TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 5, Pages 1749–1757
S 0002-9947(02)02920-3
Article electronically published on January 7, 2002

ON THE STRUCTURE OF $P(n)_*P((n))$ FOR $p = 2$

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Abstract. We show that $P(n)_*(P(n))$ for $p = 2$ with its geometrically induced structure maps is *not* an Hopf algebroid because neither the augmentation ϵ nor the coproduct Δ are multiplicative. As a consequence the algebra structure of $P(n)_*(P(n))$ is slightly different from what was supposed to be the case. We give formulas for $\epsilon(xy)$ and $\Delta(xy)$ and show that the inversion of the formal group of $P(n)$ is induced by an antimultiplicative involution $\Xi : P(n) \to P(n)$. Some consequences for multiplicative and antimultiplicative automorphisms of $K(n)$ for $p = 2$ are also discussed.

1. Introduction and statement of the results

Let BP denote the Brown-Peterson spectrum for $p = 2$ and recall that $BP_* = \mathbf{Z}_{(2)}[v_1, v_2, \dots]$ with $|v_n| = 2(2^n - 1)$. As usual we let $v_0 = 2$. BP is a commutative ring spectrum and there are related ring spectra $P(n) = BP/(v_0, v_1, \dots, v_{n-1})$, $n \geq 1$, which are connected by the Baas-Sullivan cofiber sequences

$$\Sigma^{|v_n|}P(n) \xrightarrow{\cdot v_n} P(n) \xrightarrow{\eta_n} P(n+1) \xrightarrow{\partial_n} \Sigma^{|v_n|+1}P(n).$$

From the Baas-Sullivan cofiber triangles one obtains Bockstein operations $Q_{n-1} = \partial_{n-1}\eta_{n-1} : P(n) \to \Sigma^{|v_{n-1}|+1}P(n)$.

There are essentially two multiplications m, $\overline{m}$ on $P(n)$ that are worth consideration. These are characterised recursively by the fact that they make $P(n)$ into a $P(n-1)$ algebra spectrum (cf. [W1] with corrections in [N]). Here we let $P(0) = BP$. Both of them are noncommutative: one has $\overline{m} = mT$, where $T : P(n) \wedge P(n) \to P(n) \wedge P(n)$ is the switch map. More explicitly their relation is given by (cf. [M])

$$\overline{m} = m + v_n m(Q_{n-1} \wedge Q_{n-1}). \tag{1}$$

Furthermore Q_{n-1} is a derivation with respect to both products. For all this the reader is referred to [R1], [R2], [W1] and [W2].

We prove the following lemma in section 2 which also contains a formula for $\Delta(xy)$.

Lemma 1. *Let $\epsilon : P(n)_*(P(n)) \to P(n)_*$ be the augmentation defined by $\epsilon = \pi_*(m)$. Then for all $x, y \in P(n)_*(P(n))$ we have*

$$\epsilon(xy) = \epsilon(x)\epsilon(y) + v_n\epsilon(Q_{n-1}x)\epsilon(Q_{n-1}y). \tag{2}$$

Received by the editors July 3, 2000.

1991 *Mathematics Subject Classification.* Primary 55N22; Secondary 55P43.

Key words and phrases. Hopf algebroids, Morava K-theory, bordism theory, noncommutative ring spectra.

Recall from [KW] the elements a_i, $0 \le i < n$, in $P(n)_*(P(n))$ with $|a_i| = 2^{i+1}-1$. Recall also that $BP_*(BP) = BP_*[t_1, t_2, t_3, \dots]$ with $|t_i| = 2(2^i - 1)$. We will denote the image of t_i under the canonical map $BP_*(BP) \to P(n)_*(P(n))$ again by t_i. Let $\overline{P(n)}$ denote the spectrum $P(n)$ with multiplication $\overline{m}$.

Lemma 1 leads immediately to the following modification of the main result of [KW].

