A FEW POINTS ON POINTFREE PSEUDOCOMPACTNESS

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Abstract. We present several characterizations of completely regular pseudocompact frames. The first is an extension to frames of characterizations of completely regular pseudocompact spaces given by Väänänen [19]. We follow with an embedding-type characterization stating that a completely regular frame is pseudocompact if and only if it is a P-quotient of its Stone-Čech compactification. We then give a characterization in terms of ideals in the cozero parts of the frames concerned. This characterization seems to be new and its spatial counterpart does not seem to have been observed before. We also define relatively pseudocompact quotients, and show that a necessary and sufficient condition for a completely regular frame to be pseudocompact is that it be relatively pseudocompact in its Hewitt realcompactification. Consequently a proof of a result of Banaschewski and Gilmour [6] that a completely regular frame is pseudocompact if and only if its Hewitt realcompactification is compact, is presented without the invocation of the Boolean Ultrafilter Theorem.

This paper is, in a way, a sequel to our paper [11] in which we gave several characterizations of pseudocompact frames with no separation axiom imposed. In this paper we restrict to completely regular frames. We start by extending to frames Väänänen's [19] characterizations of pseudocompactness (Proposition 2.1). Our proofs in this regard are facilitated by a recent result of Naidoo's [16] stating that a totally bounded Cauchy complete uniform frame is compact. This is the frame version of the classical result that a complete precompact uniform space is compact.

Following that, we define P-quotients of frames and give some embedding-type characterizations including one stating that a completely regular frame is pseudocompact if and only if it is a P-quotient of its Stone-Čech compactification (Corollary 2.6). We then observe, en passant, that if a completely regular frame is a P-quotient of each of its compactifications, then it admits only one uniformity, so that in fact it has only one compactification. Another embedding-type characterization is that a completely regular frame is pseudocompact if and only if whenever it is a quotient, then it is an almost coz-codense quotient (Proposition 2.8).

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We then characterize pseudocompactness in terms of ideals in cozero parts of frames, and prove that pseudocompact frames are exactly those the maximal ideals of the cozero parts of which are precisely the prime σ-ideals (Proposition 2.11). Incidentally, this result appears to be new, and its spatial counterpart does not seem to have been observed before.

A frame homomorphism transfers ideals of the cozero part of its codomain to ideals of the cozero part of its domain in a natural way. We establish that a necessary and sufficient condition for a completely regular frame to be pseudocompact is that maximal ideals in its cozero part are transferred in this natural way to maximal ideals in the cozero part of its Stone-Čech compactification (Proposition 2.12).

In section 3 we consider the frame equivalent of a subspace being relatively pseudocompact in the sense that every continuous function on the containing space is bounded on the subspace. This consideration, in conjunction with certain observations about the Hewitt realcompactification of a frame, enables us to obtain, without invocation to the Boolean Ultrafilter Theorem (BUT), a proof of a result of Banaschewski and Gilmour [6] (they used BUT) that a completely regular frame is pseudocompact if and only if its Hewitt realcompactification is compact. This in turn is true precisely when the frame is relatively pseudocompact in its Hewitt realcompactification (Proposition 3.3).

1. Preliminaries

We recall some of the definitions that we shall need and refer to Johnstone [14] and Picado, Pultr and Tozzi [17] for a general background on frames. Pultr [18] gives a more algebraic treatment of the subject. The properties of σ-frames that we need can all be culled from Banaschewski and Gilmour [5]. Good references for uniform frames include Banaschewski [3].

A *frame* is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee\{a \wedge x \mid x \in S\}$$

holds for all $a \in L$ and all $S \subseteq L$. We denote the top element and the bottom element of a frame by 1_L and 0_L respectively; omitting the subscript if no confusion may arise. If X is a topological space, $\mathfrak{O}X$ will denote the frame of its open subsets.

Throughout this commentary L will denote a frame. A *cover* C of L is a subset with $\bigvee C = 1$. A cover C *refines* a cover D if for each $c \in C$ there exists $d \in D$ such that $c \leq d$. A subset S of L is *locally finite* if there is a cover C such that each element of C meets only finitely many elements of S. We say L is *paracompact* if each cover has a locally finite refinement. It is *compact* (resp. countably compact) if each cover (resp. each countable cover) has a finite subcover, and it is *Lindelöf* if every cover has a countable subcover.

We say L is *regular* if, for each $a \in L$, $a = \bigvee\{x \in L \mid x \prec a\}$, where $x \prec a$ means that there exists $s \in L$ such that $x \wedge s = 0$ and $s \vee a = 1$. This is equivalent to $x^* \vee a = 1$ for the *pseudocomplement* $x^* = \bigvee\{w \in L \mid w \wedge x = 0\}$ of the element x. It is *completely regular* if, for each $a \in L$, $a = \bigvee\{x \in L \mid x \prec\prec a\}$, where $x \prec\prec a$ means that there is a scale $(c_q | q \in \mathbb{Q} \cap [0,1])$ such that $x = c_0, a = c_1$ and $c_q \prec c_p$ whenever $q < p$. It is *normal* if whenever $a \vee b = 1$, then there are elements u and v such that $u \wedge v = 0$, $u \vee a = v \vee b = 1$.

A *frame homomorphism* is a map between frames that preserves finite meets, including the top element, and arbitrary joins, including the bottom element. A frame homomorphism is *dense* if it maps only the bottom to the bottom. Associated with a frame homomorphism $h : L \to M$ is its *right adjoint* $h_* : M \to L$ characterized by: $h(x) \leq y \;\Leftrightarrow\; x \leq h_*(y)$.

