Maximum likelihood and Bayesian approaches to stock assessment when data are questionable

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Abstract

This study examines the use of age-structured maximum likelihood and Bayesian approaches for stock assessment of the Namibian monkfish, *Lophius vomerinus*, resource with questionable data, in which time series are short, abundance indices are variable, and research data conflict with commercial data. Bayesian approaches with both noninformative and informative priors are investigated to determine if they enhance estimation stability. Three data scenarios are assessed: commercial and research survey data, research survey data only, and commercial data only. Both statistical approaches show that resource abundance has decreased with exploitable biomass estimated at approximately 44% of pristine levels. The maximum likelihood and the Bayesian approach with noninformative priors result in similar estimates. As the abundance data contained little information pertaining to possible density dependence within the stock–recruit relationship, only a Bayesian approach with informative priors reduces uncertainty in the steepness parameter *h*. Estimated management quantities are sensitive both to the set of data sources and whether prior information was informative or not. The strengths of the Bayesian approach include the integration of prior information with uncertain data, the exploration of data conflicts, and the ability to show the uncertainty in estimates of management parameters. Its weakness is that estimation stability is dependent on the choice of priors, which alters some posterior distributions of management quantities.

Keywords: Monkfish; *Lophius vomerinus*; Age-structured production model; Markov Chain Monte-Carlo; Maximum likelihood estimation; Bayesian statistics

1. Introduction

Effective fisheries management depends largely on the ability of managers to determine levels of fishing effort, catch, and gear selectivity required for sustainable harvesting. Increasingly complex quantitative methodologies have developed for this determination (Quinn and Deriso, 1999), along with the development of methods to evaluate possible consequences of alternative harvesting policies (Francis, 1992, Hilborn et al., 1993 and Rosenberg and Restrepo, 1994). Considering that these evaluations are important “inputs” to resource managers, it is now advocated that they be incorporated into some form of probabilistic framework to account for uncertainties.

Commonly applied assessment models use a maximum likelihood approach, whereby the model is fitted statistically to the available abundance and/or catch-at-age data using a likelihood function. Likelihood methods can be extended to also incorporate *a priori* beliefs in the reliability of data sources through various weighting schemes (Merritt and Quinn, 2000). A Bayesian approach to stock assessment goes further by incorporating expert judgment, ancillary information, and (possibly) common sense into the modeling framework (Punt and Hilborn, 1997). A Bayesian approach formalizes the expression of *a priori* beliefs about an unknown quantity (in the form of prior probabilities) and then modifies them in light of available data (via the likelihood function) to arrive at some posterior probability of that quantity.

Furthermore, problems arise in stock assessment when data are questionable. By this we mean that data are too short (say, <5–10 years in length), or abundance indices do not reflect the response of the stock to harvesting pressure, or the data have large amounts of sampling and non-sampling errors, or multiple data sources have different trends or conflicts. Examples include a constantly declining trend in abundance data, also referred to as "one-way downhill trip" data that leads to negatively biased estimated levels of stock productivity (Polacheck et al., 1993) or temporally invariant indices that do not reflect the response of a population to harvesting pressure. Maximum likelihood methods tend to fail with questionable data, whereas Bayesian methods can often stabilize parameter estimation by the statistically correct inclusion of prior information (McAllister et al., 2001), as opposed to the *ad hoc* weighting or penalized maximum likelihood approaches.

Monkfish (*Lophius* spp.), targeted by the monkfish and sole directed fishery and caught in lesser amounts as bycatch in the hake (*Merluccius* spp.) directed trawl fishery, is an important resource in Namibia contributing approximately US$ 100 million in export value annually (Ministry of Fisheries and Marine Resources, Namibia, unpublished data). Two sympatric monkfish species are present in Namibian waters *Lophius vomerinus* and *L. vaillanti*. *L. vomerinus* is the more important of the two in terms of abundance (99%), landings and value. The three sources of abundance data available for assessing the resource are of varying quality and length (Table 1). The monkfish-directed biomass survey is fairly precise and shows a clear decline in abundance over its short time period. The other abundance data series, commercial catch rate data and the hake-directed biomass survey data, are longer but for the former shows high interannual variability and the latter shows relatively low precision compared to the monkfish-directed biomass survey. Catch-age data from research surveys and commercial catches are extremely limited (see below).

Table 1.

Total catch, standardized catch-per-unit effort research survey estimates of relative abundance for monkfish *Lophius vomerinus* between 1974 and 2005 in Namibia (from Kirchner and Schneider, unpublished)

Year	**Catch ($\times 10^3$ t)**	**Standardized CPUE (kg h^{-1})**	**Hake-directed survey biomass ($\times 10^3$ t ± S.E.)**	**Monk-directed survey biomass ($\times 10^3$ t ± S.E.)**
1974	0.3	–	–	–
1975	1.1	–	–	–
1976	0.9	–	–	–

Year	Catch ($\times 10^3$ t)	Standardized CPUE (kg h^{-1})	Hake-directed survey biomass ($\times 10^3$ t ± S.E.)	Monk-directed survey biomass ($\times 10^3$ t ± S.E.)
1977	5.7	–	–	–
1978	7.4	–	–	–
1979	3.5	–	–	–
1980	3.2	–	–	–
1981	15.6	–	–	–
1982	16.3	–	–	–
1983	12.9	–	–	–
1984	8.5	–	–	–
1985	8.5	–	–	–
1986	13	–	–	–
1987	11.7	–	–	–
1988	5.0	–	–	–
1989	6.6	–	–	–
1990	1.5	–	–	–
1991	4.6	77.64	–	–
1992	8.1	119	–	–
1993	9.2	123.55	–	–
1994	12.2	151.76	34.85 ± 4.54 (February)	
			22.34 ± 2.62 (May)	
			25.44 ± 2.89 (November)	
1995	10.1	125.99	13.13 ± 1.71 (May)	–
1996	9.8	94.58	21.75 ± 2.69 (February)	–
			11.37 ± 1.46 (October)	
1997	10.4	–	11.38 ± 1.27 (February)	–
1998	16.6	172.18	11.16 ± 1.46 (February)	–
1999	14.1	114.99	25.83 ± 4.62 (February)	–
2000	14.4	96.58	–	49.00 ± 9.80 (November)
2001	12.4	79.41	–	56.00 ± 11.20 (November)

Year	Catch ($\times 10^3$ t)	Standardized CPUE (kg h^{-1})	Hake-directed survey biomass ($\times 10^3$ t ± S.E.)	Monk-directed survey biomass ($\times 10^3$ t ± S.E.)
2002	15.3	74.07	–	39.00 ± 8.58 (November)
2003	12.2	95.37	–	21.00 ± 4.20 (November)
2004	9.0	71.48	–	35.00 ± 5.95 (November)
2005	6.8[a]	85.95	–	26.00 ± 4.42 (November)

The month in which the survey occurred is indicated in parentheses.
[a] Preliminary catch.

The main goal of this paper is to present a case study that determines whether Bayesian population modeling is more advantageous than a strictly maximum likelihood approach when some of the data sources are questionable (as defined earlier). We develop models for the population dynamics of the *L. vomerinus* component of the monkfish resource off Namibia within both maximum likelihood and Bayesian statistical frameworks. Because the abundance indices and age data cover varying time periods and have different precision, it is not clear which data sources should be used in the model. We address the relative importance and sensitivity of the various data sources to the assessment, together with including of noninformative and informative prior information in estimating several management quantities.