Theorem 2 ([KW]). *$P(n)_*(P(n))$ and $P(n)_*(\overline{P(n)})$ are polynomial algebras over $P(n)_*$ on the generators*

$$\{ a_0, a_1, \dots, a_{n-1} \} \cup \{ t_i \mid i > n \}.$$

There are relations

$$a_i^2 = \begin{cases} t_{i+1} + v_{i+1} & \text{in } P(n)_*(P(n)), \\ t_{i+1} & \text{in } P(n)_*(\overline{P(n)}). \end{cases}$$

Here v_k is to be interpreted as 0 if $k < n$.

For commutative ring spectra E and X that satisfy Kronecker duality it is well known that Kronecker duality sets up a one-to-one correspondence between morphisms of ring spectra $X \to E$ and algebra morphisms $E_*(X) \to E_*$. Lemma 1 shows that this is no longer true for the spectra $P(n)$ since the augmentation ϵ is the Kronecker dual of the identity $\mathrm{id} : P(n) \to P(n)$. We investigate this problem in section 3 and prove the following two theorems.

Theorem 3. *There is an antimultiplicative involution $\Xi : P(n) \to P(n)$ which on Euler classes of complex line bundles $\mathcal{L}$ is given by $\mathbf{e}(\mathcal{L}) \mapsto \mathbf{e}(\mathcal{L}^{-1})$.*

Let $K(n)$ denote the nth Morava K-theory at $p = 2$. It inherits two multiplications from $P(n)$ of which each is the opposite of the other. Let Mult^+ (resp. Mult^-) denote the set of all multiplicative (resp. antimultiplicative) automorphisms of $K(n)$. Then $\mathrm{Mult}^\pm := \mathrm{Mult}^+ \cup \mathrm{Mult}^-$ is a group. Recall from [Y] that the map $\mathbf{Z} \ni k \mapsto [k]_{F_n}(x) \in \mathrm{End}(F_n)$ extends to an isomorphism $\mathbf{Z}_2 \cong \mathrm{End}(F_n)$, where F_n is the formal group law associated to $K(n)$, $\mathrm{End}(F_n)$ is the endomorphism ring of F_n (considered as a formal group over the *graded* ring $K(n)_*$) and $\mathbf{Z}_2$ is the ring of 2-adic integers.

Theorem 4. *The canonical map $\mathrm{Mult}^\pm \to \mathrm{Aut}(F_n)$ is an isomorphism of groups. Mult^+ (resp. Mult^-) corresponds to those $x \in \mathbf{Z}_2^*$ that are congruent to 1 (resp. -1) modulo 4.*

This paper is a condensed version of my Diplomarbeit [N] which I wrote under the supervision of Professor R. Kultze. I want to take this opportunity to thank him for paying close attention to certain parts of that work. I also thank the Studienstiftung des Deutschen Volkes for support during my studies. Finally, my thanks also go to Andrey Lazarev and the referee for pointing out some minor inconsistencies.

2. Proofs

Regard n as fixed and write v instead of v_n, Q instead of Q_{n-1}. Choose once and for all an admissible multiplication $m : P(n) \wedge P(n) \to P(n)$. We will repeatedly make use of the following fact, mostly without explicitly mentioning it.

Useful fact. Left and right multiplication by v, denoted L_v and R_v, agree, that is we have $m(v \wedge \mathrm{id}) = m(\mathrm{id} \wedge v)$ as maps $P(n) \to P(n)$.

Proof. Suppressing the coherent chain of identifications $S \wedge P(n) \cong P(n) \cong P(n) \wedge S \cong S \wedge P(n)$ from the notation we can write $m(v \wedge \mathrm{id}) = mT(\mathrm{id} \wedge v) = m(\mathrm{id} \wedge v) + vm(Q \wedge Qv)$. But $Qv = 0$ since v is in the image of the canonical map $BP_* \to P(n)_*$ and $Q = \eta_{n-1}\partial_{n-1}$ annihilates this (∂_{n-1} already does). □

The benefit of this observation is that we don't have to worry about the noncomutativity of the multiplication as long as one factor is v_n.

