INSERTION OF A MEASURABLE FUNCTION

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Abstract

Some theorems on the existence of continuous real-valued functions on a topological space (for example, insertion, extension, and separation theorems) can be proved without involving uncountable unions of open sets. In particular, it is shown that well-known characterizations of normality (for example the Katětov-Tong insertion theorem, the Tietze extension theorem, Urysohn's lemma) are characterizations of normal σ-rings. Likewise, similar theorems about extremally disconnected spaces are true for σ-rings of a certain type. This σ-ring approach leads to general results on the existence of functions of class α.

1. Introduction

In this paper we shall show that some important theorems on the existence of real-valued functions on a topological space remain valid if uncountable unions of open sets are not available. For instance, the proof of Urysohn's lemma depends upon constructing a countable family of open sets $\{U_r : r \in \mathbb{Q}\}$ such that $\bar{U}_r \subset U_s$ if $r < s$. This involves the operation of closure, hence a not necessarily countable intersection. But one can easily overcome this topological ingredient by requiring the existence of a countable family of closed sets $\{K : r \in \mathbb{Q}\}$ such that $U_r \subset K_r \subset U_s$ whenever $r < s$. In fact, an intervention of this type has already been done by Speed [15] who thus proved that a normal topology on a set can be replaced by a normal σ-ring.

We shall show that the other well-known characterizations of normality (that is, the Katětov-Tong insertion theorem [4, 16] and the Tietze extension theorem) are characterizations of normal σ-rings. In fact, we shall do that for their abstract versions:

the insertion theorem of Blair [1] and Lane [10], the extension theorem of Mrówka [11], and the Urysohn extension theorem of Gillman and Jerison [3]. Also, similar theorems about extremally disconnected spaces are shown to be true for sets equipped with extremally disconnected σ-rings (that is, σ-rings in which each two disjoint members are contained in disjoint complements of members of the ring).

However, the most important advantage of those observations is that one can substitute for a σ-ring the sets of additive class α in a perfect space and obtain corresponding theorems about real-valued functions of class α on a perfect space. In fact, those theorems (insertion, extension, and separation theorems) are characterizations of a wider class of spaces which will be called $\mathscr{A}_\alpha$-normal (the case $\alpha = 1$ was discussed in [5] under the name F_σ-normal spaces, where a Urysohn type lemma and a Tietze type theorem have been obtained directly). Another corollary is a classical theorem about insertion of a real-valued function of class α between two comparable upper and lower measurable functions of class α [14].

We note that the question of when a real-valued function other than a (uniformly [13]) continuous one (for example [1, 4, 9, 10, 16]) can be inserted between two comparable real-valued functions has rarely been investigated, especially for functions of higher classes with the notable exception of [5]. Results of similar type for class one functions of a real variable can be found, for instance, in [12].

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2. Notation and terminology

Our reference on rings of subsets and measurable functions is Sikorski [14]. One good reference for generating real-valued functions by monotone families of sets is Kutateladze [8]. For easy reference we recall some basic concepts and a few relevant facts about generating real-valued functions.

A subfamily $\mathscr{A} \subset \mathscr{P}(X)$ (the power set of a set X) is a *ring* on X if $\mathscr{A}$ is closed under finite unions and finite intersections, and contains $\emptyset$ and X. If $\mathscr{A}$ is a ring, so is $\mathscr{A}_c = \{X \backslash A : A \in \mathscr{A}\}$. A ring is a σ-*ring* if it is closed under countable unions. If $S \subset X$, then $\mathscr{A}|S = \{A \cap S : A \in \mathscr{A}\}$ is a σ-ring on S whenever $\mathscr{A}$ is a σ-ring on X. A ring $\mathscr{A}$ is a δ-*ring* if $\mathscr{A}_c$ is a σ-ring. For $\mathscr{A}$ a ring, $\mathscr{A}_\sigma$ (respectively $\mathscr{A}_\delta$) is the σ-ring (respectively δ-ring) consisting of all countable unions (respectively intersections) of members of $\mathscr{A}$.