2. Material and methods

2.1. Available data

Available data for the assessment are a time-series of annual *Lophius* spp. catches from 1974 to 1999, a monkfish catch rate (CPUE) data series (standardized with a general linear model) from the commercial monkfish-and-sole directed fleet between 1991 and 1999, *L. vomerinus*-specific swept area biomass indices obtained from hake-directed biomass surveys between 1994 and 1999, and *L. vomerinus*-specific swept area biomass

indices obtained from monkfish-directed biomass surveys between 2000 and 2005. The hake-directed surveys were conducted onboard the RV Dr. Fridjof Nansen, and the monkfish-directed surveys were conducted onboard the RV Welwitchia (Table 1).

Research catch-at-age data (Table 2) are of reasonable quality as *L. vomerinus* could be identified during routine surveys. The age composition of the survey catches was estimated by transforming the annual length frequency data using a single age-length key constructed from 1997 to 1998 data. Age data from the commercial fleet data (Table 3) are questionable. Size category data were obtained from the industry between 1994 and 1999 and converted to ages from a single age-size category key constructed from 1997 to 1998 data (Maartens, 1999).

Table 2.

Relative proportion of *Lophius vomerinus* caught at age during hake-directed research surveys on the RV Dr. Fridjof Nansen between 1994 and 1999 (from Maartens and Booth, 2001)

Year	**Proportions caught at age**									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**Total number (×10^3)**
February 1994	0.046	0.161	0.244	0.128	0.157	0.126	0.095	0.031	0.008	35.405
May 1994	0.022	0.109	0.175	0.123	0.181	0.176	0.151	0.042	0.017	17.153
November 1994	0.036	0.135	0.169	0.127	0.185	0.157	0.129	0.042	0.015	18.996
May 1995	0.024	0.086	0.220	0.135	0.182	0.163	0.136	0.036	0.014	12.210
February 1996	0.121	0.123	0.173	0.162	0.163	0.115	0.098	0.030	0.011	22.382
October 1996	0.060	0.279	0.199	0.086	0.129	0.114	0.095	0.026	0.008	12.719
February 1997	0.097	0.249	0.274	0.095	0.088	0.085	0.077	0.023	0.009	13.234
February 1998	0.083	0.351	0.239	0.109	0.079	0.057	0.061	0.013	0.006	19.381
February 1999	0.036	0.188	0.273	0.158	0.155	0.096	0.068	0.019	0.006	25.386

Table 3.

Relative proportion *Lophius* spp. caught at age aggregated over both the hake directed, and monkfish and sole directed fisheries between 1994 and 1999

Year	**Proportions caught at age**									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**Total number (×10^6)**
1994	0.186	0.228	0.166	0.132	0.1	0.083	0.038	0.028	0.028	19.230
1995	0.173	0.28	0.193	0.145	0.075	0.063	0.027	0.019	0.019	18.865
1996	0.133	0.123	0.166	0.164	0.141	0.118	0.053	0.039	0.039	11.315
1997	0.259	0.278	0.164	0.116	0.065	0.054	0.023	0.016	0.016	22.126
1998	0.255	0.285	0.194	0.149	0.044	0.037	0.014	0.009	0.009	40.746
1999	0.249	0.217	0.177	0.152	0.072	0.06	0.025	0.018	0.018	27.617

Values reflect the *L. vomerinus*-specific component of the catch by reducing the total monkfish catch by 1% (from Maartens and Booth, 2001).

2.2. Population dynamics model

It is not desirable to use a simple production model for this study, because it would not take advantage of the available age-structured data from the research survey and the commercial fishery. However, the lack of annual age-structured data makes it is impossible to apply a complete age-structured model with annual estimates of recruitment (Quinn and Deriso, 1999, Chapter 8). As a middle ground, we model the dynamics of the resource by using a simplified age-structured model with deterministic recruitment from a stock–recruit relationship (Punt, 1994, Punt and Japp, 1994 and Booth and Punt, 1998).

The details of the resource dynamics and likelihood function used in the analysis are given in the Appendix A. Several parameters in the model (natural mortality, selectivity, maturity and growth), obtained from Maartens and Booth (2001), are fixed in this application (Table 4) for parsimony and to focus on the estimation of management quantities (see following section). We also focus on the three catchability parameters $q^{\text{HAKE-RES}}$, $q^{\text{MONK-RES}}$ and q^{COM} for the hake-directed research survey, monkfish-directed research survey and commercial fishery, respectively, because they play a fundamental role in both the prior and likelihood distributions.

Table 4.

Fixed parameter values used in the analyses from Maartens and Booth (2001)

Parameter description	**Parameter symbol**	**Parameter value**
Asymptotic length	L_∞ (cm)	95.04
Brody's growth coefficient	K (year^{-1})	0.10
Age at zero length	t_0 (year)	−0.31
Length–weight relationship constant	γ (g)	0.011
Length–weight relationship exponent	Φ(g cm^{-1})	3.06
Natural mortality rate	M (year^{-1})	0.3
Plus-group	Max (year)	10
Age at 50% maturity	a_{50}^{m}(year)	4.12
Inverse rate of maturation	δ^m (year^{-1})	0.99
Age at 50% selection by the commercial trawl gear	a_{50}^{com} (year)	1
Inverse rate of commercial selection	δ^{com} (year^{-1})	0.08
Age at 50% selection by the research trawl gear	a_{50}^{surv} (year)	1.92
Inverse rate of research selection	δ^{surv} (year^{-1})	0.39

The model is deterministic and has two parameters; EB_0 – pristine exploitable biomass in 1974, and *h* – the "steepness" parameter from the Beverton-Holt stock–recruitment relationship (see the Appendix A).

2.3. Management quantities

Five management-related quantities are used to assess the status, productivity, and potential yield of the monkfish resource. They are EB_0, *h*, maximum sustainable yield MSY, Depletion (the ratio of the spawner biomass at the start of 2005 to that at the start of 1974), and EB_{2005}/MSYL (the ratio of exploitable biomass in 2005 to the exploitable biomass at the MSY level).

2.4. Estimation of variability and posterior probability density functions

Estimates of parameter variability are calculated from the inverse Hessian matrix. This approach assumes that the parameter values are asymptotically normally distributed and involves replacing the log-likelihood surface at the maximum likelihood solution by the quadratic form of the Hessian.

Marginal probability density functions for each management quantity are obtained by integrating each quantity out from the joint posterior probability density function (Lee, 1997). The joint posterior probability distribution is estimated using the Metropolis-Hastings Markov Chain Monte-Carlo algorithm (Gelman et al., 1995). One hundred thousand parameter vectors were generated, with a "burn-in" of 1000 vectors, and every 25th vector saved. Geweke's (1992) diagnostic confirmed that all chains converged (p values > 0.47). The range of h values is 0.21 (as $h = 0.2$ is undefined) to 1.00, and EB_0 is bounded between 135 000 and 1 000 000 t. These values represent the smallest population size that is extant to a population size that is almost an order of magnitude larger. All analyses were conducted using AD Model Builder (Otter Research Ltd., 2000).

2.5. Data scenarios

Three data scenarios are considered: one where all available data (commercial and research) are included in the likelihood), one in which only the research CPUE and catch-age data are included, and one in which only the commercial CPUE and catch-age data are included. The latter two scenarios are common in data-limited fisheries. These two scenarios also constitute a sensitivity study of the influence of the data sources.

2.6. Prior information

Central to any Bayesian analysis is the specification of prior information. In this study, we first use noninformative priors for the parameters EB_0, $q^{\text{HAKE-RES}}$, $q^{\text{MONK-RES}}$, q^{COM}, and *h*. In an alternative scenario, we replace them with their informative equivalents to explore the sensitivity of prior specification.

For noninformative priors, EB_0, *h* and the catchability coefficients $q^{\text{HAKE-RES}}$, $q^{\text{MONK-RES}}$, and q^{COM} are considered to be uniformly distributed over their values.