A *quotient* of L is a pair (h, M) where $h : L \to M$ is an onto frame homomorphism. When we say a quotient $h : L \to M$ has a property of frames (resp. of homomorphisms) we mean that M (resp. h) has that property. The quotients $L \to \uparrow a$, given by $x \mapsto a \vee x$, for each $a \in L$ are said to be *closed.*

Because we want our study to stay in the point-free context (so that, among other things, we do not refer to classical reals), we shall follow the practice in Banaschewski and Gilmour [5] of defining pseudocompact frames and cozero elements of frames in terms of the *frame of reals* $\mathfrak{L}(\mathbb{R})$, which is the frame generated by the ordered pairs (p, q) of rational numbers $p, q \in \mathbb{Q}$ subject to the relations:

(i) $(p, q) \wedge (s, t) = (p \vee s, q \wedge t)$,

(ii) $(p, q) \vee (s, t) = (p, t)$ whenever $p \leq s < q \leq t$,

(iii) $(p, q) = \bigvee\{(s, t) \mid p < s < t < q\}$,

(iv) $1_{\mathfrak{L}(\mathbb{R})} = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

An element $a \in L$ is a *cozero* element if there is a frame homomorphism $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $a = h\big((-, 0) \vee (0, -)\big)$, where $(-, 0) = \bigvee\{(p, 0) \mid p < 0 \text{ in } \mathbb{Q}\}$ and $(0, -) = \bigvee\{(0, q) \mid 0 < q \text{ in } \mathbb{Q}\}$. The *cozero part* of L, denoted $\operatorname{Coz} L$, is the sub-σ-frame consisting of all the cozero elements of L. A useful characterization is that $a \in \operatorname{Coz} L$ if and only if there is a sequence (a_n) in L such that $a = \bigvee a_n$ and $a_k \prec\prec a_{k+1}$ for each k.

Turning to uniform frames, the *star* of an element $a \in L$ in a cover C of L is the element $Ca = \bigvee\{x \in C \mid x \wedge a \neq 0\}$. If A and B are covers of L, AB denotes the cover consisting of the stars of B in A. We then say A *star-refines* B, written $A \leq^* B$, in case AA refines B.

For a collection $\mathfrak{U}$ of covers of L we define the relation $\lhd_{\mathfrak{U}}$, or simply $\lhd$, on L by $a \lhd b$ iff $Ua \leq b$ for some $U \in \mathfrak{U}$. We say $\mathfrak{U}$ is *admissible* if, for each $a \in L$, $a = \bigvee\{x \in L \mid x \lhd a\}$. A *uniformity* on L is an admissible collection of covers which is a filter with respect to the refinement pre-order and such that each cover in the collection has a star-refinement in the collection. A *uniform frame* is a pair $(L, \mathfrak{U})$ consisting of a frame and a uniformity on it. The covers in $\mathfrak{U}$ are then called *uniform covers*. We shall at times allow notational confusion between a uniform frame and its underlying frame and speak of a uniform frame L. A frame homomorphism between uniform frames is called *uniform* if it maps uniform covers to uniform covers, and a *surjection* if it is onto both for the underlying frames and the uniformities. A uniform frame is said to be *complete* if any dense surjection to it is an isomorphism. A *completion* of a uniform frame L is a dense surjection $M \to L$ with M complete. Any uniform frame L has a completion which is denoted by $CL \to L$.

A filter F in L *converges* if it meets every cover of L. A filter in a uniform frame is *Cauchy* if it meets every uniform cover. A uniform frame is *Cauchy complete* if every Cauchy filter converges. It is *totally bounded* (or *precompact*) if every uniform cover has a finite uniform refinement.

Lastly, recall that a cover A of L is *normal* if there is a sequence (A_n) of covers of L such that $A_{n+1} \leq^* A_n$ for each n, and A_1 refines A. The collection of normal covers of a completely regular frame is a uniformity called the *fine* uniformity on L.

2. Equivalences of pseudocompactness

We start by recalling that for a frame L, a frame homomorphism $h : \mathfrak{L}(\mathbb{R}) \to L$ is said to be *bounded* if there exist $p, q \in \mathbb{Q}$ such that $h(p, q) = 1_L$. The frame is then called *pseudo-compact* in case all frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$ are bounded. Clearly, a subframe of a pseudocompact frame is pseudocompact.

Banaschewski and Pultr [8] have shown that a completely regular frame is pseudocompact if and only if it admits only totally bounded uniformities, if and only if every normal cover has a finite normal refinement. Walters-Wayland [20] has shown that a completely regular frame L is pseudocompact if and only if the Stone-Čech map $\beta : \beta L \to L$ is *coz-codense*, where the latter means that the only cozero element mapped to the top is the top.

The following characterization is the frame version of Proposition 2 in [19].

Proposition 2.1. *The following are equivalent for any completely regular frame L:*
(1) *L is pseudocompact.*
(2) *For any homomorphism $h : M \to L$ with M Lindelöf, the frame $\uparrow h_*(0)$ is compact.*
(3) *For any homomorphism $h : M \to L$ with M hereditarily Lindelöf, the frame $h[M]$ is compact.*
(4) *For any homomorphism $h : M \to L$ with M countably generated, the frame $h[M]$ is compact.*
(5) *For any homomorphism $h : M \to L$, where M is the underlying frame of a uniform frame $(M, \mathfrak{U})$, the uniform frame $(h[M], h[\mathfrak{U}])$ is totally bounded.*
(6) *For any homomorphism $h : M \to L$, where M is the underlying frame of a Cauchy complete uniform frame, the frame $\uparrow h_*(0)$ is compact.*

