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A common framework for lattice-valued uniform spaces and probabilistic uniform limit spaces

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Abstract: We study a category of lattice-valued uniform convergence spaces where the lattice is enriched by two algebraic operations. This general setting allows us to view the category of lattice-valued uniform spaces as a reflective subcategory of our category, and the category of probabilistic uniform limit spaces as a coreflective subcategory.

Keywords: L-uniform convergence space, L-uniform space, probabilistic uniform limit space, L-convergence space, L-filter.

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1 Introduction

In his thesis [6], Gutiérrez García developed a common framework for different approaches to lattice-valued uniform spaces. These approaches comprise:

- generalizations of Weil's entourage uniformities [27] such as Lowen's fuzzy uniformities [19] and Höhle's L-uniformities [8], [9];
- generalizations of a uniform operator approach: Hutton's L-uniformities [12] (see also [29]). Classically this approach has not received wide attention. It appears briefly in [17].

In [23] it is argued that Hutton's approach, though leading in the case that $L = \{0,1\}$ to a category of uniform spaces which is isomorphic to Weil's entourage uniformities [27, 1] and Tukey's *covering uniformities* [13, 26], is for $L \neq \{0,1\}$ different and more general. This approach is, consequently, developed in [23] under fairly general conditions (e.g. no distributivity requirements on the lattice L).

The approach in [7, 6] uses a more restricted lattice context but establishes links between the aforementioned approaches of lattice-valued uniform spaces. In fact, based on the notion of *lattice-valued Hutton uniformity*, all three aproaches are shown to be special cases of this general framework. In order to accommodate Lowen's [19] and Höhle's [8, 9] approaches into the new framework, the notion of L-uniformity is used. This L-uniformity is an L-filter satisfying some natural axioms. The lattice context is chosen to be enriched *cl*-premonoids, where the complete lattice $(L, \leq)$ is enriched by two further "algebraic" operations $*$ and $\otimes$, the first of which distributing over arbitrary joins. It gives thus rise to a residual impication operator. If $L = \{0,1\}$, Gutiérrez García's L-uniformities can be viewed as entourage uniformities in the definition of Weil [27].

In the classical case, the category of uniform spaces with uniformly continuous mappings as morphisms is not cartesian closed. In order to have a "nicer" category to work with, Cook and Fischer [2] (modified by Wyler [28]) defined a supercategory, the category of uniform convergence spaces. This category is topological over SET (i.e. it allows initial constructions) and cartesian closed (i.e. it has canonical function spaces) [18]. Cook and Fischer's category, as improved by Wyler [28], was generalised to a category of lattice-valued uniform convergence spaces in [16]. The lattice context was that of a complete Heyting algebra. Also this category of lattice-valued uniform convergence spaces is topological over SET and cartesian closed. Unfortunately, the restriction to complete Heyting algebras as underlying lattices misses several important examples of Gutiérrez García's L-uniform spaces. We therefore generalise the lattice context in this paper from complete Heyting algebras to enriched *cl*-premonoids as they are used in [7]. We will show that a suitable adaptation of the definition of [16] again results in a category which is topological over SET. Moreover, our new category contains many important categories of L-uniform spaces as reflective subcategories. The generalised lattice context, however, has even wider applications. It allows us to view other categories of "many-valued" uniform convergence spaces as natural examples of our definition: the probabilistic uniform limit spaces of Nusser [20], [21]. Nusser's spaces generalise the probabilistic uniform spaces of Florescu [4], and they are explicitly based on a t-norm on $[0, 1]$. Our general setting allows us to view Nusser's category of probabilistic uniform convergence spaces as a coreflective subcategory of our category of lattice-valued uniform convergence spaces when choosing the appropriate lattice context.

The paper is organised as follows: in section 2, we collect the results about lattices and lattice-valued sets that we will need later on, and we fix the notation. The next section is then devoted to lattice-valued filters. We define product filters and inverses and compositions of stratified L-filters on $X \times X$. Further, we give criteria when these constructions again yield stratified L-filters. In section 4 we define our new category of lattice-valued uniform convergence spaces and show that this category is topological over SET. Here we also mention the forgetful functor from our category to the category of lattice-valued convergence spaces [14]. Section 5 studies the most important example, namely Gutiérrez García's lattice-valued uniform spaces. We show that under a mild condition on the underlying lattice, Gutiérrez-García's category is isomorphic to a reflective subcategory of our category. Further, we mention the underlying topological space of a lattice-valued uniform space and show that there are two ways of coming to a lattice-valued convergence space when starting from a lattice-valued uniform space, and that these two ways lead to the same space. One of these methods proceeds by forgetting the uniform structure and embedding the resulting lattice-valued topological space into the category of lattice-valued convergence spaces, while the other embeds the lattice-valued uniform space into the category of lattice-valued uniform convergence spaces and then uses the underlying lattice-valued convergence space. Section 6 is then devoted to showing that Nusser's probabilistic uniform limit spaces form a category which is isomorphic to a coreflective subcategory of our category. Finally, we draw some conclusions.

2 Preliminaries

Throughout this work, we will consider $(L, \leq)$ to be a complete lattice with $\top$, the top element, and $\bot$ the bottom element such that $\top \neq \bot$. The triple $(L, \leq, *)$ is called a *quantale* [24] if

(Q1) $(L, *)$ is a semigroup,

(Q2) $*$ is distributive over arbitrary joins, i.e.

$$\left(\bigvee_{i\in J} \alpha_i\right) * \beta = \bigvee_{i\in J} (\alpha_i * \beta) \quad \text{and} \quad \beta * \left(\bigvee_{i\in J} \alpha_i\right) = \bigvee_{i\in J} (\beta * \alpha_i).$$

As consequence of the distributivity we have that when $\alpha, \beta \in L$ are such that $\alpha \le \beta$, then for any $\gamma \in L$ we will have $\alpha * \gamma \le \beta * \gamma$. When we have that $*$ is $\wedge$, the quantale $(L, \le, \wedge)$ is also called a *complete Heyting algebra*.

A commutative quantale, $(L, \le, *)$, is *divisible* if for every inequality $\beta \le \alpha$ there exists $\delta \in L$ such that $\beta = \alpha * \delta$ [24]. A quantale that is commutative, strictly two-sided ($\top$ is the unit with respect to $*$) and divisible is called a *GL-monoid* [10].

