormal column vectors must fulfil the property $gJg^\top = J$, and so is an element of $\mathsf{O}(1,2)$. Since the vectors are orthonormal, then $\det g = 1$, and finally since the orientation of the vectors complies with the positive timelike orientation of $\mathbb{R}^{1,2}$, then g_1 is such that $g_{11} > 0$, and so $\mathbf{e}_1^\top g \mathbf{e}_1 > 0$: that is, $g \in \mathsf{SO}(1,2)_0$.

Thus given any 3 vectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2$ and $\tilde{\mathbf{e}}_3$ on $\mathcal{HL}$ which are Minkowski-orthonormal and give an orthonormal basis positively oriented in time, then $\begin{bmatrix} \tilde{\mathbf{e}}_1 & \tilde{\mathbf{e}}_2 & \tilde{\mathbf{e}}_3 \end{bmatrix}$ is an element $g \in \mathsf{SO}(1,2)_0$. But then there exists a product $bk \in \mathsf{SO}(1,2)_0$ such that $bk = g$. Thus $(bk)^{-1} g = \mathbf{1}$, and $\tilde{\mathbf{e}}_1$ is mapped to $\mathbf{e}_1$, $\tilde{\mathbf{e}}_2$ is mapped to $\mathbf{e}_2$ and $\tilde{\mathbf{e}}_3$ is mapped to $\mathbf{e}_1$. Similarly, $\begin{bmatrix} \tilde{\mathbf{e}}_1' & \tilde{\mathbf{e}}_2' & \tilde{\mathbf{e}}_3' \end{bmatrix}$ is an element g' of $\mathsf{SO}(1,2)_0$ such that $g' = b'k'$. Thus $(b'k')^{-1} g' = \mathbf{1}$, and $\tilde{\mathbf{e}}_1'$ is mapped to $\mathbf{e}_1$, $\tilde{\mathbf{e}}_2'$ is mapped to $\mathbf{e}_2$ and $\tilde{\mathbf{e}}_3'$ is mapped to $\mathbf{e}_3$. Then the composition of elements $(b'k')(bk)^{-1}$ maps the orthonormal basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ to the orthonormal basis $\{\tilde{\mathbf{e}}_1', \tilde{\mathbf{e}}_2', \tilde{\mathbf{e}}_3'\}$, and the result follows. □

2.2.26 Proposition. *If a linear transformation of* $\mathbb{R}^{1,2}$ *fixes each element of an orthonormal basis for* $\mathbb{R}^{1,2}$*, then it is the identity transformation.*

Proof. Consider the linear transformation h such that for the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then $h(\mathbf{e}_1) = \mathbf{e}_1, h(\mathbf{e}_2) = \mathbf{e}_2$ and $h(\mathbf{e}_3) = \mathbf{e}_3$. Since $\mathbf{a} = a_1\mathbf{e}_1 + a_1\mathbf{e}_2 + a_3\mathbf{e}_3$, then

$$h(\mathbf{a}) = h(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1 h(\mathbf{e}_1) + a_2 h(\mathbf{e}_2) + a_3 h(\mathbf{e}_3) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \mathbf{a}$$

by assumption. Since $\mathbf{a}$ was arbitrarily chosen, then h fixes each element of $\mathbb{R}^{1,2}$ and so must be the identity transformation. □

2.2.27 Proposition. $\mathsf{SO}(1,2)_0$ *acts transitively on* $\mathcal{HL}$.

Proof. Take $\mathbf{p}$ and $\mathbf{q}$ any two points in $\mathcal{HL}$. Then for some $(u_m, v_m) \in U_i, (u_n, v_n) \in U_j$, $\mathbf{p} = \epsilon^i(u_m, v_m)$ and $\mathbf{q} = \epsilon^j(u_r, v_n)$ in the patches $\epsilon^i(u, v)$ or $\epsilon^j(u, v)$. Since in the proof of Proposition 2.2.1 the transition maps are Euclidean rotations about the $\mathbf{e}_1$-axis , we can assume that the two points lie within the same patch ϵ^i. Thus there exists a Euclidean rotation $k(v_n - v_m)$ such that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(v_n - v_m) & -\sin(v_n - v_m) \\ 0 & \sin(v_n - v_m) & \cos(v_n - v_m) \end{bmatrix} \begin{bmatrix} \cosh u_m \\ \sinh u_m \cos v_m \\ \sinh u_m \sin v_m \end{bmatrix} = \begin{bmatrix} \cosh u_m \\ \sinh u_m \cos v_n \\ \sinh u_m \sin v_n \end{bmatrix}$$

and a Lorentz boost $b_1(u_n - u_m)$ such that

$$\begin{bmatrix} \cosh(u_n - u_m) & \sinh(u_n - u_m) & 0 \\ \sinh(u_n - u_m) & \cosh(u_n - u_m) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh u_m \\ \sinh u_m \cos v_n \\ \sinh u_m \sin v_n \end{bmatrix} = \begin{bmatrix} \cosh u_n \\ \sinh u_n \cos v_n \\ \sinh u_n \sin v_n \end{bmatrix}.$$

Thus the composition $k(u_n - u_m) \circ b_1(v_n - v_m)(\mathbf{p}) = \mathbf{q}$. □

2.2.28 PROPOSITION. *The elements of* $\mathsf{SO}(1,2)_0$ *are orientation-preserving isometries on* $\mathbb{HL}$.

PROOF. Since each $g \in \mathsf{SO}(1,2)_0$ is linear, then the Jacobian matrix $J_g = g$. But clearly since $g \in \mathsf{SO}(1,2)_0$, then $\det g = 1$. Thus $\det J_g = 1$ and by (A.2.35), g is orientation-preserving. □

2.2.29 THEOREM. $\mathsf{SO}(1,2)_0$ *is the symmetry group* $\mathsf{Sym}(\mathbb{HL})$.

PROOF. From PROPOSITION 2.2.28, the elements $g \in \mathsf{SO}(1,2)_0$ are clearly orientation-preserving isometries of $\mathbb{HL}$ and so are symmetries. In PROPOSITION 2.2.19 we showed that all isometries of $\mathbb{HL}$ are linear. Assume that there exists some linear symmetry h of $\mathbb{HL}$ which is not an element of $\mathsf{SO}(1,2)_0$. Since h is a symmetry of $\mathbb{HL}$, it sends the positively-oriented orthornormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{h(\mathbf{e}_1), h(\mathbf{e}_2), h(\mathbf{e}_3)\} = \{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$. But from PROPOSITION 2.2.25, there exists g in $\mathsf{SO}(1,2)_0$ such that $(\mathbf{e}_1), g(\mathbf{e}_2), g(\mathbf{e}_3)\} = \{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$. Thus

$$g^{-1} \circ h(\mathbf{e}_1) = \mathbf{e}_1,\ g^{-1} \circ h(\mathbf{e}_2) = \mathbf{e}_2 \quad \text{and} \quad g^{-1} \circ h(\mathbf{e}_3) = \mathbf{e}_3$$

and so $g^{-1} \circ h$ is a linear transformation of $\mathbb{R}^{1,2}$ which fixes the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. But by PROPOSITION 2.2.26, then $g^{-1} \circ h = \mathbf{1}$. Thus $gg^{-1} \circ h = g \Rightarrow h = g$ and so $h \in \mathsf{SO}(1,2)_0$. The result follows. □

2.3 The upper half-plane model

2.3.1 The upper half-plane as a geometric surface

Define the open subset $\mathcal{HP} = \{(u, v) \in \mathbb{R}^2 : v > 0\} \subseteq \mathbb{R}^2$. Since $\mathcal{HP}$ is an open subset of $\mathbb{R}^2$ parametrized by the identity map $\iota : \mathbb{R}^2 \to \mathbb{R}^2$, $\iota(x, y) = (x, y)$, then by (A.2.35), $\mathcal{HP}$ is an abstract surface.

2.3.1 DEFINITION. The abstract surface $\mathcal{HP}$ equipped with the symmetric nondegenerate bilinear form $ds^2 = \dfrac{du^2 + dv^2}{v^2}$ is the **upper half-plane model** $\mathbb{HP} = (\mathcal{HP}, ds^2)$.

2.3.2 PROPOSITION. *The upper half-plane* $\mathbb{HP}$ *is a geometric surface.*

PROOF. Since $\mathcal{HP}$ is an abstract surface, we require only to show that the metric ds^2 is a Riemannian metric (A.2.18). In order to show this, we express each $ds^2|_{(u_0,v_0)}$ in matrix form

$$\mathbf{p}^\top Q|_{u_0,v_0}\mathbf{q} = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} \frac{1}{v_0^2} & 0 \\ 0 & \frac{1}{v_0^2} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

as in (2.1.1) where $\mathbf{p}$ and $\mathbf{q}$ are the vectors $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$ in $\mathbb{R}^2$ associated to the tangent vectors $\mathbf{p} = p_1\frac{\partial}{\partial u}|_{(u_0,v_0)} + p_2\frac{\partial}{\partial v}|_{(u_0,v_0)}$ and $\mathbf{q} = q_1\frac{\partial}{\partial u}|_{(u_0,v_0)} + q_2\frac{\partial}{\partial v}|_{(u_0,v_0)}$ in the fibre $\mathrm{T}_{(u_0,v_0)}\mathcal{HP}$ of $\mathrm{T}\mathcal{HP}$. Then

1. Bilinearity follows immediately from the linearity of the matrix product.
2. By the symmetry of the matrix Q, the quadratic forms $\mathbf{q}^\top Q|_{(u_0,v_0)}\mathbf{q}$ are symmetric.
3. For each (u_0, v_0) the matrix Q is diagonal with a double eigenvalue $\frac{1}{v_0^2}$ which is positive for every $(u_0, v_0) \in \mathcal{HP}$. Thus by (A.1.8), the quadratic forms $\mathbf{q}^\top Q\mathbf{q}$ are positive-definite. □

2.3.2 The symmetry group of $\mathbb{HP}$

Since (A.2.42) any isometry of an abstract surface is necessarily geodesic-preserving, we attempt to find the isometries of $\mathbb{HP}$ by first constructing maps of $\mathbb{HP}$ that map geodesics to geodesics and which preserve $\mathcal{HP}$.

2.3.3 PROPOSITION. *The intersections with* $\mathcal{HP}$ *of the lines* $\{u = c \ : \ u \in \mathbb{R}\}$ *perpendicular to the u-axis and the Euclidean circles* $\left\{v^2 + (u-k)^2 = \frac{1}{r^2} \ : \ r \in \mathbb{R}, r \neq 0, (u, v) \in \mathbb{R}^2\right\}$ *centered on the u-axis which intersect that axis perpendicularly are geodesics of* $\mathbb{HP}$.

PROOF. Since $E = \frac{1}{v^2} = G$, $F = 0$, then $E_u = G_u = F$ and the identity map is a v-Clairaut patch (A.2.50). Then by CLAIRAUT'S THEOREM (A.2.51), the paths $\gamma = \{(u(v), v) \ : \ v \in \mathbb{R}^+\}$, where $u(v)$ solves the equation

$$\frac{du}{dv} = \frac{r}{\pm\sqrt{\frac{1}{v^2} - r^2}} \quad \Rightarrow \quad \int du = \int \frac{r\,dv}{\pm\sqrt{\frac{1}{v^2} - r^2}} \tag{2.3.1}$$

for some $r \in \mathbb{R}$, are geodesics of $\mathbb{HP}$. Taking $r = 0$, this equation reduces to $du = 0$, and so the equation (A.2.42) has a solution $u = c$ for $c \in \mathbb{R}$. Taking $r \neq 0$, then

$$\int du = \int \frac{r\,dv}{\pm\sqrt{\frac{1}{v^2} - r^2}} \quad \Rightarrow \quad -v\sqrt{\frac{1}{v^2} - r^2} = \pm r(u-k).$$
$$\Rightarrow \quad v^2 + (u-k)^2 = \frac{1}{r^2},$$

for $k, c \in \mathbb{R}$. Thus the curves $\{u = c \ : \ u \in \mathbb{R}\}$ and $\left\{v^2 + (u-k)^2 = \frac{1}{r^2} \ : \ r \in \mathbb{R}, r \neq 0, u \in \mathbb{R}, v \in \mathbb{R}^+\right\}$ are geodesics of $\mathbb{HP}$. □

2.3.4 THEOREM. *The geodesics* $\{(u,v) \in \mathcal{HP} \,:\, u = 0\}$ *and* $\{(u,v) \in \mathcal{HP} \,:\, v^2 + (u-k)^2 = \frac{1}{r^2}, c \neq 0\}$ *are exactly the geodesics of* $\mathbb{HP}$.

PROOF. We have shown (PROPOSITION 2.3.3) that the intersection of these paths with $\mathcal{HP}$ are geodesics of $\mathbb{HP}$. In order to show that there exist no other geodesics, we will show that through every point $\mathbf{p}$ of $\mathbb{HP}$ and in the direction of every unit tangent vector in $\mathrm{T}_\mathbf{p}\mathcal{HP}$, there exists a geodesic of the form $\{u = c \,:\, u \in \mathbb{R}\}$ or $\{(u,v) \in \mathcal{HP} \,:\, v^2 + (u-k)^2 = \frac{1}{r^2}, r \neq 0\}$. Then we have found all geodesics of $\mathbb{HP}$, since from (A.2.40), a geodesic γ is uniquely defined by a point $\mathbf{p} \in \gamma$ and the unit direction vector u tangent to γ at $\mathbf{p}$.

The tangent plane $\mathrm{T}_\mathbf{p}\mathcal{HP}$ to any point $\mathbf{p} = (u_0, v_0)$ in $\mathbb{HP}$ is isomorphic to 2-dimensional real space $\mathbb{R}^2$. Thus any unit-length tangent vector $\mathbf{q}$ at $\mathbf{p}$ has the form $\mathbf{q} = (\cos\theta, \sin\theta)$ for some $\theta \in \mathbb{R}$. A (Euclidean) circle C passing through $\mathbf{p}$ in the direction of $(\cos\theta, \sin\theta)$ is such that $(\cos\theta, \sin\theta)$ is perpendicular to the radius of C at its point $\mathbf{p}$ of intersection with C. This radius is a (Euclidean) line that is perpendicular to $(\cos\theta, \sin\theta)$ and passes through $\mathbf{p}$, with equation

$$y = -\frac{\cos\theta}{\sin\theta}x + \left(v_0 + \frac{\cos\theta}{\sin\theta}u_0\right)$$

defined for all $\theta \neq \pi k$. This line intersects the u-axis at the point $x_0 = \frac{\sin\theta}{\cos\theta}v_0 + u_0$, which is defined for all $\theta \neq \frac{\pi}{2}k$. But then $\sqrt{\left(v_0 + (u_0 - x)\frac{\cos\theta}{\sin\theta}\right)^2}$ is the radial length of the (Euclidean) circle centered at x_0 that passes through $\mathbf{p}$ tangent to $(\sin\theta, \cos\theta)$; thus this circle exists and is uniquely defined.

In the case of $\theta = \frac{\pi}{2}k$, then $x = u_0$ is the unique Euclidean line passing through $\mathbf{p}$ in the direction of the vector $(\cos\theta, \sin\theta) = (0,1)$, while in the case of $\theta = \pi k$, then $(\cos\theta, \sin\theta) = (1,0)$, and the radius is the line $\{(u,v) \in \mathbb{R}^2 \,:\, u = u_0\}$: thus the circle with radius of length v_0 and centre u_0 is the unique geodesic passing through (u_0, v_0) in the direction of $\cos\theta, \sin\theta)$. We have thus constructed a unique geodesic passing through $\mathbf{p}$ in the direction of every unit tangent vector $(\cos\theta, \sin\theta) \in \mathrm{T}_\mathbf{p}\mathcal{HP}$. □

Using the well-known vector space isomorphism $\varsigma : \mathbb{R}^2 \to \mathbb{C}$, $\varsigma(u,v) = u + iv$, in the next sections where it will simplify calculations we are free to use the complex number $z = u + iv$ where $v > 0$ to express a point (u,v) in the upper half-plane $\mathcal{HP}$.

Having defined in (A.3.1) the complex projective line $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, we may identify $\mathcal{HP}$ with the subset $\{\mathrm{Im}(z) > 0 \,:\, z \in \mathbb{CP}^1\}$. Thus we can act on $\mathbb{HP}$ with the Möbius transformations $\mu(z) = \left\{\frac{\alpha z + \beta}{v z + \omega}, \alpha\omega - v\beta \neq 0\right\}$ which from (A.3.10) map the family of Euclidean lines and circles to itself, and thus are possibly geodesic-preserving on $\mathbb{HP}$. We use the basic Möbius transformations of (A.3.3) to express each μ as a composition of simpler transformations.

2.3.5 PROPOSITION. *The basic Möbius transformations* δ_α *for* $\mathrm{Re}(\alpha) < 0$ *and* τ_β *for* $\mathrm{Im}(\beta) < 0$ *do not preserve* $\mathcal{HP}$.

PROOF. Consider the dilation $\delta_\alpha : z \mapsto \alpha z$ where $\alpha = a_1 + ia_2$ and $\mathrm{Re}(\alpha) = a_1 < 0$. Then the image under δ_α of $i \in \mathcal{HP}$ is $\delta_\alpha(i) = \alpha i = i(a_1 + ia_2) = -a_2 + a_1 i$ which is not an element of $\mathcal{HP}$, since $\mathrm{Im}(\delta_\alpha(i)) < 0$.

Similarly, given any $\beta \in \mathbb{C}, \beta = b_1 + ib_2$, then for any $v = u + iv \in \mathcal{HP}$, $\tau_\beta(v) = (b_1 + u) + i(b_2 + v)$, where $\text{Im}(\tau_\beta(v)) < 0$ if $b_2 < -v$. Since there exists at least one point in $\mathbb{HP}$ such that $b_2 < -v$ for any negative b_2 (consider the element $(b_1, b_2) = (b_1, v + 1)$), then it follows that τ_β does not preserve $\mathcal{HP}$. □

We note that by taking $\beta \in \mathbb{R}$, then all translations τ_β preserve $\mathcal{HP}$. We introduce the **restricted Möbius transformations**

$$\mu(z) = \frac{az + b}{cz + d} \quad \text{where} \quad a, b, c, d \in \mathbb{R} \quad \text{and} \quad ad - bc \neq 0.$$

These are compositions $\widetilde{\tau} \circ \widetilde{\delta} \circ \sigma_{0,1} \circ \delta_c \circ \tau_b \circ \delta_c$, where $\tau_b : z \to z + b$ (**translations**), $\delta_c : z \to cz$ (**dilations**), and $\sigma_{0,1} : z \to \frac{1}{z}$ (**inversions**), and specifically $\widetilde{\tau}$ and $\widetilde{\delta}$ are the translations $\widetilde{\tau}(z) = \frac{a}{c} + z$ and dilations $\widetilde{\delta}(z) = (bc - ad)z$, respectively. Note that if $a < 0$, then from PROPOSITION 2.3.5, $\delta_a : z \mapsto az$ does not preserve $\mathcal{HP}$. However,

2.3.6 PROPOSITION. *If $a < 0$, then the conjugate $\bar{\delta}_a = a\bar{z}$ preserves $\mathcal{HP}$.*

PROOF. Consider the transformation $\bar{\delta}_a = a\bar{z}$ where $a < 0$. Then for each $z \in \mathcal{HP}$, that is, $z = u + iv$ where $v > 0$, then $\bar{\delta}_a(z) = a(u - iv) = u - iav$, where $\text{Im}(\bar{\delta}_a(z)) = -av > 0$. Then the imaginary part of the image of each element of $\mathcal{HP}$ under $\bar{\delta}_a$ is strictly positive, the result follows. □

2.3.7 REMARK. We will thus consider the restricted Möbius transformations which are compositions $\widetilde{\tau} \circ \widetilde{\delta} \circ \sigma_{0,1} \circ \delta_c \circ \tau_b \circ \delta_c$ of the **basic restricted Möbius transformations**

$$\tau_b : z \mapsto z + b, \quad \delta_c : z \mapsto cz \quad (c > 0), \quad \delta_c : z \mapsto c\bar{z} \quad (c < 0) \quad \text{and} \quad \sigma_{0,1} : z \mapsto \frac{1}{\bar{z}} \tag{2.3.2}$$

for $\widetilde{\tau}(z) = \frac{a}{c} + z$ and $\widetilde{\delta}(q) = (bc - ad)z$.

2.3.8 PROPOSITION. *Each restricted Möbius transformation of (2.3.2) preserves $\mathcal{HP}$.*

PROOF. Consider the arbitrary element $z = u + iv \in \mathcal{HP}$: that is, $\text{Re}(z) = v > 0$. Then $\tau_b(z) = (u + b) + iv$ and $\text{Re}(\tau_b(z)) = y > 0$. Similarly, given $c > 0$, then $\delta_c(z) = cu + icv$, and $\text{Re}(\delta_c(z)) = cv > 0$, while given $c < 0$, then $\delta_c(z) = cu - icv$, and $\text{Re}(\delta_c(z)) = -cv > 0$. Finally, $\sigma_{0,1}(z) = \frac{u+iv}{|z|^2}$, and $\text{Re}(\sigma_{0,1}(z)) = \frac{v}{|z|^2} > 0$. The result follows. □

From (A.3.14), the (restricted) Möbius transformations are conformal. Thus they will preserve the property of perpendicularity to the u-axis. We prove further that

2.3.9 PROPOSITION. *The restricted Möbius transformations send geodesics to geodesics.*

PROOF. Since all Möbius transformations are conformal (A.3.14) and preserve the family of Euclidean lines and circles (A.3.10), thus particularly the restricted Möbius transformations preserve the class of Euclidean lines and circles perpendicular to the u-axis. Since they send real numbers to real numbers, they preserve the real axis of $\mathbb{C}$ and so will map Euclidean circles centrered on the u-axis to Euclidean circles centered on the u-axis. The result follows. □

2.3.10 PROPOSITION. *The composition* $\mu = \tilde{\tau} \circ \tilde{\delta} \circ \sigma_{0,1} \circ \delta_c \circ \tau_b \circ \delta_c$ *of basic restricted Möbius transformations of (2.3.2) has the form* $\bar{\mu}(z) = \frac{az+b}{cz+d}$ *where* $bc - ad < 0$, *or* $\mu(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ *where* $bc - ad > 0$.

PROOF. Applying $\tilde{\tau} \circ \tilde{\delta} \circ \sigma_{0,1} \circ \delta_c \circ \tau_b \circ \delta_c$ to $z \in \mathbb{C}$, then

$$\tilde{\tau} \circ \tilde{\delta} \circ \sigma_{0,1} \circ \delta_c \circ \tau_b \circ \delta_c(z) = \frac{bc - ad}{c(cz + d)} + \frac{a}{c} = \frac{bc - ad + a(cz + d)}{c(cz + d)} = \frac{acz + bc + ad - ad}{c(cz + d)} = \frac{az + b}{cz + d}.$$

Since δ_c appears twice in the expansion $\tilde{\tau} \circ \tilde{\delta} \circ \sigma_{0,1} \circ \delta_c \circ \tau_b \circ \delta_c$, and conjugation is idempotent, then the sign of c can be neglected. Thus we will take the conjugate of z twice where $bc - ad < 0$ (once from $\sigma_{0,1}$ and once from $\tilde{\delta}$) and once where $bc - ad > 0$ (from $\sigma_{0,1}$). The result then follows from the idempotency of conjugation. □

2.3.11 THEOREM. *The restricted Möbius transformations* $\mu(z) = \frac{az+b}{cz+d}$, $ad - bc > 0$ *and* $\bar{\mu}(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, $ad - bc < 0$ *preserve the metric* $ds^2 = \frac{du^2 + dv^2}{v^2} = \frac{dzd\bar{z}}{z^2}$ *of* $\mathbb{HP}$.

PROOF. We express the restricted Möbius transformations $\mu(z) = \frac{az + b}{cz + d}$ for $ad - bc > 0$ in complex coordinates by $\mu(z) = \mu(u + iv) = z' = u' + iv'$. Then

$$z' = \frac{az + b}{cz + d} = \frac{acu^2 + adu + bcu + acv^2 + bd}{|cz + d|^2} + i\frac{(ad - bc)v}{|cz + d|^2} = u' + iv'$$

where $u' = \frac{acu^2 + adu + bcu + acv^2 + bd}{|cz + d|^2}$ and $v' = \frac{(ad - bc)v}{|cz + d|^2}$.
In complex variables,

$$ds^2 = \frac{dzd\bar{z}}{v^2} \quad \Rightarrow \quad dz' = \frac{\partial(\mu(z))}{\partial z} \cdot dz = \frac{(ad - bc)dz}{(cz + d)^2}$$

and correspondingly $d\bar{z}' = \frac{(ad - bc)d\bar{z}}{(c\bar{z} + d)^2}$. Thus $dz'd\bar{z}' = \frac{(ad - bc)^2 dzd\bar{z}}{((cz+d)(c\bar{z}+d))^2} = \frac{(ad - bc)^2 dzd\bar{z}}{|cz + d|^4}$, and so

$$ds' = \frac{(ad - bc)^2 dz'd\bar{z}'}{(ad - bc)^2 (v')^2} = \frac{dzd\bar{z}}{|cz + d|^4} \frac{|cz + d|^4}{v^2} = \frac{dzd\bar{z}}{v^2}.$$

Thus $\mu^*(ds^2) = ds^2$, and from (A.2.26) it follows that μ is an isometry of $\mathbb{HP}$. The case of $\bar{\mu}$ follows identical steps. □

2.3.12 PROPOSITION. *The union* $\left\{\mu(z) = \frac{az+b}{cz+d} \,:\, ad - bc > 0\right\} \cup \left\{\bar{\mu}(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \,:\, ad - bc < 0\right\}$ *of all restricted Möbius transformations, forms a group under the operation of compositon.*

PROOF. Since (A.3.13), the Möbius transformations form a group, then we require only to show that this set of restricted Möbius transformations is closed under composition and the taking of inverses. In (A.3.12) we stated that given two Möbius transformations μ, μ', where $\mu = \frac{\alpha z+\beta}{v z+\omega}$ and $\mu' = \frac{\alpha' z+\beta'}{v' z+\omega'}$ then their composition $\mu' \circ \mu$ is the Möbius transformation $\frac{Az+B}{\Upsilon z+\Omega}$, where the complex coefficients A, B, Υ, Ω are such that

$$\begin{bmatrix} \alpha & \beta \\ v & \omega \end{bmatrix} \begin{bmatrix} \alpha' & \beta' \\ v' & \omega' \end{bmatrix} = \begin{bmatrix} A & B \\ \Upsilon & \Omega \end{bmatrix}.$$

Thus particularly where $\mu = \frac{az+b}{cz+d}$ and $\mu' = \frac{a'z+b'}{c'z+d'}$, $a, b, c, d \in \mathbb{R}$, then the composition $\mu' \circ \mu$ is the Möbius transformation $\frac{Az+B}{\Upsilon z+\Omega}$ where

$$\begin{bmatrix} A & B \\ \Upsilon & \Omega \end{bmatrix} = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}.$$

Thus $\mu' \circ \mu$ has the form $\mu'' = \frac{(aa'+bc')z+(ab'+bd')}{(ca'+dc')z+(cb'+dd')}$, which is a restricted Möbius transformation since $aa' + bc', ab' + bd', ca' + dc', cb' + dd' \in \mathbb{R}$. Further, since

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} = \begin{vmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{vmatrix}$$

then $(aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc') > 0$, and the resticted Möbius transformation is of the required form. Similarly, if $\bar{\mu} = \frac{a\bar{z}+b}{c\bar{z}+d}$ and $\bar{\mu}' = \frac{a'\bar{z}+b'}{c'\bar{z}+d'}$, then the coefficients of $\bar{\mu} \circ \bar{\mu}'$ are given by $aa' + bc', ab' + bd', ca' + dc', cb' + dd' \in \mathbb{R}$, and so this transformation is a restricted Möbius transformation. Then

$$(aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc') = (ad - bc)(a'd' - b'c') > 0$$

and the resticted Möbius transformation is of the required form. Finally, if $\mu = \frac{az+b}{cz+d}$ and $\bar{\mu}' = \frac{a'\bar{z}+b'}{c'\bar{z}+d'}$, then the coefficients of $\mu \circ \bar{\mu}'$ are given by $aa' + bc', ab' + bd', ca' + dc', cb' + dd'$ in $\mathbb{R}$, and so this transformation is a restricted Möbius transformation. Further, the coefficients $(aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc') = (ad - bc)(a'd' - b'c') < 0$, and the resticted Möbius transformation is of the required form. We show that this set is closed under inversion. For any restricted Möbius transformation μ of the form $\mu = \frac{az+b}{cz+d}$, then μ^{-1} has the coefficients A, B, Ω, Υ where since $\mu \circ \mu^{-1} = \iota$,

$$\begin{bmatrix} A & B \\ \Upsilon & \Omega \end{bmatrix} = S \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

where $ad - bc > 0$. Then μ^{-1} is the transformation $z \mapsto \frac{dz+b}{-cz-a}$, which is a restricted Möbius transformation. Secondly, for any restricted Möbius transformation $\bar{\mu}$ of the form $\bar{\mu} = \frac{a\bar{z}+b}{c\bar{z}+d}$, then $\bar{\mu}^{-1}$ has the coefficients A, B, Ω, Υ where since $\bar{\mu} \circ \bar{\mu}^{-1} = \iota$,

$$\begin{bmatrix} A & B \\ \Upsilon & \Omega \end{bmatrix} = S \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

where $ad - bc < 0$. Then $\bar{\mu}^{-1}$ is the transformation $z \mapsto \frac{dz+b}{-cz-a}$, which is a restricted Möbius transformation of the required form. We have then shown that this set is closed under the taking of inverses. The result follows.

2.3.13 Corollary. *Any composition of basic restricted Möbius transformations is a transformation of the form* $\mu : z \mapsto \frac{az+b}{cz+d}$ *or* $\bar{\mu} : z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$.

Proof. Clearly, the basic restricted Möbius transformations $\delta_a, \sigma_{0,1}$ and τ_b are all transformations of the form $\mu : z \mapsto \frac{az+b}{cz+d}$, where $ad - bc > 0$, or $\bar{\mu} : z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$, where $ad - bc < 0$: δ_c, τ_b have the form μ, while $\sigma_{0,1}$ has the form $\bar{\mu}$. But we have shown in PROPOSITION 2.3.12 that the set of all such transformations is closed under the taking of compositions. Thus any composition of basic restricuted Möbius transformations has the form μ or $\bar{\mu}$. □

2.3.14 COROLLARY. *The set of restricted Möbius transformations* $\left\{\mu(z) = \frac{az+b}{cz+d} : ad - bc > 0\right\}$ *forms a group under the operation of compositon.*

PROOF. From PROPOSITION 2.3.12, the composition of $\mu(z) = \frac{az+b}{cz+d}$ and $\mu' = \frac{\alpha' z+\beta'}{v' z+\omega'}$ is the restricted Möbius transformation

$$\mu'' = \frac{(aa' + bc')z + (ab' + bd')}{(ca' + dc')z + (cb' + dd')}$$

where it follows that the coefficients $(aa'+bc')(cb'+dd') - (ab'+bd')(ca'+dc') = (ad-bc)(a'd' - b'c') > 0$. Thus this set is closed under the taking of compositions. Similarly, for any μ of the form $\mu = \frac{az+b}{cz+d}$, then we showed that μ^{-1} is the transformation $\mu : z \mapsto \frac{dz+b}{-cz-a}$, where $ad - bc = da - cb > 0$ and so this set is closed under the taking of inverses. The result follows.□

2.3.15 THEOREM. *The basic restricted Möbius transformations* $\delta : z \mapsto az$, $a > 0$ *and* $\tau_b : z \mapsto z + b$ *are orientation-preserving for all elements* $b \in \mathbb{R}$, *while the basic restricted Möbius transformations* $\delta_a : z \mapsto a\bar{z}$, $a < 0$ *and* $\sigma_{0,1} : z \mapsto \frac{1}{\bar{z}}$ *are orientation-reversing.*

PROOF. The transformation $\sigma_{0,1} : z \mapsto \frac{1}{\bar{z}}$ can be expressed in complex coordinates as the transformation $\sigma_{0,1} : u + iv \mapsto \frac{u}{(u^2+v^2)^2} + i\frac{v}{(u^2+v^2)^2} = \tilde{u} + i\tilde{v}$, which has the Jacobian matrix

$$J_{\bar{\sigma}_{0,1}} = \begin{bmatrix} \frac{v^2-u^2}{(u^2+v^2)^2} & \frac{-2uv}{(u^2+v^2)^2} \\ \frac{-2uv}{(u^2+v^2)^2} & \frac{u^2-v^2}{(u^2+v^2)^2} \end{bmatrix} \quad \Rightarrow \quad \det J_{\bar{\sigma}_{0,1}} = \frac{-(v^2-u^2)(v^2-u^2) - 4u^2v^2}{(u^2+v^2)^2} = -1.$$

Thus by (A.2.35), the transformation $\bar{\sigma}_{0,1}$ is orientation-reversing. In complex coordinates, τ_b can be expressed as the transformation $\tau_b : u + iv \mapsto (u + b) + iv$, which has the Jacobian matrix

$$J_{\tau_b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \det J_{\tau_b} = 1$$

and τ_b is orientation-preserving. The transformation $\delta_a : z \mapsto az$ for $a > 0$ can be expressed in terms of complex coordinates as the transformation $\delta_a : u + iv \mapsto au + iav$, which has the Jacobian matrix

$$J_{\delta_a} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \quad \Rightarrow \quad \det J_{\delta_a} = a^2$$

which is positive for all $a \in \mathbb{R}$ and thus the transformation δ_a is orientation-preserving. The transformation $\bar{\delta}_a : z \mapsto a\bar{z}$ for $a < 0$ can be expressed in terms of complex coordinates as the transformation $\bar{\delta}_a : u + iv \mapsto au - iav$ which has the Jacobian matrix

$$J_{\bar{\delta}_a} = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \quad \Rightarrow \quad \det J_{\bar{\delta}_a} = -a^2$$

which is negative for all $a \in \mathbb{R}$ and thus the transformation $\bar{\delta}_a$ is orientation-reversing. □

2.3.16 COROLLARY. *The restricted Möbius transformation* $\mu : \mathbb{C} \to \mathbb{C}$, $\mu(z) = \frac{az+b}{cz+d}$, $ad - bc > 0$ *are exactly the orientation-preserving isometries of* $\mathbb{HP}$.

PROOF. In PROPOSITION 2.3.10 we expressed μ as the composition $\mu = \tilde{\tau} \circ \tilde{\delta} \circ \sigma_{0,1} \circ \delta_c \circ \tau_b \circ \delta_c$ or $\tilde{\tau} \circ \tilde{\delta} \circ \sigma_{0,1} \circ \bar{\delta}_c \circ \tau_b \circ \bar{\delta}_c$ where the coefficients $ad - bc > 0$, and $\bar{\mu}$ as the composition $\bar{\mu} = \tilde{\tau} \circ \tilde{\delta} \circ \sigma_{0,1} \circ \delta_c \circ \tau_b \circ \delta_c$ or $\tilde{\tau} \circ \tilde{\delta} \circ \sigma_{0,1} \circ \bar{\delta}_c \circ \tau_b \circ \bar{\delta}_c$ where the coefficients $ad - bc < 0$.

Firstly, consider the case of μ. Where $c > 0$ the Jacobian matrix of μ has determinant given by the product $\det J_\mu = \det J_{\tilde{\tau}} \cdot \det J_{\tilde{\delta}} \cdot \det J_{\delta_c} \cdot \det J_{\tau_b} \cdot \det J_{\delta_c}$ which by PROPOSITION 2.3.15 is the product of two negative and four positive determinants. Where $c < 0$, then the Jacobian matrix of $\bar{\mu}$ has determinant $\det J_{\bar{\mu}} = \det J_{\tilde{\tau}} \cdot \det J_{\tilde{\delta}} \cdot \det J_{\bar{\delta}_c} \cdot \det J_{\tau_b} \cdot \det J_{\bar{\delta}_c}$ and so is the product of four negative and two positive determinants. Thus in both cases $\det J_\mu > 0$, and the transformation is orientation-preserving.

Secondly, consider the case of $\bar{\mu}$. Where $c > 0$ the Jacobian matrix of $\bar{\mu}$ has determinant $\det J_\mu = \det J_{\tilde{\tau}} \cdot \det J_{\tilde{\delta}} \cdot \det J_{\delta_c} \cdot \det J_{\tau_b} \cdot \det J_{\delta_c}$ which by PROPOSITION 2.3.15 is the product of one negative and five positive determinants. Where $c < 0$, then the Jacobian matrix of $\bar{\mu}$ has determinant $\det J_{\bar{\mu}} = \det J_{\tilde{\tau}} \cdot \det J_{\tilde{\delta}} \cdot \det J_{\bar{\delta}_c} \cdot \det J_{\tau_b} \cdot \det J_{\bar{\delta}_c}$ and so is the product of two positive and three negative determinants. Thus in both cases $\det J_{\bar{\mu}} < 0$, and so the transformation is orientation-reversing. Since each restricted Möbius transformation is of the form μ or $\bar{\mu}$, the result follows. □

2.3.17 PROPOSITION. *Let* C_1 *and* C_2 *be two geodesics in* $\mathcal{HP}$ *and* z_1 *an element of* C_1, z_2 *an element of* C_2. *Then there exists an isometry* μ *or* $\bar{\mu}$ *of* $\mathbb{HP}$ *which takes* C_1 *to* C_2 *and* z_1 *to* z_2.

PROOF. We will show that there exists a transformation μ taking the arbitrary circle C_1 to $\mathcal{V}$, the v-axis, and z_1 to i. Then there exists an isometry μ' taking the arbitrary circle C_2 to $\mathcal{V}$ and z_2 to i, and so the composition $(\mu')^{-1} \circ \mu$ takes C_1 to C_2 and z_1 to z_2.

Assume initially that C_1 is a Euclidean line through q on the u-axis. Then the translation τ_{-q_1} maps C_1 to $\mathcal{V}$ and z_1 to some point bi on $\mathcal{V}$. Then the transformation $\delta_{\frac{1}{b}}$ maps bi to i and preserves $\mathcal{V}$, and it follows that the composition $\mu = \delta_{\frac{1}{b}} \circ \tau_{-q_1}$ maps C_1 to $\mathcal{V}$ and z_1 to i.

We then assume that C_1 is a Euclidean circle which intersects the u-axis perpendicularly in the points a and b. But by (A.3.9), the circle inversion $\sigma_{a,1}$ will map C_1 to a Euclidean line which intersects the u-axis perpendicularly. But then we have already shown that there exists an isometry μ' which maps $\sigma_{a,r}(C_1)$ to $\mathcal{V}$ and $\sigma_{a,r}(z_1)$ to i. The result follows. □

For the next proof, we set up the (topological) metric $d : \mathcal{HP} \times \mathcal{HP} \to \mathbb{R}$ as in (A.2.44). We do not explicitly express this metric.

2.3.18 PROPOSITION. *An isometry* ρ *of* $\mathcal{HP}$ *which fixes a geodesic which is a Euclidean circle will either preserve the interior of the fixed geodesic or interchange it with the exterior. Similarly, an isometry* ρ *of* $\mathcal{HP}$ *which fixes a geodesic which is a Euclidean line will either preserve or interchange the two half-planes of the fixed geodesic.*

PROOF. Let ρ be an isometry of $\mathbb{HP}$ which fixes the Euclidean line ℓ. Assume that ρ does not preserve or interchange the half-planes of ℓ: that is, there exists at least one point $\mathbf{p}$ within one half-plane $\mathcal{L}^+$ of ℓ which remains in $\mathcal{L}^+$ if ρ interchanges all other such points (if we assume that ρ interchanges one point and fixes all others, the proof follows identical steps). Consider the family of geodesics of $\mathbb{HP}$ which pass through the point $\mathbf{p}$. By (A.2.40), given any point on ℓ there exists a unique geodesic which passes through $\mathbf{p}$ and that point. Consider the point $\mathbf{s}$ on ℓ such that the unique geodesic which passes through $\mathbf{p}$ and $\mathbf{s}$ is of minimal arc length within this family of geodesics, as defined by the (topological) metric d. Denote this minimal distance $d(\mathbf{p}, \mathbf{s})$ by d. Within the half-plane $\mathcal{L}^+$, choose a point $\mathbf{q}$ such that $d(\mathbf{p}, \mathbf{q}) = d' < d$. Such a point always exists: consider for example $\mathbf{p}$ itself. Assume that the isometry ρ maps $\mathbf{q}$ to the half-plane $\mathcal{L}^-$. Since ρ is an isometry, then $d(\mathbf{p}, \mathbf{c}) = d(\rho(\mathbf{p}), \rho(\mathbf{c})) = d > d' = d(\rho(\mathbf{p}), \rho(\mathbf{q}))$. But since $\rho(\mathbf{q})$ lies in $\mathcal{L}^-$, then $d(\rho(\mathbf{p}), \rho(\mathbf{q})) > d$, since d is the minimal distance from $\mathbf{p}$ to ℓ, a contradiction. Thus ρ either preserves or interchanges the half-planes of ℓ. □

2.3.19 PROPOSITION. *Any isometry ρ of $\mathbb{HP}$ which fixes the v-axis $\mathcal{V}$ and the geodesic $\mathcal{C}_{0,1}$ centered at the origin with raidus 1 pointwise is the identity transformation ι.*

PROOF. For any $\mathbf{p} \in \mathcal{HP}$, by THEOREM 2.3.4 we can define $\mathcal{V}'$, the unique geodesic passing through $\mathbf{p}$ and perpendicular to $\mathcal{V}$, and $\mathcal{C}'$, the unique geodesic passing through $\mathbf{p}$ and perpendicular to $\mathcal{C}_{0,1}$. Let $\mathbf{q}$ and $\mathbf{s}$ be the intersections of $\mathcal{V}'$ with $\mathcal{V}$ and $\mathcal{C}'$ with $\mathcal{C}_{0,1}$, respectively. But by definition $\rho(\mathbf{q}) = \mathbf{q}$ and $\rho(\mathbf{s}) = \mathbf{s}$, so $\rho(\mathcal{V}') = \mathcal{V}'$ and $\rho(\mathcal{C}') = \mathcal{C}'$ as $\mathcal{V}'$ and $\mathcal{C}'$ are the unique geodesics passing through $\mathbf{q}$ and $\mathbf{s}$ and perpendicular to $\mathcal{V}$ and $\mathcal{C}_{0,1}$. Since the arbitrary point $\mathbf{p}$ is the (unique) point of intersection of $\mathcal{V}'$ and $\mathcal{C}'$, thus for each $\mathbf{p} \in \mathcal{HP}$, $\rho(\mathbf{p}) = \mathbf{p}$. That is, ρ is the identity transformation on $\mathcal{HP}$. □

2.3.20 PROPOSITION. *Any isometry ρ of $\mathbb{HP}$ such that $\rho(\mathcal{V}) = \mathcal{V}$ and $\rho(\mathcal{C}_{0,1}) = \mathcal{C}_{0,1}$ is either the identity transformation, the transformation $\nu(z) = -\bar{z}$, $\sigma_{0,1}$ or $\sigma_{0,1} \circ \nu(z) = \frac{-1}{z}$.*

PROOF. Since ρ fixes $\mathcal{C}_{0,1}$, then by PROPOSITION 2.3.18, ρ may either fix or interchange $\mathcal{I}_{0,1}$ with $\mathcal{E}_{0,1}$. But then either ρ or $\sigma_{0,1} \circ \rho$ fixes $\mathcal{V}, \mathcal{C}_{0,1}$ and the interior $\mathcal{I}_{0,1}$ of the semicircle $\mathcal{C}_{0,1}$: if ρ maps $\mathcal{I}_{0,1}$ to $\mathcal{E}_{0,1}$, then the circle inversion $\sigma_{0,1}$ will map the set $\rho(\mathcal{I}_{0,1})$ exterior to $\sigma_{0,1}$ to $\mathcal{I}_{0,1}$, by (A.3.7).

Further, defining the region $\mathcal{A} = \{z \in \mathcal{HP} : \operatorname{Re}(z) > 0\}$, the half plane to the right of $\mathcal{V}$, then either $\rho, \sigma_{0,1} \circ \rho, \nu \circ \rho$ or $\nu \circ \sigma_{0,1} \circ \rho$ fixes $\mathcal{V}, \mathcal{C}_{0,1}$, $\mathcal{I}_{0,1}$ and $\mathcal{A}$: if ρ fixes $\mathcal{V}, \mathcal{C}_{0,1}$ and $\mathcal{I}_{0,1}$, then by PROPOSITION 2.3.18 ρ may either fix or interchange $\mathcal{A}$ with $-\mathcal{A}$. But if ρ interchanges these two half-planes, then $\nu \circ \rho$ will map $\mathcal{A}$ to $\mathcal{A}$. Similarly, if $\sigma_{0,1} \circ \rho$ fixes $\mathcal{V}, \mathcal{C}_{0,1}$ and $\mathcal{I}_{0,1}$, then $\sigma_{0,1} \circ \rho$ may either fix or interchange $\mathcal{A}$ with $-\mathcal{A}$. But if $\sigma_{0,1} \circ \rho$ interchanges these two half-planes, then $\nu \circ \sigma_{0,1} \circ \rho$ will map $\mathcal{A}$ to $\mathcal{A}$.

In any case, let $\tilde{\rho}$ be this isometry. Then $\tilde{\rho}$ fixes each point of $\mathcal{C}_{0,1}$ because there is a unique point of $\mathcal{C}_{0,1}$ at any given distance $d > 0$ from i in the region $\mathcal{A}$. Similarly, $\tilde{\rho}$ fixes each point of $\mathcal{V}$. Hence, $\tilde{\rho}$ is the identity by PROPOSITION 2.3.19. Then $\rho = \iota, \sigma_{0,1} \circ \rho = \iota, \nu \circ \rho = \iota$ or $\nu \circ \sigma_{0,1} \circ \rho = \iota$: thus $\rho = \iota, \rho = \sigma_{0,1}, \rho = \nu$ or $\rho = \nu \circ \sigma_{0,1}$. □

2.3.21 THEOREM. *Every isometry of* $\mathbb{HP}$ *has the form* μ *or* $\bar{\mu}$.

PROOF. Let ρ be any isometry of $\mathbb{HP}$. By PROPOSITION 2.3.17, there is an isometry $\widetilde{\rho}$ that is a composite of elementary isometries and which takes $\rho(i)$ to i and $\rho(\mathcal{V})$ to $\mathcal{V}$. Then, $\widetilde{\rho}\circ\rho$ is an isometry that fixes $\mathcal{V}$ and i. As $\mathcal{C}_{0,1}$ is the unique geodesic intersecting $\mathcal{V}$ perpendicularly at i, $\widetilde{\rho}\circ\rho$ fixes $\mathcal{C}_{0,1}$. But by PROPOSITION 2.3.15, then $\widetilde{\rho}\circ\rho$ is one of four compositions of the basic transformations: $\widetilde{\rho}\circ\rho=\iota$, $\widetilde{\rho}\circ\rho=\sigma_{0,1}$, $\widetilde{\rho}\circ\rho=\nu$ or $\widetilde{\rho}\circ\rho=\nu\circ\sigma_{0,1}$, and it follows that $\rho=\iota\circ\widetilde{\rho}^{-1}$, $\rho=\sigma_{0,1}\circ\widetilde{\rho}^{-1}$, $\rho=\nu\circ\widetilde{\rho}^{-1}$ or $\rho=\nu\circ\sigma_{0,1}\circ\widetilde{\rho}^{-1}$. Thus ρ is a composition of the basic linear fractional transformations and so from COROLLARY 2.3.13, it has the form μ or $\bar{\mu}$. □

2.3.22 THEOREM. *The symmetry group of* $\mathbb{HP}$ *is exactly the subgroup of restricted Möbius transformations*

$$\mathsf{Sym}(\mathbb{HP})=\left\{\mu(z)=\frac{az+b}{cz+d}\ :\ a,b,c,d\in\mathbb{R}, ad-bc>0\right\}.$$

PROOF. In PROPOSITION 2.3.21 we showed that all isometries of $\mathbb{HP}$ are of the form $\mu : z\mapsto \frac{az+b}{cz+d}$, where $ad-bc>0$ or $\bar{\mu} : z\mapsto\frac{a\bar{z}+b}{c\bar{z}+d}$, where $ad-bc<0$. Further, we proved in PROPOSITION 2.3.16 that each transformation in this family of restricted Möbius transformations are exactly the orientation-preserving transformations of $\mathcal{HP}$. Thus the transformations $\mu : z\mapsto\frac{az+b}{cz+d}$, where $ad-bc>0$ are exactly the symmetries of $\mathbb{HP}$. □

We now express the symmetry group of $\mathbb{HP}$ as a matrix group.

2.3.23 THEOREM. *The map*

$$\psi_1 : \begin{bmatrix} a & b\\ c & d\end{bmatrix}\mapsto\frac{az+b}{cz+d}$$

is a continuous group homomorphism from $\mathsf{GL}(2,\mathbb{R})$ *to* $\mathsf{Sym}(\mathbb{HP})$.

PROOF. Firstly, ψ_1 is surjective, since given any $\mu=\dfrac{az+b}{cz+d}\in\mathsf{Sym}(\mathbb{HP})$, then there exists $h=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ in $\mathsf{GL}(2,\mathbb{R})$ such that $\psi_1(h)=\dfrac{az+b}{cz+d}$. Secondly, from the proof of PROPOSITION 2.3.12 the composition of two restricted Möbius transformations $z\mapsto\dfrac{az+b}{cz+d}$ and $z\mapsto\dfrac{a'z+b'}{c'z+d'}$ in $\mathsf{Sym}(\mathbb{HP})$ is a restricted Möbius transformation $\frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}$ which has coefficients $\tilde{a}=aa'+bc'$, $\tilde{b}=ab'+bd'$, $\tilde{c}=ca'+dc'$, $\tilde{d}=cb'+dd'$, where

$$\begin{bmatrix}\tilde{a} & \tilde{b}\\ \tilde{c} & \tilde{d}\end{bmatrix}=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\begin{bmatrix} a' & b'\\ c' & d'\end{bmatrix}\quad\text{and}\quad\tilde{a}\tilde{d}-\tilde{b}\tilde{c}\neq 0.$$

Thus

$$\psi_1\begin{bmatrix} a & b\\ c & d\end{bmatrix}\begin{bmatrix} a' & b'\\ c' & d'\end{bmatrix}=\psi_1\begin{bmatrix}\tilde{a} & \tilde{b}\\ \tilde{c} & \tilde{d}\end{bmatrix}=\frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}=\mu(z)\circ\mu'(z)=\psi_1\begin{bmatrix} a & b\\ c & d\end{bmatrix}\psi_1\begin{bmatrix} a' & b'\\ c' & d'\end{bmatrix}$$

and the map ψ_1 from $\mathsf{GL}(2,\mathbb{R})$ to $\mathsf{Sym}(\mathbb{HP})$ sends the group operation of matrix multiplication to the group operation of composition. Thus ψ_1 is a surjective group homomorphism.

Finally, we note that the transformation $\mu : z \mapsto \frac{z+0}{1+0z}$ is the identity element of $\mathsf{Sym}(\mathbb{HP})$. But then given any $k \in \mathbb{R}, k \neq 0$, $\psi_1(k\mathbf{1}) = \frac{kz+0}{0z+k} = \psi_1(\mathbf{1})$, and $\{k\mathbf{1} : k \in \mathbb{R}\backslash\{0\}\} \subseteq \ker(\psi_1)$: thus ψ_1 is not injective. Taking h in $\mathsf{GL}(2,\mathbb{R})$ such that $\psi_1(h) = \frac{az+b}{cz+d} = \frac{z+0}{1+0z}$, then

$$\frac{az+b}{cz+d} = z \quad \Leftrightarrow \quad az+b = z(cz+d) \tag{2.3.3}$$

must be true for all $z \in \mathcal{HP}$ such that $cz+d \neq 0$. But then particularly taking $z=0$ in (2.3.3), it follows that $b = az+b = z(cz+d) = 0$ and thus $b=0$. Since $ad-bc > 0$, then consequentially $ad > 0$. Taking $z=1$ in (2.3.3), it follows that $az = z(cz+d)$ and thus $a = c+d$. Similarly, taking $z=-1$, then $-a = c-d$. Thus $c = a-d = d-a$ and so $c=0$. Since $ad > 0$, and by $c=0$ then $a=d$, it follows that $a^2 > 0$ and thus $a \in \mathbb{R}\backslash\{0\}$. Thus we have shown that if $h \in \ker(\psi_1)$, then $h \in \{k\mathbf{1} : k \in \mathbb{R}\backslash\{0\}\}$, and so $\ker(\psi_1) \subseteq \mathbb{R}^+\backslash\{0\}$. Since we showed that $\{k\mathbf{1} : k \in \mathbb{R}\backslash\{0\} \subseteq \ker(\psi_1)$, then it follows that $\ker(\psi_1) = \{k\mathbf{1} : k \in \mathbb{R}\backslash\{0\}\}$.

We show that ψ_1 is continuous. Consider some convergent sequence in $\mathsf{GL}(2,\mathbb{R})$,

$$h^t = \begin{bmatrix} a^t & b^t \\ c^t & d^t \end{bmatrix} \text{ such that } \lim_{t\to\infty} \begin{bmatrix} a^t & b^t \\ c^t & d^t \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = h.$$

Then $\lim_{t\to\infty} a^t = a$, $\lim_{t\to\infty} b^t = b$, $\lim_{t\to\infty} c^t = c$, $\lim_{t\to\infty} d^t = d$, and $\lim_{t\to\infty} \psi_1(h^t) = \lim_{t\to\infty} \frac{a^t z+b^t}{c^t z+d^t} = \frac{az+b}{cz+d} = \psi_1(h)$. Thus ψ_1 preserves the limits of convergent sequences and so must be continuous. The result follows. □

2.3.24 COROLLARY. *There exists a continuous isomorphism mappping* $\mathsf{GL}(2,\mathbb{R})/\{k\mathbf{1} : k \in \mathbb{R}\}$ *to* $\mathsf{Sym}(\mathbb{HP})$.

PROOF. In PROPOSITION 2.3.23, we defined the group homomorphism $\psi_1 : \mathsf{GL}(2,\mathbb{R}) \to \mathsf{Sym}(\mathbb{HP})$ such that $\ker(\psi_1) = \{k\mathbf{1} : k \in \mathbb{R}\}$. Then $\mathsf{Sym}(\mathbb{HP}) \cong \mathsf{GL}(2,\mathbb{R})/\{k\mathbf{1} : k \in \mathbb{R}\}$ by (A.5.8), and thus it follows that ψ_1 is a continuous isomorphism between $\mathsf{GL}(2,\mathbb{R})/\{k\mathbf{1} : k \in \mathbb{R}\}$ and $\mathsf{Sym}(\mathbb{HP})$. □

$\mathsf{GL}(2,\mathbb{R})/\{k\mathbf{1} : k \in \mathbb{R}\}$ is the **projective general linear group** $\mathsf{PGL}(2,\mathbb{R})$.

2.4 Symmetry group isomorphisms

In Appendix A,(A.4.1) - (A.4.4), we introduced the projections between the hyperbolic model $\mathbb{HL}$ and the projective disk model $\mathbb{PD}$ as well as between $\mathbb{HP}$ and $\mathbb{PD}$. In this section we use these mappings to establish the isometries between the symmetry groups we have defined in SECTIONS 2.1 and 2.2. We refer to the definition (A.4.1) of the alternate Minkowski spacetime $\mathbb{R}^{2,1}$ and the hyperboloid model $\widetilde{\mathbb{HL}}$ used in [7] and construct the mapping $\zeta : \mathbb{R}^{2,1} \to \mathbb{R}^{1,2}$, given by $\zeta(p_1,p_2,p_3) = (p_3,p_2,p_1)$.

2.4.1 PROPOSITION. *The map* $\zeta : \mathbb{R}^{2,1} \to \mathbb{R}^{1,2}$ *is continuous Riemann isometry mapping* $\widetilde{\mathbb{HL}}$ *to* $\mathbb{HL}$.

PROOF. Firstly, note that ζ is a linear map: given $\mathrm{p}, \mathrm{q} \in \mathbb{R}^{2,1}$, $\lambda \in \mathbb{R}$, then for any sum $(\mathrm{p}+\lambda\mathrm{q})$, $\zeta(\mathrm{p}+\lambda\mathrm{q}) = (p_1+\lambda q_1, p_2+\lambda q_2, p_3+\lambda q_3) = (p_3,p_2,p_1) + \lambda(q_3,q_2,q_1) = \zeta(\mathrm{p}) + \lambda\zeta(\mathrm{q})$. By the linearity of ζ, then $d\zeta(\mathbf{p})(v) = \zeta(\mathbf{v})$ for each $\mathbf{v} = (v_1, v_2, v_3) \in \mathrm{T}_\mathbf{p}(\widetilde{\mathcal{HL}})$, and it follows that $\zeta_*(\mathbf{v}\odot\mathbf{w}) = \zeta(\mathbf{v})\tilde{\odot}\zeta(\mathbf{w})$. But for each $\mathbf{v}, \mathbf{w} \in \mathrm{T}_\mathbf{p}(\widetilde{\mathcal{HL}})$, $\mathbf{v}\tilde{\odot}\mathbf{v} = v_1w_1 + v_1w_2 - v_3w_3$, whereas

$$\zeta_*(\mathbf{v}\tilde{\odot}\mathbf{w}) = \zeta(\mathbf{v})\tilde{\odot}\zeta(\mathbf{w}) = \begin{bmatrix} v_3 & v_2 & v_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} w_3 \\ w_2 \\ w_1 \end{bmatrix} = -v_1w_1 + v_2w_2 + v_3w_3 = \mathbf{v} \odot \mathbf{w}.$$

Thus ζ is a Riemann isometry mapping $\widetilde{\mathbb{HL}}$ to $\mathbb{HL}$ by (A.2.27). Since ζ is linear, then it is continuous. The result follows. □

2.4.2 PROPOSITION. *The projections $\pi_1 : \widetilde{\mathcal{HL}} \to \mathcal{PD}$ of (A.4.3), given by $\pi_1(x, y, z) = \frac{x+iy}{1+z} = u + iv$, and $\pi_2 : \mathcal{PD} \to \widetilde{\mathcal{HP}}$ of (A.4.4), given by $\pi_2(z) = \frac{z+1}{iz-i}$, are continuous. Similarly, their inverses $\pi_1^{-1}(u + iv) = \left(\frac{2u}{1-u^2-v^2}, \frac{2v}{1-u^2-v^2}, \frac{1+u^2+v^2}{1-u^2-v^2}\right)$ and $\pi_2^{-1}(z) = \frac{z-i}{z+i}$ are continuous.*

PROOF. We consider the component functions of $\pi_1 : \mathcal{HL} \to \mathcal{PD}$, where $\pi_1(x, y, z) = (\frac{x}{1+z}, \frac{y}{1+z})$. Since each component function is a rational function, then it is a continuous map. Thus it follows that π_1 is continuous. Further, the component functions of the projection $\pi_1^{-1} : \mathcal{PD} \to \mathcal{HL}$, where

$$\pi_1^{-1}(u, v) = \left(\frac{2u}{1 - u^2 - v^2}, \frac{2v}{1 - u^2 - v^2}, \frac{1 + u^2 + v^2}{1 - u^2 - v^2}\right)$$

are each rational functions which are defined on $\mathcal{PD} = \left\{(u, v) \in \mathbb{R}^2 \ : \ u^2 + v^2 < 1\right\}$ and so are continuous maps. Thus π_1^{-1} is continuous, and the result follows for π_1.

Similarly, since $\pi_2 : z \mapsto \frac{z+1}{iz-i}$ is a rational function which is defined on $\mathcal{PD} = \left\{z \in \mathbb{R}^2 \ : \ |z| < 1\right\}$, then it is a continuous map $\pi_2 : \widetilde{\mathcal{PD}} \to \mathcal{HP}$, and $\pi_2^{-1} : \mathbb{C} \to \mathbb{C}$ is the rational function $\pi_2^{-1}(z) = \frac{z-i}{z+i}$, and so continuous. The result follows for π_2. □

We use these projections in conjunction with the map ζ to prove the next theorem.

2.4.3 THEOREM. *There exists a continuous group isomorphism mapping $\mathsf{GL}(2, \mathbb{R})/\left\{k\mathbf{1} \ : \ k \in \mathbb{R}^+\right\}$ to $\mathsf{SO}(1, 2)_0$.*

PROOF. From PROPOSITION 2.4.4, the projection π_1 of (A.4.3) is a continuous Riemann isometry between the geometric surface $\widetilde{\mathbb{HL}}$ and the projective disk $\mathbb{PD}$. Since in PROPOSITION 2.4.1 we defined a continuous Riemann isometry ζ taking $\widetilde{\mathbb{HL}}$ to $\mathbb{HL}$, then $\zeta \circ \pi_1^{-1} : \mathbb{PD} \to \mathbb{HL}$ is a continuous Riemann isometry between $\mathbb{HL}$ and $\mathbb{PD}$. Further, from (A.4.4), π_2 is a Riemann isometry from $\mathbb{PD}$ to $\mathbb{HP}$. Thus for any symmetry $\mu \in \mathsf{Sym}(\mathbb{HP})$, then $\pi_1^{-1} \circ \pi_2^{-2} \circ \mu \circ \pi_2 \circ \pi_1$ is a symmetry of $\widetilde{\mathbb{HL}}$. Further, the composition $\zeta \circ \pi_2^{-2} \circ \mu \circ \pi_2 \circ \pi_1 \circ \zeta^{-1} = \varphi_1(\mu)$ is a symmetry of $\mathbb{HL}$, and so by THEOREM 2.2.29 an element of $\mathsf{SO}(1, 2)_0$. The map φ_2 is a group homomorphism, since given two elements μ_1 and μ_2 of $\mathsf{Sym}(\mathbb{HP})$, then

$$\varphi_1(\mu_1 \circ \mu_2) = \zeta \circ \pi_1^{-1} \circ \pi_2^{-2} \circ \mu_1 \circ \pi_2 \circ \pi_1 \circ \zeta^{-1} \circ \zeta \circ \pi_1^{-1} \circ \pi_2^{-2} \circ \mu_2 \circ \pi_2 \circ \pi_1 \circ \zeta^{-1} = \varphi_1(\mu_1)\varphi_1(\mu_2).$$

Further, assume that $\mu \in \ker(\varphi_1)$. Then

$$\varphi_1(\mu) = \zeta \circ \pi_1^{-1} \circ \pi_2^{-2} \circ \mu \circ \pi_2 \circ \pi_1 \circ \zeta^{-1} = \iota \Rightarrow \mu = \pi_1 \circ \pi_2 \circ \zeta^{-1} \circ \iota \circ \zeta \circ \pi_2^{-1} \circ \pi_1^{-1} = \iota$$

and so $\ker(\varphi_1) \subseteq \{\iota\}$. But clearly $\iota \subseteq \ker(\varphi_1)$, and thus $\ker(\varphi_1) = \{\iota\}$. Thus φ_1 is a continuous group isomorphism, and $\mathsf{Sym}(\mathbb{HP}) \cong \mathsf{SO}(1,2)_0$. Since by PROPOSITION 2.4.4 π_2 and π_2^{-1} are continuous on their domains, then as the composition of continuous functions is continuous, it follows that φ_1 is a continuous group isomorphism. But in THEOREM 2.3.24 we defined the continuous group isomorphism $\psi_1 : \mathsf{GL}(2,\mathbb{R})/\{k\mathbf{1} : k \in \mathbb{R}\} \to \mathsf{Sym}(\mathbb{HP})$. Thus the map $\varphi_2 = \psi_1 \circ \varphi_1 : \mathsf{GL}(2,\mathbb{R})/\{k\mathbf{1} : k \in \mathbb{R}\} \to \mathsf{SO}(1,2)_0$ is a continuous group isomorphism, and the result follows. □

2.4.4 COROLLARY. *There exists a continuous group homomorphism* φ_2 *mapping* $\mathsf{GL}(2,\mathbb{R})$ *to* $\mathsf{SO}(1,2)_0$, *where* $\ker(\varphi_2) = \{k\mathbf{1} : k \in \mathbb{R}\}$.

PROOF. From THEOREM 2.4.3, the map $\varphi_1 : \mathsf{Sym}(\mathbb{HP}) \to \mathsf{SO}(1,2)_0$ is a continuous group isomorphism. But in THEOREM 2.3.23 we defined the continuous group homomorphism ψ_1 mapping $\mathsf{GL}(2,\mathbb{R})$ to $\mathsf{Sym}(\mathbb{HP})$ such that $\ker(\psi_1) = \{k\mathbf{1} : k \in \mathbb{R}\}$. Thus the composition of maps $\varphi_2 = \varphi_1 \circ \psi_1 : \mathsf{GL}(2,\mathbb{R}) \to \mathsf{SO}(1,2)_0$ is a continuous group homomorphism, where the kernel $\ker(\varphi_2) = \{k\mathbf{1} : k \in \mathbb{R}\}$. □

Chapter 3

The Matrix Lie Group $\mathsf{SO}(1,2)_0$ and its Lie Algebra

3.1 The matrix Lie group $\mathsf{SO}(1,2)_0$

The matrix group

$$\mathsf{SO}(1,2)_0 = \left\{ g \in \mathbb{R}^{3\times 3} \ : \ gJg^\top = J,\ \det g = 1,\ \mathbf{e}_1^\top g \mathbf{e}_1 > 0 \right\}$$

is by THEOREM 2.2.29 the symmetry group of the model $\mathbb{HL}$ of hyperbolic plane geometry. We show that it is a Lie group and find its Lie algebra, for which we determine several properties. From these results, we may then derive further properties of $\mathsf{SO}(1,2)_0$, including its Iwasawa decomposition. Following [13], we establish a map between the Lie algebra $\mathfrak{so}(1,2)$ and $\mathbb{R}^{1,2}$ equipped with a Lie algebra structure $\odot$, which has an interesting interrelation with the Lorentz product, which we discuss in detail. We then establish the adjoint and co-adjoint orbits of $\mathfrak{so}(1,2)$, using its representation in $\mathbb{R}^{1,2}_{\odot}$.

3.1.1 THEOREM. *$\mathsf{SO}(1,2)_0$ is a matrix Lie group.*

PROOF. By (A.5.3), we require to show that $\mathsf{SO}(1,2)_0$ is a closed subgroup of $\mathsf{GL}(3,\mathbb{R})$. From PROPOSITION 2.2.16, $\mathsf{SO}(1,2)_0$ is a subgroup of $\mathsf{GL}(3,\mathbb{R})$: thus we show that $\mathsf{SO}(1,2)_0$ is closed in $\mathsf{GL}(3,\mathbb{R})$. Consider a convergent sequence g^t in $\mathsf{SO}(1,2)$. Then $\lim\limits_{t\to\infty}(g^t) = g$ for some matrix g. We show that g satisfies the three defining conditions $gJg^\top = J$, $\det g = 1$ and $\mathbf{e}_1^\top g \mathbf{e}_1 > 0$ of an element of $\mathsf{SO}(1,2)_0$. Define the continuous function $f(g^t) = g^t_{1i}g^t_{1j} + g^t_{2i}g^t_{2j} + g^t_{3i}g^t_{3j}$. Since g^t satisfies the property $(g^t)^\top J\, g^t = J$ for every t, and it follows that

$$gJg^\top = \begin{bmatrix} \Sigma^3_m g^t_{m1}g^t_{m1} & \Sigma^3_m g^t_{m1}g^t_{m2} & \Sigma^3_m g^t_{m1}g^t_{m3} \\ \Sigma^3_m g^t_{m2}g^t_{m1} & \Sigma^3_m g^t_{m2}g^t_{m2} & \Sigma g^t_{m3}g^t_{m2} \\ \Sigma^3_m g^t_{m3}g^t_{m1} & \Sigma^3_m g^t_{m3}a_{m2} & \Sigma^3_m g^t_{m3}g^t_{m3} \end{bmatrix} = J$$

and thus $f(g^t) = j_{ij}$, and $f(g) = \lim\limits_{t\to\infty}(f(g^t)) = \lim\limits_{t\to\infty}(g^t_{1i}g^t_{1j} + g^t_{3i}g^t_{3j}) = \lim\limits_{t\to\infty} j_{ij} = j_{ij}$, since continuous functions preserve the limits of convergent sequences. Since det is a continuous function, then $\lim_{t\to\infty}(\det g^t) = \det g$, where $\lim_{t\to\infty}(\det g^t) = \lim_{t\to\infty} 1 = 1 = \det g$. Finally, $\mathrm{e}_1^\top \left(\lim_{t\to\infty} g^t\right) \mathrm{e}_1 = \lim_{t\to\infty}\left(\mathrm{e}_1^\top g^t \mathrm{e}_1\right) = \lim_{t\to\infty}(g_{11}^t)$, where by the definition of $\mathsf{SO}(1,2)_0$, g_{11}^t is a contiunous function such that for each t, $g_{11}^t = |g_{11}^t|$. But then it follows that particularly $\lim_{t\to\infty}(g_{11}^t) = \lim_{t\to\infty}|g_{11}^t| = |\lim_{t\to\infty}(g_{11}^t)| = |g_{11}|$, and $g_{11} > 0$. Then we have shown that $g = \lim_{t\to\infty} g^t$ fulfils the three properties defining an element of $\mathsf{SO}(1,2)_0$ and so $\lim_{t\to\infty}(g^t) \in \mathsf{SO}(1,2)_0$. Thus $\mathsf{SO}(1,2)_0$ contains all its limit points and so is a closed subset of $\mathsf{GL}(3,\mathbb{R})$. □

3.1.2 THEOREM. $\mathsf{SO}(1,2)_0$ *is connected.*

PROOF. From THEOREM 2.2.23, we may express any element $g \in \mathsf{SO}(1,2)_0$ as the matrix product

$$g = \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta 1} \end{bmatrix} \begin{bmatrix} \cosh\theta_2 & \sinh\theta_2 & 0 \\ \sinh\theta_2 & \cosh\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta 1}^\top \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta 3} \end{bmatrix}.$$

for $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$. Define the curve $g(\cdot) : [0,1] \to \mathsf{SO}(1,2)_0$ where for each $t \in [0,1]$,

$$g(t) = \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_1(t)}^\top \end{bmatrix} \begin{bmatrix} \cosh\theta_2(t) & \sinh\theta_2(t) & 0 \\ \sinh\theta_2(t) & \cosh\theta_2(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_1(t)}^\top \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R_{\theta_3(t)} \end{bmatrix}$$

and $\theta_i(\cdot) : [0,1] \to [0,\theta_i], i = 1,2,3$ are continuous maps $\theta_3(1) = \theta_1(1) = \theta_2(1) = 0$ and $\theta_3(0) = \theta_3$, $\theta_2(0) = \theta_2$, $\theta_1(0) = \theta_1$. Clearly, $g(1) = 1$ and $g(0) = g$ for this curve. Thus from (A.5.12), $\mathsf{SO}(1,2)_0$ is connected. □

3.1.3 THEOREM. $\mathsf{SO}(1,2)_0$ *is not compact.*

PROOF. From THEOREM 3.1.1, $\mathsf{SO}(1,2)_0$ is a closed subgroup of $\mathsf{GL}(3,\mathbb{R})$, which from (A.5.16) is a finite-dimensional vector space equipped with the matrix norm $\|g\| = \sqrt{\mathrm{tr}(g^\top g)}$. Thus $\mathsf{SO}(1,2)_0$ is compact if and only if it is bounded. Consider the element g of $\mathsf{SO}(1,2)_0$,

$$g = \begin{bmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & 1 & 0 \\ \sinh(t) & 0 & \cosh(t) \end{bmatrix}.$$

Then

$$\mathrm{tr}(gg^\top) = \mathrm{tr}\begin{bmatrix} \cosh^2 t + \sinh^2 t & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & \cosh^2 t + \sinh^2 t \end{bmatrix} = 2(\cosh^2 t + \sinh^2 t) + 1.$$

Since $\mathsf{SO}(1,2)_0$ is a closed subgroup of $\mathsf{GL}(3,\mathbb{R})$, then $\lim_{t\to\infty}(g) \in \mathsf{SO}(1,2)_0$. But we then see that $\|\lim_{t\to\infty}(g)\| = \lim_{t\to\infty}\|g\| = \infty$. Thus $\mathsf{SO}(1,2)_0$ cannot be bounded. The result follows. □

3.2 The Lie algebra $\mathfrak{so}(1,2)$

3.2.1 THEOREM. *The Lie algebra* $\mathfrak{so}(1,2)$ *of* $\mathsf{SO}(1,2)_0$ *is the vector space*

$$\mathfrak{so}(1,2) = \left\{ X \in \mathbb{R}^{3\times 3} \ : \ X^\top J + JX = 0 \right\}$$

of 3×3 *real matrices equipped with the matrix commutator* $[\cdot,\cdot]$.

PROOF. From (A.5.18), we require to show that $\mathrm{T}_1\mathsf{SO}(1,2)_0 = \{X \in \mathbb{R}^{3\times 3} \ : \ X^\top J + JX = 0\}$. Consider the arbitrary curve $g(\cdot) : (a,b) \to \mathsf{SO}(1,2)_0$ such that $g(0) = \mathbf{1}$. Then for all $t \in (a,b)$, $g(t)^\top J g(t) = J$, and

$$\begin{aligned} \tfrac{d}{dt}(g^\top(t)(J)g(t))\,|_{t=0} &= \dot{g}(t)^\top(Jg(t)) + g(t)J\dot{g}(t)|_{t=0} &= \frac{d}{dt}J \\ \Rightarrow \quad & \dot{g}(0)^\top J + J\dot{g}(0) &= 0. \end{aligned}$$

Since by (A.5.14), $\dot{g}(0) \in \mathrm{T}_1\mathsf{SO}(1,2)_0$, then clearly $\mathrm{T}_1\mathsf{SO}(1,2) \subseteq \{X \in \mathbb{R}^{3\times 3} \ : \ X^\top J + JX = 0\}$. Define the curve $g(\cdot) : (a,b) \to \mathsf{GL}(3,\mathbb{R})$, $g(t) = \exp(tX)$ where $X^\top J + JX = 0$, or equivalently $JX^\top J = X$. Then it follows that

$$\exp(tX) = \exp(-tJX^\top J) \Leftrightarrow \quad \exp(tX) = J\exp(-tX^\top)J \tag{3.2.1}$$

$$\Leftrightarrow \quad J\exp(tX) = \exp(-tX^\top)J \tag{3.2.2}$$

$$\Leftrightarrow \quad \exp(tX^\top)\,J\exp(tX) = J \tag{3.2.3}$$

$$\Leftrightarrow \quad (\exp(tX))^\top J\,\exp(tX) = J \tag{3.2.4}$$

where (3.2.1) follows from (A.5.30), and so $g(t) = \exp(tX)$ is a curve in $\mathsf{SO}(1,2)$. But since $\dot{g}(0) = X$ where $g(0) = \mathbf{1}$, then by (A.5.18), $\dot{g}(0)$ is an element of $\mathrm{T}_1\mathsf{SO}(1,2)_0$. Thus the set $\{X \in \mathbb{R}^{3\times 3} \ : \ X^\top J + JX = 0\} \subseteq \mathrm{T}_1\mathsf{SO}(1,2)$. From (A.5.15), $\mathrm{T}_1\mathsf{SO}(1,2)_0$ is a vector space. The result follows. □

3.2.2 REMARK. Given some 3×3 matrix $X = [x_{ij}]$, the condition $XJ + JX^\top = 0$ is equivalent to the conditions $x_{11} = x_{22} = x_{33} = 0, x_{11} = x_{33}, x_{23} = -x_{32}$ and $x_{12} = x_{21}$, by matrix multiplication. Thus each element of $\mathfrak{so}(1,2)$ has the form

$$\begin{bmatrix} 0 & x_{12} & x_{13} \\ x_{12} & 0 & -x_{32} \\ x_{13} & x_{32} & 0 \end{bmatrix}$$

and we take

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as the standard basis for $\mathfrak{so}(1,2)$. The set $_1, E_2, E_3\}$ of basis elements has the commutator relations

$$[E_1, E_2] = -E_3, \quad [E_2, E_3] = E_1 \quad \text{and} \quad [E_3, E_1] = -E_2. \tag{3.2.5}$$

We use this basis to establish the algebraic results of $\mathfrak{so}(1,2)$.

