

# How many quasiplatonic surfaces?

Jan–Christoph Schlage–Puchta, Jürgen Wolfart

Mathematisches Institut der Universität, Eckerst. 1, D–79104 Freiburg

Mathematisches Seminar der Universität, Postfach 11 19 23, D–60054 Frankfurt a.M.

jcp@math.uni-freiburg.de    wolfart@math.uni-frankfurt.de

## Abstract

We show that the number of isomorphism classes of quasiplatonic Riemann surfaces of genus  $\leq g$  has a growth of type  $g^{\log g}$ . The number of non-isomorphic regular dessins of genus  $\leq g$  has the same growth type.

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Quasiplatonic Riemann surfaces  $X$  of genus  $g > 1$  can be characterized in many different ways (see e.g. [Wo2, Thm. 4], or [Si2]), e.g. by the property that the orders of their automorphism groups are isolated local maxima in the corresponding moduli space of Riemann surfaces of genus  $g$ . For the present paper we will only use the equivalent statement that their universal covering groups  $\Gamma$  are torsion free normal subgroups of finite index in some Fuchsian triangle groups  $\Delta = \Delta(p, q, r)$  of signatures  $(p, q, r)$ , and that conversely the quotient  $\Gamma \backslash \mathbf{H}$  of the upper half plane  $\mathbf{H}$  by any finite index torsion free normal subgroup  $\Gamma$  of a Fuchsian triangle group is a quasiplatonic surface  $X$ . The dessin it carries can be described as the quotient by  $\Gamma$  of a  $\Delta$ -invariant tessellation of  $\mathbf{H}$ , with  $\Delta/\Gamma$  acting as a group of automorphisms on  $X$  and on the dessin.

Notations: Let  $R(g; p, q, r)$  be the number of normal torsion free subgroups  $\Gamma$  of genus  $g$  in  $\Delta = \Delta(p, q, r)$ , and let

$$R(g) := \sum_{p,q,r} R(g; p, q, r), \quad S(g) := \sum_{1 < \gamma \leq g} R(\gamma)$$

be the number of all non-isomorphic regular dessins of this genus  $g$  and its summatory function. We will show first

**Theorem 1** *There are constants  $g_0, c_1, c_2 > 0$  such that for all genera  $g > g_0$*

$$g^{c_1 \log g} < S(g) < g^{c_2 \log g}.$$

We will use the following result of T. Müller and the first named author (a simplified version of [MSP, Theorem 1]). For a group  $\Gamma$ , denote by  $s_n^{\triangleleft}(\Gamma)$  the number of normal subgroups of index  $n$  in  $\Gamma$ .

**Lemma 1** *Let  $\Gamma$  be a finitely generated group, possessing a normal subgroup  $N$  of finite index which maps surjectively onto a non-abelian free group. Then we have for  $n > n_0$  the estimate  $\sum_{\nu \leq n} s_{\nu}^{\triangleleft}(\Gamma) \geq n^{c \log n}$ , where  $n_0$  and  $c$  are positive constants depending on  $\Gamma$ .*

Every Fuchsian triangle group has a torsion free normal subgroup of finite index, which is necessarily a surface group with at least 4 generators. Since an orientable surface group with  $2d$  generators maps surjectively onto a free group with  $d$  generators, Lemma 1 can be applied to all Fuchsian triangle groups.

We now turn to the proof of Theorem 1. To obtain the lower bound, take three different primes  $p, q, r$  giving a Fuchsian triangle group  $\Delta = \Delta(p, q, r)$  with presentation

$$\Delta = \langle \gamma_0, \gamma_1 \mid \gamma_0^p = \gamma_1^q = (\gamma_0 \gamma_1)^r = 1 \rangle.$$

All normal subgroups of index  $n > 1$  are torsion free. By the Riemann–Hurwitz formula, their genus  $g$  is related to  $n$  via

$$84(g-1) \geq n = (2g-2) \left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right)^{-1} > 2g-2,$$

and by Lemma 1 we have a lower bound for the summatory growth function

$$\sum_{1 < 2(\gamma-1) < n} R(\gamma; p, q, r) \geq |\{\Gamma \triangleleft \Delta(p, q, r) \mid 1 < (\Delta(p, q, r) : \Gamma) \leq n\}| > n^{c_1 \log n}$$

for all  $n > n_0$  for some  $n_0$  depending on  $p, q, r$ . Taking only that term in the sum  $R(\gamma) = \sum_{p, q, r} R(\gamma; p, q, r)$  coming from the triangle group  $\Delta = \Delta(p, q, r)$  for the chosen prime triple signature we obtain

$$\begin{aligned} S(g) &\geq \sum_{1 < \gamma \leq g} R(\gamma; p, q, r) \geq |\{\Gamma \triangleleft \Delta \mid 1 < (\Delta : \Gamma) \leq 2(g-1)\}| \\ &\geq |\{\Gamma \triangleleft \Delta \mid 1 < (\Delta : \Gamma) \leq g\}| > g^{c_1 \log g} \end{aligned}$$

for all  $g \geq n_0$ .