The extended reals with usual ordering are denoted by $\bar{R}$. All functions considered in this paper are real-valued, that is $\bar{R}$- or Re-valued (the latter are called finite functions). For f a function on X and $a \in Re$, $[f > a] = \{x \in X : f(x) > a\}$ and

$[f < a] = \{x \in X : f(x) < a\}$. Also, $[f \geq a] = X \backslash [f < a]$, etcetera. We write $f \leq g$ if $f(x) \leq g(x)$ for all x in X.

Let $\mathscr{A}$ be a σ-ring on X. A function f on X is called *lower* (respectively *upper*) $\mathscr{A}$-*measurable* if $[f > a] \in \mathscr{A}$ (respectively $[f < a] \in \mathscr{A}$) for each $a \in$ Re. We say that f is $\mathscr{A}$-*measurable* if it is both lower and upper $\mathscr{A}$-measurable. We shall say that two subsets A and B of X are *completely $\mathscr{A}$-separated* if there is an $\mathscr{A}$-measurable function f on X such that $A \subset [f \leq a]$ and $B \subset [f \geq b]$ for some $a < b$ in Re.

Each increasing family $\mathscr{F} = \{F_r : r \in \mathbb{Q}\} \subset \mathscr{P}(X)$ is called a *scale* on X. For future use we quote the following well-known facts.

PROPOSITION 2.1. *Let $\mathscr{F}$ be a scale on X. Then the following hold.*

(1) *For each $x \in X$, $f(x) = \inf\{r \in \mathbb{Q} : x \in F_r\}$ defines an $\bar{\mathbb{R}}$-valued function on X. (We shall say f is generated by $\mathscr{F}$.)*
(2) *If f is generated by $\mathscr{F}$, then $[f < a] = \bigcup\{F_r : r < a\}$ and $[f \leq a] = \bigcap\{F_r : r > a\}$ for all $a \in$ Re.*
(3) *If g and h are generated by $\mathscr{G}$ and $\mathscr{H}$, respectively, then $g \leq h$ if and only if $H_r \subset G_s$ whenever $r < s$.*
(4) *If f is generated by $\mathscr{F}$, then $f(x) > -\infty$ [respectively $< +\infty$] for all $x \in X$ if and only if $\bigcap \mathscr{F} = \emptyset$ [respectively $\bigcup \mathscr{F} = X$].*

(Therefore a scale $\mathscr{F}$ satisfying both $\bigcap \mathscr{F} = \emptyset$ and $\bigcup \mathscr{F} = X$ will be called a *finite* scale,)

3. Insertion and extension of a measurable function

We shall use the machinery developed by Katětov [4].

A binary relation ρ on $\mathscr{P}(X)$ will be called a *K-relation* if it satisfies the following conditions:

(i) If $A_i \rho B_j (1 \leq i \leq m, 1 \leq j \leq n)$, then there exists $C \in \mathscr{P}(X)$ such that $A_i \rho C$ and $C \rho B_j$ for all those i and j.
(ii) $A \rho B$ implies $A \subset B$.
(iii) $A \subset B$ implies $A \bar{\rho} B$, where $A \bar{\rho} B$ if and only if $B \rho C$ implies $A \rho C$, and $D \rho A$ implies $D \rho B$ for any $C, D \in \mathscr{P}(X)$

We note that (iii) is equivalent to the following statement:

(iv) If $C \subset A$, $A \rho B$, and $B \subset D$, then $C \rho D$ for every $A, B, C, D \in \mathscr{P}(X)$

LEMMA 3.1. (Katětov [4]). *Let ρ be a K-relation on $\mathscr{P}(X)$. Let $\mathscr{G} = \{G_r : r \in \mathbb{Q}\}$ and $\mathscr{H} = \{H_r : r \in \mathbb{Q}\}$ be such that $H_r \rho G_s$, $H_r \bar{\rho} H_s$, and $G_r \bar{\rho} G_s$ whenever $r < s$. Then there exists $\mathscr{F} = \{F_r : r \in \mathbb{Q}\}$ with $H_r \rho F_s$, $F_r \rho F_s$, and $F_r \rho G_s$ whenever $r < s$.*