In a commutative quantale we have the *implication operator*:

$$\alpha \to \beta = \bigvee \{\lambda \in L \,:\, \alpha * \lambda \le \beta\}.$$

This operator has the property that $\delta \le \alpha \to \beta \iff \delta * \alpha \le \beta$.

Lemma 2.1 [11] *Let* $(L, \le, *)$ *be a* GL-*monoid and* $\alpha, \beta, \delta, \alpha_i, \beta_i \in L$. *Then the following properties hold:*

(i) $\alpha \to \beta = \top \iff \alpha \le \beta$,

(ii) $\alpha \to \Big(\bigwedge_{i \in I} \beta_i\Big) = \bigwedge_{i \in I} (\alpha \to \beta_i)$,

(iii) $\Big(\bigvee_{i \in I} \alpha_i\Big) \to \beta = \bigwedge_{i \in I} (\alpha_i \to \beta)$,

(iv) $\alpha * \Big(\bigwedge_{i \in I} \beta_i\Big) = \bigwedge_{i \in I} (\alpha * \beta_i)$,

(v) $(\alpha \to \delta) * (\delta \to \beta) \le (\alpha \to \beta)$,

(vi) $\alpha \le \beta \Longrightarrow \delta \to \alpha \le \delta \to \beta$,

(vii) $\alpha \le \beta \Longrightarrow \beta \to \delta \le \alpha \to \delta$,

(viii) $\alpha \to (\beta \to \delta) = (\alpha * \beta) \to \delta$,

(ix) $\alpha * (\alpha \to \beta) = \alpha \wedge \beta$,

(x) $(\alpha \to \beta) * (\delta \to \gamma) \le (\alpha * \delta) \to (\beta * \gamma)$,

(xi) $\alpha * (\beta \to \gamma) \ \le \ \beta \to (\alpha * \gamma)$,

(xii) $\bigwedge_{i \in I} (\alpha_i * \beta_i) \ge \big(\bigwedge_{i \in I} \alpha_i\big) * \big(\bigwedge_{i \in I} \beta_i\big)$.

A *triangular norm* [25] or *t-norm* is a binary operation $*$ on the unit interval $[0, 1]$ such that the following are satisfied:

(T1) $\alpha * \beta = \beta * \alpha$ (commutativity)

(T2) $\alpha * (\beta * \delta) = (\alpha * \beta) * \delta$ (associativity)

(T3) $\alpha * \beta \le \alpha * \delta$ whenever $\beta \le \delta$ (monotonicity)

(T4) $\alpha * 1 = \alpha$ (boundary condition)

The pair $([0,1], *)$ can be considered as a quantale if the t-norm is left-continuous. The three most commonly used (left-continuous) t-norms are:

- minimum: $\alpha * \beta = \alpha \wedge \beta$,
- product: $\alpha * \beta = \alpha \cdot \beta$,
- Lukasiewicz: $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.

The triple $(L, \le, \otimes)$ is a *cl-premonoid* [11] if:

(CL1) $(L, \le)$ is a complete lattice,

(CL2) the binary operation $\otimes$ on L satisfies the *isotonicity axiom* :

$$\alpha_1 \le \alpha_2 \ , \ \beta_1 \le \beta_2 \implies \alpha_1 \otimes \beta_1 \le \alpha_2 \otimes \beta_2,$$

(CL3) for each $\alpha \in L$, $\alpha \le \alpha \otimes \top$ and $\alpha \le \top \otimes \alpha$,

(CL4) the operation $\otimes$ is distributive over *non-empty* joins, ie: for $J \neq \emptyset$,

$$\left(\bigvee_{i \in J} \alpha_i\right) \otimes \beta = \bigvee_{i \in J} (\alpha_i \otimes \beta), \quad \beta \otimes \left(\bigvee_{i \in J} \alpha_i\right) = \bigvee_{i \in J} (\beta \otimes \alpha_i).$$

An *enriched cl-premonoid* [11] is a quadruple $(L, \le, \otimes, *)$ where:

(E1) $(L, \le, \otimes)$ is a *cl*-premonoid,

(E2) $(L, \le, *)$ is a GL-monoid,

(E3) the operation $*$ is dominated by $\otimes$. That is, for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L$:

$$(\alpha_1 \otimes \beta_1) * (\alpha_2 \otimes \beta_2) \le (\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2).$$

A consequence of the domination is that $\alpha * \beta \le \alpha \otimes \beta$.

Examples 2.2

- If $(L, \le, \wedge)$ is a complete Heyting algebra, then $(L, \le, \wedge, \wedge)$ is an enriched *cl*-premonoid.

- If $(L, \leq, *)$ is a quantale, then $(L, \leq, \wedge, *)$ is an enriched *cl*-premonoid.

A GL-monoid $(L, \leq, *)$ is said to have *square roots* [10] if there exists a unary operator $S : L \longrightarrow L$ with the following properties:

(S1) for all $\alpha \in L$, $S(\alpha) * S(\alpha) = \alpha$,

(S2) $\beta * \beta \leq \alpha \Longrightarrow \beta \leq S(\alpha)$.

If $(L, \leq, *)$ is a GL-monoid with square roots, then the *monoidal mean operator*, $\circledast : L \times L \longrightarrow L$, is defined by $\alpha \circledast \beta = S(\alpha) * S(\beta)$. If $(L, \leq, *)$ also satisfies:

(S3) $S(\alpha * \beta) = \big(S(\alpha) * S(\beta)\big) \vee S(\bot)$ for all $\alpha, \beta \in L$

then if we use the monoidal mean operator as the *cl*-premonoid operation we get an enriched *cl*-premonoid: $(L, \leq, \circledast, *)$.

All of the t-norms mentioned earlier are left-continuous, and have square roots satisfying **(S3)**. Thus we can use the monoidal mean operator to form an enriched *cl*-premonoid where for the minimum t-norm $\alpha \circledast \beta = \alpha \wedge \beta$. For the product t-norm we get $\alpha \circledast \beta = \sqrt{\alpha \cdot \beta}$, the geometric mean, and for the Lukasiewicz t-norm $\alpha \circledast \beta = \frac{\alpha+\beta}{2}$, the arithmetic mean.