For informative priors, the Beverton-Holt stock–recruitment curve 'steepness' parameter is normally distributed with a mean of 0.9 and a coefficient of variation of 20%, or $p(h) = N(0.9, (0.9 \times 0.2)^2)$. Because no data are available on the "steepness" parameter for other *Lophius* species, a normal distribution is chosen to provide most weight close to, but not equal to 1, and to assign little weight to small values (Myers et al., 1995). The hake-directed research survey catchability coefficient, $q^{\text{HAKE-RES}}$, is assumed to be normally distributed with a mean of 0.4 and a coefficient of variation of 20%, or $p(q^{\text{HAKE-RES}}) = N(0.4, (0.4 \times 0.2)^2)$. The mean is taken from the estimated catchability coefficient for the Cape hake (*Merluccius* spp.) resource off Namibia (Rademeyer, 2003). The monkfish-directed research survey catchability coefficient, $q^{\text{MONK-RES}}$, is assumed to be normally distributed with a mean of 0.8 and a coefficient of variation of 40%, or $p(q^{\text{MONK-RES}}) = N(0.8, (0.8 \times 0.4)^2)$. We presume that the monkfish-directed biomass index would be close to, but not quite equal to, the absolute exploitable biomass. This is motivated by the demersal habits of monkfish and that the survey gear using “tickler” chains on the leading footrope to chase fish up into the net. Informative priors for q^{COM} or EB_0 are constructed by assuming that they are uniform across their logarithmic values. These priors are improper, proportional to their inverse, and assign less probability to high resource abundance (McAllister and Kirkwood, 1998).

We assume that all the prior probabilities are independent of each other, and hence, the joint probability is

$$p(\theta) = p(\text{EB}_0 \cap h \cap q^{\text{HAKE-RES}} \cap q^{\text{MONK-RES}} \cap q^{\text{COM}}) = p(\text{EB}_0)p(h)p(q^{\text{HAKE-RES}})p(q^{\text{MONK-RES}})p(q^{\text{COM}})$$

No prior probabilities were placed on the management quantities MSY, Depletion or EB_{2005}/MSYL.

3. Results

3.1. Maximum likelihood estimation

In all data scenarios investigated there is a general decrease in resource abundance over time (Fig. 1; Table 5), with the largest decrease occurring with the high catches during the 1980s. The decrease in catches in the early 1990s was associated with a short period of stock rebuilding. Catches, however, have increased steadily in recent years from 1500 t in 1990 to 15 300 t in 2003, with corresponding decreases in estimated abundance.

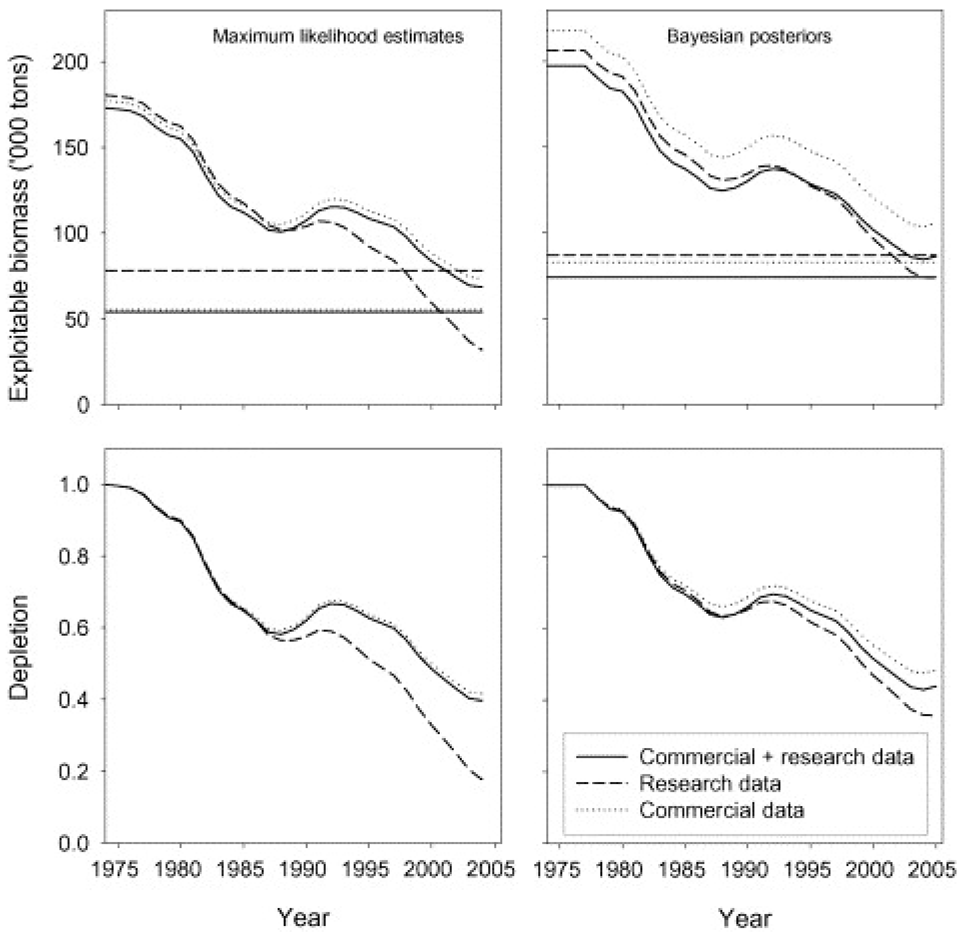

Fig. 1. Maximum likelihood and Bayesian posterior estimates of exploitable biomass (top panel) and Depletion trajectories (bottom panel) between 1974 and 2005. The horizontal lines in the top panels refer to the exploitable biomass level at MSY. Results have summarized for three data scenarios: using both the commercial and research survey data (—), using only the research survey data (- - -) and using only commercial data (· · ·).

Table 5.

Maximum likelihood estimates and their Hessian-based asymptotic coefficients of variation in parentheses, along with their log-likelihood components

	Research and commercial data	**Research data only**	**Commercial data only**
EB_0 ($\times 10^3$ t)	173.15 (7.79%)	180.58 (15.92%)	177.33 (9.21%)
h	1.00 (0.67%)	0.47 (41.28%)	1.00 (0.74%)
MSY ($\times 10^3$ t)	11.39 (7.8%)	6.14 (31.62%)	11.66 (9.22%)
Depletion	0.41 (15.52%)	0.16 (30.91%)	0.43 (16.64%)
$q^{\text{HAKE-RES}}$	0.18 (13.84%)	0.21 (17.56%)	0.17 (15.98%)

	Research and commercial data	Research data only	Commercial data only
$q^{\text{MONK-RES}}$	0.49 (22.12%)	0.94 (28.35%)	0.46 (24.72%)
q^{COM}	1.12 (16.87%)	1.57 (20.46%)	1.06 (19.19%)
$EB_{2005}/MSYL$	1.31 (15.55%)	0.38 (33.78%)	1.38 (16.68%)
$-\ln(\lambda_{\text{COM}})$	−14.64	–	−14.75
$-\ln(\lambda_{\text{HAKE-RES}})$	−3.73	−3.88	–
$-\ln(\lambda_{\text{MONK-RES}})$	−4.33	−6.46	–
$-\ln(\lambda_{\text{COM CAA}})$	−29.69	–	−29.62
$-\ln(\lambda_{\text{RES CAA}})$	−18.21	−17.55	–
$-\ln L$	−70.6	−27.89	−44.37

Results have been summarized for three data scenarios: using both commercial and research survey data, using only the research survey data, and using only the commercial data.