REMARK 3.2. We shall deal only with the case when $\mathscr{G}$ and $\mathscr{H}$ are increasing (scales). Then the conditions $G_r \bar{\rho} G_s$ and $H_r \bar{\rho} H_s$ $(r < s)$ hold trivially. Also, we shall only require $\mathscr{F}$ to satisfy $H_r \subset F_s$, $F_r \rho F_s$, and $F_r \subset G_s$, if $r < s$.

DEFINITION 3.3. Let $\mathscr{A}$ be a ring on X. An arbitrary relation ρ on $\mathscr{P}(X)$ is called an $\mathscr{A}$-separating relation if for every $A, B \in \mathscr{P}(X)$, $A \rho B$ implies there exist $C \in \mathscr{A}$ and $\mathscr{D} \in \mathscr{A}_c$ such that $A \subset C \subset B$ and $A \subset D \subset B$.

Note that ρ is $\mathscr{A}$-separating if and only if it is $\mathscr{A}_c$-separating.

EXAMPLE 3.4. (1) If $\mathscr{A}$ is a topology on X, then A is in an $\mathscr{A}$-separating relation to B if and only if A and $X \backslash B$ are topologically separated (that is, $\bar{A} \subset B$ and $A \subset \text{Int } B$).

(2) For $\mathscr{A}$ a σ-ring on X, the relation ρ on $\mathscr{P}(X)$ defined by $A \rho B$ if and only if A and $X \backslash B$ are completely $\mathscr{A}$-separated is an $\mathscr{A}$-separating K-relation. [Indeed, since $A \subset [f \leq 0] \subset [f < 1] \subset B$ with f $\mathscr{A}$-measurable, hence (ii) and (iv) hold trivially. If $A_i \rho B_j$ and f_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$ are the corresponding $\mathscr{A}$-measurable functions, then $g = \inf_i \sup_j f_{ij}$ is $\mathscr{A}$-measurable and satisfies $A_i \subset [g \leq 0] \subset [g < 1] \subset B_j$ for all i and j. Therefore $A_i \rho C \rho B_j$ with $C = [g < \frac{1}{2}]$.]

LEMMA 3.5. *If $\mathscr{A}$ is a σ-ring on X, $\mathscr{F}$ is a scale, and ρ is an $\mathscr{A}$-separating relation on $\mathscr{P}(X)$ such that $F_r \rho F_s$ whenever $r < s$, then the function f generated by $\mathscr{F}$ is $\mathscr{A}$-measurable.*

PROOF. For $r < s$, let C_{rs} and D_{rs} be the corresponding members of $\mathscr{A}$ and $\mathscr{A}_c$, respectively, such that $F_r \subset C_{rs} \subset F_s$ and $F_r \subset D_{rs} \subset F_s$. Clearly, $\bigcup \{F_r : r < a\} = \bigcup \{C_{rs} : r < s < a\} \in \mathscr{A}$ and $\bigcap \{F_r : r > a\} = \bigcap \{D_{rs} : s > r > a\} \in \mathscr{A}_c$ for all $a \in \text{Re}$. By Proposition 2.1(2), f is $\mathscr{A}$-measurable.

THEOREM 3.6. (Insertion Theorem). *Let $\mathscr{A}$ be a σ-ring on X. Let $g \leq h$ be two $\bar{\text{R}}$-valued functions on X. The following statements are equivalent*

(1) *There exists an $\mathscr{A}$-measurable $\bar{\text{R}}$-valued function f on X such that $g \leq f \leq h$.*
(2) *For each $a < b$ in Re, $[h \leq a]$ and $[g \geq b]$ are completely $\mathscr{A}$-separated.*
(3) *There exist scales $\mathscr{G}$ and $\mathscr{H}$ generating g and h, respectively such that H_r and $X \backslash G_s$ are completely $\mathscr{A}$-separated whenever $r < s$.*
(4) *There exist scales $\mathscr{G}$ and $\mathscr{H}$ generating g and h, respectively, and an $\mathscr{A}$-separating K-relation ρ on $\mathscr{P}(X)$ such that $H_r \rho G_s$ if $r < s$.*

PROOF. (1) implies (2) is trivial.