Proof. (1) $\Rightarrow$ (2): We shall use the fact that a completely regular frame is compact if and only if it is pseudocompact and Lindelöf (see [5, Remark 1]). Because closed quotients of Lindelöf frames are Lindelöf, it suffices to show that $\uparrow h_*(0)$ is pseudocompact. We claim that a frame that has a dense pseudocompact quotient is itself pseudocompact. To prove this we use Proposition 4.1(3) in [11]. So let N be a pseudocompact frame and $g : K \to N$ be a dense onto homomorphism. Then for any completely regular countable filterbase F in K, $g[F]$ is a completely regular countable filterbase in N since frame homomorphisms preserve the completely below relation. Hence $\bigvee\{g(t)^* \mid t \in F\} \neq 1$. Since dense onto homomorphisms preserve pseudocomplements, it follows that $g(\bigvee\{t^* \mid t \in F\}) \neq 1$, and therefore $\bigvee\{t^* \mid t \in F\} \neq 1$, as required. Now $h[M]$ is pseudocompact because it is a subframe of the pseudocompact frame L. Furthermore, the homomorphism $\bar{h} : \uparrow h_*(0) \to h[M]$, mapping as h, is dense onto, and so $\uparrow \bar{h}_*(0)$ is pseudocompact. But $\bar{h}_*(0) = h_*(0)$; so the result follows.

(2) $\Rightarrow$ (3): The frame $h[M]$ is Lindelöf since it is a quotient of the hereditarily Lindelöf frame M. Let $i : h[M] \to L$ be the inclusion map. By the hypothesis we have that $\uparrow i_*(0)$ is compact. Since $i_*(0) = 0$, it follows that $h[M]$ is compact.

(3) $\Rightarrow$ (4): This is so because a countably generated frame is hereditarily Lindelöf, as one checks easily.

(4) $\Rightarrow$ (1): Let $f : \mathfrak{L}(\mathbb{R}) \to L$ be a frame homomorphism. Since $\mathfrak{L}(\mathbb{R})$ is countably generated, we have, by the hypothesis, that $f[\mathfrak{L}(\mathbb{R})]$ is compact. The collection $\{(-n, n) \mid n \in \mathbb{N}\}$ is a cover of $\mathfrak{L}(\mathbb{R})$, so $\{f(-n, n) \mid n \in \mathbb{N}\}$ is a cover of the compact frame $f[\mathfrak{L}(\mathbb{R})]$. Hence $f(-k, k) = 1_L$ for some $k \in \mathbb{N}$ since the elements $f(-n, n)$ form an increasing sequence. Therefore L is pseudocompact.

(1) $\Rightarrow$ (5): If L is pseudocompact and $h : M \to L$ is a homomorphism, then $h[M]$ is pseudocompact being a subframe of a pseudocompact frame. Hence whatever uniformity it admits is totally bounded.

(5) $\Rightarrow$ (6): Consider $\uparrow h_*(0)$ as a uniform frame with the uniformity inherited from $(M, \mathfrak{U})$. By Theorem 3.3 in [16] it suffices to show that $\uparrow h_*(0)$ is totally bounded and Cauchy complete. That it is Cauchy complete holds since every closed quotient of the underlying frame of a Cauchy complete uniform frame is Cauchy complete in the inherited uniformity. Now the map $\bar{h} : \uparrow h_*(0) \to h[M]$ is a dense uniform homomorphism. Since the hypothesis is that $h[M]$ is totally bounded, it follows from Lemma 3.1 and Theorem 3.1 in [16] that $\uparrow \bar{h}_*(0)$ is totally bounded. Therefore it is compact. As before, the result is proved since $h_*(0) = \bar{h}_*(0)$.

(6) $\Rightarrow$ (1): We shall show that every normal cover of L has a finite normal refinement. To this end think of L as a uniform frame with the fine uniformity. Let $h : CL \to L$ be the completion of L. Then CL is Cauchy complete; and so, by the hypothesis, $\uparrow h_*(0)$ is compact. As above, let $\bar{h} : \uparrow h_*(0) \to L$ map as h. Let U be a normal cover of L, so that U is a uniform cover. Find a uniform cover V of CL such that $h[V]$ refines U. The set $W = \{h_*(0) \vee v \mid v \in V\}$ is a cover of $\uparrow h_*(0)$ such that $\bar{h}[W]$ refines U. By compactness W has a finite subcover T. Again by compactness, T is a normal cover of $\uparrow h_*(0)$, and therefore $\bar{h}[T]$ is a finite normal cover of L that refines U. ■

A reader who is familiar with the Hewitt realcompactification of a frame, and the fact that it is Cauchy complete in a certain uniformity, will observe that property (6) in the above proposition enables us to deduce that the Hewitt realcompactification of a pseudocompact frame is compact. We will elaborate on this in section 3.

Remark 2.2. The result of Naidoo's cited in the proof of (5) $\Rightarrow$ (6) actually tells us more.

Calling a frame *Dieudonné complete* in case it admits a Cauchy complete uniformity (terminology straight out of topology), we have that *a Dieudonné complete frame is pseudocompact iff it is compact, iff it is countably compact.* Because "compact" $\Rightarrow$ "countably compact" $\Rightarrow$ "pseudocompact" always, we verify the outstanding implication. Let L be Dieudonné complete and pseudocompact. Then, with respect to any Cauchy complete uniformity it admits, L is totally bounded since it is pseudocompact. So L is compact.

Remark 2.3. Since a complete uniform frame is Cauchy complete, and since a frame is paracompact if and only if it admits a complete uniformity, we deduce from Naidoo's result that a paracompact pseudocompact frame is compact. This was first proved by Banaschewski and Pultr [8] by a different line of reasoning.

Next, we turn our attention to embedding-type characterizations for which we shall need the following definition.