Lemma 2.3 *Let $(L, \leq, \otimes, *)$ be an enriched cl-premonoid and let $\alpha, \beta, \delta \in L$. If $\alpha \leq \alpha \otimes \alpha$, then*

$$(\alpha \to \beta) \otimes (\alpha \to \delta) \;\leq\; \alpha \to (\beta \otimes \delta).$$

PROOF: This form of the proof was suggested in communication with Javier Gutiérrez García. We will use Lemma 2.1 (ix) and the fact that the $*$ operation is dominated by the $\otimes$ to show

$$\begin{aligned}
\big((\alpha \to \beta) \otimes (\alpha \to \delta)\big) * \alpha &\leq \big((\alpha \to \beta) \otimes (\alpha \to \delta)\big) * (\alpha \otimes \alpha) \\
&\leq \big((\alpha \to \beta) * \alpha\big) \otimes \big((\alpha \to \delta) * \alpha\big) \\
&= (\alpha \wedge \beta) \otimes (\alpha \wedge \delta) \\
&\leq \beta \otimes \delta.
\end{aligned}$$

The desired result can easily be seen from the property of the implication operator.

■

For the monoidal mean operator, we will clearly have $\alpha \circledast \alpha = S(\alpha) * S(\alpha) = \alpha$. Also, for the Heyting algebra case $(L, \leq, \wedge, \wedge)$ we have $\alpha \wedge \alpha = \alpha$. So clearly the above lemma will be valid for all $\alpha \in L$ for these cases.

Let $(L, \leq, \otimes, *)$ be an enriched *cl*-premonoid. If the equation:

$$(\alpha_1 * \beta_1) \otimes (\alpha_2 * \beta_2) = \big((\alpha_1 \otimes \alpha_2) * (\beta_1 \otimes \beta_2)\big) \vee \big((\alpha_1 \otimes \bot) * (\beta_1 \otimes \top)\big) \vee \big((\bot \otimes \alpha_2) * (\top \otimes \beta_2)\big)$$

is satisfied for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L$ then $(L, \leq, \otimes, *)$ is *pseudo-bisymmetric* [11]. In the case where $\otimes = *$ and the case of the monoidal mean operator, as well as $(L, \leq, \wedge, *)$ and $(L, \leq, \wedge, \wedge)$, we have pseudo-bisymmetry.

Let $(L, \leq, \otimes, *)$ be an enriched *cl*-premonoid and X be a set. We denote the L-sets on X by $a,\ b,\ c,\ \ldots \in L^X$. For L-sets $a,\ b,\ a_j$ $(j \in J)$, we extend the operations from $(L, \leq, \otimes, *)$ pointwise by

$$\begin{aligned}
\Big(\bigwedge_{j\in J} a_j\Big)(x) &= \bigwedge_{j\in J} \big(a_j(x)\big),\\
\Big(\bigvee_{j\in J} a_j\Big)(x) &= \bigvee_{j\in J} \big(a_j(x)\big),\\
(a \otimes b)(x) &= a(x) \otimes b(x),\\
(a * b)(x) &= a(x) * b(x).
\end{aligned}$$

To extend the order relation, we say $a \leq b$ if for all $x \in X$ we have $a(x) \leq b(x)$. For any $\alpha \in L$ and $A \subset X$ we denote

$$\alpha_A(x) = \begin{cases} \alpha & \text{if } x \in A \\ \bot & \text{else.} \end{cases}$$

Two special cases of this are the charateristic function of A, $\top_A$, and the zero function, $\bot_X$.

If $\varphi : X \longrightarrow Y$ is a mapping and $a \in L^X$, $b \in L^Y$, then we denote $\varphi^{\rightarrow} : L^X \longrightarrow L^Y$ the mapping $\varphi^{\rightarrow}(a)(y) = \bigvee_{x\,:\,\varphi(x)=y} a(x)$ (with $\bigvee \emptyset = \bot$) and $\varphi^{\leftarrow} : L^Y \longrightarrow L^X$ the mapping $\varphi^{\leftarrow}(b) = b \circ \varphi$.

3 Stratified L-filters

Definition 3.1 [11] Let X be a set and $(L, \leq, \otimes, *)$ an enriched *cl*-premonoid. A map $\mathcal{F} : L^X \longrightarrow L$ is a *stratified L-filter* on X if $\mathcal{F}$ satisfies:

(LF0) $\mathcal{F}(\top_X) = \top, \quad \mathcal{F}(\bot_X) = \bot,$

(LF1) $a_1, a_2 \in L^X,\ a_1 \leq a_2 \Longrightarrow \mathcal{F}(a_1) \leq \mathcal{F}(a_2),$

(LF2) $\mathcal{F}(a_1) \otimes \mathcal{F}(a_2) \leq \mathcal{F}(a_1 \otimes a_2)$ for all $a_1, a_2 \in L^X$,

(LFS) for all $\alpha \in L$, for all $a \in L^X$, $\quad \alpha * \mathcal{F}(a) \leq \mathcal{F}(\alpha_X * a).$

The set of all stratified L-filters on X is denoted by $\mathcal{F}^S_L(X)$.

Examples 3.2

- The point filter $[x] : L^X \longrightarrow L, a \longmapsto a(x)$ is a stratified L-filter for every $x \in X$.
- For $A \subset X$, the mapping $[A] : L^X \longrightarrow L, a \longmapsto \bigwedge_{x \in A} a(x)$ is a stratified L-filter on X.

A partial ordering can be defined on the set of all stratified L-filters on X by:

$$\mathcal{F} \leq \mathcal{G} \quad \Longleftrightarrow \quad \mathcal{F}(a) \leq \mathcal{G}(a) \quad \forall\, a \in L^X.$$

Here we say that $\mathcal{F}$ is coarser than $\mathcal{G}$, or $\mathcal{G}$ is finer than $\mathcal{F}$. For a collection of stratified L-filters on X, $\{\mathcal{F}_i : i \in I\}$, the greatest lower bound is defined [6] for $a \in L^X$:

$$(\bigwedge_{i \in I} \mathcal{F}_i)(a) = \bigwedge_{i \in I} \mathcal{F}_i(a)$$

and $\bigwedge\limits_{i \in I} \mathcal{F}_i \in \mathcal{F}_L^S(X)$.