3.2.3 THEOREM. *The centre $\mathfrak{z}(\mathfrak{so}(1,2))$ of $\mathfrak{so}(1,2)$ is $\{0\}$.*

PROOF. Let $X = aE_1 + bE_2 + cE_3$ be an arbitrary element of $\mathfrak{z}(\mathfrak{so}(1,2))$. Then from (A.5.52), $[aE_1 + bE_2 + cE_3, E_n] = 0$ for $n = 1, 2, 3$, and particularly

$$[aE_1 + bE_2 + cE_3, E_1] = 0 \quad \Rightarrow \quad b(E_3) + c(-E_2) = 0 \tag{3.2.6}$$

$$[aE_1 + bE_2 + cE_3, E_2] = 0 \quad \Rightarrow \quad a(E_3) + c(-E_1) = 0 \tag{3.2.7}$$

$$[aE_1 + bE_2 + cE_3, E_3] = 0 \quad \Rightarrow \quad a(E_2) + b(E_1) = 0. \tag{3.2.8}$$

Since E_1, E_2 and E_3 are linearly independent elements, then from (3.2.6) and (3.2.8) it follows that $b(E_3) + c(-E_2) = 0 = a(E_2) + b(E_1) \Rightarrow b = 0$, and $a = c$, and from (3.2.7) and (3.2.8), $a(E_3) + c(-E_1) = 0 = a(E_2) + b(E_1)$ it follows that $a = 0$ and $b = -c$. Thus $a = b = c = 0$, and so every arbitrary $X \in \mathfrak{z}(\mathfrak{so}(1,2)$ is zero. Thus $\mathfrak{z}(\mathfrak{so}(1,2)) \subseteq \{0\}$. Since clearly $\{0\} \subseteq \mathfrak{z}(\mathfrak{so}(1,2))$, the result follows. □

3.2.4 THEOREM. *$\mathfrak{so}(1,2)$ is simple.*

PROOF. Assume $\mathfrak{so}(1,2)$ is not simple: that is, there exists a non-trivial ideal $\mathfrak{i}$ of $\mathfrak{so}(1,2)$. Then there exists some nonzero $X \in \mathfrak{i}, X = aE_1 + bE_2 + cE_3$ where for all $Y \in \mathfrak{so}(1,2), [X,Y] \in \mathfrak{i}$. Fixing any two of a,b, or c zero, then $a[E_1, E_i] + b[E_2, E_i] + c[E_3, E_i] \in \mathfrak{i}$, and by the commutator relations (3.2.5), $E_i \in \mathfrak{i}$ for $i = 1, 2$ or 3. But then $[E_i, E_1] \in \mathfrak{i}, [E_i, E_2] \in \mathfrak{i}$ and $[E_i, E_3] \in \mathfrak{i}$, and considering the commutator relations of $\mathfrak{so}(1,2)$, this implies that E_1, E_2 and E_3 in $\mathfrak{i}$. Thus $\langle E_1, E_2, E_3 \rangle = \mathfrak{so}(1,2) \subseteq \mathfrak{i}$ and the ideal $\mathfrak{i}$ is trivial. Consider the case where at least two of a, b and c are nonzero. Then

$$\begin{cases} [X, E_1] & \in \mathfrak{i} \\ [X, E_2] & \in \mathfrak{i} \\ [X, E_3] & \in \mathfrak{i} \end{cases} \quad \Rightarrow \quad \begin{cases} bE_3 - cE_2 \in \mathfrak{i} \\ a(-E_3) + c(-E_1) \in \mathfrak{i} \\ a(E_2) + b(E_1) \in \mathfrak{i}. \end{cases}$$

But since $\mathfrak{i}$ is closed under the taking of the Lie bracket, it then follows that

$$\begin{cases} [bE_3 - cE_2, E_1] \in \mathfrak{i} \\ [-aE_3 - cE_1, E_1] \in \mathfrak{i} \\ [aE_2 + bE_1, E_1] \in \mathfrak{i} \end{cases} \qquad \begin{cases} [bE_3 - cE_2, E_2] \in \mathfrak{i} \\ [-aE_3 - cE_1, E_2] \in \mathfrak{i} \\ [aE_2 + bE_1, E_2] \in \mathfrak{i} \end{cases} \qquad \begin{cases} [bE_3 - cE_2, E_3] \in \mathfrak{i} \\ [-aE_3 - cE_1, E_3] \in \mathfrak{i} \\ [aE_2 + bE_1, E_3] \in \mathfrak{i}, \end{cases}$$

and particularly by the commutator relations

$$\begin{cases} aE_2 \in \mathfrak{i} \\ -aE_3 \in \mathfrak{i} \\ (bE_2 + cE_3) \in \mathfrak{i} \end{cases} \qquad \begin{cases} bE_1 \in \mathfrak{i} \\ -bE_3 \in \mathfrak{i} \\ aE_1 - cE_3 \in \mathfrak{i} \end{cases} \qquad \begin{cases} cE_1 \in \mathfrak{i} \\ -cE_2 \in \mathfrak{i} \\ (-aE_1 - bE_2) \in \mathfrak{i}, \end{cases} \tag{3.2.9}$$

where at least two of a, b or c are nonzero. Then directly, if all of a, b and c are nonzero, it follows that E_1, E_2 and E_3 are in $\mathfrak{i}$. If one of a, b or c is zero, then by substituting into the equations (3.2.9) we see that two of E_1, E_2 or E_3 are in $\mathfrak{i}$. But since $\mathfrak{i}$ is closed under the taking of the Lie bracket, then by applying the commutator relations this implies that all of E_1, E_2 and E_3 are in $\mathfrak{i}$. Thus in both cases $\langle E_1, E_2, E_3 \rangle = \mathfrak{so}(1,2) \subseteq \mathfrak{i}$, and $\mathfrak{i}$ is trivial. The result follows.□

We use the next result to prove the nondegeneracy of the Killing form (A.5.60) on $\mathfrak{so}(1,2)$.

3.2.5 PROPOSITION. *The adjoint operators of E_1, E_2 and E_3 are expressible as the matrices*

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

respectively.

PROOF. To find the matrix representation of the adjoint operators (A.5.35), consider the values given by the commutator relations:

$$\begin{cases} \mathrm{ad}_{E_1} E_1 &= 0, \\ \mathrm{ad}_{E_2} E_1 &= E_3, \\ \mathrm{ad}_{E_3} E_1 &= -E_2, \end{cases} \quad \begin{cases} \mathrm{ad}_{E_1} E_2 &= -E_3, \\ \mathrm{ad}_{E_2} E_2 &= 0, \\ \mathrm{ad}_{E_3} E_2 &= -E_1, \end{cases} \quad \begin{cases} \mathrm{ad}_{E_1} E_3 &= E_2 \\ \mathrm{ad}_{E_2} E_3 &= E_1 \\ \mathrm{ad}_{E_3} E_3 &= 0. \end{cases}$$

Taking $X = aE_1 + bE_2 + cE_3$ an arbitrary element in $\mathfrak{so}(1,2)$, then

$$\begin{aligned} \mathrm{ad}_{E_3} X &= \mathrm{ad}_{E_3}(aE_1 + bE_2 + cE_3) &= (-a)E_2 + (-b)E_1 \\ \mathrm{ad}_{E_2} X &= \mathrm{ad}_{E_2}(aE_1 + bE_2 + cE_3) &= (c)E_1 + (a)E_3 \\ \mathrm{ad}_{E_1} X &= \mathrm{ad}_{E_1}(aE_1 + bE_2 + cE_3) &= (-b)E_3 + (c)E_2. \end{aligned}$$

Thus

$$\mathrm{ad}_{E_1} X = (0, c, -b), \quad \mathrm{ad}_{E_2} X = (c, 0, a) \quad \text{and} \quad \mathrm{ad}_{E_3} X = (-b, -a, 0)$$

and since X is arbitrarily chosen, the result follows. □

We now consider the properties of the Killing form κ on $\mathfrak{so}(1,2)$.

3.2.6 PROPOSITION. *The Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $\kappa(A, B) = \mathrm{tr}(\mathrm{ad}_A \circ \mathrm{ad}_B)$ is symmetric and bilinear.*

PROOF. Since ad is a Lie algebra automorphism (A.5.40), it is linear. Further, since the trace is linear and composition of two linear maps is bilinear, thus κ is bilinear. Further, from the relation $\exp(\mathrm{tr}(A)) = \det A$ of (A.5.29), then

$$\exp(\mathrm{tr}(\mathrm{ad}_A \circ \mathrm{ad}_B)) = \det(\mathrm{ad}_A \circ \mathrm{ad}_B) = \det(\mathrm{ad}_B \circ \mathrm{ad}_A) = \exp(\mathrm{tr}(\mathrm{ad}_B \circ \mathrm{ad}_A)).$$

Thus $\mathrm{tr}(\mathrm{ad}_A \circ \mathrm{ad}_B) = \mathrm{tr}(\mathrm{ad}_B \circ \mathrm{ad}_A)$ for all $A, B \in \mathfrak{so}(1,2)$, and κ is symmetric. □

In (A.5.61) we defined the orthogonal complement of a subset of a semisimple Lie algebra.

3.2.7 PROPOSITION. *The orthogonal complement $\mathfrak{so}(1,2)^\perp$ of $\mathfrak{so}(1,2)$ is an ideal of $\mathfrak{so}(1,2)$.*

PROOF. Firstly we note that $0 \in \mathfrak{so}(1,2)^\perp$, since $[0, E_i] = 0$ for $i = 1, 2, 3$. Assume that $X \in \mathfrak{so}(1,2)^\perp$, $X \neq 0$. Then $[X, E_i] = 0 \in \mathfrak{so}(1,2)^\perp$ for $i = 1, 2, 3$. But then for each $B \in \mathfrak{so}(1,2)$ we express $B = b_1E_1 + b_2E_2 + b_3E_3$ and it follows that $[X, B] = [X, b_1E_1] + [X, b_1E_1] + [X, b_1E_1]$ is in $\mathfrak{so}(1,2)^\perp$. Thus $\mathfrak{so}(1,2)^\perp$ is an ideal of $\mathfrak{so}(1,2)$. □

3.2.8 PROPOSITION. *κ is nondegenerate on* $\mathfrak{so}(1,2)$.

PROOF. From PROPOSITION 3.2.7, $\mathfrak{so}(1,2)^{\perp}$ is an ideal of $\mathfrak{so}(1,2)$. But $\mathfrak{so}(1,2)$ is simple: thus $\mathfrak{so}(1,2)^{\perp}$ must be either $\mathfrak{so}(1,2)$ or $\{0\}$. But using the expression of ad_{E_2} from PROPOSITION 3.2.5, we see that $\mathrm{tr}(\mathrm{ad}_{E_2} \circ \mathrm{ad}_{E_2}) = 2 \neq 0$. Thus $E_2 \notin \mathfrak{so}(1,2)^{\perp}$, and $\mathfrak{so}(1,2)^{\perp} \neq \mathfrak{so}(1,2)$. It follows that $\mathfrak{so}(1,2)^{\perp} = \{0\}$: that is, κ is nondegenerate on $\mathfrak{so}(1,2)$. □

3.3 The hat map

3.3.1 DEFINITION. The **Lorentz cross product** is the map $\odot : \mathbb{R}^{1,2} \times \mathbb{R}^{1,2} \to \mathbb{R}^{1,2}$ given by

$$\mathbf{a} \odot \mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{a} \wedge \mathbf{b} = -J\mathbf{a} \wedge \mathbf{b}.$$

Analagous to the case of $\mathbb{R}^3$ which is equipped with both the (Euclidean) scalar and vector products $\bullet$ and $\wedge$ simultaneously, we consider $\mathbb{R}^{1,2}_{\odot}$, the reduced Minkowski spacetime equipped with the Lorentz cross product.

In PROPOSITIONS 3.3.2 - 3.3.5 we determine properties of the Lorentz cross product $\odot$ on $\mathbb{R}^{1,2}_{\odot}$. The results PROPOSITION 3.3.2 and 3.3.3 are stated by Ratcliffe [27], while the others are original and analogous to properties of $\wedge$ on $\mathbb{R}^3$.

3.3.2 PROPOSITION. *Given* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^{1,2}_{\odot}$, *then* $\mathbf{a} \odot (\mathbf{b} \odot \mathbf{c}) = (\mathbf{b} \odot \mathbf{c})\mathbf{a} - (\mathbf{a} \odot \mathbf{c})\mathbf{b}$.

PROOF. Direct computation. Consider the elements $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^{1,2}_{\odot}$. Then

$$\begin{aligned}(\mathbf{a} \odot \mathbf{b}) \odot \mathbf{c} = (-c_2a_2b_1 - c_3a_3b_1 &+ c_2a_1b_2 + c_3a_1b_3, -c_1a_2b_1 + c_1a_1b_2 - c_3a_3b_2 \\ &+ c_3a_2b_3, -c_1a_3b_1 + c_2a_3b_2 + c_1a_1b_3 - c_2a_2b_3) \quad (3.3.1)\end{aligned}$$

But $\mathbf{a} \odot \mathbf{c} = (-c_1a_1 + c_2a_2 + c_3a_3)$ and $\mathbf{b} \odot \mathbf{c} = (-c_1b_1 + c_2b_2 + c_3b_3)$, and thus

$$\begin{aligned}(\mathbf{a} \odot \mathbf{c})\mathbf{b} &= (-c_1a_1b_1 + c_2a_2b_1 + c_3a_3b_1, -b_2c_1a_1 + b_2c_2a_2 + c_3a_3b_2, -b_3c_1a_1 + b_3c_2a_2 + b_3c_3a_3) \\ (\mathbf{b} \odot \mathbf{c})\mathbf{a} &= (-c_1a_1b_1 + c_2a_1b_2 + c_3a_1b_3, -a_2c_1b_1 + c_2a_2b_2 + a_2c_3b_3, -a_3c_1b_1 + a_3c_2b_2 + c_3a_3b_3)\end{aligned}$$

and so

$$\begin{aligned}(\mathbf{b} \odot \mathbf{c})\mathbf{a} - (\mathbf{a} \odot \mathbf{c})\mathbf{b} = (-c_2a_2b_1 - c_3a_3b_1 &+ c_2a_1b_2 + c_3a_1b_3, -c_1a_2b_1 + c_1a_1b_2 - c_3a_3b_2 \\ &+ c_3a_2b_3, -c_1a_3b_1 + c_2a_3b_2 + c_1a_1b_3 - c_2a_2b_3)\end{aligned}$$

But by (3.3.1), this is the same as $(\mathbf{a} \odot \mathbf{b}) \odot \mathbf{c}$. The result follows. □

3.3.3 PROPOSITION. *Given any* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^{1,2}_{\odot}$, *then*

$$\mathbf{a} \odot (\mathbf{b} \odot \mathbf{c}) = - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

PROOF. Direct computation:

$$\mathbf{b} \circledcirc \mathbf{c} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_2c_3 - c_2b_3 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - c_1b_2 \end{bmatrix}$$

and thus

$$\begin{aligned} \mathbf{a} \odot (\mathbf{b} \circledcirc \mathbf{c}) &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_2c_3 - c_2b_3 \\ b_1c_3 - b_3c_1 \\ c_1b_2 - b_1c_2 \end{bmatrix} \\ &= a_1(b_3c_2 - c_3b_2) + a_2(b_1c_3 - b_3c_1) + a_3(c_1b_2 - b_1c_2) \\ &= - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \end{aligned}$$

□

3.3.4 PROPOSITION. *Let* **a** *be a timelike vector and* **b** *a spacelike vector. Then* $\mathbf{a} \circledcirc \mathbf{b} = \mathbf{c}$, *where* **c** *is spacelike and Minkowski-orthogonal to* **a** *and* **b**.

PROOF. Firstly, we note that the timelike vector **a** and the spacelike vector **b** must be linearly-independent: if $\mathbf{a} = \lambda \mathbf{b}$, then $\mathbf{a} \odot \mathbf{a} = \lambda^2 \mathbf{b} \odot \mathbf{b}$ where **b** is spacelike and thus $\mathbf{b} \odot \mathbf{b}$ is positive. But **a** is timelike and so $\mathbf{a} \odot \mathbf{a} < 0$, a contradiction. We then show that $\mathbf{a} \circledcirc \mathbf{b}$ is Minkowski-orthogonal to **a** and to **b** and is nonzero: then by THEOREM 2.1.10, $\mathbf{c} = \mathbf{a} \circledcirc \mathbf{b}$ is a nonzero vector Minkowski-orthogonal to the timelike vector **a** and so **c** must be spacelike.

Firstly, assume that **c** is zero. Then using PROPOSITION 3.3.2, it follows that for each basis element $\mathbf{e}_i$,

$$(\mathbf{a} \circledcirc \mathbf{b}) \circledcirc \mathbf{e}_i = (\mathbf{a} \odot \mathbf{e}_i)\mathbf{b} - (\mathbf{b} \odot \mathbf{e}_i)\mathbf{a} = 0 \quad \Leftrightarrow \quad \mathbf{a} \odot \mathbf{e}_i . \mathbf{b} \odot \mathbf{e}_i = 0 \quad \text{for all } i \quad \Leftrightarrow \quad \mathbf{a} = \mathbf{b} = 0.$$

which contradicts that **a** is timelike and **b** is spacelike. Thus **c** cannot be the zero vector.

Secondly, using PROPOSITION 3.3.3, then

$$\mathbf{a} \odot (\mathbf{a} \circledcirc \mathbf{b}) = - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0 = - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{b} \odot (\mathbf{a} \circledcirc \mathbf{b})$$

by the fact that the determinant of any singular matrix is preserved under elementary row reductions. Thus $\mathbf{a} \circledcirc \mathbf{b}$ is Minkowski-orthogonal to **a** and **b**. □

As in the case of $\mathbb{R}^3$ equipped with $\wedge$, $\mathbb{R}^{1,2}_{\circledcirc}$ has additional structure:

3.3.5 PROPOSITION. $\mathbb{R}^{1,2}_{\circledcirc}$ *is a Lie algebra.*

PROOF. We show that $\circledcirc$ satisfies the properties 1-3 of A.5.17. Firstly, given $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{1,2}_{\circledcirc}$, then

$$\mathbf{p} \circledcirc \mathbf{q} = (-p_3q_2 + p_2q_3, -p_3q_1 + p_1q_3, p_2q_1 - p_1q_2) = -(p_3q_2 - p_2q_3, p_3q_1 - p_1q_3, -p_2q_1 + p_1q_2) = -\mathbf{q} \circledcirc \mathbf{p}$$

and $\circledcirc$ is skew-symmetric on $\mathbb{R}^{1,2}_{\odot}$. Secondly, for $\lambda_1, \lambda_2 \in \mathbb{R}$, then from the bilinearity of J, $(\lambda_1 \mathbf{p} + \lambda_2 \mathbf{q}) \circledcirc \mathbf{s} = -J((\lambda_1 \mathbf{p} + \lambda_2 \mathbf{q}) \wedge \mathbf{s}) = \lambda_1(-J)(\mathbf{p} \wedge \mathbf{s}) + \lambda_2(-J)(\mathbf{q} \wedge \mathbf{s}) = \lambda_1 \mathbf{p} \circledcirc \mathbf{s} + \lambda_2 \mathbf{q} \circledcirc \mathbf{s}$ by the fact that $\mathbb{R}^3$ equipped with $\wedge$ is a Lie algebra. Finally, from PROPOSITION 3.3.2, the sum

$$\begin{aligned} \mathbf{p} \circledcirc (\mathbf{q} \circledcirc \mathbf{s}) + \mathbf{q} \circledcirc (\mathbf{s} \circledcirc \mathbf{p}) + \mathbf{s} \circledcirc (\mathbf{p} \circledcirc \mathbf{q}) &= (\mathbf{p} \odot \mathbf{q})\mathbf{s} - (\mathbf{p} \odot \mathbf{s})\mathbf{q} + (\mathbf{q} \odot \mathbf{s})\mathbf{p} - (\mathbf{q} \odot \mathbf{p})\mathbf{s} \\ &\quad + (\mathbf{p} \odot \mathbf{s})\mathbf{q} + (\mathbf{s} \odot \mathbf{p})\mathbf{q} - (\mathbf{s} \odot \mathbf{q})\mathbf{p} \\ &= 0 \end{aligned}$$

by the symmetry of $\odot$. The result follows. □

In (A.5.18), we expressed $\mathbb{R}^3$ as a Lie algebra with a Lie bracket given by the cross product $\wedge$. The commntator relations on this Lie algebra are given by $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \wedge \mathbf{e}_1 = \mathbf{e}_2$, which correspond to the commutator relations $[E_1, E_2] = E_3$, $[E_2, E_3] = E_1$ and $[E_3, E_1] = E_2$ of $\mathfrak{so}(3)$. The so-called hat map $\widehat{\ } : \mathbb{R}^3 \to \mathfrak{so}(3)$, $\widehat{\mathbf{e}_i} = E_i$ is a Lie algebra isomorphism which is often used in mechanics (eg. [17], [13]). We will follow [13] in that for us hat map is described by the inverse of the map which we originally stated: $\widehat{\ } : \mathfrak{so}(3) \to \mathbb{R}^3$, such that $\widehat{E_i} = \mathbf{e}_i$. We use analogy with this case to define a corresponding hat map on the Lie algebra $\mathfrak{so}(1,2)$.

3.3.6 DEFINITION. The **hat map** is the map hat : $\mathfrak{so}(1,2) \to \mathbb{R}^{1,2}_{\odot}$ defined by

$$\mathrm{hat} \begin{bmatrix} 0 & z & y \\ z & 0 & -x \\ y & x & 0 \end{bmatrix} = (x, y, z).$$

We will denote the image hat$A$ by $\widehat{A}$ wherever this convention is appropriate.

3.3.7 THEOREM. *The hat map* hat : $\mathfrak{so}(1,2) \to \mathbb{R}^{1,2}_{\odot}$ *is a Lie algebra isomorphism.*

PROOF. Firstly, the hat map is linear: for each $A, B \in \mathfrak{so}(1,2)$,

$$\begin{aligned} \mathrm{hat} \begin{bmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & -a_1 \\ a_2 & a_1 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & b_3 & b_2 \\ b_3 & 0 & -b_1 \\ b_2 & b_1 & 0 \end{bmatrix} &= \mathrm{hat} \begin{bmatrix} 0 & a_3 + \lambda b_3 & a_2 + \lambda b_2 \\ a_3 + \lambda b_3 & 0 & -a_1 - \lambda b_1 \\ a_2 + \lambda b_2 & a_1 + \lambda b_1 & 0 \end{bmatrix} \\ &= (a_1 + \lambda b_1, a_2 + \lambda b_2, a_3 + \lambda b_3) \\ &= (a_1, a_2, a_3) + \lambda (b_1, b_2, b_3) \\ &= \mathrm{hat} \begin{bmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & -a_1 \\ a_2 & a_1 & 0 \end{bmatrix} + \lambda \mathrm{hat} \begin{bmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & -a_1 \\ a_2 & a_1 & 0 \end{bmatrix}. \end{aligned}$$

Secondly,

$$\begin{aligned} [\widehat{E}_1, \widehat{E}_2] &= \mathbf{e}_1 \circledcirc \mathbf{e}_2 = (-J)\mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_3 = \widehat{[E_i, E_j]} \\ [\widehat{E}_2, \widehat{E}_3] &= \mathbf{e}_2 \circledcirc \mathbf{e}_3 = (-J)\mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1 = \widehat{[E_i, E_j]} \\ [\widehat{E}_3, \widehat{E}_1] &= \mathbf{e}_3 \circledcirc \mathbf{e}_1 = (-J)\mathbf{e}_3 \wedge \mathbf{e}_1 = -\mathbf{e}_2 = \widehat{[E_i, E_j]} \end{aligned}$$

from the commutator relations of $\mathfrak{so}(1,2)$. By the linearity of hat, then $[\widehat{A},\widehat{B}] = \widehat{[A,B]}$for each $A, B \in \mathfrak{so}(1,2)$. □

Denote the matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.3.8 PROPOSITION. *Given any two elements $A, B \in \mathfrak{so}(1,2)$ and their corresponding images $\mathbf{a}, \mathbf{b}$ under the hat map, then $(PAP)\mathbf{b} = \mathbf{a} \circledast \mathbf{b}$.*

PROOF. Direct computation. Consider the elements

$$A = \begin{bmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & -a_1 \\ a_2 & a_1 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0 & b_3 & b_2 \\ b_3 & 0 & -b_1 \\ b_2 & b_1 & 0 \end{bmatrix}.$$

Then $\widehat{A} = (a_1, a_2, a_3), \widehat{B} = (b_1, b_2, b_3)$, and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & -a_1 \\ a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -b_2a_3 + a_2b_3 \\ -b_1a_3 + a_1b_3 \\ b_1a_2 - a_1b_2 \end{bmatrix} = \mathbf{a} \circledast \mathbf{b}.$$

□

3.3.9 PROPOSITION. *The elements $g \in SO(1,2)$ preserve $\circledast$ on $\mathbb{R}^{1,2}$: that is, for each $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{1,2}_{\circledast}$ and $g \in SO(1,2)$, then $g(\mathbf{a} \circledast \mathbf{b}) = (g\mathbf{a}) \circledast (g\mathbf{b})$.*

PROOF. Given two arbitrary elements $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^{1,2}_{\circledast}$, then for any $\mathbf{c}$ in $\mathbb{R}^{1,2}_{\circledast}$ and any $g \in SO(1,2)$ it follows from PROPOSITION 3.3.3,

$$\begin{aligned} g\mathbf{c} \odot (g\mathbf{a} \circledast g\mathbf{b}) &= -\det(g\mathbf{c} \;\; g\mathbf{a} \;\; g\mathbf{b}) \\ &= -\det(g(\mathbf{c} \;\; \mathbf{a} \;\; \mathbf{b})) \\ &= -\det g \det(\mathbf{c} \;\; \mathbf{a} \;\; \mathbf{b}) \\ &= -\det(\mathbf{c} \;\; \mathbf{a} \;\; \mathbf{b}) \\ &= \mathbf{c} \odot (\mathbf{a} \circledast \mathbf{b}) \\ &= g\mathbf{c} \odot g(\mathbf{a} \circledast \mathbf{b}), \end{aligned} \tag{3.3.2}$$

where (3.3.2) follows from the fact that g preserves $\odot$. Since $\mathbf{c}$ was arbitrarily chosen, it may be taken sucessively to be the basis elements $g^{-1}\mathbf{e}_1, g^{-1}\mathbf{e}_2$ and $g^{-1}\mathbf{e}_3$. Thus it follows that $\mathbf{e}_i \odot (g\mathbf{a} \circledast g\mathbf{b}) = \mathbf{e}_i \odot g(\mathbf{a} \circledast \mathbf{b})$ for $i = 1, 2, 3$, and so each component of the vectors $(g\mathbf{a} \circledast g\mathbf{b})$ and $g(\mathbf{a} \circledast \mathbf{b})$ is equal. Thus $g\mathbf{a} \circledast g\mathbf{b} = g(\mathbf{a} \circledast \mathbf{b})$ for all $g \in SO(1,2)$. □

3.4 Lie algebra automorphisms

We refer to (A.5.31) and (A.5.38)-(A.5.40) to determine the Lie algebra automorphisms of $\mathfrak{so}(1,2)$ using its image $\mathbb{R}^{1,2}_{\odot}$ under the hat map.

3.4.1 PROPOSITION. *The Lie algebra automorphisms of* $\mathbb{R}^{1,2}_{\odot}$ *are exactly the elements of* $\mathsf{SO}(1,2)$ *acting on* $\mathbb{R}^{1,2}$ *by left matrix multiplication.*

PROOF. By PROPOSITION 3.3.9, each element $g \in \mathsf{SO}(1,2)$ preserves $\odot$ on $\mathbb{R}^{1,2}_{\odot}$, and so for each $\mathsf{a}, \mathsf{b} \in \mathbb{R}^{1,2}_{\odot}$, then $g(\mathsf{a} \odot \mathsf{b}) = (g\mathsf{a}) \odot (g\mathsf{b})$. Since the left action of $\mathsf{SO}(1,2)$ on $\mathbb{R}^{1,2}_{\odot}$ is linear, then it follows by (A.5.31) that each $g \in \mathsf{SO}(1,2)$ is a Lie algebra automorphism of $\mathbb{R}^{1,2}_{\odot}$. Thus $\mathsf{SO}(1,2) \subseteq \mathsf{Aut}(\mathbb{R}^{1,2}_{\odot})$. Consider a Lie algebra automorphism $g \in \mathsf{Aut}(\mathbb{R}^{1,2}_{\odot})$. By definition, g is linear on $\mathbb{R}^{1,2}_{\odot}$ and for each basis element $\{\mathsf{e_1}, \mathsf{e_2}, \mathsf{e_3}\}$ of $\mathbb{R}^{1,2}$, $g(\mathsf{e}_i \odot \mathsf{e}_j) = g\mathsf{e}_i \odot g\mathsf{e}_j$ for $i,j = 1,2,3$. But then

$$\begin{aligned}
g(\mathsf{e}_i \odot \mathsf{e}_j) \odot g\mathsf{e}_k &= (g\mathsf{e}_i \odot g\mathsf{e}_j) \odot g\mathsf{e}_k \\
\Rightarrow g((\mathsf{e}_i \odot \mathsf{e}_k)\mathsf{e}_j - (\mathsf{e}_j \odot \mathsf{e}_k)\mathsf{e}_i) &= ((g\mathsf{e}_i \odot g\mathsf{e}_k)g\mathsf{e}_j - (g\mathsf{e}_j \odot g\mathsf{e}_k)g\mathsf{e}_i) \\
\Rightarrow (\mathsf{e}_i \odot \mathsf{e}_k)g\mathsf{e}_j - (\mathsf{e}_j \odot \mathsf{e}_k)g\mathsf{e}_i &= (g\mathsf{e}_i \odot g\mathsf{e}_k)g\mathsf{e}_j - (g\mathsf{e}_j \odot g\mathsf{e}_k)g\mathsf{e}_i \\
\Rightarrow (\mathsf{e}_i \odot \mathsf{e}_k) &= (g\mathsf{e}_i \odot g\mathsf{e}_k), \quad (\mathsf{e}_j \odot \mathsf{e}_k) = (g\mathsf{e}_j \odot g\mathsf{e}_k) \qquad (3.4.1)
\end{aligned}$$

where (3.4.1) follows from the linear independence of $\mathsf{e}_i, \mathsf{e}_j$ and e_k and the nondegeneracy of $\odot$. Since $\mathsf{e}_i \odot \mathsf{e}_j = J_{ij}$, then from (3.4.1),

$$\begin{aligned}
\mathsf{e}_i \odot \mathsf{e}_j = g\mathsf{e}_i \odot g\mathsf{e}_j \Rightarrow J_{ij} &= -g_{1i}g_{1j} + g_{2i}g_{2j} + g_{3i}g_{3j} \\
\Rightarrow gJg^{\mathsf{T}} &= \begin{bmatrix} \sum_{k=1}^3 g_{k1}g_{k1} & \sum_k^3 g_{k1}g_{k2} & \sum_k^3 g_{k1}g_{k3} \\ \sum_k^3 g_{k2}g_{k1} & \sum_k^3 g_{k2}g_{k2} & \sum_k^3 g_{k2}g_{k3} \\ \sum_k^3 g_{k3}g_{k1} & \sum_k^3 g_{k3}a_{k2} & \sum_k^3 g_{k3}g_{k3} \end{bmatrix} \\
&= J.
\end{aligned}$$

Thus $g\mathsf{b} \odot g\mathsf{c} = \mathsf{b} \odot \mathsf{c}$ for all $\mathsf{b}, \mathsf{c} \in \mathbb{R}^{1,2}_{\odot}$. But by assumption $g\mathsf{b} \odot g\mathsf{c} = g(\mathsf{b} \odot \mathsf{c})$. Thus using PROPOSITION 3.3.9,

$$g\mathsf{a} \odot (g\mathsf{b} \odot g\mathsf{c}) = g\mathsf{a} \odot g(\mathsf{b} \odot \mathsf{c}) = \mathsf{a} \odot (\mathsf{b} \odot \mathsf{c}) = -\det(\mathsf{a} \ \ \mathsf{b} \ \ \mathsf{c})$$

for any $\mathsf{a} \in \mathbb{R}^{1,2}_{\odot}$. But by PROPOSITION 3.3.3,

$$g\mathsf{a} \odot (g\mathsf{b} \odot g\mathsf{c}) = -\det(g\mathsf{a} \ \ g\mathsf{b} \ \ g\mathsf{c}) = -\det g \det(\mathsf{a} \ \ \mathsf{b} \ \ \mathsf{c}).$$

Thus we see that $-\det g \det(\mathsf{a} \ \ \mathsf{b} \ \ \mathsf{c}) = -\det(\mathsf{a} \ \ \mathsf{b} \ \ \mathsf{c})$, and so $\det g = 1$. But then it follows from PROPOSITION 2.2.14 that $g \in \mathsf{SO}(1,2)$: thus $\mathsf{Aut}(\mathbb{R}^{1,2}_{\odot}) \subseteq \mathsf{SO}(1,2)$. Since we have shown both containments, then $\mathsf{Aut}(\mathbb{R}^{1,2}_{\odot}) = \mathsf{SO}(1,2)$. □

3.4.2 PROPOSITION. *The two groups* $\mathsf{SO}(1,2)$ *and* $\mathsf{Aut}(\mathfrak{so}(1,2))$ *are Lie group isomorphic. Particularly, for each* $g \in \mathsf{SO}(1,2) = \mathsf{Aut}(\mathbb{R}^{1,2}_{\odot})$, *there exists a unique element* $\phi_g \in \mathsf{Aut}(\mathfrak{so}(1,2))$ *such that for each* $\mathsf{a} \in \mathbb{R}^{1,2}_{\odot}$, *then* $g(\mathsf{a}) = \mathrm{hat}\phi_g(\mathsf{a})$.

PROOF. Given $g \in \mathsf{Aut}(\mathbb{R}^{1,2}_{\odot})$, consider the map $\mathrm{hat}^{-1} \circ g \circ \mathrm{hat} : \mathfrak{so}(1,2) \to \mathfrak{so}(1,2)$. Let Ψ be the map $\Psi : g \mapsto \mathrm{hat}^{-1} \circ g \circ \mathrm{hat}$ and $\phi_g = \Psi(g)$. Then for each basis element E_i, E_j,

$$[\phi_g E_i, \phi_g E_j] = \mathrm{hat}^{-1}[g \circ \widehat{E}_i, g \circ \widehat{E}_j] = \mathrm{hat}^{-1}(g\mathbf{e}_i \odot g\mathbf{e}_j) = \mathrm{hat}^{-1} \circ g(\mathbf{e}_i \odot \mathbf{e}_j) = \phi_g\left([E_i, E_j]\right). \quad (3.4.2)$$

Since hat and g are both linear, the composition ϕ_g is linear, and so $[\phi_g A, \phi_g B] = \phi_g[A, B]$ for each $A, B \in \mathfrak{so}(1,2)$ from equation (3.4.2). Thus ϕ_g is clearly an automorphism of $\mathfrak{so}(1,2)$. Consider $g \in \mathsf{Aut}(\mathbb{R}^{1,2}_{\odot})$ such that $\Psi(g) = \mathbf{1}$. Then $g = \mathrm{hat} \circ \mathbf{1} \circ \mathrm{hat}^{-1} = \mathbf{1}$, and $\ker(\Psi) = \{\mathbf{1}\}$. Thus Ψ is injective. Given an arbitrary $\phi \in \mathsf{Aut}(\mathfrak{so}(1,2))$, then for $\mathbf{e}_i, \mathbf{e}_j$ in the standard basis,

$$\left(\mathrm{hat} \circ \phi \circ \mathrm{hat}^{-1}\mathbf{e}_i\right) \odot \left(\mathrm{hat} \circ \phi \circ \mathrm{hat}^{-1}\mathbf{e}_j\right) = \mathrm{hat}\left([\phi(E_i), \phi(E_j)]\right) = \mathrm{hat} \circ \phi \circ [E_i, E_j] = \mathrm{hat} \circ \phi \circ \mathrm{hat}^{-1}(\mathbf{e}_i \odot \mathbf{e}_j)$$

and so by linearity of ϕ, $\left(\mathrm{hat} \circ \phi \circ \mathrm{hat}^{-1}\mathbf{a}\right) \odot \left(\mathrm{hat} \circ \phi \circ \mathrm{hat}^{-1}\mathbf{b}\right) = \mathrm{hat} \circ \phi \circ \mathrm{hat}^{-1}(\mathbf{a} \odot \mathbf{b})$ for all arbitrary $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{1,2}_{\odot}$. Thus $\mathrm{hat} \circ \phi \circ \mathrm{hat}^{-1}$ is an automorphism g of $\mathbb{R}^{1,2}_{\odot}$ such that $\Psi(g) = \phi$, and the map is surjective. Given $g_1, g_2 \in \mathsf{Aut}(\mathbb{R}^{1,2}_{\odot})$, then

$$\phi_{g_1 g_2} = \mathrm{hat}^{-1} \circ g_1 \circ g_2 \circ \mathrm{hat} = \mathrm{hat}^{-1} \circ g_1 \circ \mathrm{hat} \circ \mathrm{hat}^{-1} \circ g_2 \circ \mathrm{hat} = \phi_{g_1} \circ \phi_{g_2}$$

and $\Psi : \mathsf{Aut}(\mathbb{R}^{1,2}_{\odot}) \to \mathsf{Aut}(\mathfrak{so}(1,2))$ is a group isomorphism. From THEOREM 3.3.7 hat is a Lie algebra isomorphism. Thus it is linear and so a differentiable map. Then Ψ is a composition of differentiable maps and it follows that it is itself differentiable. Thus from (A.5.9), Ψ is a Lie group isomophism. For any arbitrary $A \in \mathfrak{so}(1,2)$, we note that

$$\phi_g(A) = \mathrm{hat}^{-1} \circ g \circ \mathrm{hat}\, A = \mathrm{hat}^{-1}(g\mathbf{a}) \quad \Rightarrow \quad (g\mathbf{a})\mathrm{hat}\left(\phi_g(A)\right). \qquad \square$$

3.4.3 PROPOSITION. *The subgroup* $\mathrm{Inn}(\mathfrak{so}(1,2))$ *of* $\mathsf{Aut}(\mathfrak{so}(1,2))$ *corresponds to the subgroup* $\mathsf{SO}(1,2)_0$ *of* $\mathsf{Aut}(\mathbb{R}^{1,2}_{\odot})$ *under* Ψ.

PROOF. By (A.5.40), for any matrix Lie group G, $\mathsf{Inn}(\mathfrak{g})$ is the connected component of $\mathsf{Aut}(\mathfrak{g})$. We show that the connected component of $\mathsf{SO}(1,2)$ is $\mathsf{SO}(1,2)_0$, and then use the homeomorphism Ψ between $\mathsf{Aut}(\mathfrak{so}(1,2))$ and $\mathsf{SO}(1,2)$ to show that $\mathsf{Inn}(\mathfrak{so}(1,2))$ corresponds to $\mathsf{SO}(1,2)_0$. From PROPOSITION 2.2.14, the group $\mathsf{SO}(1,2)$ is the union of two components, $\mathsf{SO}(1,2)^- \cup \mathsf{SO}(1,2)_0$, where since $\mathsf{SO}(1,2)^-$ corresponds to subset (2.2.8), then each of its elements has the form

$$\begin{bmatrix} c & \mathbf{q}^\top \\ \mathbf{q} & \sqrt{1 + \mathbf{q}\mathbf{q}^\top} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & R_\theta \end{bmatrix} \qquad R_\theta \in \mathsf{O}(2), \quad \det R_\theta < 0, \quad c \in \mathbb{R}^+$$

and so for each element of $\mathsf{SO}(1,2)^-$, $g_{11} < 0$, and so $g \notin \mathsf{SO}(1,2)_0$. Thus the two components are disjoint. We show that the component $\mathsf{SO}(1,2)^-$ is not connected. By (A.5.12), a matrix Lie group is connected if and only if for every $g \in \mathsf{G}$ there exists a curve $g(\cdot) : (a, b) \to \mathsf{G}$ such that $g(0) = g$ and $g(1) = \mathbf{1}$. But $\mathbf{1} \in \mathsf{SO}(1,2) \backslash \mathsf{SO}(1,2)^-$, since $\mathsf{SO}(1,2)_0$ by PROPOSITION 2.2.16 is a subgroup of $\mathsf{SO}(1,2)$ and so contains $\mathbf{1}$, while $\mathsf{SO}(1,2)_0$ and $\mathsf{SO}(1,2)^-$ are disjoint. Thus for any $g \in \mathsf{SO}(1,2)^-$ there exists no curve $g(\cdot) : (a, b) \to \mathsf{SO}(1,2)^-$ such that $g(0) = g$ and $g(1) = \mathbf{1}$ which is contained in $\mathsf{SO}(1,2)^-$, and so it cannot be connected.

But by THEOREM 3.1.2, $\mathsf{SO}(1,2)_0$ is connected: thus $\mathsf{SO}(1,2)_0$ is the connected component of $\mathsf{SO}(1,2)$. Further, by (A.5.56) each curve in $\mathsf{Aut}(\mathfrak{so}(1,2))$ is an image of a curve in $\mathsf{SO}(1,2)$, $\Psi(g(\cdot)) : (a,b) \to \mathsf{Aut}(\mathfrak{so}(1,2))$. Thus since Ψ is continuous, $\Psi(\mathsf{SO}(1,2)_0)$ is the connected component of $\mathsf{Aut}(\mathfrak{so}(1,2))$ and so $\Psi(\mathsf{SO}(1,2)_0) = \mathsf{Inn}(\mathfrak{so}(1,2))$ by (A.5.40). □

3.5 Adjoint and co-adjoint orbits

3.5.1 Adjoint orbits

In (A.5.34) we defined the adjoint action of a matrix Lie group G on its Lie algebra $\mathfrak{g}$. In this section we construct the adjoint orbits (A.5.42) of $\mathfrak{so}(1,2)$.

3.5.1 THEOREM. *The adjoint orbits* $\mathcal{O}_A = \{\mathrm{Ad}_g A \,:\, g \in \mathsf{SO}(1,2)_0\}$ *of* $\mathfrak{so}(1,2)$ *are the images under* hat^{-1} *of the subsets in* $\mathbb{R}^{1,2}_C$*:*

(i) The upper or lower sheets of the cone $\mathcal{K}_L = \{(x,y,z) \in \mathbb{R}^3 \,:\, -x^2+y^2+z^2 = 0\}$,

(ii) The upper or lower sheets of the hyperboloids $\mathcal{H}^2_{\|\mathbf{a}\|} = \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2-y^2-z^2 = \|\mathbf{a}\|^2\}$ *of two sheets,*

(iii) The hyperboloids $\mathcal{H}^1_{\|\mathbf{a}\|} = \{(x,y,z) \in \mathbb{R}^3 \,:\, -x^2+y^2+z^2 = \|\mathbf{a}\|^2\}$ *of one sheet,*

(iv) The point $(0,0,0)$.

PROOF. In PROPOSITIONS 3.4.2-3.4.1, we proved that for every $\mathrm{Ad}_g \in \mathsf{Aut}(\mathfrak{so}(1,2))$ there exists an element $g' \in \mathsf{SO}(1,2)_0$ such that for every $A \in \mathfrak{so}(1,2)$ and its corresponding image $\mathbf{a}$ under the hat map, $\mathrm{hat}\,(\mathrm{Ad}_g(A)) = g'\mathbf{a}$. Thus it follows that for each $A \in \mathfrak{so}(1,2)$, then under the hat map, $\widehat{A} = \mathbf{a}$ and $\widehat{\mathcal{O}_A} = \{\mathrm{hat}\,(\mathrm{Ad}_g A) \,:\, g \in \mathsf{SO}(1,2)_0\} = \{g'\mathbf{a} \,:\, g' \in \mathsf{SO}(1,2)_0\} = \mathcal{O}_{\mathbf{a}}$. Since the Minkowski product for $\mathbf{a}$ in $\mathbb{R}^{1,2}$ can be positive, negative or zero for nonzero $\mathbf{a}$, we have the cases

(i) If $\mathbf{a} \odot \mathbf{a} = 0$ for a nonzero element $\mathbf{a}$, then by PROPOSITION 2.1.15, $\mathbf{a}$ lies on either the upper or lower sheet of the light cone $\mathcal{K}_L = \{(x,y,z) \in \mathbb{R}^3 \,:\, -x^2+y^2+z^2 = 0\}$. In PROPOSITION 3.4.3, we showed that the elements $\mathrm{Ad}_g \in \mathsf{Aut}(\mathfrak{so}(1,2)$ correspond under Ψ to the elements $g' \in \mathsf{SO}(1,2)_0$. But in PROPOSITION 3.6.6 we showed that $\mathsf{SO}(1,2)_0$ acting by left multiplication preserves the upper and lower sheets $\mathcal{K}^+_L$ and $\mathcal{K}^-_L$ of this cone and also that the action on these sheets is transitive. Thus if $\mathbf{a}$ is in $\mathcal{K}^+_L[\mathcal{K}^-_L]$, then given any $\mathbf{b} \in \mathcal{K}^+_L[\mathcal{K}^-_L]$, there exists $g \in \mathsf{SO}(1,2)_0$ such that $g\mathbf{a} = \mathbf{b}$, and so $\mathcal{O}_{\mathbf{a}} = \{g\mathbf{a} \,:\, g \in \mathsf{SO}(1,2)_0\}$ is the sheet $\mathcal{K}^+_L$ $[\mathcal{K}^-_L]$ of the light cone that contains $\mathbf{a}$.

(ii) If $\mathbf{a} \odot \mathbf{a} > 0$, then by PROPOSITION 2.1.15, $\mathbf{a}$ lies on a hyperboloid of one sheet expressed as the set $\mathcal{H}^1_{\|\mathbf{a}\|} = \{(x,y,z) \in \mathbb{R}^3 \,:\, -x^2+y^2+z^2 = \|\mathbf{a}\|^2\}$. In PROPOSITION 3.4.3, we showed that the elements $\mathrm{Ad}_g \in \mathsf{Aut}(\mathfrak{so}(1,2)$ correspond under Ψ to the elements $g' \in \mathsf{SO}(1,2)_0$. But in PROPOSITION 3.6.6 we showed that $\mathsf{SO}(1,2)_0$ acting by left multiplication preserves the hyperboloids of one sheet in $\mathbb{R}^{1,2}$, and from PROPOSITION 3.6.6 that this action is transitive. Then given any $\mathbf{b} \in \mathcal{H}^1_{\|\mathbf{a}\|}$, there exists $g \in \mathsf{SO}(1,2)_0$ such that $g\mathbf{a} = \mathbf{b}$, and so $\mathcal{O}_{\mathbf{a}} = \{g\mathbf{a} | g \in \mathsf{SO}(1,2)_0\}$ is the whole of the hyperboloid of one sheet $\mathcal{H}^1_{\|\mathbf{a}\|}$.

(iii) If $\mathbf{a} \odot \mathbf{a} < 0$, then by PROPOSITION 2.1.15, $\mathbf{a}$ lies either on the upper sheet $\mathcal{H}^{2+}_{\|\mathbf{a}\|}$ or the lower sheet $\mathcal{H}^{2-}_{\|\mathbf{a}\|}$ of the hyperboloid $\mathcal{H}^2_{\|\mathbf{a}\|} = \{(x, y, z) \in \mathbb{R}^3 \ : \ -x^2 + y^2 + z^2 = -\|a\|^2\}$. In PROPOSITION 3.4.3, we showed that the elements $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{so}(1,2)$ correspond under Ψ to the elements $g' \in \mathsf{SO}(1,2)_0$. But in PROPOSITION 2.2.27 we proved that $\mathsf{SO}(1,2)_0$ acting by left multiplication preserves these sheets, and from PROPOSITION 3.6.6 that this action is transitive. Thus given $\mathbf{a}$ in $\mathcal{H}^{2+}_{\|\mathbf{a}\|}$ $[\mathcal{H}^{2-}_{\|\mathbf{a}\|}]$, then for any $\mathbf{b} \in \mathcal{H}^{2+}_{\|\mathbf{a}\|}$ $[\mathcal{H}^{2-}_{\|\mathbf{a}\|}]$, there exists $g \in \mathsf{SO}(1,2)_0$ such that $g\mathbf{a} = \mathbf{b}$, and so the orbit $\mathcal{O}_{\mathbf{a}} = \{g\mathbf{a} \ : \ g \in \mathsf{SO}(1,2)_0\}$ is the whole sheet $\in \mathcal{H}^{2+}_{\|\mathbf{a}\|}$ or $\mathcal{H}^{2-}_{\|\mathbf{a}\|}$ of the hyperboloid $\mathcal{H}^2_{\|\mathbf{a}\|}$ containing $\mathbf{a}$.

(iv) Using the decomposition $\mathsf{SO}(1,2)_0 = \mathsf{BK}$ (THEOREM 2.2.23), we see by matrix multiplication that $\mathsf{SO}(1,2)_0$ fixes $(0,0,0)$. Thus $\mathcal{O}_0 = \{g(0,0,0) \ : \ g \in \mathsf{SO}(1,2)_0\} = \{0\}$. □

3.5.2 Co-adjoint orbits

We use the nondegenerate Killing form on $\mathfrak{so}(1,2)$ of PROPOSITIONS 3.2.6 to 3.2.8 to construct a map between $\mathfrak{so}(1,2)$ and its dual space $\mathfrak{so}(1,2)^*$. This allows us to find the co-adjoint orbits of $\mathfrak{so}(1,2)^*$ as images of the adjoint orbits under this map.

3.5.2 PROPOSITION. *The map* $\kappa^\flat : \mathfrak{so}(1,2) \to \mathfrak{so}^*(1,2)$, $\kappa^\flat(A) = \kappa(A,\cdot)$ *is a bijection.*

PROOF. Taking arbitrary elements $Y, Z \in \mathfrak{so}(1,2)$, then for each $X \in \mathfrak{so}(1,2)$,

$$Y \neq Z \quad \Leftrightarrow \mathrm{tr}(\mathrm{ad}_Y \circ \mathrm{ad}_X) \neq \mathrm{tr}(\mathrm{ad}_Z \circ \mathrm{ad}_X) \Leftrightarrow \quad \kappa(Y,X) \neq \kappa(Z,X) \quad \Leftrightarrow \quad \kappa^\flat_Y \neq \kappa^\flat_Z$$

since the identity holds for all X. Thus $\kappa^\flat$ is well-defined and injective. Further, since by PROPOSITION 3.5.1, κ is bilinear, then $\kappa^\flat$ is linear. Thus $\kappa^\flat$ is an injective linear map between two vector spaces of dimension 3, and so is a surjection. Thus $\kappa^\flat : \mathfrak{so}(1,2) \to \mathfrak{so}^*(1,2)$ is a bijection. □

3.5.3 REMARK. Thus the Killing form identifies with each $p \in \mathfrak{so}(1,2)^*$ a unique element $P \in \mathfrak{so}(1,2)$, where $\kappa(P,X) = p(X)$ for all $X \in \mathfrak{so}(1,2)$.

Using the map $\kappa^\flat$ between $\mathfrak{so}^*(1,2)$ and $\mathfrak{so}(1,2)$, we define the dual of the adjoint action:

3.5.4 DEFINITION. Given the action $\mathrm{Ad}_g : \mathfrak{so}(1,2) \to \mathfrak{so}(1,2)$, the **co-adjoint action** of $\mathsf{SO}(1,2)_0$ on $\mathfrak{so}^*(1,2)$ is its dual, given by

$$\mathrm{Ad}^* : \mathfrak{so}(1,2)^* \to \mathfrak{so}(1,2)^*, \ \mathrm{Ad}^*_{g^{-1}}p(X) = p(\mathrm{Ad}_{g^{-1}}X) = \kappa\left(P, \mathrm{Ad}_{g^{-1}}(X)\right)$$

for $p \in \mathfrak{so}^*(1,2)$, $P \in \mathfrak{so}(1,2)$ such that $\kappa^\flat(P) = p$, and $X \in \mathfrak{so}(1,2)$.

3.5.5 PROPOSITION. *For any Lie algebra automomorphism* ϕ, *then* $\phi \circ \mathrm{ad}_X \circ \phi^{-1} = \mathrm{ad}_\phi X$.

PROOF. By the definition (A.5.34) of ad_X, then for every $Y \in \mathfrak{so}(1,2)$, it follows that the composition $\phi \circ \mathrm{ad}_X \circ \phi^{-1}(Y) = \phi[X, \phi^{-1}(Y)] = [\phi(X), Y] = \mathrm{ad}_\phi(X)Y$. Since Y was arbitrarily chosen, the result follows. □

We use the map $\kappa^\flat : \mathfrak{so}(1,2) \to \mathfrak{so}(1,2)^*$ of PROPOSITION 3.5.2 and the hat map to prove

3.5.6 THEOREM. *The co-adjoint orbits $\widetilde{\mathcal{O}}_p = \{\mathrm{Ad}_g^* p \, : \, g \in \mathsf{SO}(1,2)_0\}$ of $\mathfrak{so}(1,2)^*$ are the images under the composition $\kappa^\flat \circ \mathrm{hat}^{-1}$ of the subsets in $\mathbb{R}^{1,2}$:*

(i) The upper or lower sheets of the cone $\mathcal{K}_L = \{(x,y,z) \in \mathbb{R}^3 \, : \, -x^2+y^2+z^2 = 0\}$,

(ii) The upper or lower sheets of the hyperboloids $\mathcal{H}^2_{\|\mathbf{a}\|} = \{(x,y,z) \in \mathbb{R}^3 \, : \, x^2-y^2-z^2 = \|\mathbf{a}\|^2\}$ of 2 sheets,

(iii) The hyperboloids $\mathcal{H}^1_{\|\mathbf{a}\|} = \{(x,y,z) \in \mathbb{R}^3 \, : \, -x^2+y^2+z^2 = \|\mathbf{a}\|^2\}$ of one sheet,

(iv) The point $(0,0,0)$.

PROOF. From (A.5.39), Ad_g is a Lie algebra homomorphism. Thus from PROPOSITION 3.5.5, $\kappa(\mathrm{Ad}_g A, \mathrm{Ad}_g B) = \mathrm{tr}((\mathrm{Ad}_g)^{-1}\mathrm{ad}_A(\mathrm{Ad}_g) \circ (\mathrm{Ad}_g)^{-1}\mathrm{ad}_B(\mathrm{Ad}_g)) = \mathrm{tr}(\mathrm{ad}_A \circ \mathrm{ad}_B) = \kappa(A, B)$ and κ is Ad-invariant. Since $\mathrm{Ad}^*_{g^{-1}}(p)(A) = \kappa\left(P, \mathrm{Ad}_{g^{-1}}A\right) = \kappa\left(\mathrm{Ad}_g P, \mathrm{Ad}_g \mathrm{Ad}_{g^{-1}} A\right) = \kappa\left(\mathrm{Ad}_g P, A\right)$, then as from PROPOSITION 3.2.8, κ is nondegenerate on $\mathfrak{so}(1,2)$, it follows that for each p in $\mathfrak{so}(1,2)^*$, $\mathrm{Ad}^*_{g^{-1}}p = \kappa^\flat(\mathrm{Ad}_g P)$. Since

$$\kappa^\flat\left\{\mathrm{Ad}_g P \, : \, g \in \mathsf{SO}(1,2)_0\right\} = \left\{\kappa^\flat(\mathrm{Ad}_g P) \, : \, g \in \mathsf{SO}(1,2)_0\right\} = \left\{\mathrm{Ad}_g^* p \, : \, g \in \mathsf{SO}(1,2)_0\right\},$$

by the adjoint-invariance of $\kappa^\flat$ (PROPOSITION 3.5.5), it follows that $\kappa^\flat(\mathcal{O}_P) = (\mathcal{O}_p)$, where $p = \kappa^\flat(P)$. Thus $\mathrm{hat} \circ \kappa^{\flat\,-1} : \mathfrak{so}(1,2) \to \mathbb{R}^{1,2}$ are the subsets 1 to 4 of THEOREM 3.5.1, and the result follows. □

3.6 More properties of $\mathsf{SO}(1,2)_0$

3.6.1 THEOREM. *$\mathsf{SO}(1,2)_0$ is simple.*

PROOF. In THEOREM 3.2.6, we showed that $\mathfrak{so}(1,2)$ is simple, and in THEOREM 3.1.2 that $\mathsf{SO}(1,2)_0$ is connected. Thus from (A.5.50), $\mathsf{SO}(1,2)_0$ is a simple Lie group. □

In (A.5.55) we stated that given an n-fold cover ϕ_n of a Lie group G by $\widetilde{\mathsf{G}}$ with a path α in G having initial point g, and $\widetilde{g}$ a point in $\widetilde{\mathsf{G}}$ over g, then the path α has a unique lift $\widetilde{\alpha}$ in $\widetilde{\mathsf{G}}$, such that $\widetilde{\alpha}(0) = \widetilde{g}$ and $\phi_n \circ \widetilde{\alpha} = \alpha$. We use this result to prove

3.6.2 PROPOSITION. *Given $\phi_n : \widetilde{\mathsf{G}} \to \mathsf{G}$ an n-fold cover of G by $\widetilde{\mathsf{G}}$, suppose that α and β are paths in G from g to h, and α is homotopic to β with endpoints fixed. The the lift $\widetilde{\alpha}$ of α with initial point $\widetilde{g}$ and final point $\widetilde{h}$ is homotopic to the lift $\widetilde{\beta}$, with endpoints fixed.*

PROOF. From (A.5.55), we note that $\widetilde{\alpha} = \phi_n(\alpha)$ and $\widetilde{\beta} = \phi_n(\beta)$ are the unique preimages of α and β in $\widetilde{G}$ where $\widetilde{g} = \phi_n(g) = \phi_n(\alpha(0))$. Define the homotopy $d : [0,1] \times [0,1] \to G$ from α to β. The fact that the path α is a continuous function from $[0,1]$ into G allows us to divide $[0,1]$ into a finite number of closed subintervals $i_1 = [0,k_1], i_2 = [k_1,k_2], ..., i_n = [k_{n-1},1]$, where $\alpha(0) = g = s_1.\alpha(k_1) = s_2...\alpha(k_{n-1}) = s_n$ and we restrict each i_i such that $\alpha(i_i) = [s_{i-1},s_i]$ lies within the open neighbourhood $\mathcal{N}_i$ of s_i that is mapped bijectively and bicontinuously by the homeomorphism ϕ_n onto some open neighbourhood $\phi_n^{-1}(\mathcal{N}) = \widetilde{\mathcal{N}}$ in $\widetilde{G}$. We define the restrictions of α to each i_i and denote them by α_i. We may similarly restrict the homotopy d to maps d_i, the deformations $d_i : [0,1] \times [k_{i-1},k_i] \to G$.

We then consider the composition $\phi_n^{-1} \circ d : [0,1] \times [k_{i-1},k_i] \to \widetilde{G}$ which deforms each $\widetilde{\alpha}_i$ to $\widetilde{\beta}_i$. Since ϕ_n and its inverse are continuous, we note that the preimages $\phi_n^{-1}[s_{i-1},s_i]$ are closed subsets of $\widetilde{G}$ which have a union $[s_1,s_2] \cup [s_2,s_3] \cup ... \cup [s_{n-1},s_n] = [s_1,s_n] = [\widetilde{g},\widetilde{h}]$. Thus the concatenation of the paths $\widetilde{\alpha}_i$ in $\widetilde{G}$ is the lift $\widetilde{\alpha}$ of α with initial point $\widetilde{g}$ and final point $\alpha(1) = \widetilde{h}$ which is deformed continuously by $d : [0,1][0,k_1] \cup [k_1,k_2] \cup ... [k_{n-1},1]$ to $\widetilde{\beta}(t)$ such that $\widetilde{\beta}(0) = \widetilde{g}$ and $\widetilde{\beta}(1) = \widetilde{h}$. □

3.6.3 THEOREM. $SO(1,2)_0$ *is not simply-connected.*

PROOF. In PROPOSITION 2.4.4 we defined the homomorphism $\varphi_2 : GL(2,\mathbb{R}) \to SO(1,2)_0$. Since φ_2 was shown to be continuous, it is a local homeomorphism. Consider a path $\widetilde{\alpha}$ in $GL(2,\mathbb{R})$ such that $\widetilde{\alpha}(0) = a\mathbf{1}$, $\widetilde{\alpha}(1) = b\mathbf{1}$ for $a, b \in \mathbb{R}^+$, $\alpha \neq \beta$. The image $\alpha = \varphi_2(\widetilde{\alpha})$ is a closed path in $SO(1,2)_0$, since $\varphi_2(a\mathbf{1}) = \varphi_2(b\mathbf{1})$ and thus $\alpha(0) = \alpha(1)$.

Assume that $SO(1,2)_0$ is simply-connected. Then it follows that $\alpha(s)$ is homotopic to a point, where this homotopy keeps $a\mathbf{1}$ fixed. By lifting this homotopy to $GL(2,\mathbb{R})$ as in PROPOSITION 3.6.2, it must be that the path $\widetilde{\alpha}(s)$ in $GL(2,\mathbb{R})$ can be homotopically deformed to a point, keeping its endpoints $a\mathbf{1}$ and $b\mathbf{1}$ fixed. But this is impossible, since $a\mathbf{1}$ and $b\mathbf{1}$ are distinct points in $GL(2,\mathbb{R})$. Thus $SO(1,2)_0$ cannot be simply-connected. □

3.6.4 PROPOSITION. *The group* $SO(1,2)_0$ *acting on* $\mathbb{R}^{1,2}$ *by left multiplication preserves the upper sheet and lower sheets of the light cone* $\mathcal{K}_L$. $SO(1,2)_0$ *acting on* $\mathbb{R}^{1,2}$ *by left multiplication preserves the hyperboloids of one sheet* $\mathcal{H}^1_\alpha = \{(x,y,z) \in \mathbb{R}^3 \ : \ -x^2 + y^2 + z^2 = \alpha\}$.

PROOF. We show in each case that given $\mathbf{a}$ in one of the required subsets, then its image $g(\mathbf{a})$ remains in that subset. Given any element $\mathbf{a}$ of $\mathbb{R}^{1,2}$ which lies on the light cone $\mathcal{K}_L$, then $a_2^2 + a_3^2 = a_1^2$, and so $\mathbf{a} \odot \mathbf{a} = 0$. But then for any $g \in SO(1,2)_0$, $\mathbf{a} \odot \mathbf{a} = g\mathbf{a} \odot g\mathbf{a} = 0$, and $g(\mathbf{a}) \in \mathcal{K}_L$. Thus $SO(1,2)_0$ preserves the light cone. Any element $\mathbf{a}$ on the upper sheet of $\mathcal{K}_L$ lies in the space $\{(x,y,z) \in \mathbb{R}^3 \ : \ x > 0\}$ and so is such that $\mathbf{a} \bullet \mathbf{e}_1 = a_1 > 0$, and any element $\mathbf{b}$ on the lower sheet of $\mathcal{K}$ lies in the space $\{(x,y,z) \in \mathbb{R}^3 \ : \ x < 0\}$ is such that $\mathbf{b} \bullet \mathbf{e}_1 = b_1 < 0$. But by PROPOSITION 2.2.12, each $g \in SO(1,2)_0$ preserves the orientation of the $\mathbf{e}_1$-axis, and thus if a_1, then $g\mathbf{a} \bullet \mathbf{e}_1 > 0$.

Similarly, for any $\mathbf{b}$ such that $\mathbf{b}\mathbf{e}_1 < 0$, then $g\mathbf{b} \bullet \mathbf{e}_1 < 0$. Given any element $\mathbf{c}$ of $\mathbb{R}^{1,2}$ which lies on a hyperboloid of one sheet $\mathcal{H}^1_\alpha$, then $-c_1^2 + c_2^2 + c_3^2 = \mathbf{c} \odot \mathbf{c} = \alpha$. But then for any

$g \in \mathrm{SO}(1.2)_0$, $\mathbf{c} \odot \mathbf{c} = g\mathbf{c} \odot g\mathbf{c} = \alpha$, and $g(\mathbf{c}) = \in \mathcal{H}^1_\alpha$. Thus $\mathrm{SO}(1,2)_0$ preserves the hyperboloids of one sheet $\mathcal{H}^1_\alpha$. The result follows. □

3.6.5 PROPOSITION. *We may parametrize the cone $\mathcal{K}_L$ by patches $(\epsilon^{11}(z,\theta), \mathcal{U}^{11})$ and $(\epsilon^{12}(z.\theta), \mathcal{U}^{12})$, where $\epsilon^{1i}(z,\theta) = (z, z\cos\theta, z\sin\theta)$ and $\mathcal{U}^{11} = \mathbb{R} \times (-\pi,\pi)$, $\mathcal{U}^{12} = \mathbb{R} \times (0, 2\pi)$, the hyperboloids of one sheet $\mathcal{H}^1_\alpha$ by the patches $(\epsilon^{21}(z,\theta).\mathcal{U}^{21})$ and $(\epsilon^{22}(z,\theta),\mathcal{U}^{22})$, where $\epsilon^{2i}(r,\theta) = (\alpha\sinh r. \alpha\cosh r\cos\theta, \alpha\cosh r\sin\theta)$ and $\mathcal{U}^{21} = \mathbb{R} \times (-\pi.\pi)$, $\mathcal{U}^{22} = \mathbb{R} \times (0.2\pi)$, and the hyperboloids of two sheets $\mathcal{H}^2_\alpha$ by the patches $(\epsilon^{31}(z.\theta),\mathcal{U}^{31})$ and $(\epsilon^{32}(z,\theta).\mathcal{U}^{32})$, where $\epsilon^{3i}(r.\theta) = (\sinh r, \cosh r\cos\theta, \cosh r\sin\theta)$ and $\mathcal{U}^{31} = \mathbb{R} \times (-\pi,\pi)$, $\mathcal{U}^{32} = \mathbb{R} \times (0.2\pi)$. In each case the transition maps between the two patches are rotations $k(\pi)$ about the $\mathbf{e}_1$-axis.*

PROOF. We consider first the case of the hyperboloids of one sheet $\mathcal{H}^1_\alpha$. In either patch it is clear that for $(x.y,z) \in \bigcup_{i=1,2}(\epsilon^{1i},\mathcal{U}^{1i})$, then

$$(x,y,z) = \epsilon^{1i}(r,\theta) = (\alpha\sinh r, \alpha\cosh r\cos\theta, \alpha\cosh r\sin\theta) \quad \text{and} \quad -x^2+y^2+z^2 = \alpha.$$

Thus $\bigcup_{i=1,2}(\epsilon_i,\mathcal{U}_i) \subseteq \mathcal{HL}$. We show that the condition $-p_1^2+p_2^2+p_3^2 = \alpha^2$ on $\mathbf{p} = (p_1,p_2,p_3)$ in $\mathcal{H}^1_\alpha$ defines a pair (r_0,θ_0) such that $\mathbf{p}$ lies either in the patch $(\epsilon_1,\mathcal{U}_1)$ or $(\epsilon_2,\mathcal{U}_2)$. Consider $(p_1,p_2,p_3) \in \mathbb{R}^{1,2}$ such that $-p_1^2+p_2^2+p_3^2 = \alpha^2$. Clearly, there exists some $r_0 = \sinh^{-1}(\frac{p_1}{\alpha})$ such that $\frac{p_1}{\alpha} = \sinh r_0$. Then $p_1^2 = \alpha^2\sinh^2 r_0$ and $p_1^2 - p_2^2 - p_3^2 = -\alpha^2$, so thus $p_2^2+p_3^2 = \alpha^2\cosh^2 r_0$. Further, since $\frac{p_2^2+p_3^2}{\alpha^2\cosh^2 r_0} = \frac{p_2^2+p_3^2}{\alpha^2+p_1^2} = 1$, then

$$\left|\frac{p_2}{\alpha^2\cosh^2 r_0}\right| \le 1 \quad \text{and} \quad \left|\frac{p_3}{\alpha^2\cosh^2 r_0}\right| \le 1$$

and so we can find a preimage θ_1 in $(0,2\pi)$ or $(-\pi,\pi)$ of $\frac{p_2}{\alpha^2\cosh^2(r_0)}$ under cos and a preimage θ_2 in $(0,2\pi)$ or $(-\pi,\pi)$ of $\frac{p_3}{\alpha^2\cosh^2(r_0)}$ under sin. Since

$$\left(\frac{p_2}{\alpha\sqrt{p_1^2+p_3^2}}\right)^2 + \left(\frac{p_3}{\alpha\sqrt{p_1^2+p_3^2}}\right)^2 = 1$$

then $\theta_1 = \theta_2 = \theta_0$ in (0.2π) or $(-\pi,\pi)$. Thus $p_2 = \alpha\cosh r_0\cos\theta_0$ and $p_3 = \alpha\cosh r_0\sin\theta_0$ for some (r_0,θ_0) in $\mathcal{U}_1$ or $\mathcal{U}_2$ such that $(p_1.p_2,p_3) = (\alpha\sinh r_0, \alpha\cosh r_0\cos\theta_0, \alpha\cosh r_0\sin\theta_0)$. Then (p_1,p_2,p_3) lies in either the patch $(\epsilon^{11},\mathcal{U}^{11})$ or the patch $(\epsilon^{12},\mathcal{U}^{12})$. Thus $\mathcal{H}^1_\alpha \subseteq \bigcup_{i=1,2}(\epsilon^{1i},\mathcal{U}^{1i})$. Thus the patches $(\epsilon^{11};\mathcal{U}^{11}),(\epsilon^{12},\mathcal{U}^{12})$ parametrize $\mathcal{H}^1_\alpha$.

Since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{bmatrix} \begin{bmatrix} \alpha\sinh r \\ \alpha\cosh r\cos\theta \\ \alpha\cosh r\sin\theta \end{bmatrix} = \begin{bmatrix} \alpha\sinh r \\ \alpha\cosh r\cos(\theta+\pi) \\ \alpha\cosh r\sin(\theta+\pi) \end{bmatrix}$$

then the transition maps $(\epsilon^{11})^{-1}\circ\epsilon^{12}$ and $(\epsilon^{12})^{-1}\circ\epsilon^{11}$ are smooth rotations about the origin.

We then consider the case of the light cone $\mathcal{K}_L$. In either patch it is clear that given $(x,y.z)$ in $\bigcup_{i=1,2}(\epsilon^i,\mathcal{U}_i)$, then

$$(x,y,z) = \epsilon^{2i}(z.\theta) = (z, z\cos\theta. z\sin\theta) \quad \text{and} \quad y^2+z^2 = x^2.$$

Thus $\bigcup_{i=1,2}(\epsilon^{2i},\mathcal{U}_i) \subseteq \mathcal{HL}$. We show that the condition $-p_1^2+p_2^2+p_3^2=0$ on $\mathbf{p}=(p_1,p_2,p_3)\in\mathcal{K}_L$ defines a pair (z_0,θ_0) such that $\mathbf{p}$ lies either in the patch $(\epsilon^{21},\mathcal{U}^{21})$ or $(\epsilon_{22},\mathcal{U}^{22})$. Consider $(p_1,p_2,p_3)\in\mathbb{R}^{1,2}$ such that $-p_1^2+p_2^2+p_3^2=0$. Clearly, there exists some $z_0=p_1$ in $\mathbb{R}$. But $p_2^2+p_3^2=p_1^2$, and thus $\frac{p_2^2}{z_0^2}+\frac{p_3^2}{z_0^2}=1$. Thus we may express $\frac{p_2^2}{z_0^2}=\cos\theta_0$ for θ_0 in $(0,2\pi)$ or $(-\pi,\pi)$ and $\frac{p_3^2}{z_0^2}=\sin\theta_0$ for θ_0 in $(0,2\pi)$ or $(-\pi,\pi)$. Then there always exists $z_0\in\mathbb{R}$ and $\theta_0\in(0,2\pi)\cup(-\pi,\pi)$ such that $(p_1,p_2,p_3)=(z_0,z_0\cos\theta_0,z_0\sin\theta_0)$, and so (p_1,p_2,p_3) lies in either the patch $(\epsilon^{21},\mathcal{U}^{21})$ or the patch $(\epsilon^{22},\mathcal{U}^{22})$. Thus $\mathcal{K}_L\subseteq\bigcup_{i=1,2}(\epsilon^{2i},\mathcal{U}^{2i})$.Thus the patches $(\epsilon^{21},\mathcal{U}^{21}),(\epsilon^{22},\mathcal{U}^{22})$ parametrize $\mathcal{K}_L$.
Since

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\pi & -\sin\pi\\ 0 & \sin\pi & \cos\pi\end{bmatrix}\begin{bmatrix}z\\ z\cos\theta\\ z\sin\theta\end{bmatrix}=\begin{bmatrix}z\\ z\cos(\theta+\pi)\\ z\sin(\theta+\pi)\end{bmatrix}$$

then the transition maps $(\epsilon^{21})^{-1}\circ\epsilon^{22}$ and $(\epsilon^{22})^{-1}\circ\epsilon^{11}$ are smooth rotations about the origin.

Finally, we consider the case of the hyperboloids of two sheets $\mathcal{H}_\alpha^2$. In either patch it is clear that given $(x,y,z)\in\bigcup_{i=1,2}(\epsilon^{3i},\mathcal{U}^{3i})$, then

$$(x,y,z)=\epsilon_i(u,v)=(\alpha\cosh r,\alpha\sinh r\cos\theta,\alpha\sinh r\sin\theta)\quad\text{and}\quad -x^2+y^2+z^2=-\alpha.$$

Thus $\bigcup_{i=1,2}(\epsilon^{3i},\mathcal{U}^{3i})\subseteq\mathcal{H}_\alpha^2$. We show that the condition $-p_1^2+p_2^2+p_3^2=-\alpha^2$ on $\mathbf{p}=(p_1,p_2,p_3)$ in $\mathcal{H}_\alpha^2$ defines a pair (r_0,θ_0) such that $\mathbf{p}$ lies either in the patch $(\epsilon^{31},\mathcal{U}^{31})$ or $(\epsilon^{32},\mathcal{U}^{32})$. Consider $(p_1,p_2,p_3)\in\mathbb{R}^{1,2}$ such that $-p_1^2+p_2^2+p_3^2=-\alpha^2$. Clearly, there exists some $r_0=\cosh^{-1}(\frac{m_1}{\alpha})$ such that $\frac{p_1}{\alpha}=\cosh r_0$. Then $p_1^2=\alpha^2\cosh^2 r_0$ and $p_1^2-p_2^2-p_3^2=-\alpha^2$, so thus $p_2^2+p_3^2=\alpha^2\cosh^2 r_0$. Further, since $\frac{p_2^2+p_3^2}{\alpha^2\sinh^2 r_0}=\frac{p_2^2+p_3^2}{\alpha^2+p_1^2}=1$, then

$$\left|\frac{p_2}{\alpha^2\sinh^2 r_0}\right|\leq 1\quad\text{and}\quad\left|\frac{p_3}{\alpha^2\sinh^2 r_0}\right|\leq 1$$

and so we can find a preimage θ_1 in $(0,2\pi)$ or $(-\pi,\pi)$ of $\frac{p_2}{\alpha^2\sinh^2(r_0)}$ under cos and a preimage θ_2 in $(0,2\pi)$ or $(-\pi,\pi)$ of $\frac{p_3}{\alpha^2\sinh^2(r_0)}$ under sin. Since

$$\left(\frac{p_2}{\alpha\sqrt{p_2^2+p_3^2}}\right)^2+\left(\frac{p_3}{\alpha\sqrt{p_2^2+p_3^2}}\right)^2=1$$

then $\theta_1=\theta_2=\theta_0$ in $(0,2\pi)$ or $(-\pi,\pi)$. Thus $p_2=\alpha\sinh r_0\cos\theta_0$ and $p_3=\alpha\sinh r_0\sin\theta_0$ for some (r_0,θ_0) in $\mathcal{U}_1$ or $\mathcal{U}_2$ such that $(p_1,p_2,p_3)=(\alpha\cosh r_0,\alpha\sinh r_0\cos\theta_0,\alpha\sinh r_0\sin\theta_0)$. Then (p_1,p_2,p_3) lies in either the patch $(\epsilon^{31},\mathcal{U}^{31})$ or the patch $(\epsilon^{32},\mathcal{U}^{32})$. Thus $\mathcal{H}_\alpha^1\subseteq\bigcup_{i=1,2}(\epsilon^{3i},\mathcal{U}^{3i})$. Thus the patches $(\epsilon^{31},\mathcal{U}^{31}),(\epsilon^{32},\mathcal{U}^{32})$ parametrize $\mathcal{H}_\alpha^2$.
Since

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\pi & -\sin\pi\\ 0 & \sin\pi & \cos\pi\end{bmatrix}\begin{bmatrix}\alpha\cosh r\\ \alpha\sinh r\cos\theta\\ \alpha\sinh r\sin\theta\end{bmatrix}=\begin{bmatrix}\alpha\cosh r\\ \alpha\sinh r\cos(\theta+\pi)\\ \alpha\sinh r\sin(\theta+\pi)\end{bmatrix}$$

then the transition maps $(\epsilon^{31})^{-1}\circ\epsilon^{32}$ and $(\epsilon^{32})^{-1}\circ\epsilon^{31}$ are smooth rotations about the origin.□

3.6.6 PROPOSITION. *The group* $\mathsf{SO}(1,2)_0$ *acts transitively on the upper sheet of the light cone* $\mathcal{K}_L$. *Similarly,* $\mathsf{SO}(1,2)_0$ *acts transitively on* $\mathcal{H}^1_\alpha = \{(x,y,z) \in \mathbb{R}^3 \,:\, -x^2+y^2+z^2=\alpha\}$, *the hyperboloids of one sheet, and* $\mathcal{H}^2_\alpha = \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2-y^2-z^2=\alpha\}$, *the hyperboloids of two sheets.*

PROOF. In PROPOSITION 3.6.5, we determined parametrizations for $\mathcal{K}_L, \mathcal{H}^1_\alpha$ and $\mathcal{H}^2_\alpha$.