The exploitable biomass trends are similar for the data scenarios in which the commercial data are included. In the research data only scenario a rapid depletion in exploitable biomass occurs. The depletion trend, the ratio of exploitable biomass in each year to pristine levels, mirrors the exploitable biomass trend, and is highest for the commercial data only scenario, and lowest for the research data only scenario. The combined data scenario likelihood is dominated by the commercial data. Depletion in 2005, the Depletion management quantity, is estimated at 16% and 43% for these two data scenarios, respectively. Depletion is estimated at 41% when all data are included in the analyses. The data scenarios including the commercial data show the current exploitable biomass to be over 30% higher than MSY levels. By contrast, the research data only scenario results in an overfished resource with current exploitable biomass at 70% of levels required to achieve MSY.

The contributions of the different biomass indices to the log-likelihood differ considerably (Table 5). Commercial CPUE is the longest series and provides the greatest contribution. The weakest datasets are the hake-directed biomass surveys. The strong decline noticeable in the monkfish-directed surveys accounts for its strength in the research data scenario. Owing to its length of only 5 years, it is not particularly influential when all data are included in the analysis. By contrast, the catch-at-age data are reasonably informative (Table 5).

The maximum likelihood models for the three data scenarios fit the various data sources very well (CPUE and research survey, Fig. 2; research survey catch-at-age, Fig. 3; commercial catch-at-age, Fig. 4). The maximum likelihood estimates of management quantities and their coefficients of variation are summarized in Table 5. There is little evidence for density dependence in the Beverton-Holt stock recruitment relationship, with h estimated at $\sim$1.00 for both data scenarios that include the commercial data, and 0.47 for the research only data scenario. Estimates of variability are particularly low for the commercial data scenarios (CVs < 1%) and high for research data scenario (CV = 41%). The former result is a consequence of the Hessian-based approach to estimating parameter variability, because the parameter *h* was constrained by an upper bound of 1. Maximum sustainable yield and Depletion is strongly (ρ = 0.99) correlated with EB_0 and ranges between 6000 and 12 000 t.

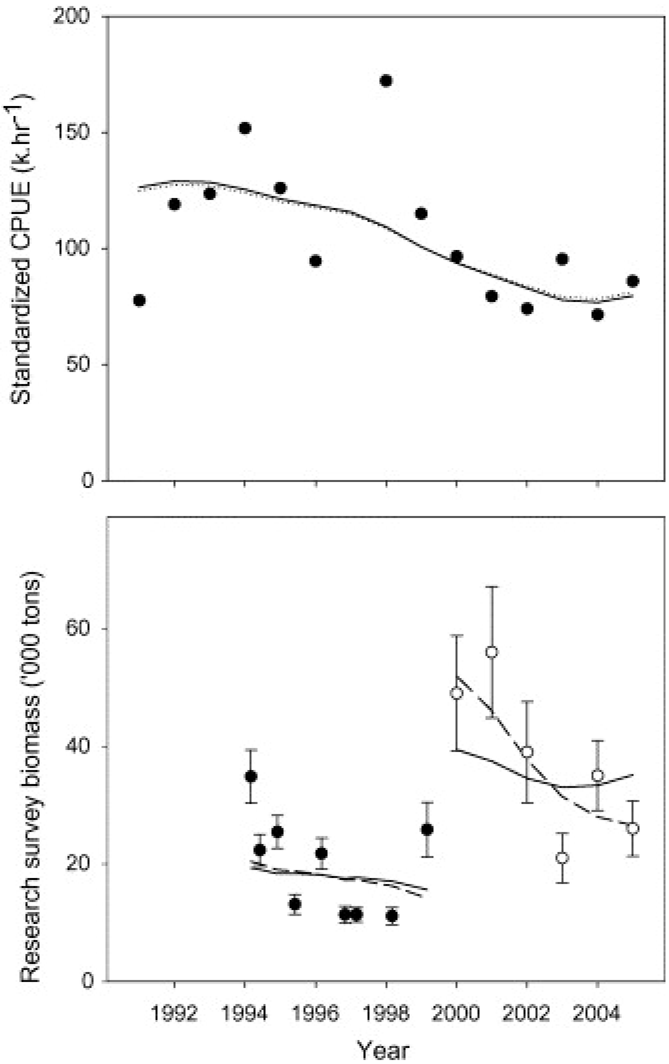

Fig. 2. Observed relative abundance indices and corresponding estimates from the maximum likelihood model. Three data scenarios are presented: using both the commercial and research survey data (—), using only the research survey data (- - -) and using only commercial data (· · ·). Model estimates have been scaled, through the different catchability coefficients, to be compatible with the abundance indices.

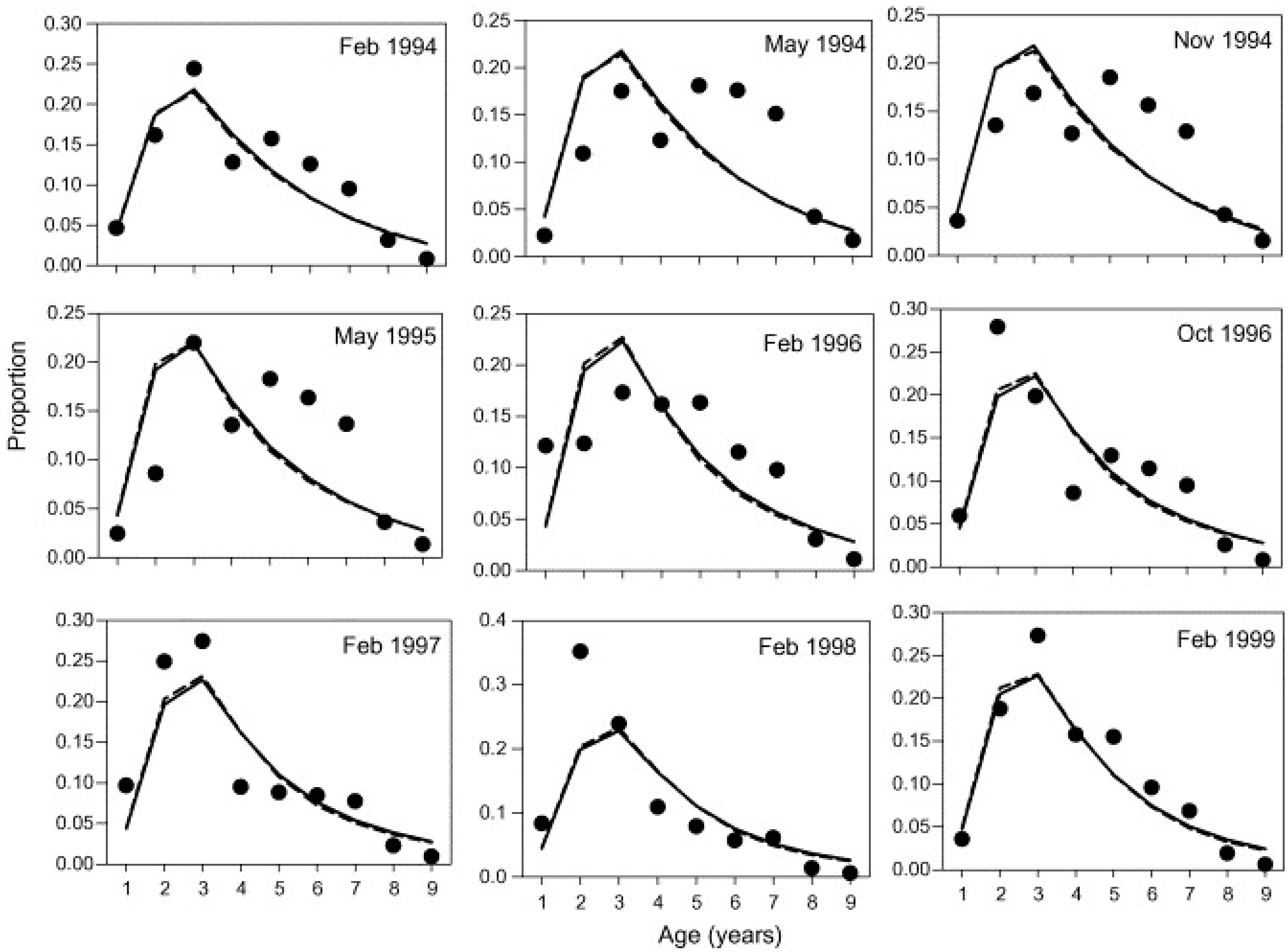

Fig. 3. Observed research survey catch-at-age and corresponding estimates from the maximum likelihood model for two data scenarios: using commercial and research data (—) and using only the research survey data only (- - -). (Research data were not used in the other scenario.)