Definition 2.4. A quotient $h : L \to M$ of L is a *P-quotient* (resp. *$P^{\aleph_0}$-quotient*) if every normal cover (resp. every countable normal cover) of M is refined by the image of some normal cover of L.

A reader with a topological outlook will recognize these definitions as nothing more than frame-theoretic articulations of some of the characterizations (see Alo and Shapiro [1]) of P-embedded (resp. $P^{\aleph_0}$-embedded) subspaces.

Before presenting the characterizations, we remind the reader that a quotient $h : L \to M$ is a *C-quotient* (resp. a *C^*-quotient*) if for every frame homomorphism (resp. every bounded frame homomorphism) $f : \mathfrak{L}(\mathbb{R}) \to M$, there is a frame homomorphism $\tilde{f} : \mathfrak{L}(\mathbb{R}) \to L$ such that $h\tilde{f} = f$. This captures precisely the notions of C-embedded and C^*-embedded subspaces. On the other hand, a quotient is *coz-onto* if every cozero element of the codomain is the image of some cozero element of the domain. This in turn captures the notion of z-embedded subspaces.

Pivotal to our proofs will be the following observation: If A and B are covers of a frame L and B star-refines A, then there is a cozero cover of L that refines A. Indeed, for each $b \in B$ choose $a_b \in A$ such that $Bb \leq a_b$. Then $b \prec\!\!\prec a_b$; and therefore there is a cozero element c_b such that $b \leq c_b \leq a_b$. This in fact shows that a cover that has a star-refinement of cardinality $\mathfrak{m}$ has a cozero refinement of cardinality at most $\mathfrak{m}$. In particular then, and this we emphasize, *a cover which is refined by a finite (resp. countable) normal cover, is also refined by a finite (resp. countable) cozero cover.*

Our embedding-type characterization will be an immediate corollary to the following proposition which we state more generally than is really needed to deduce the characterization.

Proposition 2.5. *The following are equivalent for a pseudocompact quotient $h : L \to M$ of a completely regular frame L:*
(1) *h is coz-onto.*
(2) *h is a C^*-quotient map.*
(3) *h is a C-quotient map.*
(4) *h is a P-quotient map.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): If h is coz-onto, then it is a C-quotient map by Proposition 4.10 in [12], and therefore a C^*-quotient map. On the other hand, since every C^*-quotient map is coz-onto, the equivalences hold.

(2) $\Rightarrow$ (4): Let A be a normal cover of M. Then, since M is pseudocompact, there is a finite normal cover B of M that refines A. So, by what we observed above, there is a finite cozero cover C of M that refines A. Therefore, by Proposition 7.1.1(6) in [2], there is a finite cozero cover D of L such that $h[D]$ refines C, and hence A. But now countable cozero covers of a completely regular frame are normal; so the result follows.

(4) $\Rightarrow$ (2): Let C be a finite cozero cover of M. Then C is a normal cover, and so there is a normal cover A of L such that $h[A]$ refines C. For each $c \in C$ let $a_c = \bigvee\{x \in A \mid h(x) \leq c\}$. Then $\tilde{A} = \{a_c \mid c \in C\}$ is a finite normal cover of L such that $h[\tilde{A}]$ refines C. Since finite normal covers of L generate a uniformity on L, $\tilde{A}$ is refined by a finite cozero cover. Thus C is refined by the image of a finite cozero cover, and therefore M is a C^*-quotient again by Proposition 7.1.1(6) in [2]. ■

Notice that in the proof of (4) $\Rightarrow$ (2) in the above result, the pseudocompactness of M is not used. We therefore have that, in general, every P-quotient is also a C^*-quotient.

Corollary 2.6. *A completely regular frame is pseudocompact iff it is a P-quotient of its Stone-Čech compactification.*

Proof. The forward implication follows from Proposition 2.5 because every completely regular frame is a C^*-quotient of its Stone-Čech compactification by Corollary 8.2.7 in [2].

Conversely, let A be a normal cover of L. Then there is a normal cover C of βL whose image refines A. Hence, by compactness, there is a finite cover B of βL whose image refines A. Since all covers of a compact frame are normal and images of normal covers are normal, it follows that A has a finite normal refinement, and hence L is pseudocompact. ■

Remark 2.7. The last part of the proof in the foregoing result actually shows that if a frame is a P-quotient of *any* of its compactifications, then it is pseudocompact. On the other hand though, if L is a P-quotient of each of its compactifications, then L admits only one uniformity. To see this, let $\mathfrak{U}$ be a uniformity on L. Then this uniformity is totally bounded since L is pseudocompact under the hypothesis. Now let $h : M \to L$ be the completion of the uniform frame $(L, \mathfrak{U})$. Then $h : M \to L$ is a compactification of L. For any normal cover A of L, there is a normal cover B of M such that $h[B]$ refines A. Since the uniformity on M consists of all the covers, B is a uniform cover, and therefore $A \in \mathfrak{U}$. Consequently $\mathfrak{U}$ is the fine uniformity on L.

Another embedding-type characterization of pseudocompact frames is given by the following proposition which should be contrasted with Proposition 4.10 in [12]. A frame homomorphism $h : L \to M$ is called *almost coz-codense* if for every $c \in \operatorname{Coz} L$ with $h(c) = 1$, there exists $d \in \operatorname{Coz} L$ such that $c \vee d = 1$ and $h(d) = 0$.

Proposition 2.8. *A completely regular frame L is pseudocompact iff every quotient map $h : M \to L$ is almost coz-codense.*

Proof. ($\Leftarrow$): Under this hypothesis the Stone-Čech map $\beta L \to L$ is almost coz-codense, and therefore codense since it is dense. Therefore L is pseudocompact.