We can now prove Lemma 1. The proof is straightforward: just draw the relevant diagrams and check that they commute. This diagramatic reasoning is, however, not very enlightening. (Such proofs were given in [N], and anybody who had a glimpse of that will agree instantly. This is especially true for the proof of Lemma 5 which would require at least A3 paper.) The simple facts behind it become much clearer when written out using fictitious elements of our spectra as placeholders as in the following

Proof of Lemma 1. Recall that the Pontrjagin product of $x, y \in P(n)_*(P(n))$ is the composite of $x \wedge y : S \cong S \wedge S \to P(n) \wedge P(n) \wedge P(n) \wedge P(n)$ with

$$P(n) \wedge P(n) \wedge P(n) \wedge P(n) \xrightarrow{\mathrm{id} \wedge T \wedge \mathrm{id}} P(n) \wedge P(n) \wedge P(n) \wedge P(n)$$

and

$$P(n) \wedge P(n) \wedge P(n) \wedge P(n) \xrightarrow{m \wedge m} P(n) \wedge P(n).$$

So $\epsilon(xy) = \pi_*(m)(xy)$ is naturally given as a composite of $x \wedge y : S \to P(n)^{\wedge 4}$ and a certain map ϕ from $P(n)^{\wedge 4}$ to $P(n)$ which is made up of multiplications and transpositions only. Fictitious elements $a, b, c, d \in P(n)$ can be used to keep track of these multiplications and transpositions; then, for example, ϵ is given by $(a, b) \mapsto ab$ and the Pontrjagin product of $x = (a, b)$ and $y = (c, d)$ is (ac, bd). In this notation ϕ becomes

$$P(n)^{\wedge 4} \ni (a, b, c, d) \mapsto (a, c, b, d) \mapsto (ac, bd) \mapsto acbd \in P(n).$$

We compare this with $\epsilon(x)\epsilon(y)$. This can be decomposed similarly as $\psi(x \wedge y)$ with $\psi : P(n)^{\wedge 4} \to P(n)$ given by

$$(a, b, c, d) \mapsto (ab, cd) \mapsto abcd.$$

Writing (1) as $cb = bc + vQ(b)Q(c)$ we obtain

$$\phi(a, b, c, d) = \psi(a, b, c, d) + v\psi(a, Q(b), Q(c), d).$$

If one observes that $(a, Q(b)) = Q_*(x)$ and $(Q(c), d) = Q(y)$ this can be rephrased as saying that

$$\epsilon(xy) = \epsilon(x)\epsilon(y) + v_n\epsilon(Q_*(x))\epsilon(Q(y)). \tag{3}$$

This is (2) except that we have $Q_*(x)$ instead of $Q(x)$. But we shall see later that $Q_* : P(n)_*(P(n)) \to P(n)_*(P(n))$ and $Q : P(n)_*(P(n)) \to P(n)_*(P(n))$ are identical, so this finishes the proof. □

The categorically minded reader will probably (and rightly so) frown on these "elements" since obviously their sole purpose is to hide some general nonsense facts about monoidal categories. I could not find a convenient reference, however, and it wouldn't really have improved the exposition.

Next we show how Lemma 1 afflicts the algebra structure of $P(n)_*(P(n))$.

Proof of Theorem 2. Following the proof of Theorem 1.4 in [KW] up to the middle of page 199 one finds that the additive structure of $P(n)_*(P(n))$ and enough of the multiplicative structure have been determined to leave open only the question whether $a_{n-1}^2 = t_n$ or $a_{n-1}^2 = t_n + v_n$. It is shown that

$$a_{n-1}^2 = \begin{cases} t_n & \text{if } \epsilon(a_{n-1}^2) = 0, \\ t_n + v_n & \text{if } \epsilon(a_{n-1}^2) = v_n. \end{cases}$$

In Lemma 2.2 the authors also observed that $Q_{n-1}(a_{n-1}) = 1$, and a similar argument shows that $Q_{n-1*}(a_{n-1}) = 1$, too. From (3) it follows that

$$\epsilon(a_{n-1}^2) = \epsilon(a_{n-1})^2 + v_n \epsilon(Q_{n-1*}(a_{n-1}))\epsilon(Q_{n-1}(a_{n-1})) = v_n,$$

so that $a_{n-1}^2 = t_n + v_n$ in $P(n)_*(P(n))$.