Given two points $\mathbf{p}, \mathbf{q} \in \mathcal{K}_L$, then we may assume that $\mathbf{p}$ and $\mathbf{q}$ lie in the same patch $(\epsilon^{1i}, \mathcal{U}^{1i})$, since the transition maps between the two patches are rotations. Then $\mathbf{p} = \epsilon^{1i}(\theta_m, z_m)$ and $\mathbf{q}^{1i} = \epsilon^{1i}(\theta_n, z_n)$ on $\mathcal{K}_L$, and there exists a Euclidean rotation $k(\theta_n - \theta_m)$ and a Lorentz boost $b_1(z_n - z_m)$ such that

$$k(\theta_n-\theta_m)(\epsilon^{1i}(\theta_m, z_m)) = \epsilon^{1i}(\theta_n, z_m) \quad \text{and} \quad b_1(z_n-z_m)(\epsilon^{1i}(\theta_n, z_m)) = \epsilon^{1i}(\theta_n, z_n).$$

Thus $b_1(z_n - z_m)k(\theta_n - \theta_m)(\mathbf{p}) = \mathbf{q}$ for any $\mathbf{p}, \mathbf{q}$ on $\mathcal{K}_L$.

Similarly, taking $\mathbf{p}', \mathbf{q}'$ two points in $\mathcal{H}^1_\alpha$, then we may assume that $\mathbf{p}'$ and $\mathbf{q}'$ lie in the same patch $(\epsilon^{2i}.\mathcal{U}^{2i})$, since the transition maps between the two patches are rotations. Then $\mathbf{p}' = \epsilon^{2i}(r_m, \theta_m)$ and $\mathbf{q}' = \epsilon^{2i}(r_n, \theta_n)$ on $\mathcal{H}^\alpha_1$, and there exists a Euclidean rotation $k(\theta_n - \theta_m)$ and a Lorentz boost $b_1(r_n - r_m)$ such that

$$k(\theta_n-\theta_m)(\epsilon^{2i}(\theta_m. r_m)) = \epsilon^{2i}(\theta_n, r_m) \quad \text{and} \quad b_1(r_n-r_m)(\epsilon^{2i}(\theta_n, r_m)) = \epsilon^{2i}(\theta_n, r_n).$$

Thus $b_1(r_n - r_m)k(\theta_n - \theta_m)(\mathbf{p}') = \mathbf{q}'$ for any $\mathbf{p}'. \mathbf{q}'$ on $\mathcal{H}^1_\alpha$.

Finally, taking $\mathbf{p}, \mathbf{q}$ two points in $\mathcal{H}^2_\alpha$, then we may assume that $\mathbf{p}$ and $\mathbf{q}$ lie in the same patch $(\epsilon^{3i}, \mathcal{U}^{3i})$, since the transition maps between the two patches are rotations. Then $\mathbf{p} = \epsilon^{3i}(r_m, \theta_m)$ and $\mathbf{q} = \epsilon^{3i}(r_n, \theta_n)$ on $\mathcal{H}^\alpha_2$, and there exists a Euclidean rotation $k(\theta_n - \theta_m)$ and a Lorentz boost $b_1(r_n - r_m)$ such that

$$k(\theta_n-\theta_m)(\epsilon^{3i}(\theta_m. r_m)) = \epsilon^{3i}(\theta_n, r_m) \quad \text{and} \quad b_1(r_n-r_m)(\epsilon^{3i}(\theta_n, r_m)) = \epsilon^{3i}(\theta_n, r_n).$$

Thus $b_1(r_n - r_m)k(\theta_n - \theta_m)(\mathbf{p}) = \mathbf{q}$ for any $\mathbf{p}, \mathbf{q}$ on $\mathcal{H}^2_\alpha$. □

3.6.7 PROPOSITION. *The Lorentz boosts* b_1 *in* $\mathsf{SO}(1.2)_0$ *preserve the hyperbolic hyperplanes* $\mathbf{\Gamma}_{H(\alpha)}$ *given by* $\mathbf{\Gamma}_{H(\alpha)} = \alpha\mathbf{e}_3 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle$ *in* $\mathbb{R}^{1,2}$. *The Euclidean rotations* k *in* $\mathsf{SO}(1,2)_0$ *preserve the elliptic hyperplanes* $\mathbf{\Gamma}_{E(\alpha)}$ *given by* $\mathbf{\Gamma}_{E(\alpha)} = \alpha\mathbf{e}_1 + \langle \mathbf{e}_2. \mathbf{e}_3\rangle$ *in* $\mathbb{R}^{1,2}$.

PROOF. Firstly, $b_1\mathbf{e}_3 = (0,0,1) = \mathbf{e}_3$, $b_1\mathbf{e}_1 = (\cosh t. \sinh t. 1)$ and $b_1\mathbf{e}_2 = (\sinh t, \cosh t, 1)$ by matrix multiplication. Since the boosts b_1 are linear, then it follows that

$$b_1(\mathbf{\Gamma}_{H(\alpha)}) = b_1(\alpha\mathbf{e}_3 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle) = \alpha b_1(\mathbf{e}_3) + \langle b_1(\mathbf{e}_1). b_1(\mathbf{e}_2)\rangle = \alpha\,(\mathbf{e}_3) + \langle \mathbf{e}_1. \mathbf{e}_2\rangle = \mathbf{\Gamma}_{H(\alpha)}.$$

Similarly, $k\mathbf{e}_3 = (0, -\sin\theta, \cos\theta)$, $k\mathbf{e}_1 = (1.0,0) = \mathbf{e}_1$ and $k\mathbf{e}_2 = (\cos\theta, \sin\theta t. 1)$, and since the Euclidean rotations k are linear, then

$$k(\mathbf{\Gamma}_{E(\alpha)}) = k(\alpha\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle) = \alpha k(\mathbf{e}_1) + \langle k(\mathbf{e}_2), k(\mathbf{e}_3)\rangle = \alpha\,(\mathbf{e}_1) + \langle \mathbf{e}_2, \mathbf{e}_3\rangle = \mathbf{\Gamma}_{E(\alpha)}.$$ □

Finally, we refer to (A.6.1) - (A.6.8) in order to obtain an Iwasawa decomposition (A.6.7) of $SO(1,2)_0$.

3.6.8 PROPOSITION. *There exists a direct sum decomposition* $\mathfrak{so}(1,2) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ *where* $\mathfrak{k} = \langle E_1 \rangle$ *is a compact subalgebra,* $\mathfrak{a} = \langle E_2 \rangle$ *is an Abelian subalgebra and* $\mathfrak{n} = \langle E_1 - E_3 \rangle$ *is a nilpotent subalgebra.*

PROOF. Firstly, $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$ are subalgebras of $\mathfrak{so}(1,2)$, since for any $a, b \in \mathbb{R}$, $[aE_1, bE_1] = 0$ and thus $\mathfrak{k}$ is closed under the Lie bracket, $[aE_2, bE_2] = 0$ and thus $\mathfrak{a}$ is closed under the Lie bracket, and $[a(E_1 - E_3), b(E_1 - E_3)] = 0$ and thus $\mathfrak{n}$ is closed under the Lie bracket.

Secondly, any element $X \in \mathfrak{so}(1,2)$ may be written as a sum $K + A + N$ where $K \in \mathfrak{k}, A \in \mathfrak{a}$ and $N \in \mathfrak{n}$: for $X \in \mathfrak{so}(1,2)$, $X = aE_1 + bE_2 + cE_3 = (a+c)E_1 + bE_2 + (-c)(E_1 - E_3)$.

Thirdly, we show that the subalgebras $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$ intersect only in the zero element. Assume that there is a nonzero intersection $\mathfrak{k} \cap \mathfrak{a}$. Then there exists some $X \in \mathfrak{so}(1,2)$ such that $X \in \mathfrak{k}, X \in \mathfrak{a}$. But then $X = aE_1 = bE_2$, which implies that $a = b = 0$ by the fact that E_1 and E_2 are linearly independent, and thus $X = 0$. Similarly, consider some $X \in \mathfrak{a} \cap \mathfrak{n}$. Then $X = aE_2 = b(E_1 - E_3)$, which implies that $a = b = 0$ by the fact that E_1, E_2, E_3 are linearly independent, and thus $X = 0$. Finally, consider some $X \in \mathfrak{k} \cap \mathfrak{n}$. Then $X = aE_1 = b(E_1 - E_3)$, which implies that $a = b = 0$ by the fact that E_1 and E_3 are linearly independent, and thus $X = 0$.

We have then proved that $\mathfrak{so}(1,2) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a direct sum decomposition. We then show that $\mathfrak{k}$ is a compact Lie algebra. In PROPOSITIONS 3.2.6 and 3.2.8 we showed that the Killing form $\kappa(\cdot) = \mathrm{tr}(\mathrm{ad}(\cdot) \circ \mathrm{ad}(\cdot))$ is a nondegenerate symmetric bilinear form on $\mathfrak{so}(1,2)$. We show that κ is negative-definite on $\mathfrak{k}$. Given any $K \in \mathfrak{k}$, then $K = tE_1$ for some $t \in \mathbb{R}$, and

$$\mathrm{tr}(\mathrm{ad}_K \circ \mathrm{ad}_K) = \mathrm{tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & -t & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & -t & 0 \end{bmatrix}\right) = -2t^2$$

which is negative for all $t \in \mathbb{R}$. Thus by (A.2.17), κ is negative-definite on $\mathfrak{k}$. Thus by (A.6.1), $\mathfrak{k}$ is a compact subalgebra. Given any elements A_1 and A_2 of $\mathfrak{a}$, then

$$A_1 A_2 = \begin{bmatrix} 0 & 0 & t_1 \\ 0 & 0 & 0 \\ t_1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & t_2 \\ 0 & 0 & 0 \\ t_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & t_1 + t_2 \\ 0 & 0 & 0 \\ t_1 + t_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & t_2 + t_1 \\ 0 & 0 & 0 \\ t_2 + t_1 & 0 & 0 \end{bmatrix} = A_2 A_1$$

and thus by (A.6.2), $\mathfrak{a}$ is an Abelian subalgebra. Finally, any element N of $\mathfrak{n}$ has the form

$$N = \begin{bmatrix} 0 & -t & 0 \\ -t & 0 & -t \\ 0 & t & 0 \end{bmatrix} \Rightarrow N^3 = \begin{bmatrix} t^2 & 0 & t^2 \\ 0 & 0 & 0 \\ -t^2 & 0 & -t^2 \end{bmatrix}\begin{bmatrix} 0 & -t & 0 \\ -t & 0 & -t \\ 0 & t & 0 \end{bmatrix} = 0.$$

Thus by (A.6.3), $\mathfrak{n}$ is a nilpotent subalgebra. □

Let KAN be an Iwasawa decomposition of the semisimple Lie group G. We will denote the diffeomorphic multiplication map $K \times A \times N \to KAN$ of (A.6.6) by ϕ, and prove from (A.6.6), 3.6.9 COROLLARY. *Given* KAN *an Iwasawa decomposition of the semisimple Lie group* G, *then for any* $g \in G$, *there is a unique expression* $g = kan$ *for* $k \in K, a \in A$ *and* $n \in N$.

PROOF. Consider $g \in G$. By the surjectivity of the map ϕ there exist $k \in K, a \in A$ and $n \in N$ such that $g = k \cdot a \cdot n$. Assume that $g = k_1 \cdot a_1 \cdot n_1 = k_2 \cdot a_2 \cdot n_2$. Then by the injectivity of ϕ,

$$k_1 \cdot a_1 \cdot n_1 = k_2 \cdot a_2 \cdot n_2 \Rightarrow (k_1, a_1, n_1) = (k_2, a_2, n_2) \in K \times A \times N$$

where $K \cap A = A \cap N = K \cap N = \{1\}$. Thus $k_1 = k_2, a_1 = a_2$ and $n_1 = n_2$, and g is uniquely expressible as a product $k \cdot a \cdot n$. □

3.6.10 THEOREM. (IWASAWA DECOMPOSITION OF $SO(1,2)_0$) *Given the one-parameter subgroups*

$$K = \{\exp(tE_1) : t \in \mathbb{R}\}, \quad A = \{\exp(tE_2) : t \in \mathbb{R}\} \quad \textit{and} \quad N = \{\exp(t(E_1 - E_2)) : t \in \mathbb{R}\}$$

of $SO(1,2)_0$, *then* $SO(1,2)_0 = KAN$ *in the sense that for each* $g \in SO(1.2)_0$, *then there exist unique elements* $k \in K, a \in A$ *and* $n \in N$ *such that* $g = k \cdot a \cdot n$.

PROOF. In PROPOSITION 3.6.11, we expressed $\mathfrak{so}(1,2)$ as the direct sum of its subalgebras $\mathfrak{so}(1,2) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{k}$ is compact, $\mathfrak{a}$ is Abelian and $\mathfrak{n}$ is nilpotent. The one-parameter subgroups K, A and N as expressed in the statement of the theorem respectively have $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$ as their Lie algebras. Thus by THEOREM A.6.6 the map $K \times A \times N \to SO(1,2)_0$ given by $(k, a.n) \mapsto kan$ is a diffeomorphism onto. Thus by COROLLARY 3.6.9, for each $g \in SO(1.2)$, there exists a unique $k \in K, a \in A$ and $n \in N$ such that $g = kan$. The result follows.

3.6.11 REMARK. We express the connected subgroups K, A and N:

$$K = \{\exp(tE_1) : t \in \mathbb{R}\} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix} \theta \in \mathbb{R} \right\} \tag{3.6.1}$$

$$A = \{\exp(tE_2) : t \in \mathbb{R}\} = \left\{ \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix} : t \in \mathbb{R} \right\} \tag{3.6.2}$$

$$N = \{\exp(t(E_1 - E_3)) : t \in \mathbb{R}\} = \left\{ \begin{bmatrix} \frac{1}{2}(2+t^2) & -t & \frac{t^2}{2} \\ -t & 1 & -t \\ -\frac{t^2}{2} & t & \frac{1}{2}(2-t^2) \end{bmatrix} : t \in \mathbb{R} \right\} \tag{3.6.3}$$

using Mathematica (C.2).

Chapter 4

Left-Invariant Control Systems on $\mathsf{SO}(1,2)_0$

4.1 Classification under local detached feedback equivalence

We refer to (A.8.3) for the definition of local detached feedback equivalence (l.d.f.e.). In this section we will classify all affine left-invariant control systems on $\mathsf{SO}(1,2)_0$ under l.d.f.e.

4.1.1 DEFINITION. A system Σ is a **reparametrization** of a system $\widetilde{\Sigma}$ if their corresponding traces $\Gamma = \{\Xi(\mathbf{1}, u) \,:\, u \in \mathbb{R}^\ell\}$ and $\widetilde{\Gamma} = \left\{\widetilde{\Xi}(\mathbf{1}, u) \,:\, u \in \mathbb{R}^\ell\right\}$ are equal (as sets).

If the two systems Σ and $\widetilde{\Sigma}$ are both homogeneous [inhomogeneous], we will say that they have the same homogeneity.

4.1.2 REMARK. By (A.8.5), the two left-invariant control systems Σ and $\widetilde{\Sigma}$ on G are l.d.f.e. if there exist open neighbourhoods $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathbf{1}$ and a diffeomorphism $\Phi : \mathcal{N}_1 \to \mathcal{N}_2$ such that for each $u \in \mathbb{R}^\ell$, $T_{\mathbf{1}}\Phi \cdot \Xi(\mathbf{1}, u) = \widetilde{\Xi}((1), \psi(u))$ where ψ is an affine map $\psi : \mathbb{R}^\ell \to \mathbb{R}^\ell$. Since clearly $\Gamma = \{\Xi(\mathbf{1}, u) \,:\, u \in \mathbb{R}^\ell\} = \{\Xi(\mathbf{1}, \psi(u)) \,:\, u \in \mathbb{R}^\ell\}$, this is equivalent to stating that two left-invariant control systems Σ and $\widetilde{\Sigma}$ on G are l.d.f.e if Σ is a reparametrization of $\widetilde{\Sigma}$ which is local state-space equivalent (A.8.2) to $\widetilde{\Sigma}$: that is,

$$\text{there exists a Lie algebra automorphism } \phi \text{ such that } \phi(\Gamma) = \widetilde{\Gamma}. \tag{4.1.1}$$

This observation will provide our method of showing l.d.f.e.

In the rest of this section, we identify the traces Γ of systems Σ in $\mathfrak{so}(1,2)$ with the affine subspaces $\mathbf{\Gamma}$ in $\mathbb{R}^{1,2}_{\wedge}$ under the hat map. We denote the image $\Xi(\mathbf{1}, u)$ by Ξ_u, and will use this simplified notation where convenient throughout this thesis .

We will use the group of Lie algebra automorphisms of $\mathbb{R}^{1,2}_{\wedge}$ established in PROPOSITION 4.1.4 to determine distinct classes of l.d.f.e. systems using the method stated in (4.1.1). Note that we limit this classification to systems of *full rank*. The systems which can be l.d.f.e. are restricted by

4.1.3 PROPOSITION. *If two systems Σ and $\widetilde{\Sigma}$ are l.d.f.e., then their traces Γ and $\widetilde{\Gamma}$ have the same dimension and the same homogeneity.*

PROOF. If Σ is l.d.f.e. to $\widetilde{\Sigma}$, then by (4.1.1) there exists a Lie algebra isomorphism ϕ such that $\phi(\Gamma) = \widetilde{\Gamma}$. But ϕ is bijective and bilinear. Consider the expansion $\Gamma = A + \langle B_1 ... B_\ell\rangle$, where $\widetilde{\Gamma} = \phi(\Gamma) = \phi(A) + \langle\phi(B_1)...\phi(B_\ell)\rangle$ and $\phi(A) = 0 \Leftrightarrow A = 0$. Thus Γ and $\widetilde{\Gamma}$ have the same homogeneity. Similarly, $\phi(B_i) = 0 \Leftrightarrow B_i = 0$, and since $\dim(\Gamma)$ is given by the number of nonzero linearly independent elements $B_1, B_2 ... B_n$, then $\dim(\Gamma) = \dim(\phi(\Gamma)) = \dim(\widetilde{\Gamma})$ and the result follows. □

We may thus restrict our investigation of equivalent systems to systems which have the same homogeneity and input number. We begin with systems Σ having traces Γ whose images $\mathbf{\Gamma}$ under the hat map are 2-dimensional linear subspaces of $\mathbb{R}^{1,2}_{\cdot}$.

4.1.1 Two-input homogeneous systems

We refer for the terminology of elliptic, hyperbolic and parabolic planes to DEFINITION 2.1.12. Define the hyperplanes $\mathbf{\Gamma}_{H(0)}$, $\mathbf{\Gamma}_{E(0)}$ and $\mathbf{\Gamma}_{P(0)}$ in $\mathbb{R}^{1,2}$, where $\mathbf{\Gamma}_{H(0)} = \{(x, y, z) \in \mathbb{R}^3_{\circ} \; : \; z = 0\}$ is hyperbolic, since $ds^2|_{\mathbf{\Gamma}_{H(0)}} = -dx^2 + dy^2$ is represented by a quadratic form with signature $(1,1,0)$, $\mathbf{\Gamma}_{E(0)} = \{(x, y, z) \in \mathbb{R}^3 \; : \; x = 0\}$ is elliptic, since $ds^2|_{\mathbf{\Gamma}_{E(0)}} = dy^2 + dz^2$ is represented by a quadratic form with signature $(0,2,0)$, and $\mathbf{\Gamma}_{P(0)} = \{(x, y, z) \in \mathbb{R}^3 \; : \; x - y = 0\}$ is parabolic, since $ds^2|_{\mathbf{\Gamma}_{P(0)}} = dz^2$ is represented by a quadratic form with signature $(0,1,1)$. These planes can be expressed as the spans $\mathbf{\Gamma}_{H(0)} = \langle \mathrm{e}_1, \mathrm{e}_2\rangle$, $\mathbf{\Gamma}_{E(0)} = \langle \mathrm{e}_2, \mathrm{e}_3\rangle$ and $\mathbf{\Gamma}_{P(0)} = \langle \mathrm{e}_1 - \mathrm{e}_2, \mathrm{e}_3\rangle$. From PROPOSITION (2.1.14), we prove

4.1.4 LEMMA. *Each 2-dimensional linear subspace of $\mathbb{R}^{1,2}_{\cdot}$ can be mapped to one of the subspaces $\mathbf{\Gamma}_{H(0)}, \mathbf{\Gamma}_{E(0)}$ or $\mathbf{\Gamma}_{P(0)}$ by an element of* $\mathsf{O}(1,2)$.

PROOF. From PROPOSITION 2.1.11, a hyperplane of $\mathbb{R}^{1,2}$ may be either parabolic, hyperbolic or elliptic, where we noted that these clasess are distinct. That is, an arbitrary linear subspace is a hyperplane $\mathbf{\Gamma}_E, \mathbf{\Gamma}_H$ or $\mathbf{\Gamma}_P$, where the subscripts E, H and P denote an elliptic, hyperbolic or parabolic plane, respectively. If $\mathbf{\Gamma}$ is elliptic, then by COROLLARY 2.1.14 there exists an element of $\mathsf{O}(1,2)$ which maps $\mathbf{\Gamma}$ to $\mathbf{\Gamma}_{E(0)}$. Similarly, if $\mathbf{\Gamma}$ is hyperbolic, then by this corollary there exists an element of $\mathsf{O}(1,2)$ which maps $\mathbf{\Gamma}$ to $\mathbf{\Gamma}_{H(0)}$, and finally if $\mathbf{\Gamma}$ is parabolic, then by this corollary there exists an element of $\mathsf{O}(1,2)$ which maps Γ to $\mathbf{\Gamma}_{P(0)}$. The result follows. □

4.1.5 LEMMA. *The matrices $g_1 = \mathrm{diag}(-1,1,1)$ and $g_3 = \mathrm{diag}(-1,-1,-1)$ are elements of* $\mathsf{O}(1,2)$.

PROOF. By definition, $\mathsf{O}(1,2) = \{g \in \mathbb{R}^{3\times 3} \; : \; gJg^\top = J\}$. But

$$g_1^\top J g_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = J$$

and

$$g_2^\top J g_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = J.$$

Thus $g_1, g_2 \in \mathsf{O}(1,2)$. □

4.1.6 LEMMA. *Any arbitrary 2-dimensional linear subspace* $\mathbf{\Gamma}$ *of* $\mathbb{R}^{1,2}_{\circlearrowleft}$ *can be mapped to one of the subspaces* $\mathbf{\Gamma}_{H(0)}, \mathbf{\Gamma}_{E(0)}$ *or* $\mathbf{\Gamma}_{P(0)}$ *by an element of* $\mathsf{SO}(1,2)$.

PROOF. From PROPOSITION 2.1.11, $\mathbf{\Gamma}$ must be either elliptic, hyperbolic or parabolic. From LEMMA 4.1.4, any arbitrary hyperbolic plane $\mathbf{\Gamma}_H$ may be mapped to $\mathbf{\Gamma}_{H(0)}$ by an element $g \in \mathsf{O}(1,2)$: that is, $g(\mathbf{\Gamma}_H) = \mathbf{\Gamma}_{H(0)}$. Assuming that $\det g = 1$, then $g \in \mathsf{SO}(1,2)$ and there is nothing to prove. Thus assume $\det g = -1$. We note that $g_1(\mathbf{\Gamma}_{H(0)}) = \mathbf{\Gamma}_{H(0)}$, since by the linearity of g_1, $g_1(\mathbf{\Gamma}_{H(0)}) = \langle g_1(\mathrm{e}_1), g_1(\mathrm{e}_2)\rangle = \langle \mathrm{e}_1, \mathrm{e}_2\rangle$. Then $\mathbf{\Gamma}_{H(0)} = g_1(\mathbf{\Gamma}_{H(0)})$, and so $g_1 g(\mathbf{\Gamma}_H) = \mathbf{\Gamma}_{H(0)}$, where $\det(g_1 g) = -1 \cdot \det g > 0$. Thus there exists an element $g' = g_1 g$ in $\mathsf{SO}(1,2)$ such that $g'(\mathbf{\Gamma}_H) = \mathbf{\Gamma}_{H(0)}$, and the result follows for hyperbolic planes.

Similarly, by LEMMA 4.1.4 any arbitrary elliptic plane $\mathbf{\Gamma}_E$ may be mapped to $\mathbf{\Gamma}_{E(0)}$ by an element $g \in \mathsf{O}(1,2)$. Assume that $\det g = -1$. But $\mathbf{\Gamma}_{E(0)} = g_1(\mathbf{\Gamma}_{E(0)})$, since the image $g_1(\mathbf{\Gamma}_{P(0)}) = \langle g_1(\mathrm{e}_2), g_1(\mathrm{e}_3)\rangle = \langle \mathrm{e}_2, \mathrm{e}_3\rangle$ by the linearity of g_1. Thus there exists $g' = g_1 g \in \mathsf{SO}(1,2)$ such that $g'(\mathbf{\Gamma}_E) = \mathbf{\Gamma}_{E(0)}$, and the result follows for elliptic planes.

Finally, by LEMMA 4.1.4 given any arbitrary parabolic plane $\mathbf{\Gamma}_P$, it may be mapped to $\mathbf{\Gamma}_{P(0)}$ by an element $g \in \mathsf{O}(1,2)$. Assume $\det g = -1$. But $\mathbf{\Gamma}_{P(0)} = \langle \mathrm{e}_1 - \mathrm{e}_2, \mathrm{e}_3\rangle$ is invariant under g_3, since $g_3(\mathbf{\Gamma}_{P(0)}) = \langle g_3(\mathrm{e}_1) - g_3(\mathrm{e}_2), g_2(\mathrm{e}_3)\rangle = \langle \mathrm{e}_1 - \mathrm{e}_2, \mathrm{e}_3\rangle$ by the linearity of g_3. Then $\mathbf{\Gamma}_{P(0)} = g_3(\mathbf{\Gamma}_{P(0)})$, and so $g_3 g(\mathbf{\Gamma}_P) = \mathbf{\Gamma}_{P(0)}$, where $\det(g_3 g) = -1 \cdot \det g > 0$. Thus there exists an element $g' = g_3 g \in \mathsf{SO}(1,2)$ such that $g'(\mathbf{\Gamma}_P) = \mathbf{\Gamma}_{P(0)}$, and the result follows for parabolic planes.

4.1.7 REMARK. We use the hat map to identify $\langle \mathrm{e}_1, \mathrm{e}_2\rangle$ with $\langle E_1, E_2\rangle$, $\langle \mathrm{e}_2, \mathrm{e}_3\rangle$ with $\langle E_2, E_3\rangle$ and $\langle \mathrm{e}_2 + \mathrm{e}_1, \mathrm{e}_3\rangle$ with $\langle E_3, E_1 - E_2\rangle$. But for any system with trace $\langle E_3, E_1 - E_2\rangle$, then

$$[E_1 - E_2, E_1 - E_2] = 0, \quad [E_3, E_3] = 0 \quad \text{and} \quad [E_1 - E_2, E_3] = E_2 - E_1. \tag{4.1.2}$$

Thus $\mathsf{Lie}(\Gamma) \subseteq \langle E_1 - E_2, E_3\rangle \subsetneq \mathbb{R}^{1,2}_{\circlearrowleft}$, and systems with trace $\Gamma = \langle E_1 - E_2, E_3\rangle$ are not full rank. Consequently under all Lie algebra automorphisms ϕ, the images $\phi(\Gamma_{P(0)}) = \Gamma_P$ cannot be full rank: thus we do not consider the class of parabolic planes in this classification.

4.1.8 PROPOSITION. *Each full-rank 2-input control affine left-invariant homogeneous system* Σ *is l.d.f.e to a system* $\Sigma_1^{(2,0)}$ *where the parametrization map* $\Xi_1^{(2,0)}(1, u) = u_1 E_2 + u_2 E_1$ *or a system* $\Sigma_2^{(2,0)}$ *where the parametrization map* $\Xi_2^{(2,0)}(1, u) = u_1 E_3 + u_2 E_2$.

PROOF. Given the full-rank system $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ with trace $\Gamma = \langle B_1, B_2\rangle \in \mathfrak{so}(1,2)$, then under the hat map, $\mathbf{\Gamma} = \langle \mathbf{a}, \mathbf{b}\rangle$ is an arbitrary 2-dimensional linear subspace in $\mathbb{R}^{1,2}_{\circlearrowleft}$, which by

(4.1.1) and (4.1.2) is either hyperbolic or elliptic. By LEMMA 4.1.6 there exists $g \in \mathsf{SO}(1,2)$ such that

$$g(\Gamma) = \langle e_2, e_1 \rangle \quad \text{or} \quad g(\Gamma) = \langle e_3, e_2 \rangle .$$

Under the inverse of the hat map, $\text{hat}^{-1}(g(\Gamma)) = \langle E_2, E_1 \rangle$ or $\text{hat}^{-1}(g(\Gamma)) = \langle E_3, E_2 \rangle$. Thus given the abitrary 2-input homogeneous control affine system Σ with trace $\Gamma = \langle B_1, B_2 \rangle \in \mathfrak{so}(1,2)$, it follows from PROPOSITION 3.4.2 that there exists a Lie algebra automorphism $\phi_g = \Psi(g)$ such that $\phi_g(\Gamma) = u_1 E_2 + u_2 E_1$ or $\phi_g(\Gamma) = u_1 E_3, u_2 E_2$. The result follows. □

4.1.9 PROPOSITION. *In PROPOSITION 4.1.8, each full-rank 2-input control affine left-invariant homogeous system $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ is is l.d.f.e to exactly one of system $\Sigma_1^{(2,0)}$ or system $\Sigma_2^{(2,0)}$.*

PROOF. Assume there exists a system $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ which is l.d.f.e to both systems $\Sigma_1^{(2,0)}$ and $\Sigma_2^{(2,0)}$. Then there exists a Lie algebra automorphism ϕ_1 such that $\phi_1(\Gamma) = \langle E_1, E_2 \rangle$, and a second such automorphism ϕ_2 such that $\phi_2(\Gamma) = \langle E_2, E_3 \rangle$. Thus by PROPOSITION 3.4.2, there exists an automorphism g in $\mathsf{SO}(1,2)$ such that $g(\Gamma) = \langle e_1, e_2 \rangle$, and an automorphism g' in $\mathsf{SO}(1,2)$ such that $g'(\Gamma) = \langle e_2, e_3 \rangle$. Then it follows that

$$\Gamma = g^{-1}(\langle e_1, e_2 \rangle) = (g')^{-1}(\langle e_2, e_3 \rangle)$$

and so $g \circ (g')^{-1}$ is a symmetry of $\mathbb{R}^{1,2}$ such that $g \circ (g')^{-1}(\langle e_2, e_3 \rangle) = \langle e_1, e_2 \rangle$. But this is impossible, since the restriction of ds^2 to the hyperbolic plane $\langle e_1, e_2 \rangle$ is $-dx^2 + dy^2$ which corresponds to a quadratic form of signature $(1,1,0)$, while the restriction of ds^2 to the elliptic plane $\langle e_2, e_3 \rangle$ is given by $dy^2 + dz^2$ which corresponds to a quadratic form of signature $(0,2,0)$, and thus these two planes cannot be isometric, a contradiction. The result follows. □

We state the results PROPOSITION 4.1.8 and 4.1.9 together as

4.1.10 THEOREM. *Under l.d.f.e, there exist two distinct classes of full-rank 2-input control affine left-invariant homogeous systems $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$, represented by the systems $\Sigma_1^{(2,0)}$ and $\Sigma_2^{(2,0)}$.*

4.1.2 Two-input inhomogeneous systems

We refer to (A.3.15)- (A.3.20) for a definition and statement of some general properties of conics.

4.1.11 REMARK. For the cone $\mathcal{K}_L = \{(x,y,z) \in \mathbb{R}^3 : x^2 = y^2 + z^2\}$, the angle θ_n of (A.3.20) is $\frac{\pi}{4}$. Then $\sin\theta_n = \frac{1}{\sqrt{2}}$ and (A.3.20) the eccentricity of the projective conics is $e = \sqrt{2}\sin\theta_p$ where θ_p (measured in the direction of positive e_1 from the positive e_2-axis) is the angle between the plane containing the conic and the plane $\langle e_2, e_3 \rangle$ of rotation of $\mathcal{K}_L$.

In the remaining sections it will be necessary for us to make a distinction between **displacement vectors** of $\mathbb{R}^{1,2}_C$, which are the vectors $\overrightarrow{ab} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$ where (a_1, a_2, a_3) is the initial point of the displacement vector and (b_1, b_2, b_3) is its final point, and the vectors $\mathbf{a}, \mathbf{b}$ identified with the points (a_1, a_2, a_3) and (b_1, b_2, b_3) in $\mathbb{R}^{1,2}_C$. In cases where the displacement vector is a unit normal vector to a given plane, we will denote it by $\overrightarrow{n}_{\mathbf{a}}$, where n gives the direction as normal to the given plane and the fact that the vector is unit-length, and a gives the initial point of the unit normal.

We continue to consider the images $\mathbf{\Gamma}$ under hat of the traces Γ of arbitrary systems Σ. Firstly we discuss the case where $\mathbf{\Gamma}$ is a parabolic plane, and then discuss the hyperbolic and elliptic cases together. We express some results of projective geometry in the context of $\mathbb{R}^{1,2}$.

4.1.12 PROPOSITION. *An elliptic plane of* DEFINITION *2.1.12 intersects the light cone $\mathcal{K}_L$ in an elliptic conic. A parabolic plane of* DEFINITION *2.1.12 intersects the light cone $\mathcal{K}_L$ in a parabolic conic.*

PROOF. Consider the arbitrary plane $\mathbf{\Gamma}$ and the intersection of the direction subspace $\mathbf{\Gamma}^0$ of $\mathbf{\Gamma}$ with the plane $\mathbf{\Gamma}_Z = \{(x,y,z) \in \mathbb{R}^3 \,:\, z = 0\}$. Since $\mathbf{\Gamma}^0$ passes through the origin, then it always has a nonempty intersection with $\mathbf{\Gamma}_Z$, which is a line $\ell = \langle \sin\theta_p \mathbf{e}_1 + \cos\theta_p \mathbf{e}_2 \rangle$ where θ_p is the angle between the plane $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ and the plane $\mathbf{\Gamma}$. Since $\langle \sin\theta_p \mathbf{e}_1 + \cos\theta_p \mathbf{e}_2 \rangle \subseteq \mathbf{\Gamma}^0$, then the vector$(r\sin\theta_p, r\cos\theta_p, 0)$ is an element of $\mathbf{\Gamma}^0$ for all $r \in \mathbb{R}$. We consider the Minkowski product

$$(r\sin\theta_p, r\cos\theta_p, 0) \odot (r\sin\theta_p, r\cos\theta_p, 0) \;=\; r^2\cos\theta_p^2 - r^2\sin\theta_p^2$$

where clearly $f(\theta_p) = r^2\cos\theta_p^2 - r^2\sin\theta_p^2$ is a monotonic decreasing function of θ_p on the interval $\theta_p \in [0, \frac{\pi}{2}]$ which is zero at $\theta_p = \frac{\pi}{4}$, since $\cos\theta$ is monotonic decreasing on this interval, and $\sin\theta$ is monotonic increasing.

Consider first the case when $\mathbf{\Gamma}$ is a parabolic plane $\mathbf{\Gamma}_P$. Then by DEFINITION 2.1.12, the restriction $ds^2|_{\mathbf{\Gamma}_P} = ds^2|_{\mathbf{\Gamma}_P^0}$ is represented by a quadratic form with siganture $(0,1,1)$: that is, for all $\mathbf{q} \in \mathbf{\Gamma}_P^0$, $\mathbf{q} \odot \mathbf{q} \geq 0$. But then if $\mathbf{\Gamma}_P^0$ makes an angle of $\theta_p \in (\frac{\pi}{4}, \frac{\pi}{2}]$ with the plane $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$, it follows that $\mathbf{\Gamma}_P^0$ contains the vector $(r\sin\theta_p, r\cos\theta_p, 0)$, where

$$(r\sin\theta_p, r\cos\theta_p, 0) \odot (r\sin\theta_p, r\cos\theta_p, 0) \;=\; r^2\cos\theta_p^2 - r^2\sin\theta_p^2 < 0. \qquad (4.1.3)$$

But then $\mathbf{\Gamma}_P^0$ admits a timelike vector, a contradiction of the fact that $ds^2|_{\mathbf{\Gamma}_P^0}$ is represented by a quadratic form with siganture $(0,1,1)$. Thus it follows that $\mathbf{\Gamma}_P$ intersects $\mathcal{K}_L$ in an elliptic or parabolic conic. Assume $\mathbf{\Gamma}_P$ intersects $\mathcal{K}_L$ in an elliptic conic; that is, the angle θ_p between $\mathbf{\Gamma}_P^0$ and the plane of rotation of $\mathcal{K}_L$ is strictly less than $\frac{\pi}{4}$. But then the angle between the plane of rotation $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ of $\mathcal{K}_L$, and the normal vector of $\mathbf{\Gamma}_P^0$ is strictly greater than $\frac{\pi}{4}$, and so the normal vector is a vector $(r\sin\theta_p, r\cos\theta_p, 0)$ such that $\theta_p \in (\frac{\pi}{4}, \frac{\pi}{2}]$, which by (4.1.3) makes it timelike. But then by PROPOSITION 2.1.10, every vector of $\mathbf{\Gamma}_P$ is spacelike, and $ds^2|_{\mathbf{\Gamma}_P}$ is represented by a quadratic form with siganture $(0,2,0)$, a contradiction. Thus the angle θ_p between $\mathbf{\Gamma}_P^0$ and the plane of rotation of $\mathcal{K}_L$ is exactly $\frac{\pi}{4}$, and $\mathbf{\Gamma}_P$ intersects $\mathcal{K}_L$ in a parabolic conic.

Next consider the elliptic planes $\mathbf{\Gamma}_E$. Since $ds^2|_{\mathbf{\Gamma}_E}$ is represented by a quadratic form with siganture $(0,2,0)$, then every vector of $\mathbf{\Gamma}_E^0$ is spacelike and so $\mathbf{\Gamma}_E^0$ admits two spacelike orthonormal vectors $\mathbf{s}, \mathbf{s}'$. Thus by COROLLARY 2.1.8 there exists a timelike vector $\mathbf{t}$ such that $\{\mathbf{s}, \mathbf{s}', \mathbf{t}\}$ is an orthonormal basis for $\mathbb{R}^{1,2}$ such that $\mathbf{t}$ is timelike. Since this vector is normal to $\mathbf{s}$ and $\mathbf{s}'$, then it is normal to to $\mathbf{\Gamma}_E^0$ and may be represented by a vector $(r\sin\theta, r\cos\theta, 0)$ where

$$(r\sin\theta, r\cos\theta, 0) \odot (r\sin\theta, r\cos\theta, 0) \;=\; \cos\theta^2 - \sin\theta^2 < 0.$$

Thus the angle θ between the plane of rotation of $\mathcal{K}_L$ and the normal vector $\mathbf{t}$ of $\mathbf{\Gamma}_E^0$ is strictly greater than $\frac{\pi}{4}$ and so the angle θ_p between the plane of rotation $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ of $\mathcal{K}_L$ and $\mathbf{\Gamma}_E^0$ is strictly less than $\frac{\pi}{4}$. Thus $\mathbf{\Gamma}_E$ intersects $\mathcal{K}_L$ in an elliptic conic. □

4.1.13 COROLLARY. *The angle θ_p between the plane $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ and any spacelike vector* $\mathbf{s}$ *is always in the interval* $[0, \frac{\pi}{4})$.

PROOF. We begin by noting that the angle between the spacelike vector $\mathbf{e}_2$ and the plane $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ is zero, and so lies in the interval $[0, \frac{\pi}{4})$. But we showed in PROPOSITION 2.1.15 that all spacelike vectors lie on hyperboloids of one sheet $\mathcal{H}_\alpha^1$, and in PROPOSITION 3.6.6 that $\mathrm{SO}(1,2)_0$ acts transitively on these hyperboloids. Thus for each $\mathbf{b} \in \mathcal{H}_\alpha^1$, there exists an element g in $\mathrm{SO}(1,2)_0$ such that $\mathbf{b} = g\mathbf{e}_2$. Further, the elements g in $\mathrm{SO}(1,2)_0$ are isometries and so conformal transformations; thus the angle between $\mathbf{b}$ and $g(\langle \mathbf{e}_2, \mathbf{e}_3 \rangle)$ is zero. But $g(\langle \mathbf{e}_2, \mathbf{e}_3 \rangle)$ is also an elliptic plane, by PROPOSITION 2.1.14. Thus the angle between $g(\langle \mathbf{e}_2, \mathbf{e}_3 \rangle)$ and $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ is in the interval $[0, \frac{\pi}{4})$, and so it follows that the angle between $\mathbf{b}$ and $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ similary lies in this interval. The result follows. □

We use this result to prove

4.1.14 PROPOSITION. *A hyperbolic plane of* DEFINITION *2.1.12 intersects the light cone $\mathcal{K}_L$ in a hyperbolic conic.*

PROOF. Given the hyperbolic plane $\mathbf{\Gamma}_H$, the restriction $ds^2|_{\mathbf{\Gamma}_H^0}$ is represented by a quadratic form with signature $(1,1,0)$. Thus $\mathbf{\Gamma}_H^0$ admits a timelike vector $\mathbf{t}$. Since the normal vector $\vec{n}_0$ to $\mathbf{\Gamma}_H^0$ at the origin must be perpendicular to every vector in $\mathbf{\Gamma}_H^0$, it follows from PROPOSITION 2.1.10 that $\vec{n}_0$ is a spacelike vector $\mathbf{s}$. But then by COROLLARY 4.1.13, the angle θ between $\mathbf{s}$ and the plane $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ must lie in the interval $[0, \frac{\pi}{4})$. But $\mathbf{s}$ is normal to every vector in $\mathbf{\Gamma}_H^0$: thus the angle between $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ and every vector admitted by $\mathbf{\Gamma}_H^0$ must be given by $-\theta + \frac{\pi}{2}$ and so lies in the interval $(\frac{\pi}{4}, \frac{\pi}{2}]$. Thus the angle θ_p between $\mathbf{\Gamma}_H^0$ and $\langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ lies in this interval, and so from (A.3.20), $\mathbf{\Gamma}_H$ intersects $\mathcal{K}_L$ in a hyperbolic conic. □

Parabolic planes

4.1.15 REMARK. From PROPOSITION 4.1.12, the parabolic planes are those planes having rulings parallel to one of the generators of the cone $\mathcal{K}$, or alternately, the parabolic planes have normal vectors which are perpendicular to one of the generators of $\mathcal{K}_L$

4.1.16 PROPOSITION. *Each parabolic plane $\mathbf{\Gamma}_P$ has a unique lightlike normal vector which lies on the cone $\mathcal{K}_L$.*

PROOF. From PROPOSITION 4.1.12, the plane $\mathbf{\Gamma}_P$ intersects $\mathcal{K}_L$ in a nondegenerate parabolic conic and as we stated in the previous remark is parallel to a generator ℓ of $\mathcal{K}_L$. Further, the axis of the parabola of intersection is a line $\ell \subseteq \mathbf{\Gamma}_P$ parallel to the generator ℓ, while then the tangent to the parabola of intersection at its turning point $\mathbf{p}$ is perpendicular to ℓ' and is tangent to the parallel of the cone $\mathcal{K}_L$ at $\mathbf{p}$. But in PROPOSITION 3.6.5, we parametrized the cone $\mathcal{K}_L$ by the patches $(\epsilon^{1i}, \mathcal{U}^{1,i})$, $i = 1, 2$, where $\epsilon^{1i}(\theta, z) = (z, z\cos\theta, z\sin\theta)$ for $i = 1, 2$. Thus for all $z \in \mathbb{R}$ and some $\theta_0 \in \mathbb{R}$, we see that

$$\epsilon^i_\theta(z, \theta_0) \bullet \epsilon(z, \theta_0) = (0, -z\sin\theta_0, z\cos\theta_0) \bullet (z, z\cos\theta_0, z\sin\theta_0) = 0$$

and the generating line $\epsilon^i(z, \theta_0)|_{z\in\mathbb{R}}$ of the cone through a point $\epsilon^i(z_0, \theta_0)$ is always perpendicular to the tangent of the parallel at that point. Thus the generator ℓ'' of $\mathcal{K}_L$ at $\mathbf{p}$ is perpendicular to the tangent line ℓ'. The generator ℓ'' cannot be parallel to the generator ℓ since rotations preserve the orgin and all generators intersect at the origin. Thus ℓ'' is perpendicular to ℓ and lies on the cone $\mathcal{K}_L$. But then the unit direction vector of ℓ'' at $\mathbf{p}$ is the unique normal to Γ_P at $\mathbf{p}$ and lies on the cone $\mathcal{K}_L$. □

Since the pair $(\vec{n}_{\mathbf{p}}, \mathbf{p})$ together can be used to construct Γ_P, and for any parabolic plane Γ_P, the point $\mathbf{p}$ of PROPOSITION 4.1.16 uniquely identifies the pair $(\vec{n}_{\mathbf{p}}, \mathbf{p})$, it follows that each parabolic plane Γ_P can be uniquely identified with the lightlike vector $\mathbf{p}$ on $\mathcal{K}_L$ described in PROPOSITION 4.1.16.

4.1.17 PROPOSITION. *The transformation* $g_2 = \operatorname{diag}(-1, -1, 1)$ *is an element of* $\mathsf{SO}(1, 2)$.

PROOF. By definition, $\mathsf{SO}(1,2) = \{g \in \mathbb{R}^{3\times 3} \, : \, gJg^\top = J, \det g = 1\}$. But

$$g_2^\top J g_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = J$$

and also expanding along the bottom row, then $\det g_2 = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} (1) = 1.$ □

4.1.18 PROPOSITION. *For any parabolic plane* Γ_P, *there exists some element* $g \in \mathsf{SO}(1,2)$ *such that* $g(\Gamma_P) = \Gamma_{P(1)}$, *where* $\Gamma_{P(1)} = \{(x, y, z) \in \mathbb{R}^3 \, : \, x - y = 1\}$.

PROOF. Since we have identified each parabolic plane Γ_P with the lightlike element $\mathbf{p}$ of PROPOSITION 4.1.16 which uniquely defines the pair $(\vec{n}_{\mathbf{p}}, \mathbf{p})$ where $\vec{n}_{\mathbf{p}}$ is lightlike, then we uniquely identify $\Gamma_{P(1)} = \{(x, y, z) \in \mathbb{R}^3 \, : \, x - y = 1\}$ with the element $\left(\frac{1}{2}, \frac{-1}{2}, 0\right) = \mathbf{p}_1$. But from PROPOSITION 3.6.6, $\mathsf{SO}(1,2)$ acts transitively on the two sheets of $\mathcal{K}_L$. Thus if $\mathbf{p}$ lies on the upper sheet, there exists some $g \in \mathsf{SO}(1,2)$ such that $g(\mathbf{p}) = \mathbf{p}_1$. Further, from the decomposition of PROPOSITION 2.2.15 of $\mathsf{SO}(1,2)$, then if $\mathbf{p}$ lies on the lower sheet, the image $g_2(\mathbf{p})$ lies on the upper sheet and there exists some $g \in \mathsf{SO}(1,2)_0$ such that $g(g_2(\mathbf{p})) = \left(\frac{1}{2}, \frac{-1}{2}, 0\right)$. Since g is linear, it follows that $g(\Gamma_P)$ is a hyperplane containing $g(\mathbf{p})$. But the unique lightlike unit normal vector $\vec{n}_{\mathbf{p}}$ of $\Gamma_{\mathbf{p}}$ is sent to a unit vector passing through $g(\mathbf{p})$ and lying within the cone $\mathcal{K}_L$. Since g is conformal, then $g(\vec{n}_{\mathbf{p}})$ is normal to the image hyperplane $g(\Gamma_P)$, and so $g(\Gamma_P)$ is a hyperplane passing through $g(\mathbf{p}) = \mathbf{p}_1$ with the unique lightlike normal vector $\vec{n}_{\mathbf{p}_1}$. By the uniquness of the parabolic plane $\Gamma_{P(1)}$ determined by the pair $(\vec{n}_{\mathbf{p}_1}, (\mathbf{p}_1))$, the result follows. □

4.1.19 THEOREM. *Each full-rank 2-input control affine left-invariant inhomogeous system Σ with trace Γ such that the image $\mathbf{\Gamma}$ of Γ under the hat map is a parabolic plane in $\mathbb{R}^{1,2}$, is l.d.f.e to the system* $\Sigma_1^{(2,1)} = (\mathsf{SO}(1,2)_0, \Xi_1^{(2,1)})$, *where* $\Xi_1^{(2,1)} = E_1 + u_1(E_1 - E_2) + u_2 E_3$.

PROOF. Given the parabolic plane $\mathbf{\Gamma} = \mathbf{a} + \langle \mathbf{b}_1, \mathbf{b}_2 \rangle$ corresponding to the trace Γ of Σ. then by PROPOSITION 4.1.18 there exists $g \in \mathsf{SO}(1,2)$ such that $g(\mathbf{\Gamma}) = \mathbf{e}_1 + \langle \mathbf{e}_3, \mathbf{e}_1 - \mathbf{e}_2 \rangle$. Under the inverse of the hat map, $\mathrm{hat}^{-1}(g(\mathbf{\Gamma})) = E_1 + \langle E_3 . E_1 - E_2 \rangle$. Thus by PROPOSITION 3.4.2 there exists a Lie algebra automorphism ϕ_g such that $\phi_g(\Gamma) = E_1 + u_1(E_1 - E_2) + u_2 E_3 = \mathrm{hat}^{-1}(g(\mathbf{\Gamma}))$. The result follows. □

Hyperbolic and elliptic planes

Given a hyperbolic or an elliptic hyperplane $\mathbf{\Gamma}$ in $\mathbb{R}^{1,2}$, we show that we can always construct the unique hyperboloid to which the plane is tangent. In the next proof we use Mathematica (C.3)

4.1.20 PROPOSITION. *An arbitrary hyperbolic plane $\mathbf{\Gamma}_H$ is a tangent plane to exactly one hyperboloid of one sheet $\mathcal{H}^1_\alpha$. An arbitrary elliptic plane is a tangent plane to exactly one hyperboloid of 2 sheets $\mathcal{H}^2_\alpha$.*

PROOF. Denote the hyperplane $\{(x, z, y) \in \mathbb{R}^3 \ : \ z = 0\}$ by $\mathbf{\Gamma}_Z$. We show that every arbitrary hyperbolic hyperplane $\mathbf{\Gamma}_H$ is tangent to exactly one hyperboloid $\mathcal{H}^1_\alpha$ by showing that $\mathbf{\Gamma}_H \cap \mathbf{\Gamma}_Z$ has the same gradient as exactly one generator $\mathcal{H}^1_\alpha \cap \mathbf{\Gamma}_Z$ of an $\mathcal{H}^i_\alpha$, considered as a surface of rotation about the $\mathbf{e}_1$-axis, at a point of intersection. We use identical steps for the case of $\mathbf{\Gamma}_E$ and $\mathcal{H}^2_\alpha$.

In the case that $\mathbf{\Gamma}$ is hyperbolic, $\mathbf{\Gamma}_H = \{(x, y. z) \in \mathbb{R}^3 \ : \ ax + by + cz = h\}$, then the intersection $\mathbf{\Gamma}_H \cap \mathbf{\Gamma}_Z = \{(x, y, z) \in \mathbb{R}^3 \ : \ ax + by = h\}$, where by PROPOSITION 4.1.14, the gradient of $\mathbf{\Gamma}_H \cap \mathbf{\Gamma}_Z$ is such that the angle of inclination of $\mathbf{\Gamma}_H$ to the $y - z$ plane is θ radians, $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$. Thus the gradient $\frac{-a}{b}$ of $\mathbf{\Gamma}_H \cap \mathbf{\Gamma}_Z$ is such that $\tan\left(\frac{\pi}{4}\right) < \frac{-a}{b} < \tan\left(\frac{3\pi}{4}\right)$.

The hyperboloids $\mathcal{H}^1_\alpha$ considered as surfaces of revolution have as generators the hyperbolas $\gamma_1 = \mathcal{H}^1_\alpha \cap \mathbf{\Gamma}_Z = \{-x^2 + y^2 = \alpha^2 \ : \ (x, y) \in \mathbb{R}^2\}$ and axes of rotation the $\mathbf{e}_1$- axis. The gradients of these generators are given by $\frac{dy}{dx} = \frac{x}{y} = \frac{x}{\pm\sqrt{\alpha^2 + x^2}}$, which lies in the interval $(-1, 1)$: the codomain of $\tan(\theta)$ where $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$. Thus we may equate the gradients of γ_1 with the gradients of $\mathbf{\Gamma}_H \cap \mathbf{\Gamma}_Z$,

$$\frac{x}{\pm\sqrt{\alpha^2 + x^2}} = \frac{dy}{dx} = \frac{-a}{b}, \quad \text{where then} \quad x = \sqrt{\frac{\alpha^2 a^2}{(b^2 - a^2)}} \text{ or } x = -\sqrt{\frac{\alpha^2 a^2}{(b^2 - a^2)}}$$

which are both defined, since $\frac{-a}{b} \in (-1. 1)$ and thus $b > a$. We choose here to consider the case where the intersection lies on the positive branch of the hyperbola, and so choose the solution $x = \sqrt{\frac{\alpha^2 a^2}{(b^2 - a^2)}}$. However, if we chose the negative square root, we could find the value α^2 which uniquely defines $\mathcal{H}^1_\alpha$ using identical calculations.

We solve for y by substituting into the equation of $\mathcal{H}^1_\alpha \cap \mathbf{\Gamma}_Z$, to give $y = \sqrt{\alpha^2 - \frac{\alpha^2 a^2}{(b^2 - a^2)}}$. Then we substitute back into the equation of $\mathbf{\Gamma}_H \cap \mathbf{\Gamma}_Z$ to give the unique value of α^2, $\alpha^2 = \frac{a^2h^2 - b^2h^2}{+a^4 + 2a^2b^2 + b^4}$. Thus we have defined the unique hyperboloid

$$\mathcal{H}^1_\alpha = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ -x^2 + y^2 + z^2 = \frac{a^2h^2 - b^2h^2}{+a^4 + 2a^2b^2 + b^4}\right\}$$

at which $\mathbf{\Gamma}_H$ has the same gradient as $\mathcal{H}^1_\alpha$ at a point of intersection. Then it follows that $\mathbf{\Gamma}_H$ is tangent to $\mathcal{H}^1_\alpha$ at that point.

In the case that $\mathbf{\Gamma}$ is elliptic, $\mathbf{\Gamma}_E = \{(x, y, z) \in \mathbb{R}^3 \ : \ ax + by + cz = h\}$, then the intersection $\mathbf{\Gamma}_E \cap \mathbf{\Gamma}_Z = \{(x, y, z) \in \mathbb{R}^3 \ : \ ax + by = h\}$, where by PROPOSITION 4.1.12, the gradient of $\mathbf{\Gamma}_E \cap \mathbf{\Gamma}_Z$ is such that the angle of inclination of $\mathbf{\Gamma}_E$ to the $y - z$ plane is θ radians, where $0 < \theta < \frac{\pi}{4}$ or $\frac{3\pi}{4} < \theta < \pi$. Thus the gradient $\frac{-a}{b}$ of $\mathbf{\Gamma}_E \cap \mathbf{\Gamma}_Z$ is such that $\frac{-a}{b} < \tan\left(\frac{\pi}{4}\right)$ or $\tan\left(\frac{3\pi}{4}\right) < \frac{-a}{b}$.

The hyperboloids $\mathcal{H}^2_\alpha$ considered as surfaces of revolution have as generators the hyperbolas $\gamma_2 = \mathcal{H}^1_\alpha \cap \mathbf{\Gamma}_Z = \{x^2 - y^2 = \alpha^2 \ : \ (x, y) \in \mathbb{R}^2\}$ and axes of rotation the $\mathbf{e}_1$-axis. The gradients of these generators are given by $\frac{dy}{dx} = \frac{x}{y} = \frac{-x}{\pm\sqrt{x^2 - \alpha^2}}$, which lies in the interval $(-\infty, -1) \cup (1, \infty)$, the codomain of $\tan(\theta)$ where $\theta \in (0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$. Thus we may equate the gradients of γ_2 with the gradients of $\mathbf{\Gamma}_E \cap \mathbf{\Gamma}_Z$,

$$\frac{-x}{\pm\sqrt{x^2 - \alpha^2}} = \frac{dy}{dx} = \frac{-a}{b}, \qquad \text{where then} \qquad x = \sqrt{\frac{\alpha^2 a^2}{(a^2 - b^2)}} \text{ or } x = -\sqrt{\frac{\alpha^2 a^2}{(a^2 - b^2)}}$$

which is always defined, since $\frac{-a}{b} \in (\infty, -1) \cup (1, \infty)$ and thus $a > b$. We choose here to consider the case where the intersection lies on the positive branch of the generating hyperbola γ_2, and so choose the solution $x = \sqrt{\frac{\alpha^2 a^2}{(a^2 - b^2)}}$. However, if we chose the negative square root, we could find the value α^2 which uniquely defines $\mathcal{H}^2_\alpha$ using identical calculations.

We solve for y by substituting into the equation of $\mathcal{H}^1_\alpha \cap \mathbf{\Gamma}_Z$, to give $y = \sqrt{\frac{\alpha^2 a^2}{(a^2 - b^2)} - \alpha^2}$. Then we substitute back into the equation of $\mathbf{\Gamma}_E \cap \mathbf{\Gamma}_Z$ to give the unique value of α^2, $\alpha^2 = \frac{b^2h^2 - a^2h^2}{-a^4 - 2a^2b^2 - b^4}$. Thus we have defined the unique hyperboloid

$$\mathcal{H}^2_\alpha = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 - y^2 - z^2 = \frac{b^2h^2 - a^2h^2}{-a^4 - 2a^2b^2 - b^4}\right\}$$

for which $\mathbf{\Gamma}_E$ has the same gradient as $\mathcal{H}^2_\alpha$ at a point of intersection. Then it follows that $\mathbf{\Gamma}_E$ is tangent to $\mathcal{H}^2_\alpha$ at that point. □

4.1.21 REMARK. For any given hyperbolic or elliptic plane $\mathbf{\Gamma} = \{(x, y, z) \in \mathbb{R}^3 \ : \ ax + by + cz = h\}$ in $\mathbb{R}^{1,2}_{\circ}$, in order to find the point $\mathbf{p}_\alpha$ of tangency to the hyperboloids of revolution of PROPOSITION 4.1.20, we substitute the calculated values for x and y in terms of α into the equation of $\mathbf{\Gamma}$, giving one of the four points

$$\left(\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}}, \sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}-\alpha^2}, \frac{h-a\sqrt{\frac{a^2\alpha^2}{b^2-a^2}}-b\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}-\alpha^2}}{c}\right) \quad (4.1.4)$$

$$\left(-\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}}, \sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}-\alpha^2}, \frac{h+a\sqrt{\frac{a^2\alpha^2}{b^2-a^2}}-b\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}-\alpha^2}}{c}\right) \quad (4.1.5)$$

$$\left(\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}}, -\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}-\alpha^2}, \frac{h-a\sqrt{\frac{a^2\alpha^2}{b^2-a^2}}+b\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}-\alpha^2}}{c}\right) \quad (4.1.6)$$

$$\left(-\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}}, -\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}-\alpha^2}, \frac{h+a\sqrt{\frac{a^2\alpha^2}{b^2-a^2}}+b\sqrt{\frac{\alpha^2a^2}{(b^2-a^2)}-\alpha^2}}{c}\right) \quad (4.1.7)$$

in the case of the hyperbolic planes Γ_H, or one of the four points

$$\left(\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}}, -\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}-\alpha^2}, \frac{h-a\sqrt{\frac{a^2\alpha^2}{(a^2-b^2)}}+b\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}-\alpha^2}}{c}\right) \quad (4.1.8)$$

$$\left(-\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}}, \sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}-\alpha^2}, \frac{h+a\sqrt{\frac{a^2\alpha^2}{a^2-b^2}}-b\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}-\alpha^2}}{c}\right) \quad (4.1.9)$$

$$\left(\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}}, \sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}-\alpha^2}, \frac{h-a\sqrt{\frac{a^2\alpha^2}{a^2-b^2}}-b\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}-\alpha^2}}{c}\right) \quad (4.1.10)$$

$$\left(-\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}}, -\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}-\alpha^2}, \frac{h+a\sqrt{\frac{a^2\alpha^2}{a^2-b^2}}+b\sqrt{\frac{\alpha^2a^2}{(a^2-b^2)}-\alpha^2}}{c}\right) \quad (4.1.11)$$

in the case of the elliptic planes Γ_E, where in both cases the positive and negative signs of the x-coefficients derive from whether the point lies on the positive or negative branch of the hyperbola of revolution, as in the proof of PROPOSITION 4.1.20.
Thus particularly to each elliptic or hyperbolic plane we can uniquely associate the point of tangency $\mathbf{p}_\alpha$ to the appropriate hyperboloid $\mathcal{H}^i_\alpha$. The point $\mathbf{p}_\alpha$ uniquely determines the pair given by $(\vec{n}_{\mathbf{p}_\alpha}, \mathbf{p}_\alpha)$ where $\vec{n}_{\mathbf{p}_\alpha}$ is the unit normal vector to $\mathcal{H}^i_\alpha$ at the point $\mathbf{p}_\alpha$. This pair in turn uniquely identifies Γ, since given $(\vec{n}_{\mathbf{p}_\alpha}, \mathbf{p}_\alpha)$ we may construct the unique plane passing through $\mathbf{p}_\alpha$ with normal $\vec{n}_{\mathbf{p}_\alpha}$, which is the plane Γ.

But this gives us a direct way to classify all systems whose traces correspond to the hyperbolic and elliptic planes in $\mathbb{R}^{1,2}_{\cdot}$. In order to express this classification, we need

4.1.22 PROPOSITION. *The hyperboloids* $\mathcal{H}^1_\alpha = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ -x^2 + y^2 + z^2 = \alpha^2\right\}$ *have tangent planes* $\alpha\mathbf{e}_2 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle$ *and the hyperboloids* $\mathcal{H}^2_\alpha = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 - y^2 - z^2 = \alpha^2\right\}$ *have tangent planes* $\alpha\mathbf{e}_2 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle$ *for* $\alpha \in \mathbb{R}$.

PROOF. As in PROPOSITION 3.6.5, we may parametrize the hyperboloids of two sheets $\mathcal{H}^2_\alpha$ by the patches $(\epsilon^{31}(z, \theta), \mathcal{U}^{31})$ and $(\epsilon^{32}(z, \theta), \mathcal{U}^{32})$, where $\epsilon^{3i}(r, \theta) = (\cosh r, \sinh r \cos\theta, \sinh r \sin\theta)$

and $\mathcal{U}^{31} = \mathbb{R} \times (-\pi, \pi)$, $\mathcal{U}^{32} = \mathbb{R} \times (0, 2\pi)$. The tangent plane at $(u, v) = (0, 0)$ is the span of orthogonal vectors

$$\epsilon_u^{31}(0,0) \quad \text{and} \quad \epsilon_v^{31}(0,0), \quad \text{where } \epsilon_u^{31}(0,0) = (0,1,0) \quad \text{and} \quad \epsilon_v^{31}(0,0) = (0,0,1)$$

and the plane passes through $\epsilon^{31}(0,0) = (\alpha, 0, 0)$. Thus the plane $\alpha\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ is tangent to $\mathcal{H}_\alpha^2$ at the point $(\alpha, 0, 0)$. In PROPOSITION 3.6.5, we parametrized $\mathcal{H}_\alpha^1$ by the patches $(\epsilon^{21}(z, \theta), \mathcal{U}^{21})$ and $(\epsilon^{22}(z, \theta), \mathcal{U}^{22})$, where $\epsilon^{2i}(r, \theta) = (\alpha \sinh r, \alpha \cosh r \cos\theta, \alpha \cosh r \sin\theta)$ and $\mathcal{U}^{21} = \mathbb{R} \times (-1, \pi)$, $\mathcal{U}^{22} = \mathbb{R} \times (0, 2\pi)$. Thus the tangent space at $(u, v) = (0, 0)$ is spanned by the orthogonal vectors

$$\epsilon_u^{21}(0,0) \quad \text{and} \quad \epsilon_v^{21}(0,0), \quad \text{where } \epsilon_u^{21}(0,0) = (1,0,0), \quad \text{and} \quad \epsilon_v^{21}(0,0) = (0,0,1)$$

and the plane passes through the point $\epsilon^{21}(0,0) = (0, \alpha, 0)$. Thus the plane $\alpha\mathbf{e}_2 + \langle \mathbf{e}_1, \mathbf{e}_3 \rangle$ is tangent to $\mathcal{H}_\alpha^1$ at the point $(0, \alpha, 0)$. □

Denote the specific hyperbolic planes $\alpha\mathbf{e}_2 + \langle \mathbf{e}_1, \mathbf{e}_3 \rangle$ by $\mathbf{\Gamma}_{H(\alpha)}$, and the specific elliptic planes $\alpha\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ by $\mathbf{\Gamma}_{E(\alpha)}$.

4.1.23 PROPOSITION. *Any hyperbolic [elliptic] hyperplane of $\mathbb{R}^{1,2}_{\circ}$ may be mapped by an element of* $\mathsf{SO}(1,2)$ *to exactly one of the planes $\mathbf{\Gamma}_{H(\alpha)}$ [$\mathbf{\Gamma}_{E(\alpha)}$] where $\alpha > 0$.*

PROOF. In PROPOSITION 4.1.20 we showed that each hyperbolic [elliptic] plane is tangent to exactly one hyperboloid $\mathcal{H}_\alpha^1$ [$\mathcal{H}_\alpha^2$], and denoted the point of tangency by $\mathbf{p}_\alpha$ (4.1.4)-(4.1.11). But by the transitivity of $\mathsf{SO}(1,2)_0$ on $\mathcal{H}_\alpha^1$ [$\mathcal{H}_\alpha^2$] by PROPOSITION 2.2.27, [PROPOSITION 3.6.6], there exists an element g of $\mathsf{SO}(1,2)_0$ mapping $\mathbf{p}_\alpha$ to the point $(\alpha, 0, 0)$ [$(0, \alpha, 0)$] on this hyperboloid.We note that we may restrict to values $\alpha > 0$, since if $g(\mathbf{\Gamma})$ intersects $\mathcal{H}_\alpha^2$ [$\mathcal{H}_\alpha^1$] in the point $(-\alpha, 0, 0)$ [$(0, -\alpha, 0)$] for positive α, then the reflection g_2 of PROPOSITION 4.1.17 maps $(-\alpha, 0, 0)$ to $(\alpha, 0, 0,)$ [$(0, -\alpha, 0)$ to $(0, \alpha, 0)$], where $g_2 g \in \mathsf{SO}(1,2)$ is an automorphism of $\mathbb{R}^{1,2}_{\circ}$. Since $\mathsf{SO}(1,2)$ is linear, then the images $g(\mathbf{\Gamma})$ and $g_2 \circ g(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$ are hyperplanes of $\mathbb{R}^{1,2}$, where $g(\mathbf{\Gamma})$ contains $g(\mathbf{p}_\alpha)$ and $g_2 g(\mathbf{\Gamma})$ contains $g_2 g(\mathbf{p}_\alpha)$, and either $g(\mathbf{p}_\alpha)$ or $g_2 g(\mathbf{p}_\alpha)$ is the point $(\alpha, 0, 0)$ [$(0, \alpha, 0)$] for positive α. Finally, by the linearity of g and $g_2 g$ and the fact that they preserve the hyperboloid $\mathcal{H}_\alpha^2$ [$\mathcal{H}_\alpha^1$], it follows that

$$g(\mathcal{H}_\alpha^2 \cap \mathbf{\Gamma}) = g(\mathcal{H}_\alpha^2) \cap g(\mathbf{\Gamma}) = \{g(\mathbf{p}_\alpha)\} \quad \text{and} \quad g_2 g(\mathcal{H}_\alpha^1 \cap \mathbf{\Gamma}) = g_2 g(\mathcal{H}_\alpha^1) \cap g_2 g(\mathbf{\Gamma}) = \{g_1 \circ g(\mathbf{p}_\alpha)\}$$

that is, either $g(\mathbf{\Gamma})$ or $g_2 g(\mathbf{\Gamma})$ is tangent to $\mathcal{H}_\alpha^1$ [$\mathcal{H}_\alpha^2$] at $(\alpha, 0, 0)$ [$(0, \alpha, 0)$] for positive α. But the tangent plane to $\mathcal{H}_\alpha^1$ [$\mathcal{H}_\alpha^2$] at $(\alpha, 0, 0)$ [$(0, \alpha, 0)$] is unique, and by PROPOSITION 4.1.22 it is the hyperplane $\alpha\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ [$\alpha\mathbf{e}_2 + \langle \mathbf{e}_1, \mathbf{e}_3 \rangle$]. Thus $g(\mathbf{\Gamma}) = \alpha\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ [$g(\mathbf{\Gamma}) = \alpha\mathbf{e}_2 + \langle \mathbf{e}_1, \mathbf{e}_3 \rangle$] or $g_2 \circ g(\mathbf{\Gamma}) = \alpha\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$ [$g_2 \circ g(\mathbf{\Gamma}) = \alpha\mathbf{e}_2 + \langle \mathbf{e}_1, \mathbf{e}_3 \rangle$] and it follows that there exists an element of $\mathsf{SO}(1,2)$ mapping $\mathbf{\Gamma}$ to one of the planes $\mathbf{\Gamma}_{H(\alpha)}$ [$\mathbf{\Gamma}_{E(\alpha)}$] where $\alpha > 0$. □

We use this result to prove

4.1.24 PROPOSITION. *Any full-rank 2-input inhomogeneous control affine left-invariant system Σ having a trace Γ such that the image $\mathbf{\Gamma}$ of Γ under the hat map is a hyperbolic or an elliptic plane* *is l.d.f.e to one of the systems of the family* $\Sigma^{(2,1)}_{2,\alpha} = (\mathsf{SO}(1,2)_0, \Xi^{(2,1)}_{2,\alpha})$ *where the dynamics* $\Xi^{(2,1)}_{2,\alpha} = \alpha E_3 + u_1E_2 + u_2E_1$, *or of the family* $\Sigma^{(2,1)}_{3,\alpha} = (\mathsf{SO}(1,2)_0, \Xi^{(2,1)}_{3,\alpha})$ *where the dynamics* $\Xi^{(2,1)}_{3,\alpha} = \alpha E_1 + u_1E_3 + u_2E_2$ *for* $\alpha \in \mathbb{R}^+$.

PROOF. Given the hyperbolic [elliptic] plane $\mathbf{\Gamma} = \mathbf{a} + \langle \mathbf{b}_1, \mathbf{b}_2\rangle$ corresponding to Γ, then by PROPOSITION 4.1.23 there exists $g \in \mathsf{SO}(1,2)$ such that $g(\mathbf{\Gamma}) = \alpha\mathbf{e}_3 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle$ $[\alpha\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle]$. Under the inverse of the hat map, $\text{hat}^{-1}(g(\mathbf{\Gamma})) = \alpha E_3 + \langle E_1, E_2\rangle$ $[\text{hat}^{-1}(g(\mathbf{\Gamma})) = \alpha E_1 + \langle E_2, E_3\rangle]$, and so by PROPOSITION 3.4.2 there exists an element $\phi_g = \Psi(g) \in \mathsf{Aut}(\mathfrak{so}(1,2)$ such that $\phi_g(\Gamma) = \alpha E_3 + \langle E_1, E_2\rangle = \text{hat}^{-1}(g(\mathbf{\Gamma}))$ $[\ \phi_g(\Gamma) = \alpha E_1 + \langle E_2, E_3\rangle = \text{hat}^{-1}(g(\mathbf{\Gamma}))]$. The result follows. □

4.1.25 PROPOSITION. *In* PROPOSITION *4.1.27, each full-rank 2-input control affine left-invariant inhomogeous system* $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ *is is l.d.f.e to* exactly *one of system* $\Sigma^{(2,1)}_1, \Sigma^{(2,1)}_{2,\alpha}$ *or* $\Sigma^{(2,1)}_{3,\alpha}$.

PROOF. The representation of the restrictions $ds^2|_{\mathbf{\Gamma}_{\mathbf{H}(\alpha)}}$, $ds^2|_{\mathbf{\Gamma}_{\mathbf{E}(\alpha)}}$ and $ds^2|_{\mathbf{\Gamma}_{\mathbf{P}(1)}}$ as quadratic forms have signatures $(1,1,0)$, $(0,0,2)$ and $(0,1,1)$, respectively. Assume intially that there exists a system $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ which is l.d.f.e to both a system $\Sigma^{(2,1)}_{1,\alpha}$ and $\Sigma^{(2,1)}_{2,\alpha'}$. Then there exists a Lie algebra automorphism ϕ_1 such that $\phi_1(\Gamma) = \alpha E_3 + \langle E_1, E_2\rangle$, and a second such automorphism ϕ_2 such that $\phi_2(\Gamma) = \alpha' E_1 + \langle E_2, E_3\rangle$, $\alpha, \alpha' > 0$. Then by PROPOSITION 3.4.2, there exist elements $g, g' \in \mathsf{SO}(1,2)$ such that

$$g(\mathbf{\Gamma}) = \alpha\mathbf{e}_3 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle \quad \text{and} \quad g'(\mathbf{\Gamma}) = \alpha'\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle .$$

Then $\mathbf{\Gamma} = g(\alpha\mathbf{e}_3 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle) = g'(\alpha'\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle)$, and so $g \circ (g')^{-1}$ is a symmetry of $\mathbb{R}^{1,2}$ such that $g \circ (g')^{-1}(\alpha\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle = \alpha'\mathbf{e}_3 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle$.