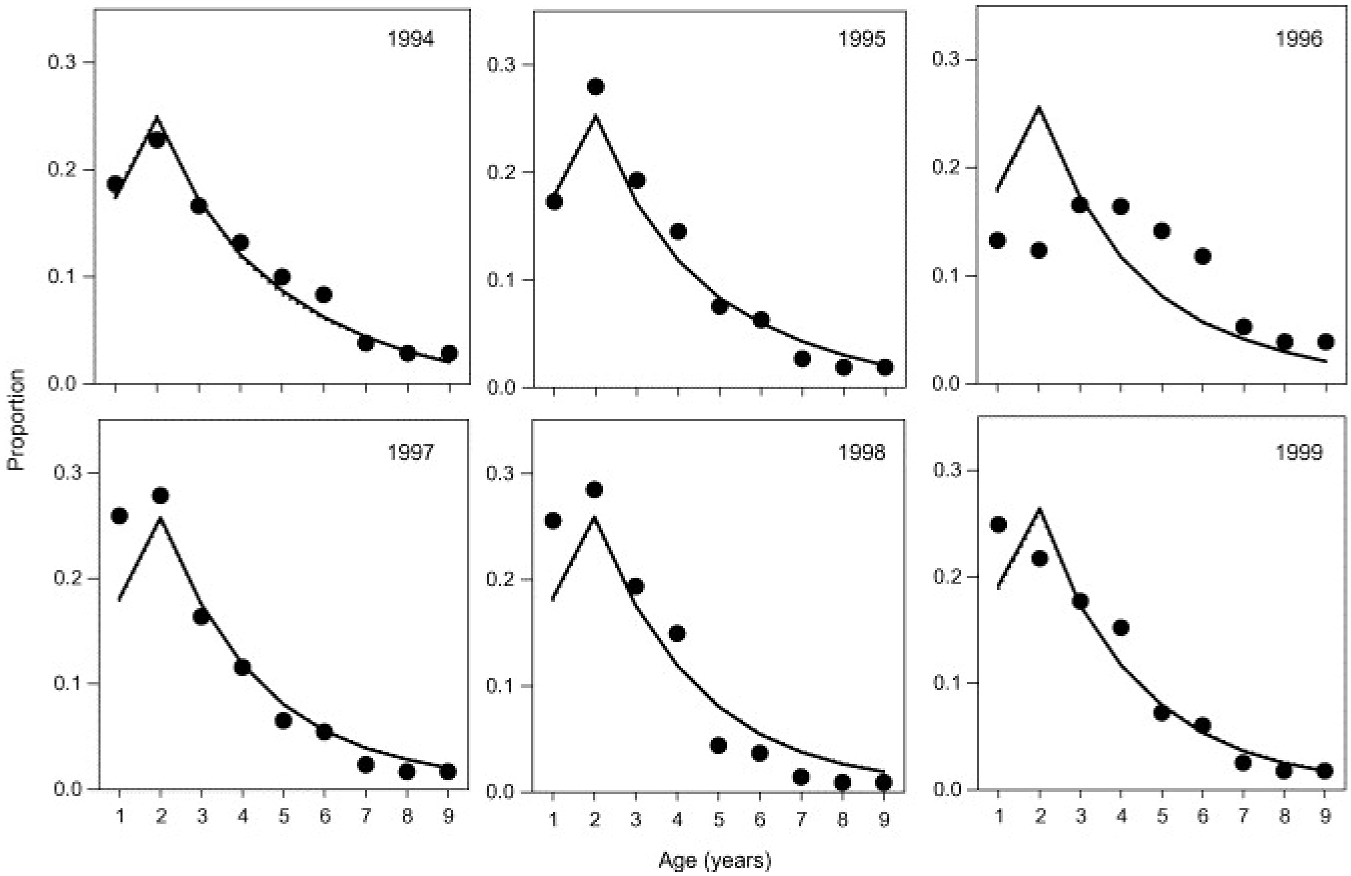

Fig. 4. Observed commercial catch-at-age and corresponding estimates from the maximum likelihood model for two data scenarios: using commercial and research indices (—) and using only commercial data (· · ·). (Commercial data were not used in the other scenario.)

Parameter variability is linked to how informative are the data included in the analyses (Table 5). That is, the coefficients of variation for the noninformative research data only scenario are all >15%.

3.2. Bayesian estimation

3.2.1. Scenarios with noninformative priors

The Bayesian approach with noninformative priors results in similar estimates to those from the maximum likelihood approach (Fig. 1; Table 5 and Table 6). The main difference lay with a more thorough investigation of the posterior surface around a broader range of values for *h*. As a consequence *h* is estimated between 0.64 and 0.81 with corresponding higher estimates of EB_0. The estimates of MSY and Depletion mirror the general productivity of the resource, as measured by *h* and EB_0. The results show that the resource has been depleted to between 36% and 44% of pristine levels, and that estimated MSY is between 9000 and 11 000 t.

Table 6.

Marginal posterior expected values and coefficients of variation (in parentheses) for five management quantities and three catchability coefficients from the Bayesian approach with both noninformative and informative priors

	Noninformative priors	**Informative priors**
Research and commercial data		
EB_0 ($\times 10^3$ t)	201.58 (12.04%)	177.85 (6.95%)
h	0.81 (22.7%)	0.88 (12.33%)
MSY ($\times 10^3$ t)	10.74 (14.89%)	10.27 (8.83%)
Depletion	0.44 (17.03%)	0.36 (16.45%)
$q^{HAKE\text{-}RES}$	0.17 (15.85%)	0.20 (10.97%)
$q^{MONK\text{-}RES}$	0.48 (23.67%)	0.64 (19.75%)
q^{COM}	1.08 (18.73%)	1.35 (14.06%)
EB_{2005}/MSYL	1.21 (19.99%)	1.04 (18.54%)
Research data only		
EB_0 ($\times 10^3$ t)	225.07 (36.77%)	165.83 (7.08%)
h	0.64 (51.43%)	0.83 (17.67%)
MSY ($\times 10^3$ t)	8.62 (35.78%)	9.05 (10.22%)
Depletion	0.36 (51.94%)	0.25 (26.11%)
$q^{HAKE\text{-}RES}$	0.18 (31.59%)	0.24 (10.43%)
$q^{MONK\text{-}RES}$	0.66 (50.82%)	0.95 (22.83%)
q^{COM}	–	–
EB_{2005}/MSYL	0.92 (56.18%)	0.70 (28.7%)
Commercial data only		
EB_0 ($\times 10^3$ t)	221.71 (21.53%)	201.39 (13.17%)
h	0.78 (27.17%)	0.87 (13.56%)
MSY ($\times 10^3$ t)	11.18 (18.47%)	11.49 (13.29%)
Depletion	0.48 (19.52%)	0.46 (17.35%)
$q^{HAKE\text{-}RES}$	–	–

	Noninformative priors	**Informative priors**
$q^{MONK\text{-}RES}$	–	–
q^{COM}	0.96 (24.12%)	1.06 (19.82%)
EB_{2005}/MSYL	1.31 (21.77%)	1.31 (18.91%)

The results are presented for three data scenarios: using both commercial and research survey data, using only the research survey data, and using only the commercial data.