($\Rightarrow$): Let $c \in \operatorname{Coz} M$ such that $h(c) = 1$. Let (c_n) be a sequence of cozero elements of M such that $c_n \prec\!\!\prec c_{n+1}$ for each n, and $c = \bigvee c_n$. Then $h(c_n) \prec\!\!\prec h(c_{n+1})$ for each n and

$\bigvee h(c_n) = 1$. By pseudocompactness there exists $k \in \mathbb{N}$ such that $h(c_k) = 1$. Since $c_k \prec\!\!\prec c$, there exists $s \in \operatorname{Coz} M$ such that $s \wedge c_k = 0$ and $s \vee c = 1$. Thus, $h(s) = h(s) \wedge h(c_k) = 0$; showing that h is almost coz-codense. ■

Remark 2.9. The foregoing result appears in the "pointed version" in [9], where the spatial counterparts of the statements involved are also shown to be equivalent to what the authors call property (β). We have not been able to determine if the three equivalences can be extended to frames. The frame translation of property (β) is that a quotient $h : L \to M$ *satisfies property* (β) if for every $c, d \in \operatorname{Coz} L$ with $h(c) \vee h(d) = 1$, there exist $u, v \in \operatorname{Coz} L$ such that $u \vee v = 1$, $h(u) \leq h(c)$ and $h(v) \leq h(d)$.

Now we characterize pseudocompact frames in terms of ideals. Recall that an ideal in a lattice L is called *prime* if whenever the meet of two elements belongs to the ideal, then at least one of the elements also belongs to the ideal. Further, an ideal in a lattice with countable joins is said to be a *σ-ideal* if it contains all the joins of its countable subsets.

Given a frame homomorphism $h : L \to M$ and an ideal I in $\operatorname{Coz} M$, we denote by $h^{\#}I$ the ideal of $\operatorname{Coz} L$ given by $h^{\#}I = \{x \in \operatorname{Coz} L \mid h(x) \in I\}$. This, of course, is an adaptation of Gillman and Jerison's [13] notation. We aim to show that a completely regular frame L is pseudocompact if and only if for each maximal ideal I in $\operatorname{Coz} L$, the ideal $\beta^{\#}I$ is maximal in βL where $\beta : \beta L \to L$ is the Stone-Čech compactification of L. To this end we shall need the following result from [12].

Lemma 2.10. *Let L be a frame. If I is a prime σ-ideal in $\operatorname{Coz} L$, then I is a maximal ideal in $\operatorname{Coz} L$.*

Based on this we have the following characterization of pseudocompactness. It should be contrasted with Marcus' [15] result that *a completely regular frame is pseudocompact iff every maximal ideal in its cozero part is σ-proper*, where the latter means that each countable subset of the ideal has join strictly below the top.

Proposition 2.11. *A frame is pseudocompact iff maximal ideals in its cozero part are precisely the prime σ-ideals.*

Proof. ($\Rightarrow$): Let L be pseudocompact. Then prime σ-ideals of $\operatorname{Coz} L$ are maximal by Lemma 2.10. Now let I be a maximal ideal in $\operatorname{Coz} L$. Then I is prime. To show that I is a σ-ideal let S be a countable subset of I. If we assume that $\bigvee S \notin I$, then there exists $a \in I$ such that $a \vee \bigvee S = 1$ since I is maximal. So by pseudocompactness there is a finite $T \subseteq S$ such that $a \vee \bigvee T = 1$. But this is not possible since $T \cup \{a\}$ is a finite subset of the proper ideal I.

($\Leftarrow$): Let L be a frame with the hypothesized property. Let (c_n) be a sequence of cozero elements of L with $\bigvee c_n = 1$. If we assume that no finitely many c_n cover the frame, then the ideal I of $\operatorname{Coz} L$ generated by the c_n is proper. Let K be a maximal ideal in $\operatorname{Coz} L$ such that $I \subseteq K$. Then $1 = \bigvee c_n \in K$, which is false. Hence L is pseudocompact. ■

Part of the proof of the following characterization, although restricted to a special case, shows that, in fact, if L is pseudocompact and $h : M \to L$ is an onto homomorphism, then $h^{\#}I$ is a maximal ideal in $\operatorname{Coz} M$ for each maximal ideal I in $\operatorname{Coz}\ L$.

Proposition 2.12. *A completely regular frame L is pseudocompact iff $\beta^{\#}I$ is a maximal ideal in $\operatorname{Coz}\beta L$ for each maximal ideal I in $\operatorname{Coz} L$.*

Proof. ($\Rightarrow$): Clearly $\beta^{\#}I$ is prime since I is prime. For any countable $S \subseteq \beta^{\#}I$ we have that $\beta(s) \in I$ for each $s \in S$, so that $\beta(\bigvee S) = \bigvee \beta[S] \in I$ by Proposition 2.11. Hence $\bigvee S \in \beta^{\#}I$, showing that $\beta^{\#}I$ is a maximal ideal in $\operatorname{Coz} L$ by Lemma 2.10.