To prove the claim about the structure of $P(n)_*(\overline{P(n)})$ we will first prove that

$$x \,\overline{\star}\, y = x \star y + v_n Q_*(x) \star Q_*(y) \tag{4}$$

for $x, y \in P(n)_*(P(n))$. Here $Q = Q_{n-1}$ and (just for this proof) $\star$ (resp. $\overline{\star}$) denotes Pontrjagin multiplication in $P(n)_*(P(n))$ (resp. $P(n)_*(\overline{P(n)})$). This is easily accomplished using fictitious elements again. With $x = (a, b)$, $y = (c, d)$ we have

$$x \,\overline{\star}\, y = (a, b) \,\overline{\star}\, (c, d) = (ac, db), \qquad x \star y = (a, b) \star (c, d) = (ac, bd).$$

Using (1) we get

$$(ac, db) = (ac, bd) + v_n(ac, Q(b)Q(d)) = x \star y + v_n Q_*(x) \star Q_*(y).$$

From this we get $a_{n-1} \,\overline{\star}\, a_{n-1} = t_n + v_n + v_n = t_n$ as claimed. (4) shows also that $x \star y = x \,\overline{\star}\, y$ whenever x or y lies in the subalgebra generated by $a_0, \ldots, a_{n-2}$ and the t_i; since this subalgebra comes from $P(n-1)_*(P(n-1))$ it is annihilated by Q and Q_*. So the rest of the multiplicative structure is not affected. □

Note that both $Q : P(n)_*(P(n)) \to P(n)_*(P(n))$ and $Q_* : P(n)_*(P(n)) \to P(n)_*(P(n))$ are derivations. In the proof just given we noted that they agree on the algebra generators of Theorem 2, so we obtain another

Useful fact. $Q, Q_* : P(n)_*(P(n)) \to P(n)_*(P(n))$ are equal.

This has been used in the proof of Lemma 1 above and will be used henceforth without explicit reference.

To give a formula for $\Delta(xy)$ we first have to recall the definition of the coproduct

$$\Delta : P(n)_*(P(n)) \to P(n)_*(P(n)) \otimes_{P(n)_*} P(n)_*(P(n)).$$

There are two ingredients: firstly the map

$$\overline{\Delta} : P(n)_*(P(n)) \cong \pi_*(P(n) \wedge S \wedge P(n)) \xrightarrow{\pi_*(\mathrm{id} \wedge i \wedge \mathrm{id})} \pi_*(P(n) \wedge P(n) \wedge P(n)),$$

where $i : S \to P(n)$ is the unit. Secondly

$$\chi : P(n)_*(P(n)) \otimes_{P(n)_*} P(n)_*(P(n)) \to \pi_*(P(n) \wedge P(n) \wedge P(n))$$

which is given by $x \otimes y \mapsto (\mathrm{id} \wedge m \wedge \mathrm{id})(x \wedge y)$ and which is an isomorphism since $P(n)_*(P(n))$ is $P(n)_*$-flat. Δ is defined to be $\chi^{-1}\overline{\Delta}$.

Recall that $P(n)_*(P(n))$ is a bilateral $P(n)_*$-module, courtesy of the left and right unit maps $\eta_L, \eta_R : P(n)_* \to P(n)_*(P(n))$. The "$\otimes$"s above are to be understood with respect to this bimodule structure. Luckily, v_n is invariant in $P(n)_*$, so we don't need to discriminate between $v_n x \otimes y$, $x v_n \otimes y$, $x \otimes v_n y$ and $x \otimes y v_n$.