3.2.2. Scenarios with informative priors

The inclusion of the informative priors has a directional, and stabilizing, effect on the posterior distributions when compared to the use of noninformative priors (Fig. 5; Table 6). The inclusion of the informative prior for $q^{MONK\text{-}RES}$ effectively places additional weight on the monkfish-directed trawl survey data by assuming that the survey biomass is close to total biomass. In this scenario, the outlook for the monkfish resource is more pessimistic with lower Depletion levels, lower predicted MSY and with exploitable biomass in 2005 being at, or slightly over, the biomass required to achieve MSY.

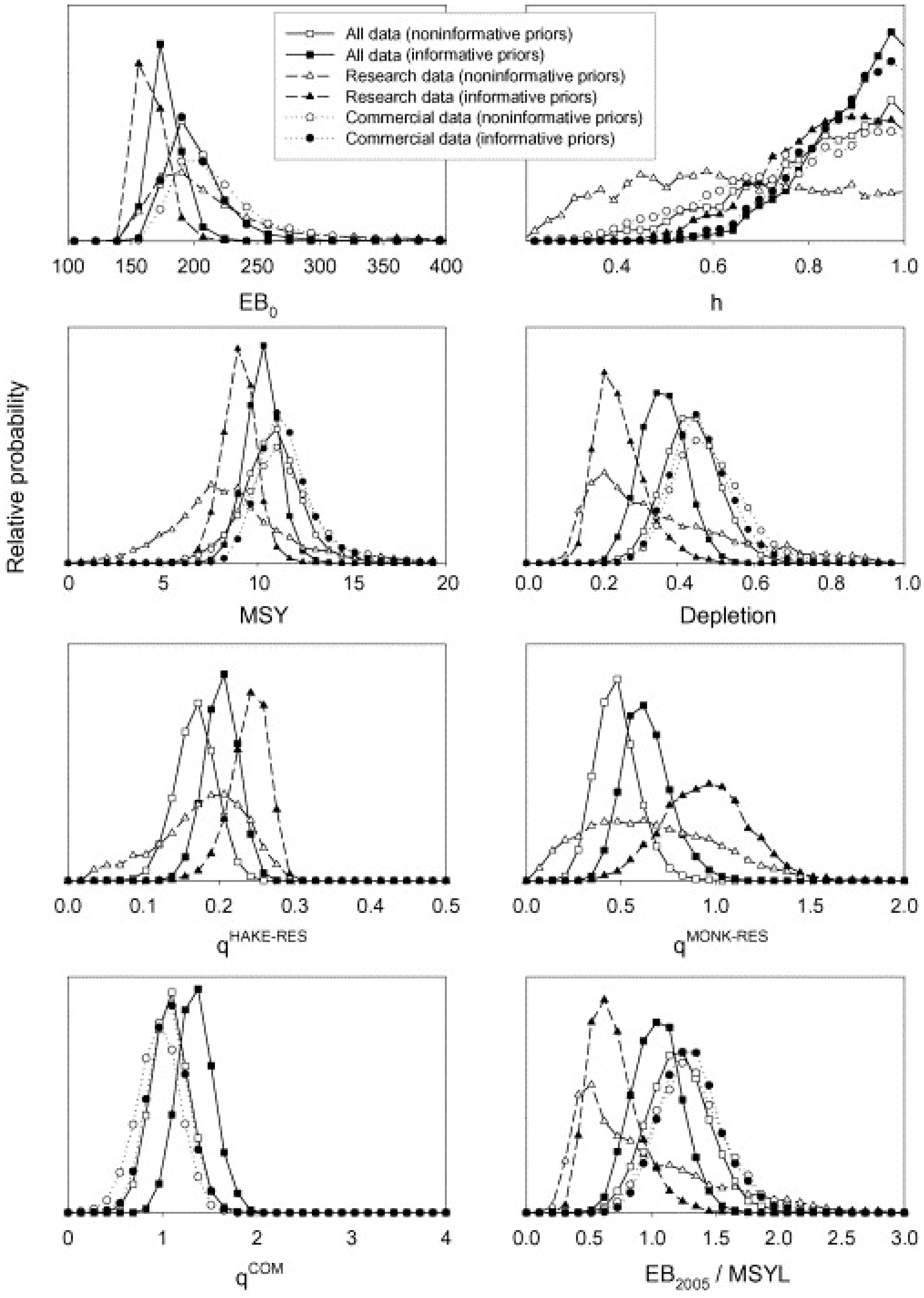

Fig. 5. Marginal posterior density functions for the parameters EB_0 and h, the management quantities MSY, Depletion and EB_{2005}/MSYL and the catchability coefficients $q^{HAKE\text{-}RES}$, $q^{MONK\text{-}RES}$ and q^{COM}. The results have been presented for three data scenarios: using both the commercial and research survey data, using only the research survey data and using only commercial data. Open symbols refer to the use of noninformative priors and filled symbols refer to the use of informative priors.

The inclusion of informative priors within the research data only scenario increases estimation precision considerably. For the other data scenarios that included the commercial data, there is only a marginal effect on improving parameter precision, because there is sufficient information contained in the likelihood not to be influenced by the priors. The inclusion of informative priors suggests that the resource has been fished to between 25% and 46% of pristine levels, and that estimated MSY lies between 9000 and 11 000 t.

4. Discussion

Trends in exploitable biomass and depletion were qualitatively similar, no matter what statistical approach was used or which data sources (Fig. 1). The results of the assessment provide evidence for a decrease in monkfish abundance from the 1970s to the late 1980s, a weak recovery in the early 1990s when catches were the lowest, and then a further decrease in abundance as catches increased with an expansion of the industry in year after Namibia's independence from South Africa. The only issue is the relative magnitude of the population decline (and other related parameters), which is estimated to be larger for the research only data scenario than the scenarios with commercial data. If commercial data are used in the assessment, then the monkfish resource appears to be healthy, with exploitable biomass above the MSY level and MSY near 10 000 t, regardless of whether research data are used.

The data sources and most of the other parameters were well-estimated by either the maximum likelihood approach or the Bayesian approach (Fig. 2, Fig. 3 and Fig. 4 and Table 5 and Table 6). Uncertainties in parameter estimates from the likelihood approach and the Bayesian approach with noninformative priors were similar, as expected. The Bayesian approach with informative priors produced lower uncertainties in general. However, misspecification of the priors would lead to biases in parameter estimates and uncertainties. Furthermore, there have been several studies that note that catchability in commercial CPUE often increases over time due to increases in fishing knowledge and technological development. Because this potential increase could not be estimated in this model due to data limitations, an increase in catchability would create a direct and nonconservative bias in all parameter estimates. Finally, the sensitivity to the choice of priors (Fig. 5) suggests that considerable uncertainty remains in the knowledge of Namibian monkfish.

The most difficult parameter to estimate in this case study was the steepness parameter h in the stock recruitment relationship. The maximum likelihood method failed to estimate h or its uncertainly with any consistency (Table 5) In data-poor situations like Namibian monkfish, we suggest that an alternative resampling approach, such as bootstrapping, should be employed instead of the Hessian method, if a maximum likelihood framework must be used. The Bayesian approaches used in this study, with and without the inclusion of informative priors, produced more stable estimates of steepness and its uncertainty (Table 6). Furthermore, the use of informative priors produced higher estimated precision than noninformative priors. Nevertheless, estimation bias cannot be addressed in a case study, so that a definitive statement about which is better cannot be made.