It is clear that for $A \subset X$ we have $[A] = \bigwedge_{x \in A}[x]$. Further, $[X]$ is the coarsest stratified L-filter on X [11]. The least upper bound of two stratified L-filters does not always exist but it has been shown [7] that an upper bound for two L-filters will exist when they satisfy certain conditions.

Proposition 3.3 [7] *Let $(L, \leq, \otimes, *)$ be an enriched cl-premonoid that is pseudo-bisymmetric. Further let $\mathcal{F}$ and $\mathcal{G}$ be two stratified L-filters on X. If $\mathcal{F}(a_1) * \mathcal{G}(a_2) = \bot$ for all $a_1, a_2 \in L^X$ such that $a_1 * a_2 = \bot_X$, then there exists an upper bound for both $\mathcal{F}$ and $\mathcal{G}$.*

Gutiérrez García [7] has shown that if $\otimes = *$ and the condition above is satisfied, then the least upper bound of two stratified L-filters is given by:

$$(\mathcal{F} \vee \mathcal{G})(a) = \bigvee \{\mathcal{F}(a_1) * \mathcal{G}(a_2) \mid a_1, a_2 \in L^X \text{ and } a_1 * a_2 \leq a\}.$$

Let X and Y be sets, $\varphi : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_L^S(X)$. The *image of $\mathcal{F}$ under φ*, $\varphi^{\rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$, is always a stratified L-filter on Y and is defined [11] for $a \in L^Y$:

$$\varphi^{\rightarrow}(\mathcal{F})(a) = \mathcal{F}\big(\varphi^{\leftarrow}(a)\big) = \mathcal{F}(a \circ \varphi).$$

From this definition it is straightforward to deduce that $\varphi\big([x]\big) = \big[\varphi(x)\big]$.

Let X and Y be sets, and suppose $\varphi : X \longrightarrow Y$ and let $\mathcal{F} \in \mathcal{F}_L^S(Y)$. For $a \in L^X$ define [11] $\varphi^{\leftarrow}(\mathcal{F}) : L^X \longrightarrow L$ by

$$\varphi^{\leftarrow}(\mathcal{F})(a) = \bigvee \big\{\mathcal{F}(b) \mid \varphi^{\leftarrow}(b) \leq a\big\}.$$

The mapping $\varphi^{\leftarrow}(\mathcal{F})$ is a stratified L-filter on X if and only if, for $b \in L^Y$, $\mathcal{F}(b) = \bot$ whenever $\varphi^{\leftarrow}(b) = b \circ \varphi = \bot_X$ [14].

If we have X, Y and Z as sets, and $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$, $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$, then it is easy to see that $(\varphi \circ \psi)^{\rightarrow}(\mathcal{F}) = \varphi^{\rightarrow}\big(\psi^{\rightarrow}(\mathcal{F})\big)$, and $\varphi(\mathcal{F}) \wedge \varphi(\mathcal{G}) = \varphi(\mathcal{F} \wedge \mathcal{G})$. As a consequence, if we have φ and $\mathcal{F}, \mathcal{G}$ as above with $\mathcal{F} \leq \mathcal{G}$, then $\varphi(\mathcal{F}) \leq \varphi(\mathcal{G})$.

Lemma 3.4 *Let X and Y be sets and $\varphi : X \longrightarrow Y$. With $\mathcal{F} \in \mathcal{F}_L^S(X)$ and $\mathcal{G} \in \mathcal{F}_L^S(Y)$ the following hold:*

(i) $\varphi^{\leftarrow}\big(\varphi^{\rightarrow}(\mathcal{F})\big) \in \mathcal{F}_L^S(X)$ *and* $\varphi^{\leftarrow}\big(\varphi^{\rightarrow}(\mathcal{F})\big) \leq \mathcal{F}$,

(ii) *if* $\varphi^{\leftarrow}(\mathcal{G}) \in \mathcal{F}_L^S(X)$, *then* $\mathcal{G} \leq \varphi^{\rightarrow}\big(\varphi^{\leftarrow}(\mathcal{G})\big)$,

(iii) *if* $\mathcal{G} \leq \varphi^{\rightarrow}(\mathcal{F})$ *then* $\varphi^{\leftarrow}(\mathcal{G}) \in \mathcal{F}_L^S(X)$.

PROOF: the proofs for **(i)** and **(ii)** are straightforward, while **(iii)** follows from **(i)** and the fact that $\varphi^{\leftarrow}(\mathcal{G}) \leq \varphi^{\leftarrow}(\varphi^{\rightarrow}(\mathcal{F}))$.

■

From here on, we consider our lattice L to be an enriched *cl*-premonoid $(L, \leq \otimes, *)$ that is pseudo-bisymmetric. The pseudo-bisymmetry is required as it will guarantee the existence of upper bounds (see Proposition 3.3), and this is required in the definition of a product L-filter.

Consider now the *projection mappings*:

$$P_1 : \begin{cases} X \times Y \longrightarrow X \\ (x, y) \longmapsto x \end{cases} \qquad \text{and} \qquad P_2 : \begin{cases} X \times Y \longrightarrow Y \\ (x, y) \longmapsto y. \end{cases}$$

Definition 3.5 Let X and Y be sets and let $\mathcal{F} \in \mathcal{F}_L^S(X)$ and $\mathcal{G} \in \mathcal{F}_L^S(Y)$. We define their product $\mathcal{F} \times \mathcal{G}$ by:

$$\mathcal{F} \times \mathcal{G} = P_1^{\leftarrow}(\mathcal{F}) \vee P_2^{\leftarrow}(\mathcal{G}).$$

Proposition 3.6 *The mapping $\mathcal{F} \times \mathcal{G}$ is a stratified L-filter on $X \times Y$.*

PROOF: Here we use the result of Proposition 3.3. That is, we must show that $P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b) = \perp$ for all $a, b \in L^{X \times Y}$ such that $a * b = \perp_{X \times Y}$. This will show that there exists an upper bound for $P_1^{\leftarrow}(\mathcal{F})$ and $P_2^{\leftarrow}(\mathcal{G})$, and hence there must exist a least upper bound as the meet of all upper bounds.