But this is impossible, since the restriction of ds^2 to the hyperbolic plane $\alpha\mathbf{e}_3 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle$ is given by $-dx^2 + dy^2$ which is represented by a quadratic form of signature $(1,1,0)$, while the restriction of ds^2 to the elliptic plane $\alpha'\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle$ is given by $dy^2 + dz^2$ which is represented by a quadratic form of signature $(0,1,1)$. Thus these two planes cannot be isometric, a contradiction. The remaining cases ($\Sigma^{(2,1)}_1$ is l.d.f.e to $\Sigma^{(2,1)}_{3,\alpha}$ or $\Sigma^{(2,1)}_{2,\alpha}$ is l.d.f.e to $\Sigma^{(2,1)}_{3,\alpha}$) follow identical steps.

Secondly assume that $\Sigma^{(2,1)}_{2,\alpha}$ is l.d.f.e. to $\Sigma^{(2,1)}_{2,\alpha}$ for $\alpha, \alpha' \in \mathbb{R}^+, \alpha \neq \alpha'$. Then there exists a $g \in \mathsf{SO}(1,2)$ such that $g\Xi^{(2,1)}_{2,\alpha} = \Xi^{(2,1)}_{2,\alpha'}$ and so $g(\mathbf{\Gamma}) = \alpha\mathbf{e}_3 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle = \alpha'\mathbf{e}_3 + \langle \mathbf{e}_1, \mathbf{e}_2\rangle$. Then particularly $g\alpha\mathbf{e}_3 = \alpha'\mathbf{e}_3$, where $\|\alpha\mathbf{e}_3\| = \alpha^2 \neq (\alpha')^2 = \|\alpha'\mathbf{e}_3\|$ since $\alpha, \alpha' > 0, \alpha \neq \alpha'$. But this contradicts the fact that g is an isometry. Thus $\Sigma^{(2,1)}_{1,\alpha}$ cannot be l.d.f.e. to $\Sigma^{(2,1)}_{1,\alpha}$ if $\alpha \neq \alpha'$. The case of the family $\Sigma^{(2,1)}_{3,\alpha}$ follows identical steps. □

We may then state the results of PROPOSITION 4.1.26 and PROPOSITION 4.1.25 together as

4.1.26 THEOREM. *Any full-rank 2-input inhomogeneous control affine left-invariant system* Σ *having a trace* Γ *such that the image* $\mathbf{\Gamma}$ *of* Γ *under the hat map is a hyperbolic or an elliptic plane is l.d.f.e to* exactly *one of the systems of the family* $\Sigma^{(2,1)}_{2,\alpha} = (\mathsf{SO}(1,2)_0, \Xi^{(2,1)}_{2,\alpha})$ *where the dynamics* $\Xi^{(2,1)}_{2,\alpha} = \alpha E_3 + u_1E_2 + u_2E_1$, *or one of the systems of the family* $\Sigma^{(2,1)}_{3,\alpha} = (\mathsf{SO}(1,2)_0, \Xi^{(2,1)}_{3,\alpha})$ *where the dynamics* $\Xi^{(2,1)}_{3,\alpha} = \alpha E_1 + u_1E_3 + u_2E_2$ *for* $\alpha \in \mathbb{R}^+$.

We collect the results of THEOREMS 4.1.19 and 4.1.26 in

4.1.27 THEOREM. *Any full-rank 2-input inhomogeneous control affine left-invariant system Σ is l.d.f.e to one of the systems of the family $\Sigma^{(2,0)}_{2,\alpha}$ the family $\Sigma^{(2,0)}_{3,\alpha}$ or the system $\Sigma^{(2,0)}$, for $\alpha \in \mathbb{R}^+$.*

PROOF. From PROPOSITION 2.1.11, the image $\mathbf{\Gamma}$ of the trace Γ of an arbitrary full-rank 2-input inhomogeneous control affine system Σ is a hyperplane of $\mathbb{R}^{1,2}$ not passing through the origin which is exactly hyperbolic, elliptic or parabolic. If $\mathbf{\Gamma}$ is hyperbolic or elliptic, then by THEOREM 4.1.26, $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ is l.d.f.e to one of the systems $\Sigma^{(2,0)}_{2,\alpha}$ or $\Sigma^{(2,0)}_{3,\alpha}$. Similarly, if $\mathbf{\Gamma}$ is parabolic, then by THEOREM 4.1.19 it follows that $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ is l.d.f.e to the system $\Sigma^{(2,0)}_1$, where $\Xi^{(2,0)}_1 = E_1 + \langle E_3, E_1 - E_2\rangle$. The result follows. □

4.1.3 Single-input inhomogeneous systems

4.1.28 REMARK. In THEOREMS 4.1.10 and 4.1.27 we showed that all hyperplanes of $\mathbb{R}^{1,2}$ are of the form $g\mathbf{\Gamma}_{H(\alpha)}, g\mathbf{\Gamma}_{E(\alpha)}, g\mathbf{\Gamma}_{P(1)}, g\mathbf{\Gamma}_{H(0)}$ or $g\mathbf{\Gamma}_{E(0)}$, where $\alpha > 0$ and $g \in \mathsf{SO}(1,2)$. From PROPOSITION 2.2.24, each element of $\mathsf{SO}(1,2)$ is a product bkg_2 or bk, and each element of $\mathsf{SO}(1,2)_0$ is a product bk. Consider the hyperplanes $g\mathbf{\Gamma}_{H(\alpha)}$ and $g\mathbf{\Gamma}_{E(\alpha)}$, where $\alpha > 0$. Taking $g \in \mathsf{SO}(1,2)_0$, then

$$g(\alpha \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle) = \alpha g(\mathbf{e}_1) + \langle g(\mathbf{e}_2), g(\mathbf{e}_3)\rangle)$$

where $\alpha g(\mathbf{e}_1)$ lies on the upper sheet of the hyperboloid $\mathcal{H}^2_\alpha$, since by REMARK 2.2.13 the elements of $\mathsf{SO}(1,2)_0$ preseve the sheets of these hyperboloids. Since this is true for all $g \in \mathsf{SO}(1,2)_0$, then the only element of $\mathsf{SO}(1,2)$ which sends $\alpha g(\mathbf{e}_1)$ to the lower sheet of $\mathcal{H}^2_\alpha$ is the element g_2. Thus given the plane $\mathbf{\Gamma}_{E(\alpha)}$ expressed as $\alpha\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle$ where $\alpha > 0$, the image $g(\mathbf{\Gamma}_{E(\alpha)}) = \alpha'\mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{e}_3\rangle$ where $\alpha' < 0$ is possible if and only if $g = bkg_2 \in \mathsf{SO}(1,2)$. Thus we may obtain all elliptic planes as images $g(\mathbf{\Gamma}_{E(\alpha)})$ where $g \in \mathsf{SO}(1,2)_0$ if and only if we allow $\alpha \in \mathbb{R}$.

For parabolic planes, $g(\lambda\mathbf{e}_1 + \langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3\rangle) = \lambda g(\mathbf{e}_2) + \langle g(\mathbf{e}_1 + \mathbf{e}_2), g(\mathbf{e}_3)\rangle$ where $\lambda = \pm 1$, g in $\mathsf{SO}(1,2)_0$ and where $g(\mathbf{e}_1)$ lies on the positive branch of the hyperbola of revolution of $\mathcal{H}^2_\alpha$, since we noted in REMARK 2.2.13 that the elements of $\mathsf{SO}(1,2)_0$ preseve the sheets of these hyperboloids. Since this is true for all $g \in \mathsf{SO}(1,2)_0$, then the only element of $\mathsf{SO}(1,2)$ which sends $g(\mathbf{e}_1)$ to the lower sheet of $\mathcal{H}^2_1$ is the element g_2. Given $\mathbf{\Gamma}_{P(1)} = \mathbf{e}_1 + \langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3\rangle$, then the image $g(\mathbf{\Gamma}_{P(1)}) = -\mathbf{e}_1 + \langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3\rangle$ where $\alpha' < 0$ is possible if and only if $g = bkg_2 \in \mathsf{SO}(1,2)$. Thus we may obtain all parabolic planes as images $g(\mathbf{\Gamma}_{P(1)})$ or $g(\mathbf{\Gamma}_{P(-1)})$ where $g \in \mathsf{SO}(1,2)_0$.

For the hyperbolic planes, the element $k(\pi)$ maps $\mathbf{e}_2$ to $-\mathbf{e}_2$, and so all hyperbolic planes are images $g(\mathbf{\Gamma}_{H(\alpha)})$ where $g \in \mathsf{SO}(1,2)_0$ and $\alpha > 0$.

Thus we may modify the statements of THEOREMS 4.1.10 and 4.1.27 to state that *all hyperplanes of $\mathbb{R}^{1,2}$ are of the form $g\mathbf{\Gamma}_{H(\alpha)}, g\mathbf{\Gamma}_{E(\alpha)}, g\mathbf{\Gamma}_{P(1)}$ or $g\mathbf{\Gamma}_{P(-1)}$ where $\alpha \in \mathbb{R}$ and $g \in \mathsf{SO}(1,2)_0$.* We will use this alternative classification in the proofs of this section.

In order to begin classification of single-input systems using this classification of 2-input systems, we require the well-known result of Euclidean geometry, that

4.1.29 PROPOSITION. *In $\mathbb{R}^3$, any line $\ell = \mathbf{a} + \langle \mathbf{b} \rangle$ can be expressed as the intersection of two hyperplanes.*

We then use the expression as images $g\mathbf{\Gamma}_{H(\alpha)}, g\mathbf{\Gamma}_{E(\alpha)}, g\mathbf{\Gamma}_{P(1)}$ or $g\mathbf{\Gamma}_{P(-1)}$ where $\alpha \in \mathbb{R}$ and $g \in \mathsf{SO}(1,2)_0$ of hyperplanes in $\mathbb{R}^{1,2}$ as in REMARK 4.1.28 to establish

4.1.30 PROPOSITION. *Any line ℓ in $\mathbb{R}^{1,2}$ can be mapped by an element of $\mathsf{SO}(1,2)_0$ to one of the intersections*

$$\begin{array}{llll} \text{(Case 1):} & \mathbf{\Gamma}_{E(\alpha)} \cap g(\mathbf{\Gamma}_{E(\beta)}) & \text{(Case 4):} & \mathbf{\Gamma}_{P(\pm1)} \cap g(\mathbf{\Gamma}_{H(\alpha)}) \\ \text{(Case 2):} & \mathbf{\Gamma}_{H(\alpha)} \cap g(\mathbf{\Gamma}_{H(\beta)}) & \text{(Case 5):} & \mathbf{\Gamma}_{P(\pm1)} \cap g(\mathbf{\Gamma}_{E(\alpha)}) \\ \text{(Case 3):} & \mathbf{\Gamma}_{H(\alpha)} \cap g(\mathbf{\Gamma}_{E(\beta)}) & \text{(Case 6):} & \mathbf{\Gamma}_{P(\pm1)} \cap g(\mathbf{\Gamma}_{P(1)}) \end{array}$$

for some $\alpha, \beta \in \mathbb{R}, g \in \mathsf{SO}(1,2)_0$.

PROOF. By PROPOSITION 4.1.29, each line $\ell \subseteq \mathbb{R}^{1,2}_{\circlearrowleft}$ may be expressed as the intersection of two hyperplanes $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$. But as we noted in the previous remark, by PROPOSITIONS 4.1.18 and 4.1.23 there exist elements $g, g' \in \mathsf{SO}(1,2)_0$ such that

$$\mathbf{\Gamma}_1 = g(\mathbf{\Gamma}_{H(\alpha)}), g(\mathbf{\Gamma}_{E(\alpha)}) \text{ or } g\left(\mathbf{\Gamma}_{P(\pm1)}\right) \quad \text{and} \quad \mathbf{\Gamma}_2 = g'(\mathbf{\Gamma}_{H(\beta)}), g'(\mathbf{\Gamma}_{E(\beta)}) \text{ or } g'(\mathbf{\Gamma}_{P(\pm1)})$$

where $\alpha, \beta \in \mathbb{R}$. Let $\mathbf{\Gamma}_M$ denote $g^{-1}(\mathbf{\Gamma}_1)$ and $\mathbf{\Gamma}_N$ denote $(g')^{-1}(\mathbf{\Gamma}_2)$. Thus it follows that the intersection $\ell = g(\mathbf{\Gamma}_M) \cap g'(\mathbf{\Gamma}_N) = g'(g'^{-1}g\mathbf{\Gamma}_M \cap \mathbf{\Gamma}_N)$. Thus there exists $g'' = g'^{-1}g \in \mathsf{SO}(1,2)$ such that the image $g'^{-1}(\ell) = g''(\mathbf{\Gamma}_M) \cap \mathbf{\Gamma}_N$. But then since $\mathbf{\Gamma}_M = \mathbf{\Gamma}_{H(\alpha)}, \mathbf{\Gamma}_{E(\alpha)}$ or $\mathbf{\Gamma}_{P(\pm1)}$ and similarly $\mathbf{\Gamma}_N = \mathbf{\Gamma}_{H(\beta)}, \mathbf{\Gamma}_{E(\beta)}$ or $\mathbf{\Gamma}_{P(\pm1)}$, it follows that there exists an element g'' of $\mathsf{SO}(1,2)_0$ such that $g''(\ell)$ lies in one of the given intersections. □

We find explicit expressions for the lines in each of the 6 cases.

Case 1 - $\mathbf{\Gamma}_{H(\alpha)} \cap g\left(\mathbf{\Gamma}_{H(\beta)}\right)$

From THEOREM 2.2.23 each element in $\mathsf{SO}(1,2)_0$ is a product bk where b is some Lorentz boost in a plane containing $\mathbf{e}_1$ and k is a Euclidean rotation about the $\mathbf{e}_1$-axis. Further, in PROPOSITION 3.6.7 we showed that the planes $\mathbf{\Gamma}_{H(\beta)}$ are preserved by b_1. Thus the intersection $\mathbf{\Gamma}_{H(\alpha)} \cap g(\mathbf{\Gamma}_{H(\beta)})$ will not vary if $g = b_1$. Since from THEOREM 2.2.23 each Lorentz boost is a product $k^{-1}b_1k$, we consider the case $g = k$ only since the image of $\mathbf{\Gamma}_{H(\alpha)}$ under all other Lorentz boosts may be found by applying b_1 and then k^{-1} to the images $k\left(\mathbf{\Gamma}_{H(\alpha)}\right)$. The plane $k\left(\mathbf{\Gamma}_{H(\beta)}\right) = \beta(k(\mathbf{e}_3)) + \langle k(\mathbf{e}_1), k(\mathbf{e}_2)\rangle$ can be expressed as the image given by set $k\left(\mathbf{\Gamma}_{H(\beta)}\right) = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ z\cos\theta - y\sin\theta = x\right\}$ where $\theta \neq n\pi$ (for $k = \pm\mathbf{1}$, then $k\left(\mathbf{\Gamma}_{H(\beta)}\right)$ is parallel to $\mathbf{\Gamma}_{H(\alpha)}$). Thus

$$(x, y, z) \in k\left(\mathbf{\Gamma}_{H(\beta)}\right) \cap \mathbf{\Gamma}_{H(\alpha)} \quad \Leftrightarrow \quad -y\sin\theta = \beta - \alpha\cos\theta \quad \Rightarrow \quad y = \frac{\alpha\cos\theta - \beta}{\sin\theta}.$$

Thus the line $\mathbf{\Gamma}_{H(\alpha)} \cap k\left(\left(\mathbf{\Gamma}_{H(\beta)}\right)\right)$ passes through the point $\left(0, \frac{\alpha\cos\theta - \beta}{\sin\theta}, \alpha\right)$ with direction vector $(0, -\sin\theta, \cos\theta) \wedge (0, 0, 1) = (-\sin\theta, 0, 0)$. Thus the line $\ell = \left(0, \frac{\alpha\cos\theta - \beta}{\sin\theta}, \alpha\right) + \langle \mathbf{e}_1 \rangle$. However, using Mathematica (C.3) to apply a rotation $k(\widetilde{\theta})$ to ℓ, then

$$k(\widetilde{\theta})(\ell) = (t, \cos\widetilde{\theta}(-\beta + \alpha\cos\theta)\csc\theta - \alpha\sin\widetilde{\theta}, \alpha\cos\widetilde{\theta} + (-\beta + \alpha\cos\theta)\csc\theta\sin\widetilde{\theta})|_{t\in\mathbb{R}}.$$

In the code (C.3) we note that for each $\alpha, \beta \in \mathbb{R}$ there exists $\widetilde{\theta}_1 \in \mathbb{R}$ such that the image under $k(\widetilde{\theta})$ of ℓ is

$$(t, \cos\widetilde{\theta}_2(-\beta + \alpha\cos\theta)\csc\theta - \alpha\sin\widetilde{\theta}_2, \alpha\cos\widetilde{\theta}_2 + (-\beta + \alpha\cos\theta)\csc\theta\sin\widetilde{\theta}_2)|_{t\in\mathbb{R}} = t\mathrm{e}_1 + c'\mathrm{e}_3|_{t\in\mathbb{R}}.$$

Thus there always exists a $g \in \mathsf{SO}(1,2)_0$ such that $g(\ell) = g(\mathbf{\Gamma}_{H(\alpha)} \cap g(\mathbf{\Gamma}_{H(\beta)}))$ is the line $g(\ell) = c\mathrm{e}_1 + \langle \mathrm{e}_3 \rangle$, for $c \in \mathbb{R}$.

Case 2 - $\mathbf{\Gamma}_{E(\alpha)} \cap g(\mathbf{\Gamma}_{E(\beta)})$

As in the previous case, we express each element of $\mathsf{SO}(1,2)$ as bk. Since from PROPOSITION 4.1.19 the planes $\mathbf{\Gamma}_{E(\alpha)}$ are preserved by any Euclidean rotation k, then the intersection $\mathbf{\Gamma}_{E(\alpha)} \cap g(\mathbf{\Gamma}_{E(\beta)})$ will not vary if $g = k$. Thus we consider g as a Lorentz boost only, particularly the case where b is the Lorentz boost b_1 of DEFINITION 3.1.12, since the image of $\mathbf{\Gamma}_{E(\beta)}$ under all other Lorentz boosts can be seen from THEOREM 2.2.23 to be obtained by applying first a rotation k to $\mathbf{\Gamma}_{E(\beta)}$, then applying b_1 and finally k^{-1}. We express $b_1(\mathbf{\Gamma}_{E(\beta)}) = \beta(b_1(\mathrm{e}_1)) + \langle b_1(\mathrm{e}_2), b_2(\mathrm{e}_3)\rangle$ as the set $b_1(\mathbf{\Gamma}_{E(\beta)}) = \{(x,y,z) \in \mathbb{R}^3 \ : \ \cosh\theta x - \sinh\theta y = \beta\}$, where $\theta \neq 0$ (since if $b_1 = \mathbf{1}$ and then $b_1(\mathbf{\Gamma}_{E(\beta)})$ is parallel to $\mathbf{\Gamma}_{E(\alpha)}$). We express the plane $\mathbf{\Gamma}_{E(\beta)} = \{(x,y,z) \in \mathbb{R}^3 \ : \ x = \beta\}$. Thus it follows that

$$(x,y,z) \in b_1(\mathbf{\Gamma}_{E(\alpha)}) \cap \mathbf{\Gamma}_{E(\beta)} \quad \Leftrightarrow \quad -\sinh\theta y + \cosh\theta\beta = \alpha \quad \Rightarrow \quad y = \frac{\beta\cosh\theta - \alpha}{\sinh\theta}$$

and the line $\mathbf{\Gamma}_{E(\alpha)} \cap b_1(\mathbf{\Gamma}_{E(\beta)})$ passes through the point $\left(\beta, \frac{\beta\cosh\theta-\alpha}{\sinh\theta}, 0\right)$ with direction vector $(\cosh\theta, \sinh\theta, 0) \wedge (1,0,0) = (0,0,-\sinh\theta)$. Thus $\ell = \left(\beta, \frac{\beta\cosh\theta-\alpha}{\sinh\theta}, 0\right) + \langle \mathrm{e}_3\rangle$. Consider the image $b_1(\widetilde{\theta})(\ell) = (\beta\cosh\widetilde{\theta} + (-\alpha + \beta\cos\theta)\csc\theta\sinh\widetilde{\theta}, (-\alpha + \beta\cos\theta)\cosh\widetilde{\theta}\csc\theta + \beta\sinh\widetilde{\theta}, t)|_{t\in\mathbb{R}}$. Using Mathematica (C.3), we note that for each α, β there exists either $\widetilde{\theta}_1 \in \mathbb{R}$ such that

$$(\beta\cosh\widetilde{\theta}_1 + (-\alpha + \beta\cos\theta)\csc\theta\sinh\widetilde{\theta}_1, (-\alpha + \beta\cos\theta)\cosh\widetilde{\theta}_1\csc\theta + \beta\sinh\widetilde{\theta}_1, t)|_{t\in\mathbb{R}} = \langle \mathrm{e}_3\rangle + c\mathrm{e}_1$$

or

$$(\beta\cosh\widetilde{\theta}_2 + (-\alpha + \beta\cos\theta)\csc\theta\sinh\widetilde{\theta}_2, (-\alpha + \beta\cos\theta)\cosh\widetilde{\theta}_2\csc\theta + \beta\sinh\widetilde{\theta}_2, t)|_{t\in\mathbb{R}} = \langle \mathrm{e}_3\rangle + c'\mathrm{e}_2$$

and further that these angles $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ never exist simultaneously for the same combination of α, β. Thus there always exists a $g' \in \mathsf{SO}(1,2)_0$ such that $g'(\ell) = \mathbf{\Gamma}_{E(\alpha)} \cap g(\mathbf{\Gamma}_{E(\beta)})$ is the line $c\mathrm{e}_1 + \langle \mathrm{e}_3\rangle$, or $c'\mathrm{e}_2 + \langle \mathrm{e}_3\rangle$ for $c, c' \in \mathbb{R}$.

Case 3 - $\mathbf{\Gamma}_{H(\alpha)} \cap g(\mathbf{\Gamma}_{E(\beta)})$

In PROPOSITION 4.1.19, we showed that the planes $\mathbf{\Gamma}_{E(\beta)}$ are preserved by any Euclidean rotation k, and so the intersection $\mathbf{\Gamma}_{H(\alpha)} \cap g(\mathbf{\Gamma}_{E(\beta)})$ will not vary if $g = k$. Thus we consider g only as the Lorentz boost b_1. The image of $\mathbf{\Gamma}_{E(\beta)}$ under all other Lorentz boosts from THEOREM 2.2.23 may be obtained by applying first k to $\mathbf{\Gamma}_{E(\beta)}$, then applying b_1 and finally k^{-1}. The plane $b_1(\mathbf{\Gamma}_{E(\beta)}) = \beta(b_1(\mathrm{e}_1)) + \langle b_1(\mathrm{e}_2), b_1(\mathrm{e}_3)\rangle$ is the set $b_1(\mathbf{\Gamma}_{E(\beta)}) = \{(x,y,z) \in \mathbb{R}^3 \ : \ \cosh\theta x - \sinh\theta y = \beta\}$.

Since $\Gamma_{H(\alpha)} = \{(x,y,z) \in \mathbb{R}^3 : z = \alpha\}$, then $k(\frac{\pi}{2})(\Gamma_{H(\alpha)}) = \{(x,y,z) \in \mathbb{R}^3 : y = \alpha\}$, where the corresponding intersection $k(\frac{\pi}{2})(\Gamma_{H(\alpha)}) \cap g(\Gamma_{E(\beta)})$ can be mapped by the element $k(-\frac{\pi}{2})$of $\mathrm{SO}(1,2)$ to the intersection $\Gamma_{H(\alpha)} \cap k(-\frac{\pi}{2})\, g(\Gamma_{E(\beta)}) = \Gamma_{H(\alpha)} \cap g'(\Gamma_{E(\beta)})$ for g, g' arbitrary elements of $\mathrm{SO}(1,2)$. Thus considering the intersection of $g(\Gamma_{E(\beta)})$ with the image set $k(\frac{\pi}{2})(\Gamma_{H(\alpha)})$ does not change the case. Then it follows that for each point in the intersection,

$$(x,y,z) \in k\left(\frac{\pi}{2}\right)\left(\Gamma_{H(\alpha)} \cap g(\Gamma_{E(\beta)}\right) \quad \Leftrightarrow \quad \cosh\theta x = \alpha - \beta\sinh\theta \quad \Rightarrow \quad x = \frac{\alpha - \beta\sinh\theta}{\cosh\theta}$$

and the line $\ell = k(\frac{\pi}{2})(\Gamma_{H(\alpha)}) \cap g(\Gamma_{E(\beta)})$ passes through the point $(\frac{\alpha-\beta\sinh\theta}{\cosh\theta}, \alpha, 0)$ with direction vector $(0,1,0) \wedge (\cosh\theta, \sinh\theta, 0) = (0,0,\cosh\theta)$. Thus the line $\ell = c(\frac{\alpha-\beta\sinh\theta}{\cosh\theta}, \alpha, 0) + \langle e_3 \rangle$. Consider the image

$$b_1(\widetilde{\theta})(\ell) = (\cosh\widetilde{\theta}\sec\theta(\alpha - \beta\sin\theta) + \alpha\sinh\widetilde{\theta}, \alpha\cosh\widetilde{\theta} + \sec\theta(\alpha - \beta\sin\theta)\sinh\widetilde{\theta}, t)|_{t\in\mathbb{R}}.$$

Using Mathematica (C.3), we note that for each α, β there exists either $\widetilde{\theta}_1 \in \mathbb{R}$ such that

$$(\cosh\widetilde{\theta}_1\sec\theta(\alpha - \beta\sin\theta) + \alpha\sinh\widetilde{\theta}_1, \alpha\cosh\widetilde{\theta}_1 + \sec\theta(\alpha - \beta\sin\theta)\sinh\widetilde{\theta}_1, t)|_{t\in\mathbb{R}} = \langle e_3 \rangle + ce_1$$

or $\widetilde{\theta}_2$,

$$(\cosh\widetilde{\theta}_2\sec\theta(\alpha - \beta\sin\theta) + \alpha\sinh\widetilde{\theta}_2, \alpha\cosh\widetilde{\theta}_2 + \sec\theta(\alpha - \beta\sin\theta)\sinh\widetilde{\theta}_2, t)|_{t\in\mathbb{R}} = \langle e_3 \rangle + c'e_2$$

and further that these angles $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ never exist simultaneously for the same combination of α, β. Thus there always exists a $g' \in \mathrm{SO}(1,2)_0$ such that $g'(\ell) = \Gamma_{H(\alpha)} \cap g(\Gamma_{E(\beta)})$ is the line $ce_1 + \langle e_3 \rangle$, or $c'e_2 + \langle e_3 \rangle$ for $c, c' \in \mathbb{R}$.

Case 4 - $\Gamma_{P(\pm1)} \cap g(\Gamma_{H(\alpha)})$

From PROPOSITION 3.6.7, the planes $\Gamma_{H(\alpha)}$ are preserved by the Lorentz boost b_1. Thus the intersection $\Gamma_{P(1)} \cap g(\Gamma_{H(\beta)})$ will not vary if $g = b_1$. Since from THEOREM 2.2.23 each Lorentz boost is a product $k^{-1}b_1k$, we consider the case $g = k$ only, since the image of $\Gamma_{H(\beta)}$ under all other Lorentz boosts may be found by applying b_1 and then k^{-1} to the image $k(\Gamma_{H(\alpha)})$. The plane $k(\Gamma_{H(\alpha)}) = \beta k(e_3) + \langle k(e_1), k(e_2)\rangle = \{(x,y,z) \in \mathbb{R}^3 : z\cos\theta - y\sin\theta = \alpha\}$ and the planes $\Gamma_{P(\pm1)} = \{(x,y,z) \in \mathbb{R} : x + y \pm 1 = 0\}$. Thus $(x,y,z) \in \Gamma_{P(\pm1)} \cap k(\Gamma_{H(\alpha)})$ if and only if $z\cos\theta - (x \pm 1)\sin\theta = \alpha$. This introduces two cases: $\theta = \frac{(2n+1)\pi}{2}$ or $\theta \neq \frac{(2n+1)\pi}{2}$.

Case 4a $\left(\theta = \frac{(2n+1)\pi}{2}\right)$

Taking $\theta = \frac{(2n+1)\pi}{2}$, then $k(\Gamma_{H(\alpha)}) = \{(x,y,z) \in \mathbb{R}^3 : y = \alpha\}$, and $\Gamma_{P(1)} \cap k(\Gamma_{H(\alpha)})$ passes through $(\alpha+1, 0, 0) = (c, 0, 0)$ with direction vector $(1,1,0) \wedge (0, -\sin\frac{\pm\pi}{1}, \cos\frac{\pm\pi}{1}) = (0,0,1)$. Similarly, $\Gamma_{P(-1)} \cap k(\Gamma_{H(\alpha)})$ passes through $(\alpha+1, \alpha, 0)$ with direction vector $(0,0,-1)$. Thus $\ell = \Gamma_{P(\pm1)} \cap k(\Gamma_{H(\alpha)})$ is the span $(\alpha+1, \alpha, 0) + \langle e_3 \rangle$. Consider the image of the line ℓ,

$$b_1(\widetilde{\theta})(\ell) = ((1+\alpha)\cosh\widetilde{\theta} + \alpha\sinh\widetilde{\theta}, \alpha\cosh\widetilde{\theta} + (1+\alpha)\sinh\widetilde{\theta}, t)|_{t\in\mathbb{R}}.$$

Using Mathematica (C.3), there exist $\widetilde{\theta}_1, \widetilde{\theta}_2 \in \mathbb{R}$ such that

$$(\cosh\widetilde{\theta}_1\sec\theta(\alpha - \beta\sin\theta) + \alpha\sinh\widetilde{\theta}_1, \alpha\cosh\widetilde{\theta}_1 + \sec\theta(\alpha - \beta\sin\theta)\sinh\widetilde{\theta}_1, t)|_{t\in\mathbb{R}} = \langle e_3 \rangle + ce_1$$

or

$$(\cosh\widetilde{\theta}_2\sec\theta(\alpha-\beta\sin\theta)+\alpha\sinh\widetilde{\theta}_2, \alpha\cosh\widetilde{\theta}_2+\sec\theta(\alpha-\beta\sin\theta)\sinh\widetilde{\theta}_2, t)|_{t\in\mathbb{R}} = \langle \mathbf{e}_3\rangle + c'\mathbf{e}_2$$

and further these angles $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ never exist simultaneously for the same combination of α, β. Thus there always exists a $g' \in \mathsf{SO}(1,2)_0$ such that the image $g'(\ell) = g'(\mathbf{\Gamma}_{P(\pm 1)} \cap k(\theta)\left(\Gamma_{H(\alpha)}\right))$ for $\theta = \frac{(2n+1)\pi}{2}$ is a line $g'(\ell) = c\mathbf{e}_1 + \langle \mathbf{e}_3\rangle$, or $g'(\ell) = c'\mathbf{e}_2 + \langle \mathbf{e}_3\rangle$ for $c, c' \in \mathbb{R}$.

Case 4b $\left(\theta \neq \frac{(2n+1)\pi}{2}\right)$

Taking $\theta \in \mathbb{R}, \theta \neq \frac{(2n+1)\pi}{2}$, then the set $k(\theta)(\Gamma_{H(\alpha)}) = \left\{(x, y, z) \in \mathbb{R}^3 \;:\; z = \alpha\cos\theta + y\sin\theta\right\}$, and so $\ell = \mathbf{\Gamma}_{P(1)} \cap k(\theta)(\Gamma_{H(\alpha)}) = \{(t, 1-t, \alpha\cos\theta + (1-t)\sin\theta) \;:\; t \in \mathbb{R}\}$. Thus ℓ passes through $(1, 0, \alpha\cos\theta)$ with direction vector $(0, -\sin\theta, \cos\theta) \wedge (1, 1, 0) = (\cos\theta, \cos\theta, \sin\theta)$. Thus the line $\ell = \mathbf{\Gamma}_{P(1)} \cap k(\theta)\left(\Gamma_{H(\alpha)}\right) = (1, 0, \alpha\cos\theta) + \langle\cos\theta, \cos\theta, \sin\theta\rangle$. Using Mathematica (C.3), then

$$\begin{bmatrix}\cosh\widetilde{\theta} & 0 & \sinh\widetilde{\theta}\\ 0 & 1 & 0\\ \sinh\widetilde{\theta} & 0 & \cosh\widetilde{\theta}\end{bmatrix}\begin{bmatrix}\cos\theta\\ \cos\theta\\ \sin\theta\end{bmatrix} = \begin{bmatrix}\cos\theta\cosh\widetilde{\theta} + \sin\theta\sinh\widetilde{\theta}\\ \cos\theta\\ \cosh\widetilde{\theta}\sin\theta + \cos\theta\sinh\widetilde{\theta}\end{bmatrix}$$

where there always exists a value $\widetilde{\theta}_1$ of $\widetilde{\theta}$ such that $\cosh\widetilde{\theta}_1\sin\theta + \cos\theta\sinh\widetilde{\theta}_1 = 0$. Taking this value of $\widetilde{\theta}$, then

$$\begin{bmatrix}\cosh\widetilde{\theta} & \sinh\widetilde{\theta} & 0\\ \sinh\widetilde{\theta} & \cosh\widetilde{\theta} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\widetilde{\theta}\cosh\widetilde{\theta}_1 + \sin\widetilde{\theta}\sinh\widetilde{\theta}_1\\ \cos\widetilde{\theta}\\ 0\end{bmatrix} = \begin{bmatrix}\cos\widetilde{\theta}\cosh\widetilde{\theta}_1^2 + (\cos\widetilde{\theta} + \cosh\widetilde{\theta}_1\sin\widetilde{\theta})\sinh\widetilde{\theta}_1\\ \sin\widetilde{\theta}\sinh\widetilde{\theta}_1^2 + \cos\widetilde{\theta}\cosh\widetilde{\theta}_1(1+\sinh\widetilde{\theta}_1)\\ 0\end{bmatrix}$$

where $\cos\widetilde{\theta}\cosh\widetilde{\theta}_1^2 + (\cos\widetilde{\theta} + \cosh\widetilde{\theta}_1\sin\widetilde{\theta})\sinh\widetilde{\theta}_1 = \sin\widetilde{\theta}\sinh\widetilde{\theta}_1^2 + \cos\widetilde{\theta}\cosh\widetilde{\theta}_1(1+\sinh\widetilde{\theta}_1)$ for some value of $\widetilde{\theta}_1$. Correspondingly, under $b_1(\widetilde{\theta})$ and $b_2(\widetilde{\theta}_1)$, the point $\mathbf{a} = (1, 0, \alpha\cos\theta)$ is mapped to

$$(\cosh\widetilde{\theta}_1(\cosh\widetilde{\theta} + \alpha\cos\theta\sinh\widetilde{\theta}), (\cosh\widetilde{\theta} + \alpha\cos\widetilde{\theta}\sinh\widetilde{\theta})\sinh\widetilde{\theta}_1, \alpha\cos\theta\cosh\widetilde{\theta} + \sinh\widetilde{\theta}).$$

But the transformations $n(t) = \exp(t(E_1 - E_2))$ in $\mathsf{SO}(1,2)_0$ preserve the sum $\mathbf{e}_1 + \mathbf{e}_2$. Thus the homogeneous part of the intersection is invariant under $n(\theta_2)$, and $\mathbf{a}$ is mapped to a vector with $\mathbf{e}_3$-term

$$\alpha\cos\theta\cosh\widetilde{\theta} + \sinh\widetilde{\theta} - \widetilde{\theta}_2\cosh\widetilde{\theta}_1(\cosh\widetilde{\theta} + \alpha\cos\theta\sinh\widetilde{\theta}) + \widetilde{\theta}_2(\cosh\widetilde{\theta} + \alpha\cos\widetilde{\theta}\sinh\widetilde{\theta})\sinh\widetilde{\theta}_1$$

where there always exists a value of $\widetilde{\theta}_2$ such that this value is zero. For the $\mathbf{e}_1$ and $\mathbf{e}_2$ entries of this image, we may subtract the second entry from the first, to determine that $\mathbf{a}$ is a sum $e^{\widetilde{\theta}_1}\cosh\widetilde{\theta} + \alpha\sinh\widetilde{\theta}(\cos\theta\cosh\widetilde{\theta}_1 - \cos\widetilde{\theta}\sinh\widetilde{\theta}_1)$. Thus there always exists some $g' \in \mathsf{SO}(1,2)_0$ such that the intersection $g'(\ell)$ is the line $c\mathbf{e}_1 + \langle\mathbf{e}_1 + \mathbf{e}_2\rangle$. The result follows.

Case 5 - $\mathbf{\Gamma}_{P(\pm 1)} \cap g\left(\Gamma_{E(\alpha)}\right)$

Since from PROPOSITION 3.6.7 the planes $\Gamma_{E(\alpha)}$ are preserved by any Euclidean rotation k, then the intersection $\Gamma_{P(\pm1)} \cap g\left(\Gamma_{H(\beta)}\right)$ will not vary if $g = k$. Thus we consider g as a Lorentz boost only, particularly the case where b is the Lorentz boost b_1. The image of $\Gamma_{E(\alpha)}$ under all other Lorentz boosts can be seen from THEOREM 2.2.23 to be obtained by applying first a rotation k to $\Gamma_{E(\alpha)}$, then applying b_1 and finally k^{-1}. The plane $b_1\left(\Gamma_{E(\alpha)}\right) = \alpha(b_1(\mathbf{e}_1)) + \langle b_1(\mathbf{e}_2), b_1(\mathbf{e}_3)\rangle$ is the set $b_1\left(\Gamma_{E(\alpha)}\right) = \left\{(x, y, z) \in \mathbb{R}^3 \,:\, \cosh\theta x - \sinh\theta y = \alpha\right\}$ and the planes $\Gamma(\pm1)$ are the sets $\Gamma_{P(1)} = \{(x, y, z) \in \mathbb{R} \,:\, x - y - 1 = 0\}$ and $\Gamma_{P(-1)} = \{(x, y, z) \in \mathbb{R} \,:\, x - y + 1 = 0\}$, respectively. Thus for each point

$$(x, y, z) \in \Gamma_{P(\pm1)} \cap b_1(\Gamma_{E(\alpha)} \quad \Leftrightarrow \quad \cosh\theta x = \alpha + \sinh\theta(x \pm 1) \quad \Leftrightarrow \quad x = \frac{\alpha \pm \sinh\theta}{\cosh\theta}$$

and the lines $\ell = \Gamma_{P(\pm1)} \cap b_1\left(\Gamma_{E(\alpha)}\right)$ pass through the point $(\frac{\alpha\pm\sinh\theta}{\cosh\theta}, \alpha, 0)$ with direction vector $(-1, 1, 0) \wedge (\cosh\theta, \sinh\theta, 0) = (0, 0, -\sinh\theta - \cosh\theta)$. Thus the lines $\ell = (\frac{\alpha\pm\sinh\theta}{\cosh\theta}, \alpha, 0) + \langle \mathbf{e}_3\rangle$. Using Mathematica (C.3), consider the image of ℓ under the Lorentz boost $b_1(\widetilde{\theta})$, given in that appendix. For each α, β we show in (C.3) that there exists either $\widetilde{\theta}_1 \in \mathbb{R}$ such that

$$b_1(\widetilde{\theta})\left(\frac{\alpha \pm \sinh\theta}{\cosh\theta}, \alpha, 0\right) = c\mathbf{e}_1 \quad \text{or} \quad b_1(\widetilde{\theta})\left(\frac{\alpha \pm \sinh\theta}{\cosh\theta}, \alpha, 0\right) = c'\mathbf{e}_2$$

and further that these angles never exist simultaneously for the same combination of α, β. The boosts $b_1(\widetilde{\theta})$ preserve the span $\langle \mathbf{e}_3\rangle$ in each case. Thus there always exists a g' in $\mathsf{SO}(1,2)_0$ such that $g'(\ell) = g'(\Gamma_{P(\pm1)} \cap g\left(\Gamma_{E(\alpha)}\right))$ is the line $g'(\ell) = c\mathbf{e}_1 + \langle \mathbf{e}_3\rangle$, or $g'(\ell) = c'\mathbf{e}_2 + \langle \mathbf{e}_3\rangle$ for $c, c' \in \mathbb{R}$.

Case 6 - $\Gamma_{P(\pm1)} \cap g(\Gamma_{P(\pm1)})$

We note firstly that since clearly $\Gamma_{P(1)} \cap g(\Gamma_{P(-1)}) = g(g^{-1}\Gamma_{P(1)} \cap \Gamma_{P(-1)})$, then it is necessary only to consider the cases $\Gamma_{P(\pm1)} \cap g(\Gamma_{P(1)})$. We will consider first the case $\Gamma_{P(\lambda1)} \cap g(\Gamma_{P(\lambda1)})$ and then the case $\Gamma_{P(-\lambda1)} \cap g(\Gamma_{P(\lambda1)})$.

Note that the Lorentz boost b_1 sends the normal vector $(1, 1, 0)$ of $\Gamma_{P(0)} = \langle \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3\rangle$ to the normal vector

$$(0, 0, 1) \wedge (\sinh\theta + \cosh\theta, \sinh\theta + \cosh\theta, 0) = (-\sinh\theta - \cosh\theta, \sinh\theta + \cosh\theta, 0)$$

and thus sends $\Gamma^0_{P(1)}$ to $b_1(\Gamma^0_{P(1)}) = \langle \mathbf{e}_2 - \mathbf{e}_1, \mathbf{e}_3\rangle$. Thus b_1 preserves the parallelness of the parabolic planes $\Gamma_{P(1)}$ and $\Gamma_{P(1)} \cap b_1(\Gamma_{P(1)}) = \emptyset$ or the whole plane $\Gamma_{P(1)}$. Since from THEOREM 2.2.23 each Lorentz boost is a product $k^{-1}b_1k$, we consider the case $g = k$ only, since the image of $\Gamma_{P(1)}$ under all other Lorentz boosts may then be found by applying b_1 and then k^{-1} to the image $k\left(\Gamma_{P(1)}\right)$.

The image of the plane $\Gamma_{P(\lambda1)}$ under $k(\theta)$ has normal vector $(-1, -\cos\theta, -\sin\theta)$ and passes through the point $(\lambda1, 0, 0)$. Thus the intersection $\ell = \Gamma_{P(\lambda1)} \cap k(\theta)\left(\Gamma_{P(\lambda1)}\right)$ has normal vector $(1, 1, 0) \wedge (-1, -\cos\theta, -\sin\theta) = (-\sin\theta, \sin\theta, 1 - \cos\theta)$ and passes through the point $(\lambda1, 0, 0)$ common to both $\Gamma_{P(\lambda1)}$ and $k(\theta)\left(\Gamma_{P(\lambda1)}\right)$. Thus the line of intersection ℓ is given by $\lambda\mathbf{e}_1 + \langle(\sin(-\theta), \sin\theta, 1 - \cos\theta)\rangle$. Using Mathematica (C.3), we take the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\widetilde{\theta} & -\sin\widetilde{\theta} \\ 0 & \sin\widetilde{\theta} & \cos\widetilde{\theta} \end{bmatrix} \begin{bmatrix} -\sin\widetilde{\theta} \\ \sin\widetilde{\theta} \\ 1 - \cos\widetilde{\theta} \end{bmatrix} = \begin{bmatrix} -\sin\widetilde{\theta} \\ -\sin\widetilde{\theta} + \sin(\widetilde{\theta} + \theta) \\ \cos\widetilde{\theta} - \cos(\widetilde{\theta} + \theta) \end{bmatrix}$$

where there always exists some value $\widetilde{\theta}_1$ such that $\cos\widetilde{\theta}_1 - \cos(\widetilde{\theta}_1 + \theta) = 0$ or some value of $\widetilde{\theta}_2$ such that $-\sin\widetilde{\theta}_2 + \sin(\widetilde{\theta}_2 + \theta) = 0$. Thus we may always take either the $\mathbf{e}_2$ or the $\mathbf{e}_3$ terms of the direction vector to be zero. Since $\mathbf{e}_1$ is invariant under the transformations $k(\theta)$, then it follows that the intercept is always $\mathbf{e}_1$. We then apply the transformation

$$\begin{bmatrix} \cosh\widetilde{\theta} & \sinh\widetilde{\theta} & 0 \\ \sinh\widetilde{\theta} & \cosh\widetilde{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\widetilde{\theta}_1 \\ -\sin\widetilde{\theta}_1 + \sin[\widetilde{\theta}_1 + \theta] \\ 0 \end{bmatrix} = \begin{bmatrix} -\cosh\sin\widetilde{\theta}\sin\theta + (-\sin\widetilde{\theta}_1 + \sin(\widetilde{\theta}_1 + \theta))\sinh\widetilde{\theta} \\ \cosh\widetilde{\theta}(-\sin\widetilde{\theta}_1 + \sin(\widetilde{\theta}_1 + \theta)) - \sin(\theta)\sinh\widetilde{\theta} \\ 0 \end{bmatrix}$$

where (C.3) there exists a value $\widetilde{\theta}_2$ of $\widetilde{\theta}$ such that $\cosh\widetilde{\theta}_2(-\sin\widetilde{\theta}_1 + \sin(\widetilde{\theta}_1 + \theta)) - \sin(\theta)\sinh\widetilde{\theta}_2 = 0$, and no value for $\widetilde{\theta}$ such that $-\cosh\widetilde{\theta}\sin\theta + (-\sin\widetilde{\theta}_1 + \sin(\widetilde{\theta}_1 + \theta))\sinh\widetilde{\theta} = 0$. Under b_1, the image of the intercept is $(\lambda\cosh\widetilde{\theta}_2, \lambda\sinh\widetilde{\theta}_2, 0)$, and thus there always exists some $g = b_1(\widetilde{\theta}_2)k(\widetilde{\theta}_1)$ such that the intersection ℓ is sent to $g(\ell) = \lambda\cosh\widetilde{\theta}_2\mathbf{e}_1 + \langle\mathbf{e}_2\rangle$.

We now consider the case of the intersection $\ell = \mathbf{\Gamma}_{P(\lambda 1)} \cap k(\theta)\left(\mathbf{\Gamma}_{P(-\lambda 1)}\right)$. The image of the plane $\mathbf{\Gamma}_{P(\lambda 1)}$ under $k(\theta)$ has normal vector $(-1, -\cos\theta, -\sin\theta)$ and passes through the point $(\lambda 1, 0, 0)$. Thus $(1,1,0) \wedge (-1, -\cos\theta, -\sin\theta) = (-\sin\theta, \sin\theta, 1 - \cos\theta)$ is the normal vector of the line of intersection $\ell = \mathbf{\Gamma}_{P(-\lambda 1)} \cap k(\theta)\left(\mathbf{\Gamma}_{P(\lambda 1)}\right)$. Expressing $\mathbf{\Gamma}_{P(-\lambda 1)} = \left\{(x,y,z) \in \mathbb{R}^3 \,:\, x - y = -\lambda\right\}$ and $k(\theta)\left(\mathbf{\Gamma}_{P(\lambda 1)}\right) = \left\{(x,y,z) \in \mathbb{R}^3 \,:\, x + y\cos\theta + z\sin\theta = \lambda\right\}$, it follows that these two planes intersect in the point $\left(y - \lambda, y, \frac{-(y - 2\lambda + y\cos\theta)}{\sin\theta}\right) = (0, \lambda, 0)$ at $y = \lambda$. Thus the line of intersection $\ell = (0, \lambda, 0) + t(-\sin\theta, \sin\theta, 1 - \cos\theta)|_{t\in\mathbb{R}}$. Taking the image

$$\begin{bmatrix} \cosh\widetilde{\theta} & \sinh\widetilde{\theta} & 0 \\ \sinh\widetilde{\theta} & \cosh\widetilde{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \sin\theta \\ 1 - \cos\theta \end{bmatrix} = \begin{bmatrix} -\cosh\widetilde{\theta}\sin\theta + \sin\widetilde{\theta} - \cos\theta\sinh\widetilde{\theta} \\ \sin\theta \\ \cosh\widetilde{\theta} - \cos\theta\cosh\widetilde{\theta} - \sin\theta\sinh\widetilde{\theta} \end{bmatrix}$$

where (C.3) there always exists a value $\widetilde{\theta}_1$ such that $-\cosh\widetilde{\theta}_1\sin\theta + \sinh\widetilde{\theta}_1 - \cos\theta\sinh\widetilde{\theta}_1 = 0$. So thus we may use an $\mathbf{e}_2$- preserving transformation to get the $\mathbf{e}_1$-entries to zero. We may then take the image of this vector $\left(0, \sin\theta, \cosh\widetilde{\theta}_1 - \cos\theta\cosh\widetilde{\theta}_1 - \sin\theta\sinh\widetilde{\theta}_1\right)$ under $k(\widetilde{\theta})$, to get

$$\begin{bmatrix} 0 \\ (-1 + \cos\theta)\cosh\widetilde{\theta}_1\sin\theta + \sin\theta(\cos\theta + \sin\theta\sinh\theta_1) \\ -\cos\theta(-1 + \cos\theta)\cosh\widetilde{\theta}_1 + \sin\theta(\sin\theta - \cos\theta\sinh\widetilde{\theta}_1) \end{bmatrix}$$

and determine $\widetilde{\theta}_2$ such that $(-1 + \cos\theta)\cosh\widetilde{\theta}_1\sin\widetilde{\theta}_2 + \sin\theta(\cos\widetilde{\theta}_2 + \sin\widetilde{\theta}_2\sinh\widetilde{\theta}_1) = 0$. Thus we may immediately take the direction subspace as $\langle\mathbf{e}_3\rangle$. But the image of the intercept under $k(\widetilde{\theta}_2)$ is given by $(0, \lambda\cos\widetilde{\theta}_2, \lambda\sin\widetilde{\theta}_2)$. Thus there always exists some $g' = b_2(\widetilde{\theta}_2)k(\widetilde{\theta}_1)$ in $\mathsf{SO}(1,2)_0$ such that the intersection ℓ is sent to $g'(\ell) = \mathbf{e}_2 + \langle\mathbf{e}_3\rangle$.

We use these expressions of cases 1-6 to show

4.1.31 Proposition. *Each line ℓ of $\mathbb{R}^{1,2}$ may be mapped by an element of $\mathsf{SO}(1,2)$ to one of the lines $c\mathbf{e}_2 + \langle\mathbf{e}_1\rangle$, $c\mathbf{e}_2 + \langle\mathbf{e}_3\rangle$, $c\mathbf{e}_1 + \langle\mathbf{e}_3\rangle$ or $\mathbf{e}_2 + \langle\mathbf{e}_1 + \mathbf{e}_2\rangle$ where $c > 0$.*

Proof. In Proposition 4.1.30 we stated that the arbitrary line ℓ could be mapped by an element of $\mathsf{SO}(1,2)_0$ to one of the lines of cases 1-6, which we expressed as the lines $c\mathbf{e}_2 + \langle\mathbf{e}_1\rangle$,

$c\mathbf{e}_2 + \langle \mathbf{e}_3 \rangle$, $c\mathbf{e}_1 + \langle \mathbf{e}_3 \rangle$ and $\mathbf{e}_2 + \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$, where $c \in \mathbb{R}$. Consider now the element $g_2 \in \mathsf{SO}(1,2)$. Taking $c > 0$, the image

$$g_2(c\mathbf{e}_2 + \langle \mathbf{e}_1 \rangle) = cg_2(\mathbf{e}_2) + \langle g_2(\mathbf{e}_1) \rangle = -c\mathbf{e}_2 + \langle \mathbf{e}_1 \rangle = c'\mathbf{e}_2 + \langle \mathbf{e}_1 \rangle$$

where $c' < 0$. Similarly, for $c > 0$,

$$g_2(c\mathbf{e}_2 + \langle \mathbf{e}_3 \rangle) = cg_2(\mathbf{e}_2) + \langle g_2(\mathbf{e}_3) \rangle = -c\mathbf{e}_2 + \langle \mathbf{e}_3 \rangle = c'\mathbf{e}_2 + \langle \mathbf{e}_3 \rangle$$

where $c' < 0$ and finally for $c > 0$, then

$$g_2(c\mathbf{e}_1 + \langle \mathbf{e}_3 \rangle) = cg_2(\mathbf{e}_1) + \langle g_2(\mathbf{e}_3) \rangle = -c\mathbf{e}_1 + \langle \mathbf{e}_3 \rangle = c'\mathbf{e}_1 + \langle \mathbf{e}_3 \rangle$$

where $c' < 0$. Thus it follows that for any line ℓ there exists an element $g' = g_2 g \in \mathsf{SO}(1,2)$ such that $g'(\ell) = c\mathbf{e}_2 + \langle \mathbf{e}_1 \rangle$, $c\mathbf{e}_2 + \langle \mathbf{e}_3 \rangle$, $c\mathbf{e}_1 + \langle \mathbf{e}_3 \rangle$ or $\mathbf{e}_2 + \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$. □

4.1.32 REMARK. Consider the images under the inverse hat map $cE_2 + \langle E_1 \rangle$, $cE_2 + \langle E_3 \rangle$, $cE_1 + \langle E_3 \rangle$ and $E_2 + \langle E_1 + E_2 \rangle$ of the lines $c\mathbf{e}_2 + \langle \mathbf{e}_1 \rangle$, $c\mathbf{e}_2 + \langle \mathbf{e}_3 \rangle$, $c\mathbf{e}_1 + \langle \mathbf{e}_3 \rangle$ and $\mathbf{e}_2 + \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ in PROPOSITION 4.1.31. Then in the case $cE_2 + \langle E_1 \rangle$, the Lie bracket of elements of the span $[E_2, E_1] = E_3$, and thus any system Σ with trace $\Gamma = cE_2 + \langle E_1 \rangle$ for some c is full rank, since $E_1, E_2, E_3 \in \mathrm{Lie}(\Gamma)$.

Similarly, in the case $cE_2 + \langle E_3 \rangle$, the Lie bracket of elements of the span $[E_2, E_3] = E_1$, and thus any system Σ with trace $\Gamma = cE_2 + \langle E_3 \rangle$ for some c is full rank, since $E_1, E_2, E_3 \in \mathrm{Lie}(\Gamma)$.

In the case $cE_1 + \langle E_3 \rangle$, the Lie bracket of elements of the span $[E_1, E_3] = E_2$, and thus any system Σ with trace $\Gamma = cE_1 + \langle E_3 \rangle$ for some c is full rank, since $E_1, E_2, E_3 \in \mathrm{Lie}(\Gamma)$.

Finally, in the case $cE_2 + \langle E_1 + E_2 \rangle$, the Lie bracket of elements of the span $[E_2, E_1 + E_2] = E_3$, and thus any system Σ with trace $\Gamma = cE_2 + \langle E_1 + E_2 \rangle$ is full rank, since $E_1, E_2, E_3 \in \mathrm{Lie}(\Gamma)$. Thus each of the lines $c\mathbf{e}_2 + \langle \mathbf{e}_1 \rangle$, $c\mathbf{e}_2 + \langle \mathbf{e}_3 \rangle$, $c\mathbf{e}_1 + \langle \mathbf{e}_3 \rangle$ and $\mathbf{e}_2 + \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ of PROPOSITION 4.1.31 correspond to full-rank systems under the hat map.

4.1.33 PROPOSITION. *Each full-rank single-input control affine left-invariant inhomogeneous system Σ is l.d.f.e to a system of the family $\Sigma^{(1,1)}_{1,c}$ which has parametrization map $\Xi^{(1,1)}_{1,c}(\mathbf{1}, u) = cE_2 + uE_1$, a system of the family $\Sigma^{(1,1)}_{2,c}$ which has parametrization map $\Xi^{(1,1)}_{2,c}(1, u) = cE_2 + uE_3$, a system of the family $\Sigma^{(1,1)}_{3,c}$ which has parametrization map $\Xi^{(1,1)}_{3,c}(\mathbf{1}, u) = cE_1 + uE_3$ or a system $\Sigma^{(1,1)}_4$ which has parametrization map $\Xi^{(1,1)}_4(\mathbf{1}, u) = E_2 + u(E_1 + E_2)$. In each case $c \in \mathbb{R}^+$.*

PROOF. Consider the image $\mathbf{\Gamma}$ of the trace Γ of the arbitrary single-input system Σ, given by $\mathbf{\Gamma} = \mathbf{a} + \langle \mathbf{b} \rangle$ in $\mathbb{R}^{1,2}_{c}$. Then by PROPOSITION 4.1.30, there exists some $g \in \mathsf{SO}(1,2)_0$ such that $g(\Gamma)$ is in one of the cases 1-6. In PROPOSITION 4.1.31, we expressed each of these cases as $g(\mathbf{\Gamma}) = c\mathbf{e}_2 + \langle \mathbf{e} \rangle$, $g(\mathbf{\Gamma}) = c\mathbf{e}_2 + \langle \mathbf{e}_3 \rangle$, $g(\mathbf{\Gamma}) = c\mathbf{e}_1 + \langle \mathbf{e}_3 \rangle$ and $g(\mathbf{\Gamma}) = \mathbf{e}_2 + \langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ where $c \in \mathbb{R}^+$ and $g \in \mathsf{SO}(1,2)$. Then by PROPOSITION 3.4.2 there exists $\Psi(g) = \phi_g \in \mathsf{Aut}(\mathfrak{so}(1,2))$ such that $\phi_g(\Gamma)$ is one of $cE_2 + uE_1$, $cE_2 + uE_3$, $cE_1 + uE_3$ or $E_2 + u(E_1 + E_2)$. The result follows. □

4.1.34 THEOREM. *Each full-rank single-input control affine left-invariant inhomogeous system Σ is is l.d.f.e to* exactly *one of the systems in the families $\Sigma^{(1,1)}_{1,c}$, $\Sigma^{(1,1)}_{2,c}$, $\Sigma^{(1,1)}_{3,c}$ or $\Sigma^{(1,1)}_4$.*

PROOF. Consider the image Γ of the trace Γ of the arbitrariy single-input system Σ. Initially, we let the image $g(\Gamma)$ be in each of the Cases 1-3 and 5b sucessively, and express $ds^2|_{g(\Gamma)}$. Firstly, for $g(\Gamma)$ in case 1, then

$$g(\Gamma) = \{(x,y,z) \in \mathbb{R}^3 \ : \ (x,y,z) = (t,c,0)\} \quad \Rightarrow \quad ds^2|_{g(\Gamma)} = -dt^2$$

for $g(\Gamma)$ in case 2, then

$$g(\Gamma) = \{(x,y,z) \in \mathbb{R}^3 \ : \ (x,y,z) = (0,c,t)\} \quad \Rightarrow \quad ds^2|_{g(\Gamma)} = dt^2$$

for $g(\Gamma)$ in case 3, then

$$g(\Gamma) = \{(x,y,z) \in \mathbb{R}^3 \ : \ (x,y,z) = (c,0,t)\} \quad \Rightarrow \quad ds^2|_{g(\Gamma)} = dt^2$$

and finally, for $g(\Gamma)$ an intersection of the form of case 4, then

$$g(\Gamma) = \{(x,y,z) \in \mathbb{R}^3 \ : \ (x,y,z) = (t,1+t,0)\} \quad \Rightarrow \quad ds^2|_{g(\Gamma)} = 0.$$

Thus the lines of cases 1 and 2, 1 and 3 and 1 and 5b cannot be isometric: that is, there exists no element $g \in \mathsf{SO}(1,2)$ which maps a line in one of the cases to another. Similarly, lines of cases 2 and 3 cannot be isometric to 5b, which means that it is only possible for lines of cases 2 and 3 to be isometric. But the lines of case 2 admit exactly the vectors $(0,c,t)$, while those of case 3 admit exactly the vectors $(c,0,t)$, where for $t < c$,

$$(c,0,t) \odot (c,0,t) = -c^2 + t^2 < 0$$

and so the lines of case 3 admit timelike vectors. However, for every vector $(0,c,t)$ of the lines of case 2,

$$(0,c,t) \odot (0,c,t) = c^2 + t^2 > 0.$$

Since no element of $\mathsf{SO}(1,2)$ may map a timelike vector to a spacelike vector, thus there exists no bijective isometry $g' \in \mathsf{SO}(1,2)$ which maps a line of case 2 which admits only spacelike vectors to a line of case 3 which admits spacelike and timelike vectors. Thus no line Γ may be isometric to a line which is not in the same case as Γ, as there exists no element $g' \in \mathsf{SO}(1,2)$ which maps Γ to a line in another of the cases 1-3, 5b. But then corrrespondingly by PROPOSITION 3.4.2, there exists no element $\Psi(g') = \phi_{g'} \in \mathsf{Aut}(\mathfrak{so}(1,2))$ which maps a system of one family to another.

Secondly, assume for example that $\Sigma_{1,c}^{(1,1)}$ is l.d.f.e. to $\Sigma_{1,c'}^{(1,1)}$ for $c, c' \in \mathbb{R}^+, c \neq c'$. Then there exists a $g \in \mathsf{SO}(1,2)$ such that $g\Xi_{1,c}^{(1,1)} = \Xi_{1,c'}^{(1,1)}$ and so $g(\Gamma) = cg\mathrm{e}_2 + \langle g\mathrm{e}_1\rangle = c\mathrm{e}_2 + \langle \mathrm{e}_1 \rangle$. Then particularly $cg\mathrm{e}_2 = c'\mathrm{e}_2$, where $\|c\mathrm{e}_2\| = c^2 \neq (c')^2 = \|c'\mathrm{e}_2\|$ since $c, c' > 0, c \neq c'$. But this contradicts the fact that g is an isometry. Thus $\Sigma_{1,c}^{(1,1)}$ cannot be l.d.f.e. to $\Sigma_{1,c'}^{(1,1)}$ if $c \neq c'$. The cases of the families $\Sigma_{2,c}^{(1,1)}$ and $\Sigma_{3,c}^{(1,1)}$ follow identical steps. □

4.1.4 Three-input homogeneous and inhomogeneous systems

For any arbitrary three-input control affine system Σ with trace $\Gamma = A + \langle B_1, B_2, B_3\rangle$, the elements B_1, B_2 and B_3 are linearly independent (A.7.4) and thus we may express the drift term A (which may be zero or nonzero) as $A = a_1B_1 + a_2B_a + a_3B_3$. Thus in any case A is in the span $\langle B_1, B_2, B_3\rangle$, and $\Gamma = a_1B_1 + a_2B_2 + a_3B_3 + \langle B_1, B_2, B_3\rangle = \langle B_1, B_2, B_3\rangle$. Thus we may consider the homogeneous and inhomogeneous cases simultaneously. We establish

4.1.35 THEOREM. *Any homogeneous or inhomogeneous three-input control affine left-invariant system* $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ *is l.d.f.e to the control system* $\Sigma^{(3,1)} = (\mathsf{SO}(1,2), \Xi^{(3,1)})$ *with parametrization map* $\Xi^{(3,1)}(1,u) = u_{E1} + u_2E_2 + u_3E_3$.

PROOF. We have shown that any arbitrary 3-input homogeneous or inhomogeneous system Σ has trace $\Gamma = \langle B_1, B_2, B_3\rangle$. But then $\Gamma = \langle B_1, B_2, B_3\rangle = \mathfrak{so}(1,2) = \langle E_1, E_2, E_3\rangle = \Gamma^{(3,1)}$, and the trace Γ is mapped to the trace $\Gamma^{(3,1)}$ by the Lie algebra automorphism $\mathbf{1} \in \mathsf{SO}(1,2)$. Thus Σ is l.d.f.e. to $\Sigma^{(3,1)}$. □

4.2 A controllability criterion for systems on $\mathsf{SO}(1,2)_0$

4.2.1 Control on a connected, semisimple (matrix) Lie group

In this section we refer to the definitions and results (A.7.1)-(A.7.20). Firstly we note that from (A.7.18) we already have a necessary and sufficient condition for controllability of homogeneous affine systems:

4.2.1 PROPOSITION. *All homogeneous control affine systems* $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ *are controllable if and only if they are full rank.*

PROOF. From PROPOSITION 3.1.2, the group $\mathsf{SO}(1,2)_0$ is connected. From (A.7.18), a symmetric system on connected matrix Lie group is controllable if and only if it is full rank. Since each homogeneous system has a trace $\Gamma = \langle B_1, B_2, B_3\rangle$ for at least one B_i nonzero, $i = 1, 2, 3$, then $\Gamma = \langle B_1, B_2, B_3\rangle = \langle -B_1, -B_2, -B_3\rangle = -\mathbf{1}\langle B_1, B_2, B_3\rangle = -\Gamma$, and from (A.7.17) each such system is symmetric. Thus each homogeneous system is controllable if and only if it is full rank. □

Next, we derive some properties of $\mathsf{SO}(1,2)_0$ from its Iwasawa decomposition (THEOREM 2.5.10).

4.2.2 PROPOSITION. *Any map of the form*

$$t \mapsto \exp\left(t(\pm\alpha g E_1 g^{-1})\right) \quad g \in \mathsf{SO}(1,2)_0, \alpha \in \mathbb{R}^+$$

is periodic.

PROOF. In (3.6.1) we expressed the elements of $\mathsf{K} = \{\exp(tE_1) \ : \ t \in \mathbb{R}\}$, which allows us to express a trajectory of the form $\{\exp(t \pm \alpha E_1) \ : \ t \in \mathbb{R}^+\}$ as a curve in K:

$$k(\cdot) : \mathbb{R}^+ \to \mathsf{K}, \quad k(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pm\alpha t) & -\sin(\pm\alpha t) \\ 0 & \sin(\pm\alpha t) & \cos(\pm) \end{bmatrix} \text{ for each } t \in \mathbb{R}^+$$

which is periodic, since for any $t_0 > 0$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pm\alpha t_0) & -\sin(\pm\alpha t_0) \\ 0 & \sin(\pm\alpha t_0) & \cos(\pm\alpha t_0) \end{bmatrix} = k \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pm\alpha t_0 + \frac{2\pi}{\pm\alpha}) & -\sin(\pm\alpha t_0 + \frac{2\pi}{\pm\alpha}) \\ 0 & \sin(\pm\alpha t_0 + \frac{2\pi}{\pm\alpha}) & \cos(\pm\alpha t_0 + \frac{2\pi}{\pm\alpha}) \end{bmatrix} = k$$

by the periodicty of sin and cos. But then for any $g \in \mathsf{SO}(1,2)_0$, each point $g(t_0)$ of the trajectory $\{\exp\left(t(gE_1g^{-1}\right) \ : \ t \in [0,T], g \in \mathsf{SO}(1,2)_0\}$ is given by

$$g\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\pm\alpha t_0 & -\sin\pm\alpha t_0 \\ 0 & \sin\pm\alpha t_0 & \cos\pm\alpha t_0 \end{bmatrix} g^{-1} = gkg^{-1} \quad \Rightarrow \quad g\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pm\alpha t_0 + \frac{2\pi}{\pm\alpha}) & -\sin(\pm\alpha t_0 + \frac{2\pi}{\pm\alpha}) \\ 0 & \sin(\pm\alpha t_0 + \frac{2\pi}{\pm\alpha}) & \cos(\pm\alpha t_0 + \frac{2\pi}{\pm\alpha}) \end{bmatrix} g^{-1} = gkg^{-1},$$

and it follows that any trajectory of the form $\{\exp(t \pm \alpha g E_1 g^{-1}) \ : \ t \in [0,T], g \in \mathsf{SO}(1,2)_0\}$ is periodic. □

4.2.3 PROPOSITION. *In the Iwasawa decomposition of* $\mathsf{SO}(1,2)_0$*, the generators in* $\mathfrak{so}(1,2)$ *of the subgroup* $K = g\mathsf{K}g^{-1}$ *under exponentation have a timelike image under the hat map; those of* $A = g\mathsf{A}g^{-1}$ *have a spacelike image under the hat map, and those of* $N = g\mathsf{N}g^{-1}$ *have a lightlike image under the hat map.*

PROOF. From THEOREM 3.6.10, K, A and N are subgroups intersecting only in identity. Consider first the case of K. For the generator E_1 of K, $\widehat{E_1} = \mathbf{e}_1 = (1,0,0)$ which is timelike in $\mathbb{R}^{1,2}$. Further, given the generator gE_1g^{-1} of K for any $g \in \mathsf{SO}(1,2)_0$, then from PROPOSITION 3.4.2 there exists an element g' of $\mathsf{SO}(1,2)_0$ such that $\text{hat}\left(gE_1g^{-1}\right) = g'\mathbf{e}_1 = g'(1,0,0)$, which is timelike, since $g \in \mathsf{SO}(1,2)$ is an isometry of $\mathbb{R}^{1,2}$.

Similarly for the generator E_2 of A, $\widehat{E_2} = \mathbf{e}_2 = (0,1,0)$ which is spacelike in $\mathbb{R}^{1,2}$. Further, given the generator gE_2g^{-1} of A for any $g \in \mathsf{SO}(1.2)_0$, then from PROPOSITION 3.4.2 there exists an element g' of $\mathsf{SO}(1,2)_0$ such that the image $\text{hat}\left(gE_2g^{-1}\right) = g'\mathbf{e}_2 = g'(0,1,0)$, which is spacelike, since $g' \in \mathsf{SO}(1,2)$ is an isometry of $\mathbb{R}^{1,2}$.

Finally, for the generator $(E_1 - E_3)$ of N, $\text{hat}\,(E_1 - E_3) = (\mathbf{e}_1 - \mathbf{e}_3) = (1,0,-1)$ which is lightlike in $\mathbb{R}^{1,2}$. Further, given the generator $g(E_1 - E_3)g^{-1}$ of N for any $g \in \mathsf{SO}(1,2)_0$, then from PROPOSITION 3.4.2 there exists an element g' of $\mathsf{SO}(1,2)_0$ such that the image $\text{hat}\left(g(E_1 - E_3)g^{-1}\right) = g'(\mathbf{e}_1 - \mathbf{e}_3) = g'(1,0,-1)$, which is lightlike, since $g' \in \mathsf{SO}(1.2)$ is an isometry of $\mathbb{R}^{1,2}$. □

We will henceforth refer to the elements T of $\mathfrak{so}(1,2)$ such that $\widehat{T} = \mathbf{t}$ for $\mathbf{t}$ timelike as timelike elements of the Lie algebra, the elements S of $\mathfrak{so}(1,2)$ such that $\widehat{S} = \mathbf{s}$ for $\mathbf{s}$ spacelike as spacelike elements, and the elements N of $\mathfrak{so}(1,2)$ such that $\widehat{N} = \mathbf{n}$ for $\mathbf{n}$ lightlike as lightlike elements.

4.2.4 PROPOSITION. *Any spacelike elements of* $\mathfrak{so}(1,2)$ *have the form* $\alpha g E_2 g^{-1}$ *for some* $g \in \mathsf{SO}(1,2)_0$. *Any lightlike elements of* $\mathfrak{so}(1,2)$ *have the form* $\pm g(E_1 - E_3)g^{-1}$ *for some* $g \in \mathsf{SO}(1,2)_0$, $\alpha \in \mathbb{R}$. *Finally, any timelike elements of* $\mathfrak{so}(1,2)$ *have the form* $\pm \alpha g E_1 g^{-1}$ *for some* $g \in \mathsf{SO}(1,2)_0$, $\alpha \in \mathbb{R}^+$.

PROOF. In PROPOSITION 2.1.15 we showed that all timelike elements of $\mathbb{R}^{1,2}$ lie on the upper and lower sheets of the hyperboloids of two sheets $\mathcal{H}^2_\alpha$, all spacelike elements lie on the hyperboloid of one sheet $\mathcal{H}^1_\alpha$ and all lightlike elements lie on the upper and lower sheets of the cone $\mathcal{K}_L$. But in PROPOSITION 3.6.6 we showed that $\mathsf{SO}(1,2)_0$ acts transitively on the upper and lower sheets of $\mathcal{H}^2_\alpha$ and $\mathcal{K}_L$, and on $\mathcal{H}^1_\alpha$.

Thus firstly given any spacelike element $\mathbf{s} \in \mathbb{R}^{1,2}_{\circlearrowright}$, there exists an element $g \in \mathsf{SO}(1,2)_0$ such that $g(\mathbf{s}) = \alpha \mathbf{e}_2$. Thus $\mathbf{s} = \alpha g^{-1}(\mathbf{e}_2)$, and it follows from PROPOSITIONS 3.4.2 and 3.4.1 that there exists some $g' \in \mathsf{SO}(1,2)_0$ such that under the hat map $S = \alpha\phi_{g'}(E_2) = \alpha \mathrm{Ad}_{g'}(E_2)$.

Similarly, given any timelike element $\mathbf{t} \in \mathbb{R}^{1,2}_{\circlearrowright}$, then $\mathbf{t}$ lies on either the upper or the lower sheet of $\mathcal{H}^2_\alpha$. If $\mathbf{t}$ lies in $\mathcal{H}^{2+}_\alpha$, then there exists an element $g \in \mathsf{SO}(1,2)_0$ such that $g(\mathbf{t}) = \alpha\mathbf{e}_1$. Further, if $\mathbf{t}$ lies in $\mathcal{H}^{2-}_\alpha$, then there exists an element $g \in \mathsf{SO}(1,2)_0$ such that $g(\mathbf{t}) = -\alpha\mathbf{e}_1$. Thus

$$\mathbf{t} = \alpha g^{-1}(\mathbf{e}_1) \quad \text{or} \quad \mathbf{t} = -\alpha g^{-1}(\mathbf{e}_1)$$

and it follows from PROPOSITIONS 3.4.2 and 3.4.1 that there exists some $g' \in \mathsf{SO}(1,2)_0$ such that the image under the hat map, $T = \phi_{g'}(E_1) = \alpha\mathrm{Ad}_{g'}(E_1)$ or $T = -\phi_{g'}(E_1) = -\alpha\mathrm{Ad}_{g'}(E_1)$.

Finally, given any lightlike element $\mathbf{n} \in \mathbb{R}^{1,2}_{\circlearrowright}$, then $\mathbf{n}$ lies on either the upper or the lower sheet of $\mathcal{K}_L$. If $\mathbf{n}$ lies in $\mathcal{K}^+_L$, then there exists an element $g \in \mathsf{SO}(1,2)_0$ such that $g(\mathbf{n}) = \mathbf{e}_1 - \mathbf{e}_3$. Further, if $\mathbf{n}$ lies in $\mathcal{K}^-_L$, then there exists an element $g \in \mathsf{SO}(1,2)_0$ such that $g(\mathbf{n}) = -(\mathbf{e}_1 - \mathbf{e}_3)$. Thus

$$\mathbf{n} = g^{-1}(\mathbf{e}_1 - \mathbf{e}_3) \quad \text{or} \quad \mathbf{n} = -g^{-1}(\mathbf{e}_1 - \mathbf{e}_3) \qquad \square$$

and it follows from PROPOSITIONS 3.4.2 and 3.4.1 that there exists some $g' \in \mathsf{SO}(1,2)_0$ such that under the hat map, $N = \phi_{g'}(E_1 - E_3) = \mathrm{Ad}_{g'}(E_1 - E_3)$ or $T = -\phi_{g'}(E_1 - E_3) = -\mathrm{Ad}_{g'}(E_1 - E_3)$.

4.2.5 PROPOSITION. *Given any inhomogeneous left-invariant affine system* $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$, *then the image* $\mathbf{\Gamma}$ *in* $\mathbb{R}^{1,2}_{\circlearrowright}$ *of the trace* Γ *admits at least one spacelike vector: that is, there exists some* $u \in \mathbb{R}^\ell$ *such that* $\mathbf{s} = \mathrm{hat}\,(\Xi(1,u))$ *is spacelike.*

PROOF. For the 3-input case, the result is clear, since we showed in THEOREM 4.1.35 that for each 3-input system Σ the trace Γ has the form $\langle B_1, B_2, B_3\rangle$ which is clearly full rank and therefore the image $\mathbf{\Gamma}$ admits an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_i$, $i = 2, 3$ are spacelike from PROPOSITION 2.1.7 and COROLLARY 2.1.8.