The standard stock assessment approach is to integrate all data sources about a population into a comprehensive model containing a likelihood function (Quinn, 2003). The use of the Bayesian approach is a natural extension, and one that is relatively straightforward to implement, because it generalizes model parameters to be random variables, which are initially specified as prior probabilities and then modified given the information available within the likelihood. The strength of a Bayesian (as opposed to a strictly maximum likelihood) approach lies with the explicit inclusion of these prior probabilities. Within a strict maximum likelihood context, there is no inclusion of *a priori* data. In the context of data-poor/limited situations or when available data are questionable, the use of prior specification is one of the few ways of dealing with such problems. In many cases, data-limited fisheries only have one abundance index. By including some prior information into the modeling framework, biological plausibility can be enhanced.

In many Bayesian analyses, the choice of priors can have a large effect on the outputs of the stock assessment model (Adkison and Peterman, 1996, Chen et al., 2000 and McAllister et al., 2001). In the absence of prior knowledge it has been common practice to seek an appropriate noninformative prior distribution, referred to as the reference prior. Both Lee (1997) and Millar (2002) note that flat, or uniform, priors are not necessarily noninformative, particularly if data are log-normally distributed or shape parameters such as variances are considered. The use of a Jeffreys' prior (Jeffreys, 1961) has therefore been proposed as a general solution in fisheries data (Millar, 2002). Derivation of a reference prior is not a trivial task, particularly with respect to complex fisheries models. Millar (2002) notes that the Jeffreys' prior for a simple biomass dynamic model, such as that used by Polacheck et al. (1993), cannot be written in closed form. In this study, as in both the biomass dynamic and sequential population analysis models outlined by Millar (2002), the catchability coefficients are independent of the other model parameters, so that the inverse priors used are therefore suitable. However, we were unable to construct Jeffreys' priors for other parameters in our hybrid age-structured surplus production mode, because biomass in each year is dependent on the two model parameters h and EB_0 and the previous year's spawner biomass (see Appendix A). There is usually a strong correlation between h and EB_0, because increased prior density is given to higher values of h as EB_0 increases. Alternatively, confidence distributions, as proposed by Schweder (1998) could be used in conjunction with other priors that exhibit desirable properties such as invariance and good coverage. A better set of reference priors for this model will be the subject of future work.

There is both potential and danger in the use of Bayesian stock assessment models in data-poor situations. First, there may be a desire to obtain a constrained solution for a parameter or a management quantity. One concern is that the model could be "massaged" to obtain a preconceived solution and therefore the specification of priors requires careful justification and scrutiny. Second, some form of stock assessment model is frequently needed by managers to make some form of decision. In data-poor situations, the only available option may be to use "best guesses" explicitly specified as priors (and then updated as more data become available). But errors in these best guesses could affect decisions, because the choice of priors affects model results. Thirdly, decision analysis, a formal way of including all known (and unknown) states of nature within a framework of various management choices, could be the strongest use of the Bayesian approach. The key in this approach is to expose the uncertainties due to alternative states of nature being possible. Quantitative methods can then be brought in to attempt to achieve societal objectives.

This study has shown that a biologically realistic stock assessment model can be developed in a 'questionable' data situation using maximum likelihood and Bayesian methodology. Our case study shows that the Bayesian approach is complementary to (and a generalization of) the maximum likelihood approach. The development of plausible prior distributions in the Bayesian setting produced a more credible estimate of steepness than in the maximum likelihood setting, which led to more defensible estimates of management parameters.

No longer is it particularly difficult to put together a stock assessment model that contains at least partial age structure information to estimate management quantities. The inevitable conflicts between different data sources can be explored by investigating different scenarios in which some data sources are omitted. Further, noninformative and informative priors can be explored within the same modeling context. While this approach cannot singularly identify which scenario is most appropriate, it can highlight the advantages and disadvantages of different choices, and in the spirit of openness in stock assessment modeling, to fully and explicitly identify the parameter and data uncertainties and model and data alternatives of our understanding of natural systems.

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Appendix A. The age-structured surplus production model

A.1. Resource dynamics

Population abundance, in numbers, is governed by the following equations:

$$N_{y,0} = \frac{\mathrm{SB}_{y-1}}{\alpha + \beta \mathrm{SB}_{y-1}},$$

$$N_{y,a} = (N_{y-1,a-1}\mathrm{e}^{-M/2} - C_{y-1,a-1})\mathrm{e}^{-M/2},$$

$$N_{y,\max} = (N_{y-1,\max-1}\mathrm{e}^{-M/2} - C_{y-1,\max-1})\mathrm{e}^{-M/2} + (N_{y-1,\max}\mathrm{e}^{-M/2} - C_{y-1,\max})\mathrm{e}^{-M/2}$$

where $N_{y,a}$ is the number of fish at age a at the start of year y, M the rate of natural mortality, $C_{y,a}$ the numbers of fish caught at age a in year y, and max is the maximum age considered and treated as lumped plus-group.

Trawl-based selectivity for gear i (for either commercial or research trawling) as a function of age a is described by a logistic function of the form $S_a^i = (1 + \exp(-(a - a_{50}^i)/\delta^i))^{-1}$ where $a_{50}^i$is the age at 50% selection and δ is the curvature parameter.

The model is initiated in the year preceding fishing activity and the initial number of age-0 recruits, R_0, is calculated from the mid-year exploitable biomass, the commercial selectivity at age a S_a^{COM} and EB_0, and the exploitable biomass per-recruit such that

$$\sqrt{1} \exp(-[a + 1/2]M] + W_{max+1/2} S_{max}^{COM} \frac{1}{1 - \exp(-M)} \exp(-[max + 1/2]M) \}$$

Spawner biomass in year y, SB_y, is defined as $SB_y = \sum_{a=0}^{max} N_{y,a} W_a \psi_a$, with ψ_a the proportion of fish at age a that are sexually mature, calculated from the logistic function $\psi_a = (1 + \exp(-(a - a_{50}^m)/\delta^m))^{-1}$, and W_a the begin-year weight of a fish of age a, such that $W_a = \gamma[L_\infty(1 - e^{-K(a - t_0)})]^\Phi$, with L_∞, K and t_0 the von Bertalanffy growth equation parameters, γ and Φ the length–weight parameters, and a_{50}^m and δ^m, the logistic parameters.

The stock recruitment relationship is reparameterized to contain a single "steepness" parameter h (that proportion of pristine recruitment when spawner biomass is reduced to 20% of pristine levels) such that

$$\alpha = \frac{SB_0(1 - h)}{4R_0 h} \quad \text{and} \quad \beta = \frac{(5h - 1)}{4R_0 h}.$$

A.2. Catches

Annual catches are assumed to occur mid-year and estimated as

$$C_y = \mu_y \sum_{a=0}^{max} W_{a+1/2} N_{y,a} S_a^{COM} e^{-M/2}$$

where $W_{a+1/2}$ is the weight-at-age of a fish at the middle of the year and μ_y is the exploitation fraction for year y.

A.3. The likelihood function

The model is fitted to the commercial CPUE and research survey abundance indices, and the commercial and research survey catch-at-age data. The likelihood function is, therefore, composed of five components, such that $L = \lambda_{CPUE} \times \lambda_{HAKE\text{-}RES\ BIOMASS} \times \lambda_{MONK\text{-}RES\ BIOMASS} \times \lambda_{COM\ CAA} \times \lambda_{RES\ CAA}$.

A.3.1. Biomass data

The first three likelihood components, λ_{CPUE}, $\lambda_{HAKE\text{-}RES\ BIOMASS}$ and $\lambda_{MONK\text{-}RES\ BIOMASS}$, corresponding to the biomass indices, assume that each observed abundance index is log-normally distributed about its expected value such that $O_y^i = q^i I_y^i \exp(\varepsilon_y^i)$ and $\varepsilon_y^i \sim N(0, (\sigma_y^i)^2)$, where O_y^i and I_y^i are the observed and model predicted biomasses for index i, q^i the catchability coefficient for index i, σ_y^i is the standard deviation for index i in year y.