Suppose that $a, b \in L^{X \times Y}$ are such that $a * b = \perp_{X \times Y}$. Then

$$\begin{aligned}
&P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b) \\
&= \bigvee \big\{\mathcal{F}(c) \mid c \in L^X, P_1^{\leftarrow}(c) \leq a\big\} * \bigvee \big\{\mathcal{G}(d) \mid d \in L^Y, P_2^{\leftarrow}(d) \leq b\big\} \\
&\leq \bigvee \big\{\mathcal{F}(c) * \mathcal{G}(d) \mid c \in L^X, d \in L^Y, P_1^{\leftarrow}(c) * P_2^{\leftarrow}(d) \leq \perp_{X \times Y}\big\}.
\end{aligned}$$

Now, $P_1^{\leftarrow}(c)(x,y) = c \circ P_1(x,y) = c(x)$. Similarly $P_2^{\leftarrow}(d)(x,y) = d(y)$. Therefore

$$P_1^{\leftarrow}(\mathcal{F})(a) * P_Y^{\leftarrow}(\mathcal{G})(b) \leq \bigvee \big\{\mathcal{F}(c) * \mathcal{G}(d) \mid c(x) * d(y) = \bot,\ \forall\, x \in X,\ \forall\, y \in Y\big\}.$$

Now we note that if $c(x) * d(y) = \bot$ for all $x \in X, y \in Y$, then since $*$ is a quantale operation, and using **(Q2)** we get

$$\bot = \bigvee_{x \in X} \bigvee_{y \in Y} \big(c(x) * d(y)\big) = \big(\bigvee_{x \in X} c(x)\big) * \big(\bigvee_{y \in Y} d(y)\big).$$

Together with **(LF1)**, this yields

$$P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b) \leq \bigvee \big\{\mathcal{F}(\alpha_X) * \mathcal{G}(\beta_Y) \mid \alpha * \beta = \bot\big\}.$$

Now, using **(LFS)**, the stratification of the L-filters $\mathcal{F}$ and $\mathcal{G}$, we have that for all $\alpha, \beta \in L$,

$$\mathcal{F}(\alpha_X) * \mathcal{G}(\beta_Y) \leq \mathcal{F}\Big(\alpha_X * \big(\mathcal{G}(\beta_Y)\big)_X\Big).$$

Then consider

$$\begin{aligned}\big[\alpha_X * \big(\mathcal{G}(\beta_Y)\big)_X\big](x) &= \alpha_X(x) * \big(\mathcal{G}(\beta_Y)\big)_X(x) \\ &= \alpha * \mathcal{G}(\beta_Y) \\ &\leq \mathcal{G}(\alpha_Y * \beta_Y) \\ &= \mathcal{G}(\bot_Y) = \bot.\end{aligned}$$

From this we get that $\mathcal{F}(\alpha_X) * \mathcal{G}(\beta_Y) \leq \mathcal{F}(\bot_X) = \bot$. Therefore, since

$$P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b) \leq \bigvee \big\{\mathcal{F}(\alpha_X) * \mathcal{G}(\beta_Y) \mid \alpha * \beta = \bot\big\}$$

we get that $P_1^{\leftarrow}(\mathcal{F})(a) * P_2^{\leftarrow}(\mathcal{G})(b) = \bot$.

$\blacksquare$

Lemma 3.7 [16] *Let X and Y be sets and let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X), \mathcal{H}, \mathcal{K} \in \mathcal{F}_L^S(Y)$. If $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{H} \leq \mathcal{K}$ then $\mathcal{F} \times \mathcal{H} \leq \mathcal{G} \times \mathcal{K}$.*

Lemma 3.8 [15] *Let $\mathcal{F} \in \mathcal{F}_L^S(X \times Y)$. Then $P_1^{\rightarrow}(\mathcal{F}) \times P_2^{\rightarrow}(\mathcal{F}) \leq \mathcal{F}$. Further, if $\mathcal{G} \in \mathcal{F}_L^S(X)$ and $\mathcal{H} \in \mathcal{F}_L^S(Y)$ then $P_1^{\rightarrow}(\mathcal{G} \times \mathcal{H}) \geq \mathcal{G}$ and $P_2^{\rightarrow}(\mathcal{G} \times \mathcal{H}) \geq \mathcal{H}$.*

Now we propose the definition of a new mapping from $L^{X \times X} \longrightarrow L$, one that is in fact a stratified L-filter on the product space $X \times X$. This is later used when inducing a stratified L-limit space from a stratified L-uniform convergence space.

Definition 3.9 Let X be a set, $\mathcal{F} \in \mathcal{F}_L^S(X), x \in X$. We define $\mathcal{F}_x : L^{X \times X} \longrightarrow L$ by

$$\mathcal{F}_x(d) = \mathcal{F}\big(d(\cdot, x)\big),\ \text{for } d \in L^{X \times X}.$$

Proposition 3.10 *Let X be a set, $\mathcal{F} \in \mathcal{F}_L^S(X)$ and $x \in X$. Then $\mathcal{F}_x \in \mathcal{F}_L^S(X \times X)$.*

The proof is left to the reader.

Lemma 3.11 [16] *Let X be set and $(L, \leq, \wedge, \wedge)$ a complete Heyting algebra. If $\mathcal{F} \in \mathcal{F}_L^S(X)$ and $x \in X$, then*

$$\mathcal{F}_x = \mathcal{F} \times [x].$$

Lemma 3.12 *Let $x, y \in X$ and $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X)$ and let $\varphi : X \longrightarrow Y$. Then*

(i) $[x]_y = [(x, y)]$,

(ii) $\mathcal{F}_x \wedge \mathcal{G}_x = (\mathcal{F} \wedge \mathcal{G})_x$,

(iii) $(\varphi \times \varphi)^{\rightarrow}(\mathcal{F}_x) = \varphi^{\rightarrow}(\mathcal{F})_{\varphi(x)}$.

Proof: We show only **(iii)** as **(i)** and **(ii)** are straightforward. Let $a \in L^{Y \times Y}$. Then

$$\begin{aligned}
(\varphi \times \varphi)^{\rightarrow}(\mathcal{F}_x)(a) &= \mathcal{F}_x\big(a \circ (\varphi \times \varphi)\big) \\
&= \mathcal{F}\Big(\big(a \circ (\varphi \times \varphi)\big)(\cdot, x)\Big) \\
&= \mathcal{F}\Big(a\big(\varphi(\cdot), \varphi(x)\big)\Big) \\
&= \mathcal{F}\Big(a\big(\cdot, \varphi(x)\big) \circ \varphi\Big) \\
&= \mathcal{F}\Big(\varphi^{\leftarrow}\Big(a\big(\cdot, \varphi(x)\big)\Big)\Big) \\
&= \varphi^{\rightarrow}(\mathcal{F})\Big(a\big(\cdot, \varphi(x)\big)\Big) \\
&= \varphi^{\rightarrow}(\mathcal{F})_{\varphi(x)}(a).
\end{aligned}$$

■

For a stratified L-filter $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ and $d \in L^{X \times X}$, define $\mathcal{F}^{-1}(d) = \mathcal{F}(d^{-1})$, where $d^{-1}(x, y) = d(y, x)$ for $(x, y) \in X \times X$. From [16] we know that $\mathcal{F}^{-1} \in \mathcal{F}_L^S(X \times X)$.