($\Leftarrow$): We prove this by showing that β is coz-codense. Think of βL as $R\mathcal{J}(\operatorname{Coz} L)$, the frame of regular ideals of $\operatorname{Coz} L$ (see, for instance, [7] p.4). Now let $\beta(I) = 1$ for some $I \in \operatorname{Coz}\beta L$. Suppose, by way of contradiction, that $I \neq 1_{\beta L}$. Then $1 \notin I$, and hence by Lemma 5 in [7] there is a maximal ideal K in $\operatorname{Coz} L$ such that $I \subseteq K$. Then $\beta^{\#}K$ is a maximal ideal in $\operatorname{Coz}\beta L$. Since $\bigvee I = 1 \notin K$, it follows that $I \notin \beta^{\#}K$. Since $\beta^{\#}K$ is maximal, this implies that there exists $U \in \beta^{\#}K$ such that $I \vee U = 1_{\beta L}$. Hence there exist $a \in I$ and $b \in U$ such that $a \vee b = 1$. But now $U \in \beta^{\#}K$ implies $\bigvee U \in K$. So a and $\bigvee U$ are elements of the proper ideal K that have join equal to the top; which is not possible. Consequently, $I = 1_{\beta L}$. ■

3. Pseudocompactness and relative pseudocompactness in frames

In this section we look briefly, from a frame-theoretic vantage point, at the concept of relative pseudocompactness (see Blair and Swardon [10]) which, spatially, is defined as follows: A subspace S of a topological space X is *relatively pseudocompact* if for every continuous function $f : X \to \mathbb{R}$, the restriction of f to S is bounded. We shall take as our definition a direct translation of this.

Definition 3.1. A quotient $h : L \to M$ of L is *relatively pseudocompact* (in L) if for every frame homomorphism $f : \mathfrak{L}(\mathbb{R}) \to L$, the composite hf is bounded.

To demonstrate that this concept (just like pseudocompactness) can be treated without invoking the reals, we shall establish a criterion for relative pseudocompactness which, if one had wished to avoid the usage of maps from $\mathfrak{L}(\mathbb{R})$, could have been adopted as the definition, and then use that in our computations.

The criterion in question is lifted from spaces [10] and states that a subspace is relatively pseudocompact if and only if its closure in the Hewitt realcompactification of the containing space is compact.

We remind the reader how υL, the Hewitt realcompactification of a completely regular frame L, is constructed (see [6] or [15] for details). Recall first that a *nucleus* on a frame L is a map $j : L \to L$ such that, for all $a, b \in L$, (1) $a \leq j(a) = j(j(a))$, and (2) $j(a \wedge b) = j(a) \wedge j(b)$. The set $\mathrm{Fix}(j) = \{a \in L \mid j(a) = a\}$ is then a frame with meet as in L and, for any $S \subseteq \mathrm{Fix}(j)$, $\bigvee_{\mathrm{Fix}(j)} S = j(\bigvee_L S)$. Furthermore, $j : L \to \mathrm{Fix}(j)$ is an onto frame homomorphism.

The frame of σ-ideals of $\mathrm{Coz}\, L$ is denoted by $\mathcal{H}\mathrm{Coz}\, L$. The map $\ell : \mathcal{H}\mathrm{Coz}\, L \to \mathcal{H}\mathrm{Coz}\, L$ given by $J \mapsto [\bigvee J] \cap \bigcap\{P \in \mathcal{H}\mathrm{Coz}\, L \mid P \supseteq J,\ P \,\text{maximal}\}$, where $[t] = \{x \in \mathrm{Coz}\, L \mid x \leq t\}$, is a nucleus. Then υL is defined to be the frame $\mathrm{Fix}(\ell)$, and the *Hewitt realcompactification* of L is the dense onto homomorphism $\upsilon : \upsilon L \to L$ given by join. For each $c \in \mathrm{Coz}\, L$, $[c] \in \mathrm{Coz}\, \upsilon L$. In fact, the cozero elements of υL are precisely the $[c]$ for $c \in \mathrm{Coz}\, L$.

Notice that the quotient map $\upsilon : \upsilon L \to L$ is a C-quotient map. We establish this claim by using Theorem 8.2.12 in [2]. If $\{c_n \mid n \in \mathbb{N}\}$ is a countable cozero cover of L, then $\{[c_n] \mid n \in \mathbb{N}\}$ is a countable cozero cover of υL for which $\upsilon([c_n]) = c_n$ for each n. To see that the $[c_n]$ do indeed cover υL, let J be their join in υL. Since $c_n \in J$ for each n and $\bigvee c_n = 1$, J cannot be a proper ideal as it is a σ-ideal.

Using the definition, one sees immediately that if L or M is pseudocompact, then the quotient $L \to M$ is relatively pseudocompact.

In the proof of the following result we shall need the well-known fact that a uniform frame is totally bounded if and only if every uniform cover has a finite (not necessarily uniform) subcover. The non-trivial implication is verified as follows: Given a uniform cover U of a uniform frame L with the latterly stated property, let V be a uniform star-refinement of U and $v_1, \ldots, v_k$ be finitely many elements of V such that $v_1 \vee \cdots \vee v_k = 1$. For each $i \in \{1, \ldots, k\}$ let u_i be an element of U such that $Vv_i \le u_i$. Then $\{v_i^*, u_i\}$ is a uniform cover of L for each i as it is refined by V. Thus $\{v_1^*, u_1\} \wedge \cdots \wedge \{v_k^*, u_k\}$ is a uniform cover of L which refines $\{u_1, \ldots, u_k\}$ since $v_1^* \wedge \cdots \wedge v_k^* = 0$. Thus U has a finite uniform refinement.