To correct for possible negative bias in the estimates of variance, and prevent giving undue weight to either index, a lower variance bound (σ_A^i) was input into the model (Rademeyer, 2003), such that $(\sigma_y^i)^2 = \sigma_A^2 + (CV_y^i)^2$. This was specified at 0.15 for the commercial index and 0.20 for both biomass indices. Sampling error associated with each annual abundance estimate in index i in year y is also included in the annual variance estimate. For the commercial CPUE index a coefficient of variation of 15% is assumed (Francis et al., 2003). For the research surveys, the coefficient of variation associated with each annual survey was used.

Each likelihood component is, therefore, expressed as

$$\lambda_i = \prod_y \frac{1}{\sigma_y^i \sqrt{2\pi}} \exp\left(\frac{1}{2(\sigma_y^i)^2} \ln\left(\frac{O_y^i}{\hat{q}^i I_y^i} \right)^2 \right)$$

The model estimates of biomass for the commercial indices are calculated as

$$I_y^i = \sum_{a=0}^{\max} W_{a+1/12} S_a^{\mathrm{COM}} N_{y,a} e^{-M/2} (1 - S_a^{\mathrm{COM}} \mu_{y/2}),$$

for the hake-directed research survey index as

$$I_y^i = \sum_{a=0}^{\max} W_{a+m/12} S_a^{\mathrm{RES}} N_{y,a} e^{-mM/12} \left(1 - S_a^{\mathrm{RES}} \frac{m}{12} \mu_y\right)$$

And for the monkfish-directed research survey index as

$$I_y^i = \sum_{a=0}^{\max} W_{a+m/12} S_a^{\mathrm{COM}} N_{y,a} e^{-mM/12} \left(1 - S_a^{\mathrm{COM}} \frac{m}{12} \mu_y\right)$$

where m is the month when the survey occurred and S_a^{RES} is the age-specific research survey selectivity at age a.

The residuals are assumed to be homoscedastic such that $\sigma_y^i = \hat{\sigma}^i$. The standard deviation is then calculated as its maximum likelihood estimate as

$$\hat{\sigma}^i = \sqrt{\frac{\sum_y (\ln(O_y^i) - \ln(I_y^i))^2}{\sum_y 1} - (\sigma_A^i)^2}$$

The catchability coefficients are replaced by their maximum likelihood equivalents, such that

$$\hat{q}^i = \exp\left(\frac{\sum_y [(\sigma_A^i)^2 + (\mathrm{CV}_y^i)^2]^{-1} [\ln(O_y^i) - \ln(I_y^i)]}{\sum_y [(\sigma_A^i)^2 + (\mathrm{CV}_y^i)^2]^{-1}}\right)$$

A.3.2. Catch-at-age data

The third and forth components of the likelihood, $\lambda_{\mathrm{COM\ CAA}}$ and $\lambda_{\mathrm{RES\ CAA}}$, correspond to the commercial and research survey catch-at-age data, respectively.

As with the biomass data, it is assumed that the proportion of fish of age *a* caught in year *y*, $p^i_{a,y}$, are log-normally distributed about their expected value, $\hat{p}^i_{a,y}$, such that $p^i_{a,y} = \hat{p}^i_{a,y} \exp(\varepsilon^i_{a,y})$and a standard deviation of $\sigma^i / \sqrt{\hat{p}^i_{a,y}}$such that $\varepsilon^i_{a,y} \sim N\left(0, \left(\sigma^i / \sqrt{\hat{p}^i_{a,y}}\right)^2\right)$(Smith and Punt, 1998 and Rademeyer, 2003). This "adjusted" form the of the log-normal likelihood weights the residuals based on the expected proportions, thereby ensuring that undue importance is not placed on small proportions.

The form of the likelihood for each series *i* is therefore:

$$\lambda_i = \prod_a \prod_y \frac{\sqrt{\hat{p}^i_{a,y}}}{\hat{\sigma}^i_y \sqrt{2\pi}} \exp\left(\frac{\hat{p}^i_{a,y}}{2(\hat{\sigma}^i_y)^2} \ln\left(\frac{p^i_{a,y}}{\hat{p}^i_{a,y}}\right)^2\right)$$

with the predicted proportions at age calculated as:

$$\hat{p}^{\mathrm{COM}}_{a,y} = \frac{N_{y,a} S^{\mathrm{COM}}_a \mathrm{e}^{-M/2}}{\sum_a N_{y,a} S^{\mathrm{COM}}_a \mathrm{e}^{-M/2}}$$

$$\hat{p}^{\mathrm{RES}}_{a,y} = \frac{N_{y,a} S^{\mathrm{RES}}_a \mathrm{e}^{-mM/12} \left(1 - S^{\mathrm{RES}}_a \frac{m}{12} \mu_y\right)}{\sum_a N_{y,a} S^{\mathrm{RES}}_a \mathrm{e}^{-mM/12} \left(1 - S^{\mathrm{RES}}_a \frac{m}{12} \mu_y\right)}.$$

The standard deviation is calculated as its maximum likelihood estimate as

$$\hat{\sigma}^i = \sqrt{\frac{\sum_a \sum_y \hat{p}^i_{a,y} \ln\left(\frac{p^i_{a,y}}{\hat{p}^i_{a,y}}\right)^2}{\sum_a \sum_y 1}}$$

A.4. Estimating yield

Yield, as a function exploitation fraction μ, was calculated as $Y(\mu) = \text{YPR}(\mu) \times R(\mu)$, where $\text{YPR}(\mu)$ is the yield per recruit as a function of the exploitation fraction μ, and $R(\mu)$ is the equilibrium recruitment as a function of the exploitation fraction μ.

Yield per recruit, as a function of the exploitation fraction, was estimated as

$$\text{YPR}(\mu) = \mu \sum_{a=0}^{\max} \tilde{N}_a W_{a+1/2} S_a^{\text{COM}} \mathrm{e}^{-M/2}$$

where

$$\tilde{N}_a = \begin{cases} 1 & \text{if } a = 0 \\ \tilde{N}_{a-1} \mathrm{e}^{-M}(1 - \mu S_a^{\text{COM}}) & \text{if } 0 < a < \max \\ \dfrac{\tilde{N}_{a-1} \mathrm{e}^{-M}(1 - \mu S_a^{\text{COM}})}{1 - \mathrm{e}^{-M}(1 - \mu S_{\max}^{\text{COM}})} & \text{if } a = \max \end{cases}$$

Equilibrium recruitment μ is calculated as

$$R(\mu) = \frac{\text{SBR}(\mu) - \alpha}{\text{SBR}(\mu)\beta}$$

where $\text{SBR}(\mu) = \sum_{a=0}^{\max} \tilde{N}_a W_a \psi_a$, and α and β the maximum likelihood estimates of the stock recruitment parameters.

Yield is maximised iteratively by calculating the first and second finite difference numerical derivatives of $Y(\mu)$ with respect to μ. The first and second derivatives were calculated as

$$f' = \frac{Y(\mu + h) - Y(\mu)}{h}$$

and

$$f'' = \frac{Y(\mu + h) - 2Y(\mu) + Y(\mu - h)}{h^2}$$

where $h = 0.00001$.

The yield curve was maximised by updating μ iteratively as $\mu_i = \mu_{i-1} - (f'/f'')$, where i is the ith iteration, until a tolerance of $|(\mu_i - \mu_{i+1})/\mu_{i+1}| > 0.00001$ is reached.