Consider the 2-input [1-input] inhomogeneous systems Σ such that $\mathbf{\Gamma}$ is a plane [line] not parallel to the plane $\langle \mathbf{e}_2, \mathbf{e}_3\rangle$. Each such $\mathbf{\Gamma}$ has both [either] an $\mathbf{e}_2$-intercept $\lambda_1\mathbf{e}_2$ and [or] an $\mathbf{e}_3$-intercept $\lambda_2\mathbf{e}_3$, for λ_1, λ_2 real scalars which cannot both be zero since Σ is inhomogeneous.

Then there exist some $u, u' \in \mathbb{R}^2[\mathbb{R}]$ such that $\lambda_1 e_2 = \Xi_u^{\natural}$ and [or] $\lambda_2 e_3 = \Xi_{u'}^{\natural}$. But since $\lambda_1 e_2$, $\lambda_2 e_3$ are spacelike ($\lambda_1 e_2 \odot \lambda_1 e_2 = \lambda_1^2$, $\lambda_2 e_3 \odot \lambda_2 e_3 = \lambda_2^2$) , then Γ admits a spacelike vector.

Consider the 2-input [1-input] inhomogeneous systems Σ such that Γ is a plane [line] parallel to the plane $\langle e_2, e_3 \rangle$. Then Γ is a plane [ruling of a plane] $\Gamma_{\alpha'} = \alpha' e_1 + \langle e_2, e_3 \rangle$, which intersects any hyperboloid of one sheet $\mathcal{H}_\alpha$ $(\alpha > 0)$ in the set

$$\mathcal{H}_\alpha \cap \Gamma = \mathcal{C}_{\alpha',\alpha} = \left\{ (\alpha', \sqrt{\alpha + (\alpha')^2}\cos(\theta), \sqrt{\alpha + (\alpha')^2}\sin(\theta)) \ : \ \theta \in \mathbb{R} \right\}$$

of spacelike vectors. If Γ is the plane Γ_α, then it admits a spacelike vector, since there exists some $u \in \mathbb{R}^\ell$ such that $s = \Xi_u^{\natural} \in \mathcal{C}_{\alpha',\alpha}$. Assume that Γ is an arbitrary ruling of this plane, $\Gamma = \{(\alpha', \frac{-cx_3 - a\alpha'}{b}, x_3) \ : \ x_3 \in \mathbb{R}\}$, where $b^2 + c^2 \neq 0$. We assume here that $b \neq 0$, but the proof in the case that $c \neq 0$ follows identical steps. If a is zero, then this line passes through the centre $(\alpha', 0, 0)$ of $\mathcal{C}_{\alpha',\alpha}$ and so it intersects the circle $\mathcal{C}_{\alpha',\alpha}$ at the spacelike point

$$s = \left(\alpha', \frac{-c\sqrt{b^2}\sqrt{\alpha + (\alpha')^2}}{b\sqrt{c^2 + b^2}}, \frac{\sqrt{b^2}\sqrt{\alpha + (\alpha')^2}}{\sqrt{c^2 + b^2}} \right).$$

where $\cos(\tan^{-1}(\frac{c}{b})) = \frac{\sqrt{b^2}}{\sqrt{c^2+b^2}}, \sin(\tan^{-1}(\frac{c}{b})) = \frac{\sqrt{b^2}c}{b\sqrt{c^2+b^2}}$. From the parametrization of Γ, there exists some $u \in \mathbb{R}$ such that $s = \Xi_u^{\natural}$ and so Γ admits a spacelike vector. If $a \neq 0$, then the line Γ intersects $\mathcal{C}_{\alpha',\alpha}$ in the point $(\alpha', \frac{-cx_3' - a\alpha'}{b}, x_3')$ where

$$x_3' = \frac{-ac\alpha' - \sqrt{b^4\alpha + b^2c^2\alpha - a^2b^2(\alpha')^2 + b^4(\alpha')^2 + b^2c^2(\alpha')^2}}{b^2 + c^2} \qquad \square$$

and we choose α such that $c > \sqrt{\frac{a^2(\alpha')^2}{\alpha + (\alpha')^2}}$ or $c < -\sqrt{\frac{a^2(\alpha')^2}{\alpha + (\alpha')^2}}$ so that this value is real. But each element of $\mathcal{C}_{\alpha',\alpha}$ is spacelike. Thus the element $s = (\alpha', \frac{-cx_3' - a\alpha'}{b}, x_3')$ in Γ is spacelike, and there exists some $u \in \mathbb{R}$ such that $s = \Xi_u^{\natural}$, Thus Γ admits a spacelike vector.

4.2.6 LEMMA. *No element $k(\theta) = \exp(\theta E_1) \in \mathsf{SO}(1,2)_0$ for $\theta \in \mathbb{R}$ can be expressed as a finite product of the exponentials of spacelike and lightlike elements of $\mathfrak{so}(1,2)$.*

PROOF. Consider $k = k(\theta_1) = \exp(\theta_1 E_1)$ for some $\theta_1 \in \mathbb{R}$ and assume that there exists a finite product of exponentials of spacelike and lightlike elements $g_1, g_2, ... g_p$ in $\mathsf{SO}(1,2)_0$ such that $k = g_1 g_2 ... g_p$. But by the KAN-decomposition of $\mathsf{SO}(1,2)_0$, the elements $g_1, g_2, ... g_p$ can each be expressed as products $g_1 = k_1 a_1 n_1, g_2 = k_2 a_2 n_2 ... g_p = k_p a_p n_p$, where $a_i n_i \neq \mathbf{1}$ for any $i = 1, 2 ... p$, since we have assumed that the elements are exponentials of spacelike or lightlike elements. Then

$$\begin{aligned}
k &= k_1 a_1 n_1 \cdot k_2 a_2 n_2 \ldots \cdot k_p a_p n_p && (4.2.1)\\
\Rightarrow k_1^{-1} \cdot k &= a_1 n_1 \cdot k_2 a_2 n_2 \ldots \cdot k_p a_p n_p && (4.2.2)\\
\Rightarrow k_1^{-1} \cdot k &= (a_1 n_1) \cdot k_2 a_2 n_2 \ldots \cdot k_p (a_1 n_1)^{-1} (a_{p_1} n_{p_1}), && (4.2.3)\\
\Rightarrow k_1^{-1} \cdot k &= k_2' a_2' n_2' \ldots \cdot k_p' (a_{p_1} n_{p_1}), && (4.2.4)
\end{aligned}$$

where in (4.2.4) $k_i' = (a_1 n_1) k_i (a_1 n_1)^{-1}$, $a_i' = (a_1 n_1) a_i (a_1 n_1)^{-1}, n_i' = (a_1 n_1) n_i (a_1 n_1)^{-1}$ and from the group property of AN (A.6.8), the element $a_{p_1} n_{p_1}$ introduced in (4.2.3) such that $(a_1 n_1)^{-1}(a_{p_1} n_{p_1}) = a_p n_p$ always exists: $a_{p_1} n_{p_1} = a_1 n_1 a_p n_p$. Then similarly

$$\begin{aligned} k_1^{-1} \cdot k &= k_2' a_2' n_2' \ldots \cdot k_p'(a_p' n_p'), && (4.2.5) \\ \Rightarrow (k_2')^{-1} k_1^{-1} \cdot k &= a_2' n_2' \ldots \cdot k_p'(a_p' n_p') && (4.2.6) \\ \Rightarrow (k_2')^{-1} k_1^{-1} \cdot k &= (a_2' n_2') \ldots \cdot k_p' (a_2' n_2')^{-1} (a_{p_2} n_{p_2}). && (4.2.7) \end{aligned}$$

We continue until we obtain $(k_p^{'' \cdots '})^{-1} \cdot \ldots (k_2')^{-1} k_1^{-1} \cdot k(\theta_1) = a_{p_p} n_{p_p}(\theta_2)$, which exists since $a_i n_i \neq \mathbf{1}$ for $i = 1, 2 \ldots p$. But $\left\| \left(k_p^{'' \cdots '}\right)^{-1} \cdot \ldots k_2' k_1^{-1} \right\| \leq \left\| \left(k_p^{'' \cdots '}\right)^{-1} \right\| \ldots \left\| (k_2')^{-1} \right\| \|k_1^{-1}\| \cdot \|k\| = u_1 \cdot u_2 \ldots u_p$, where u_i are the finite upper bounds of $\left\{ (k_i^{'' \cdots '})^{-1}(t) \ : \ t \in \mathbb{R} \right\}$, which are compact one-parameter subgroups . Using Mathematica (C.3),

$$a_{p_p}(\theta_3) n_{p_p}(\theta_2) = \begin{bmatrix} \frac{1}{2}\left((2+\theta_2^2)\cosh\theta_3 + \theta_2^2 \sinh\theta_3\right) & -\frac{1}{2} e^{\theta_3} \theta_2^2 + \sinh\theta_3 & -e^{\theta_3}\theta_2 \\ \frac{e^{\theta_3}\theta_2^2}{2} + \sinh\theta_3 & \cosh\theta_3 - \frac{1}{2}\theta_2^2 \cosh\theta_3 & -\frac{1}{2}\theta_2^2 \sinh\theta_3 - e^{\theta_3}\theta_2 \\ -\theta_2 & \theta_2 & 1 \end{bmatrix}$$

and so $\|a_{p_p}(\theta_3) n_{p_p}(\theta_2)\| = \sqrt{1 + e^{-2\theta_3} + 2\theta_2^2 + e^{2\theta_3}\left(1+\theta_2^2\right)^2}$. Let $\theta_1, \theta_2, \theta_3$ be continuous, monotonic increasing functions of t, $\theta_i(\cdot) : \mathbb{R} \to \mathbb{R}$. Then it follows from the from the equality $(k_p^{'' \cdots '})^{-1} \cdot \ldots (k_2')^{-1} \cdot k_1^{-1} \cdot k(\theta_1) = a_{p_p}(\theta_3) n_{p_p}(\theta_2)$ that

$$\lim_{t\to\infty} \left\| (k_p^{'' \cdots '})^{-1} \cdot \ldots k_1^{-1} \cdot k(\theta_1(t)) \right\| = \lim_{t\to\infty} \|a_{p_p}(\theta_3) n_{p_p} \theta_2(t)\| \quad \Rightarrow \quad u_1 \cdot u_2 \ldots \cdot u_p = \infty$$

which is impossible, since each of $u_1, u_2 \ldots u_p$ are finite and we have assumed that p is finite. Thus it follows that k cannot be expressed as a finite product of exponentials of spacelike or lightlike elements of $\mathfrak{so}(1,2)_0$. □

4.2.2 A controllability criterion on $\mathfrak{so}(1,2)$

We are now in a position to state and prove

4.2.7 THEOREM. *An inhomogeneous control affine system $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ is controllable if and only if the image $\mathbf{\Gamma} \subseteq \mathbb{R}^{1,2}$ of its trace Γ under the hat map admits at least one timelike vector.*

PROOF. ($\Rightarrow$) In order to show that the image under the hat map of the trace of any controllable system Σ admits a timelike vector, we prove the equivalent statement that if $\mathbf{\Gamma}$ admits no timelike vector, then Σ cannot be controllable. Assume Σ is such a system: that is, there exists no $u^0 \in \mathbb{R}^\ell$ such that $\mathbf{t} \in \mathbf{\Gamma}, \mathbf{t} = \text{hat}\left(\Xi(\mathbf{1}, u^0)\right)$ is timelike. Equivalently, every element $A_i \in \Gamma$ is either lightlike or spacelike. Particularly, $E_1 \notin \Gamma$.

In (A.7.16), we stated that each element of the attainable set $\mathcal{A}$ is expressible as a finite product of exponentials $g = \exp(t_n A_n) \ldots \exp(t_1 A_1), A_i \in \Gamma$. But in LEMMA 4.2.6 we showed that the element $\exp(tE_1)$ is not expressible as a finite product of exponentials of spacelike or lightlike elements A_j. Then since $E_1 \notin \Gamma$. it follows that for each $t \in \mathbb{R}^+$, $k = \exp(tE_1) \notin \mathcal{A}$.

From the Iwasawa decomposition $\mathsf{SO}(1,2)_0 = \mathsf{KAN}$ (THEOREM 3.6.10), then $\mathcal{A} \cap \mathsf{K} = \{\mathbf{1}\}$, and thus $\mathcal{A} \subsetneq \mathsf{SO}(1,2)_0$. Thus from (A.7.13), Σ cannot be controllable on $\mathsf{SO}(1,2)_0$.
($\Leftarrow$) We stated in (A.7.20) that for a system Σ on $\mathsf{SO}(1,2)_0$ to be controllable, the trace Γ must be full rank, and there must exist some $u^0 \in \mathbb{R}^\ell$ such that the trajectory $\{\exp(t\Xi(\mathbf{1}, u^0)) \ : \ t > 0\}$ is periodic. We assume the image $\mathbf{\Gamma}$ of the trace Γ of Σ admits a timelike vector and show how both of these conditions is met by the presence of a timelike vector in $\mathbf{\Gamma}$.

In PROPOSITION 4.2.5, we showed that for each inhomogeneous system Σ the image $\mathbf{\Gamma}$ of the trace Γ admits a spacelike vector $\mathbf{s}$. Since we have assumed that $\mathbf{\Gamma}$ admits a timelike vector $\mathbf{t}$, then $\mathbf{t} \odot \mathbf{s}$ is an element of $\mathsf{Lie}(\mathbf{\Gamma})$ and from PROPOSITION 2.1.10 is a spacelike vector $\mathbf{s}'$ orthogonal to $\mathbf{s}$. Thus $\{\mathbf{s}, \mathbf{s}', \mathbf{t}\}$ is an orthogonal basis of $\mathbb{R}^{1,2}$, and $\langle \mathbf{s}, \mathbf{s}', \mathbf{t}\rangle = \mathbb{R}^{1,2}_{\odot}$. Then under the Lie algebra isomorphism hat^{-1}, it follows that S, S', T are in $\mathsf{Lie}(\Gamma)$, and $\langle S, S, T\rangle = \mathfrak{so}(1,2)$. Thus Γ is full rank. By assumption, there exists some $u^0 \in \mathbb{R}^\ell$ such that $\mathbf{t} = \text{hat}\left(\Xi(\mathbf{1}, u^0)\right) \in \mathbf{\Gamma}$ is timelike. Correspondingly, under the inverse of the hat map, then $T = \Xi(\mathbf{1}, u^0) \in \Gamma$, where by PROPOSITION 4.2.4, $\Xi(\mathbf{1}, u^0) = \pm\alpha g E_1 g^{-1}$ for some $\alpha \in \mathbb{R}^+, g \in \mathsf{SO}(1,2)_0$. Then taking $u = u^0$, the trajectory $\left\{e^{t\Xi(\mathbf{1}, u^0)} \ : \ t \in \mathbb{R}^+\right\} = \left\{\exp(t(\alpha g E_1 g^{-1})) \ : \ t \in \mathbb{R}^+\right\}$ is periodic by PROPOSITION 4.2.2. □

Thus with PROPOSITION 4.2.1 we have necessary and sufficient conditions for the controllability of any affine control system on $\mathsf{SO}(1,2)_0$:

4.2.8 THEOREM. *A homogeneous control affine system $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ is controllable if and only if it is full rank.*

PROOF. Firstly, since $\Gamma = \langle B_1, ..., B_\ell\rangle = \langle B_1, ..., B_\ell\rangle = \langle B_1, ..., B_\ell\rangle = -\Gamma$, then all homogeneous systems are symmetric. Thus since by THEOREM 3.1.2 $\mathsf{SO}(1,2)_0$ is connected, it follows from PROPOSITION 4.2.1 that these systems are controllable on $\mathsf{SO}(1,2)_0$ if and only if they are full rank. □

We state THEOREM 4.2.7 and Theorem 4.2.8 together as

4.2.9 THEOREM. (A CONTROLLABILITY CRITERION ON $\mathsf{SO}(1,2)_0$) *An inhomogeneous control affine system $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ is controllable if and only if the image $\mathbf{\Gamma} \subseteq \mathbb{R}^{1,2}$ of its trace Γ under the hat map admits at least one timelike vector. A homogeneous control affine system $\Sigma = (\mathsf{SO}(1,2)_0, \Xi)$ is controllable if and only if it is full rank*

Chapter 5

Optimal Control on $\mathsf{SO}(1,2)_0$

5.1 Introduction

Using the classification results of CHAPTER 4, we consider the optimal control problem with quadratic costs (A.10.6)

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \qquad g \in \mathsf{SO}(1,2)_0,\ (u_1, u_2) \in \mathbb{R}^2$$
$$g(0) = g_0, \quad g(T) = g_T$$
$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + c_2 u_2^2(t)\right) dt \to \min, \quad c_1, c_2 > 0$$

where in THEOREMS 5.2.2-5.2.5 we consider $\Xi_2^{(2,0)}(\mathbf{1}, u) = u_1 E_2 + u_2 E_1$ and in THEOREMS 5.3.1-5.3.5 $\Xi_3^{(2,0)}(\mathbf{1}, u) = u_1 E_3 + u_2 E_2$, the two representative systems of all 2-input homogeneous control affine systems on $\mathsf{SO}(1,2)_0$ (THEOREM 4.1.10). We establish the reduced extremal equations (A.10.9) and their solutions the extremal pairs $(g(\cdot), p(\cdot), u(\cdot))$ by working in the trivialization (A.5.28) $\mathsf{T}^*\mathsf{SO}(1,2)_0 = \mathsf{SO}(1,2)_0 \times \mathfrak{so}(1,2)^*$ of the cotangent bundle, and express the projections of the extremal curves on $\mathfrak{so}(1,2)^*$ in terms of Jacobi elliptic functions, and the projection $g(\cdot)$ of the extremal curve onto $\mathsf{SO}(1,2)_0$ as a product of exponentials such that for each $t \in [0, T]$

$$g(t) = \exp\left(\phi_3(t) N\right) \exp(\phi_2(t) E_2) \exp(\phi_1(t) E_1)$$

for $\phi_i(\cdot) : [0, T] \to \mathbb{R}$ continuous functions of t. This expression follows from the Iwasawa decomposition (3.6.8) of $\mathsf{SO}(1,2)_0$.

While expression in terms of Jacobi elliptic functions can be used analagously to give solutions for each of the 10 typical systems under l.d.f.e. classification, we have found that the solutions which arise in the inhomogeneous cases are unintuitive and unhelpful in terms of applications, and so we choose to concentrate only on the homogeneous cases in this thesis.

Note that as the PMP gives a set of necessary conditions for a trajectory to be optimal, it provides us with only a family of possible candidates for optimal controls and their corresponding optimal trajectories. However, since in both cases (THEOREMS 5.2.1, 5.3.1) the PMP gives rise to exactly one optimal control, for the purposes of this thesis we will assume that there always exists an optimal trajectory in the family of the projections of extremals onto $\mathsf{SO}(1,2)_0$, and that it is the projection of the extremal curve which arises using the given control.

5.1.1 REMARK. We identify $\mathfrak{so}(1,2)$ with $\mathfrak{so}(1,2)^*$ via the pairing $\kappa(P,X) = p(X)$. Particularly, $\kappa^\flat(E_1) = E_1^*, \kappa^\flat(E_2) = E_2^*$ and $\kappa^\flat(E_3) = E_3^*$ where $\{E_1^*, E_2^*, E_3^*\}$ is the dual basis. Thus the projection $p(\cdot) : [0,T] \to \mathfrak{so}(1,2)^*$ of each extremal curve may be identified with a curve $P(\cdot)$ in $\mathfrak{so}(1,2)$: $\kappa(P(\cdot),X) = p(\cdot)X$. Expressing $p(\cdot) = p_1(\cdot)E_1^* + p_2(\cdot)E_2^* + p_3(\cdot)E_3^*$, then for every $t \in [0,T]$,

$$\kappa^\flat(p(t)) = \kappa^\flat(p_1(t))E_1 + \kappa^\flat(p_2(t))E_2 + \kappa^\flat(p_3(t))E_3 = P_1(t)E_1 + P_2(t)E_2 + P_3(t)E_3 = P(t)$$

where $P(\cdot)$ is a curve in $\mathfrak{so}(1,2)$. Thus in both of the remaining sections we consider the image $P(\cdot)$ in $\mathfrak{so}(1,2)$ of $p(\cdot)$ in the discussion of the solutions of the reduced extremal equations. Particularly, since from (A.9.7) $P_i(\cdot) = p(\cdot)(E_i) = H_{E_i}(p(\cdot))$, we may then use (4) in (A.9.4) to write $\{P_i, P_j\} = \{H_{E_i}, H_{E_j}\} = H_{[E_i,E_j]}$. Thus the image $\{P_1, P_2\} = -P_3, \{P_2, P_3\} = P_1$ and $\{P_3, P_1\} = -P_2$, which we use to set up the system (A.10.9) of reduced extremal equations.

5.1.2 PROPOSITION. *The function $K : \mathfrak{so}^*(1,2) \to \mathbb{R}, K(p) = -\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2$ is a Casimir function.*

PROOF. If for each co-adjoint orbit $\widetilde{\mathcal{O}}_p$, $K(\widetilde{\mathcal{O}}_p) = \text{const.}$ then for all $p \in \mathfrak{so}(1,2)^*$,

$$K(\{\mathrm{Ad}_g^*(p) \;:\; g \in \mathsf{SO}(1,2)_0\}) = \text{const.} = (\mathrm{Ad}_g^*(p)) \;:\; g \in \mathsf{SO}(1,2)_0\}$$

and so for each $p \in \mathfrak{so}(1,2)^*, g \in \mathsf{SO}(1,2)_0$, $K(\mathrm{Ad}_g^*(p)) = \text{const.}$ and K is a Casimir function (A.10.13). Thus we require only to show that K is constant on every co-adjoint orbit. For every vector $\mathbf{a} \in \mathbb{R}^{1,2}_{\otimes}$,

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \quad \Leftrightarrow \quad \left(\kappa^\flat\right)^{-1}\widehat{\mathbf{a}} = p_1E_1^* + p_2E_2^* + p_3E_3^*$$

since $\left(\kappa^\flat\right)^{-1} \circ$ hat is a linear map. Thus for any $p \in \mathfrak{so}^*(1,2)$, the co-adjoint orbit $\widetilde{\mathcal{O}}_p$ through p of THEOREM 3.5.6 is given by $-p_1^2 + p_2^2 + p_3^2 = \text{const}$. Then for any $p' \in \widetilde{\mathcal{O}}_p$, $K(p') = \frac{1}{2}\text{const}$. Thus K is constant on $\widetilde{\mathcal{O}}_p$, and the result follows. □

5.1.3 REMARK. In PROPOSITION (5.1.2) we determined the Casimir function $K(p) = -\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2$. As we stated in the previous remark, in the proofs of THEOREMS 5.2.2 and 5.3.3 we will consider the image of $K_{\mathfrak{so}^*}$ under the map $\kappa^\flat$, which for each $P \in \mathfrak{so}(1,2)$ is given by $K(P) = -\frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2}P_3^2$. Clearly, this function is constant on the images under $\kappa^\flat$ of the co-adjoint orbits in $\mathfrak{so}(1,2)^*$, which by THEOREM 3.5.6 are the adjoint orbits of $\mathfrak{so}(1,2)$.

5.2 The case $\Gamma = \langle E_1, E_2 \rangle$

5.2.1 THEOREM. *Given the left-invariant control problem $(\Sigma, \mathcal{L}, (g_0, g_T, T))$*

$$\dot{g} = g(u_1E_2 + u_2E_1), \qquad g \in \mathsf{SO}(1,2)_0,\ (u_1, u_2) \in \mathbb{R}^2$$

$$g(0) = g_0, \quad g(T) = g_T$$

$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1u_1^2(t) + c_2u_2^2(t)\right) dt \to \min, \quad c_1, c_2 > 0$$

then the optimal controls are

$$u_1 = \frac{P_2}{c_1} \quad \textit{and} \quad u_2 = \frac{P_1}{c_2}$$

and the optimal Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left(\frac{P_2^2}{c_1} + \frac{P_1^2}{c_2}\right) \tag{5.2.1}$$

where P_1 and P_2 are solutions of the reduced extremal equations

$$\begin{cases} \dot{P}_1 = -\dfrac{1}{c_1}P_2P_3 & (5.2.2)\\ \dot{P}_2 = \dfrac{1}{c_2}P_1P_3 & (5.2.3)\\ \dot{P}_3 = -\left(\dfrac{c_2+c_1}{c_1c_2}\right)P_1P_2. & (5.2.4)\end{cases}$$

PROOF. By (A.10.1), the control Hamiltonian of the optimal control problem is

$$\mathcal{H} = -\frac{1}{2}\left(c_1u_1^2 + c_2u_2^2\right) + p(u_1E_2 + u_2E_1).$$

As we have stated, we may identify $P_i = p(E_i)$ via the Killing form; then

$$\mathcal{H} = -\frac{1}{2}\left(c_1u_1^2 + c_2u_2^2\right) + u_1P_2 + u_2P_1. \tag{5.2.5}$$

By the PMP, the optimal Hamiltonian satisfies

$$\frac{\partial\mathcal{H}}{\partial u_1} = -c_1u_1 + P_2 = 0 \Leftrightarrow u_1 = \frac{1}{c_1}P_2 \tag{5.2.6}$$

$$\frac{\partial\mathcal{H}}{\partial u_2} = -c_2u_2 + P_1 = 0 \Leftrightarrow u_2 = \frac{1}{c_2}P_1. \tag{5.2.7}$$

Thus u_1 and u_2 are the optimal controls, and the optimal Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\left(\frac{P_2^2}{c_1} + \frac{P_1^2}{c_2}\right)$$

by substitution of (5.2.6) and (5.2.7) into (5.2.5). Then from (A.10.1), the system of reduced extremal equations is given by

$$\begin{cases} \dot{P}_1 = P_2\left\{P_1, \frac{P_2}{c_1}\right\} + P_1\left\{P_1, \frac{P_1}{c_2}\right\} = -\frac{1}{c_1}P_2P_3\\ \dot{P}_2 = P_2\left\{P_2, \frac{P_2}{c_1}\right\} + P_1\left\{P_2, \frac{P_1}{c_2}\right\} = \frac{1}{c_2}P_1P_3\\ \dot{P}_3 = P_2\left\{P_3, \frac{P_2}{c_1}\right\} + P_1\left\{P_3, \frac{P_1}{c_2}\right\} = -\left(\frac{c_1+c_2}{c_1c_2}\right)P_1P_2.\end{cases}$$

□

5.2.1 Explicit integration of the extremal curve $(p(\cdot), g(\cdot))$

In PROPOSITION 5.1.2 we determined a Casimir function K which we have noted we may express in $\mathfrak{so}(1,2)$ as $K = -\frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2}P_3^2$. In the next theorem we use this function in conjunction with the optimal Hamlitonian $\mathcal{H}$ to determine the solutions of system (5.2.3), (5.2.2), (5.2.4), as described in the introductory section. Note that the values of K may be positive, negative or zero. This separation of the level surfaces of the Casimir function K appears naturally in the process of solving the reduced extremal equations, as we illustrate in fig. C.31.

5.2.2 THEOREM. *The reduced extremal equations can be solved in terms of Jacobi elliptic functions by*
Case 1 *($K < 0$)*

$$\begin{cases} P_1(t) &= \lambda C_2\sqrt{b^2 + b^2\left(\mathrm{sn}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_2(t) &= -\lambda C_1\sqrt{a^2 - b^2\left(\mathrm{sn}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_3(t) &= b \cdot \mathrm{sn}\left(Ct, \frac{b}{a}\right) \end{cases}$$

or

$$\begin{cases} P_1(t) &= \lambda C_2\sqrt{b^2 + b^2\left(\mathrm{cd}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_2(t) &= -\lambda C_1\sqrt{a^2 - b^2\left(\mathrm{cd}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_3(t) &= b \cdot \mathrm{cd}\left(Ct, \frac{b}{a}\right) \end{cases}$$

where $a = \lambda\sqrt{2\mathcal{H}c_2 + 2K}$, $b = \lambda\sqrt{2\mathcal{H}c_1 - 2K}, C_1 = \sqrt{\frac{c_1}{(c_1+c_2)}}, C_2 = \sqrt{\frac{c_2}{(c_1+c_2)}}, C = \frac{a}{\sqrt{c_1c_2}}$ *and* $\lambda = \pm 1$,
Case 2 *($K = 0$)*

$$\begin{cases} P_1(t) &= \lambda C_2\sqrt{b^2 + b^2\left(\mathrm{sn}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_2(t) &= -\lambda C_1\sqrt{a^2 - b^2\left(\mathrm{sn}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_3(t) &= b \cdot \mathrm{sn}\left(Ct, \frac{b}{a}\right) \end{cases}$$

or

$$\begin{cases} P_1(t) &= \lambda C_2\sqrt{b^2 + b^2\left(\mathrm{cd}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_2(t) &= -\lambda C_1\sqrt{a^2 - b^2\left(\cdot\mathrm{cd}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_3(t) &= b \cdot \mathrm{cd}\left(Ct, \frac{b}{a}\right) \end{cases}$$

where $a = \lambda\sqrt{2\mathcal{H}c_2 + 2K}$, $b = \lambda\sqrt{2\mathcal{H}c_1 - 2K}, C_1 = \sqrt{\frac{c_1}{(c_1+c_2)}}, C_2 = \sqrt{\frac{c_2}{(c_1+c_2)}}, C = \frac{a}{\sqrt{c_1c_2}}$ *and* $\lambda = \pm 1$, *or finally*

Case 3a *(if $K > 0$, $\frac{c_1+c_2}{c_1}P_1^2 < P_3^2$)*

$$\begin{cases} P_1(t) &= \lambda C_2\sqrt{b^2\left(\mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)\right)^2 - b^2} \\ P_2(t) &= -\lambda C_1\sqrt{a^2 - b^2\left(\mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)\right)^2} \\ P_3(t) &= b \cdot \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \end{cases}$$

or

$$\begin{cases} P_1(t) &= \lambda C_2\sqrt{a^2\left(\mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)\right)^2 - b^2} \\ P_2(t) &= -\lambda C_1\sqrt{a^2 - a^2\left(\mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)\right)^2} \\ P_3(t) &= a \cdot \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \end{cases}$$

where $a = \sqrt{2\mathcal{H}c_2 + 2K}$, $b = \lambda\sqrt{2K - 2\mathcal{H}c_1}, C_1 = \sqrt{\frac{c_1}{(c_1+c_2)}}, C_2 = \sqrt{\frac{c_2}{(c_1+c_2)}}, C = \frac{a}{\sqrt{c_1c_2}}$ *and* $\lambda = \pm 1$, *or*

Case 3b *(if $K > 0$, $\left(\frac{c_1+c_2}{c_1}P_1^2\right) > P_3^2$)*

$$\begin{cases} P_1(t) &= \lambda C_2\sqrt{b^2 + b^2\left(\mathrm{sn}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_2(t) &= -\lambda C_1\sqrt{a^2 - b^2\left(\mathrm{sn}\left(Ct, \frac{b}{a}\right)\right)^2} \\ P_3(t) &= b \cdot \mathrm{sn}\left(Ct, \frac{b}{a}\right) \end{cases}$$

or

$$\begin{cases} P_1(t) &= \lambda C_2\sqrt{b^2+b^2\left(\mathrm{cd}\left(Ct,\frac{b}{a}\right)\right)^2} \\ P_2(t) &= -\lambda C_1\sqrt{a^2-b^2\left(\cdot\mathrm{cd}\left(Ct,\frac{b}{a}\right)\right)^2} \\ P_3(t) &= b\cdot\mathrm{cd}\left(Ct,\frac{b}{a}\right) \end{cases}$$

where $a=\lambda\sqrt{2\mathcal{H}c_2+2K}$, $b=\lambda\sqrt{2\mathcal{H}c_1-2K}, C_1=\sqrt{\frac{c_1}{(c_1+c_2)}}, C_2=\sqrt{\frac{c_2}{(c_1+c_2)}}, C=\frac{a}{\sqrt{c_1c_2}}$ *and* $\lambda=\pm 1$.

PROOF. From (A.10.13), the extremal curves lie in the intersection of the level surfaces of the optimal Hamiltonian (5.2.1) with the level surfaces of the Casimir function K. We use this fact to express the variables P_1 and P_2 in terms of the variable P_3. From the optimal Hamiltonian, then $c_1P_1^2=2c_1c_2\mathcal{H}-c_2P_2^2$. But $P_1^2=P_2^2+P_3^2-2K$ from the Casimir function K of PROPOSITION 5.1.2 expressed in $\mathfrak{so}(1,2)$. Thus $c_1(P_2^2+P_3^2-2K)=2c_1c_2\mathcal{H}-c_2P_2^2$. Solving for P_2^2, then

$$P_2^2=\left(\frac{c_1}{c_1+c_2}\right)(2K+2\mathcal{H}c_2-P_3^2) \tag{5.2.8}$$

Also from the optimal Hamiltonian, $c_2P_2^2=2c_1c_2\mathcal{H}-c_1P_1^2$. But $P_2^2=2K+P_1^2-P_3^2$ from the Casimir function. Thus $c_2(2K+P_1^2-P_3^2)=2c_1c_2\mathcal{H}-c_1P_1^2$. Solving for P_1^2, then

$$P_1^2=\left(\frac{c_2}{c_1+c_2}\right)(2\mathcal{H}c_1+P_3^2-2K). \tag{5.2.9}$$

Substituting (5.2.8) and (5.2.9) into the extremal equation (5.2.4), then

$$\dot{P}_3^2=\frac{(c_1+c_2)^2}{c_1^2c_2^2}P_1^2P_2^2=\frac{(c_1+c_2)^2}{c_1^2c_2^2}\frac{c_2c_1}{(c_1+c_2)^2}(2K+2\mathcal{H}c_2-P_3^2)(2\mathcal{H}c_1+P_3^2-2K)$$

and we have the system of equations

$$\begin{cases} P_2^2 = \left(\frac{c_1}{c_1+c_2}\right)(K+2\mathcal{H}c_2-P_3^2) & (5.2.10)\\ P_3^2 = \left(\frac{c_2}{c_1+c_2}\right)(2\mathcal{H}c_1+P_3^2-2K) & (5.2.11)\\ \dot{P_3}^2 = \left(\frac{1}{c_1c_2}\right)(2K+2\mathcal{H}c_2-P_3^2)(2\mathcal{H}c_1+P_3^2-2K) & (5.2.12)\end{cases}$$

where from (5.2.12)

$$\begin{aligned} \left(\frac{dP_3}{dt}\right)^2 &= \left(\frac{1}{c_1c_2}\right)(2\mathcal{H}c_2+2K-P_3^2)(2\mathcal{H}c_1+P_3^2-2K) \\ \Rightarrow \frac{dt}{dP_3} &= \frac{\sqrt{c_1c_2}}{\sqrt{(2\mathcal{H}c_2+2K-P_3^2)(2\mathcal{H}c_1+P_3^2-2K)}} \\ \Rightarrow \int_0^t d\tau &= \sqrt{c_1c_2}\int_0^{P_3}\frac{dP_3}{\sqrt{(2\mathcal{H}c_2+2K-P_3^2)(2\mathcal{H}c_1+P_3^2-2K)}} \end{aligned}$$

and thus

$$t = \sqrt{c_1c_2}\int_0^{P_3}\frac{dP_3}{\sqrt{(a^2-P_3^2)(P_3^2-b^2)}} \tag{5.2.13}$$

where $a^2=(2\mathcal{H}c_2+2K)$ and $b^2=(2K-2\mathcal{H}c_1)$. Using equations (5.2.8) and (5.2.9), then

$$P_2^2=\left(\frac{c_1}{c_1+c_2}\right)(2K+2\mathcal{H}c_2-P_3^2) \quad\Leftrightarrow\quad 2K+2\mathcal{H}c_2=P_3^2+\left(\frac{c_1+c_2}{c_1}\right)P_2^2=a^2$$

$$P_1^2 = \left(\frac{c_2}{c_1+c_2}\right)(2\mathcal{H}c_1 + P_3^2 - 2K) \quad \Leftrightarrow \quad 2K - 2\mathcal{H}c_1 = P_3^2 - \left(\frac{c_1+c_2}{c_2}\right)P_1^2 = b^2$$

and thus

$$a^2 > 0 \quad \Leftrightarrow -\left(\frac{c_1+c_2}{c_1}\right)P_2^2 < P_3^2 \quad \text{and} \quad b^2 > 0 \quad \Leftrightarrow \left(\frac{c_1+c_2}{c_2}\right)P_1^2 < P_3^2.$$

Note that since $c_1, c_2 > 0$, then the condition $-\left(\frac{c_1+c_2}{c_1}\right)P_2^2 < P_3^2$ for $a^2 > 0$ is always satisfied; thus we consider the two cases

$$-\left(\frac{c_1+c_2}{c_1}\right)P_2^2 < P_3^2, \quad \text{and} \quad \left(\frac{c_1+c_2}{c_2}\right)P_1^2 < P_3^2 \tag{5.2.14}$$

$$-\left(\frac{c_1+c_2}{c_1}\right)P_2^2 < P_3^2, \quad \text{and} \quad \left(\frac{c_1+c_2}{c_2}\right)P_1^2 > P_3^2 \tag{5.2.15}$$

where correspondingly we express (5.2.13) as an elliptic integral

$$\begin{cases} t = \sqrt{c_1c_2}\displaystyle\int_0^{P_3} \frac{dP_3}{\sqrt{(a^2-P_3^2)(P_3^2-b^2)}} & (5.2.16) \\ t = \sqrt{c_1c_2}\displaystyle\int_0^{P_3} \frac{dP_3}{\sqrt{(a^2-P_3^2)(b^2-P_3^2)}} & (5.2.17) \end{cases}$$

where in the first case $b^2 = 2K - 2\mathcal{H}c_1$, and in the second case $b^2 = 2\mathcal{H}c_1 - 2K$.

We have already stated that the Casimir function K may take positive or negative values or may be zero (corresponding to when the level surfaces of K take the form of hyperboloids of one sheet, hyperboloids of two sheets or the right cone, respectively). We consider each of the cases $K > 0, K < 0$ and $K = 0$ separately. However, since in case (5.2.14) condition $b^2 > 0$ corresponds to requiring that $2K - 2\mathcal{H}c_1 > 0$ and $\mathcal{H}$ and c_1 are everywhere positive, this circumstance is possible only where $K > 0$. Thus for the cases $K < 0$ and $K = 0$ we use only the value $b^2 = 2\mathcal{H}c_1 - 2K$ and express (5.2.13) as elliptic integrals of the form (5.2.17). Note also that the requirement $a > P_3 > b$ (A.11) for expressing the solutions in terms of the elliptic functions dc and ns (A.11) gives the condition

$$a > b \quad \Rightarrow \quad 2\mathcal{H}c_2 + 2K > 2K - 2\mathcal{H}c_1 \quad \Rightarrow \quad c_1 > -c_2$$

which is always satisfied since $c_1, c_2 > 0$ by definition. For the elliptic functions cn and sn (A.11), the requirements $a > 0, b > 0$ (A.11) are always satisfied when we choose $a^2 = 2\mathcal{H}c_2 + 2K, b^2 = 2\mathcal{H}c_1 - K$ and $K \leq 0$, since $\mathcal{H}, c_1$ and c_2 are always positive. Thus we have the full solutions

Case 1 $\left(K < 0,\ -\left(\frac{c_1+c_2}{c_1}\right)P_2^2 < P_3^2,\ \left(\frac{c_1+c_2}{c_2}\right)P_1^2 > P_3^2\right)$

$$\begin{aligned} t &= \sqrt{c_1c_2}\textstyle\int_0^{P_3} \frac{dP_3}{\sqrt{(a^2-P_3^2)(b^2-P_3^2)}} \\ &= \frac{\sqrt{c_1c_2}}{a}\cdot \mathrm{sn}^{-1}\left(\frac{P_3}{b}, \frac{b}{a}\right) \\ \Rightarrow P_3 &= b\cdot \mathrm{sn}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{b}{a}\right) \end{aligned}$$

or

$$\begin{aligned} t &= \sqrt{c_1c_2}\textstyle\int_{P_3}^{b} \frac{dP_3}{\sqrt{(a^2-P_3^2)(b^2-P_3^2)}} \\ &= \frac{\sqrt{c_1c_2}}{a}\cdot \mathrm{cd}^{-1}\left(\frac{P_3}{b}, \frac{b}{a}\right) \\ \Rightarrow P_3 &= b\cdot \mathrm{cd}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{b}{a}\right) \end{aligned}$$

where $a^2 = 2\mathcal{H}c_2 + 2K$, $b^2 = 2\mathcal{H}c_1 - 2K$, and we take the square roots $a = \lambda\sqrt{2\mathcal{H}c_2 + 2K}$ and $b = \lambda\sqrt{2\mathcal{H}c_1 - 2K}$ where $\lambda = \pm 1$, since we require the modulus $k = \frac{b}{a}$ to be positive.

Case 2 $\left(K = 0,\ -\left(\frac{c_1+c_2}{c_1}\right)P_2^2 < P_3^2,\ \left(\frac{c_1+c_2}{c_2}\right)P_1^2 > P_3^2\right)$

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_0^{P_3} \frac{dP_3}{\sqrt{(a^2-P_3^2)(b^2-P_3^2)}} \\ &= \frac{\sqrt{c_1c_2}}{a}\cdot \operatorname{sn}^{-1}\left(\frac{P_3}{b}, \frac{b}{a}\right) \\ \Rightarrow P_3 &= b\cdot \operatorname{sn}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{b}{a}\right) \end{aligned}$$

or

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_{P_3}^{b} \frac{dP_3}{\sqrt{(a^2-P_3^2)(b^2-P_3^2)}} \\ &= \frac{\sqrt{c_1c_2}}{a}\cdot \operatorname{cd}^{-1}\left(\frac{P_3}{b}, \frac{b}{a}\right) \\ \Rightarrow P_3 &= b\cdot \operatorname{cd}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{b}{a}\right) \end{aligned}$$

where $a^2 = 2\mathcal{H}c_2 + 2K$, $b^2 = 2\mathcal{H}c_1 - 2K$ and we take the square roots $a = \lambda\sqrt{2\mathcal{H}c_2 + 2K}$ and $b = \lambda\sqrt{2\mathcal{H}c_1 - 2K}$ where $\lambda = \pm 1$, since we require the modulus $k = \frac{b}{a}$ to be positive.

Case 3a $\left(K > 0,\ -\left(\frac{c_1+c_2}{c_1}\right)P_2^2 < P_3^2,\ \left(\frac{c_1+c_2}{c_2}\right)P_1^2 < P_3^2\right)$

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_b^{P_3} \frac{dP_3}{\sqrt{(a^2-P_3^2)(P_3^2-b^2)}} \\ &= \frac{\sqrt{c_1c_2}}{a}\cdot \operatorname{nd}^{-1}\left(\frac{P_3}{b}, \frac{\sqrt{a^2-b^2}}{a}\right) \\ \Rightarrow P_3 &= b\cdot \operatorname{nd}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{\sqrt{a^2-b^2}}{a}\right) \end{aligned}$$

or

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_{P_3}^{a} \frac{dP_3}{\sqrt{(a^2-P_3^2)(P_3^2-b^2)}} \\ &= \frac{\sqrt{c_1c_2}}{a}\cdot \operatorname{dn}^{-1}\left(\frac{P_3}{a}, \frac{\sqrt{a^2-b^2}}{a}\right) \\ \Rightarrow P_3 &= a\cdot \operatorname{dn}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{a^2-b^2}{a}\right) \end{aligned}$$

where $a^2 = 2\mathcal{H}c_2 + 2K$, $b^2 = 2K - 2\mathcal{H}c_1$ and we take the sqnare roots $a = \sqrt{2\mathcal{H}c_2 + 2K}$ and $b = \lambda\sqrt{2K - 2\mathcal{H}c_1}$ where $\lambda = \pm 1$, since we require the modulus $k = \frac{\sqrt{a^2-b^2}}{a}$ to be positive.

Case 3b $\left(K > 0,\ -\left(\frac{c_1+c_2}{c_1}\right)P_2^2 < P_3^2,\ \left(\frac{c_1+c_2}{c_2}\right)P_1^2 > P_3^2\right)$

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_0^{P_3} \frac{dP_3}{\sqrt{(a^2-P_3^2)(b^2-P_3^2)}} \\ &= \frac{\sqrt{c_1c_2}}{a}\cdot \operatorname{sn}^{-1}\left(\frac{P_3}{b}, \frac{b}{a}\right) \\ \Rightarrow P_3 &= b\cdot \operatorname{sn}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{b}{a}\right) \end{aligned}$$

or

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_{P_3}^{b} \frac{dP_3}{\sqrt{(a^2-P_3^2)(b^2-P_3^2)}} \\ &= \frac{\sqrt{c_1c_2}}{a}\cdot \operatorname{cd}^{-1}\left(\frac{P_3}{b}, \frac{b}{a}\right) \\ \Rightarrow P_3 &= b\cdot \operatorname{cd}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{b}{a}\right) \end{aligned}$$

where $a^2 = 2\mathcal{H}c_2 + 2K$, $b^2 = 2\mathcal{H}c_1 - 2K$ and we take the square roots $a = \lambda\sqrt{2\mathcal{H}c_2 + 2K}$ and $b = \lambda\sqrt{2\mathcal{H}c_1 - 2K}$ where $\lambda = \pm 1$, since we require the modulus $k = \frac{b}{a}$ to be positive.

In order to solve for P_1 and P_2, we subsitute back into equations (5.2.9) and (5.2.8). But then $P_1 = \lambda_1 C_2 \sqrt{(2\mathcal{H}c_1 + P_3^2 - 2K)}. P_2 = \lambda_2 C_1 \sqrt{(2K + 2\mathcal{H}c_2 - P_3^2)}$, where $\lambda_1 = \pm 1, \lambda_2 = \pm 1$. In order to find which solutions are valid, we make a simplifying assumption on a and b in the expressions of $P_1, P2$ and P_3, and subsitute these values for P_1 and P_2 back into the equation $f(P_1, P_2 \dot{P}_3) = \dot{P}_3 - \frac{c_1 - c_2}{c_1 c_2} P_1 P_2$ in order to see which sign combiuations (the values of λ_1 and λ_2) are valid, that is, for which combinations $\dot{P}_3 - \frac{c_1 - c_2}{c_1 c_2} P_1 P_2 = 0$. Since the two alternative elliptic functions (cd and dn, or sn and cd) for each case each represent the same solution and thus parametrize the same portions of the intersection of the level surfaces of K and $\mathcal{H}$, we may consider only one of the two alternatives in order to find the combinations of λ_1 and λ_2 for that solution.

Cases 1,2 and 3b: Let $a \to 1, b \to 1$. Then by (A.11.3), the modulus $k = \frac{b}{a} \to 1$, and correspondingly $\text{sn}(t, k) \to \tanh t$. Thus at the limit, we have the functions $P_1 = \lambda_1 C_1 \sqrt{1 + \tanh^2 t}$, $P_3 = \tanh t, P_2 = \lambda_2 C_2 \sqrt{1 - \tanh^2 t} = \lambda_2 C_2 \text{sech} t$ and thus $\dot{P}_3 = \text{sech}^2 t$. Then

$$\begin{aligned}\dot{P}_3 + \left(\frac{c_1 + c_2}{c_1 c_2}\right) P_1 P_2 &= \text{sech}^2 t + \left(\frac{c_1 + c_2}{c_1 c_2}\right) \left(\lambda_1 C_1 \sqrt{1 + \tanh^2 t}\right) (\lambda_2 C_2 \text{sech} t) \\ &= \text{sech} t \left(\text{sech} t + \left(\frac{c_1 + c_2}{c_1 c_2}\right) \left(\lambda_1 C_1 \sqrt{1 + \tanh^2 t}\right) (\lambda_2 C_2)\right).\end{aligned}$$

Since $\text{sech} t > 0$ for all $t \in \mathbb{R}$, this value may be zero if and only if $\lambda_1 \lambda_2 < 0 \Leftrightarrow \lambda_1 = -\lambda_2 = \lambda$, and the valid solutions are as stated.

Case 3a: Let $a \to 1, b \to 0$. Then by (A.11.3), the modulus $k = \frac{a^2 - b^2}{a} \to 1$, and correspondingly $\text{dn}(t, k) \to \text{sech} t$. Thus at the limit, we have the functions $P_1 = \lambda_1 C_1 \sqrt{1 + \tanh^2 t}$, $P_3 = \text{sech} t$, $P_2 = \lambda_2 C_2 \sqrt{1 - \text{sech}^2 t} = \lambda_2 C_2 \tanh t$ and thus $\dot{P}_3 = \text{sech} t \tanh t$. Then

$$\begin{aligned}\dot{P}_3 + \left(\frac{c_1 + c_2}{c_1 c_2}\right) P_1 P_2 &= \text{sech} t \tanh t + \left(\frac{c_1 + c_2}{c_1 c_2}\right) \left(\lambda_1 C_1 \sqrt{1 + \tanh^2 t}\right) (\lambda_2 C_2 \tanh t) \\ &= \tanh t \left(\text{sech} t + \left(\frac{c_1 + c_2}{c_1 c_2}\right) \left(\lambda_1 C_1 \sqrt{1 + \tanh^2 t}\right) (\lambda_2 C_2)\right).\end{aligned}$$

Since $\text{sech} t > 0$ for all $t \in \mathbb{R}$, this value may be zero if and only if $\lambda_1 \lambda_2 < 0 \Leftrightarrow \lambda_1 = -\lambda_2 = \lambda$, and the valid solutions are as stated. □

In PROPOSITION 3.4.2 we proved that the two groups $\mathsf{SO}(1,2)$ and $\mathsf{Aut}(\mathfrak{so}(1,2))$) are Lie group isomorphic, and particularly, for each $g \in \mathsf{SO}(1,2) = \mathsf{Aut}(\mathbb{R}^{1,2}_{\circ})$, there exists a unique element $\phi_g \in \mathsf{Aut}(\mathfrak{so}(1,2))$ such that for each $\mathrm{x} \in \mathbb{R}^{1,2}_{\circ}$, then $g(\mathrm{x}) = \text{hat}(\phi_g(\mathrm{x}))$. In the next lemma we apply this result to the automorphism $\text{Ad}_{\exp(tE_1)}$; we require the image of this automorphism to determine the projection $g(t)$ onto $\mathsf{SO}(1,2)_0$ of the extremal curves of THEOREMS 5.2.1 and 5.2.2.

5.2.3 LEMMA. *For each $X \in \mathfrak{so}(1,2)$ the adjoint action* $\exp(\phi(t)E_1) X \exp(\phi(t)E_1)^{-1}$ *maps to* $(\phi(t)E_1)\mathrm{x}$ *under the hat map.*

PROOF. In PROPOSITION 3.4.2 we defined the map $\Psi : \mathsf{SO}(1,2) \to \mathsf{Aut}(\mathfrak{so}(1,2))$, where further, from PROPOSITION 3.4.3 it follows that $\Psi : \mathsf{SO}(1,2)_0 \to \mathsf{Inn}(\mathfrak{so}(1,2))$ is a bijection. Thus the image under the hat map of $\mathrm{Ad}_{\exp(tE_1)}$ is some $g(t) \in \mathsf{SO}(1,2)_0$ for every t. But from (A.5.39) $\mathrm{Ad} : \mathsf{SO}(1,2)_0 \to \mathsf{Inn}(\mathfrak{so}(1,2))$, $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g$ is a group isomorphism, and so $\mathrm{Ad}_{\exp(tE_1)}$ is a one-parameter subgroup of $\mathsf{Inn}(\mathfrak{so}(1,2))$.

Since Ψ is a group homomorphism, then the image hat $\left(\{\mathrm{Ad}_{\exp(tE_1)} \ : \ {}_{t\in\mathbb{R}} = (t) \ : \ t \in \mathbb{R}\}\right)$ is a one-parameter subgroup of $\mathsf{SO}(1,2)_0$. But from the Iwasawa decomposition 3.6.10, the one-parameter subgroups up to conjugacy of $\mathsf{SO}(1,2)_0$ are K, A and N. Thus $(t) \ : \ t \in \mathbb{R}\}$ must coincide with a subgroup $g\mathsf{K}g^{-1}, g\mathsf{A}g^{-1}$ or $g\mathsf{N}g^{-1}$ for some $g \in \mathsf{SO}(1,2)_0$. But we note particularly that for every $t \in \mathbb{R}$, then

$$\exp(tE_1)E_1\exp(tE_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus under the hat map, $\mathrm{Ad}_{\exp(tE_1)}$ must correspond to an element in $\mathsf{SO}(1,2)_0$ which preserves $\mathbf{e}_1$. But from the decomposition $\mathsf{SO}(1,2)_0 = \mathsf{BK}$, the only elements of $\mathsf{SO}(1,2)_0$ which preserve $\mathbf{e}_1$ are the Euclidean rotations $k(t) = \exp(tE_1)$ themselves.

Thus for $t, t' \in \mathbb{R}$, $\Psi(\exp(tE_1)) = \mathrm{Ad}_{\exp(t'E_1)}$. But under the hat map,

$$\text{hat}\left(\mathrm{Ad}_{\exp(t'E_1)}E_i\right) = \exp(tE_1)\mathbf{e}_i, \ i = 1,2,3.$$

Thus

$$\begin{aligned}
\text{hat}\left(\exp(t'E_1)E_1\exp(t'E_1)^{-1}\right) &= \text{hat}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} &= \exp(-t'E_1)\mathbf{e}_1 \\
\text{hat}\left(\exp(t'E_1)E_2\exp(t'E_1)^{-1}\right) &= \text{hat}\begin{bmatrix} 0 & -\sin t' & \cos t' \\ -\sin t' & 0 & 0 \\ \cos t' & 0 & 0 \end{bmatrix} &= \exp(-t'E_1)\mathbf{e}_2 \\
\text{hat}\left(\exp(t'E_1)E_3\exp(t'E_1)^{-1}\right) &= \text{hat}\begin{bmatrix} 0 & \cos t' & \sin t' \\ \cos t' & 0 & 0 \\ \sin t' & 0 & 0 \end{bmatrix} &= \exp(-t'E_1)\mathbf{e}_3.
\end{aligned}$$

Since the linear map $\exp(tE_1)$ is uniquely determined by its image on each element of an orthonormal basis, it follows that hat $\left(\mathrm{Ad}_{\exp(t'E_1)}X\right) = \exp(-tE_1)\mathrm{x} = \exp(-tE_1)^{\mathsf{T}}\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^{1,2}$. □

5.2.4 THEOREM. *The projection $g(\cdot)$ onto $\mathsf{SO}(1,2)_0$ of the extremal curve $(g(\cdot), p(\cdot))$ of the left-invariant control problem of* THEOREMS *5.2.1 and 5.2.2 can be expressed as the product*

$$g(t) = \exp(\phi_3(t)N) \cdot \exp(\phi_2(t)E_2) \cdot \exp(\phi_1(t)E_1)$$

where $N = E_1 - E_3$ and ϕ_1. ϕ_2 and ϕ_3 solve the system of differential equations

$$\begin{cases} \dot{\phi}_1 &= \frac{P_2}{c_1} + \frac{P_1^2 \tan\phi_1}{c_2 P_3 - c_2 P_2 \tan\phi_1} \\ \dot{\phi}_2 &= \frac{P_1 P_3 \sec\phi_1}{c_2 P_3 - c_2 P_2 \tan\phi_1} \\ \dot{\phi}_3 &= -\frac{P_1 \tan\phi_1}{c_2(P_3 - P_2 \tan\phi_1)}. \end{cases}$$

PROOF. In SECTION 5.1, we expressed the projection $g(t)$ of the extremal pair $(g(\cdot), p(\cdot))$ as the product $g(t) = \exp(\phi_3(t)N)\exp(\phi_2(t)E_2)\exp(\phi_1(t)E_1)$. For ease of notation, we will supress the t's in the expression of $\phi_i(t)$. Since N is not linearly dependent on E_2, then the derivative of $g(t)$ is given by

$$\begin{aligned} \dot{g}(t) &= \tfrac{d}{dt}(\exp(\phi_3 N)\cdot\exp(\phi_2 E_2)\cdot\exp(\phi_1 E_1)) \\ &= \left(\dot{\phi}_3 N\exp(\phi_3 N)\right)\exp(\phi_2 E_2)\exp(\phi_1 E_1) + \exp(\phi_3 N)(\exp(\phi_2 E_2)\dot{\phi}_2 E_2)\exp(\phi_1 E_1) \\ &\quad + \exp(\phi_3 N)\exp(\phi_2 E_2)(\exp(\phi_1 E_1)\dot{\phi}_1 E_1) \\ &= ((\dot{\phi}_3 N)g(t) + g(t)\exp(-\phi_1 E_1)\dot{\phi}_2 E_2)\exp(\phi_1 E_1) + g(t)(\dot{\phi}_1 E_1) \\ &= g(t)\left(g(t)^{-1}(\dot{\phi}_3 N)g(t) + \exp(-\phi_1 E_1)(\dot{\phi}_2 E_2)\exp(\phi_1 E_1) + \dot{\phi}_1 E_1\right) \end{aligned}$$

where for a given t, $g(t)^{-1}(\dot{\phi}_3 N)g(t) + \exp(-\phi_1 E_1)(\dot{\phi}_2 E_2)\exp(\phi_1 E_1) + \dot{\phi}_1 E_1$ is an element of $\mathfrak{so}(1,2)$. We may apply the hat map to this element, where by PROPOSITION 3.4.1 for every $t \in \mathbb{R}$ there exists an element $g'(t) \in \mathsf{SO}(1,2)_0$ such that hat $\left(g(t)^{-1}(\dot{\phi}_3 N)g(t)\right) = g(t)'(\dot{\phi}_3 \mathbf{n})$. Then, using PROPOSITION 5.2.3, it follows that

$$g(t)'(\dot{\phi}_3 \mathbf{n}) + \exp(\phi_1 E_1)^\top(\dot{\phi}_2 \mathbf{e}_2) + \dot{\phi}_1 \mathbf{e}_1 = (\dot{\phi}_3 \tilde{\mathbf{n}} + \exp(E_1\phi_1)^\top(\dot{\phi}_2 \mathbf{e}_2) + \dot{\phi}_1 \mathbf{e}_1$$

where $\tilde{\mathbf{n}}$ is the arbitrary lightlike element $\tilde{\mathbf{n}} = P_1\mathbf{e}_1 + P_2\mathbf{e}_2 + P_3\mathbf{e}_3$. Then

$$\begin{aligned} &(\dot{\phi}_3\tilde{\mathbf{n}}) + \exp(\phi_1 E_1)^\top(\dot{\phi}_2 \mathbf{e}_2) + \dot{\phi}_1 \mathbf{e}_1 && (5.2.18) \\ &\quad = (\dot{\phi}_3(P_1\mathbf{e}_1 + P_2\mathbf{e}_2 + P_3\mathbf{e}_3) + \exp(\phi_1 E_1)^\top(\dot{\phi}_2 \mathbf{e}_2) + \dot{\phi}_1 \mathbf{e}_1 && (5.2.19) \\ &\quad = ((\dot{\phi}_3 P_1 + \dot{\phi}_1)\mathbf{e}_1 + (\dot{\phi}_3 P_2 + \exp(\phi_1 E_1)^\top \dot{\phi}_2)\mathbf{e}_2 + \dot{\phi}_3 P_3 \mathbf{e}_3. && (5.2.20) \end{aligned}$$

Further, since

$$\exp(\phi_1 E_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & -\sin\phi_1 \\ 0 & \sin\phi_1 & \cos\phi_1 \end{bmatrix}$$

then $\exp(\phi_1 E_1^\top)\mathbf{e}_2 = \cos\phi_1\mathbf{e}_2 + \sin\phi_1\mathbf{e}_3$. Note that for this system, $\Xi(1.u) = u_1E_2 + u_2E_1$. Thus taking $u_1 = \frac{1}{c_1}P_2, u_2 = \frac{1}{c_2}P_1$ so that $g(t)$ corrresponds to the optimal trajectory of the control problem of THEOREM 5.2.1, then $\frac{1}{c_1}P_2E_1 + \frac{1}{c_2}P_1E_2$ is an element of Γ. and correspondingly

$$\frac{1}{c_1}P_2\mathbf{e}_1 + \frac{1}{c_2}P_1\mathbf{e}_2 \tag{5.2.21}$$

is an element of Γ. Equating the two expressions (5.2.20) and (5.2.21) for this element of $\mathbb{R}^{1,2}$ and rearranging, then

$$(\dot\phi_3 P_1 + \dot\phi_1)\mathrm{e}_1 + (\phi_3 P_2 + \dot\phi_2 \cos\phi_1)\mathrm{e}_2 + (\dot\phi_2 \sin\phi_1 + \phi_3 P_3)\mathrm{e}_3 = \frac{1}{c_2} P_1 \mathrm{e}_2 + \frac{1}{c_1} P_2 \mathrm{e}_1$$

and thus by the linear independence of $\mathrm{e}_1, \mathrm{e}_2$ and e_3, it follows that

$$\begin{cases} \dot\phi_3 P_1 + \dot\phi_1 = \dfrac{1}{c_1} P_2 & (5.2.22) \\ \dot\phi_3 P_2 + \dot\phi_2 \cos\phi_1 = \dfrac{1}{c_2} P_1 & (5.2.23) \\ \dot\phi_2 \sin\phi_1 + \dot\phi_3 P_3 = 0. & (5.2.24) \end{cases}$$

From (5.2.22), we find

$$\dot\phi_1 = \frac{P_2 - c_1 P_1 \dot\phi_3}{c_1} \tag{5.2.25}$$

and from (5.2.23),

$$\dot\phi_2 = \frac{(P_1 - c_2 P_2 \dot\phi_3)\sec\phi_1}{c_2} \tag{5.2.26}$$

finally, substituting for $\dot\phi_2$ from (5.2.26) in (5.2.24) and solving, gives

$$\dot\phi_3 = -\frac{P_1 \tan\phi_1}{c_2(P_3 - P_2\tan\phi_1)} \tag{5.2.27}$$

and substituting (5.2.27) back into (5.2.25) and (5.2.26) for $\dot\phi_3$ gives

$$\dot\phi_1 = \frac{P_2}{c_1} + \frac{P_1^2 \tan\phi_1}{c_2 P_3 - c_2 P_2 \tan\phi_1} \quad \text{and} \quad \dot\phi_2 = \frac{P_1 P_3 \sec\phi_1}{c_2 P_3 - c_2 P_2 \tan\phi_1} \tag{5.2.28}$$

respectively, and the result follows. Equations (5.2.25) to (5.2.28) were determined using Mathematica (C.4). □

5.2.2 Equilibrium points and stability

We find the equilibruim points for the system of reduced extremal equations (5.2.3),(5.2.2) and (5.2.4). We then investigate the non-linear stability of each of the equilibrium points using the extended energy-Casimir (A.12.4) method as well as (A.12.6).

5.2.5 THEOREM. *Given the left-invariant control problem of* THEOREM *5.2.1, then the equilibrium points of the system of the reduced extremal equations are*

$$P_{e_1}^M = (M, 0, 0), \quad P_{e_2}^M = (0, M, 0) \quad \textit{and} \quad P_{e_3} = (0, 0, 0) \quad M \in \mathbb{R}\setminus\{0\}$$

where P_{e_3} and $P_{e_1}^M$ are nonlinear stable and $P_{e_2}^M$ is unstable.

PROOF. The equilbrium points of the system of reduced extremal equations (5.2.3), (5.2.2) and (5.2.4) are the solutions of the system

$$\begin{cases} \frac{1}{c_1} P_2 P_3 = 0 \\ \frac{1}{c_2} P_1 P_3 = 0 \\ \left(\frac{c_1 + c_2}{c_1 c_2}\right) P_1 P_2 = 0 \end{cases}$$

which are exactly the points

$$(M, 0, 0), \quad (0, M, 0), \quad (0, 0, 0) \quad \text{and} \quad (M, N, 0)|_{c_1=-c_2}$$

where $M, N \in \mathbb{R}\setminus\{0\}$. Note that since we require that $c_1 > 0, c_2 > 0$, we do not consider the equilibrium points $(M, N, 0)|_{c_1=-c_2}$. The matrix corresponding to the linearized operator of the system of extremal equations is

$$\begin{bmatrix} 0 & 0 & -\frac{1}{c_1}P_2 \\ 0 & 0 & \frac{1}{c_2}P_1(t) \\ -\frac{1}{c_1}P_2 & -\frac{1}{c_2}P_1(t) & 0 \end{bmatrix}.$$

At the equilibrium points $P^M_{e_2}$, then

$$\begin{bmatrix} 0 & 0 & -\frac{1}{c_1}P_2 \\ 0 & 0 & \frac{1}{c_1}P_2(t) \\ -\frac{1}{c_2}P_1 & -\frac{1}{c_1}P_2(t) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{M}{c_1} \\ 0 & 0 & 0 \\ -\frac{M}{c_1} & 0 & 0 \end{bmatrix}$$

which has the real eigenvalues $\lambda_1 = 0, \lambda_2 = \frac{M}{c_1}, \lambda_3 = -\frac{M}{c_1}$. Since for $M > 0$, $\lambda_2 > 0$, and for $M < 0$, then $\lambda_3 > 0$, the linearized system always has one positive eigenvalue and so the points $P^M_{e_2}$ are linearly unstable by (A.12.5). Thus by (A.12.6), the equilibrium points $P^M_{e_2}$ are unstable.

We use the extended energy-Casimir method (A.12.4) to show that the equilibrium points $P^M_{e_1}$ are nonlinear stable. Construct the function $L = 2c_2\mathcal{H} + 2K = \left(1 + \frac{c_2}{c_1}\right)P_2^2 + P_3^2$. Then

$$dL = \begin{bmatrix} 0 & 2\left(1 + \frac{c_2}{c_1}\right)P_2 & 2P_3 \end{bmatrix}$$

and so $dL \cdot P^M_{e_1} = 0$. The Hessian of L is given by

$$d^2L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\left(1 + \frac{c_2}{c_1}\right) & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

where $d\mathcal{H} = \begin{bmatrix} \frac{1}{c_2}P_1 & \frac{1}{c_1}P_2 & 0 \end{bmatrix}$, and $\ker(d\mathcal{H}) \cdot P^M_{e_1} = \langle e_2, e_3 \rangle$. Also, $dK = \begin{bmatrix} -P_1 & P_2 & P_3 \end{bmatrix}$, and $\ker(dK) \cdot P^M_{e_1} = \langle e_2, e_3 \rangle$. Then $W = \ker(d\mathcal{H} \cdot P^M_{e_1}) \cap \ker(dK \cdot P^M_{e_1}) = \langle e_2, e_3 \rangle$, and we consider the restriction of dL^2 to $W \times W$, which is given by

$$d^2L|_{W\times W} = \begin{bmatrix} 2\left(1 + \frac{c_2}{c_1}\right) & 0 \\ 0 & 2 \end{bmatrix} = Q.$$

The quadratic form $\mathbf{p}^\top Q\,\mathbf{p}$ is clearly positive-definite since Q has two positive eigenvalues (A.1.10). Thus there exist constants $\lambda_0 = 2c_2, \lambda_1 = 2$ such that $L = \lambda_0\mathcal{H} + \lambda_1 K$ fulfils the requirements of the extended energy-Casimir method and the points $P^M_{e_1}$ are nonlinear stable.

Finally, we use the extended energy-Casimir method to show that the equilibrium point P_{e_3} is nonlinear stable. Construct the function $L = 2c_2\mathcal{H} + K = \frac{1}{2}P_1^2 + \left(\frac{1}{2} + \frac{c_2}{c_1}\right)P_2^2 + \frac{1}{2}P_3^2$. Then

$$dL = \begin{bmatrix} P_1 & \left(1 + \frac{2c_2}{c_1}\right)P_2 & P_3 \end{bmatrix}$$

and so $dL \cdot P_{e_3} = 0$. The Hessian of L is given by

$$d^2L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(1 + \frac{2c_2}{c_1}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where since $d\mathcal{H} = \begin{bmatrix} \frac{1}{c_2}P_1 & \frac{1}{c_1}P_2 & 0 \end{bmatrix}$, we see that $\ker(d\mathcal{H}\cdot P_{e_3}) = \mathbb{R}^3$. Similarly, $dK = \begin{bmatrix} -P_1 & P_2 & P_3 \end{bmatrix}$, and $\ker(dK \cdot P_{e_3}) = \mathbb{R}^3$. Then $W = \ker(d\mathcal{H} \cdot P_{e_3}) \cap \ker(dK \cdot P_{e_3}) = \mathbb{R}^3$, and we consider the restriction of dL^2 to $W \times W$, which is given by

$$d^2L|_{W\times W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(1 + \frac{2c_2}{c_1}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} = Q.$$

□

The quadratic form $\mathbf{p}^\top Q\,\mathbf{p}$ is clearly positive-definite since Q has three positive eigenvalues (A.1.10). Thus there exist constants $\lambda_0 = 2c_2, \lambda_1 = 1$ such that $L = \lambda_0\mathcal{H} + \lambda_1 K$ fulfils the requirements of the extended energy-Casimir method, and so the equilibrium point P_{e_3} is nonlinear stable.

5.3 The case $\Gamma = \langle E_3, E_2\rangle$

5.3.1 THEOREM. *Given the left-invariant control problem* $(\Sigma, \mathcal{L}, (g_0, g_T, T))$

$$\dot{g} = g(u_1E_3 + u_2E_2), \qquad g \in \mathsf{SO}(1,2)_0,\ (u_1, u_2) \in \mathbb{R}^2$$
$$g(0) = g_0, \quad g(T) = g_T$$
$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1u_1^2(t) + c_2u_2^2(t)\right)dt \to \min, \quad c_1, c_2 > 0$$

then the optimal controls are

$$u_1 = \frac{P_3}{c_1} \quad \text{and} \quad u_2 = \frac{P_2}{c_2}$$

and the optimal Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left(\frac{P_3^2}{c_1} + \frac{P_2^2}{c_2}\right) \tag{5.3.1}$$

where P_1 and P_2 are solutions of the reduced extremal equations

$$\begin{cases} \dot{P}_1 = \left(\dfrac{c_2 - c_1}{c_1c_2}\right)P_2P_3 & (5.3.2) \\ \dot{P}_2 = \dfrac{1}{c_1}P_3P_1 & (5.3.3) \\ \dot{P}_3 = -\dfrac{1}{c_2}P_2P_1. & (5.3.4) \end{cases}$$

PROOF. By (A.10.1), the control Hamiltonian is

$$\mathcal{H} = -\frac{1}{2}(c_1 u_1^2 + c_2 u_2^2) + p(u_1 E_3 + u_2 E_2).$$

As we have stated, we may identify $P_i = p(E_i)$ via the Killing form; then

$$\mathcal{H} = -\frac{1}{2}(c_1 u_1^2 + c_2 u_2^2) + u_1 P_3 + u_2 P_2. \tag{5.3.5}$$

By the PMP, the optimal Hamiltonian satisfies

$$\frac{\partial \mathcal{H}}{\partial u_1} = -c_1 u_1 + P_3 = 0 \iff u_1 = \frac{1}{c_1} P_3 \tag{5.3.6}$$

$$\frac{\partial \mathcal{H}}{\partial u_2} = -c_2 u_2 + P_2 = 0 \iff u_2 = \frac{1}{c_2} P_2. \tag{5.3.7}$$

Thus u_1 and u_2 are the optimal controls, and the optimal Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\left(\frac{P_3^2}{c_1} + \frac{P_2^2}{c_2}\right)$$

by substitution of (5.3.6) and (5.3.7) into (5.3.5). From (A.10.9), the system of extremal equations is given by

$$\begin{aligned}
\dot{P}_1 &= P_3\left\{P_1, \tfrac{P_3}{c_1}\right\} + P_2\left\{P_1, \tfrac{P_2}{c_2}\right\} &&= \left(\tfrac{c_2 - c_1}{c_1 c_2}\right) P_2 P_3 \\
\dot{P}_2 &= P_3\left\{P_2, \tfrac{P_3}{c_1}\right\} + P_2\left\{P_2, \tfrac{P_2}{c_2}\right\} &&= \tfrac{1}{c_1} P_3 P_1 \\
\dot{P}_3 &= P_3\left\{P_3, \tfrac{P_3}{c_1}\right\} + P_2\left\{P_3, \tfrac{P_2}{c_2}\right\} &&= -\tfrac{1}{c_2} P_2 P_1.
\end{aligned}$$

□

5.3.1 Explicit integration of the extremal curve $(p(\cdot), g(\cdot))$

5.3.2 THEOREM. *Under the condition $c_1 = c_2 = c$, the reduced extremal equations (5.3.2), (5.3.3) and (5.3.4) can be solved in terms of trigenometric functions to give*

$$\begin{cases}
P_1(t) &= P_1(0) \\
P_2(t) &= P_2(0)\cos\left(\frac{P_1}{c}t\right) + P_3(0)\sin\left(\frac{P_1}{c}t\right) \\
P_3(t) &= -P_2(0)\sin\left(\frac{P_1}{c}t\right) + P_3(0)\cos\left(\frac{P_1}{c}t\right).
\end{cases}$$

PROOF. Taking $c_1 = c_2 = c$, then the system of reduced extremal equations (5.3.2), (5.3.3) and (5.3.4) collapses to

$$\dot{P}_1 = 0, \quad \dot{P}_2 = \frac{1}{c} P_3 P_1 \quad \text{and} \quad \dot{P}_3 = -\frac{1}{c} P_2 P_1$$

These equations can be expressed in matrix form by

$$\begin{bmatrix} \dot{P}_2 \\ \dot{P}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{c}P_1 \\ \frac{-1}{c}P_1 & 0 \end{bmatrix} \begin{bmatrix} P_2 \\ P_3 \end{bmatrix}$$

where by (A.7.15), the Cauchy problem $\dot{P} = AP, P(0)$ has the solution $P(t) = \exp(tA)P(0)$. Thus using Mathematica (C.3) to calculate the matrix exponential,

$$\begin{bmatrix} P_2 \\ P_3 \end{bmatrix} = \exp\begin{bmatrix} 0 & \frac{P_1}{c}t \\ -\frac{P_1}{c}t & 0 \end{bmatrix} \begin{bmatrix} P_2(0) \\ P_3(0) \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{P_1}{c}t\right) & \sin\left(\frac{P_1}{c}t\right) \\ -\sin\left(\frac{P_1}{c}t\right) & \cos\left(\frac{P_1}{c}t\right) \end{bmatrix} \begin{bmatrix} P_2(0) \\ P_3(0) \end{bmatrix}.$$

Since $\dot{P}_1 = 0$, then P_1 is the constant curve $P_1(0)$, and the result follows. □

We illustrate this case in fig.C.34.

Again using the intersection of the optimal Hamlitonian $\mathcal{H}$ and the Casimir function K to give the solution curves, the values of K may be positive, negative or zero. This separation of the level surfaces of the Casimir function K appears naturally in the process of solving the reduced extremal equations, as we illustrate in figs. C.32 and C.33.