Lemma 3.13 [16] *Let X and Y be sets, $\varphi : X \longrightarrow Y$ and $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X \times X)$. If $\mathcal{F} \leq \mathcal{G}$ then*

(i) $(\mathcal{F}^{-1})^{-1} = \mathcal{F}$,

(ii) $\mathcal{F}^{-1} \leq \mathcal{G}^{-1}$,

(iii) $(\varphi \times \varphi)^{\rightarrow}(\mathcal{F}^{-1}) = \big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F})\big)^{-1}$.

Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X \times X)$. We then define the mapping $\mathcal{F} \circ \mathcal{G} : L^{X \times X} \longrightarrow L$ by:

$$\mathcal{F} \circ \mathcal{G}(d) = \bigvee \big\{\mathcal{F}(a) * \mathcal{G}(b) : a, b \in L^{X \times X}, a \circ b \leq d\big\}$$

with $a \circ b(x,y) = \bigvee_{z \in X} a(x,z) * b(z,y)$. We call $\mathcal{F} \circ \mathcal{G}$ the *composition* of $\mathcal{F}$ and $\mathcal{G}$.

Below we will give a condition that, when satisfied, will give us $\mathcal{F} \circ \mathcal{G} \in \mathcal{F}_L^S(X \times X)$. In order to prove the main condition which ensures that the composition $\mathcal{F} \circ \mathcal{G}$ is a stratified L-filter, we will make use of the following results.

Lemma 3.14 *Let $f, g, \bar{f}, \bar{g} \in L^{X \times X}$ and $a, b \in L^{X \times X}$. If $f \circ g \leq a$ and $\bar{f} \circ \bar{g} \leq b$, and with $\alpha \in L$, we have*

(i) $(f \otimes \bar{f}) \circ (g \otimes \bar{g}) \leq a \otimes b$,

(ii) $(f \otimes \perp_{X \times X}) \circ (g \otimes \top_{X \times X}) \leq a \otimes b$,

(iii) $(\perp_{X \times X} \otimes \bar{f}) \circ (\top_{X \times X} \otimes \bar{g}) \leq a \otimes b$,

(iv) $(\alpha_{X \times X} * f) \circ g \leq \alpha_{X \times X} * a$.

These proofs are technical, but not difficult and so are omitted.

Proposition 3.15 *Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^S(X \times X)$ and let $(L, \leq, \otimes, *)$ be a pseudo-bisymmetric enriched cl-premonoid. For any $f, g \in L^{X \times X}$, the following are equivalent:*

(i) *the mapping $\mathcal{F} \circ \mathcal{G} \in \mathcal{F}_L^S(X \times X)$,*

(ii) *if $f \circ g = \perp_{X \times X}$, then $\mathcal{F}(f) * \mathcal{G}(g) = \perp$.*

PROOF:

Suppose **(i)** and let $f \circ g = \perp_{X \times X}$. From **(LF0)** we have $\mathcal{F} \circ \mathcal{G}(\perp_{X \times X}) = \perp$. This will only be the case if

$$\perp = \bigvee_{\substack{h,k \in L^{X \times X} \\ h \circ k = \perp_{X \times X}}} \mathcal{F}(h) * \mathcal{G}(k).$$

Since $\mathcal{F}(f) * \mathcal{G}(g) \leq \bigvee_{\substack{h,k \in L^{X \times X} \\ h \circ k = \perp_{X \times X}}} \mathcal{F}(h) * \mathcal{G}(k)$, we get $\mathcal{F}(f) * \mathcal{G}(g) = \perp$.

Conversely, suppose **(ii)**. We check the axioms for a stratified L-filter.

LF0: It is easily checked that $\top_{X\times X} \circ \top_{X\times X} \leq \top_{X\times X}$. With this we conclude that $\mathcal{F} \circ \mathcal{G}(\top_{X\times X}) = \top$. Further we have

$$\begin{aligned}\mathcal{F} \circ \mathcal{G}(\perp_{X\times X}) &= \bigvee_{f\circ g=\perp_{X\times X}} \mathcal{F}(f) * \mathcal{G}(g)\\ &= \bigvee_{f\circ g=\perp_{X\times X}} \perp \qquad \text{by } \textbf{(ii)}\\ &= \perp.\end{aligned}$$

LF1: This is easy and is left to the reader.

LF2: Let $a, b \in L^{X\times X}$. Then

$$\begin{aligned}\mathcal{F} \circ \mathcal{G}(a) \otimes \mathcal{F} \circ \mathcal{G}(b) &= \Big(\bigvee_{f\circ g\leq a} \mathcal{F}(f) * \mathcal{G}(g)\Big) \otimes \Big(\bigvee_{\bar{f}\circ\bar{g}\leq b} \mathcal{F}(\bar{f}) * \mathcal{G}(\bar{g})\Big)\\ &= \bigvee_{\substack{f\circ g\leq a,\\ \bar{f}\circ\bar{g}\leq b}} \Big(\big(\mathcal{F}(f) * \mathcal{G}(g)\big) \otimes \big(\mathcal{F}(\bar{f}) * \mathcal{G}(\bar{g})\big)\Big) = P.\end{aligned}$$

The above equality is as a result of the distributivity of the $\otimes$ operation over non-empty joins. We now use the property of a pseudo-bisymmetric subset to produce the following inequality.