(2) implies (3): for each $r \in \mathbb{Q}$, put $G_r = [g < r]$ and $H_r = [h \leq r]$.

(3) implies (4): by Example 3.4(2).

(4) implies (1): by Lemma 3.1 (see also Remark 3.2), there exists $\mathscr{F} = \{F_r : r \in \mathbb{Q}\}$ such that $F_r \rho F_s$ (hence $\mathscr{F}$ is a scale by (ii)), $H_r \subset F_s$, and $F_r \subset G_s$ if $r < s$. Thus, by Proposition 2.1(3), the function f generated by $\mathscr{F}$ satisfies $g \leq f \leq h$. Since ρ is $\mathscr{A}$-separating, f is $\mathscr{A}$-measurable by Lemma 3.5.

REMARK 3.7. Of course, if g and h are finite functions, so is f. For this case and with $\mathscr{A}$ a topology on X (then $\mathscr{A}$-measurability = continuity), the equivalence of (1) and (2) in Theorem 3.6 was proved by Blair [1, 3, 5], and (1) if and only if (2) if and only if (3) was proved independently by Lane [10, Theorem 2.1]; (4) if and only if (1) is in Lane [9, Theorem 1]. Note in passing that Lane [9, 10] uses the concept of lower indefinite cut sets: if f is an Re-valued function on X and $[f < a] \subset A(f, a) \subset [f \leq a]$ for all $a \in$ Re, then $A(f, a)$ is called a lower indefinite cut set of f at a. Of course, $\{A(f, r) : r \in \mathbb{Q}\}$ is a finite scale that generates f. Also, any other scale generating f, $\{F_r : r \in \mathbb{Q}\}$ say, must satisfy $[f < r] \subset F_r \subset [f \leq r]$ for all $r \in \mathbb{Q}$. Therefore, Lane's families of cut sets are actually finite scales.

Note also that Theorem 3.6 partially improves [13, Theorem 3.4], which in turn is an $\bar{\text{R}}$-valued version of the mentioned results of Blair [1] and Lane [10].

The following extension theorem for bounded functions is an immediate consequence of Theorem 3.6. We include a proof (whose technique is well-known) for completeness.

THEOREM 3.8. (Mrówka Extension Theorem). *Let $\mathscr{A}$ be a σ-ring on X and $S \subset X$. Let $f : S \to [0, 1]$ be $(\mathscr{A}|S)$-measurable. Then the following are equivalent:*

(1) *f has an $\mathscr{A}$-measurable extension over X.*
(2) *If $a < b$ in $[0, 1]$, then $[f \leq b]$ and $[f \geq b]$ are completely $\mathscr{A}$-separated.*

PROOF. (1) implies (2): let g be an extension of f. Then $[f \leq a] \subset [g \leq a]$ and $[f \geq b] \subset [g \geq b]$, that is, g separates these sets.

(2) implies (1): define $g, h : X \to [0, 1]$ by $g = h = f$ on S, $g = 0$ on $X \backslash S$, and $h = 1$ on $X \backslash S$. Then $[g \geq b] = [f \geq b]$ if $b > 0$, and $[h \leq a] = [f \leq a]$ if $a < 1$. Therefore if $a < b$ in $[0, 1]$, then $[h \leq a]$ and $[g \geq b]$ are completely $\mathscr{A}$-separated, that is, Statement 3.6(2) holds. Since $g \leq h$, by Statement 3.6(1) there exists an $\mathscr{A}$-measurable function f^* on X such that $g \leq f^* \leq h$, which is therefore the required extension of f.