Lemma 5. *For all $x, y \in P(n)_*(P(n))$ we have*

$$\Delta(xy) = \Delta(x)\Delta(y) + v_n \left([(\mathrm{id} \otimes Q_{n-1})\Delta(x)][(Q_{n-1} \otimes \mathrm{id})\Delta(y)] \right).$$

Proof. First note that there is an obvious algebra structure on $\pi_*(P(n) \wedge P(n) \wedge P(n))$. With elements (a, b, c) and (d, e, f) this is given by

$$(a, b, c) \cdot (d, e, f) = (ad, be, cf).$$

$\overline{\Delta}$ is given by $(a, b) \mapsto (a, 1, b)$ so it is obviously multiplicative.

χ, however, is not multiplicative. Let $x = (a, b) \otimes (c, d)$, $y = (e, f) \otimes (g, h)$ be elements of $P(n)_*(P(n)) \otimes_{P(n)_*} P(n)_*(P(n))$. Then $xy = (ae, bf) \otimes (cg, dh)$ and

$$\chi(xy) = (ae, bfcg, dh).$$

On the other hand $\chi(x) = (a, bc, d)$, $\chi(y) = (e, fg, h)$ so that

$$\chi(x)\chi(y) = (ae, bcfg, dh).$$

Using $fc = cf + v_n Q(c)Q(f)$ we get

$$\begin{aligned} \chi(xy) &= (ae, bcfg, dh) + (ae, bQ(c)v_n Q(f)g, dh) \\ &= \chi(x)\chi(y) + M_v \chi((\mathrm{id} \otimes Q)x)\chi((Q_* \otimes \mathrm{id})y). \end{aligned} \tag{5}$$

Here we used that $(\mathrm{id} \otimes Q)x = (a, b) \otimes (Q(c), d)$ and $(Q_* \otimes \mathrm{id})y = (e, Q(f)) \otimes (g, h)$. M_v denotes multiplication by v_n in the middle.

Using (5) we can verify the formula for $\Delta(xy)$ given in the lemma: we have to show that

$$\chi \left(\Delta(x)\Delta(y) + v_n \left([(\mathrm{id} \otimes Q)\Delta(x)][(Q_* \otimes \mathrm{id})\Delta(y)] \right) \right) = \overline{\Delta}(xy).$$

We compute

$$\begin{aligned} & \chi \left(\Delta(x)\Delta(y) + v_n \left([(\mathrm{id} \otimes Q)\Delta(x)][(Q_* \otimes \mathrm{id})\Delta(y)] \right) \right) \\ = \;& \chi(\Delta(x)\Delta(y)) + M_v \chi \left([(\mathrm{id} \otimes Q)\Delta(x)][(Q_* \otimes \mathrm{id})\Delta(y)] \right) \\ = \;& \chi(\Delta(x))\chi(\Delta(y)) \\ + \;& M_v \chi \left([(\mathrm{id} \otimes Q)\Delta(x)] \right) \chi \left([(Q_* \otimes \mathrm{id})\Delta(y)] \right) \\ + \;& M_v \chi \left([(\mathrm{id} \otimes Q)\Delta(x)] \right) \chi \left([(Q_* \otimes \mathrm{id})\Delta(y)] \right) \\ + \;& M_v^2 \chi \left([(\mathrm{id} \otimes Q)^2 \Delta(x)] \right) \chi \left([(Q_* \otimes \mathrm{id})^2 \Delta(y)] \right) \\ = \;& \chi(\Delta(x))\chi(\Delta(y)) \qquad \text{(since } Q^2 = Q_*^2 = 0) \\ = \;& \overline{\Delta}(x)\overline{\Delta}(y) \\ = \;& \overline{\Delta}(xy). \end{aligned}$$

□

Using the identities $(Q \otimes \mathrm{id})\Delta(x) = \Delta(Qx)$ and $(\mathrm{id} \otimes Q_*)\Delta(x) = \Delta(Q_*(x))$ one can check that Lemma 5 says that Δ is an algebra homomorphism

$$P(n)_*(\overline{P(n)}) \to P(n)_*(P(n)) \otimes_{P(n)_*} P(n)_*(P(n)).$$

This observation might at least mnemonically be useful.