To prove the upper bound, recall Lubotzky’s estimate  $\nu^{6(\Omega(\nu)+1)}$  for the number of index  $\nu$  normal subgroups in the free group with two generators ([LS, Thm. 2.7]) where  $\Omega(\nu)$  denotes the number of prime divisors of  $\nu$  counted with multiplicities. For any fixed Fuchsian triangle group  $\Delta = \Delta(p, q, r)$  it implies

$$|\{\Gamma \triangleleft \Delta \mid (\Delta : \Gamma) \leq n\}| \leq \sum_{\nu \leq n} \nu^{6(\Omega(\nu)+1)} < n^{c_3 \log n}$$

for some constant  $c_3$ , since  $\Omega(\nu) \leq \log_2 \nu$ . This upper bound is *a fortiori* valid for the torsion free normal subgroups, hence

$$\sum_{1 < \gamma \leq g} R(\gamma; p, q, r) < (84g)^{c_3 \log(84g)}.$$

Since  $\Delta/\Gamma$  has generators of orders  $p, q, r$ , we have moreover  $R(\gamma; p, q, r) = 0$  for  $p, q$  or  $r > 84g$  ( $> |\Delta/\Gamma|$ ), therefore

$$S(g) = \sum_{1 < \gamma \leq g} R(\gamma) < \sum_{p, q, r \leq 84g} (84g)^{c_3 \log(84g)} < (84g)^{3+c_3 \log(84g)} < g^{c_2 \log g}$$

for all  $g > g_0$  with suitable  $c_2$  and  $g_0$ .

**Theorem 2** *Let  $Q(g)$  denote the number of isomorphism classes of quasiplatonic Riemann surfaces of genera  $\gamma$ ,  $1 < \gamma \leq g$ . With the same constants  $g_0, c_1, c_2 > 0$  as in Theorem 1 we have for all  $g > g_0$*

$$\frac{1}{120} g^{c_1 \log g} < Q(g) < g^{c_2 \log g}.$$

*Proof.* Since every quasiplatonic surface is obtained from a regular dessin (equivalently, from a torsion free normal subgroup in a triangle group), and is uniquely determined by that dessin, the upper bound follows from Theorem 1. The lower bound follows similarly, but a quasiplatonic surface can be obtained by up to five different types of regular dessins ([Gi]), and another overcount can happen: in a fixed triangle group  $\Delta = \Delta(p, q, r)$  several torsion free normal subgroups can be  $\mathrm{PSL}_2(\mathbf{R})$ -conjugate, leading to isomorphic surfaces. In [GW], Thms. 5, 6, 7, it is shown that such conjugations take place in a finite extension of  $\Delta$  which is again a triangle group. By Singerman's work [Si1] the maximal possible index between Fuchsian triangle groups is known to be 24 (occurring for  $\Delta(2, 3, 7) \supset \Delta(7, 7, 7)$ ), so we have at most 120 normal subgroups counted in the proof of Theorem 1 leading to isomorphic surfaces (by a more detailed analysis, this number can considerably decreased). This gives the lower bound for  $Q(g)$ .

Another consequence of Thm. 1 is

**Theorem 3** *With the same constants  $g_0, c_1, c_2 > 0$  as in Theorem 1, the number of non-isomorphic regular dessins in genus  $g > g_0$  is  $R(g) < g^{c_2 \log g}$ . Infinitely often we have*

$$g^{-1+c_1 \log g} < R(g).$$

*An analogous statement holds for the number of quasiplatonic surfaces of genus  $g$ .*

*Remarks.* 1) From the tables in [Wo1, Sec. 6] of regular dessins in genera  $g \leq 4$  and work of Kuribayashi and Kimura [KK] for  $g = 5$  one may deduce

$$S(5) = 104 \quad \text{and} \quad Q(5) = 37 ,$$

to be compared with  $5^{\log 5} \approx 13$ . [GAP] calculations [Sc] indicate that also for  $5 < g \leq 10$  one has always

$$g^{\log g} < S(g) < g^{2 \log g} .$$

2) Counting regular dessins in genera 0 and 1 is different from higher genera. In genus 0 the Riemann sphere is the only surface, however having an infinity of regular dessins defined by the cyclic and dihedral triangle groups of signatures  $(1, n, n)$ ,  $(2, 2, n)$  and those corresponding to the platonic bodies, i.e.  $(2, 3, 3)$ ,  $(2, 3, 4)$ ,  $(2, 3, 5)$ .

In genus 1 the triangle groups of signatures  $(3, 3, 3)$ ,  $(2, 3, 6)$ ,  $(2, 4, 4)$  have infinitely many torsion free normal subgroups of finite index, acting by translations on the complex plane. As quotients one obtains infinitely many non-isomorphic elliptic curves with regular dessins, but all fall in two isogeny classes only (see [SiS]) having complex multiplication by the fields of third or fourth roots of unity, respectively.

3) The most famous quasiplatonic surfaces are the *Hurwitz curves* whose automorphism groups attain the (according to Hurwitz) maximal possible order  $84(g-1)$ . Their universal covering groups are the torsion free normal subgroups of finite index in  $\Delta(2, 3, 7)$ . By the same or even easier arguments as above one can deduce that the number of non-isomorphic Hurwitz curves of genera  $\gamma \leq g$  lies between  $g^{c_4 \log g}$  and  $g^{c_5 \log g}$  for all  $g \geq g_1$  with suitable constants  $g_1, c_4, c_5$ .

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