COROLLARY 3.9. (Urysohn Extension Theorem). *For $\mathscr{A}$ a σ-ring on X and $S \subset X$, the following statements are equivalent:*

(1) *Each $(\mathscr{A}|S)$-measurable function $f : S \to [0, 1]$ has an $\mathscr{A}$-measurable extension over X.*

(2) *Each two completely $(\mathscr{A}|S)$-separated (in S) subsets of S are completely $\mathscr{A}$-separated in X.*

PROOF. (1) implies (2): if f completely $(\mathscr{A}|S)$-separates A and B in S, and if g is an $\mathscr{A}$-measurable extension of f, then $A \subset [f = 0] \subset [g = 0]$ and $B \subset [f = 1] \subset [g = 1]$.

(2) implies (1): let $f : S \to [0, 1]$ be $(\mathscr{A}|S)$-measurable. Then for each $a < b$ in $[0, 1]$, $[f \leq a]$ and $[f \geq b]$ are completely $(\mathscr{A}|S)$-separated (by f). Therefore, f has a continuous extension to X by Theorem 3.8.

REMARK 3.10. With $\mathscr{A}$ a topology on X, Theorem 3.8 is due to Mrówka [11, 4.11], and Corollary 3.9 is known as the Urysohn Extension Theorem [3, 1.17].

4. Perfect rings, normal rings, and extremally disconnected rings

The concepts of normality and perfectness of a topology of a space can easily be extended to rings of subsets. Recall that a perfect space is one in which each open set is F_σ.

A ring $\mathscr{A}$ on a set X is *normal* if, given any two disjoint members of $\mathscr{A}_c$, there are disjoint members of $\mathscr{A}$ containing them respectively. A ring $\mathscr{A}$ is called *perfect* if $\mathscr{A} \subset \mathscr{A}_{c\sigma}$ (that is, if each member of $\mathscr{A}$ is a countable union of complements of $\mathscr{A}$); equivalently, if $\mathscr{A}_c \subset \mathscr{A}_\delta$. We shall say that a ring $\mathscr{A}$ is *extremally disconnected* if $\mathscr{A}_c$ is normal. (We note that in [15] a ring $\mathscr{A}$ is called normal if the ring $\mathscr{A}_c$ is normal in the sense above. Such a terminology, however, is inconsistent with extremal disconnectedness of general topology; see the example which follows.)

EXAMPLE 4.1. For $(X, \mathscr{A})$ a topological space, X is normal [extremally disconnected] if and only if $\mathscr{A}$ is normal [extremally disconnected]. For further examples, see Remarks 4.7 and 5.2, and Proposition 5.3. (Recall that a space is extremally disconnected if and only if every open set has an open closure if and only if disjoint open sets have disjoint closures, and the latter is clearly equivalent to the statement that each two disjoint open sets are contained in disjoint closed sets.)

We now specialize Theorem 3.6 for the case when $\mathscr{A}$ is normal or extremally disconnected.

LEMMA 4.2. *Let $\mathscr{A}$ be a normal ring on X. Then the relation ρ on $\mathscr{P}(X)$, defined by $A\rho B$ if and only if there exist $C \in \mathscr{A}_c$ and $D \in \mathscr{A}$ with $A \subset C \subset D \subset B$, is an $\mathscr{A}$-separating K-relation.*

PROOF. Each $\mathscr{A}$-separating ρ satisfies (ii), and (iv) holds trivially. So, we verify (i). For if $A_i \rho B_j$, and C_{ij}, D_{ij} are the corresponding members of $\mathscr{A}_c$ and $\mathscr{A}$, respectively $(1 \leq i \leq m, 1 \leq j \leq n)$, then $C = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} C_{ij} \in \mathscr{A}_c$, $D = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} D_{ij} \in \mathscr{A}$, and $\bigcup_{i=1}^{m} A_i \subset C \subset D \subset \bigcap_{j=1}^{n} B_j$. Since $\mathscr{A}$ is normal there exist $D_1 \in \mathscr{A}$ and $C_1 \in \mathscr{A}_c$ such that $C \subset D_1 \subset C_1 \subset D$. Therefore $A_i \rho D_1 \rho B_j$ for all those i and j.