3. Multiplicative and antimultiplicative maps

To get at multiplicative or antimultiplicative maps $P(n) \to P(n)$ or $K(n) \to K(n)$ we need to be able to characterise them in terms of their Kronecker duals. The following lemma is a first step.

Lemma 6. *Let E and X be ring spectra and suppose that the Kronecker homomorphism*

$$E^*(X \wedge X) \longrightarrow \mathrm{Hom}_{E_*}(E_*(X \wedge X), E^*)$$

is an isomorphism. Assume that $E_*(X)$ is a flat E_*-module. Then a $\theta : X \to E$ is multiplicative iff

$$\epsilon(\theta_*(xy)) = \epsilon(\theta_*(x)\theta_*(y)) \tag{6}$$

holds for all $x, y \in E_*(X)$.

Proof. θ is multiplicative iff $\theta m_X = m_E(\theta \wedge \theta)$. Both are maps from $X \wedge X$ to E, so by Kronecker duality this is equivalent to

$$\epsilon((\theta m_X)_*(z)) = \epsilon((m_E(\theta \wedge \theta))_*(z))$$

for all $z \in E_*(X \wedge X)$. Since $E_*(X)$ is a flat E_*-module, the exterior homology product $\underline{\triangle} : E_*(X) \otimes_{E_*} E_*(X) \to E_*(X \wedge X)$ is an isomorphism. Hence we can assume that $z = x \,\underline{\triangle}\, y$ for $x, y \in E_*(X)$. The lemma follows because $\theta_*(x \,\underline{\triangle}\, y) = \theta_*(x) \,\underline{\triangle}\, \theta_*(y)$, $xy = (m_X)_*(x \,\underline{\triangle}\, y)$ and $\theta_*(x)\theta_*(y) = (m_E)_*(\theta_*(x) \,\underline{\triangle}\, \theta_*(y))$. □

For the $P(n)$ this gives

Lemma 7. *Let $\theta : P(n) \to P(n)$ be any map and denote its Kronecker dual by $\bar{\theta}$. Then θ is multiplicative iff*

$$\bar{\theta} : P(n)_*(\overline{P(n)}) \to P(n)_*$$

is an algebra homomorphism. It is antimultiplicative iff

$$\bar{\theta} : P(n)_*(P(n)) \to P(n)_*.$$

is an algebra homomorphism.

Proof. We prove the assertion about multiplicative maps only, the antimultiplicative case being completely analogous.

According to Lemma 6 θ is multiplicative iff (6) holds. Using (2) this may be rewritten

$$\epsilon(\theta_*(xy)) = \epsilon(\theta_*(x))\epsilon(\theta_*(y)) + v_n\epsilon(Q\theta_*(x))\epsilon(Q\theta_*(y)). \tag{7}$$

Let $\overline{\star}$ denote the multiplication in $P(n)_*(\overline{P(n)})$. We recall from the proof of Theorem 2 that $x \,\overline{\star}\, y = xy + v_n Q(x)Q(y)$. So $\bar{\theta} : P(n)_*(\overline{P(n)}) \to P(n)_*$ is multiplicative iff

$$\epsilon(\theta_*(xy + v_n Q(x)Q(y))) = \epsilon(\theta_*(x))\epsilon(\theta_*(y)). \tag{8}$$

From both (7) and (8) we can conclude that

$$\epsilon(\theta_*(x))\epsilon(\theta_*(y)) = \epsilon(\theta_*(xy)) \qquad \text{if } Qx = 0 \text{ or } Qy = 0. \tag{9}$$

So to show the equivalence of (7) and (8) we may assume (9). But from (9) it follows easily that

$$v_n\epsilon(Q\theta_*(x))\epsilon(Q\theta_*(y)) = v_n\epsilon(\theta_*(Q(x)Q(y))),$$

which is the difference between (7) and (8). (Use that $QQ = 0$ and $Q\theta_* = \theta_* Q$.) □

To prove Theorems 3 and 4 we have to recall the relation of the $P(n)$ to formal group laws. [R1], especially Appendix 2, is a good reference for most of this material.