$$\begin{aligned}P \leq &\bigvee_{f\circ g\leq a, \bar{f}\circ\bar{g}\leq b} \Big(\Big(\big[\mathcal{F}(f) \otimes \mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g) \otimes \mathcal{G}(\bar{g})\big]\Big)\\ &\vee \Big(\big[\mathcal{F}(f) \otimes \perp\big] * \big[\mathcal{G}(g) \otimes \top\big]\Big)\\ &\vee \Big(\big[\perp \otimes \mathcal{F}(\bar{f})\big] * \big[\top \otimes \mathcal{G}(\bar{g})\big]\Big)\Big)\\ = &\bigvee_{f\circ g\leq a, \bar{f}\circ\bar{g}\leq b} \Big(\Big(\big[\mathcal{F}(f) \otimes \mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g) \otimes \mathcal{G}(\bar{g})\big]\Big)\\ &\vee \Big(\big[\mathcal{F}(f) \otimes \mathcal{F}(\perp_{X\times X})\big] * \big[\mathcal{G}(g) \otimes \mathcal{G}(\top_{X\times X})\big]\Big)\\ &\vee \Big(\big[\mathcal{F}(\perp_{X\times X}) \otimes \mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(\top_{X\times X}) \otimes \mathcal{G}(\bar{g})\big]\Big)\Big)\\ = &\,Q.\end{aligned}$$

The equality above comes as a result of the fact that both $\mathcal{F}$ and $\mathcal{G}$ are stratified L-filters and from property **(LF0)** described earlier. We can further produce another inequality by, instead of taking the join over a single small set, we take the join of the joins of three larger sets:

$$\begin{aligned}
Q \leq & \bigvee_{f\circ g\leq a,\bar{f}\circ\bar{g}\leq b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big] \Big) \\
& \vee \bigvee_{f\circ g\leq a,\bar{f}\circ\bar{g}\leq b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bot_{X\times X})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\top_{X\times X})\big] \Big) \\
& \vee \bigvee_{f\circ g\leq a,\bar{f}\circ\bar{g}\leq b} \Big(\big[\mathcal{F}(\bot_{X\times X})\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(\top_{X\times X})\otimes\mathcal{G}(\bar{g})\big] \Big) \qquad = R.
\end{aligned}$$

Now we use the results from Lemma 3.14 to choose larger sets for each of the joins shown above:

$$\begin{aligned}
R \leq & \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big] \Big) \\
& \vee \bigvee_{(f\otimes\bot_{X\times X})\circ(g\otimes\top_{X\times X})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bot_{X\times X})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\top_{X\times X})\big] \Big) \\
& \vee \bigvee_{(\bot_{X\times X}\otimes\bar{f})\circ(\top_{X\times X}\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(\bot_{X\times X})\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(\top_{X\times X})\otimes\mathcal{G}(\bar{g})\big] \Big) \\
& = S.
\end{aligned}$$

Now we again choose larger sets by allowing any $f, g, \bar{f}$ or $\bar{g}$ instead of $\bot_{X\times X}$ and $\top_{X\times X}$.

$$\begin{aligned}
S \leq & \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big] \Big) \\
& \vee \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big] \Big) \\
& \vee \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big] \Big).
\end{aligned}$$

Since each of these sups is the same, we have:

$$\begin{aligned}
S = & \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\big[\mathcal{F}(f)\otimes\mathcal{F}(\bar{f})\big] * \big[\mathcal{G}(g)\otimes\mathcal{G}(\bar{g})\big] \Big) \\
\leq & \bigvee_{(f\otimes\bar{f})\circ(g\otimes\bar{g})\leq a\otimes b} \Big(\mathcal{F}(f\otimes\bar{f}) * \mathcal{G}(g\otimes\bar{g}) \Big) \\
\leq & \bigvee_{h\circ k\leq a\otimes b} \Big(\mathcal{F}(h) * \mathcal{G}(k) \Big) \\
= & \mathcal{F}\circ\mathcal{G}(a\otimes b).
\end{aligned}$$

LFS: This follows from Lemma 3.14(iv).

■

We have thus provided a condition that, if satisfied, will guarantee that the composition of two stratified L-filters will again be a stratified L-filter. We now show some further results relating to the composition of L-filters that will be needed later on.

Lemma 3.16 *Let X and Y be sets and let $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K} \in \mathcal{F}_L^S(X \times X)$ and $\mathcal{M}, \mathcal{N} \in \mathcal{F}_L^S(Y \times Y)$. Further suppose that $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{G} \leq \mathcal{K}$ and that $\varphi : X \longrightarrow Y$. Then*

(i) *if $\mathcal{H} \circ \mathcal{K} \in \mathcal{F}_L^S(X \times X)$, then $\mathcal{F} \circ \mathcal{G} \in \mathcal{F}_L^S(X \times X)$ and $\mathcal{F} \circ \mathcal{G} \leq \mathcal{H} \circ \mathcal{K}$,*

(ii) $(\varphi \times \varphi)^{\rightarrow}(\mathcal{F}) \circ (\varphi \times \varphi)^{\rightarrow}(\mathcal{G}) \leq (\varphi \times \varphi)^{\rightarrow}(\mathcal{F} \circ \mathcal{G})$,

(iii) $(\varphi \times \varphi)^{\leftarrow}(\mathcal{M} \circ \mathcal{N}) \leq (\varphi \times \varphi)^{\leftarrow}(\mathcal{M}) \circ (\varphi \times \varphi)^{\leftarrow}(\mathcal{N})$.

PROOF: The proof of (i) can be found in [16]. For (ii), we first show that for $d_1, d_2, a \in L^{X \times X}$, if $d_1 \circ d_2 \leq a$, then

$$\big(d_1 \circ (\varphi \times \varphi)\big) \circ \big(d_2 \circ (\varphi \times \varphi)\big) \leq a \circ (\varphi \times \varphi).$$

Suppose $d_1 \circ d_2 \leq a$ and let $(x, y) \in X \times X$. Then

$$\begin{aligned}
\big(d_1 \circ (\varphi \times \varphi)\big) \circ \big(d_2 \circ (\varphi \times \varphi)\big)(x, y) &= \bigvee_{z \in X} d_1 \circ (\varphi \times \varphi)(x, z) * d_2 \circ (\varphi \times \varphi)(z, y) \\
&= \bigvee_{z \in X} d_1\big(\varphi(x), \varphi(z)\big) * d_2\big(\varphi(z), \varphi(y)\big) \\
&\leq \bigvee_{w \in Y} d_1\big(\varphi(x), w\big) * d_2\big(w, \varphi(y)\big) \\
&= (d_1 \circ d_2)\big(\varphi(x), \varphi(y)\big) \\
&\leq a\big(\varphi(x), \varphi(y)\big) \\
&= a \circ (\varphi \times \varphi)(x, y).
\end{aligned}$$