COROLLARY 4.3. *Let $\mathscr{A}$ be an extremally disconnected ring on X. Then the relation ρ on $\mathscr{P}(X)$, defined by $A\rho B$ if and only if there exist $C \in \mathscr{A}$ and $D \in \mathscr{A}_c$ with $A \subset C \subset D \subset B$, is an $\mathscr{A}$-separating K-relation.*

PROOF. $\mathscr{A}_c$ is normal and $\mathscr{A}_{cc} = A$, so Lemma 4.2 applies.

Note that this sort of 'duality' between normality and extremal disconnectedness, discussed in [6], appears to be a very efficient source of deriving new results about normality from extremal disconnectedness and vice versa. The following provides examples of such dualities.

THEOREM 4.4. *Let $\mathscr{A}$ be a σ-ring on X. Then the following statements are equivalent:*

(1) *A is normal (extremally disconnected).*
(2) *(Insertion) If $g \leq h$ [respectively $h \leq g$] are real-valued functions on X such that g is upper and h is lower $\mathscr{A}$-measurable, then there exists an $\mathscr{A}$-measurable function f on X such that $g \leq f \leq h$ [respectively $g \leq f \leq h$].*
(3) *(Separation) Each two disjoint A and B in $\mathscr{A}_c$ [respectively $\mathscr{A}$] are completely $\mathscr{A}$-separated.*
(4) *(Extension) For every $S \in \mathscr{A}_c$ [respectively $\mathscr{A}$] each $(\mathscr{A}|S)$-measurable function $f : S \to [0, 1]$ has an $\mathscr{A}$-measurable extension to all of X.*

PROOF. (1) implies (2): define $G_r = [g < r]$ and $H_r = [h \leq r]$ for all $r \in \mathbb{Q}$. Clearly, $\{G_r\}$ and $\{H_r\}$ are scales generating g and h (finite scales if g and h are finite functions), respectively. Also, $H_r \subset G_s$ (respectively $G_r \subset H_s$) if $r < s$. And since $H_r \in \mathscr{A}_c$, $G_r \in \mathscr{A}$ for all r, in fact, $H_r \rho G_s$ [respectively $G_r \rho H_s$] if $r < s$, where ρ is the relation of Lemma 4.2 (respectively Corollary 4.3). Thus, by Theorem 3.6(4), there exists an $\mathscr{A}$-measurable function f such that $g \leq f \leq h$ (respectively $h \leq f \leq g$).

(2) implies (3): if $A, B \in \mathscr{A}_c$ (respectively $\mathscr{A}$) are disjoint, then $\chi_A \leq \chi_{X \setminus B}$, χ_A is upper (respectively lower) and $\chi_{X \setminus B}$ is lower (respectively upper) $\mathscr{A}$-measurable. Then the $\mathscr{A}$-measurable function f between χ_A and $\chi_{X \setminus B}$ has the required property.

(3) implies (4), for $\mathscr{A}_c$: for any $a < b$ in $[0, 1]$, $[f \leq a]$ and $[f \geq b]$ are in $(\mathscr{A}|S)_c$. Since $S \in \mathscr{A}_c$ we have $(\mathscr{A}|S)_c \subset \mathscr{A}_c$. By Theorem 3.8, f has an $\mathscr{A}$-measurable extension.

(3) implies (4), for $\mathscr{A}$: let $a < b$ in $[0, 1]$. If $a < c < d < b$, then $C = [f < c]$ and $D = [f > d]$ are in $\mathscr{A}|S$ and $\mathscr{A}|S \subset \mathscr{A}$ since $S \in \mathscr{A}$. Thus C and D are completely $\mathscr{A}$-separated and so are $[f \leq a] \subset C$ and $[f \geq b] \subset D$. By Theorem 3.8, f is extendable to an $\mathscr{A}$-measurable function.