$(P(n)_*, P(n)_*(BP))$ inherits from $(BP_*, BP_*(BP))$ an Hopf algebroid structure. Recall that $P(n)_*(BP)$ is the $P(n)_*$-subalgebra of $P(n)_*(P(n))$ generated by the t_i. For a ring R of characteristic 2 the set of ring homomorphisms $P(n)_* \to R$ may naturally be identified with the set $\mathcal{FG}_n(R)$ of all 2-typical formal group laws of height $\geq n$ over R. Similarly, the ring $P(n)_*(BP)$ corepresents the set $\mathcal{SI}_n(R)$ of triples (F, f, G) with $F, G \in \mathcal{FG}_n(R)$ and $f : G \to F$ a strict isomorphism. Given $\phi : P(n)_*(BP) \to R$ the triple (F, f, G) is obtained as follows: F (resp. G) is the formal group law classified by $\phi\eta_L$ (resp. $\phi\eta_R$). The isomorphism $f(x)$ is then defined by

$$f^{-1}(x) = x +_F \sum_{i\geq 1}^{F} \phi(t_i)x^{2^i}.$$

Proof of Theorem 3. We are particularly interested here in the natural transformation $\mathcal{FG}_n(R) \ni F \mapsto (F, [-1]_F(x), F) \in \mathcal{SI}_n(R)$. This is induced by a ring homomorphism $\delta_n : P(n)_*(BP) \to P(n)_*$. These δ_n are obviously compatible as n varies and, obviously again, $\delta_n\eta_L = \delta_n\eta_R = \mathrm{id}$, so δ_n is $P(n)_*$-linear. If we can extend this to an algebra map

$$\overline{\delta_n} : P(n)_*(P(n)) \to P(n)_*$$

its Kronecker dual $\Xi : P(n) \to P(n)$ will be antimultiplicative by Lemma 7 and will by construction have the stated effect on Euler classes.

We don't have much choice in extending δ_n: since $|a_i|$ is odd for every i and $P(n)_*$ does not have nonzero odd dimensional elements we have to let $\overline{\delta_n}(a_i) = 0$, so the required multiplicativity implies that there is at most one such extension. All we have to do is to check that this is consistent with the relations $a_i^2 = t_{i+1}$, $0 \leq i < n-1$ and $a_{n-1}^2 = t_n + v_n$. So we have to show that $\delta_n(t_i) = 0$ for $1 \leq i \leq n-1$ and that $\delta_n(t_n) = v_n$.

The first requirement follows easily from dimensional considerations. To see that $\delta_n(t_n) = v_n$ recall that with Araki's v_k we have

$$[2]_F(x) = 2x +_F v_1x^2 +_F v_2x^4 +_F \cdots +_F v_kx^{2^k} +_F \cdots$$

for the formal group law F of BP. For the formal group P_n of $P(n)$ this gives $[2]_{P_n}(x) = v_nx^{2^n} +$ higher terms. Comparing coefficients in $[-1]_{P_n}(x) +_{P_n} [2]_{P_n}(x) = x$ then gives the result.

That Ξ is an involution is quite clear: since Ξ^2 is multiplicative its Kronecker dual $\overline{\epsilon}$ is an algebra homomorphism $P(n)_*(\overline{P(n)}) \to P(n)_*$. Since it is the identity on Euler classes of complex line bundles its restriction to $P(n)_*(BP)$ agrees with ϵ. Since $\overline{\epsilon}$ and ϵ both are multiplicative extensions of this restriction, uniqueness implies $\overline{\epsilon} = \epsilon$, i.e. $\Xi^2 = \mathrm{id}$. □