Proposition 3.2. *A quotient $h : L \to M$ is relatively pseudocompact iff $\uparrow(hv)_*(0)$ is compact.*

Proof. ($\Leftarrow$): Let $f : \mathfrak{L}(\mathbb{R}) \to L$ be a frame homomorphism. Since $\mathfrak{L}(\mathbb{R})$ is realcompact, there is a frame homomorphism $\tilde{f} : \mathfrak{L}(\mathbb{R}) \to \upsilon L$ such that $\upsilon\tilde{f} = f$. Since $\{(p, q) \mid p, q \in \mathbb{Q}\}$ is a cover of $\mathfrak{L}(\mathbb{R})$, $\{\tilde{f}(p, q) \vee \upsilon_* h_*(0) \mid p, q \in \mathbb{Q}\}$ is a (directed) cover of $\uparrow(h\upsilon_*(0)$. By compactness there exists $s, t \in \mathbb{Q}$ such that $\tilde{f}(s, t) \vee \upsilon_* h_*(0) = 1_{\upsilon L}$. This implies that $\upsilon\tilde{f}(s, t) \vee h_*(0) = 1_L$, which in turn implies that $f(s, t) \vee h_*(0) = 1_L$, whence $hf(s, t) = 1_M$ as desired.

($\Rightarrow$): We will show that $\uparrow(hv)_*(0)$ admits a uniformity in which it is Cauchy complete and totally bounded. The uniformity in question is that inherited from υL equipped with its real uniformity (see Banaschewski [4] for the definition and properties of the real uniformity). Since υL is realcompact, it is Cauchy complete in its real uniformity by Proposition 4 in [4]. Hence $\uparrow(hv)_*(0)$ is Cauchy complete in the inherited uniformity. To show total boundedness it suffices to show that, for a given positive integer m and some frame homomorphism $f : \mathfrak{L}(\mathbb{R}) \to \upsilon L$, the subbasic uniform cover $\{(hv)_*(0) \vee f(p, q) \mid (p, q) \in C_m\}$ has a finite subcover. Recall that $C_m = \{(p, q) \mid q - p < \frac{1}{m}\}$. Applying the hypothesis to the composite hvf, we can produce $p, q \in \mathbb{Q}$ such that $hvf(p, q) = 1_M$. Now if x is an element of C_m that misses (p, q), then $(hv)f(x) = 0$, so that $f(x) \le (hv)_*(0)$, and hence $(hv)_*(0) \vee f(x) = (hv)_*(0)$. Clearly, there are finitely many elements $x_1, \ldots, x_n$ of C_m such that $(p, q) \le x_1 \vee \cdots \vee x_n$. As a consequence, $\{(hv)_*(0), (hv)_*(0) \vee f(x_1), \ldots, (hv)_*(0) \vee f(x_n)\}$ is a finite subcover extracted from $\{(hv)_*(0) \vee f(p, q) \mid (p, q) \in C_m\}$. ■

Now we demonstrate via the characterization in Proposition 3.2 that a pseudocompact quotient is relatively pseudocompact, and that a quotient of a pseudocompact frame is relatively pseudocompact. Having observed these from the definition, this might seem to be a pointless endeavor (no pun intended), but that is not so because from the series of observations in this regard will emerge a proof, not using the Boolean Ultrafilter Theorem, that a frame is pseudocompact if and only if its Hewitt realcompactification is compact. This latter fact was proved by Banaschewski and Gilmour [6] using the Boolean Ultrafilter Theorem.

We start by showing that a closed quotient of a realcompact frame is realcompact. Actually we do a little more. Call a quotient $h : L \to M$ of L *extension-closed* in case for every cover C of M, $h_*[C]$ is a cover of L. Closed quotients are obviously extension-closed. We then have that an extension-closed quotient $L \to M$ of a realcompact frame is realcompact. Indeed, if F is a filter of M that meets every countable cozero cover of M, then the inverse image of F is a filter of L that meets every countable cozero cover of L. Therefore, by Proposition 5 in [6], it meets every cover of L. But then this implies that F meets every cover of M; showing M to be realcompact again by Proposition 5 in [6].

Next, recall that we observed earlier that a frame with a dense pseudocompact quotient is itself pseudocompact. Now if the quotient $h : L \to M$ is pseudocompact, then $\uparrow(h\upsilon)_*(0)$ is pseudocompact because the map $h\upsilon : \uparrow(h\upsilon)_*(0) \to M$ is dense. Since υL is realcompact, it follows that $\uparrow(h\upsilon)_*(0)$ is realcompact and pseudocompact, and is therefore compact.

On the other hand, if L is pseudocompact then so is υL because υ is dense. Thus, υL is compact, so that if $h : L \to M$ is a quotient of L, then $\uparrow(h\upsilon)_*(0)$ is compact.

Proposition 3.3. *The following are equivalent for a completely regular frame L:*

(1) *L is pseudocompact.*

(2) *υL is compact.*

(3) *L is relatively pseudocompact in υL.*

(4) *For every maximal ideal I in $\mathrm{Coz}\,L$, there is a σ-proper maximal ideal $\mathcal{I}$ in $\mathrm{Coz}\,\upsilon L$ such that $\upsilon[\mathcal{I}] = I$.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are immediate from the discussion above.

(3) $\Rightarrow$ (1): Let $\zeta : K \to \upsilon L$ denote the Hewitt realcompactification of υL. Then ζ is an isomorphism since υL is realcompact. Now the hypothesis implies that $\uparrow(\upsilon\zeta)_*(0)$ is compact.

Since both ζ and υ are dense, this implies that $\uparrow 0_K = K$ is compact, and hence υL is compact. Therefore $\upsilon : \upsilon L \to L$ is a compactification of L for which the compactification map is a C^*-quotient map. Thus, by Corollary 8.2.7 in [2], υL is the Stone-Čech compactification of L. But the map $\upsilon : \upsilon L \to L$ is coz-codense, so L is pseudocompact.

(1) $\Rightarrow$ (4): If (1) holds then so does (2), and hence υL is (isomorphic to) βL. Hence by Proposition 2.12 we have that $\upsilon^{\#} I$ is a maximal ideal in $\mathrm{Coz}\, \upsilon L$. By pseudocompactness, $\upsilon^{\#} I$ is σ-proper. Since υ is coz-onto, we have that $\upsilon[\upsilon^{\#} I] = I$ by Proposition 3.7(2) in [12].