5.3.3 THEOREM. *Under the condition $c_1 \neq c_2$, the reduced extremal equations (5.3.2), (5.3.3) and (5.3.4) can be solved in terms of Jacobi elliptic functions to give*

Case 1 $(K < 0)$

Case 1a $\left(c_2 > c_1, \left(\frac{c_1-c_2}{c_1}\right) P_3^2 < P_1^2, -\left(\frac{c_1-c_2}{c_2}\right) P_2^2 < P_1^2\right)$

$$\begin{cases} P_1(t) &= b \cdot \operatorname{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_1 \sqrt{b^2 \operatorname{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_2 \sqrt{a^2 - b^2 \operatorname{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

or

$$\begin{cases} P_1(t) &= a \cdot \operatorname{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_1 \sqrt{a^2 \operatorname{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_2 \sqrt{a^2 - a^2 \operatorname{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

where $a = \sqrt{2\mathcal{H}c_2 - 2K}, b = \lambda\sqrt{2\mathcal{H}c_1 - 2K}, C_1 = \sqrt{\frac{c_1}{(c_2-c_1)}}, C_2 = \sqrt{\frac{c_2}{(c_2-c_1)}}, C = \frac{a}{\sqrt{c_1c_2}}$
and $\lambda = \pm 1, \lambda' = -\lambda$

or

Case 1b $\left(c_1 > c_2, \left(\frac{c_2-c_1}{c_2}\right) P_3^2 < P_1^2, -\left(\frac{c_2-c_1}{c_1}\right) P_2^2 < P_1^2\right)$

$$\begin{cases} P_1(t) &= b \cdot \operatorname{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_2 \sqrt{b^2 \operatorname{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_1 \sqrt{a^2 - b^2 \operatorname{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

or

$$\begin{cases} P_1(t) &= a \cdot \operatorname{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_2 \sqrt{a^2 \operatorname{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_1 \sqrt{a^2 - a^2 \operatorname{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

where $a = \sqrt{2\mathcal{H}c_1 - 2K}, b = \lambda\sqrt{2\mathcal{H}c_2 - 2K}, C_1 = \sqrt{\frac{c_1}{(c_1-c_2)}}, C_2 = \sqrt{\frac{c_2}{(c_1-c_2)}}, C = \frac{a}{\sqrt{c_1c_2}}$
and $\lambda = \pm 1, \lambda' = -\lambda$.

Case 2 $(K = 0)$

Case 2a $\quad (c_2 > c_1, \left(\frac{c_1-c_2}{c_1}\right) P_3^2 < P_1^2, -\left(\frac{c_1-c_2}{c_2}\right) P_2^2 < P_1^2)$

$$\begin{cases} P_1(t) &= b \cdot \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_1 \sqrt{b^2 \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_2 \sqrt{a^2 - b^2 \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

or

$$\begin{cases} P_1(t) &= a \cdot \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_1 \sqrt{a^2 \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_2 \sqrt{a^2 - a^2 \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

where $a = \lambda\sqrt{2\mathcal{H}c_2 - 2K}, b = \lambda\sqrt{2\mathcal{H}c_1 - 2K}, C_1 = \sqrt{\frac{c_1}{(c_2-c_1)}}, C_2 = \sqrt{\frac{c_2}{(c_2-c_1)}}, C = \frac{a}{\sqrt{c_1c_2}}$
and $\lambda = \pm 1, \lambda' = -\lambda,$
or

Case 2b $\quad (c_1 > c_2, \left(\frac{c_2-c_1}{c_2}\right) P_3^2 < P_1^2, -\left(\frac{c_2-c_1}{c_1}\right) P_2^2 < P_1^2)$

$$\begin{cases} P_1(t) &= b \cdot \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_2 \sqrt{b^2 \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3 &= \lambda' C_1 \sqrt{a^2 - b^2 \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

or

$$\begin{cases} P_1(t) &= a \cdot \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_2 \sqrt{a^2 \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_1 \sqrt{a^2 - a^2 \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

where $a = \sqrt{2\mathcal{H}c_1 - 2K}, b = \lambda\sqrt{2\mathcal{H}c_2 - 2K}, C_1 = \sqrt{\frac{c_1}{(c_1-c_2)}}, C_2 = \sqrt{\frac{c_2}{(c_1-c_2)}}, C = \frac{a}{\sqrt{c_1c_2}}$
and $\lambda = \pm 1, \lambda' = -\lambda$.

Case 3 $\quad (K > 0)$

Case 3a $\quad (c_2 > c_1, \left(\frac{c_1-c_2}{c_1}\right) P_3^2 < P_1^2, -\left(\frac{c_1-c_2}{c_2}\right) P_2^2 < P_1^2)$

$$\begin{cases} P_1(t) &= b \cdot \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_1 \sqrt{b^2 \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_2 \sqrt{a^2 - b^2 \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

or

$$\begin{cases} P_1(t) &= a \cdot \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_1 \sqrt{a^2 \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_2 \sqrt{a^2 - a^2 \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

where $a = \sqrt{2\mathcal{H}c_2 - 2K}, b = \lambda\sqrt{2\mathcal{H}c_1 - 2K}, C_1 = \sqrt{\frac{c_1}{(c_2-c_1)}}, C_2 = \sqrt{\frac{c_2}{(c_2-c_1)}}, C = \frac{a}{\sqrt{c_1c_2}}$
and $\lambda = \pm 1, \lambda' = -\lambda$,
or Case 3b $\left(c_1 > c_2, \left(\frac{c_2-c_1}{c_2}\right)P_3^2 < P_1^2, -\left(\frac{c_2-c_1}{c_1}\right)P_2^2 < P_1^2\right)$

$$\begin{cases} P_1(t) &= b\cdot \mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_2\sqrt{b^2\mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_1\sqrt{a^2 - b^2\mathrm{nd}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

or

$$\begin{cases} P_1(t) &= a\cdot \mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right) \\ P_2(t) &= \lambda C_2\sqrt{a^2\mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2 - b^2} \\ P_3(t) &= \lambda' C_1\sqrt{a^2 - a^2\mathrm{dn}\left(Ct, \frac{\sqrt{a^2-b^2}}{a}\right)^2} \end{cases}$$

where $a = \sqrt{2\mathcal{H}c_1 - 2K}, b = \lambda\sqrt{2\mathcal{H}c_2 - 2K} C_1 = \sqrt{\frac{c_1}{(c_1-c_2)}}, C_2 = \sqrt{\frac{c_2}{(c_1-c_2)}}, C = \frac{a}{\sqrt{c_1c_2}}$
and $\lambda = \pm 1, \lambda' = -\lambda$,
or
Case 3c $\left(c_2 > c_1, \left(\frac{c_1-c_2}{c_1}\right)P_3^2 < P_1^2, -\left(\frac{c_1-c_2}{c_2}\right)P_2^2 > P_1^2\right)$

$$\begin{cases} P_1(t) &= b\cdot \mathrm{sn}\left(Ct, \frac{b}{a}\right) \\ P_2(t) &= \lambda_1 C_1\sqrt{b^2 - b^2\mathrm{sn}\left(Ct, \frac{b}{a}\right)^2} \\ P_3(t) &= \lambda_2 C_2\sqrt{a^2 - b^2\mathrm{sn}\left(Ct, \frac{b}{a}\right)^2} \end{cases}$$

or

$$\begin{cases} P_1(t) &= b\cdot \mathrm{cd}\left(Ct, \frac{b}{a}\right) \\ P_2(t) &= \lambda_1 C_1\sqrt{b^2 - b^2\mathrm{cd}\left(Ct, \frac{b}{a}\right)^2} \\ P_3(t) &= \lambda_2 C_2\sqrt{a^2 - b^2\mathrm{cd}\left(Ct, \frac{b}{a}\right)^2} \end{cases}$$

where $a = \lambda\sqrt{2\mathcal{H}c_2 - 2K}, b = \lambda\sqrt{2K - 2\mathcal{H}c_1}, C_1 = \sqrt{\frac{c_1}{(c_2-c_1)}}, C_2 = \sqrt{\frac{c_2}{(c_2-c_1)}}, C = \frac{a}{\sqrt{c_1c_2}}$
and $\lambda = \pm 1, \lambda' = -\lambda$,
or
Case 3d $\left(c_1 > c_2, \left(\frac{c_2-c_1}{c_2}\right)P_3^2 < P_1^2, -\left(\frac{c_2-c_1}{c_1}\right)P_2^2 > P_1^2\right)$

$$\begin{cases} P_1(t) &= a\cdot \mathrm{sn}\left(Ct, \frac{b}{a}\right) \\ P_2(t) &= \lambda_1 C_2\sqrt{b^2 - a^2\mathrm{sn}\left(Ct, \frac{b}{a}\right)^2} \\ P_3(t) &= \lambda_2 C_1\sqrt{a^2 - a^2\mathrm{sn}\left(Ct, \frac{b}{a}\right)^2} \end{cases}$$

or

$$\begin{cases} P_1(t) &= b\cdot \mathrm{cd}\left(Ct, \frac{b}{a}\right) \\ P_2(t) &= \lambda_1 C_2\sqrt{b^2 - b^2\mathrm{cd}\left(Ct, \frac{b}{a}\right)^2} \\ P_3(t) &= \lambda_2 C_1\sqrt{a^2 - b^2\mathrm{cd}\left(Ct, \frac{b}{a}\right)^2} \end{cases}$$

where $a = \lambda\sqrt{2\mathcal{H}c_1 - 2K}, b = \lambda\sqrt{2K - 2\mathcal{H}c_2}, C_1 = \sqrt{\frac{c_1}{(c_1-c_2)}}, C_2 = \sqrt{\frac{c_2}{(c_1-c_2)}}, C = \frac{a}{\sqrt{c_1c_2}}$,
$\lambda = \pm 1$, $\lambda_1 = \pm 1$ *and* $\lambda_2 = \pm 1$.

PROOF. From (A.10.13) the extremal curves lie in the intersection of the level surfaces of the optimal Hamiltonian (5.3.1) with the level surfaces of the Casimir function of PROPOSITION 5.1.2, which we express in $\mathfrak{so}(1,2)$ via the Killing form. We use this fact to express the variables P_2 and P_3 in terms of P_1. From (5.3.1), then $c_2P_3^2 = 2c_1c_2\mathcal{H} - c_1P_2^2$. But $P_3^2 = 2K + P_1^2 - P_2^2$ from the Casimir function. Thus $c_2(2K + P_1^2 - P_2^2) = 2c_1c_2\mathcal{H} - c_1P_2^2$. Solving for P_2^2, then

$$P_2^2 = \left(\frac{c_2}{c_2 - c_1}\right)(2K + P_1^2 - 2\mathcal{H}c_1). \tag{5.3.8}$$

Also from (5.3.1), $c_1P_2^2 = 2c_1c_2\mathcal{H} - c_2P_3^2$. But $c_1(2K + P_1^2 - P_3^2) = c_1P_2^2$ from the Casimir function. Thus $c_1(2K + P_1^2 - P_3^2) = 2c_1c_2\mathcal{H} - c_2P_3^2$. Solving for P_3^2, then

$$P_3^2 = \left(\frac{c_1}{c_2 - c_1}\right)(2\mathcal{H}c_2 - P_1^2 - 2K). \tag{5.3.9}$$

Substituting (5.3.8) and (5.3.9) into the reduced extremal equation (5.3.4), then

$$\dot{P}_1^2 = \frac{(c_2-c_1)^2}{c_1^2c_2^2}P_2^2P_3^2 = \frac{(c_2-c_1)^2}{c_1^2c_2^2}\frac{c_2c_1}{(c_2-c_1)^2}(2K + P_1^2 - 2\mathcal{H}c_1)(2\mathcal{H}c_2 - P_1^2 - 2K)$$

and we have the system of equations

$$\begin{cases} P_2^2 = \left(\dfrac{c_2}{c_2 - c_1}\right)(2K + P_1^2 - 2\mathcal{H}c_1) & (5.3.10) \\ P_3^2 = \left(\dfrac{c_1}{c_2 - c_1}\right)(2\mathcal{H}c_2 - P_1^2 - 2K) & (5.3.11) \\ \dot{P}_1^{\,2} = \left(\dfrac{1}{c_1c_2}\right)(2K + P_1^2 - 2\mathcal{H}c_1)(2\mathcal{H}c_2 - P_1^2 - 2K). & (5.3.12) \end{cases}$$

From (5.3.12)

$$\begin{aligned} \frac{dP_1^2}{dt^2} &= \left(\frac{1}{c_1c_2}\right)(2K + P_1^2 - 2\mathcal{H}c_1)(2\mathcal{H}c_2 - P_1^2 - 2K) \\ \Rightarrow \frac{dt}{dP_1} &= \sqrt{c_1c_2}\frac{1}{\sqrt{(2K + P_1^2 - 2\mathcal{H}c_1)(2\mathcal{H}c_2 - P_1^2 - 2K)}} \\ \Rightarrow \int_0^t d\tau &= \sqrt{c_1c_2}\int_0^{P_1}\frac{dP_1}{\sqrt{(2K + P_1^2 - 2\mathcal{H}c_1)(2\mathcal{H}c_2 - P_1^2 - 2K)}} \end{aligned}$$

and thus

$$t = \sqrt{c_1c_2}\int_0^{P_1}\frac{dP_1}{\sqrt{(a^2 - P_1^2)(P_1^2 - b^2)}} \tag{5.3.13}$$

where $a^2 = (2\mathcal{H}c_2 - 2K), b^2 = (2\mathcal{H}c_1 - 2K)$.
Using equations (5.3.8) and (5.3.9), we see that

$$P_3^2 = \left(\frac{c_1}{c_2 - c_1}\right)(2\mathcal{H}c_2 - P_1^2 - 2K) \quad \Leftrightarrow \quad 2\mathcal{H}c_2 - 2K = P_1^2 + \left(\frac{c_2 - c_1}{c_1}\right)P_3^2$$

$$P_2^2 = \left(\frac{c_2}{c_2 - c_1}\right)(2K + P_1^2 - 2\mathcal{H}c_1) \quad \Leftrightarrow \quad 2\mathcal{H}c_1 - 2K = P_1^2 - \left(\frac{c_2 - c_1}{c_2}\right)P_2^2$$

and thus

$$a^2 > 0 \quad \Leftrightarrow \left(\frac{c_1 - c_2}{c_1}\right)P_3^2 < P_1^2 \quad \text{and} \quad b^2 > 0 \quad \Leftrightarrow -\left(\frac{c_1 - c_2}{c_2}\right)P_2^2 < P_1^2.$$

Then we have the cases

$$\left(\frac{c_1 - c_2}{c_1}\right) P_3^2 < P_1^2, \quad \text{and} \quad \left(\frac{c_2 - c_1}{c_2}\right) P_2^2 < P_1^2 \tag{5.3.14}$$

$$\left(\frac{c_1 - c_2}{c_1}\right) P_3^2 > P_1^2, \quad \text{and} \quad \left(\frac{c_2 - c_1}{c_2}\right) P_2^2 < P_1^2 \tag{5.3.15}$$

$$\left(\frac{c_1 - c_2}{c_1}\right) P_3^2 < P_1^2, \quad \text{and} \quad \left(\frac{c_2 - c_1}{c_2}\right) P_2^2 > P_1^2 \tag{5.3.16}$$

$$\left(\frac{c_1 - c_2}{c_1}\right) P_3^2 > P_1^2, \quad \text{and} \quad \left(\frac{c_2 - c_1}{c_2}\right) P_2^2 > P_1^2 \tag{5.3.17}$$

where we consider the corresponding elliptic integrals

$$\begin{cases} \sqrt{c_1 c_2} \displaystyle\int_0^{P_1} \frac{dP_1}{\sqrt{(a^2 - P_1^2)(P_1^2 - b^2)}} & (5.3.18) \\ \sqrt{c_1 c_2} \displaystyle\int_0^{P_1} \frac{dP_1}{\sqrt{(P_1^2 - a^2)(P_1^2 - b^2)}} & (5.3.19) \\ \sqrt{c_1 c_2} \displaystyle\int_0^{P_1} \frac{dP_1}{\sqrt{(a^2 - P_1^2)(b^2 - P_1^2)}} & (5.3.20) \\ \sqrt{c_1 c_2} \displaystyle\int_0^{P_1} \frac{dP_1}{\sqrt{(P_1^2 - a^2)(b^2 - P_1^2)}}. & (5.3.21) \end{cases}$$

where in (5.3.18), $a^2 = \sqrt{2\mathcal{H}c_2 - 2K}$, $b^2 = \sqrt{2\mathcal{H}c_1 - 2K}$, in (5.3.19), $a^2 = \sqrt{2K - 2\mathcal{H}c_2}$, $b^2 = \sqrt{2\mathcal{H}c_1 - 2K}$, in (5.3.20), $a^2 = \sqrt{2\mathcal{H}c_2 - 2K}$, $b^2 = \sqrt{2K - 2\mathcal{H}c_1}$ and finally in (5.3.21), $a^2 = \sqrt{2K - 2\mathcal{H}c_2}$, $b^2 = \sqrt{2K - 2\mathcal{H}c_1}$.
Consider the condition of (A.11) for these integrals, $b < P_1 < a$:

$$b < a \quad \Rightarrow \quad b^2 < a^2 \quad \Rightarrow \quad 2\mathcal{H}c_1 - 2K < 2\mathcal{H}c_2 - 2K \quad \Rightarrow \quad c_1 < c_2.$$

Under this condition $c_1 < c_2$, we note that conditions (5.3.14) and (5.3.16) are satisfied, while (5.3.15) and (5.3.17) are invalid, since in this case $c_1 - c_2 < 0$. Thus we consider the cases (5.3.18) and (5.3.20) only. Note that if we swap c_1 and c_2, we interchange a and b, and so the cases (5.3.18) and (5.3.21) are swapped. Similarly so are the cases (5.3.19) and (5.3.20), and the condition $b < a$ is preserved. Thus we may immediatly state the solutions in the case $c_2 < c_1$ by swapping c_1 and c_2 in the solutions given for $c_1 < c_2$.

We have already stated that the Casimir function K may take positive or negative values or may be zero (corresponding to when the level surfaces of K take the form of hyperboloids of one sheet, hyperboloids of two sheets or the right cone, respectively). We consider each of the cases $K > 0, K < 0$ and $K = 0$ separately. However, since $a^2 = 2\mathcal{H}c_2 - K$ and $b^2 = 2\mathcal{H}c_1 - K$, where $\mathcal{H}$ and c_1 are everywhere positive, we note that where $K < 0, K = 0$, the conditions $a^2 > 0$ and $b^2 > 0$ of (A.11) for the elliptic integrals $\text{ns}^{-1}, \text{dc}^{-1}, \text{dn}^{-1}$ and nd^{-1} are satisfied everywhere. Thus we require to consider the possibility (5.3.19) only where $K > 0$. Thus for the cases $K < 0$ and $K = 0$ we use only the value $b^2 = 2\mathcal{H}c_1 - 2K$ and express (A.12.1) as an elliptic integral of the form (5.3.18). We then have the solutions

Case 1 a, Case 2a, Case 3a $(c_2 > c_1, \left(\frac{c_1-c_2}{c_1}\right) P_3^2 < P_1^2, -\left(\frac{c_1-c_2}{c_2}\right) P_2^2 < P_1^2)$

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_b^{P_1} \frac{dP_1}{\sqrt{((a^2-P_1^2)(P_1^2-b^2)}} \\ &= \sqrt{c_1c_2}\frac{1}{a}\cdot \mathrm{nd}^{-1}\left(\frac{P_1}{b}, \frac{\sqrt{a^2-b^2}}{a}\right) \\ \Rightarrow P_1 &= b\cdot \mathrm{nd}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{\sqrt{a^2-b^2}}{a}\right) \end{aligned}$$

or

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_{P_1}^{a} \frac{dP_1}{\sqrt{((a^2-P_1^2)(P_1^2-b^2)}} \\ &= \sqrt{c_1c_2}\frac{1}{a}\cdot \mathrm{dn}^{-1}\left(\frac{P_1}{a}, \frac{\sqrt{a^2-b^2}}{a}\right) \\ \Rightarrow P_1 &= a\cdot \mathrm{dn}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{\sqrt{a^2-b^2}}{a}\right) \end{aligned}$$

where $a^2 = 2\mathcal{H}c_2 - 2K, b^2 = 2\mathcal{H}c_1 - 2K$ and we take the square roots $a = \sqrt{2\mathcal{H}c_2 - 2K}$ and $b = \lambda\sqrt{2\mathcal{H}c_1 - 2K}$ where $\lambda = \pm 1$, since we require the modulus $k = \frac{\sqrt{a^2-b^2}}{a}$ to be positive.

Case 3c $(c_2 > c_1. \left(\frac{c_1-c_2}{c_1}\right) P_3^2 < P_1^2, -\left(\frac{c_1-c_2}{c_2}\right) P_2^2 > P_1^2)$

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_0^{P_1} \frac{dP_1}{\sqrt{(a^2-P_1^2)(b^2-P_1^2)}} \\ &= \sqrt{c_1c_2}\frac{1}{a}\cdot \mathrm{sn}^{-1}\left(\frac{P_1}{b}, \frac{b}{a}\right) \\ \Rightarrow P_1 &= b\cdot \mathrm{sn}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{b}{a}\right) \end{aligned}$$

or

$$\begin{aligned} t &= \sqrt{c_1c_2}\int_0^{P_1} \frac{dP_1}{\sqrt{(a^2-P_1^2)(b^2-P_1^2)}} \\ &= \sqrt{c_1c_2}\frac{1}{a}\cdot \mathrm{cd}^{-1}\left(\frac{P_1}{b}, \frac{b}{a}\right) \\ \Rightarrow P_1 &= b\cdot \mathrm{cd}\left(\frac{a\cdot t}{\sqrt{c_1c_2}}, \frac{b}{a}\right). \end{aligned}$$

where $a^2 = 2\mathcal{H}c_2 - 2K. b^2 = 2K - 2\mathcal{H}c_1$ and we take the square roots $a = \lambda\sqrt{2\mathcal{H}c_2 - 2K}$ and $b = \lambda\sqrt{2K - 2\mathcal{H}c_1}$ where $\lambda = \pm 1$, since we require the modulus $k = \frac{b}{a}$ to be positive.

Cases 1b, 2b and **3b** follow directly from **1a, 2a** and **3a** by swapping c_1 and c_2. Similarly, case **3d** follows from case **3c** by swapping c_1 and c_2.

In order to solve for P_2 and P_3, we subsitute back into equations (5.3.8) and (5.3.9). But then

$$P_2 = \lambda_1\sqrt{\frac{c_2^2}{c_2-c_1}(2K + P_1^2 - 2\mathcal{H}c_1)},\ P_3 = \lambda_2\sqrt{\frac{c_1}{c_2-c_1}(2\mathcal{H}c_2 - P_1^2 - 2K)}$$

where $\lambda_1 = \pm 1, \lambda_2 = \pm 1$.

In order to find which solutions are valid, we make a simplifying assumption on a and b in the expression of $P_1. P_2$ and P_3 and subsitute these values of P_1 and P_2 back into the equation $f(\dot{P}_1, P_2, P_3) = \dot{P}_1 - \frac{c_2-c_1}{c_1c_2}P_2P_3$ in order to see which sign combinations (the values of λ_1 and λ_2) are valid: that is, for which combinations $\dot{P}_1 - \frac{c_2-c_1}{c_1c_2}P_2P_3 = 0$. Since the two alternative elliptic functions (nd or dn, and sn or cd) for each case each represent the same solution, and thus parametrize the same portions of the intersection of the level surfaces of K and $\mathcal{H}$, we may consider only one of the two alternatives in order to find the combinations of λ_1 and λ_2 for that solution.

Cases 1a, 2a and **3a**: Let $a \to 1, b \to 0$. Then by (A.11.3), the modulus $k = \frac{b}{a} \to 1$, and correspondingly dn $\to$ sech$t$. Thus at the limit the functions $P_1 = \text{sech}t$, $P_3 = \lambda_1 C_1 \sqrt{1+\text{sech}^2 t}$ and $P_2 = \lambda_2 C_2 \sqrt{1-\text{sech}^2 t} = \lambda_2 C_2 \tanh t$ and thus $\dot{P}_1 = \text{sech}t \tanh t$. Then

$$\begin{aligned}\dot{P}_1 - \left(\frac{c_2-c_1}{c_1c_2}\right) P_1P_2 &= \text{sech}t \tanh t - \left(\frac{c_2-c_1}{c_1c_2}\right)(\lambda_2 C_2 \tanh t)\left(\lambda_1 C_1 \sqrt{1+^2 t}\right)\\ &= \tanh t\left(\text{sech}t + \left(\frac{c_2-c_1}{c_1c_2}\right)(\lambda_2 C_2)\left(\lambda_1 C_1\sqrt{1+^2 t}\right)\right).\end{aligned}$$

Since sech$t > 0$ for all $t \in \mathbb{R}$, this value may be zero if and only if $\lambda_1\lambda_2 > 0 \Leftrightarrow \lambda_1 = \lambda_2 = \lambda$, and the valid solutions are as stated. **Cases 1b, 2b** and **3b** follow directly from **1a**, **2a** and **3a** by swapping c_1 and c_2.

Case 3c: Let $a \to 1, b \to 0$. Then by (A.11.3), the modulus $k = \frac{a^2-b^2}{a} \to 1$, and correspondingly sn $\to \tanh t$. Thus at the limit we have the functions $P_1 = \tanh t$, $P_3 = \lambda_1 C_1\sqrt{1+\text{sech}^2 t}$ and $P_2 = \lambda_2 C_2 \sqrt{1-\tanh^2 t} = -\lambda_2 C_2 \text{sech}t$ and thus $\dot{P}_1 = -t^2$. Then

$$\begin{aligned}\dot{P}_1 - \left(\frac{c_2-c_1}{c_1c_2}\right) P_1P_2 &= -\text{sech}t^2 + \left(\frac{c_2-c_1}{c_1c_2}\right)(\lambda_2 C_2 \text{sech})\left(\lambda_1 C_1\sqrt{1+\tanh^2 t}\right)\\ &= -\text{sech}\left(\text{sech}t - \left(\frac{c_2-c_1}{c_1c_2}\right)(\lambda_2C_2)\left(\lambda_1C_1\sqrt{1+\tanh^2 t}\right)\right).\end{aligned}$$

Since sech$t > 0$ for all $t \in \mathbb{R}$, this value may be zero if and only if $\lambda_1\lambda_2 > 0 \Leftrightarrow \lambda_1 = \lambda_2 = \lambda$, and the valid solutions are as stated. **Case 3d** follows from **Case 3c** by swapping c_1 and c_2. □

5.3.4 THEOREM. *The projection $g(t)$ onto $SO(1,2)_0$ of the extremal curve $(g(\cdot), p(\cdot))$ of the left-invariant control problem of* THEOREMS *5.3.1 and 5.3.2 can be expressed as the product*

$$g(t) = \exp(\phi_3(t)N)\cdot\exp(\phi_2(t)E_2)\cdot\exp(\phi_1(t)E_1)$$

where $N = E_1 - E_3$ and ϕ_1, ϕ_2 and ϕ_3 solve the system of differential equations

$$\begin{cases}\dot{\phi}_1 = \dfrac{-c_2P_1P_3 + c_1P_1P_2\tan\phi_1}{c_1c_2P_3 - c_1c_2P_2\tan\phi_1}\\ \dot{\phi}_2 = \dfrac{(c_1-c_2)P_2P_3\sec\phi_1}{c_1c_2(P_3-P_2\tan\phi_1)}\\ \dot{\phi}_3 = \dfrac{(c_2P_3 - c_1P_2\tan\phi_1)}{c_1c_2(P_3-P_2\tan\phi_1)}.\end{cases}$$

PROOF. In the introduction, we expressed the projection $g(\cdot)$ of the extremal curve $(g(\cdot), p(\cdot))$ onto $SO(1,2)_0$ as the product $g(t) = \exp(\phi_3(t)N)\exp(\phi_2(t)E_2)\exp(\phi_1(t)E_1)$. For ease of notation we supress the t in $\phi_i(t)$. Since N is not linearly dependent on E_2, then the derivative of

$g(t)$ is given by

$$\begin{aligned}\dot{g}(t) &= \tfrac{d}{dt}(\exp(\phi_3 N)\cdot\exp(\phi_2 E_2)\cdot\exp(\phi_1 E_1)))\\ &= \left(\dot\phi_3 N\exp(\phi_3 N)\right)\exp(\phi_2E_2)\exp(\phi_1E_1) + \exp(\phi_3N)(\exp(\phi_2E_2)\dot\phi_2E_2)\exp(\phi_1E_1)\\ &\quad+\exp(\phi_3N)\exp(\phi_2E_2)(\exp(\phi_1E_1))\dot\phi_1E_1)\\ &= ((\dot\phi_3N)g + g(t)\exp(-\phi_1E_1)\dot\phi_2E_2)\exp(\phi_1E_1) + g(t)(\dot\phi_1E_1)\\ &= g(t)\left(g(t)^{-1}(\dot\phi_3N)g(t) + \exp(-\phi_1E_1)(\dot\phi_2E_2)\exp(\phi_1E_1) + \dot\phi_1E_1\right)\end{aligned}$$

where for a given t, then $g(t)^{-1}(\dot\phi_3N)g(t) + \exp(-\phi_1E_1)(\dot\phi_2E_2)\exp(\phi_1E_1) + \dot\phi_1E_1$ is an element of $\mathfrak{so}(1,2)$. We may apply the hat map to this element, where by PROPOSITION 3.4.1 for every $t \in \mathbb{R}$ there exists an element $g'(t) \in \mathsf{SO}(1,2)_0$ such that hat $\left(g(t)^{-1}(\dot\phi_3N)g(t)\right) = g(t)'(\dot\phi_3\mathbf{n})$, and by PROPOSITION 5.2.3, then hat$\left(\exp(-E_1\phi_1)\left(\dot\phi_2E_2\right)\right) = \exp\left(\phi_1E_1^\top\right)\left(\dot\phi_2\mathbf{e}_2\right)$. Thus

$$(g(t)^{-1})^\top(\dot\phi_3\mathbf{n}) + \exp(\phi_1E_1^\top)(\dot\phi_2\mathbf{e}_2) + \dot\phi_1\mathbf{e}_1 = (\dot\phi_3\tilde{\mathbf{n}} + \exp(E_1\phi_1)^\top(\dot\phi_2\mathbf{e}_2) + \dot\phi_1\mathbf{e}_1$$

where $\tilde{\mathbf{n}}$ is an arbitrary lightlike element, $\tilde{\mathbf{n}} = P_1\mathbf{e}_1 + P_2\mathbf{e}_2 + P_3\mathbf{e}_3$. Then

$$(\dot\phi_3\tilde{\mathbf{n}}) + \exp(\phi_1E_1)^\top(\dot\phi_2\mathbf{e}_2) + \dot\phi_1\mathbf{e}_1 \tag{5.3.22}$$
$$= (\dot\phi_3(P_1\mathbf{e}_1 + P_2\mathbf{e}_2 + P_3\mathbf{e}_3) + \exp(\phi_1E_1)^\top(\dot\phi_2\mathbf{e}_2) + \dot\phi_1\mathbf{e}_1 \tag{5.3.23}$$
$$= ((\dot\phi_3P_1 + \dot\phi_1)\mathbf{e}_1 + (\dot\phi_3P_2 + \exp(\phi_1E_1)^\top\dot\phi_2)\mathbf{e}_2 + \dot\phi_3P_3\mathbf{e}_3. \tag{5.3.24}$$

Further, since

$$\exp(\phi_1E_1^\top) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi_1 & -\sin\phi_1\\ 0 & \sin\phi_1 & \cos\phi_1\end{bmatrix}$$

then $\exp(\phi_1E_1^\top)\mathbf{e}_2 = \cos\phi_1\mathbf{e}_2 + \sin\phi_1\mathbf{e}_3$. Note that for this system, $\Xi(\mathbf{1},u) = u_1E_3 + u_2E_2$. Thus taking $u_1 = \frac{1}{c_1}P_3, u_2 = \frac{1}{c_2}P_2$ so that $g(t)$ corrresponds to the optimal trajectory of the control problem, then $\frac{1}{c_1}P_3E_3 + \frac{1}{c_2}P_2E_2$ is an element of Γ, and correspondingly

$$\frac{1}{c_1}P_3\mathbf{e}_3 + \frac{1}{c_2}P_2\mathbf{e}_2 \tag{5.3.25}$$

is an element of $\mathbf{\Gamma}$. Equating the two expressions (5.3.24) and (5.3.25) for this element of $\mathbb{R}^{1,2}$ and rearranging, then

$$(\dot\phi_3P_1 + \dot\phi_1)\mathbf{e}_1 + (\dot\phi_3P_2 + \dot\phi_2\cos\phi_1)\mathbf{e}_2 + (\dot\phi_2\sin\phi_1 + \dot\phi_3P_3)\mathbf{e}_3 = \frac{1}{c_1}P_3\mathbf{e}_3 + \frac{1}{c_2}P_2\mathbf{e}_2.$$

From the linear independence of $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$, it follows that

$$\begin{cases}\dot\phi_3P_1 + \dot\phi_1 = 0 & (5.3.26)\\ \dot\phi_3P_2 + \dot\phi_2\cos\phi_1 = \dfrac{1}{c_2}P_2 & (5.3.27)\\ \dot\phi_2\sin\phi_1 + \dot\phi_3P_3 = \dfrac{1}{c_1}P_3. & (5.3.28)\end{cases}$$

From (5.3.26), we find

$$\dot{\phi}_1 = -P_1\dot{\phi}_3 \tag{5.3.29}$$

and from (5.3.27),

$$\dot{\phi}_2 = \frac{(P_2 - c_2P_2\dot{\phi}_3)\sec\phi_1}{c_2} \tag{5.3.30}$$

finally, substituting for $\dot{\phi}_1$ from (5.3.29) in (5.3.28) and solving, gives

$$\dot{\phi}_3 = \frac{(c_2P_3 - c_1P_2\tan\phi_1)}{c_1c_2(P_3 - P_2\tan\phi_1)} \tag{5.3.31}$$

and substituting (5.3.31) back into (5.3.29) and (5.3.30) gives

$$\dot{\phi}_1 = \frac{-c_2P_1P_3 + c_1P_1P_2\tan\phi_1}{c_1c_2P_3 - c_1c_2P_2\tan\phi_1} \text{ and } \dot{\phi}_2 = \frac{(c_1 - c_2)P_2P_3\sec\phi_1}{c_1c_2(P_3 - P_2\tan\phi_1)} \tag{5.3.32}$$

respectively. The result follows. Equations (5.3.29) to (5.3.32) were determined using Mathematica (C.3). □

5.3.2 Equilibrium points and stability

We find the equilibruim points for the system of reduced extremal equations (5.3.2),(5.3.3) and (5.3.4). We then investigate the non-linear stability of each of the equilibrium points using the energy-Casimir (A.12.3) and extended energy-Casimir (A.12.4) methods as well as (A.12.6).

5.3.5 THEOREM. *Given the left-invariant control problem of* THEOREM *5.3.1, then the equilibrium points of the reduced extremal equations are*

$$P_{e_1}^M(0,0,M),\ P_{e_2}^M = (0,M,0),\ P_{e_3} = (0,0,0) \quad \textit{and} \quad P_{e_4}^{M,N}(0,M,N)|_{c_1=c_2} \quad M,N \in \mathbb{R}\setminus\{0\}$$

where P_{e_3} is nonlinear stable and $P_{e_1}^M$, $P_{e_2}^M$ and $P_{e_4}^{M,N}$ are unstable.

PROOF. The equilbrium points of the system of reduced extremal equations (5.3.2),(5.3.3) and (5.3.4) are the solutions of the system

$$\begin{cases} \frac{1}{c_1}P_2P_3 - \frac{1}{c_2}P_2P_3 &= 0 \\ \frac{1}{c_1}P_3P_1 &= 0 \\ -\frac{1}{c_2}P_2P_1 &= 0 \end{cases}$$

which are exactly the points

$$P_{e_1}^M = (0,0,M),\ P_{e_2}^M = (0,M,0),\ P_{e_3} = (0,0,0) \quad \text{and} \quad P_{e_4}^{M,N} = (0,M,N)|_{c_1=c_2}$$

where $M, N \in \mathbb{R}\setminus\{0\}$. The matrix corresponding to the linearized operator of the system of reduced extremal equations is

$$\begin{bmatrix} 0 & \frac{1}{c_1}P_3 & -\frac{1}{c_2}P_2 \\ \frac{1}{c_1}P_3 & 0 & 0 \\ -\frac{1}{c_2}P_2 & 0 & 0 \end{bmatrix}.$$

At the equilibrium point $P_{e_4}^{M,N}$, then

$$\begin{bmatrix} 0 & \frac{1}{c}P_3 & -\frac{1}{c}P_2 \\ \frac{1}{c}P_3 & 0 & 0 \\ -\frac{1}{c}P_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b & -a \\ b & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 0, \lambda_2 = -\frac{\sqrt{M^2+N^2}}{c}, \lambda_3 = \frac{\sqrt{M^2+N^2}}{c}$. Since $M^2 + N^2 > 0$ for all $M, N \in \mathbb{R}\backslash\{0\}$, then all roots are real and $\lambda_3 = \frac{\sqrt{M^2+N^2}}{c} > 0$. Then the linearized system always has one positive real eigenvalue and so the points $P_{e_4}^{M,N}$ are linearly unstable by (A.12.5). Thus by (A.12.6) the points $P_{e_4}^{M,N}$ are unstable.

At the equilibrium points $P_{e_1}^{M}$, then

$$\begin{bmatrix} 0 & \frac{1}{c_1}P_3 & -\frac{1}{c_2}P_2 \\ \frac{1}{c_1}P_3 & 0 & 0 \\ -\frac{1}{c_2}P_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{M}{c_1} & 0 \\ \frac{M}{c_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 0, \lambda_2 = -\frac{M}{c_1}$ and $\lambda_3 = \frac{M}{c_1}$. Since all roots are real and $\lambda_2 > 0$ for $M < 0$, while $\lambda_2 > 0$ for $M > 0$, then the linearized system always has one positive real eigenvalue and so the points $(0, 0, M)$ are linearly unstable by (A.12.5). Thus by (A.12.6) the points $P_{e_1}^{M}$ are unstable. At the equilibrium points $P_{e_2}^{M}$, then

$$\begin{bmatrix} 0 & \frac{1}{c_1}P_3 & -\frac{1}{c_2}P_2 \\ \frac{1}{c_1}P_3 & 0 & 0 \\ -\frac{1}{c_2}P_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{M}{c_2} \\ 0 & 0 & 0 \\ -\frac{M}{c_2} & 0 & 0 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 0, \lambda_2 = -\frac{M}{c_2}$ and $\lambda_3 = \frac{M}{c_2}$. Since all roots are real and $\lambda_2 > 0$ for $M < 0$, while $\lambda_3 > 0$ for $M > 0$, then the linearized system always has one positive eigenvalue and so the points $P_{e_2}^{M}$ are linearly unstable by (A.12.5). Thus by (A.12.6) the points $P_{e_2}^{M}$ are unstable. Finally, we use the energy-Casimir method (A.12.3) to show that the point P_{e_3} is nonlinear stable. Construct the energy-Casimir function

$$H_\psi = \frac{P_3^2}{2c_1} + \frac{P_2^2}{2c_2} + \psi\left(-\frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2}P_3^2\right)$$

for $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$. The first variation of H_ψ is given by

$$\begin{aligned} \delta H_\psi &= \frac{d}{dt}\left(\frac{1}{2c_1}(P_3 + \delta_3 t)^2 + \frac{1}{2c_2}(P_2 + \delta_2 t)^2\right)\Big|_{t=0} \\ &\quad + \frac{d}{dt}\left(\psi\left(-\frac{1}{2}(P_1 + \delta_1 t)^2 + \frac{1}{2}(P_2 + \delta_2 t)^2 + \frac{1}{2}(P_3 + \delta_3 t)^2\right)\right)\Big|_{t=0} \\ &= \frac{1}{c_1}\delta_3 P_3 + \frac{1}{c_2}\delta_2 P_2 + (-\delta_1 P_1 + \delta_2 P_2 + \delta_3 P_3)\,\dot{\psi}\left(-\frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2}P_3^2\right). \end{aligned}$$

At the equilibrium point P_{e_3}, then $\delta H_\psi|_{(0,0,0)} = 0 \cdot \dot{\psi}(0) = 0$. Thus the first variation is zero at the equlibrium point for any function ψ, and we may take $\dot{\psi}(0) = C$ for any $C \in \mathbb{R}$. The second variation $\delta^2 H_\psi = \delta(\delta H_\psi)$ is then given by

$$\begin{aligned}
\delta^2 H_\psi &= \frac{d}{dt}\left(\frac{1}{c_1}\delta_3(P_3+\delta_3 t) + \frac{1}{c_2}\delta_2(P_2+\delta_2 t)\right)\Big|_{t=0} \\
&+ \frac{d}{dt}\left(-\delta_1(P_1+\delta_1 t) + \delta_2(P_2+\delta_2 t) + \delta_3(P_3+\delta_3 t)\right) \\
&\cdot\dot{\psi}\left(-\frac{1}{2}(P_1+\delta_1 t)^2 + \frac{1}{2}(P_2+\delta_2 t)^2 + \frac{1}{2}(P_3+\delta_3 t)^2\right)\Big|_{t=0} \\
&= \frac{1}{c_3}\delta_3^2 + \frac{1}{c_2}\delta_2^2 + (-\delta_1^2+\delta_2^2+\delta_3^2)\dot{\psi}\left(-\frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2}P_3^2\right) \\
&+ (-\delta_1 P_1 + \delta_2 P_2 + \delta_3 P_3)^2\,\ddot{\psi}\left(-\frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2}P_3^2\right).
\end{aligned}$$

At the equlibrium point P_{e_3}, then

$$\delta^2 H_\psi|_{(0,0,0)} = \frac{1}{c_2}\delta_1^2 + \frac{1}{c_1}\delta_2^2 + (-\delta_1^2+\delta_2^2+\delta_3^2)\dot{\psi}\,(0)$$

where additionally $\dot{\psi}\,(0)$ is any real number C. Thus

$$\delta^2 H_\psi|_{(0,0,0)} = \frac{1}{c_1}\delta_3^2 + \frac{1}{c_2}\delta_2^2 + C(-\delta_1^2+\delta_2^2+\delta_3^2)$$

which can be written as the quadratic form

$$\begin{bmatrix}\delta_1 & \delta_2 & \delta_3\end{bmatrix}\begin{bmatrix}-C & 0 & 0 \\ 0 & \frac{1}{c_2}+C & 0 \\ 0 & 0 & \frac{1}{c_1}+C\end{bmatrix}\begin{bmatrix}\delta_1 \\ \delta_2 \\ \delta_3\end{bmatrix}$$

which by (A.1.8) is positive-definite if $C < 0, |C| < \frac{1}{c_1}$ and $|C| < \frac{1}{c_2}$. Choosing $c_2 > c_1$, the negative real number $-\left(\frac{1}{c_2+1}\right)$ fulfils this requirement; similarly, choosing $c_1 > c_2$, then the negative real number $-\left(\frac{1}{c_1+1}\right)$ fulfils this requirement. Then in the case $c_2 > c_1$ we choose ψ to be the function $\psi(x) = Cx$ for $C = -\left(\frac{1}{c_2+1}\right)$, and the energy-Casimir function H_ψ satisfies the conditions of the energy-Casimir method, while in the case $c_1 > c_2$, we choose the function $\psi(x) = Cx$ for $C = -\left(\frac{1}{c_1+1}\right)$ and the energy-Casimir function H_ψ satisfies the conditions of the energy-Casimir method. Thus in each case we have found a function H_ψ which satisfies these conditions, and so the equilibrium point P_{e_3} is nonlinear stable. □

Chapter 6

Conclusion

In this thesis, we have used two distinct approaches to the problems of geometric control theory on a matrix Lie group. Firstly, we considered the group structure of the state space as a symmetry group of a model of hyperbolic plane geometry. Since in both cases we determined that the model was a homogeneous space of its symmetry group (the symmetry group's action was found to be transitive) this allowed us to access the control-theoretic results in a geometric way. Secondly, since by this transitive action of the symmetry group it is a matrix Lie group (Myers and Steenrod (1951)), we accessed these results via the structure of its Lie algebra. In both approaches, we began with a metric on a particular structured space - in the first the abstract surface $\mathcal{HL}$ or $\mathcal{HP}$, and in the second the Lie algebra, where the symmetric bilinear form $\odot$ estabilished with $\odot$ on its image under the hat map made it an inner product space.

In the initial approach, we established two models of hyperbolic geometry as geometric surfaces $\mathbb{HL}$ and $\mathbb{HP}$, beginning with the space itself and a metric of constant negative Gaussian curvature $\mathrm{K} = -1$. The spaces $\mathcal{HL}$ and $\mathcal{HP}$ we expressed as abstract surfaces. We then used the metric to construct the symmetry group, in the case of $\mathbb{HL}$ explicitly, and in the case of $\mathbb{HP}$ by constructing its family of geodesics and then using the well-established theorems of projective geometry to construct the symmetry group, considering geodesics as paths invariant under the action of this group. We established in both cases a matrix representation of the symmetry group.

We extended this metric approach by considering the Lie algebra of the symmetry group $\mathsf{SO}(1,2)_0$ of the hyperboloid model $\mathbb{HL}$. We determined its topological and structural properties as well as properties of its action on $\mathbb{HL}$. Many of these properties, for example the simply-connectedness result of THEOREM 3.6.3 were referred back to properties of the homogeneous space of the group and the isomorphism with $\mathsf{PGL}(2, \mathbb{R})$. This isomorphism itself was established as a result of the projections between the two homogeneous spaces of these groups.

We went on to discuss an Iwasawa decomposition of the group $\mathsf{SO}(1,2)_0$. The hat map between $\mathfrak{so}(1,2)$ and $\mathbb{R}^{1,2}_{\odot}$ linked the partitioning of Minkowski spacetime into spacelike, lightlike and timelike vectors and the Iwasawa decomposition of the symmetry group's Lie algebra into elements conjugate to elements in Abelian, nilpotent and compact subalgebras. This result was determined as a conseqence of the transitive action of the symmetry group on the given model. We used this decomposition extensively in the construction of the controlability criterion for control affine systems on this group. As in the seminal work of Milnor [20], we chose to consider the Lie algebra as an inner product space. This was done indirectly via the hat map between $\mathfrak{so}(1,2)$ and $\mathbb{R}^{1,2}_{\otimes}$, where $\otimes$ is the Lie bracket on $\mathbb{R}^{1,2}$. We chose to do this because of the rich and well-established terminology and theorems already in place for Minkowski spacetime, which made expressing the results more simple and direct, and gave a geometric intuition to the problems.

Using the structures of Minkowski spacetime, the interpretation of Witt's theorem as expressed by Berger [4] and some results of projective geometry, we were able to develop a simple and intuitive classification up to local detached feedback equivalence of all full-rank control affine systems on $\mathsf{SO}(1,2)_0$. Here, the traces of the control affine systems were considered as corresponding under the hat map to lines and planes in Minkowski spacetime. The restriction of the Minkowski metric to the affine subsets corresponding to hyperplanes $\mathbf{\Gamma}$ gave three distinct types of hyperplanes, depending on the signature of the restriction of $\odot$. We termed elliptic, hyperbolic and parabolic planes those for which the restriction $J_{\mathbf{\Gamma}}$ had signature $(0,2,0)$, $(1,1,0)$ and $(0,1,1)$, respectively. In correlation with projective geometry, these planes were seen respectively to correspond to the planes which intersect the light cone in elliptic, hyperbolic and parabolic conics. Using Witt's theorem on the orbits of vector subspaces under the action of the isometry group of an inner product space, we were able immediately to classify all full-rank 2-input systems (whose traces correspond to 2-dimensional vector subspaces under hat) into elliptic and hyperbolic classes represented by systems $\Sigma_1^{(2,0)}$ and $\Sigma_2^{(2,0)}$. Results drawn from projective geometry which determined a correlation between the elliptic, hyperbolic and parabolic hyperplanes with the planes intersecting the light cone $\mathcal{K}_L$ in elliptic, hyperbolic and parabolic conics allowed for the classification of the 2-input inhomogeneous and 1-input inhomogeneous cases. For the 2-input inhomogeneous systems we determined a classification into 2 infinte families of classes represented by $\Sigma_{2,\alpha}^{(2,1)}, \Sigma_{3,\alpha}^{(2,1)}$ and a class represented by the system $\Sigma_1^{(2,1)}$ whose traces have images under hat which are parabolic, hyperbolic and elliptic hyperplanes, respectively. For the 1-input inhomogeneous systems we determined a classification into 3 infinte families of classes represented by the system $\Sigma_{1,\alpha}^{(1,1)}, \Sigma_{2,\alpha}^{(1,1)}, \Sigma_{3,\alpha}^{(1,1)}$ and a class represented by system $\Sigma_4^{(1,1)}$. The 3-input inhomogeneous and homogeneous systems were determined always to be equivalent to each other under l.d.f.e., and we determined one class of these systems, represented by $\Sigma^{(3,0)}$. These results are recorded in table B.1.

The metric approach was used to good effect in determining a controllability criterion on all control affine systems on $\mathsf{SO}(1,2)_0$. Using a result of Jurdjevic and Sussman [30] requiring the periodicity of a trajectory of any controllable system, we were able to determine that it is both sufficient and necessary for the controllability of any inhomogeneous system that there exists in the image $\mathbf{\Gamma}$ of its trace Γ under the hat map a timelike element of $\mathbb{R}^{1,2}$. In the homogeneous case, this condition could be extended to the requirement that $\mathbf{\Gamma}$ admit two linearly independent vectors of any kind. This requirement is similarly necessary and sufficient for controllability.

Using the l.d.f.e. classification of all full-rank control affine systems, we then considered the optimal control problem with quadratic costs on the two representative systems $\Sigma_1^{(2,0)}$ and $\Sigma_2^{(2,0)}$ of the 2-input homogeneous systems. We used the Pontryagin Maximum Principle to determine the optimal cost-extended Hamiltonians. Following Jurdjevic [13], we then worked in the trivialization of the cotangent bundle $\mathsf{T}\mathsf{G}^* = \mathsf{G} \times \mathfrak{g}^*$ to set up and solve reduced extremal equations to find the projection onto $\mathfrak{so}(1,2)^*$ of the extremal pairs $(g(t), p(t), u(t))$ in both cases. These solutions were determined in terms of Jacobi elliptic functions. We consequently made use of the Iwasawa decomposition $\mathsf{SO}(1,2)_0 = \mathsf{KAN}$ to determine the projection of the extremal curve onto $\mathsf{SO}(1,2)_0$, which we expressed as the solution to a system of differential equations. In the course of proving this result we made use of the homeomorphic group isomorphism Ψ established between the groups $\mathsf{Aut}(\mathfrak{so}(1,2))$ and $\mathsf{Aut}(\mathbb{R}^{1,2})$ in order to express these differential equations in a simple way.
While the necessary condition of the Pontryagin Maximum Principle provides us with only a family of possible candidates for optimal controls and their corresponding optimal trajectories, in the case of both systems $\Sigma_1^{(2,0)}$ and $\Sigma_2^{(2,0)}$ the principle gave rise to exactly one control candidate, which we assumed to be the optimal control. We note that in the inhomogeneous cases, the expressions obtained in terms of Jacobi elliptic functions proved to be very complex and obscured rather than clarified the underlying geometry, and so were not included in this thesis. Considering the separation of cases which naturally occured in correlation with the intersection of different level surfaces of the optimal Hamiltonian and the Casimir function as shown in figs. D.1-D.15, we suspect that this was due to the much more complex intersections resulting from the intersection of level surfaces of optimal Hamiltonians of these systems which were parabolic cylinders or parabolic sheets, with the level surfaces of the Casimir function.

Finally, we established the nonlinear stability properties of the equilibrium points of the reduced extremal equations of each of the 10 l.d.f.e. classes of systems on the Lorentz group. This was done using the energy-Casimir and extended energy-Casimir methods. In this thesis we provided explicit calculations of these results for the 2-input homogeneous control affine systems, while the results for the other 8 systems are provided in tabular form in tables B.2-5. For the 2-input homogeneous systems we determined that the system of reduced extremal equations of $\Sigma_1^{(2,0)}$ has 2 infinite families of equilibrium points, where the infinite family on the timelike axis of $\mathbb{R}^{1,2}$ and the equilbrium point at the origin are stable and the infinite family covering the spacelike $\mathbf{e}_2$-axis is unstable. The system of reduced extremal equations of $\Sigma_2^{(2,0)}$ has 3 infinite families of equilibrium points on the span of the spacelike axes of $\mathbb{R}^{1,2}$, all three of which are unstable, and a stable equilibrium at the origin. The distribution of the unstable equilibrium points, in each case on or in the span of the $\mathbf{e}_2, \mathbf{e}_3$-axes, may also give insight into which of the choices of expression in terms of Jacobi elliptic functions is most suitable for use in computation; the function sn solves a differential equation involving elements on the spacelike axes, which are unstable, while the functions cn and dn solve differential equations involving variables on the spacelike axis $\mathbf{e}_2$, which is unstable, and the timelike axis $\mathbf{e}_1$, which is a stable axis. These considerations leave room for further research.

In this thesis we discussed the problems of controllability, equivalence classification and optimality of left-invariant control affine systems on $\mathsf{SO}(1,2)_0$ in the geometric context, establishing first the properties of $\mathsf{SO}(1,2)_0$ as a symmetry group of a model of hyperbolic plane geometry, and then expressing the Lie algebra as an inner product space via the hat map, an expression which we used to discuss the problems of controllability, equivalence classification and optimality.

Topic	**Results**
Hyperbolic plane geometry	Expression of $\mathbb{HL}$ and $\mathbb{HP}$ as geometric surfaces
Equivalence classification	L.d.f.e. classification of control affine systems
Controllability of affine systems	Controllability criterion for all control affine systems
Optimal control of affine systems	Extremal curves for optimal control problem on $\Sigma_1^{(2,0)}$ and $\Sigma_1^{(2,0)}$
	Stability of equilibrium points of reduced extremal equations

Bibliography

[1] E. Artin, *Geometric Algebra*, Interscience, 1957.

[2] A.A. Agrachev and Y. Sachkov, *Control Theory from the Geometric Viewpoint*, Springer-Verlag, 2003.

[3] A. Baker, *Matrix Lie Groups*, Springer-Verlag, 2001.

[4] M. Berger, *Geometry II*, Springer-Verlag, 1996.

[5] R. Biggs and C. C. Remsing, On the equivalence of control systems on Lie groups (submitted)

[6] A.A. Brannan, M.F. Esplen and J.I. Grey, *Geometry*, Cambridge University Press, 1991.

[7] J.W. Cannon, W.J. Floyd, R. Kenyon and W. R. Parry, Hyperbolic geometry, *Flavors of Geometry*, 31 (1997), 59-108.

[8] M.P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976.

[9] K. Erdmann and M.J. Wildon, *Lie Algebras*, Springer-Verlag, 2002.

[10] J. Gallier, *Geometric Methods and Applications: For Computer Science and Engineering*, Springer-Verlag, 2011.

[11] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries*, Freeman, 1973.

[12] A. Grey, E. Abbena and S. Salamon, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, Chapman and Hall, 2006.

[13] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, 1997.

[14] V. Jurdjevic, Hamiltonian point of view of non-Euclidean geometry and elliptic functions, *Systems Control Lett.*, 43(2001), 25-41.

[15] W. Knapp, *Lie Theory: Beyond an Introduction*, Birkhauser, 1996.

[16] J. M. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 1989.

[17] J. E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Springer-Verlag, 1999.

[18] J. McCleary, *Geometry from the Differentiable Viewpoint*, Cambridge University Press, 1994.

[19] K.R. Meyer, Jacobi elliptic functions from a dynamical systems point of view, *Amer. Math. Monthly*, 108(2001), 729-737.

[20] J. Milnor, Curvatures of left-invariant metrics on Lie groups, *Adv. Math.*, 21(1979), 293-329.

[21] G.L. Naber, *The Geometry of Minkowski Spacetime*, Dover, 2003.

[22] P.T. Nagy, Dynamical invariants of rigid motions on the hyperbolic plane, *Geom. Dedicata*, 37(1991), 125-139.

[23] J-P Ortega, V. Planas-Bielsa and T.S. Ratiu, Asymptotic and Lyapunov stability of constrained and Poisson equilibria, *J. Diff. Equations*, 214(2005), 92-127.

[24] E. Vinberg, *Geometry II*, Springer-Verlag, 1991.

[25] A. Pressley, *Elementary Differential Geometry*, Springer-Verlag, 2001.

[26] M.Puta, *Hamiltonian Mechanical Systems and Geometric Quantization*, Kluwer, 1993.

[27] J.G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer-Verlag, 2006.

[28] W. Rossman, *Lie Groups. An Introduction through Linear Groups*, Oxford University Press, 2002.

[29] Yu. L. Sachkov, Control theory on Lie groups, *J. Math. Sci.*, 156(2009), 381-439.

[30] H.J. Sussman and V. Jurdjevic, Controllability of nonlinear systems, *J. Diff. Equations*, 12(1972), 95-116.

[31] J. Stillwell, *Geometry of Surfaces*, Springer-Verlag, 1992.

[32] J. Stillwell, *The Four Pillars of Geometry*, Springer-Verlag, 2001.

[33] J. Stillwell, *Naive Lie Theory*, Springer-Verlag, 2008.

[34] E.T. Whittaker and G.N. Watson, *Course of Modern Analysis*, Cambridge University Press, 1958.

Appendix A

Review of Prerequisites

A.1 Quadratic forms

References used include [1] and [4]. Let V be an n-dimensional (real) vector space, and $Q = [q_{ij}]$ be a symmetric matrix. Here, $\mathbb{R}^{n\times n}$ denotes the set of all real $n \times n$ matrices, and $\mathbb{R}^+$ the set of all real numbers strictly greater than 0.

A.1.1 DEFINITION. A **quadratic form** associated to the symmetric matrix Q is a function $q : \mathsf{V} \to \mathbb{R}$ defined by

$$q(\mathbf{p}) = \mathbf{p}^\top Q\mathbf{p} = \sum_{i,j=1}^{n} q_{ij}p_ip_j.$$

We define a basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ on V and express each element $\mathbf{p} \in \mathsf{V}$ as $\mathbf{p} = (p_1, ..., p_n)$. We henceforth identify $(p_1, ..., p_n)$ with the column matrix $\begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$.

A.1.2 DEFINITION. The group of all invertible $n \times n$ real matrices is denoted by

$$\mathsf{GL}(n, \mathbb{R}) = \left\{Q \in \mathbb{R}^{n\times n} \,:\, \det Q \neq 0\right\}.$$

A.1.3 DEFINITION. A **quadratic form** q is **nondegenerate** if $q(\mathbf{p}) = 0$ implies that $\mathbf{p} = 0$.

If q is nondegenerate, then the matrix Q is an element of $\mathsf{GL}(n, \mathbb{R})$.

A.1.4 DEFINITION. An **inner product** is a symmetric bilinear form $\phi : \mathsf{V} \times \mathsf{V} \to \mathbb{R}$ such that if $\phi(\mathbf{p}, \mathbf{q}) = 0$ for all $\mathbf{q} \in \mathsf{V}$, then $\mathbf{p} = 0$.

Particularly, we associate the nondegenerate quadratic form q with the inner product

$$\phi : \mathsf{V} \times \mathsf{V} \to \mathbb{R}, \qquad \phi(\mathbf{p}, \mathbf{q}) = \mathbf{p}^\top Q\mathbf{q}$$

where clearly $\phi(\mathbf{p}, \mathbf{p}) = q(\mathbf{p})$ for each $\mathbf{p} \in \mathsf{V}$.

A.1.5 DEFINITION. A vector space V equipped with a nondegenerate quadratic form q is called an **inner product space**, denoted by (V, q).

A.1.6 PROPOSITION. (SYLVESTER'S LAW OF INERTIA) *Given a symmetric square matrix Q, then there exists some S such that Q is similar to a diagonal matrix $SQS^{-1} = D$ having along the diagonal n_0 entries which are 0, $n+$ entries which are $+1$ and $n-$ which are -1 . The triple (n_-, n_+, n_0) depends only on Q and not on the basis used to define it: that is, for any matrix Q' similar to Q there exists some S' such that $S'Q'(S')^{-1} = D$.*

A.1.7 DEFINITION. In Sylvester's law, the triple (n_-, n_+, n_0) is the **signature** of the quadratic form q associated to the symmetric matrix $Q = [q_{ij}]$.

A.1.8 PROPOSITION. *The quadratic form q has signature $(n, 0, 0)$ if and only if all the eigenvalues of Q are negative. The quadratic form q has signature $(k, 0, m)$ if and only if all the eigenvalues of Q are non-positive.*

A.1.9 COROLLARY. *If Q has one positive eigenvalue, then the quadratic form q has signature (n_1, n_2, n_3) where $n_i \in \mathbb{N}$, $n_2 \neq 0$.*

A.1.10 DEFINITION. A quadratic form q with signature $(n, 0, 0)$ is said to be **negative-definite** (on V). The quadratic form q with signature $(0, n, 0)$ is **positive-definite**.

A.1.11 DEFINITION. A linear bijection $\phi : \mathsf{V} \to \mathsf{V}'$ such that $\phi^*(q') = q$ is said to be **isometric**.

A.1.12 DEFINITION. The **orthogonal group** of (V, q) is the group

$$\mathsf{O}(\mathsf{V}, q) = \{g \in \mathsf{GL}(\mathsf{V}) \ : \ g^*(Q) = Q\}$$

A.1.13 DEFINITION. Given an n-dimensional vector space V, and a subspace W of V, then for each $v \in V$, we define the set $g(\mathsf{W}) = \{gv \ : \ v \in V\}$. Then the subset

$$\mathcal{O}_{\mathsf{W}} = \{g\mathsf{W} \ : \ g \in \mathsf{O}(\mathsf{V}, Q)\}$$

is the **orbit** of W under the action of $\mathsf{O}(\mathsf{V}, Q)$.

A.1.14 DEFINITION. The **restriction** of the quadratic form q to a subspace W of V is the inner product space $(\mathsf{W}, q|_{\mathsf{W}})$.

A.1.15 DEFINITION. Two vector subspaces W, W' of (V, q) are **isometric subspaces** if $q|_{\mathsf{W}} = q|_{\mathsf{W}'}$.

We will correspondingly call two planes Γ and Γ' isometric if their direction subspaces are isometric.

A.1.16 THEOREM. (WITT'S THEOREM) *The orbits of the set of subspaces of V under the action of $\mathsf{O}(\mathsf{V}, Q)$ are exactly the sets of (inner product space) isometric subspaces of (V, Q).*

A.2 Abstract and geometric surfaces

References used include [8], [29], [16], [18], [25] and [31].

A.2.1 DEFINITION. An **abstract surface** is a set $\mathcal{S}$ equipped with a countable collection of injective functions $\epsilon_a : U_a \to \mathcal{S}$ indexed by $a \in A$ such that

1. U_a is an open subset of $\mathbb{R}^2$
2. $\bigcup_a \epsilon_a(U_a) = \mathcal{S}$
3. Given a and b in A and $\epsilon_a(U_a) \cap \epsilon_b(U_b) = V_{ab} \subseteq \mathcal{S} \neq \emptyset$, then the composition given by $\epsilon_a^{-1} \circ \epsilon_b(\cdot) : \epsilon_b^{-1}(V_{ab}) \to \epsilon_a^{-1}(V_{ab})$ is a smooth map.

The injective functions ϵ_a are the **surface patches** of $\mathcal{S}$. The map $\epsilon_a^{-1} \circ \epsilon_b(\cdot)$ is the **transition map** between these open sets of $\mathbb{R}^2$.

A.2.2 DEFINITION. The function $x_i : \epsilon_u(\mathcal{U}_u) \to \mathbb{R}$, $x_i = u_i \circ \epsilon_a^{-1}$ is the i**-th coordinate function**, and (x_1, x_2) is the **system of local coordinates** on $\mathcal{S}$.

A.2.3 DEFINITION. The differentiable map $\alpha(\cdot) = \epsilon(u(\cdot), v(\cdot)) : (a, b) \to \mathcal{S}$ is a **curve** on the abstract surface $\mathcal{S}$.

We will refer to the image set $\alpha = \{\alpha(t) \ : \ t \in (a, b)\}$ as a **path** in $\mathcal{S}$.

A.2.4 DEFINITION. Given the curve $\alpha(\cdot) = \epsilon(u(\cdot), v(\cdot)) : (a, b) \to \mathcal{S}$, let $\alpha(0) = \mathbf{p}$ and f be a function on $\mathcal{S}$ differentiable at $\mathbf{p}$. The **tangent vector** to the curve $\alpha(\cdot)$ at $\mathbf{p}$ is the function

$$\dot{\alpha}(0)(f) = \left. \frac{df \circ \alpha}{dt} \right|_{t=0}.$$

A.2.5 DEFINITION. For each $\mathbf{p} \in \mathcal{S}$, the **tangent space** to $\mathcal{S}$ at $\mathbf{p}$ is the set of all tangent vectors to curves in $\mathcal{S}$ at $\mathbf{p}$, denoted by $\mathrm{T}_\mathbf{p}\mathcal{S}$.

A.2.6 DEFINITION. Two subsets $\mathcal{S}_1 \subseteq \mathbb{R}^m$ and $\mathcal{S}_2 \subseteq \mathbb{R}^n$ are **homeomorphic** to one another if there exists a continuous bijective map $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$. The map Φ which satsifies these properties is a **homeomorphism**. If in addition Φ and Φ^{-1} are smooth, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are **diffeomorphic** to one another and Φ is a **diffeomorphism**. Φ is a **local diffeomorphism** if there exists a neighbourhood of any point of $\mathcal{S}_1$, restricted to which Φ is a diffeomorphism onto its image. Φ is a **local homeomorphism** if there exists a neighbourhood of any point of $\mathcal{S}_1$, restricted to which Φ is a homeomorphism onto its image.

A.2.7 DEFINITION. The **tangent bundle** of the abstract surface $\mathcal{S}$ is the disjoint union $\bigcup_{\mathbf{p} \in \mathcal{S}} \mathrm{T}_\mathbf{p}\mathrm{S}$.

A.2.8 DEFINITION. A **vector field** X on an abstract surface $\mathcal{S}$ is a map $X : \mathcal{S} \to \mathrm{T}\mathcal{S}$ such that $X(\mathbf{p}) \in \mathrm{T}_\mathbf{p}\mathrm{G}$. If the map X is smooth, then X is a **smooth vector field**. The real vector space of all smooth vector fields on $\mathcal{S}$ is denoted by $\mathfrak{X}(\mathcal{S})$.

A.2.9 DEFINITION. Suppose Φ is a smooth mapping $\Phi : G_1 \to G_2$ between two abstract surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$. The **tangent map** of Φ is the linear mapping $d\Phi : T\mathcal{S}_1 \to T\mathcal{S}_2$ such that for each $\mathbf{p} \in \mathcal{S}_1$ and any map $\alpha(\cdot) : (a, b) \to \mathcal{S}_1$, $\alpha(0) = \mathbf{p}$, then the map $d\Phi(g) : T_{\mathbf{p}}\mathcal{S}_1 \to T_{\Phi(\mathbf{p})}\mathcal{S}_2$ is given by $d\Phi(\mathbf{p})\,(\dot{\alpha}(0)) = \frac{d}{dt}|_{t=0}\Phi(\alpha(t))$.

A.2.10 DEFINITION. The **pushforward of a vector field** X by Φ is the map $\Phi_* X(\Phi(\mathbf{m}) = d\Phi(\mathbf{p}) \cdot X(\mathbf{p})$. The **pullback of a vector field** X by Φ is the map $\Phi^* X = (\Phi^{-1})_*(X)$.

A.2.11 DEFINITION. A **covariant tensor field of degree** r on the abstract surface $\mathcal{S}$ is a multilinear map $\chi : \mathfrak{X}(\mathcal{S}) \times \mathfrak{X}(\mathcal{S}) \times ... \times \mathfrak{X}(\mathcal{S}) \to \mathbb{R}$ such that

$$\chi(X_1 ... a_i Y_i + b_i X_i, ..., X_r) = a_i \chi(X_1, ..., Y_i, ..., X_r) + b_i \chi(X_1, ..., X_i, ..., X_r)$$

for each $a_i, b_i \in \mathbb{R}$ and $X_i \in \mathfrak{X}(\mathcal{S})$.

A.2.12 DEFINITION. A **covariant tensor of degree** r **at** $\mathbf{p}$ in the abstract surface $\mathcal{S}$ is a multilinear map $\chi : T_{\mathbf{p}}\mathcal{M} \times T_{\mathbf{p}}\mathcal{S} \times ... \times T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}$ such that

$$\chi(\mathbf{v}_1, ..., a_i \mathbf{w}_i + b_i \mathbf{v}_i, ..., \mathbf{v}_r) = a_i \chi(\mathbf{v}_1, ..., \mathbf{w}_i, ..., \mathbf{v}_r) + b_i \chi(\mathbf{v}_1, ..., \mathbf{v}_i, ..., \mathbf{v}_r)$$

for each $a_i, b_i \in \mathbb{R}$ and $\mathbf{v}_i, \mathbf{w}_i \in T_{\mathbf{p}}\mathcal{M}$.

A.2.13 DEFINITION. A covariant tensor field of degree 2 on $\mathcal{S}$ is a **symmeteric bilinear form** χ^s if $\chi^s(X, Y) = \chi^s(Y, X)$ for each $X, Y \in \mathfrak{X}(\mathcal{S})$, where the superscript s denotes symmetry.

A.2.14 REMARK. We may define using this covariant tensor field χ^s a tensor $\chi^s_{\mathbf{p}}$ on each tangent space $T_{\mathbf{p}}\mathcal{S}$: for $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$, then $\chi^s_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) = \chi^s(X, Y)(\mathbf{p})$, where $X, Y \in \mathfrak{X}$ are any vector fields on $\mathcal{S}$ chosen such that $X(\mathbf{p}) = \mathbf{v}$ and $Y(\mathbf{p}) = \mathbf{w}$.

A.2.15 DEFINITION. A symmetric bilinear form χ^s on $\mathcal{S}$ is **nondegenerate** if $\chi^s_{\mathbf{p}}$ is nondegenerate on each tangent space $T_{\mathbf{p}}\mathcal{S}$ for each $\mathbf{p} \in \mathcal{S}$: that is, if $\chi^s_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$, then $\mathbf{v} = 0$.

A.2.16 DEFINITION. A nondegenerate symmetric bilinear form is a **pseudo-Riemannian metric**.

We may denote the Riemannian metric by $\langle \cdot, \cdot \rangle$, and $\chi^s_{\mathbf{p}}$ by $\langle \cdot, \cdot \rangle|_{\mathbf{p}}$.

A.2.17 DEFINITION. A symmetric bilinear form is **positive-definite** on $T_{\mathbf{p}}\mathcal{S}$ if $\chi^s_{\mathbf{p}}(\mathbf{v}, \mathbf{v}) > 0$ for all $\mathbf{v} \in T_{\mathbf{p}}\mathcal{S}$. Similarly, a symmetric bilinear form is **negative-definite** on $T_{\mathbf{p}}\mathcal{S}$ if $\chi^s_{\mathbf{p}}(\mathbf{v}, \mathbf{v}) < 0$ for all $\mathbf{v} \in T_{\mathbf{p}}\mathcal{S}$.

A.2.18 DEFINITION. A symmetric bilinear form is a **Riemannian metric** on $\mathcal{S}$ if $\chi^s_{\mathbf{p}}$ is positive-definite on $T_{\mathbf{p}}\mathcal{S}$ for each $\mathbf{p} \in \mathcal{S}$.

A.2.19 DEFINITION. Let $\epsilon_a : \mathcal{U}_a \to \mathcal{S}$ be a patch on $\mathcal{S}$, $\epsilon_a^{-1} = (x_1, x_2)$ and $\mathbf{p} \in \mathcal{S}$ where $\mathbf{p} = \epsilon_a(u_0, v_0)$. Then the functions $\left.\frac{\partial}{\partial x_i}\right|_{(u_0,v_0)} : C^\infty(\mathcal{S}) \to \mathbb{R}$ such that for each $f \in C^\infty(\mathcal{S})$,

$$\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{p}} = \left.\frac{\partial f \circ \epsilon_a}{\partial u_i}\right|_{(u_0,v_0)}$$

are the elements of the **local basis for the tangent space** $\mathrm{T}_{\mathbf{p}}\mathcal{S}$, given by

$$\left\{ \left.\frac{\partial}{\partial x_i}\right|_{\mathbf{p}}, \left.\frac{\partial}{\partial x_i}\right|_{\mathbf{p}} \right\}.$$

A.2.20 REMARK. Using the local basis $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$ for $\mathrm{T}\mathcal{S}$, denote $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$. The local basis $\{dx_1, dx_2\}$ of $\mathrm{T}^*\mathcal{S}$ corresponds to the local basis $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$ of $\mathrm{T}\mathcal{S}$ in the sense that $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$.

A.2.21 THEOREM. *The metric* $\langle \cdot, \cdot \rangle$ *can be expressed as* $ds^2 = \sum_{i,j=1}^n g_{ij} dx_i dx_j$.

A.2.22 DEFINITION. The functions $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$ are the **components** of the metric ds^2 in the local coordinate system $\{x_1, x_2\}$.

A.2.23 DEFINITION. An abstract surface $\mathcal{S}$ equipped with a Riemannian metric $ds^2 = \langle \cdot, \cdot \rangle$ is a **geometric surface** $\mathbb{S} = (\mathcal{S}, ds^2)$.

A.2.24 DEFINITION. Given a diffeomorphism $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ between two geometric surfaces $\mathbb{S}_1$ and $\mathbb{S}_2$ with Riemmanian metrics $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively, then the **pullback of** $\langle \cdot, \cdot \rangle_2$ **by** Φ is the metric $\Phi^* \langle \cdot, \cdot \rangle_2$ on $\mathcal{S}^1$ given by

$$\Phi^* \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_2 = \langle d\Phi(p) \cdot (\mathbf{v}_1), d\Phi(p) \cdot (\mathbf{v}_2) \rangle_2$$

for each $\mathbf{p} \in \mathcal{S}_1$ and tangent vector $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{T}_p\mathcal{S}_1$.

Note that in the ds^2-notation we may write $\Phi^* ds^2(\mathbf{v}_1, \mathbf{v}_2) = ds^2(d\Phi(p) \cdot (\mathbf{v}_1), d\Phi(p) \cdot (\mathbf{v}_2))$.

A.2.25 THEOREM. *Given a chart* $\epsilon^i : \mathbb{R}^2 \to \mathcal{S}$ *of the geometric surface* $\mathbb{S} = (\mathcal{S}, \langle \cdot, \cdot \rangle)$ *where* $\langle \cdot, \cdot \rangle$ *has component functions* g_{ij}, *then* $g_{11} \circ \epsilon^i = E$, $g_{12} \circ \epsilon^i = F$, $g_{21} \circ \epsilon^i = F$ *and* $g_{22} \circ \epsilon^i = G$ *where* $E = \langle \epsilon^i_u, \epsilon^i_u \rangle$, $F = \langle \epsilon^i_u, \epsilon^i_v \rangle$, $G = \langle \epsilon^i_v, \epsilon^i_v \rangle$ *and* u *and* v *are the standard basis elements* $u = (1, 0)$ *and* $v = (0, 1)$ *of* $\mathbb{R}^2$. *The pullback of the Riemannian metric* $\langle \cdot, \cdot \rangle$ *by the patch* $\epsilon^i(\cdot) : \mathbb{R}^2 \to \mathcal{S}$ *is given by* $\epsilon^{i*}(g_{11}dx_1^2 + g_{12}dx_1dx_2 + g_{21}dx_1dx_2 + g_{22}dx_2^2) = Edu^2 + 2Fdudv + Gdv$.

A.2.26 DEFINITION. A **local isometry** Φ between two geometric surfaces $\mathbb{S}_1$ and $\mathbb{S}_2$ with Riemannian metrics $ds_1^2 = \langle \cdot, \cdot \rangle_1$ and $ds_2^2 = \langle \cdot, \cdot \rangle_2$, respectively, is a map $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ such that for all $\mathbf{v}, \mathbf{w}$ in the tangent space $\mathrm{T}_{\mathbf{p}}\mathcal{S}_1$, the pullback $\Phi^* \langle \mathbf{v}, \mathbf{w} \rangle_2 = \langle d \cdot \Phi(\mathbf{p})(\mathbf{v}), d \cdot \Phi(\mathbf{p})(\mathbf{w}) \rangle_2 = \langle \mathbf{v}, \mathbf{w} \rangle_1$: that is, the pullback $\Phi^*(ds_2) = ds_1$.