To prove Theorem 4 we first have to carry over some of the results on $P(n)$ to $K(n)$. Let $K(n)_* = \mathbf{F}_2[v_n, v_n^{-1}]$ as a module over $P(n)_*$. Then $X \mapsto K(n)_* \otimes_{P(n)_*} P(n)_*(X) =: K(n)_*(X)$ is a homology theory on the stable homotopy category and $X \mapsto \mathrm{Hom}_{K(n)_*}(K(n)_*(X), K(n)_*)$ a cohomology theory, and both are represented by the same spectrum $K(n)$. The two multiplications m and $\overline{m}$ on $P(n)$ give similar ring spectrum structures on $K(n)$ that will be denoted by the same symbols. The Bockstein Q_{n-1}, too, carries over to $K(n)$ and (1) still holds. If we denote the augmentation of $K(n)_*(K(n))$ again by ϵ then (2) continues to hold, too. Let $\overline{K(n)}$ have the obvious meaning.

To get at the structure of $K(n)_*(K(n))$ and $K(n)_*(\overline{K(n)})$ let

$$\Sigma_n = K(n)_* \otimes_{P(n)_*} P(n)_*(BP) \otimes_{P(n)_*} K(n)_*$$

be the nth Morava stabiliser algebra (cf. [R1], ch. 6). The following lemma follows easily from Theorem 2 and the definitions.

Lemma 8. *We have*

$$K(n)_*(K(n)) = \Sigma_n[a_0, \dots, a_{n-1}]$$

with relations $a_i^2 = t_{i+1} + v_{i+1}$ *for* $0 \le i \le n-1$ *and*

$$K(n)_*(\overline{K(n)}) = \Sigma_n[a_0, \dots, a_{n-1}]$$

with relations $a_i^2 = t_{i+1}$ *for* $0 \le i \le n-1$. □

Lemma 7 carries over, too, as does its proof.

Lemma 9. *Let* $\theta : K(n) \to K(n)$ *be any map and denote its Kronecker dual by* $\bar{\theta}$. *Then* θ *is multiplicative iff*

$$\bar{\theta} : K(n)_*(\overline{K(n)}) \to K(n)_*$$

is an algebra homomorphism. It is antimultiplicative iff

$$\bar{\theta} : K(n)_*(K(n)) \to K(n)_*.$$

is an algebra homomorphism. □

Finally recall (and there will be no more recollections, I promise) that graded ring homomorphisms $\Sigma_n \to K(n)_*$ classify strict graded automorphisms of the canonical formal group law F_n over $K(n)_*$ and that the group $\mathrm{Aut}(F_n)$ of such automorphisms is isomorphic to $\mathbf{Z}_2^*$, the isomorphism being given by $\mathbf{Z}_2^* \ni k \mapsto [k]_{F_n}(x) \in \mathrm{Aut}(F_n)$.

Proof of Theorem 4. For each $\phi \in \mathrm{Aut}(F_n)$, $\phi : \Sigma_n \to K(n)_*$, we have to provide a multiplicative extension to either $K(n)_*(K(n)) \to K(n)_*$ or $K(n)_*(\overline{K(n)}) \to K(n)_*$. As in the proof of Theorem 3, an extension to the former will exist (and be unique) iff $\phi(t_i) = 0$ for $1 \le i \le n-1$ and $\phi(t_n) = v_n$. The first condition is automatically satisfied for dimensional reasons. Similarly, an extension to the latter exists iff $\phi(t_n) = 0$. Since 0 and v_n are the only elements in $K(n)_*$ of degree $|t_n|$ exactly one of these conditions is fulfilled. (We leave it to the interested reader to verify that $\phi(t_n)$ depends on the congruence class of ϕ modulo 4 in the way claimed.) Thus we obtain a well-defined map

$$\mathrm{Aut}(F_n) \to \mathrm{Mult}^{\pm}$$

which is an inverse to the geometrically defined map

$$\mathrm{Mult}^{\pm} \to \mathrm{Aut}(F_n)$$

that gives the effect on Euler classes. Since the latter is obviously a group homomorphism, we are done. □

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