(4) $\Rightarrow$ (1): Let I be a maximal ideal $\mathrm{Coz}\, L$ and $\mathcal{I}$ be a σ-proper maximal ideal in $\mathrm{Coz}\, \upsilon L$ such that $\upsilon[\mathcal{I}] = I$. Then I is σ-proper, lest there exist a sequence (a_n) in I such that $\bigvee a_n = 1$, leading to a contradiction as follows: For each n let $[c_n] \in \mathcal{I}$ such that $\upsilon([c_n]) = a_n$, that is $c_n = a_n$. Consequently, $\bigvee \{[c_n] \mid n \in \mathbb{N}\} = 1_{\upsilon L}$, contradicting the fact that $\mathcal{I}$ is σ-proper. ∎

Blair and Swardson [10] have shown that a z-embedded subspace is pseudocompact if and only if it is relatively pseudocompact and well embedded. This result extends to frames. To prove that, we shall need the following lemma.

Lemma 3.4. *Every C-quotient is a $P^{\aleph_0}$-quotient.*

Proof. Let $h : L \to M$ be a C-quotient. Let A be a countable normal cover of L and $B = \{b_1, b_2, \ldots\}$ be a countable cozero cover of M that refines A. Since h is coz-onto, there exists, for each n, $c_n \in \mathrm{Coz}\, L$ such that $h(c_n) = b_n$. Put $c = \bigvee c_n$ and notice that c is a cozero element of M such that $h(c) = \bigvee b_n = 1$. Since h is almost coz-codense, there exists $d \in \mathrm{Coz}\, L$ such that $c \vee d = 1$ and $h(d) = 0$. Thus $C = \{d, b_1, b_2, \ldots\}$ is a countable cozero cover, and hence a countable normal cover of L the image of which refines A. ∎

It is apposite to remark that the proof of this lemma is the only instance, in this paper, of an almost verbatim translation of a proof in spaces.

Proposition 3.5. *Let $h : L \to M$ be a coz-onto quotient map. Then M is pseudocompact iff it is relatively pseudocompact and h is almost coz-codense.*

Proof. The forward implication follows from Proposition 2.5 (since C-quotient $\Leftrightarrow$ coz-onto plus almost coz-codense) and the fact that a pseudocompact quotient is relatively pseudocompact.

Conversely, let $\{c_n \mid n \in \mathbb{N}\}$ be a countable cozero cover of M. Then, since h is a C-quotient map by the hypothesis, it is a $P^{\aleph_0}$-quotient map. Therefore there is a countable cozero cover $\{s_n \mid n \in \mathbb{N}\}$ of L such that $h(s_n) \leq c_n$ for each n. As observed earlier, $\{[s_n] \mid n \in \mathbb{N}\}$ is a cozero cover of υL. Consequently $\bigvee_{\uparrow(h\upsilon)_*(0)}\{(h\upsilon)_*(0) \vee [s_n] \mid n \in \mathbb{N}\} = 1_{\upsilon L}$. Since $\uparrow(h\upsilon)_*(0)$ is compact, there are finitely many indices $\{n_1, \ldots, n_k\}$ such that $\upsilon_*\big(h_*(0)\big) \vee [s_{n_1}] \vee \cdots \vee [s_{n_k}] = 1_{\upsilon L}$. Acting υ on this equality yields $h_*(0) \vee s_{n_1} \vee \cdots \vee s_{n_k} = 1_L$, and acting h on this latter equality yields $c_{n_1} \vee \cdots \vee c_{n_k} = 1_M$, whence we deduce that M is pseudocompact. ■

Combining this result with Proposition 2.1(6) we get:

Corollary 3.6. *The closure of a relatively pseudocompact C-quotient of a Dieudonné complete frame is compact.*

In [21] Weir proves (a result he attributes to A.W. Hager and D.G. Johnson) that if U is a relatively pseudocompact cozero set of a completely regular space X, then the closure of U is pseudocompact. In the case of normal regular frames (or spaces) more can be said as we show below.

Proposition 3.7. *The closure of a relatively pseudocompact quotient of a normal regular frame is pseudocompact.*

Proof. Let $\{c_1, c_2, \ldots\}$ be a countable cozero cover of $\uparrow h_*(0)$ where $h : L \to M$ is a relatively pseudocompact quotient of a normal regular frame L. Since L is normal, the closed quotient $L \to \uparrow h_*(0)$ is a C-quotient, and therefore a $P^{\aleph_0}$-quotient. There is therefore a countable cozero cover $\{s_1, s_2, \ldots\}$ of L such that $s_n \vee h_*(0) \leq c_n$ for each n. Since $h_*(0) \leq c_n$ for each n, we have that $\upsilon_*(c_n) \in \uparrow(h\upsilon)_*(0)$ for each n. We claim that $\bigvee\{\upsilon_*(c_n) \mid n \in \mathbb{N}\} = 1_{\upsilon L}$. If we denote this join by J, then, in view of the fact that $s_n \leq c_n$ for each n, we have that $s_n \in J$ for each n. So J cannot be a proper ideal as it is a σ-ideal and $\bigvee s_n = 1$. So, by compactness, there are finitely many indices $n_1, \ldots, n_k$ such that $\upsilon_*(c_{n_1}) \vee \cdots \vee \upsilon_*(c_{n_k}) = 1_{\upsilon L}$. Now act υ to obtain $c_{n_1} \vee \cdots \vee c_{n_k} = 1$, which then shows that $\uparrow h_*(0)$ is pseudocompact. ■

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