A.2.27 DEFINITION. An **Riemann isometry** Φ between two geometric surfaces $\mathbb{S}_1$ and $\mathbb{S}_2$ with Riemannian metrics ds_1^2 and ds_2^2, respectively, is a map $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ that is simultaneously a local isometry and a local diffeomorphism.

A.2.28 DEFINITION. The group of all Riemann isometries between $\mathbb{S}$ and itself is the **isometry group** of $\mathbb{S}$, $\mathsf{Isom}(\mathbb{S})$.

A.2.29 DEFINITION. Given a diffeomorphism $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$\Phi(\mathbf{p}) = \Phi(p_1, p_2) = (f(p_1), f(p_2) = (\widetilde{p}_1, \widetilde{p}_2),$$

then the **Jacobian matrix** of Φ is the matrix

$$J_\Phi = \begin{bmatrix} \frac{\partial \widetilde{p}_1}{\partial p_1} & \frac{\partial \widetilde{p}_1}{\partial p_2} \\ \frac{\partial \widetilde{p}_2}{\partial p_1} & \frac{\partial \widetilde{p}_2}{\partial p_2} \end{bmatrix}$$

A.2.30 DEFINITION. In an n-dimensional abstract vector space V for which there is no canonical basis, two bases $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ and $\{\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, ..., \widetilde{\mathbf{e}}_n\}$ are **consistently oriented** if the transition matrix $[B_{ij}]$ such that $\mathbf{e}_i = [B_{ij}]\widetilde{\mathbf{e}}_j$ has a positive determinant.

A.2.31 PROPOSITION. *Orientation is an equivalence relation on the set of all ordered bases of* V*; there are exactly two equivalence classes.*

A.2.32 DEFINITION. An **orientation** on V is the equivalence class of all positively-oriented bases on V. V equipped with this equivalence class is said to be an **oriented vector space**.

A.2.33 DEFINITION. For a smooth abstract surface $\mathcal{S}$, the **pointwise orientation** of $\mathcal{S}$ is the choice of orientation of each tangent space $\mathrm{T}_\mathbf{p}\mathcal{S}$ at $\mathbf{p}$.

A.2.34 DEFINITION. A transformation $\Phi : \mathsf{V} \to \mathsf{V}$ of the vector space V is **orientation-reversing** if Φ is such that given a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of V, then $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\Phi(\mathbf{e}_1), \Phi(\mathbf{e}_2)\}$ are not consitently oriented. The transformation Φ is **orientation-preserving** if Φ is such that $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\Phi(\mathbf{e}_1), \Phi(\mathbf{e}_2)\}$ are consistently oriented for each basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of V.

A.2.35 PROPOSITION. *Given a diffeomorphism* $\Phi : \mathcal{S} \to \mathcal{S}$ *such that* $\det J_\Phi > 0$*, then* Φ *is orientation-preserving. If* $\det J_\Phi < 0$*, then it is orientation-reversing.*

A.2.36 DEFINITION. If a map $\Phi : \mathcal{S} \to \mathcal{S}$ is both angle-preserving and orientation-preserving, then it is **conformal**.

A.2.37 DEFINITION. Given a surface patch $\epsilon^i : \mathcal{U}_i \to \mathcal{S}$ of $\mathbb{S} = (\mathcal{S}, ds^2)$ where $ds^2 = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2$ as in (A.2.22), the **Christoffel symbols relative to the patch** ϵ^i are the functions $\gamma_{ij}^k : \mathcal{S} \to \mathbb{R}$,

$$\gamma_{ij}^k = \frac{1}{2}\Sigma_{n=1}^2 \left(\frac{\partial g_{nj}}{\partial x_k} + \frac{\partial g_{nk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_n} \right).$$

A.2.38 THEOREM. *Given a surface patch* $\epsilon^i : \mathcal{U}_i \to \mathcal{S}$ *of* $\mathbb{S}$*, then* $\gamma_{ij}^k \circ \epsilon^i = \Gamma_{ij}^k$.

A.2.39 DEFINITION. Given a surface patch $\epsilon^i : \mathcal{U}_i \to \mathcal{S}$ of $\mathbb{S}$, then the **geodesics** on $\mathbb{S}$ are the paths $\gamma \subseteq \mathcal{S}$ of the curves $\gamma(\cdot) = \epsilon(u(\cdot), v(\cdot))$ which solve the **geodesic equations**

$$\ddot{u}(t) + \Gamma_{11}^1\dot{u}(t) + 2\Gamma_{12}^1\dot{u}(t)\dot{v}(t) + \Gamma_{22}^1\dot{v}(t) = 0 \quad \text{(A.2.1)}$$

$$\ddot{v}(t) + \Gamma_{11}^2\dot{u}(t) + 2\Gamma_{12}^2\dot{u}(t)\dot{v}(t) + \Gamma_{22}^2\dot{v}(t) = 0. \quad \text{(A.2.2)}$$

A.2.40 THEOREM. *Given any point* $\mathbf{p} \in \mathbb{S}$ *and any tangent vector* $\mathbf{v} \in T_{\mathbf{p}}S$*, then there exists a unique geodesic* $\gamma(t)$ *of* $\mathbb{S}$ *passing through* $\mathbf{p}$ *in the direction* $\mathbf{v}$.

A.2.41 COROLLARY. *Given any 2 points* $\mathbf{p}, \mathbf{q} \in S$*, there exists a unique geodesic* γ *such that* $\mathbf{p} \in \gamma, \mathbf{q} \in \gamma$.

A.2.42 PROPOSITION. *The isometries of a geometric surface* $\mathbb{S}$ *map geodesics of* $\mathbb{S}$ *to geodesics of* $\mathbb{S}$.

A.2.43 DEFINITION. The subgroup of $\mathsf{Isom}(\mathbb{S})$ of all orientation-preserving Riemann isometries of $\mathbb{S}$ is its **symmetry group** $\mathsf{Sym}(\mathbb{S})$.

A.2.44 DEFINITION. The **arc length** of a curve $\alpha(\cdot) : (a,b) \to S$ in $\mathbb{S} = (S, ds^2)$ is the integral $\int_\alpha ds = \int_a^b \alpha^*(ds)$.

A.2.45 DEFINITION. A **(topological) metric** $d(\cdot,\cdot) : S \times S \to \mathbb{R}$ on $\mathbb{S}$ is the integral $d(\mathbf{p},\mathbf{q}) = \int_{\mathbf{p}}^{\mathbf{q}} \gamma^*(ds)$ where $\gamma(\cdot)$ is a map parametrizing the unique geodesic of (A.2.41) which passes through $\mathbf{p}$ and $\mathbf{q}$ for all $\mathbf{p}, \mathbf{q} \in S$.

A.2.46 THEOREM. (THE EULER-LAGRANGE EQUATIONS) *On the set* X*, given a functional* F *which acts on a function* α *of a real argument* t*,* $F(\alpha) = \int_{t_0}^{t_1} L(t, \alpha(t), \dot\alpha(t))dt$ *where* $\alpha : [t_0, t_1] \to X$ *is a differentiable function such that* $\alpha(t_0) = \mathbf{p}_0, \alpha(t_1) = \mathbf{p}_1$*, and* $L : [t_0, t_1] \times X \times TX \to \mathbb{R}$*, then* F *is minimised between* $\mathbf{p}_0$ *and* $\mathbf{p}_1$ *along the paths* α *whose curves* $\alpha(\cdot)$ *satisfy the* ***Euler-Lagrange equations*** $\mathcal{L}_\alpha(t, \alpha, \dot\alpha) - \frac{d}{dt}\mathcal{L}_{\dot\alpha}(t, f, \dot\alpha) = 0$.

A.2.47 DEFINITION. Given surface patches $(\epsilon^i(u,v).\mathcal{U}_i)$ on $\mathbb{S} = (S, ds^2)$, the **Riemannian problem** for $\mathbb{S}$ consists of finding the paths $\alpha = \{\epsilon^i(u(t), v(t)) \; : \; t \in (a,b)\}$ such that given any two points $\mathbf{p}_0 = \epsilon^i(u(t_0), v(t_0))$ and $\mathbf{p}_1 = \epsilon^i(u(t_1).v(t_1))$ in S, their curves $\alpha(t)$ have the minimal arc length (A.2.44) between $\mathbf{p}_0$ and $\mathbf{p}_1$.

A.2.48 PROPOSITION. *The Riemannian problem may be solved using the Euler-Lagrange equations, where the Lagrangian is given by* $\mathcal{L}(t, u(t), v(t), \dot u(t), \dot v(t)) = E\dot u^2 + 2F\dot u\dot v + G\dot v^2$.

A.2.49 DEFINITION. A curve $\alpha(\cdot)$ of arc-length (A.2.44) 1, is an **arc-length parametrized curve.** The parametrization (ϵ, U) of S such that $\alpha(\cdot)$ is an arc-length curve is said to be an **arc-length parametrization** for the curve $\alpha(\cdot)$.

A.2.50 DEFINITION. Given a surface patch $\epsilon^i : \mathcal{U}_i \to S$ of $\mathbb{S}$, then ϵ^i is a v**-Clairaut patch** if the coefficients $E_u = G_u = F = 0$.

A.2.51 THEOREM. (CLAIRAUT'S THEOREM) *Let* $\epsilon : \mathcal{U} \to S$ *be a* v*-Clairaut patch of* $\mathbb{S}$*. Then a path* γ *in* S*,* $\gamma = \{\epsilon^i(u(v), v) \; : \; v \in (a,b)\}$*, is a geodesic if and only if there exists a constant* $r \in \mathbb{R}$ *such that*

$$\frac{du}{dv} = \pm r\sqrt{\frac{G}{E^2 - r^2 E}}.$$

A.2.52 PROPOSITION. *The* ***Gaussian curvature*** K *of a geometric surface* $\mathbb{S}$ *is the product*

$$\mathrm{K} = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{EG - F^2}$$

A.2.53 PROPOSITION. *Given a surface patch* $\epsilon^i : \mathcal{U}_i \to S$ *of* $\mathbb{S}$ *such that* $F = 0$*, then the equation for the Gaussian curvature of* $\mathbb{S}$ *reduces to*

$$\mathrm{K} = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right).$$

A.3 Möbius transformations

References used include [5], [11] and [32].

A.3.1 DEFINITION. Denote the value $\frac{1}{0}$ by ∞. The **complex projective plane** $\mathbb{CP}^1$ is the union $\mathbb{C} \cup \infty$.

A.3.2 DEFINITION. Given $\alpha, \beta, \omega, v \in \mathbb{C}$ such that $\alpha v - \beta\omega \neq 0$, a **Möbius transformation** is a function $\mu : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that for each $z \in \mathbb{CP}^1$, $\mu : z \mapsto \frac{\alpha z+\beta}{\omega z+v}$.

A.3.3 DEFINITION. The transformations

$$\tau_b(z) = z + \beta \quad \text{(translation)}, \quad \delta_\omega(z) = \omega z \quad \text{(dilation)} \quad \text{and} \quad \sigma_{0,1}(z) = \frac{1}{\bar{z}} \quad \text{(inversion)}$$

are the **basic Möbius transformations.**

A.3.4 PROPOSITION. *The Möbius transformation* $\mu(z) = \frac{\alpha z+\beta}{\omega z+v}$ *can be expressed as the composition* $\widetilde{\tau}\circ\widetilde{\delta}\circ\sigma_{0,1}\circ\delta_\omega\circ\tau_\beta\circ\delta_v$ *of basic Möbius transformations, where* $\widetilde{\tau}(x) = \frac{\alpha}{\omega}+z$ *and* $\widetilde{\delta}(z) = (\beta\omega - \alpha v)z$.

A.3.5 DEFINITION. Let $\xi_{\alpha,r}$ denote the composition $\delta_r \circ \tau_\alpha$ for $r \in \mathbb{R}^+$. Then the composition of transformations $\xi_{\alpha,r}^{-1} \circ \sigma_{0,1} \circ \xi_{\alpha,r}$ is a **circle inversion** $\sigma_{\alpha,r}$ in the circle $C_{\alpha,r}$ with centre α and radius r.

A.3.6 PROPOSITION. *The circle inversions* $\sigma_{\alpha,r}$ *are involutions.*

A.3.7 COROLLARY. *A circle inversion* $\sigma_{\alpha,r}$ *maps the interior* $\mathcal{I}_{\alpha,r}$ *of* $C_{\alpha,r}$ *to the exterior* $\mathcal{E}_{\alpha,r}$ *of* $C_{\alpha,r}$*, maps* $\mathcal{E}_{\alpha,r}$ *to* $\mathcal{I}_{\alpha,r}$*, and preserves the circle* $C_{\alpha,r}$*.*

A.3.8 PROPOSITION. (IMAGES OF EUCLIDEAN LINES UNDER CIRCLE INVERSIONS) *Given a Euclidean line* ℓ *and a circle inversion* $\sigma_{\alpha,r}$ *in the circle* $C_{\alpha,r}$*, then*

1. *If* ℓ *does not pass through* α*, the image of* ℓ *under* $\sigma_{\alpha,r}$ *is a Euclidean line which passes through* α

2. *If ℓ passes through α, it is mapped to itself under $\sigma_{\alpha,r}$.*

A.3.9 PROPOSITION. (IMAGES OF EUCLIDEAN CIRCLES UNDER CIRCLE INVERSIONS) *Given a Euclidean circle C and a circle inversion $\sigma_{\alpha,r}$ in the circle $C_{\alpha,r}$, then*

1. *If C does not pass through α, the image of C under $\sigma_{\alpha,r}$ is a Euclidean circle which passes through α*

2. *If C passes through α, it is mapped to a Euclidean line which does not pass through α.*

A.3.10 REMARK. By (A.3.8) and (A.3.9), the circle inversions can be seen to preserve the family of Euclidean lines and Euclidean circles, although they may map lines to circles or vice-versa.

A.3.11 PROPOSITION. *Circle inversions preserve intersections within the family of Euclidean lines and Euclidean circles.*

A.3.12 PROPOSITION. *The composition $\mu \circ \widetilde{\mu}(z)$ transformations $\mu(z) = \frac{\alpha z+\beta}{\omega z+\upsilon}$ and $\widetilde{\mu}(z) = \frac{\widetilde{\alpha} z+\widetilde{\beta}}{\widetilde{\omega} z+\widetilde{\upsilon}}$ gives the transformation*

$$\mu'(z) = \frac{z(\alpha\widetilde{\alpha} + \widetilde{\omega}\beta) + (\alpha\widetilde{\beta} + \beta\widetilde{\upsilon})}{z(\omega\widetilde{\alpha} + \upsilon\widetilde{\omega}) + (\omega\widetilde{\beta} + \upsilon\widetilde{\upsilon})} = \frac{Az+B}{\Omega z + \Upsilon}$$

which is again a Möbius transformation. The coefficients of $\mu'(z) = \frac{Az+B}{\Omega z+\Upsilon}$ are expressed by the matrix product

$$\begin{bmatrix} A & B \\ \Omega & \Upsilon \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \omega & \upsilon \end{bmatrix} \begin{bmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\omega} & \widetilde{\upsilon} \end{bmatrix}.$$

A.3.13 THEOREM. *The Möbius transformations form a group under composition.*

We denote this group by Möb. Note that the requirement $\alpha\upsilon - \beta\omega \neq 0$ corresponds to the condition on the determinant of the elements of the complex general linear group (??). Indeed, Möb can be identified with $\mathsf{GL}(2,\mathbb{C})$ acting on $\mathbb{CP}^1$, where the action $(\cdot,\cdot) : \text{Möb} \to \mathsf{GL}(2,\mathbb{C})$ is given by $(A,z) = \frac{\alpha z+\beta}{\omega z+\upsilon}$.

A.3.14 THEOREM. *All Möbius transformations are conformal transformations of $\mathbb{CP}^1$.*

A.3.15 DEFINITION. Given $\mathbf{p},\mathbf{q} \in \mathbb{R}^3$, a **(projective) cone** is any surface of revolution in $\mathbb{R}^3$ with generator a (Euclidean) line $\ell = \langle \mathbf{p} \rangle$ and axis of rotation the Euclidean line $\langle \mathbf{q} \rangle$.

A.3.16 DEFINITION. A **conic** in $\mathbb{R}^3$ is the subset of $\mathbb{R}^3$ which is the intersection of a projective cone with a hyperplane $\mathbf{\Gamma} = \{ax + by + cz = k \ : \ (x,y,z) \in \mathbb{R}^3\}$.

A conic has an equation of the form $ax^2 + bxy + cy^2 + dx + ky = h$ for $a,b,c,d,k,h \in \mathbb{R}$, where $a^2 + b^2 + c^2 \neq 0$.

A.3.17 DEFINITION. The **eccentricity** of a conic $\{(u,v) \in \mathbb{R} | u^2 + cv^2 + kv = h\}$ is given by $e = \sqrt{1-c}$.

A.3.18 DEFINITION. A conic for which $e > 1$ is a hyperbola; a conic for which $e < 1$ is an ellipse and a conic for which $e = 1$ is a parabola.

A.3.19 THEOREM. *A conic is either an ellipse, a parabola or a hyperbola: that is, these classes of conics do not intersect.*

A.3.20 THEOREM. *Given a cone $\mathcal{K}$ with generator the Euclidean line $\ell = \langle \mathrm{p} \rangle$ and axis of rotation the Euclidean line $\ell' = \langle \mathrm{q} \rangle$ such that θ_c is the angle between the generator and the plane of rotation of the cone (the unique plane orthogonal to ℓ' and passing through the origin), the eccentricity of a conic that is the intersection of $\mathcal{K}$ with a plane θ_p degrees from the plane of rotation is given by $e = \frac{\sin\theta_p}{\sin\theta_c}$.*

A.4 Models of hyperbolic geometry

References used include [7] and [32]. We discussed the history and some of the major results of hyperbolic geometry in CHAPTER 1. Here, we state the Riemann isometries (A.2.27) between the geometric surfaces representing three hyperbolic models $\mathbb{HL}, \mathbb{HP}$ and $\mathbb{PD}$, where we discuss $\mathbb{HL}$ and $\mathbb{HP}$ in CHAPTER 2. Since the reference [7] uses a different inner product for reduced Minkowski spacetime to the one used in this thesis (which is more widely used in applications in physics), we make

A.4.1 DEFINITION. **Alternate reduced Minkowski spacetime** $\mathbb{R}^{2,1}$ is the 3-dimensional real space $\mathbb{R}^3$ equipped with the **alternate Minkowski metric** $ds^2 = dx^2 + dy^2 - dz^2$. Particulaly, we may define a hyperboloid model $\widetilde{\mathbb{HL}}$ analagously to the hyperboloid model $\mathbb{HL}$ discussed in **section** 2.1, which is the hyperboloid $\widetilde{\mathcal{HL}} = \left\{x^2 + y^2 - z^2 = -1 \ : \ (x, y, z) \in \mathbb{R}^3\right\}$ equipped with the induced metric $ds^2 = dx^2 + dy^2 - dz^2$ of $\mathbb{R}^{2,1}$.

A.4.2 DEFINITION. The open subset $\mathcal{PD} = \left\{(u, v) \in \mathbb{R}^2 \ : \ u^2 + v^2 < 1\right\}$ of $\mathbb{R}^2$ eqnipped with the **Poincaré metric tensor** $ds^2 = \dfrac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2}$ is the **Poincaré disk model** $\mathbb{PD} = (\mathcal{PD}, ds^2)$

A.4.3 PROPOSITION. *The projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}^{2,1}$*

$$\pi_1(x, y, 0) = \left(\frac{2x}{1 - x^2 - y^2}, \frac{2y}{1 - x^2 - y^2}, \frac{1 + x^2 + y^2}{1 - x^2 - y^2}\right)$$

is a Riemann isometry mapping $\mathbb{PD}$ to $\widetilde{\mathbb{HL}}$, where. The inverse of this map is the projection $\pi_1^{-1} : \mathbb{R}^{2,1} \to \mathbb{R}^2$

$$\pi_1^{-1}(x, y, 1) = \left(\frac{x}{z+1}, \frac{y}{z+1}, z\right)$$

which is a Riemann isometry mapping $\widetilde{\mathbb{HL}}$ to $\mathbb{PD}$.

A.4.4 PROPOSITION. *The projection $\pi_2 : \mathbb{C} \to \mathbb{C}$*

$$\pi_2(z) = \frac{z+1}{iz - i}$$

is a Riemann isometry mapping $\mathbb{PD}$ *to* $\mathbb{HP}$. *The inverse of this map is the projection* $\pi_2^{-1} : \mathbb{C} \to \mathbb{C}$

$$\pi_2^{-1}(z) = \frac{z-i}{z+i}$$

which is a Riemann isometry mapping $\mathbb{PD}$ *to* $\mathbb{HP}$.

A.5 Matrix Lie groups

References used include [33], [13], [9], [3], [16] and [28]. Let $\mathbb{R}^{n\times n}$ denote the set of all real $n \times n$ matrices. $\mathbb{R}^{n\times n}$ is a normed vector space equipped with the (matrix) norm $\|g\| = \sqrt{\mathrm{tr}(g^\top g)}$. $\mathsf{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$. We denote the $n \times n$ identity matrix by $\mathbf{1}$.

A.5.1 DEFINITION. A **Lie group** is a smooth manifold G which is also a group such that the mappings $\mathsf{G}\times\mathsf{G}, (g_1, g_2) \mapsto g_1g_2$ and $\mathsf{G}\times\mathsf{G}, g \mapsto g^{-1}$ are smooth.

A.5.2 PROPOSITION. *The real general linear group* $\mathsf{GL}(n,\mathbb{R})$ *is a Lie group.*

A.5.3 DEFINITION. A **matrix Lie group** is a closed subgroup of $\mathsf{GL}(n,\mathbb{R})$.

It can be shown that any matrix Lie group is a Lie subgroup of $\mathsf{GL}(n,\mathbb{R})$. Let G denote a (matrix) Lie group.

A.5.4 EXAMPLES. The following (matrix) Lie groups are used in this thesis:

(i) The **complex general linear group** $\mathsf{GL}(n,\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} \ : \ \det A \neq 0\}$.

(ii) The **projective general linear group** $\mathsf{PGL}(n,\mathbb{R}) = \mathsf{GL}(n,\mathbb{R})/\{k\mathbf{1} \ : \ k \in \mathbb{R}\}$.

(iii) The **orthogonal group** $\mathsf{O}(n) = \{g \in \mathbb{R}^{n\times n} \ : \ g^\top g = \mathbf{1}\}$.

(iv) The **special orthogonal group** $\mathsf{SO}(n) = \{g \in \mathbb{R}^{n\times n} \ : \ g^\top g = 1. \det g = 1\}$.

A.5.5 REMARK. The property $g^\top g = \mathbf{1}$ for each g in $\mathsf{O}(n)$ is equivalent to stating that for each element $g = [g_{ij}]$ in $\mathsf{O}(n)$, then $\Sigma_{k=1}^{2} g_{ki}g_{kj} = \delta_{ij}$: that is, the orthogonal group $\mathsf{O}(n)$ may equivalently be considered as the group of all $n \times n$ real matrices whose columns $[g_{i,1}], [g_{i,2}], \ldots, [g_{i,n}]$ are orthonormal vectors.

The matrices $g \in \mathsf{O}(n)$ are referred to as orthogonal matrices.

A.5.6 DEFINITION. The matrices $g \in \mathsf{GL}(n,\mathbb{R})$ are **positive-definite** if their eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}^+$. The matrices $g \in \mathsf{GL}(n,\mathbb{R})$ are **positive semi-definite** if their eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all real and either positive or zero.

A.5.7 DEFINITION. Given $A \in \mathsf{GL}(n,\mathbb{R})$, a **polar decomposition** of A expresses A as the product $A = UP$, where U is orthogonal and P is symmetric positive (semi)-definite.

A.5.8 THEOREM. (THE FIRST ISOMORPHISM THEOREM) *Given any two groups* G, H *and an* n*-to-1 group homomorphism* $\Phi : \mathsf{G} \to \mathsf{H}$, *then* $\mathsf{H} \cong \mathsf{G}/\ker(\Phi)$, *and* Φ *is a group isomorphism between* H *and* $\mathsf{G}/\ker(\Phi)$.

A.5.9 DEFINITION. Given any two Lie groups G, H, a map $\Phi : \mathsf{G} \to \mathsf{H}$ is a **Lie group homomorphism** if it is simultaneously a group homomorphism and a differentiable map. A map $\Phi : \mathsf{G} \to \mathsf{H}$ is a **Lie group isomorphism** if it is simultaneously a group isomorphism and a differentiable map.

A.5.10 PROPOSITION. *If* $A \in \mathbb{R}^{2\times 2}$, *then its characteristic equation* $\mathrm{char}_A(\lambda) = \det(A - \lambda \mathbf{1})$ *has the form*

$$\mathrm{char}_A(\lambda) = \lambda^2 - (\mathrm{tr}\, A)\lambda + \det A.$$

A.5.11 DEFINITION. Let $I \subseteq \mathbb{R}$ be an interval. A **curve** in a matrix Lie group G is a map $\alpha(\cdot) : I \to \mathbb{R}^{n\times n}$ such that for every $t \in I$, $\alpha(t) \in \mathsf{G}$.

We use the term **path** to refer to the image set $\alpha = \{\alpha(t) \,:\, t \in I\}$.

A.5.12 DEFINITION. G is **connected** if for any $g \in \mathsf{G}$, there exists a continuous curve $\alpha(\cdot) : [0,1] \to \mathsf{G}$ such that $\alpha(0) = g$ and $\alpha(1) = \mathbf{1}$.

A.5.13 DEFINITION. Given a smooth curve α in G, then $\dot\alpha(0) = \lim_{h\to 0} \frac{\alpha(t+h)-\alpha(t)}{h} \in \mathbb{R}^{n\times n}$ is the **tangent vector** to G at $\alpha(0)$.

A.5.14 DEFINITION. For each g an element of G, the **tangent space to** G **at** g is the set of all tangent vectors to G at g: $\mathrm{T}_g\mathsf{G} = \{\dot\alpha(0) \,:\, \alpha \text{ a smooth curve in } \mathsf{G},\ \alpha(0) = g\}$.

A.5.15 PROPOSITION. *The tangent space* $\mathrm{T}_g\mathsf{G}$ *of* G *is a vector subspace of* $\mathbb{R}^{n\times n}$.

A.5.16 DEFINITION. The **dimension** of G is the dimension of its tangent space $\mathrm{T}_{\mathbf{1}}\mathsf{G}$.

A.5.17 DEFINITION. A (real) **Lie algebra** $\mathfrak{g}$ is a real vector space equipped with a product $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, $(X,Y) \mapsto [X,Y]$, such that for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $X, Y, Z \in \mathfrak{g}$, then

1. $[X,Y] = -[Y,X]$
2. $[\lambda_1 X + \lambda_2 Y, Z] = \lambda_1[X,Z] + \lambda_2[Y,Z]$
3. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

The product $[\cdot,\cdot]$ is the **Lie bracket** on $\mathfrak{g}$.

A.5.18 EXAMPLES. The following Lie algebras are referred to in this thesis:

(i) The vector space $\mathbb{R}^{n\times n}$ equipped with the **matrix commutator** $[X,Y] = XY - YX$ for all $X, Y \in \mathbb{R}^{n\times n}$.

(ii) The vector space $\mathbb{R}^3$ equipped with the **cross product** $\mathbf{p}\wedge\mathbf{q} = [\mathbf{p},\mathbf{q}] = (p_2q_3 - p_3q_2, p_1q_3 - p_1q_3, p_1q_2 - p_2q_1)$ for all $\mathbf{p} = (p_1,p_2,p_3), \mathbf{q} = (q_1,q_2,q_3)$ in $\mathbb{R}^3$.

A.5.19 PROPOSITION. *The tangent space at identity* T_1G *equipped with the matrix commutator, is a Lie algebra.*

A.5.20 DEFINITION. The **Lie algebra of the Lie group** G is its tangent space at identity T_1G equipped with the matrix commutator.

We will denote the Lie algebra of G by $\mathfrak{g}$.

A.5.21 DEFINITION. The map $L_g : G \to G$, $g : x \mapsto gx$ denotes the action of **left translation** on G by $g \in G$. The map $R_g : G \to G$, $g : x \mapsto gx$ denotes the action of **right translation** on G by $g \in G$.

A.5.22 PROPOSITION. *The tangent map of a left translation* L_g *is invertible: in particular, the map* $dL_g : \mathfrak{g} \to T_gG$, $X \mapsto dL_gX$ *is a linear isomorphism.*

A.5.23 DEFINITION. For any vector space V, the **dual space** of V, denoted V^*, is the space of all linear functionals on V.

Particularly, for any $g \in G$, $(T_gG)^*$ is the dual space of T_gG.

A.5.24 DEFINITION. For a linear mapping $F : V \to W$ between vector spaces, the **dual mapping** of F is given by $F^* : W^* \to V^*$ such that $F^*(\mathrm{w})\mathrm{v} = \mathrm{w} \circ F(\mathrm{v})$ for each $\mathrm{v}, \mathrm{w} \in V$.

A.5.25 DEFINITION. The **dual of the left translation** is the map $L_g^* : G^* \to G^*$ such that for each $y^* \in G^*$ and $x \in G$, then $(L_g^*(y^*))(x) = y^*(L_g(x)) = y^*(gx)$.

A.5.26 DEFINITION. The **cotangent bundle** of G is the disjoint union $\bigcup_{g \in G}(T_gG)^*$.

A.5.27 PROPOSITION. *The tangent map of the dual of the left translation* L_g *is invertible: in particular, given* $dL_g^* : T_{gx}^*G \to T_x^*G$, *then* $dL_{g^{-1}}^* : \mathfrak{g}^* \to T_g^*G$, *where* $\left(dL_{g^{-1}}^*(Y)\right)(X_g) = Y\left(dL_g^{-1}(X_g)\right)$ *is a linear isomorphism.*

A.5.28 REMARK. (THE TRIVIALIZATION OF THE COTANGENT BUNDLE)
Since the tangent map of the dual of the left translation $dL_{g^{-1}}^*$ maps T_1^*G to T_g^*G bijectively, then every fibre of TG^* is the image under $dL_{g^{-1}}^*$ of some $Y \in \mathfrak{g}$, and so we can make the identification of every $dL_{g^{-1}}^*(Y) \in T_g^*G \subseteq T^*G$ with (g, Y) for $g \in G, Y \in T_1^*G$ and so express $T^*G = G \times \mathfrak{g}^*$.

A.5.29 DEFINITION. The **matrix exponential** of $X \in \mathbb{R}^{n \times n}$ is the (invertible) matrix $\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!}X^k$. The **exponential mapping** is the map $\exp : \mathbb{R}^{n \times n} \to GL(n, \mathbb{R})$ defined by $\exp : X \mapsto \exp(X)$.

A.5.30 PROPOSITION. *For each* $X, Y \in \mathbb{R}^{n \times n}$ *and* $t \in \mathbb{R}$,

1. $\frac{d}{dt}\exp(tX) = X\exp(tX) = \exp(tX)X$.

2. *If X and Y commute, then* $\exp(X+Y)=\exp(X)\exp(Y)$.

3. $\det \exp(tX)=\exp(\mathrm{tr}(X))$.

4. $Y\exp(X)Y^{-1}=\exp(YXY^{-1})$.

A.5.31 DEFINITION. For two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, the linear mapping $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi([X,Y]) = [\phi(X),\phi(Y)]$ for each $X,Y \in \mathfrak{g}_1$ is a **Lie algebra homomorphism**. If in addition ϕ is a bijection, then ϕ is a **Lie algebra isomorphism**. If $\mathfrak{g}_1=\mathfrak{g}_2=\mathfrak{g}$, then the Lie algebra isomorphism $\phi:\mathfrak{g}\to\mathfrak{g}$ is a **Lie algebra automorphism**.

A.5.32 PROPOSITION. *Let Φ be a smooth group homomorphism $\Phi : \mathsf{G}_1 \to \mathsf{G}_2$ between G_1 and G_2 which have Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Then the tangent map $d\Phi : \mathfrak{g}_1\to\mathfrak{g}_2$ is a Lie algebra homomorphism.*

If $\Phi : \mathsf{G}_1 \to \mathsf{G}_2$ is a linear map, then $d\Phi=\Phi$. Note that from (A.5.32), smoothly isomorphic Lie groups have isomorphic Lie algebras.

A.5.33 PROPOSITION. *The **conjugation map** $L_g \circ R_g^{-1} : \mathsf{G}\to\mathsf{G}$, $L_g\circ R_g^{-1}(h) = ghg^{-1}$ is a smooth group isomorphism.*

A.5.34 DEFINITION. For any g in G, the tangent map of the conjugation map $L_g\circ R_g^{-1}$ is the **adjoint action of g on $\mathfrak{g}$**, $\mathrm{Ad}_g:\mathfrak{g}\to\mathfrak{g}$, where $\mathrm{Ad}_gX=gXg^{-1}$ for all $X\in\mathfrak{g}$.

A.5.35 DEFINITION. For any $X \in \mathfrak{g}$, the derivative of the adjoint map is given by $\mathrm{ad}_X:\mathfrak{g}\to\mathfrak{g}$, $\mathrm{ad}_XY=[X,Y]$ for all $X,Y\in\mathfrak{g}$. ad_X is the **adjoint operator** of X.

A.5.36 PROPOSITION. *For each $g\in\mathsf{G}$, $\mathrm{Ad}_g:\mathfrak{g}\to\mathfrak{g}$ is a Lie algebra automorphism.*

A.5.37 DEFINITION. The group of all isomorphisms $\Phi:\mathsf{G}\to\mathsf{G}$ of G is its the **automorphism group** $\mathsf{Aut}(\mathsf{G})$. The subgroup $\{L_g\circ R_{g^{-1}} : g\in\mathsf{G}\}$ of $\mathsf{Aut}(\mathfrak{g})$ is the group of **inner automorphisms** $\mathsf{Inn}(\mathsf{G})$.

A.5.38 DEFINITION. The group of all Lie algebra isomorphisms $\phi:\mathfrak{g}\to\mathfrak{g}$ of $\mathfrak{g}$ is its **automorphism group** $\mathsf{Aut}(\mathfrak{g})$. The subgroup $\{\mathrm{Ad}_g : g\in\mathsf{G}\}$ of $\mathsf{Aut}(\mathfrak{g})$ is the group of **inner automorphisms** $\mathsf{Inn}(\mathfrak{g})$.

A.5.39 PROPOSITION. *The mapping* $\mathrm{Ad}:\mathsf{G}\to\mathsf{Aut}(\mathfrak{g})$ *such that* $\mathsf{Aut}(g)=\mathrm{Ad}_g$, *is a smooth group homomorphism.*

A.5.40 PROPOSITION. *If $\mathfrak{g}$ is semisimple, then* $\mathsf{Inn}(\mathfrak{g})=\mathsf{Aut}(\mathfrak{g})_0$, *the connected component of* $\mathsf{Aut}(\mathfrak{g})$.

A.5.41 DEFINITION. For each $g\in\mathsf{G}$, the **co-adjoint action of g on $\mathfrak{g}^*$** is the map $\mathrm{Ad}_g^{*-1}:\mathfrak{g}\to\mathfrak{g}$ such that $(\mathrm{Ad}_g^{*-1}p)(X)=p(\mathrm{Ad}_g(X))$ for all $X\in\mathfrak{g}$.

A.5.42 DEFINITION. The set $\mathcal{O}_X=\{\mathrm{Ad}_gX : g\in\mathsf{G}\}$ is the **adjoint orbit of G through** $X\in\mathfrak{g}$.

A.5.43 DEFINITION. The set $\widetilde{\mathcal{O}}_p = \left\{ \mathrm{Ad}^*_{g^{-1}} p \ : \ g \in \mathsf{G} \right\}$ is the **co-adjoint orbit of** G **through** $p \in \mathfrak{g}^*$.

A.5.44 DEFINITION. A subspace $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ is a **ideal** of $\mathfrak{g}$ if for each $X \in \mathfrak{g}, Y \in \mathfrak{i}$, then $[X, Y] \in \mathfrak{i}$.

A.5.45 DEFINITION. An ideal $\mathfrak{i}$ of $\mathfrak{g}$ is **Abelian** if for each $X, Y \in \mathfrak{i}$, then $XY = YX$.

A.5.46 DEFINITION. A **semisimple Lie algebra** is a Lie algebra for which there are no nonzero Abelian ideals.

A.5.47 DEFINITION. A **semisimple Lie group** is a Lie group G with a semisimple Lie algebra.

A.5.48 DEFINITION. A **simple Lie algebra** is a non-Abelian Lie algebra with no non-trivial ideals.

A.5.49 DEFINITION. A (connected) **simple Lie group** is a connected non-Abelian Lie group for which there are no non-trivial normal subgroups.

A.5.50 PROPOSITION. *A connected matrix Lie group is simple if its Lie algebra is simple.*

A.5.51 PROPOSITION. *A simple Lie group* G *is semisimple.*

A.5.52 DEFINITION. The **centre** of a Lie algebra $\mathfrak{g}$ is the set $\mathfrak{z}(\mathfrak{g}) = \{A \in \mathfrak{g} \ : \ [A, X] = 0 \text{ for all } X \in \mathfrak{g}\}$.

A.5.53 DEFINITION. A **homotopy** of a path α to a path β in a group G is a continuous function $d : [0, 1] \times [0, 1] \to \mathsf{G}$ such that for each $t \in [0, 1]$, $d(0, t) = \alpha(t)$ and $d(1, t) = \beta(t)$.

A.5.54 DEFINITION. G is **simply-connected** if it is connected, and for any two paths α and β where $\alpha(0) = g, \alpha(1) = h$ and $\beta(0) = g, \beta(1) = h$ there is a homotopy d of the path α to the path β such that $d(s, 0) = g$ and $d(s, 1) = h$: that is, the endpoints are preserved.

A.5.55 DEFINITION. A **cover** of a group G by a group $\widetilde{\mathsf{G}}$ is a local homeomorphism $\Phi : \widetilde{\mathsf{G}} \to \mathsf{G}$ which is also a group homomorphism. Particularly, if the local homeomorphism Φ is an n-to-1 homomorphism, then Φ is an **n-fold cover** of G by $\widetilde{\mathsf{G}}$. The preimages $\widetilde{g} \in \widetilde{\mathsf{G}}$ of $g \in \mathsf{G}$ under Φ are said to be **over** $g \in \mathsf{G}$. Particularly, a 2-to-1 local homeomorphism between matrix Lie groups G and $\widetilde{\mathsf{G}}$ is a **double cover** of $\widetilde{\mathsf{G}}$ by G.

A.5.56 THEOREM. *Given* $\Phi : \widetilde{\mathsf{G}} \to \mathsf{G}$ *an n-fold cover of* G *by* $\widetilde{\mathsf{G}}$, *suppose that α is a path in* G *with initial point g, and $\widetilde{g}$ is a point in* $\widetilde{\mathsf{G}}$ *over g. Then there is a unique path $\widetilde{\alpha}$ in* $\widetilde{\mathsf{G}}$ *such that* $\widetilde{\alpha}(0) = \widetilde{g}$ *and* $\Phi \circ \widetilde{\alpha} = \alpha$.

A.5.57 DEFINITION. In (A.5.56), the unique path $\widetilde{\alpha}$ in the n-fold cover $\widetilde{\mathsf{G}}$ of G such that $\widetilde{\alpha}(0) = \widetilde{g}$ and $\Phi \circ \widetilde{\alpha} = \alpha$ is the **lift** of the path α.

A.5.58 DEFINITION. A vector field X on G is **left-invariant** if $dL_g(X) = gX$ for each $X \in \mathfrak{g}$ and each $g \in \mathsf{G}$.

A.5.59 DEFINITION. The **integral curve** of a smooth vector field X on G with initial condition $g_0 \in G$ is a curve $g(\cdot) : (a, b) \to G$ such that $g(0) = g_0$ and $\dot{g}(t) = X(g(t))$ for all $t \in (a, b)$.

The integral curve of X is often referred to as the **flow** of X.

A.5.60 DEFINITION. The **Killing form** of $\mathfrak{g}$ is the symmetric bilinear mapping $\kappa : \mathfrak{g} \times \mathfrak{g}$ such that $\kappa(X, Y) = \text{tr}(\text{ad}_X \circ \text{ad}_Y)$ for each $X, Y \in \mathfrak{g}$.

A.5.61 DEFINITION. Let $\mathfrak{g}$ be equipped with a Killing form. Then for any subset $S \subseteq \mathfrak{g}$, the **orthogonal complement** to the subset S is $S^{\perp} = \in \mathfrak{g} \; : \; \kappa(X, Y) = 0. \forall Y \in S\} \subseteq \mathfrak{g}$.

A.5.62 DEFINITION. If $\mathfrak{g}^{\perp} = \{0\}$, then the Killing form κ on $\mathfrak{g}$ is **nondegenerate**.

A.5.63 PROPOSITION. *If $\mathfrak{g}$ is semisimple, then the Killing form κ is nondegenerate.*

A.6 Iwasawa decomposition

References used include [15]. Let G be a *semisimple* matrix Lie group and $\mathfrak{g}$ its Lie algebra.

A.6.1 DEFINITION. A subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is a **compact subalgebra** if the Killing form κ of $\mathfrak{g}$ is negative-definite on $\mathfrak{k}$.

A.6.2 DEFINITION. A subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is an **Abelian subalgebra** if $[A_1, A_2] = 0$ for any $A_1, A_2 \in \mathfrak{a}$.

A.6.3 DEFINITION. A subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ is a **nilpotent** subalgebra if the series

$$\mathfrak{n} \supseteq [\mathfrak{n}, \mathfrak{n}] \supseteq [[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] \supseteq [[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}], \mathfrak{n}] \ldots$$

terminates in zero.

A.6.4 DEFINITION. A direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{k}$ is a compact subalgebra of $\mathfrak{g}$, $\mathfrak{a}$ is an Abelian subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ is an **Iwasawa decomposition of the (semisimple) Lie algebra** $\mathfrak{g}$.

A.6.5 THEOREM. *An Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ is unique up to conjugacy: that is, the subalgebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are unique up to an inner automorphism of $\mathfrak{g}$.*

A.6.6 THEOREM. *Given an Iwasawa decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of G, let K be a subgroup of G with Lie algebra $\mathfrak{k}$, A be a subgroup of G with Lie algebra $\mathfrak{a}$ and N be a subgroup of G with Lie algebra $\mathfrak{n}$. Then the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism onto.*

A.6.7 DEFINITION. A decomposition $G = KAN$ of (A.6.6) is an **Iwasawa decomposition of the (semisimple) Lie group** G.

A.6.8 PROPOSITION. *In any Iwasawa decomposition $G = KAN$, the product AN is a subgroup of G.*

A.7 Left-invariant control systems

References used include [30],[29], [2], [13] and [14] . Let G be a (matrix) Lie group, and $\mathfrak{g}$ its Lie algebra.

A.7.1 DEFINITION. An **admissible control** is a piecewise-constant mapping $u(\cdot) : [0, T_u] \to \mathbb{R}^\ell$.

The components $u_1(\cdot), u_2(\cdot), ..., u_\ell(\cdot)$ of $u(\cdot)$ will be referred to as **input functions**.

A.7.2 DEFINITION. A **control system** Σ (on G) is a pair $\Sigma = (\mathsf{G}, \Xi)$, where $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is a smooth mapping.

We refer to Ξ as the **dynamics** of the system Σ.

A.7.3 DEFINITION. The control system Σ is **left-invariant** if the dynamics is invariant under left translation: that is, for all $g \in \mathsf{G}$, $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$.

We refer to the map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ as the **parametrization map**. Where convienient, we denote the image $\Xi(\mathbf{1}, u)$ by Ξ_u.

A.7.4 DEFINITION. A **control affine left-invariant system** Σ is a left-invariant control system such that for each $u \in \mathbb{R}^\ell$ the parametrization map has the form $\Xi(\mathbf{1}, u) = A + \Sigma_{i=1}^n u_i B_i$ (the elements $B_1, B_2, ..., B_\ell$ are assumed to be linearly independent).

In the classical notation, we express such a system Σ as

$$\dot{g} = g\left(A + \sum_{i=1}^{\ell} u_i B_i\right), \quad g \in \mathsf{G},\ u = (u_1, u_2, ..., u_\ell) \in \mathbb{R}^\ell.$$

A.7.5 DEFINITION. The **trace** of Σ is the image set $\Gamma = \{\Xi_u \ : \ u \in \mathbb{R}\} \subseteq \mathfrak{g}$.

Let $\Sigma = (\mathsf{G}, \Xi)$ be a control affine left-invariant system. We express the trace as $\Gamma = A + \langle B_1, ..., B_\ell\rangle = A + \Gamma^0$.

A.7.6 DEFINITION. The system Σ is **homogeneous** if A is linearly dependent on $B_1, ..., B_\ell$; otherwise, Σ is **inhomogeneous**.

Let $\mathsf{Lie}(\Gamma)$ denote the Lie algebra generated by the trace Γ.

A.7.7 DEFINITION. A **full-rank** system is a system Σ such that $\mathsf{Lie}(\Gamma) = \mathfrak{g}$.

A.7.8 DEFINITION. A **trajectory** of a left-invariant control system Σ on G is the absolutely-continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(\cdot)$ satisfies $\dot{g}(t) = \Xi\left((g(t), u(t))\right)$ almost everywhere for some admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$.

A.7.9 DEFINITION. The **attainable set** for Σ from $g \in \mathsf{G}$ is the set $\mathcal{A}(g)$ of all points reachable along trajectories of Σ from g for positive time: $\mathcal{A}(g) = \{g(T) \ : \ g(\cdot) \text{ a trajectory of } \Sigma, g(0) = g, T \geq 0\}$.

A.7.10 PROPOSITION. *For any left-invariant control system* $\Sigma = (\mathsf{G}, \Xi)$, $\mathcal{A}(g) = g\mathcal{A}(\mathbf{1})$

For the left-invariant control systems $\Sigma = (\mathsf{G}, \Xi)$ we denote $\mathcal{A}(\mathbf{1})$ by $\mathcal{A}$ and refer to it as the attainable set of Σ.

A.7.11 PROPOSITION. *For any left-invariant control system* $\Sigma = (\mathsf{G}, \Xi)$, $\mathcal{A}$ *is a subsemigroup of* G.

A.7.12 DEFINITION. A left-invariant control system Σ is **controllable** if given any $g_0, g_1 \in \mathsf{G}$, there exists $t \geq 0$ such that $g_1 \in \mathcal{A}(g_0)$.

A.7.13 PROPOSITION. *A left-invariant control system* Σ *is controllable if given any* $g \in \mathsf{G}$, *then the attainable set* $\mathcal{A}(g) = \mathsf{G}$.

A.7.14 DEFINITION. The set $\mathcal{O}(g) = \{g(t) \,:\, g(t)$ a trajectory of Σ, $t \in \mathbb{R}$, $g(t) \in \mathsf{G}$, $g(0) = g\}$ is the **orbit** of Σ through $g \in \mathsf{G}$.

A.7.15 PROPOSITION. *The Cauchy problem* $\dot{g} = gA$, $g(0) = \mathbf{1}$ *has the solution* $g(t) = \exp(tA)$.

A.7.16 PROPOSITION. *Given a left-invariant control affine system* Σ *with piecewise-constant controls expressed in the classical notation as* $\dot{g} = g\left(A + \Sigma_{k=1}^{\ell} u_k B_k\right)$, $u \in \mathbb{R}^{\ell}$, *with trajectory* $g(\cdot) : [0, T] \to \mathsf{G}$, $g(0) = g_0$, *then there exist* $N \in \mathbb{N}$, *real numbers* $\tau_1, \tau_2, ..., \tau_N > 0$ *and* $A_1, A_2, ..., A_n \in \Gamma$ *such that for each* $t \in [0, T]$, $g(t)$ *is the product of matrix exponentials*

$$g(t) = g_0 \exp \tau_N A_N ... \exp \tau_2 A_2 \exp \tau_1 A_1$$

where $\tau_1 + \tau_2 + ... + \tau_N = T$.

A.7.17 DEFINITION. The system Σ is called **symmetric** if $\Gamma = -\Gamma$.

Let G be a *connected* matrix Lie group.

A.7.18 THEOREM. *A symmetric system* $\Sigma = (\mathsf{G}, \Xi)$ *is controllable if and only if it is full rank.*

A.7.19 DEFINITION. A map $\alpha(\cdot) : I \to \mathsf{G}$ is **periodic** (in t) if there exists some $p \in \mathbb{R}$ such that for any $t_0 \in \mathbb{R}, \alpha(t_0) = g_0$, then $\alpha(t_0 + p) = g_0$.

A.7.20 THEOREM. *The full-rank system* $\Sigma = (\mathsf{G}, \Xi)$ *is controllable if there exists some* $u^0 \in \mathbb{R}^{\ell}$ *such that the map* $t \mapsto \exp(t\,\Xi(\mathbf{1}, u^0))$ *is periodic.*

A.8 Equivalences of control systems

The reference used is [5]. Let $\Sigma = (\mathsf{G}, \Xi)$ and $\widetilde{\Sigma} = (\widetilde{\mathsf{G}}, \widetilde{\Xi})$ be two control systems, where $\dim(\mathsf{G}) = \dim(\widetilde{\mathsf{G}})$.

A.8.1 DEFINITION. Σ and $\widetilde{\Sigma}$ are **local state-space equivalent** (l.s.s.e.) if for every $g \in \mathsf{G}$, $\widetilde{g} \in \widetilde{\mathsf{G}}$ there exists a local diffeomorphism $\Phi : \mathcal{N} \to \widetilde{\mathcal{N}}$ (where $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are neighbourhoods of g and $\widetilde{g}$, respectively) such that the tangent map $d\Phi(\mathbf{1}) \cdot \Xi(g, u) = \widetilde{\Xi}(\widetilde{g}, u)$ for all $u \in \mathbb{R}^{\ell}$.

A.8.2 THEOREM. *Σ and $\widetilde{\Sigma}$ are l.s.s.e. if and only if there exists a Lie algebra isomorphism ϕ such that for each $u \in \mathbb{R}^\ell$, $\phi(\Xi(\mathbf{1}, u)) = \widetilde{\Xi}(\mathbf{1}, u)$.*

A.8.3 DEFINITION. Σ and $\widetilde{\Sigma}$ are **local feedback equivalent** (l.f.e.) at points g and $\widetilde{g}$ if there exists a local diffeomorphism $\Phi : \mathcal{N} \times \mathbb{R}^\ell \to \widetilde{\mathcal{N}} \times \mathbb{R}^\ell$ (where $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are neighbourhoods of g and $\widetilde{g}$, respectively) such that $\Phi(g, u) = (\phi(g), \psi(g, u)) = (\widetilde{g}, \widetilde{u})$ and $d\Phi(g) \cdot \Xi(g, u) = \widetilde{\Xi}(\phi(g), \psi(g, u))$.

A.8.4 DEFINITION. Σ and $\widetilde{\Sigma}$ are **local detached feedback equivalent** (l.d.f.e.) at points g and $\widetilde{g}$ if there exists a local diffeomorphism $\Phi : \mathcal{N} \times \mathbb{R}^\ell \to \widetilde{\mathcal{N}} \times \mathbb{R}^\ell$ (where $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are neighbourhoods of g and $\widetilde{g}$, respectively) such that $\Phi(g, u) = (\phi(g), \psi(u)) = (\widetilde{g}, \widetilde{u})$ and $d\Phi(g) \cdot \Xi(g, u) = \widetilde{\Xi}(\phi(g), \psi(u))$.

A.8.5 PROPOSITION. *Two control systems $\Sigma = (\mathsf{G}, \Xi)$ and $\widetilde{\Sigma} = (\mathsf{G}, \widetilde{\Xi})$ are l.d.f.e. at $g_1, g_2 \in \mathsf{G}$ if and only if they are l.d.f.e at $\mathbf{1} \in \mathsf{G}$.*

A.9 Hamiltonian formalism

References used include [26] and [17].

A.9.1 DEFINITION. A **Poisson bracket** on a vector space V is a bilinear operator $\{\cdot, \cdot\}$ on $C^\infty(\mathsf{V})$ such that for all $F, G, H \in C^\infty(\mathsf{V})$, $\{FG, H\} - \{F, H\}G - F\{G, H\} = 0$. The vector space V equipped with the Poisson bracket $\{\cdot, \cdot\}$ is a **Poisson space**, denoted by $(\mathsf{V}, \{\cdot, \cdot\})$.

Let H be a function in $C^\infty(\mathsf{V})$ and $(\mathsf{V}, \{\cdot, \cdot\})$ be a Poisson space.

A.9.2 DEFINITION. The vector field $\vec{H}$ defined by $\vec{H}[F] = \{H, F\}$ for all $F \in C^\infty(\mathsf{V})$ is a **Hamiltonian vector field** associated to H.

We will refer to H as the **Hamiltonian function** of the vector field $\vec{H}$ associated to it.

A.9.3 DEFINITION. A **Hamilton-Poisson system** is a triple $(\mathsf{V}, H, \{\cdot, \cdot\})$.

A.9.4 PROPOSITION. *Let $(\mathsf{V}, H, \{\cdot, \cdot\})$ be a Hamilton-Poisson system with flow $\phi_t = \exp(t\vec{H})$. Then*

1. $H \circ \phi_t = H$
2. $\frac{d}{dt}(F \circ \phi_t) = \{F, H\} \circ \phi_t = \{F \circ \phi_t, H \circ \phi_t\} = \{F \circ \phi_t, H\}$
3. *The Lie bracket of two Hamiltonian vector fields $\vec{F}, \vec{G}$ is a Hamiltonian vector field, and* $[\vec{F}, \vec{G}] = \overrightarrow{\{F, G\}}$
4. *If H_X and H_Y are Hamiltonian functions on $\mathsf{T}^*\mathsf{G}$ which correspond to smooth vector fields X and Y, then $\{H_X, H_Y\} = H_{[X,Y]}$.*

We will denote $\frac{d}{dt}(F \circ \phi_t)$ by $\dot{F}$. Let G be a (matrix) Lie group and $\mathfrak{g}$ its Lie algebra.

A.9.5 EXAMPLE. The dual space $\mathfrak{g}^*$ equipped with the **Lie-Poisson bracket** $\{F, G\}(p) = -p[dF(p), dG(p)]$ for all $p \in \mathfrak{g}^*, F, G \in C^\infty(\mathsf{V})$ is a Hamilton-Poisson system.

A.9.6 REMARK. Given a basis $\{E_1, ..., E_n\}$ for $\mathfrak{g}$, the **dual basis** $\{E_1^*, E_2^*, ..., E_n^*\}$ is the set of elements $E_i^* \in \mathfrak{g}^*$ such that $E_i^*(E_j) = \delta_{ij}$. We may thus express any $p \in \mathfrak{g}^*$ as the sum $p_1 E_1^* + p_2 E_2^* + ... + p_n E_n^*$.

A.9.7 DEFINITION. For each left-invariant vector field X on a matrix Lie group G, we define the corresponding Hamiltonian function H_X by $H_X(p) = p(X(g))$ for each $g \in \mathsf{G}$, $p \in \mathfrak{g}^*$. H_X is the **reduced Hamiltonian of** X.

A.10 Optimal control

References used include [13] and [17]. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and g_0, g_T arbitrary fixed points in G, and $\Sigma = (\mathsf{G}, \Xi)$ a left-invariant control affine system.

A.10.1 DEFINITION. If $u(\cdot)$ is an admissible control and $g(\cdot)$ is the corresponding trajectory, then $(g(\cdot), u(\cdot))$ is a **trajectory-control pair** of Σ.

Given $g_0, g_T \in \mathsf{G}$, a trajectory-control pair **transfers** the point g_0 to g_T if there exists an interval $[0, T]$ contained in the domain of (g, u) such that $g(0) = g_0, g(T) = g_T$.

A.10.2 DEFINITION. The **cost** of the transfer of g_0 to g_T by a trajectory-control pair $(g(\cdot), u(\cdot))$ is the functional $\mathcal{J} = \int_0^T \mathcal{L}(g(t), u(t))dt$.

A.10.3 DEFINITION. A trajectory-control pair $(g(\cdot), u(\cdot))$ is **optimal** with respect to the given points g_0 and g_T if $g(0) = g_0$ and $g(T) = g_T$ and if for any other trajectory-control pair $(g'(\cdot), u'(\cdot))$ which transfers g_0 to g_T, it follows that $\mathcal{J}' = \int_0^T \mathcal{L}(g'(t), u'(t))dt > \int_0^T \mathcal{L}(g(t), u(t))dt = \mathcal{J}$.

$\mathcal{L}$ is the Lagrangian of this optimization.

A.10.4 DEFINITION. A Lagrangian $\mathcal{L} \in C^\infty(\mathsf{G} \times \mathbb{R}^\ell)$ is **left-invariant** if $L(g_2 g_1, u) = \mathcal{L}(g_1, u)$ for all g_1, g_2 in G.

Note that a left-invariant Lagrangian $\mathcal{L}$ is constant over G and depends only on the controls.

A.10.5 DEFINITION. If the Lagrangian has the form $\mathcal{L} = \Sigma_{i=1}^{\ell} c_i u_i^2$, then $\mathcal{J} = \int_0^T \mathcal{L}(g(t), u(t))dt$ is a **quadratic cost**.

In this thesis we will always use quadratic costs.

A.10.6 DEFINITION. In this thesis an **optimal control problem on** G associated with the control system $\Sigma = (\mathsf{G}, \Xi)$ is the problem of finding the trajectory-control pair $(g(\cdot), u(\cdot))$ relative to the given points g_0, g_T such that

$$\dot{g} = g\,\Xi(1, u), \qquad g \in \mathsf{SO}(1,2)_0, \; (u_1, u_2, ..., u_\ell) \in \mathbb{R}^\ell$$

$$g(0) = g_0, \quad g(T) = g_T$$

$$\mathcal{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + c_2 u_2^2(t) + ... + c_\ell u_\ell^2(t)\right) dt \to \min \quad c_1, c_2, ..., c_\ell > 0.$$

We denote this control problem by $(\Sigma, \mathcal{L}, (g_0, g_T, T))$.

A.10.7 DEFINITION. A **left-invariant control problem** is an optimal control problem for which both the Lagrangian $\mathcal{L}$ and the control system $\Sigma = (\mathsf{G}, \Xi)$ are left-invariant.

Note that the optimal control problem of (A.10.6) is left-invariant.

A.10.8 REMARK. (LIFTING THE CONTROL PROBLEM TO $\mathrm{T}^*\mathsf{G}$) Given the particular control problem of (A.10.6), we may determine the Hamiltonian for this problem as in (A.9.7): for each $u \in \mathbb{R}^\ell$, the reduced Hamiltonian H of the left-invariant vector field $\Xi_u = A + \Sigma_{i=1}^{\ell} u_i B_i$ is given by the function $H(p_g) = p_g\left(g(A + \Sigma_{i=1}^{\ell} u_i B_i)\right)$. By the trivialization (A.5.28), we express each $p_g \in \mathrm{T}_g^*\mathsf{G} \subseteq \mathrm{T}^*\mathsf{G}$ by $p_g = dL_{g^{-1}}^* p = (g, p)$: then

$$\begin{aligned} H(g,p) = dL_{g^{-1}}^*(p)\left(g\left(A + \Sigma_{i=1}^{\ell} u_i B_i\right)\right) &= p\left(g^{-1}g\left(A + \Sigma_{i=1}^{\ell} u_i B_i\right)\right) \\ &= p\left(A + \Sigma_{i=1}^{\ell} u_i B_i\right) \end{aligned}$$

where then H is a linear function on $\mathfrak{g}^*$ only.

The Hamiltonian of the left-invariant optimal control problem (A.10.6) is the function

$$\mathcal{H}^\lambda(p,u) = -\lambda L(u) + p\left(A + \Sigma_{i=1}^{\ell} u_i B_i\right) \tag{A.10.1}$$

where $p \in \mathfrak{g}^*$, $u \in \mathbb{R}^\ell$ and $\lambda = 0, 1$.
From (A.9.6), the component functions p_i satisfy $\dot{p}_i = \{p_i, \mathcal{H}^\lambda\}$.

A.10.9 DEFINITION. The **reduced extremal equations** for the optimal control problem (A.10.6) with Hamiltonian $\mathcal{H}^\lambda$ is the system of differential equations $\dot{p}_i = \{p_i, \mathcal{H}^\lambda\}$.

A.10.10 PROPOSITION. *Suppose that $(g(\cdot), p(\cdot))$ is an integral curve of the Hamiltonian vector field $\vec{\mathcal{H}}^\lambda(p, u(\cdot))$ for some control function $u(\cdot)$, with $\mathcal{H}^\lambda(p,u) = -\lambda L(u) + p\left(A + \Sigma_{i=1}^{\ell} u_i B_i\right)$. Then $\dot{g} = g\left(\frac{\partial \mathcal{H}^\lambda}{\partial p}(p,u)\right)$ and $p(t) = \mathrm{Ad}_{g(t)}^* p(0)$, for some $p(0) \in \mathfrak{g}^*$. Consequently, for each $t \in (a,b)$, $p(t)$ is contained in the co-adjoint orbit of G through $p(0)$.*

A.10.11 COROLLARY. *For each integral curve $(g(\cdot), p(\cdot))$ of $\mathcal{H}(p, u(t))$, $\mathrm{Ad}_{g(t)^{-1}}^* p(t)$ is constant.*

A.10.12 DEFINITION. A function K is $\mathrm{Ad}_{\mathsf{G}}^*$-invariant if $K(p) = K(\mathrm{Ad}_g^*(p))$ for all $g \in \mathsf{G}$ and $p \in \mathfrak{g}^*$.

A.10.13 DEFINITION. A **Casimir function** is any $\mathrm{Ad}_{\mathsf{G}}^*$-invariant smooth function K on $\mathfrak{g}^*$.

A.10.14 PROPOSITION. *A Casimir function is a constant of motion for any Hamiltonian function H on $\mathfrak{g}^*$: that is, $\{H, K(p)\} = 0$ for all $p \in \mathfrak{g}^*$.*

A.10.15 REMARK. Since from (A.10.10) the integral curves $p(\cdot)$ of the reduced extremal equations develop on the co-adjoint orbits $\widetilde{O}_{p(0)}$ (which are the level surfaces of the Casimir function K) and from (A.9.4) (1) it follows that $\frac{d}{dt}H(p(t)) = 0$, then the extremal curves lie on the intersections of the level surfaces $H_{u(0)} \cap K_{u(0)}$.

We now give a statement of the Pontryagin Maximum principle (PMP), which gives a set of necessary conditions for a trajectory to be optimal.

A.10.16 THEOREM. (PMP) *If $(g(\cdot), u(\cdot))$ is an optimal trajectory-control pair of the left-invariant optimal control problem associated to system Σ on an interval $[0,T]$, then $g(\cdot)$ is the projection of the integral curve $(g(\cdot), p(\cdot))$ of the Hamiltonian vector field $\vec{\mathcal{H}}^\lambda, \lambda = 0, 1$, such that*

1. *If $\lambda = 0$, then $(g(\cdot), p(\cdot))$ is not identically zero on $[0, T]$.*
2. *$\mathcal{H}^\lambda(g(\cdot), p(\cdot), u(\cdot)) \geq \mathcal{H}((g(\cdot), p(\cdot), u))$ for any $u \in \mathbb{R}^\ell$ and a.e. $t \in [0,T]$*
3. *$\mathcal{H}^\lambda(g(\cdot), p(\cdot), u(\cdot))$ is constant for all $t \in [0,T]$.*

A.10.17 DEFINITION. A pair of curves $(g(\cdot), p(\cdot), u(\cdot)) \subseteq \mathsf{G} \times \mathfrak{g}^* \times \mathbb{R}^\ell$ on an interval $[0,T]$ is an **extremal pair** if $(g(\cdot), p(\cdot))$ is an integral curve of $\vec{\mathcal{H}}^\lambda$ for either $\lambda = 0$ or 1 such that the first two statements of the PMP hold. A projection $(g(\cdot), p(\cdot))$ of the extremal pair is called an **extremal curve**. The extremal curves corresponding to $\lambda = 1$ are **normal extremals**, while those corresponding to $\lambda = 0$ are **abnormal extremals**.

Note that in this thesis we will always restrict to normal extremals, and so refer to the normal extremals as extremal curves. Thus we denote the Hamiltonian vector field $\vec{\mathcal{H}}^\lambda$ by $\vec{\mathcal{H}}$ and its Hamiltonian function by $\mathcal{H}$.

A.11 Elliptic functions

References used include [34] and [19].

A.11.1 DEFINITION. For a real $k \in (0,1)$, the **Jacobi elliptic functions** $\mathrm{sn}(\cdot,k), \mathrm{cn}(\cdot,k)$ and $\mathrm{dn}(\cdot,k)$ are defined as the solutions to the system of differential equations

$$\dot{x} = yz, \quad \dot{y} = -zx \quad \text{and} \quad \dot{z} = -k^2xy$$

satisfying initial condtions $\mathrm{sn}(0,k) = x(0) = 0$, $\mathrm{cn}(0,k) = y(0) = 1$ and $\mathrm{dn}(0,k) = z(0) = 1$.

The real parameter $k \in (0,1)$ is the **modulus** of the elliptic function. Nine other elliptic functions are defined by taking reciprocals and quotients:

$$\begin{aligned}
\mathrm{ns}(\cdot,k) &= \tfrac{1}{\mathrm{sn}(\cdot,k)}; & \mathrm{nc}(\cdot,k) &= \tfrac{1}{\mathrm{cn}(t,k)}; & \mathrm{nd}(\cdot,k) &= \tfrac{1}{\mathrm{dn}(\cdot,k)} \\
\mathrm{sc}(\cdot,k) &= \tfrac{\mathrm{sn}(t,k)}{\mathrm{cn}(\cdot,k)}; & \mathrm{cd}(\cdot,k) &= \tfrac{\mathrm{cn}(t,k)}{\mathrm{dn}(\cdot,k)}; & \mathrm{ds}(\cdot,k) &= \tfrac{\mathrm{dn}(t,k)}{\mathrm{sn}(\cdot,k)} \\
\mathrm{cs}(\cdot,k) &= \tfrac{\mathrm{cn}(t,k)}{\mathrm{sn}(\cdot,k)}; & \mathrm{dc}(\cdot,k) &= \tfrac{\mathrm{dn}(t,k)}{\mathrm{cn}(\cdot,k)}; & \mathrm{sd}(\cdot,k) &= \tfrac{\mathrm{sn}(t,k)}{\mathrm{dn}(\cdot,k)}.
\end{aligned}$$

Of the given elliptic functions, only 6 (sn, cd, dc, ns, nd and dn) are used in this thesis.

A.11.2 PROPOSITION. *The derivatives of the Jacobi elliptic functions* sn, cn *and* dn *are given by*

$$\frac{d}{dt}\mathrm{sn}(t,k) = \mathrm{cn}(d,k)\mathrm{dn}(d,k), \quad \frac{d}{dt}\mathrm{cn}(t,k) = -\mathrm{dn}(d,k)\mathrm{sn}(d,k), \quad \frac{d}{dt}\mathrm{dn}(t,k) = k^2\mathrm{cn}(d,k)\mathrm{sn}(d,k).$$

The derivatives of the other 9 elliptic functions may be obtained from these via the product and quotient rules.

A.11.3 PROPOSITION. *As k approaches 0 from the right, correspondingly*

$$\mathrm{sn}(t,k) \to \sin(t),\ \mathrm{cn}(t,k) \to \cos(t) \quad \textit{and} \quad \mathrm{dn}(t,k) \to 1.$$

As k approaches 1 from the left, correspondingly

$$\mathrm{sn}(t,k) \to \tanh(t),\ \mathrm{cn}(t,k) \to (t) \quad \textit{and} \quad \mathrm{dn}(t,k) \to (t).$$

The convergence is uniform on compact sets.

A.11.4 DEFINITION. An **elliptic integral** is an integral of the form $\int R(t, P(t))dt$ where R is a rational function, P is the square root of a polynomial of degree three or four with no repeated roots, and $c \in \mathbb{R}$ is a constant.

In this thesis, we make use of the elliptic integrals

$$\begin{aligned}
\int_0^x \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} &= \mathrm{sn}^{-1}\left(\frac{x}{b}, \frac{b}{a}\right) && 0 \le x \le b < a \\
\int_x^b \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} &= \frac{1}{a}\mathrm{cd}^{-1}\left(\frac{x}{b}, \frac{b}{a}\right) && 0 \le x \le b < a \\
\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} &= \frac{1}{a}\mathrm{dc}^{-1}\left(\frac{x}{a}, \frac{b}{a}\right) && b < a \le x \\
\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} &= \frac{1}{a}\mathrm{ns}^{-1}\left(\frac{x}{b}, \frac{b}{a}\right) && b < a \le x \\
\int_b^x \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} &= \frac{1}{a}\mathrm{nd}^{-1}\left(\frac{x}{b}, \frac{\sqrt{a^2-b^2}}{a}\right) && b \le x \le a \\
\int_x^a \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} &= \frac{1}{a}\mathrm{dn}^{-1}\left(\frac{x}{a}, \frac{\sqrt{a^2-b^2}}{a}\right) && b \le x \le a
\end{aligned}$$

A.12 Stability

References used include [17], [26] and [23]. Let V be a real vector space and $\mathsf{W} \subseteq \mathsf{V}$ open. Given a smooth map $X : \mathsf{W} \to \mathsf{V}$, consider the differential equation

$$\dot{\mathbf{p}} = X(\mathbf{p}). \tag{A.12.1}$$

Here, $\dot{\mathbf{p}}$ denotes $\frac{d\mathbf{p}}{dt}$.

A.12.1 DEFINITION. A **equilibrium point** of the differential equation (A.12.1) is a point $\mathbf{p}_e$ such that $X(\mathbf{p}_e) = 0$.

A.12.2 DEFINITION. An equilibrium point $\mathbf{p}_e$ is **nonlinearly stable** if for every neighbourhood $\mathcal{N}_1$ of $\mathbf{p}_e$, there exists a neighbourhood $\mathcal{N}_2$ of $\mathbf{p}_e$ such that the trajectories $\mathbf{p}(\cdot)$ passing through $\mathbf{p}_e$ and intially in $\mathcal{N}_2$ never leave $\mathcal{N}_1$. If the point $\mathbf{p}_e$ is not nonlinearly stable, then it is **(nonlinearly) unstable**.

The **energy-Casimir method** gives sufficient conditions for the nonlinear stability of equilibrium points of a differential equation of the form (A.12.1):

A.12.3 THEOREM. (THE ENERGY-CASIMIR METHOD) *Let* $(V, \{\cdot,\cdot\}, H)$ *be a Hamilton-Poisson system, and* $\mathbf{p}_e$ *be an equilibrium point of the system* $\dot{\mathbf{p}} = \{\mathbf{p}, H\}$. *Then*

1. *Find a family of constants of motion for the Hamiltonian system. These are generally Casimir functions.*

2. *Select a constant of motion* K *from the family determined in step 1 such that the* ***energy-Casimir function*** $H + K$ *has a critical point at* $\mathbf{p}_e$.

3. *Determine the second derivative of the energy-Casimir function at* $\mathbf{p}_e$.

If the second derivative of the energy-Casimir function is positive-definite or negative-definite at $\mathbf{p}_e$, *then* $\mathbf{p}_e$ *is nonlinearly stable. If not, the test is inconclusive.*

An extention of this method by Ortega, Ratiu and Planas-Bielsa [23] also gives sufficient conditions for nonlinear stability for a differential equation of the form (A.12.1):

A.12.4 THEOREM. (EXTENDED ENERGY-CASIMIR METHOD) *Let* $(M, \{\cdot,\cdot\}, H)$ *be a Hamilton-Poisson system,* $\mathbf{p}_e$ *be an equilibrium point of the system* $\dot{\mathbf{p}} = \overrightarrow{H}(\mathbf{p})$ *and* $C_1, C_2, ..., C_k$ *conserved quantities, that is,* $\{C_i, H\} = 0$ *for* $i = 1, 2, ..., k$. *If there exist constants* $\lambda_0, \lambda_1, ..., \lambda_k$ *such that for the function* $L = \lambda_0 H + \lambda_1 C_1 + \lambda_2 C_2 +, ..., \lambda_k C_k$, *then*

$$dL(\mathbf{p}_e) = d(\lambda_0 H + \lambda_1 C_1 + \lambda_2 C_2 + ... + \lambda_k C_k)(\mathbf{p}_e) = 0$$

and the quadratic form

$$d^2 L|_{W\times W}(\mathbf{p}_e) = d^2(\lambda_0 H + \lambda_1 C_1 + \lambda_2 C_2 + ... + \lambda_k C_k)|_{W\times W}(\mathbf{p}_e) = 0$$

is positive-definite, where

$$W = \ker(dH(\mathbf{p}_e)) \cap \ker(dC_1(p_e)) \cap ... \cap \ker(dC_k(\mathbf{p}_e))$$

then $\mathbf{p}_e$ *is non-linear stable.*

A.12.5 DEFINITION. An equilibrium point $\mathbf{p}_e$ of the system $\dot{\mathbf{p}} = X(\mathbf{p})$ is **linearly stable** at $\mathbf{p}_e$ if the eigenvalues of $dX(\mathbf{p}_e)$ have no positive real parts. If $\mathbf{p}_e$ is not linearly stable, then it is **(linearly) unstable**.

A.12.6 PROPOSITION. *Nonlinear stability implies linear stability. However, linear stability does not necessarily imply nonlinear stability.*

By the contrapositive, (A.12.6) will provide our method for showing that equilibrium points are unstable, since if a point is linearly unstable, then it is nonlinearly unstable. We will refer to linearly unstable points as **unstable**.

Appendix B

Tables

Type	Representative	controllability
1-input (inhomogeneous)	$\alpha(E_2) + uE_1$	controllable
	$\alpha E_2 + uE_3$	not controllable
	$\alpha E_1 + uE_3$	controllable
	$E_3 + u(E_1 + E_3)$	controllable
2-input homogeneous	$u_1E_2 + u_2E_1$	controllable
	$u_1E_3 + u_2E_2$	controllable
2-input inhomogeneous	$\alpha E_3 + u_1E_2 + u_2E_1$	controllable
	$\alpha E_1 + u_1E_3 + u_2E_2$	controllable
	$E_1 + u_1E_3 + u_2(E_1 + E_2)$	controllable
3-input (homogeneous)	$u_1E_1 + u_2E_2 + u_3E_3$	controllable

Table B.1: Classification of left-invariant control affine systems under l.d.f.c. and controllability of the representative elements.

(Results used: CHAPTER 4, THEOREMS 4.1.10, 4.1.27, 4.1.34 and 4.1.35, the controllability criterion THEOREM 4.2.9.)

Representative	Equilibrium points	Stability value
$u_1E_2 + u_2E_1$	$(0,0,0)$	stable
	$(M,0,0)$	stable
	$(0,M,0)$	unstable
$u_1E_3 + u_2E_2$	$(0,0,0)$	stable
	$(0,0,M)$	unstable
	$(0,M,0)$	unstable
	$(0,M,N)\|_{c_1=c_2}$	unstable

Table B.2: Stability of equilibrium points: 2-input homogenous systems. In each case $M \in \mathbb{R}\setminus\{0\}$.

(Results used:THEOREMS 5.2.5 and 5.3.5.)