(4) implies (1): for each two disjoint A and B in $\mathscr{A}_c$ (respectively $\mathscr{A}$), the function f from $S = A \cup B$ to $[0, 1]$, defined by and $f = 0$ on A and $f = 1$ on B, is $(\mathscr{A}|S)$-measurable, hence extendable to an $\mathscr{A}$-measurable function f^*. Clearly, $A \subset [f^* < \frac{1}{2}]$ and $B \subset [f^* > \frac{1}{2}]$ (respectively $A \subset [f^* \leq \frac{1}{2}]$ and $B \subset [f^* \geq \frac{3}{4}]$), which proves the assertion.

REMARK 4.5. (1) First note that (4) above is easily obtainable from (2) by avoiding (3) and, thus, Theorem 3.8 as well. Indeed, g and h defined as in the proof of Theorem 3.8 are, respectively, upper and lower (respectively lower and upper) $\mathscr{A}$-measurable whenever $S \in \mathscr{A}_c$ (respectively $\mathscr{A}$). Therefore the $\mathscr{A}$-measurable function, which does exist by Theorem 4.4(2), is the required extension. In other words, one can derive extension from insertion by avoiding separation.

(2) If $\mathscr{A}$ is an extremally disconnected topology, then (1) if and only if (2) for finite functions is in Lane [9, Corollary 4], and (1) if and only if (3) if and only if (4) is in Gillman and Jerison [3, 1.H].

(3) That (1) if and only if (3) for $\mathscr{A}$ a normal σ-ring is proved by Speed [15, Theorem 3.1] (cf. 4.1) by modifying the usual Urysohn's procedure (see Section 1).

(4) Of course, if $\mathscr{A}$ is a normal topology, the statements (2), (3), and (4) become, respectively, the Katětov-Tong theorem [4, 16], Urysohn's lemma, and the Tietze extension theorem.

REMARK 4.6. Another area which is covered by the normality case of Theorem 4.4 are F_σ-normal spaces of [5] (this class of spaces includes all perfect spaces, that is well-behaved spaces in the terminology of [5]): a topological space is called F_σ-normal if for each two disjoint G_δ-sets there exist two disjoint F_σ-sets containing them. In other words, a space is F_σ-normal if and only if $\mathscr{F}_\sigma$, the σ-ring of all F_σ-sets, is normal. With $\mathscr{A} = \mathscr{F}_\sigma$, the normality cases of (1) if and only if (3) and (1) if and only if (4) of Theorem 4.4 become Theorem 2.3 and 2.10 of [5], respectively. Note that real valued $\mathscr{F}_\sigma$-measurable functions are class one functions. Those two results of [5] are of course simple consequences (compare Remark 4.5(1)) of the following insertion theorem (that is, (1) if and only if (2) of Theorem 4.4 with $\mathscr{A} = \mathscr{F}_\sigma$): A topological space X is F_σ-normal if and only if, whenever $g \leq h$ are upper and lower $\mathscr{F}_\sigma$-measurable functions on X respectively, there exists a class one function f on X such that $g \leq f \leq h$. (In [5], upper and lower $\mathscr{F}_\sigma$-measurable functions are called upper semi-K_1 and lower semi-K_1, respectively.) These ideas can be performed for σ-rings of (higher than 1) additive class α in a topological space, at least in a perfect space.

5. Insertion and extension of functions of class α

As mentioned in Remark 4.6, each result of Section 4 goes over to measurable functions of class α on a perfect space; also see Remark 5.4. Therefore recall that the classification of Borel sets in a metric space is still valid in a perfect space (see [2, 1.7.5]). In fact, the same classification can be performed in a set equipped with a perfect σ-ring as actually done in [14, Chapter V, Section 9]:

Let $\mathscr{A}$ be a perfect σ-ring on a set X, let $\mathscr{A}_0 = \mathscr{A}$ and $\mathscr{B}_0 = \mathscr{A}_c$, and let for each countable ordinal $\alpha > 0$, $\mathscr{A}_\alpha = (\bigcup_{\beta < \alpha} \mathscr{B}_\beta)_\sigma$ and $\mathscr{B}_\alpha = (\bigcup_{\beta < \alpha} \mathscr{A}_\beta)_\delta$. Then for each $\alpha > 0$

(I) $\mathscr{A}_\alpha$ is a σ-ring and $\mathscr{B}_\alpha = (\mathscr{A}_\alpha)_c$ (hence a δ-ring).