Representative	Equilibrium points	Stability value
$\alpha E_2 + uE_1$	$(0, M, 0)\|_{M<0}$	stable
	$(0, M, 0)\|_{M>0}$	unstable
	$(M, -\alpha c_1, 0)\|_{\|M\|<c_1\alpha}$	stable
	$(M, -\alpha c_1, 0)\|_{\|M\|>c_1\alpha}$	unstable
	$(0, 0, 0)$	stable
$\alpha E_1 + uE_3$	$(0, M, 0)\|_{M>0}$	unstable
	$(0, M, 0)\|_{M<0}$	stable
	$(-\alpha c_1, 0, M)\|_{\|M\|>c_1\alpha}$	stable
	$(-\alpha c_1, 0, M)\|_{\|M\|<c_1\alpha}$	unstable
	$(0, 0, 0)$	unstable
$\alpha E_2 + uE_3$	$(0, M, 0)\|_{M>0}$	unstable
	$(0, M, 0)\|_{M<0}$	stable
	$(0, \alpha c_1, M)\|_{\|M\|>c_1\alpha}$	stable
	$(0, \alpha c_1, M)\|_{\|M\|<c_1\alpha}$	unstable
	$(0, 0, 0)$	unstable
$-E_2 + u(E_1 + E_2)$	$\left(M, -\sqrt{cM} - M, 0\right)$	unstable
	$\left(M, \sqrt{cM} - M, 0\right)\|_{M>\frac{c}{4}}$	unstable
	$\left(M, \sqrt{cM} - M, 0\right)\|_{0<M<\frac{c}{4}}$	stable
	$(c, 0, 0)$	stable
	$(0, 0, 0)$	unstable

Table B.3: Stability of equilibrium points: 1-input (inhomogenous) systems. In each case $M \in \mathbb{R} \setminus \{0\}$.

Representative	Equilibrium points	Stability value
$\alpha E_3 + u_1E_2 + u_2E_1$	$(0,0,0)$	unstable
	$(0,0,M)\|_{M>0}$	stable
	$(0,0,M)\|_{M<0}$	unstable
	$(M,0,-\alpha c_2)\|_{\|M\|>\alpha c_2}$	stable
	$(M,0,-\alpha c_2)\|_{\|M\|<\alpha c_1}$	unstable
	$(0,M,\alpha c_1)\|_{\|M\|<\alpha c_1}$	unstable
$\alpha E_1 + u_1E_3 + u_2E_2$	$(0,0,0)$	stable
	$(M,0,0)$	stable
	$(-\alpha c_1,0,M)\|_{\|M\|>\alpha c_1}$	stable
	$(-\alpha c_1,0,M)\|_{\|M\|<\alpha c_1}$	unstable
	$(-\alpha c_2,0,M)\|_{\|M\|>\alpha c_2}$	stable
	$(-\alpha c_2,0,M)\|_{\|M\|<\alpha c_2}$	unstable
	$(\alpha c_1,M,N)\|_{c_1=c_2}$	unstable
$-E_2 + u_1E_3 + u_2(E_1+E_2)$	$(0,0,0)$	unstable
	$(0,c_2,0)$	unstable
	$(M,-\sqrt{c_2M}-M,0)$	unstable
	$(M,\sqrt{c_2M}-M,0)$	unstable

Table B.4: Stability of equilibrium points: 2-input inhomogenous systems. In each case $M, N \in \mathbb{R}\backslash\{0\}$.

Representative	Equilibrium points	Stability value
$u_1E_1 + u_2E_2 + u_3E_3$	$(0,0,0)$	stable
	$(A,0,0)$	stable
	$(0,0,C)$	unstable
	$(0,B,0)$	unstable
	$(0,B,C)_{c_2=c_3}$	unstable

Table B.5: Stability of equilibrium points: 3-input systems. In each case $A, B, C \in \mathbb{R}\backslash\{0\}$.

Appendix C

Mathematica codes

C.1

The codes used in **Chapter 2**

Code used to find the eigenvalues in THEOREM 2.1.11

Eigenvalues$[\{\{1-(b\hat{}2/a\hat{}2),(-bc)/a\hat{}2\},$
$\{(-bc)/a\hat{}2,1-(c\hat{}2/a\hat{}2)\}\}]\%$
Eigenvalues$[\{\{(a\hat{}2/b\hat{}2)-1,(ac)/b\hat{}2\},$
$\{(\mathrm{ac})/b\hat{}2,(c\hat{}2/b\hat{}2)+1\}\}]\%$
Eigenvalues$[\{\{(a\hat{}2/c\hat{}2)-1,(ab)/c\hat{}2\},$
$\{(\mathrm{ab})/c\hat{}2,(b\hat{}2/c\hat{}2)+1\}\}]\%$

Codes used to find the Gaussian curvatures in THEOREMS 2.1.15 and 2.3.3

$e=-(\mathrm{Sinh}[v])\hat{}2$
$f=0$
$g=1$
$\mathrm{s1}=\partial_u g$
$\mathrm{s2}=\partial_v e$
$k=(1/(2\mathrm{Sqrt}[eg]))\,(\partial_u(\mathrm{s1}/\mathrm{Sqrt}[eg])$
$\quad -\partial_v(\mathrm{s2}/\mathrm{Sqrt}[eg]))$

$-\mathrm{Sinh}[v]^2$
0
1
0
$-2\mathrm{Cosh}[v]\mathrm{Sinh}[v]$

$$\frac{\frac{2\mathrm{Cosh}[v]^2\mathrm{Sinh}[v]^2}{(-\mathrm{Sinh}[v]^2)^{3/2}}+\frac{2\mathrm{Cosh}[v]^2}{\sqrt{-\mathrm{Sinh}[v]^2}}+\frac{2\mathrm{Sinh}[v]^2}{\sqrt{-\mathrm{Sinh}[v]^2}}}{2\sqrt{-\mathrm{Sinh}[v]^2}}$$

$$\mathbf{FullSimplify}\left[\frac{\frac{2\mathrm{Cosh}[v]^2\mathrm{Sinh}[v]^2}{(-\mathrm{Sinh}[v]^2)^{3/2}}+\frac{2\mathrm{Cosh}[v]^2}{\sqrt{-\mathrm{Sinh}[v]^2}}+\frac{2\mathrm{Sinh}[v]^2}{\sqrt{-\mathrm{Sinh}[v]^2}}}{2\sqrt{-\mathrm{Sinh}[v]^2}}\right]$$

-1

$e=1/v\hat{}2$
$\mathrm{s1}=\partial_u g$
$\mathrm{s2}=\partial_v e$
$\mathrm{Sqrt}[eg]$
$1/(\mathrm{Sqrt}[eg])$
$k=(1/(2\mathrm{Sqrt}[eg]))\,(\partial_u(\mathrm{s1}/\mathrm{Sqrt}[eg])$
$\quad -\partial_v(\mathrm{s2}/\mathrm{Sqrt}[eg]))$

$\frac{1}{v^2}$
0
$-\frac{2}{v^3}$
$\sqrt{\frac{1}{v^4}}\ \frac{1}{\sqrt{\frac{1}{v^4}}}$

$$\frac{-6\sqrt{\frac{1}{v^4}}+\frac{4}{\left(\frac{1}{v^4}\right)^{3/2}v^8}}{2\sqrt{\frac{1}{v^4}}}$$

$$\mathbf{FullSimplify}\left[\frac{-6\sqrt{\frac{1}{v^4}}+\frac{4}{\left(\frac{1}{v^4}\right)^{3/2}v^8}}{2\sqrt{\frac{1}{v^4}}}\right]$$

-1

C.2

The codes used in **Chapter 3**

Code used to find the matrix exponentials in REMARK 3.6.11

E1 = {{0,0,0}, {0,0,−1}, {0,1,0}}

E2 = {{0,0,1}, {0,0,0}, {1,0,0}}

E3 = {{0,1,0}, {1,0,0}, {0,0,0}}

FullSimplify[MatrixExp[*t*E1]]

FullSimplify[MatrixExp[*t*E2]]

FullSimplify[MatrixExp[*t*E3]]

{{0,0,0}, {0.0,−1}, {0,1,0}}

{{0,0,1}, {0,0,0}, {1,0,0}}

{{0,1,0}, {1,0,0}, {0,0,0}}

{{1,0,0}, {0. Cos[*t*], −Sin[*t*]}, {0. Sin[*t*], Cos[*t*]}}

{{Cosh[*t*], 0, Sinh[*t*]}, {0,1,0}, {Sinh[*t*], 0, Cosh[*t*]}}

{{Cosh[*t*], Sinh[*t*], 0}, {Sinh[*t*], Cosh[*t*], 0}, {0.0,1}}

C.3

The codes used in **Chapter 4**

Code used to find the expression of the unique hyperboloids in PROPOSITION 4.1.20

Solve $\left[a\sqrt{(t\text{^}2a\text{^}2)/(a\text{^}2-b\text{^}2)}\right.$

$+b\sqrt{(t\text{^}2a\text{^}2)/(a\text{^}2-b\text{^}2)-t\text{^}2} == h,t]\%$

Solve $\left[a\sqrt{(t\text{^}2a\text{^}2)/(b\text{^}2-a\text{^}2)}\right.$

$+b\sqrt{(t\text{^}2a\text{^}2)/(b\text{^}2-a\text{^}2)+t\text{^}2} == h,t]\%$

Code used to determine the line ℓ in **Case 1** of PROPOSITION 4.1.30

Case 1

Solve[Cos[OverTilde[θ]]($-\beta+\alpha$Cos[θ])Csc[θ]

$-\alpha$Sin[OverTilde[θ]] == 0,OverTilde[θ]]

Solve[αCos[OverTilde[θ]] + ($-\beta+\alpha$Cos[θ])

Csc[θ]Sin[OverTilde[θ]] == 0,OverTilde[θ]]

$$\left\{\left\{\tilde\theta\to-\text{ArcCos}\left[-\frac{\alpha}{\sqrt{\alpha^2+\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}\right]\right\},\right.$$
$$\left\{\tilde\theta\to\text{ArcCos}\left[-\frac{\alpha}{\sqrt{\alpha^2+\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}\right]\right\},$$
$$\left\{\tilde\theta\to-\text{ArcCos}\left[\frac{\alpha}{\sqrt{\alpha^2+\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}\right]\right\},$$
$$\left.\left\{\tilde\theta\to\text{ArcCos}\left[\frac{\alpha}{\sqrt{\alpha^2+\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}\right]\right\}\right\}$$

$$\left\{\left\{\tilde\theta\to-\text{ArcCos}\left[-\frac{\sqrt{\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}{\sqrt{\alpha^2+\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}\right]\right\},\right.$$
$$\left\{\tilde\theta\to\text{ArcCos}\left[-\frac{\sqrt{\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}{\sqrt{\alpha^2+\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}\right]\right\},$$
$$\left\{\tilde\theta\to-\text{ArcCos}\left[\frac{\sqrt{\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}{\sqrt{\alpha^2+\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}\right]\right\},$$
$$\left.\left\{\tilde\theta\to\text{ArcCos}\left[\frac{\sqrt{\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}{\sqrt{\alpha^2+\alpha^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\beta^2\text{Csc}[\theta]^2}}\right]\right\}\right\}$$

Code used to determine the line ℓ in **Case 2** of PROPOSITION 4.1.30

"Case 2"

Solve[βCosh[OverTilde[θ]] + ($-\alpha+\beta$Cos[θ])

Csc[θ]Sinh[OverTilde[θ]] == 0, OverTilde[θ]]

Solve[($-\alpha+\beta$Cos[θ])Cosh[OverTilde[θ]]Csc[θ]

$+\beta$Sinh[OverTilde[θ]] == 0, OverTilde[θ]]

$$\left\{\left\{\tilde\theta\to-\text{ArcCosh}\left[-\frac{\sqrt{\beta^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\alpha^2\text{Csc}[\theta]^2}}{\sqrt{-\beta^2+\beta^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\alpha^2\text{Csc}[\theta]^2}}\right]\right\},\right.$$
$$\left\{\tilde\theta\to\text{ArcCosh}\left[-\frac{\sqrt{\beta^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\alpha^2\text{Csc}[\theta]^2}}{\sqrt{-\beta^2+\beta^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\alpha^2\text{Csc}[\theta]^2}}\right]\right\},$$
$$\left\{\tilde\theta\to-\text{ArcCosh}\left[\frac{\sqrt{\beta^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\alpha^2\text{Csc}[\theta]^2}}{\sqrt{-\beta^2+\beta^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\alpha^2\text{Csc}[\theta]^2}}\right]\right\},$$
$$\left.\left\{\tilde\theta\to\text{ArcCosh}\left[\frac{\sqrt{\beta^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\alpha^2\text{Csc}[\theta]^2}}{\sqrt{-\beta^2+\beta^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\alpha^2\text{Csc}[\theta]^2}}\right]\right\}\right\}$$

$$\left\{\left\{\tilde\theta\to-\text{ArcCosh}\left[-\frac{\beta}{\sqrt{\beta^2-\beta^2\text{Cot}[\theta]^2+2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]-\alpha^2\text{Csc}[\theta]^2}}\right]\right\},\right.$$
$$\left\{\tilde\theta\to\text{ArcCosh}\left[\frac{\beta}{\sqrt{\beta^2-\beta^2\text{Cot}[\theta]^2+2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]-\alpha^2\text{Csc}[\theta]^2}}\right]\right\},$$
$$\left\{\tilde\theta\to-\text{ArcCosh}\left[\frac{\beta}{\sqrt{\beta^2-\beta^2\text{Cot}[\theta]^2+2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]-\alpha^2\text{Csc}[\theta]^2}}\right]\right\},$$
$$\left.\left\{\tilde\theta\to\text{ArcCosh}\left[\frac{\beta}{\sqrt{\beta^2-\beta^2\text{Cot}[\theta]^2+2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]-\alpha^2\text{Csc}[\theta]^2}}\right]\right\}\right\}$$

In this code we note immediately that in the denominator

$$\sqrt{-\beta^2+\beta^2\text{Cot}[\theta]^2-2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]+\alpha^2\text{Csc}[\theta]^2}$$

common to all the first set of solutions, the interior of the square root is exactly the negative of the interior of the square root in the denominator

$$\sqrt{\beta^2-\beta^2\text{Cot}[\theta]^2+2\alpha\beta\text{Cot}[\theta]\text{Csc}[\theta]-\alpha^2\text{Csc}[\theta]^2}$$

common to all the second set of solutions. Thus if the solutions in the first set are real, the solutions in the second set must be the image under ArcCosh of an imaginary number, which is not real. Thus there is no real $\widetilde{\theta}$ which solves both the first and the second equation simultaneously.

Code used to find the unique transformation b_1 in THEOREM 4.1.30
= **Solve[Cosh[OverTilde[θ]]Sec[ϕ]($\alpha - \beta$Sin[ϕ])**

$$\left\{\left\{\tilde\theta \to -\text{ArcCosh}\left[-\frac{\alpha}{\sqrt{\alpha^2-\alpha^2\text{Sec}[\theta]^2+2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]-\beta^2\text{Tan}[\theta]^2}}\right]\right\},\right.$$
$$\left\{\tilde\theta \to \text{ArcCosh}\left[-\frac{\alpha}{\sqrt{\alpha^2-\alpha^2\text{Sec}[\theta]^2+2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]-\beta^2\text{Tan}[\theta]^2}}\right]\right\},$$
$$\left\{\tilde\theta \to -\text{ArcCosh}\left[\frac{\alpha}{\sqrt{\alpha^2-\alpha^2\text{Sec}[\theta]^2+2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]-\beta^2\text{Tan}[\theta]^2}}\right]\right\},$$
$$\left.\left\{\tilde\theta \to \text{ArcCosh}\left[\frac{\alpha}{\sqrt{\alpha^2-\alpha^2\text{Sec}[\theta]^2+2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]-\beta^2\text{Tan}[\theta]^2}}\right]\right\}\right\}$$
$$\left\{\left\{\tilde\theta \to -\text{ArcCosh}\left[-\frac{\sqrt{\alpha^2\text{Sec}[\theta]^2-2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]+\beta^2\text{Tan}[\theta]^2}}{\sqrt{-\alpha^2+\alpha^2\text{Sec}[\theta]^2-2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]+\beta^2\text{Tan}[\theta]^2}}\right]\right\},\right.$$
$$\left\{\tilde\theta \to \text{ArcCosh}\left[-\frac{\sqrt{\alpha^2\text{Sec}[\theta]^2-2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]+\beta^2\text{Tan}[\theta]^2}}{\sqrt{-\alpha^2+\alpha^2\text{Sec}[\theta]^2-2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]+\beta^2\text{Tan}[\theta]^2}}\right]\right\},$$
$$\left\{\tilde\theta \to -\text{ArcCosh}\left[\frac{\sqrt{\alpha^2\text{Sec}[\theta]^2-2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]+\beta^2\text{Tan}[\theta]^2}}{\sqrt{-\alpha^2+\alpha^2\text{Sec}[\theta]^2-2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]+\beta^2\text{Tan}[\theta]^2}}\right]\right\},$$
$$\left.\left\{\tilde\theta \to \text{ArcCosh}\left[\frac{\sqrt{\alpha^2\text{Sec}[\theta]^2-2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]+\beta^2\text{Tan}[\theta]^2}}{\sqrt{-\alpha^2+\alpha^2\text{Sec}[\theta]^2-2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]+\beta^2\text{Tan}[\theta]^2}}\right]\right\}\right\}$$

In this code we note immediately that in the denominator

$$\sqrt{\alpha^2-\alpha^2\text{Sec}[\theta]^2+2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]-\beta^2\text{Tan}[\theta]^2}$$

common to all the first set of solutions, the interior of the square root is exactly the negative of the interior of the square root in the denominator

$$\sqrt{-\alpha^2+\alpha^2\text{Sec}[\theta]^2-2\alpha\beta\text{Sec}[\theta]\text{Tan}[\theta]+\beta^2\text{Tan}[\theta]^2}$$

common to all the second set of solutions. Thus if the solutions in the first set are real, the solutions in the second set must be the image under ArcCosh of an imaginary number, which is not real. Vice versa, if the solutions in the second set are real, then the solutions in the first set are not real. Thus there is no real $\widetilde{\theta}$ which solves both the first and the second equation simultaneously.

Code used to determine the line ℓ in **Case 4 a** of PROPOSITION 4.1.30

Solve[αCosh[OverTilde[θ]] + (1 + α) Sinh[OverTilde[θ]] == 0, OverTilde[θ]]
Solve[(1 + α)Cosh[OverTilde[θ]]+ αSinh[OverTilde[θ]] == 0, OverTilde[θ]]

$$\left\{\left\{\text{OverTilde}[\theta] \to -\text{ArcCosh}\left[-\frac{\sqrt{1+2\alpha+\alpha^2}}{\sqrt{1+2\alpha}}\right]\right\},\left\{\text{OverTilde}[\theta] \to \text{ArcCosh}\left[-\frac{\sqrt{1+2\alpha+\alpha^2}}{\sqrt{1+2\alpha}}\right]\right\},\left\{\text{OverTilde}[\theta] \to -\text{ArcCosh}\left[\frac{\sqrt{1+2\alpha+\alpha^2}}{\sqrt{1+2\alpha}}\right]\right\},\left\{\text{OverTilde}[\theta] \to \text{ArcCosh}\left[\frac{\sqrt{1+2\alpha+\alpha^2}}{\sqrt{1+2\alpha}}\right]\right\}\right\}$$

$$\left\{\left\{\text{OverTilde}[\theta] \to -\text{ArcCosh}\left[-\frac{\alpha}{\sqrt{-1-2\alpha}}\right]\right\},\left\{\text{OverTilde}[\theta] \to \text{ArcCosh}\left[-\frac{\alpha}{\sqrt{-1-2\alpha}}\right]\right\},\left\{\text{OverTilde}[\theta] \to -\text{ArcCosh}\left[\frac{\alpha}{\sqrt{-1-2\alpha}}\right]\right\},\left\{\text{OverTilde}[\theta] \to \text{ArcCosh}\left[\frac{\alpha}{\sqrt{-1-2\alpha}}\right]\right\}\right\}$$

Code used to determine the line ℓ in **Case 4b** of PROPOSITION 4.1.30

MatrixExp[{{0, 0, 0}, {0, 0, −OverTilde[θ]}, {0, OverTilde[θ], 0}}−{{0, 0, OverTilde[θ]}, {0, 0, 0}, {OverTilde[θ], 0, 0}}]

$$\left\{\left\{\frac{1}{2}\left(2+\tilde\theta^2\right),-\frac{\tilde\theta^2}{2},-\tilde\theta\right\},\left\{\frac{\tilde\theta^2}{2},\frac{1}{2}\left(2-\tilde\theta^2\right),-\tilde\theta\right\},\{-\tilde\theta,\tilde\theta,1\}\right\}$$

FullSimplify[
$$\left\{\left\{\frac{1}{2}\left(2+\tilde\theta^2\right),-\frac{\tilde\theta^2}{2},-\tilde\theta\right\},\left\{\frac{\tilde\theta^2}{2},\frac{1}{2}\left(2-\tilde\theta^2\right),-\tilde\theta\right\},\{-\tilde\theta,\tilde\theta,1\}\right\}.\{1,1,0\}]$$

$$\{1,1,0\}$$

$$\left\{\frac{1}{2}\left(2+\tilde\theta^2\right)-\alpha\tilde\theta\text{Cos}[\theta],\frac{\tilde\theta^2}{2}-\alpha\tilde\theta\text{Cos}[\theta],-\tilde\theta+\alpha\text{Cos}[\theta]\right\}$$

Solve[$-\widetilde{\theta} + \alpha$Cos[$\theta$] == 0, $\widetilde{\theta}$]

$$\{\{\tilde\theta \to \alpha\text{Cos}[\theta]\}\}$$

$\widetilde{\theta} = \alpha$Cos[$\theta$]

$$\left\{\frac{1}{2}\left(2+\tilde\theta^2\right)-\alpha\tilde\theta\text{Cos}[\theta],\frac{\tilde\theta^2}{2}-\alpha\tilde\theta\text{Cos}[\theta],-\tilde\theta+\alpha\text{Cos}[\theta]\right\}$$

αCos[θ]

$$\left\{-\alpha^2\text{Cos}[\theta]^2+\frac{1}{2}\left(2+\alpha^2\text{Cos}[\theta]^2\right),-\frac{1}{2}\alpha^2\text{Cos}[\theta]^2,0\right\}$$

$\{-\alpha^2\text{Cos}[\theta]^2 + \frac{1}{2}\left(2 + \alpha^2\text{Cos}[\theta]^2\right), -\frac{1}{2}\alpha^2\text{Cos}[\theta]^2, 0\}$

FullSimplify $[-\alpha^2\text{Cos}[\theta]^2 + \frac{1}{2}\left(2 + \alpha^2\text{Cos}[\theta]^2\right) - \left(-\frac{1}{2}\alpha^2\text{Cos}[\theta]^2\right)]$

1

Code used to determine the line ℓ in **Case 5** of Proposition 4.1.30

Solve[
Sinh[OverTilde[θ]](−1 + Sech[θ](α − Sinh[θ]))+
Cosh[OverTilde[θ]]Sech[θ](α − Sinh[θ]) ==
0, OverTilde[θ]]
Solve[Cosh[OverTilde[θ]](−1 + Sech[θ](α − Sinh[θ]))
+Sech[θ]Sinh[OverTilde[θ]](α − Sinh[θ]) ==
0, OverTilde[θ]]

$$\left\{\left\{\tilde{\theta} \to -\text{ArcCosh}\left[-\frac{\left(\sqrt{\left(-1+2\alpha\text{Sech}[\theta]+2\alpha\text{Sech}[\theta]\text{Tanh}[\theta]-\text{Tanh}[\theta]^2\right)}\right)}{\left(\sqrt{-1+2\alpha\text{Sech}[\theta]-2\text{Tanh}[\theta]}\right)}\right]\right\}, \left\{\tilde{\theta} \to \text{ArcCosh}\left[-\frac{\left(\sqrt{\left(-1+2\alpha\text{Sech}[\theta]+2\alpha\text{Sech}[\theta]\text{Tanh}[\theta]-\text{Tanh}[\theta]^2\right)}\right)}{\left(\sqrt{-1+2\alpha\text{Sech}[\theta]-2\text{Tanh}[\theta]}\right)}\right]\right\}, \left\{\tilde{\theta} \to -\text{ArcCosh}\left[\frac{\left(\sqrt{\left(-1+2\alpha\text{Sech}[\theta]+2\alpha\text{Sech}[\theta]\text{Tanh}[\theta]-\text{Tanh}[\theta]^2\right)}\right)}{\left(\sqrt{-1+2\alpha\text{Sech}[\theta]-2\text{Tanh}[\theta]}\right)}\right]\right\}, \left\{\tilde{\theta} \to \text{ArcCosh}\left[\frac{\left(\sqrt{\left(-1+2\alpha\text{Sech}[\theta]+2\alpha\text{Sech}[\theta]\text{Tanh}[\theta]-\text{Tanh}[\theta]^2\right)}\right)}{\left(\sqrt{-1+2\alpha\text{Sech}[\theta]-2\text{Tanh}[\theta]}\right)}\right]\right\}\right\}$$

$$\left\{\left\{\tilde{\theta} \to -\text{ArcCosh}\left[-\frac{\sqrt{-\alpha^2\text{Sech}[\theta]^2+2\alpha\text{Sech}[\theta]\text{Tanh}[\theta]-\text{Tanh}[\theta]^2}}{\sqrt{1-2\alpha\text{Sech}[\theta]+2\text{Tanh}[\theta]}}\right]\right\}, \left\{\tilde{\theta} \to \text{ArcCosh}\left[-\frac{\sqrt{-\alpha^2\text{Sech}[\theta]^2+2\alpha\text{Sech}[\theta]\text{Tanh}[\theta]-\text{Tanh}[\theta]^2}}{\sqrt{1-2\alpha\text{Sech}[\theta]+2\text{Tanh}[\theta]}}\right]\right\}, \left\{\tilde{\theta} \to -\text{ArcCosh}\left[\frac{\sqrt{-\alpha^2\text{Sech}[\theta]^2+2\alpha\text{Sech}[\theta]\text{Tanh}[\theta]-\text{Tanh}[\theta]^2}}{\sqrt{1-2\alpha\text{Sech}[\theta]+2\text{Tanh}[\theta]}}\right]\right\}, \left\{\tilde{\theta} \to \text{ArcCosh}\left[\frac{\sqrt{-\alpha^2\text{Sech}[\theta]^2+2\alpha\text{Sech}[\theta]\text{Tanh}[\theta]-\text{Tanh}[\theta]^2}}{\sqrt{1-2\alpha\text{Sech}[\theta]+2\text{Tanh}[\theta]}}\right]\right\}\right\}$$

In this code we note immediately that in the denominator

$$\sqrt{-1 + 2\alpha\text{Sech}[\theta] - 2\text{Tanh}[\theta]}$$

common to all the first set of solutions, the interior of the square root is exactly the negative of the interior of the square root in the denominator

$$\sqrt{1 - 2\alpha\text{Sech}[\theta] + 2\text{Tanh}[\theta]}$$

common to all the second set of solutions. Thus if the solutions in the first set are real, the solutions in the second set must be the image uuder ArcCosh of an imaginary number, which is not real. Vice versa, if the solutions in the second set are real, then the solutions in the first set are not real. Thus there is no real $\widetilde{\theta}$ which solves both the first and the second equation simultaneously.

Code used to determine the line ℓ in **Case 6** of Proposition 4.1.30

Note- in this code, t10 and t11 denote $\widetilde{\theta}$.

$\{-\text{Sin}[\theta], -\text{Sin}[\text{t10}] + \text{Sin}[\text{t10} + \theta], \text{Cos}[\text{t10}] - \text{Cos}[\text{t10} + \theta]\}$

Solve[Cos[t10] − Cos[t10 + θ] == 0, t10]

$$\left\{\left\{\text{t10} \to -\text{ArcCos}\left[-\frac{\text{Sin}[\theta]}{\sqrt{1-2\text{Cos}[\theta]+\text{Cos}[\theta]^2+\text{Sin}[\theta]^2}}\right]\right\}, \left\{\text{t10} \to \text{ArcCos}\left[-\frac{\text{Sin}[\theta]}{\sqrt{1-2\text{Cos}[\theta]+\text{Cos}[\theta]^2+\text{Sin}[\theta]^2}}\right]\right\}, \left\{\text{t10} \to -\text{ArcCos}\left[\frac{\text{Sin}[\theta]}{\sqrt{1-2\text{Cos}[\theta]+\text{Cos}[\theta]^2+\text{Sin}[\theta]^2}}\right]\right\}, \left\{\text{t10} \to \text{ArcCos}\left[\frac{\text{Sin}[\theta]}{\sqrt{1-2\text{Cos}[\theta]+\text{Cos}[\theta]^2+\text{Sin}[\theta]^2}}\right]\right\}\right\}$$

FullSimplify[{{Cosh[t11], Sinh[t11], 0},
{Sinh[t11], Cosh[t11], 0}, {0, 0, 1}}.
{−Sin[θ], −Sin[t10] + Sin[t10 + θ], 0}]

$\{-\text{Cosh}[\text{t11}]\text{Sin}[\theta] + (-\text{Sin}[\text{t10}] + \text{Sin}[\text{t10}+\theta])\text{Sinh}[\text{t11}], \text{Cosh}[\text{t11}](-\text{Sin}[\text{t10}]+\text{Sin}[\text{t10}+\theta]) - \text{Sin}[\theta]\text{Sinh}[\text{t11}], 0\}$

Solve[−Cosh[t11]Sin[θ] + (−Sin[t10]
+Sin[t10 + θ])Sinh[t11] == 0,
t11]

$$\left\{\left\{\text{t11} \to -\text{ArcCosh}\left[-\frac{\sqrt{-\text{Sin}[\text{t10}]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}{\sqrt{-\text{Sin}[\text{t10}]^2+\text{Sin}[\theta]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}\right]\right\}, \left\{\text{t11} \to \text{ArcCosh}\left[-\frac{\sqrt{-\text{Sin}[\text{t10}]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}{\sqrt{-\text{Sin}[\text{t10}]^2+\text{Sin}[\theta]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}\right]\right\}, \left\{\text{t11} \to -\text{ArcCosh}\left[\frac{\sqrt{-\text{Sin}[\text{t10}]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}{\sqrt{-\text{Sin}[\text{t10}]^2+\text{Sin}[\theta]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}\right]\right\}, \left\{\text{t11} \to \text{ArcCosh}\left[\frac{\sqrt{-\text{Sin}[\text{t10}]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}{\sqrt{-\text{Sin}[\text{t10}]^2+\text{Sin}[\theta]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}\right]\right.$$

Solve[Cosh[t11](−Sin[t10] + Sin[t10 + θ])
−Sin[θ]Sinh[t11] == 0,
t11]

$$\left\{\left\{\text{t11} \to -\text{ArcCosh}\left[-\frac{\text{Sin}[\theta]}{\sqrt{-\text{Sin}[\text{t10}]^2+\text{Sin}[\theta]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}\right]\right.$$
$$\left\{\text{t11} \to \text{ArcCosh}\left[-\frac{\text{Sin}[\theta]}{\sqrt{-\text{Sin}[\text{t10}]^2+\text{Sin}[\theta]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}\right]\right\}$$
$$\left\{\text{t11} \to -\text{ArcCosh}\left[\frac{\text{Sin}[\theta]}{\sqrt{-\text{Sin}[\text{t10}]^2+\text{Sin}[\theta]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}\right]\right\}$$

$\left\{\text{t11} \to \text{ArcCosh}\left[\frac{\text{Sin}[\theta]}{\sqrt{-\text{Sin}[\text{t10}]^2+\text{Sin}[\theta]^2+2\text{Sin}[\text{t10}]\text{Sin}[\text{t10}+\theta]-\text{Sin}[\text{t10}+\theta]^2}}\right]\right.$

But immediately we note that the numerator of t11 in the first set of solutions is always negative or zero, since in the function $f(\theta) = 2\sin\pi\sin(2\pi+\theta) - \sin(2\pi+\theta)^2 - \sin(2\pi)^2$, $\max_\theta - \sin(2\pi+\theta)^2 - \sin(2\pi)^2 = -2$ and $\max_\theta 2\sin\pi\sin(2\pi+\theta) = -2$, and since these two maxima occur at the same value θ, then $\max_\theta f(\theta) = 0$. Since arccosh is not defined at 0, then t11 in the first set of solutions is never a real number. Thus only the second set of solutions is valid.

FullSimplify[{{Cosh[t11], Sinh[t11], 0}, {Sinh[t11], Cosh[t11], 0}, {0, 0, 1}}. {λ, 0, 0}]

{λCosh[t11], λSinh[t11], 0}

FullSimplify[{{Cosh[t10], 0, Sinh[t10]}, {0, 1, 0}, {Sinh[t10], 0, Cosh[t10]}}. {−Sin[θ], Sin[θ], 1 − Cos[θ]}] {−Cosh[t10]Sin[θ]+ Sinh[t10] − Cos[θ]Sinh[t10], Sin[θ], Cosh[t10]−Cos[θ]Cosh[t10]−Sin[θ]Sinh[t10]}

Solve[−Cosh[t10]Sin[θ] + Sinh[t10] −Cos[θ]Sinh[t10] == 0, t10]

$\left\{\left\{\text{t10} \to -\text{ArcCosh}\left[-\frac{\sqrt{1+2\text{Cos}[\theta]\ \ \text{Cos}[\theta]^2}}{\sqrt{-1+2\text{Cos}[\theta]-\text{Cos}[\theta]^2+\text{Sin}[\theta]^2}}\right]\right\}\right.$

$\left\{\text{t10} \to \text{ArcCosh}\left[-\frac{\sqrt{-1+2\text{Cos}[\theta]-\text{Cos}[\theta]^2}}{\sqrt{-1+2\text{Cos}[\theta]-\text{Cos}[\theta]^2+\text{Sin}[\theta]^2}}\right]\right\}$

$\left\{\text{t10} \to -\text{ArcCosh}\left[\frac{\sqrt{-1+2\text{Cos}[\theta]-\text{Cos}[\theta]^2}}{\sqrt{\ \ 1+2\text{Cos}[\theta]\ \ \text{Cos}[\theta]^2+\text{Sin}[\theta]^2}}\right]\right\}$

$\left.\left\{\text{t10} \to \text{ArcCosh}\left[\frac{\sqrt{-1+2\text{Cos}[\theta]-\text{Cos}[\theta]^2}}{\sqrt{-1+2\text{Cos}[\theta]-\text{Cos}[\theta]^2+\text{Sin}[\theta]^2}}\right]\right\}\right\}$

FullSimplify[{{1, 0, 0}, {0, Cos[t11], −Sin[t11]}, {0, Sin[t11], Cos[t11]}}.{0, Sin[θ], Cosh[t10] − Cos[θ]Cosh[t10] − Sin[θ]Sinh[t10]}]%

Solve[(−1 + Cos[θ])Cosh[t10]Sin[t11] + Sin[θ] (Cos[t11] + Sin[t11]Sinh[t10]) == 0, t11]%

FullSimplify[{{1, 0, 0}, {0, Cos[t11], −Sin[t11]}, {0, Sin[t11], Cos[t11]}}.{0, λ, 0}]

{0, λCos[t11], λSin[t11]}

Code used to express the product an and find the limit of $\|an\|_F$ as $t \to \infty$ in LEMMA 4.2.6

FullSimplify[Sqrt[Tr[
$\left\{\left\{\frac{1}{2}\left(2+\phi5^2\right), -\frac{\phi5^2}{2}, -\phi5\right\}, \left\{\frac{\phi5^2}{2}, \frac{1}{2}\left(2-\phi5^2\right), -\phi5\right\}, \{-\phi5, \phi5, 1\}\right\}.$
Transpose[
$\left\{\left\{\frac{1}{2}\left(2+\phi5^2\right), -\frac{\phi5^2}{2}, -\phi5\right\}, \left\{\frac{\phi5^2}{2}, \frac{1}{2}\left(2-\phi5^2\right), -\phi5\right\}, \{-\phi5, \phi5, 1\}\right\}$**]]]]**

$\sqrt{3+4\phi5^2+\phi5^4}$

MatrixForm[FullSimplify[{{Cosh[t10], Sinh[t10], 0}, {Sinh[t10], Cosh[t10], 0}, {0, 0, 1}}.
$\left\{\left\{\frac{1}{2}\left(2+\phi5^2\right), -\frac{\phi5^2}{2}, -\phi5\right\}, \left\{\frac{\phi5^2}{2}, \frac{1}{2}\left(2-\phi5^2\right), -\phi5\right\}, \{-\phi5, \phi5, 1\}\right\}$**]]%**

FullSimplify[Sqrt[Tr[
$\{\{\frac{1}{2}\left(\left(2+\phi5^2\right)\text{Cosh[t10]}+ \phi5^2\text{Sinh[t10]}\right), -\frac{1}{2}e^{\text{t10}}\phi5^2 + \text{Sinh[t10]}, -e^{\text{t10}}\phi5\}, \{\frac{e^{\text{t10}}\phi5^2}{2} + \text{Sinh[t10]}, \text{Cosh[t10]}- \frac{1}{2}\phi5^2\text{Cosh[t10]}- \frac{1}{2}\phi5^2\text{Sinh[t10]}, -e^{\text{t10}}\phi5\}, \{-\phi5, \phi5, 1\}\}.$
Transpose[
$\{\{\frac{1}{2}\left(\left(2+\phi5^2\right)\text{Cosh[t10]}+ \phi5^2\text{Sinh[t10]}\right), -\frac{1}{2}e^{\text{t10}}\phi5^2 + \text{Sinh[t10]}, -e^{\text{t10}}\phi5\}, \{\frac{e^{\text{t10}}\phi5^2}{2} + \text{Sinh[t10]}, \text{Cosh[t10]}-$

$\frac{1}{2}\phi5^2$Cosh[t10]−
$\frac{1}{2}\phi5^2$Sinh[t10],
$-e^{t10}\phi5\}$,
$\{-\phi5, \phi5, 1\}\}]]]]$
$\sqrt{}\left(1 + e^{-2t10} + 2\phi5^2 + e^{2t10}\left(1 + \phi5^2\right)^2\right)$
Limit[
$\sqrt{}\left(1 + e^{-2t10} + 2\phi5^2+\right.$
$\left.e^{2t10}\left(1 + \phi5^2\right)^2\right)$,
t10 → Infinity]
$\sqrt{\left(1 + \phi5^2\right)^2}\infty$
Limit $\left[\sqrt{\left(1 + \phi5^2\right)^2}\infty,\right.$
$\phi5 \to$ Infinity]
∞

C.4

The codes used in **Chapter 5**

Code used to express the trajectories in THEOREM 5.2.4

Solve[ϕ3P1 + ϕ1==(1/c1)P2, ϕ1]
Solve[ϕ3P2 + ϕ2Cos[ϕ1]==(1/c2)P1, ϕ2]

$\left\{\left\{\phi1 \to \frac{\text{P2}-\text{c1P1}\phi3}{\text{c1}}\right\}\right\}$
$\left\{\left\{\phi2 \to \frac{(\text{P1}-\text{c2P2}\phi3)\text{Sec}[\phi1]}{\text{c2}}\right\}\right\}$
Solve $\left[\left(\frac{(\text{P1}-\text{c2P2}\phi3)\text{Sec}[\phi1]}{\text{c2}}\right)\text{Sin}[\phi1] + \phi3\text{P3}==0, \phi3\right]$
$\left\{\left\{\phi3 \to -\frac{\text{P1Tan}[\phi1]}{\text{c2(P3}-\text{P2Tan}[\phi1])}\right\}\right\}$
FullSimplify $\left[\frac{\text{P2}-\text{c1P1}\left(\frac{(\text{P1}-\text{c2P2}\phi3)\text{Sec}[\phi1]}{\text{c2}}\right)}{\text{c1}}\right]$
$\frac{\text{P2}}{\text{c1}} + \frac{\text{P1}(-\text{P1}+\text{c2P2}\phi3)\text{Sec}[\phi1]}{\text{c2}}$
FullSimplify $\left[\frac{\left(\text{P1}-\text{c2P2}\left(-\frac{\text{P1Tan}[\phi1]}{\text{c2(P3}-\text{P2Tan}[\phi1])}\right)\right)\text{Sec}[\phi1]}{\text{c2}}\right]$
$\frac{\text{P1P3Sec}[\phi1]}{\text{c2P3}-\text{c2P2Tan}[\phi1]}$

Code used to express the matrix exponential in THEOREM 5.3.4

MatrixExp[{{(1/c1)P1, 0}, {0, (−1/c2)P1}}]

$\left\{\left\{e^{\frac{\text{P1}}{\text{c1}}}, 0\right\}, \left\{0, e^{-\frac{\text{P1}}{\text{c2}}}\right\}\right\}$

Code used to express the trajectories in THEOREM 5.2.4

Solve[ϕ3P1 + ϕ1==0, ϕ1]
Solve[ϕ3P2 + ϕ2Cos[ϕ1]==(1/c2)P2, ϕ2]

$\{\{\phi1 \to -\text{P1}\phi3\}\}$
$\left\{\left\{\phi2 \to -\frac{(-\text{P2}+\text{c2P2}\phi3)\text{Sec}[\phi1]}{\text{c2}}\right\}\right\}$
Solve[ϕ2Sin[ϕ1] + ϕ3P3== (1/c1)P3, ϕ3]
$\left\{\left\{\phi3 \to \frac{\text{P3}-\text{c1}\phi2\text{Sin}[\phi1]}{\text{c1P3}}\right\}\right\}$
FullSimplify $\left[-\text{P1}\left(\frac{\text{P3}-\text{c1}\phi2\text{Sin}[\phi1]}{\text{c1P3}}\right)\right]$
FullSimplify $\left[-\frac{\left(-\text{P2}+\text{c2P2}\left(\frac{\text{P3}-\text{c1}\phi2\text{Sin}[\phi1]}{\text{c1P3}}\right)\right)\text{Sec}[\phi1]}{\text{c2}}\right]$
$-\frac{\text{P1}}{\text{c1}} + \frac{\text{P1}\phi2\text{Sin}[\phi1]}{\text{P3}}$
$\text{P2}\left(\left(-\frac{1}{\text{c1}} + \frac{1}{\text{c2}}\right)\text{Sec}[\phi1] + \frac{\phi2\text{Tan}[\phi1]}{\text{P3}}\right)$

C.5

A sample of the codes used to generate figures D1 to D15

Code used to generate Figure D1

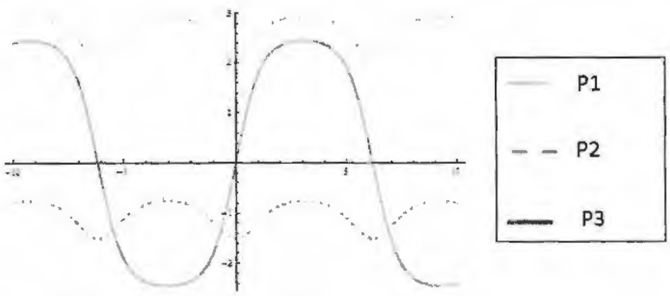
c1 = 0.5;
c2 = 6;
h = 1;
k = 1;
Cap = ParametricPlot3D[
$\left\{\sqrt{\text{Abs}[k]}\text{Sinh}[u],\right.$
$\sqrt{\text{Abs}[k]}\text{Cosh}[u]\text{Cos}[\theta]$,
$\left.\sqrt{\text{Abs}[k]}\text{Cosh}[u]\text{Sin}[\theta]\right\}$,
$\{\theta, -\pi, \pi\}, \{u, -2, 2\}$,

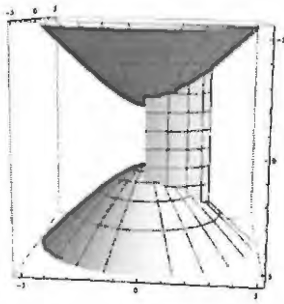
```
Mesh → 10];
HHa = ParametricPlot3D[
{z, √hc1 Cos[θ], √hc2 Sin[θ]},
{θ, 0, π}, {z, −2.5, 2.5},
Mesh → 8];
H = ParametricPlot3D[
{√hc1 Cos[θ], √hc2 Sin[θ], z},
{θ, −π, π}, {z, −2.5, 2.5},
Mesh → 8];

VView = {2Pi, Pi/4, 0};
VViewv = {1, 1, 1};
Opts = {ViewVertical → VViewv,
ViewPoint → VView, Axes → True,
BoxRatios → {1, 1, 1},
PlotRange →
{{−3, 3}, {−3, 3}, {−3, 3}},
Boxed → False,
ImageSize → Medium,
AxesLabel → {"E1*", "E2*", "E3*"},
LabelStyle →
Directive[Medium]};
Show[Cap, H]
    "Case 1 - K < 0"
"B^2" Plot[2t-2,{t,-10,10}]
"A^2"Plot[2t + 2,
{t,-10,10}]
Plot[{2 -t,2t + 2},
{t,-10,10}]
c1 = 2
c2 = 5
H = 1
K=- 1
a = Sqrt[2 H c2 + 2 K]
b = Sqrt[2H c1 - 2K]
```

Appendix D

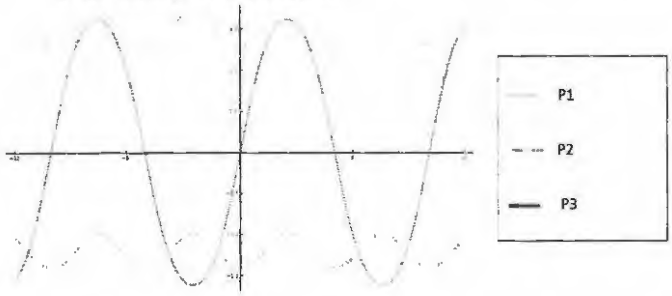
Figures

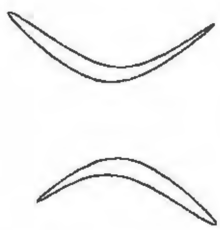
Here we include Mathematica plots of the Hamiltonian and Casimir functions of the cases 1-3b of THEOREM 5.2.2 and 1-3d of THEOREMS 5.3.2-5.3.3. For each case, we also plot the solution using Mathematica's functions `JacobiSN[]`, `JacobiND[]` and `JacobiDC[]`, and then use the function `NDSolve[]` to give the numerical solution of the appropriate reduced extremal equations. A sample of the codes used to produce the plots is given in (C.5).





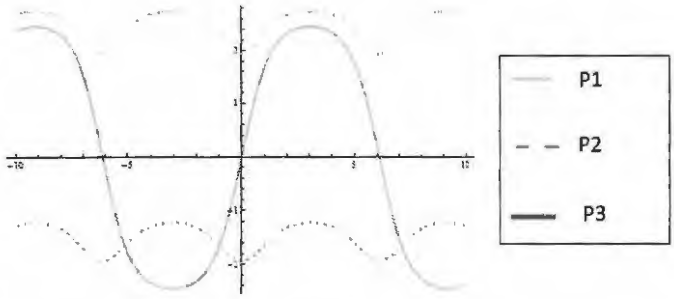
Plot using `JacobiSN[]` Intersection of $\mathcal{H}$ and K

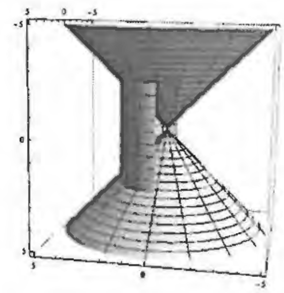




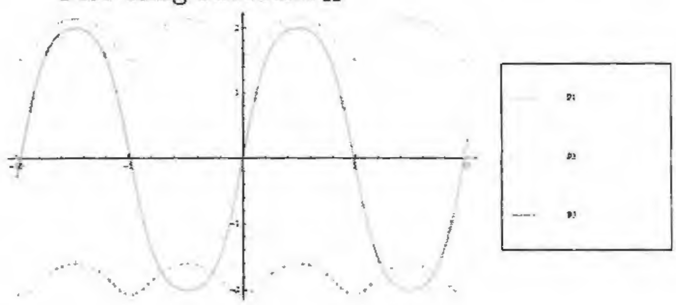
Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

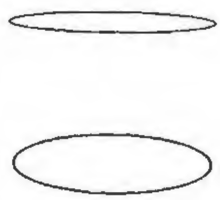
Figure D.1: Case 1 ($K < 0$) of THEOREM 5.2.2. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.





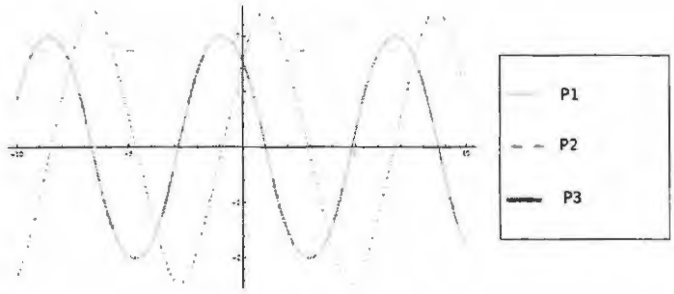
Plot using `JacobiSN[]` Intersection of $\mathcal{H}$ and K

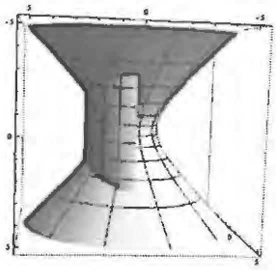




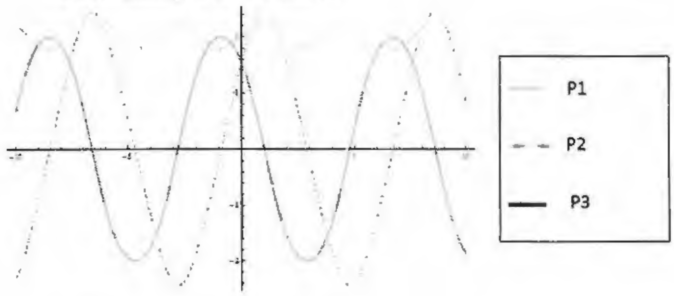
Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

Figure D.2: Case 2 ($K = 0$) of THEOREM 5.2.2. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.



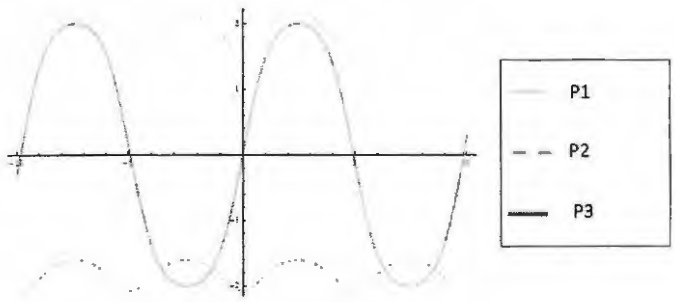


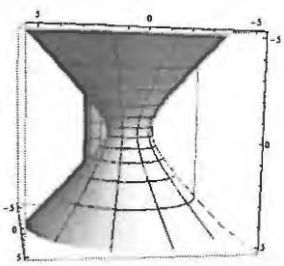
Plot using `JacobiND[]` Intersection of $\mathcal{H}$ and K



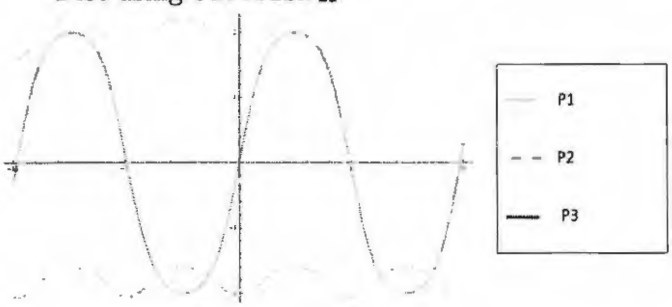
Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

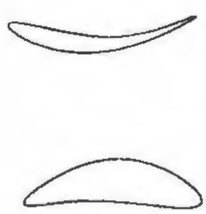
Figure D.3: Case 3a ($K > 0$, $\left(\frac{c_1+c_2}{c_1}P_1^2\right) < P_3^2$) of THEOREM 5.2.2. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.





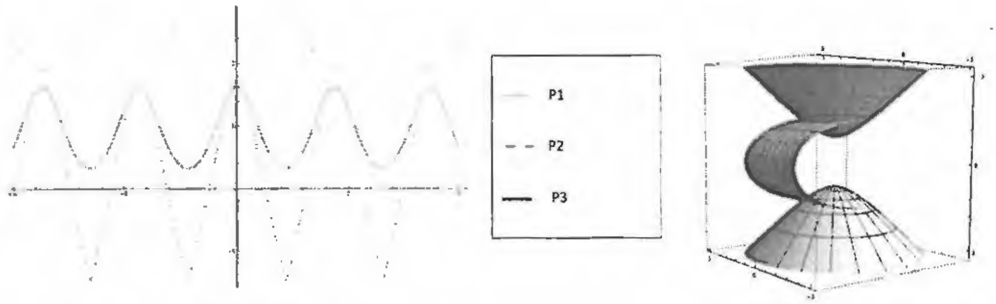
Plot using `JacobiSN[]` Intersection of $\mathcal{H}$ and K



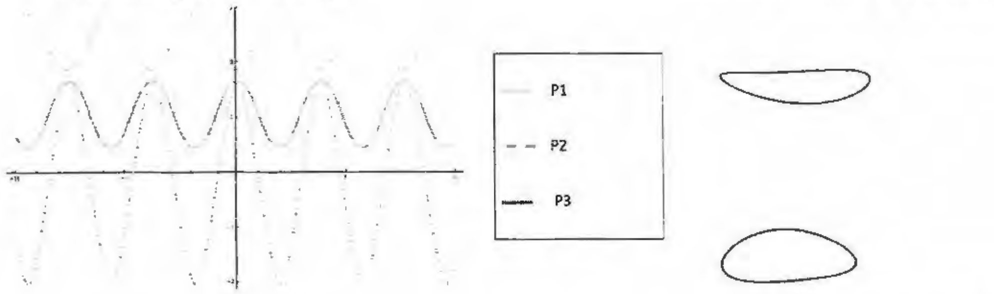


Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

Figure D.4: Case 3b ($K > 0$, $\left(\frac{c_1+c_2}{c_1}P_1^2\right) > P_3^2$) of THEOREM 5.2.2. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.

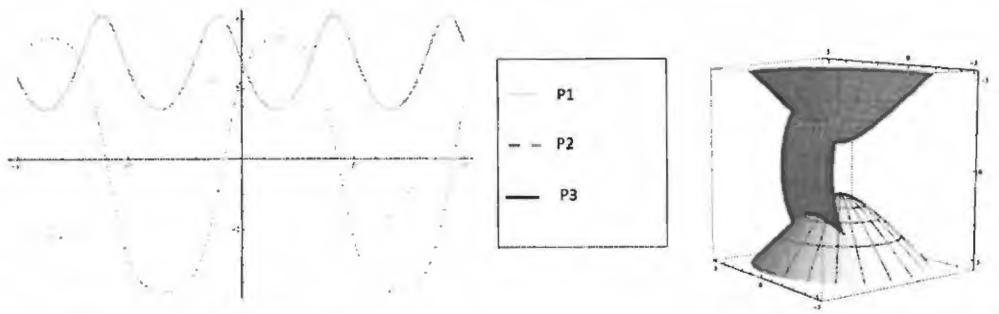


Plot using `JacobiND[]` Intersection of $\mathcal{H}$ and K

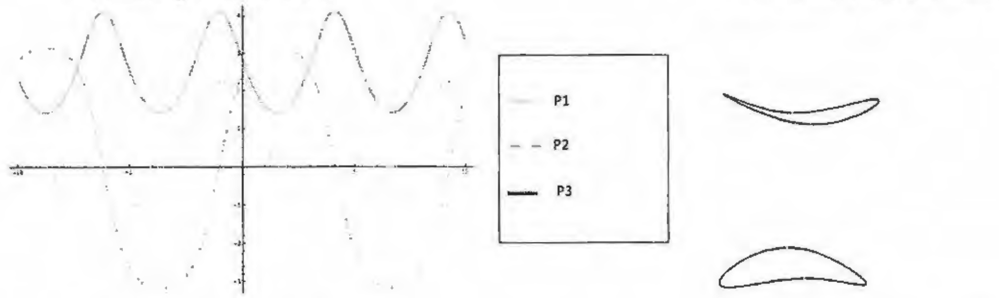


Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

Figure D.5: Case 1a ($K < 0$, $c_2 > c_1$ $\left(\frac{c_1-c_2}{c_1}P_3^2\right) < P_1^2$ $\left(\frac{c_2-c_1}{c_1}P_1^2\right) < P_1^2$) of THEOREM 5.3.3. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.

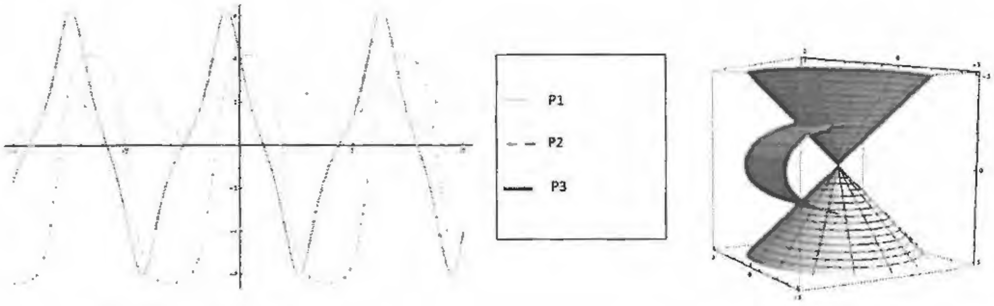


Plot using `JacobiND[]` Intersection of $\mathcal{H}$ and K

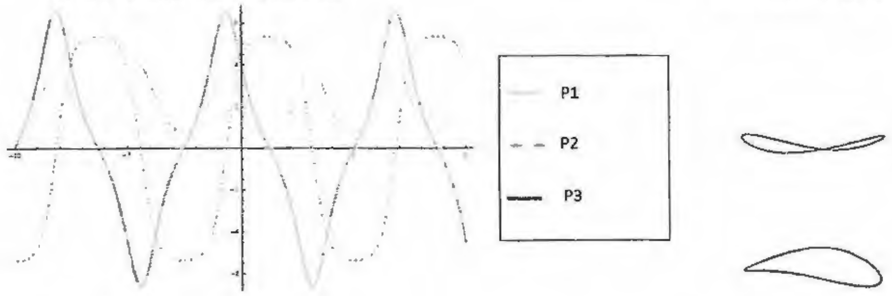


Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

Figure D.6: Case 1b ($K < 0$, $c_1 > c_2$ $\left(\frac{c_2-c_1}{c_1}P_3^2\right) < P_1^2$ $\left(\frac{c_1-c_2}{c_1}P_1^2\right) < P_1^2$) of THEOREM 5.3.3. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.

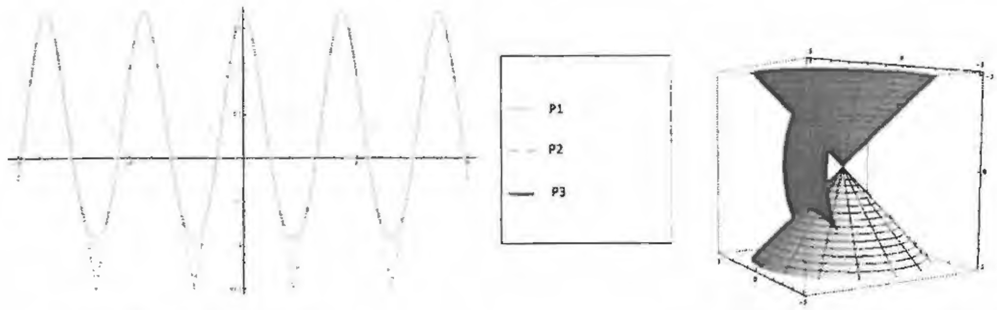


Plot using `JacobiND[]` Intersection of $\mathcal{H}$ and K

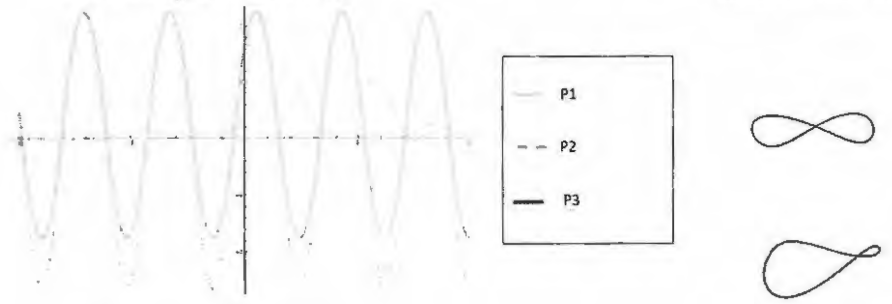


Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

Figure D.7: Case 2a ($K = 0$, $c_2 > c_1 \left(\frac{c_1-c_2}{c_1}P_3^2\right) < P_1^2 \left(\frac{c_2-c_1}{c_1}P_1^2\right) < P_1^2$) of THEOREM 5.3.3. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.

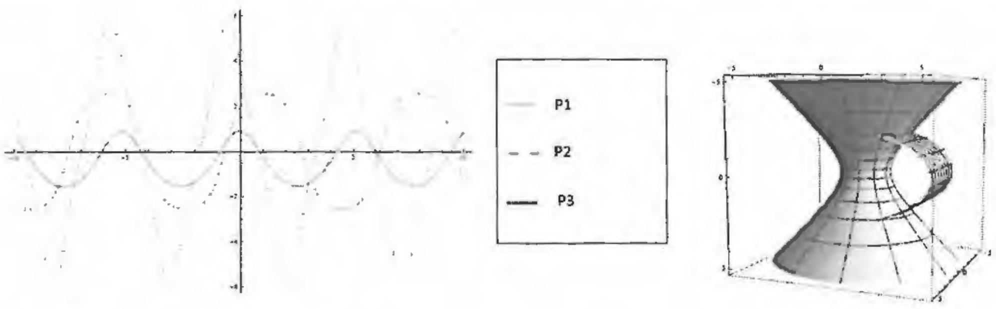


Plot using `JacobiND[]` Intersection of $\mathcal{H}$ and K

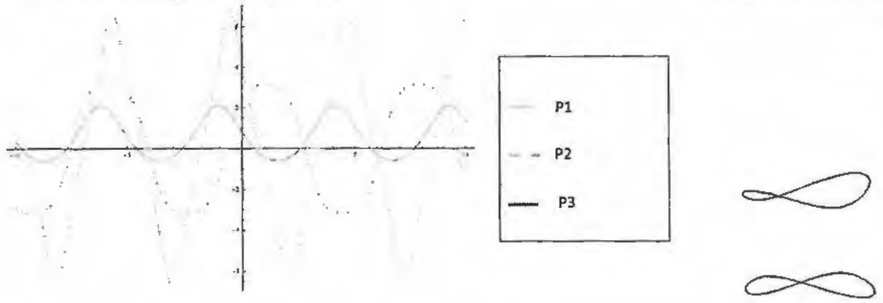


Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

Figure D.8: Case 2b ($K = 0$, $c_1 > c_2 \left(\frac{c_2-c_1}{c_1}P_3^2\right) < P_1^2 \left(\frac{c_1-c_2}{c_1}P_1^2\right) < P_1^2$) of THEOREM 5.3.3. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.

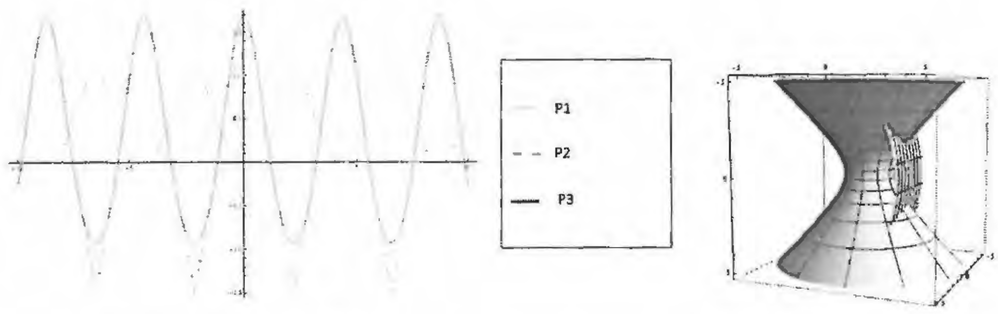


Plot using `JacobiND[]` Intersection of $\mathcal{H}$ and K

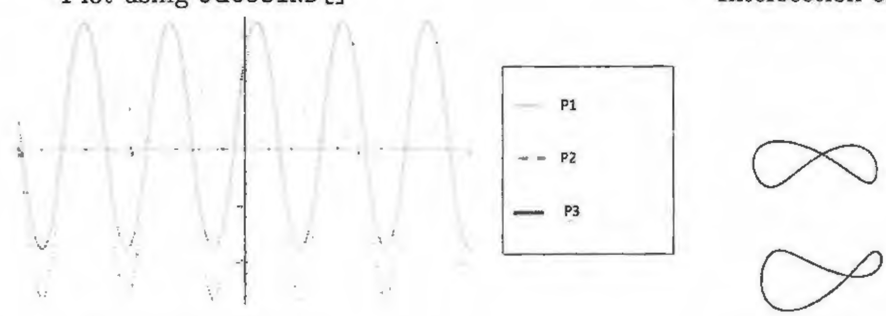


Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

Figure D.9: Case 3a $(K > 0,\, c_2 > c_1 \left(\frac{c_1-c_2}{c_1}P_3^2\right) < P_1^2 \left(\frac{c_2-c_1}{c_1}P_1^2\right) < P_1^2)$ of THEOREM 5.3.3. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.

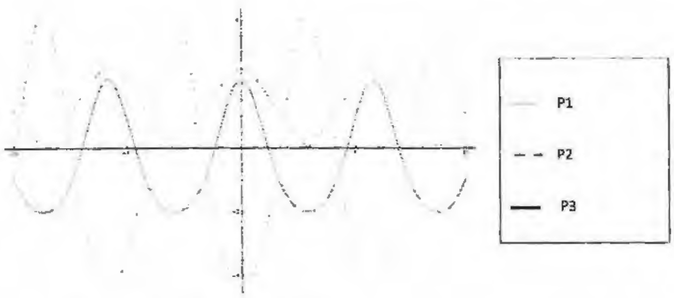


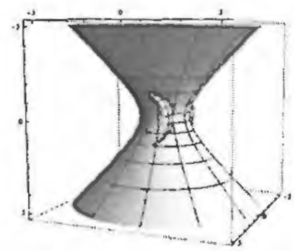
Plot using `JacobiND[]` Intersection of $\mathcal{H}$ and K



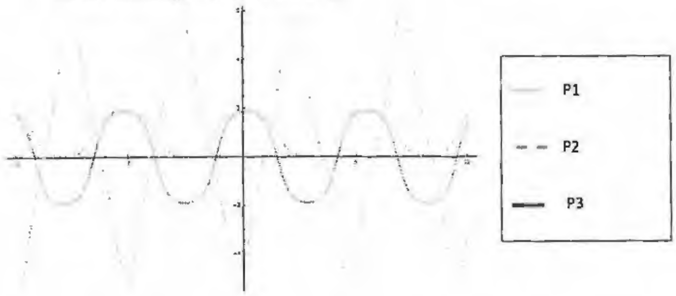
Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

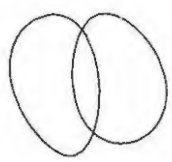
Figure D.10: Case 3b $(K > 0,\, c_1 > c_2 \left(\frac{c_2-c_1}{c_1}P_3^2\right) < P_1^2 \left(\frac{c_1-c_2}{c_1}P_1^2\right) < P_1^2)$ of THEOREM 5.3.3. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.





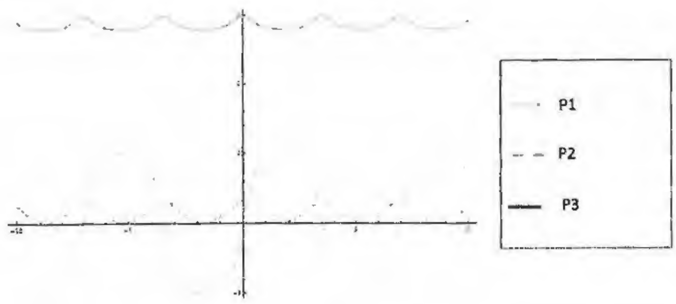
Plot using `JacobiDC[]` Intersection of $\mathcal{H}$ and K

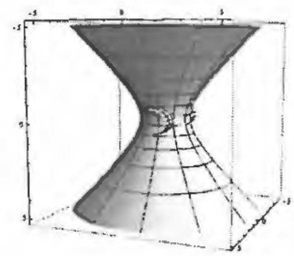




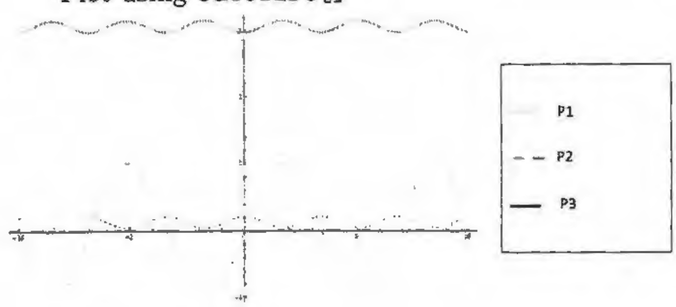
Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

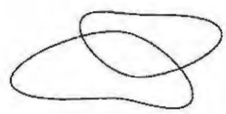
Figure D.11: Case 3c ($K > 0$, $c_1 > c_2$ $\left(\frac{c_2-c_1}{c_1}P_3^2\right) > P_1^2$ $\left(\frac{c_1-c_2}{c_1}P_1^2\right) < P_1^2$) of THEOREM 5.3.3. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.





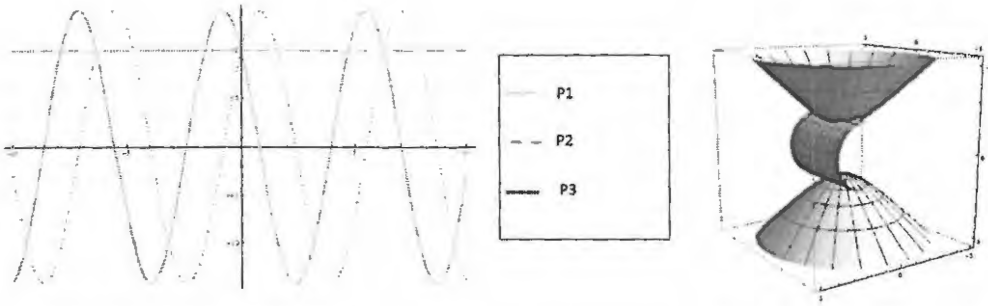
Plot using `JacobiDC[]` Intersection of $\mathcal{H}$ and K



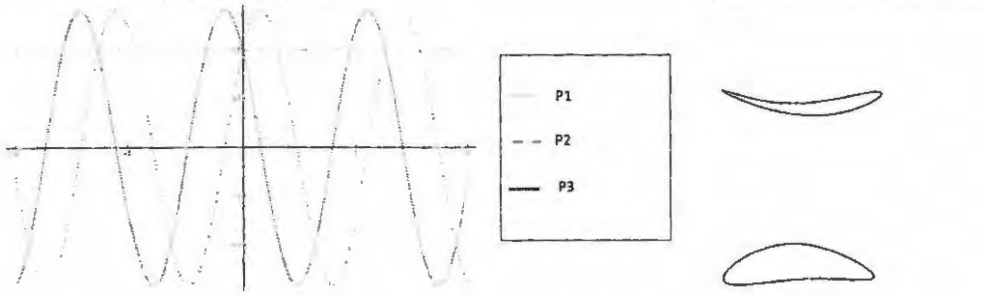


Plot using `NDSolve[]` Intersection of $\mathcal{H}$ and K

Figure D.12: Case 3d ($K > 0$, $c_1 > c_2$ $\left(\frac{c_2-c_1}{c_1}P_3^2\right) > P_1^2$ $\left(\frac{c_1-c_2}{c_1}P_1^2\right) < P_1^2$) of THEOREM 5.3.3. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.

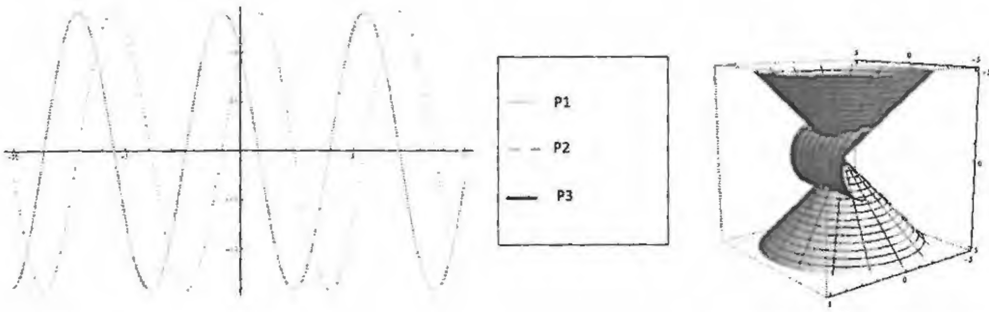


Plot using Cos[]/Sin[] Intersection of $\mathcal{H}$ and K

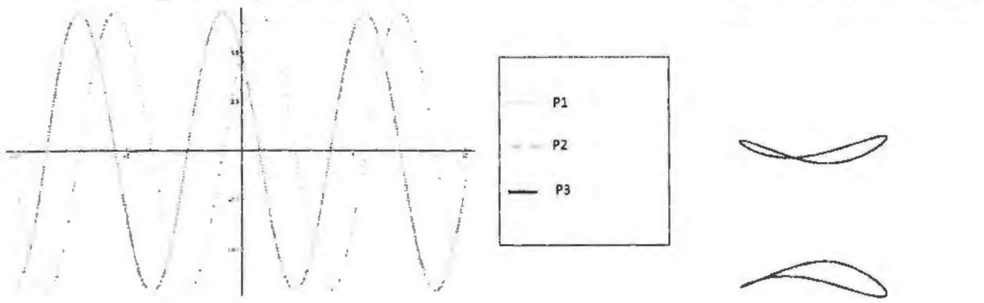


Plot using NDSolve[] Intersection of $\mathcal{H}$ and K

Figure D.13: Case 1 ($K < 0$) of THEOREM 5.3.2. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.

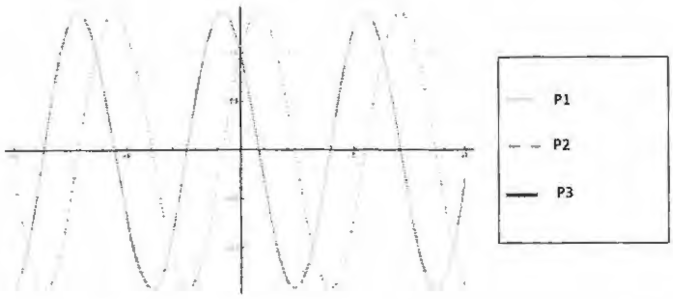


Plot using Cos[]/Sin[] Intersection of $\mathcal{H}$ and K

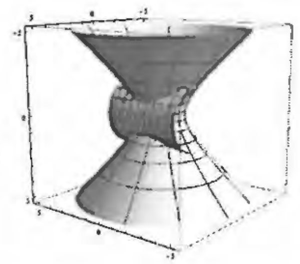


Plot using NDSolve[] Intersection of $\mathcal{H}$ and K

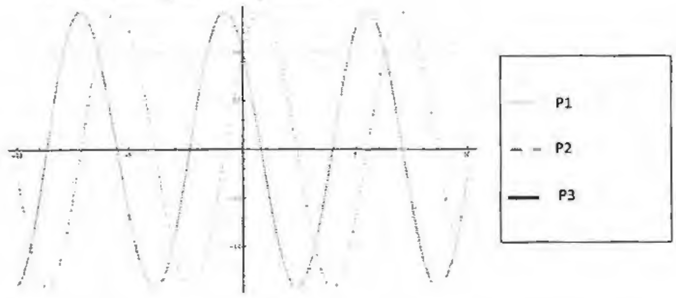
Figure D.14: Case 2 ($K = 0$) of THEOREM 5.3.2. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.



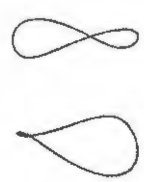
Plot using Cos[]/Sin[]



Intersection of $\mathcal{H}$ and K



Plot using NDSolve[]



Intersection of $\mathcal{H}$ and K

Figure D.15: Case 3 ($K > 0$) of THEOREM 5.3.2. The Hamiltonian $\mathcal{H}$ is shown over half its period of revolution.