(II)   If $0 \le \beta < \alpha$, then $\mathscr{A}_\beta \cup \mathscr{B}_\beta \subset \mathscr{A}_\alpha \cap \mathscr{B}_\alpha$.

Note that the above construction does not use the perfectness of $\mathscr{A}$ except for proving that $\mathscr{A} \subset \mathscr{A}_\alpha$ and $\mathscr{A}_c \subset \mathscr{B}_\alpha$ for each $\alpha > 0$, that is for proving a part of (II) with $\beta = 0$. Therefore the following definition makes sense.

DEFINITION 5.1. A σ-ring $\mathscr{A}$ on X is $\mathscr{A}_\alpha$-normal ($\alpha \ge 0$) if $\mathscr{A}_\alpha$ is a normal ring, that is, for any two disjoint members of $\mathscr{B}_\alpha$ there exist disjoint members of $\mathscr{A}_\alpha$ containing them.

REMARK 5.2. Of course, a topological space $(X, \mathscr{A})$ is normal (respectively F_σ-normal) if and only if $\mathscr{A}$ is $\mathscr{A}_0$-normal (respectively $\mathscr{A}_1$-normal). As proved in [5], each perfect space is $\mathscr{A}_1$-normal. In fact each perfect σ-ring (in particular, each perfect space) is $\mathscr{A}_\alpha$-normal for any $\alpha > 0$, as follows from the following observation.

PROPOSITION 5.3. *Each perfect σ-ring $\mathscr{A}$ on a set X is $\mathscr{A}_\alpha$-normal for arbitrary countable $\alpha > 0$.*

PROOF. [7, Theorem 2, p. 350] shows, in particular, that in a metric space X, every two disjoint members of $\mathscr{B}_\alpha$ are contained in disjoint members of $\mathscr{A}_\alpha$ ($\alpha > 0$). (In fact, they are contained in disjoint members of $\mathscr{A}_\alpha \cap \mathscr{B}_\alpha$, but this is irrelevant for our purposes.) The argument for proving this uses merely the fact that open sets of X form a perfect σ-ring.

REMARK 5.4. In addition to Remark 4.6, note that the normality case of Theorem 4.4 characterizes $\mathscr{A}_\alpha$-normal spaces ($\alpha \ge 0$) too. In particular, since each perfect space $(X, \mathscr{A})$ is $\mathscr{A}_\alpha$-normal ($\alpha > 0$) and $\mathscr{A}_\alpha$-measurable functions on a perfect space are just functions of class α, Theorem 4.4 provides properties of perfect spaces in terms of inserting and extending functions of class α as well as in terms of separating disjoint sets of multiplicative class α by functions of class α ($\alpha > 0$). We state one of them explicitly.

THEOREM 5.5. *If X is a perfect space and $\alpha > 0$ is a countable ordinal, then every class α function from a subset of multiplicative class α to $[0, 1]$ has a class α extension to the whole of X.*

REMARK 5.6. The above theorem partially improves one classical result of [7, Theorem, p. 434], in which X is a metric space and the function takes values in a complete separable metric space. Note that the proof of [7] is not valid for nonmetrizable spaces.

With $\alpha = 1$, Theorem 5.5 is [5, Corollary 2.11].

REMARK 5.7. For $\mathscr{A}$ a perfect σ-ring on a set X, the normality case of the Insertion Theorem of Theorem 4.4 was previously known under the hypothesis that there exists a ring $\mathscr{R}$ of real-valued functions on X such that each lower (respectively upper) $\mathscr{A}$-measurable function on X is a pointwise limit of an increasing (respectively decreasing) sequence of functions from $\mathscr{R}$ (see [14, Chapter V, 10.1]).

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