Now we let $b \in L^{Y \times Y}$ and show that

$$\begin{aligned}
\big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F}) \circ (\varphi \times \varphi)^{\rightarrow}(\mathcal{G})\big)(b) &= \bigvee_{\substack{d_1, d_2 \in L^{Y \times Y} \\ d_1 \circ d_2 \leq b}} \big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F})\big)(d_1) * \big((\varphi \times \varphi)^{\rightarrow}(\mathcal{G})\big)(d_2) \\
&= \bigvee_{\substack{d_1, d_2 \in L^{Y \times Y} \\ d_1 \circ d_2 \leq b}} \mathcal{F}\big(d_1 \circ (\varphi \times \varphi)\big) * \mathcal{G}\big(d_2 \circ (\varphi \times \varphi)\big) \\
&\leq \bigvee_{\substack{c_1, c_2 \in L^{X \times X} \\ c_1 \circ c_2 \leq b \circ (\varphi \times \varphi)}} \mathcal{F}(c_1) * \mathcal{G}(c_2) \\
&= (\mathcal{F} \circ \mathcal{G})\big(b \circ (\varphi \times \varphi)\big) \\
&= \big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F} \circ \mathcal{G})\big)(b).
\end{aligned}$$

For (iii), as a first step consider $m, n \in L^{Y\times Y}$. Then with $(x,y) \in X \times X$, we have

$$\begin{aligned}(\varphi\times\varphi)^{\leftarrow}(m)\circ(\varphi\times\varphi)^{\leftarrow}(n)(x,y) &= \bigvee_{z\in X} m(\varphi(x),\varphi(z)) * n(\varphi(z),\varphi(y))\\ &\le \bigvee_{u\in Y} m(\varphi(x),u) * n(u,\varphi(y))\\ &= m\circ n(\varphi(x),\varphi(y)) \;=\; (\varphi\times\varphi)^{\leftarrow}(m\circ n)(x,y).\end{aligned}$$

With this we conclude

$$\begin{aligned}(\varphi\times\varphi)^{\leftarrow}(\mathcal{M}\circ\mathcal{N})(a) &= \bigvee_{(\varphi\times\varphi)^{\leftarrow}(d)\le a} \mathcal{M}\circ\mathcal{N}(d)\\
&= \bigvee_{d:\,(\varphi\times\varphi)^{\leftarrow}(d)\le a}\Big(\bigvee_{b\circ c\le d}\mathcal{M}(b)*\mathcal{N}(c)\Big)\\
&\le \bigvee_{\substack{b,c:\\(\varphi\times\varphi)^{\leftarrow}(b\circ c)\le a}} \mathcal{M}(b)*\mathcal{N}(c)\\
&\le \bigvee_{\substack{b,c:\\(\varphi\times\varphi)^{\leftarrow}(b)\circ(\varphi\times\varphi)^{\leftarrow}(c)\le a}} \mathcal{M}(b)*\mathcal{N}(c)\\
&\le \bigvee_{\substack{h,k:\\h\circ k\le a}}\Big(\bigvee_{\substack{b,c:\\(\varphi\times\varphi)^{\leftarrow}(b)\le h\\(\varphi\times\varphi)^{\leftarrow}(c)\le k}}\mathcal{M}(b)*\mathcal{N}(c)\Big)\\
&= \bigvee_{\substack{h,k:\\h\circ k\le a}}\Big(\bigvee\{\mathcal{M}(b)\;:\;(\varphi\times\varphi)^{\leftarrow}(b)\le h\} * \bigvee\{\mathcal{N}(c)\;:\;(\varphi\times\varphi)^{\leftarrow}(c)\le k\}\Big)\\
&= \bigvee_{\substack{h,k:\\h\circ k\le a}}\Big((\varphi\times\varphi)^{\leftarrow}(\mathcal{M})(h) * (\varphi\times\varphi)^{\leftarrow}(\mathcal{N})(k)\Big)\\
&= (\varphi\times\varphi)^{\leftarrow}(\mathcal{M})\circ(\varphi\times\varphi)^{\leftarrow}(\mathcal{N})(a).\end{aligned}$$

■

4 Lattice-Valued Uniform Convergence Spaces

Here we propose a new definition of a lattice-valued uniform convergence structure on a set X, generalising the work of Jäger and Burton [16]. We show that our category is topological over SET and we present the induced stratified L-limit structure. It is then shown that we can define a forgetful functor to the subcategory $SL\text{-}LIM$ [14] that will preserve the initial structures. Unless otherwise stated, our lattice L will be a pseudo-bisymmetric enriched *cl*-premonoid.

Definition 4.1 Let X be a non-empty set, and $(L, \le, \otimes, *)$ a pseudo-bisymmetric enriched *cl*-premonoid. A mapping $\Lambda : \mathcal{F}_L^S(X\times X) \longrightarrow L$ is called a *stratified L-uniform convergence structure* if Λ satisfies the following:

(LUC1) for all $x \in X$, $\Lambda([(x,x)]) = \top$,

(LUC2) $\mathcal{F} \leq \mathcal{G} \Longrightarrow \Lambda(\mathcal{F}) \leq \Lambda(\mathcal{G})$,

(LUC3) $\Lambda(\mathcal{F}) \leq \Lambda(\mathcal{F}^{-1})$,

(LUC4) $\Lambda(\mathcal{F}) \wedge \Lambda(\mathcal{G}) \leq \Lambda(\mathcal{F} \wedge \mathcal{G})$,

(LUC5) $\Lambda(\mathcal{F}) * \Lambda(\mathcal{G}) \leq \Lambda(\mathcal{F} \circ \mathcal{G})$ whenever $\mathcal{F} \circ \mathcal{G}$ exists.

The pair (X, Λ) is called a *stratified L-uniform convergence space.*

The original definition proposed by Jäger and Burton [16] was for the case where L is a complete Heyting algebra. Their **(LUC1)** stated that for all $x \in X$, $\Lambda([x] \times [x]) = \top$. For the case of L a complete Heyting algebra, it can be seen from Lemmas 3.11 and 3.12(i) that $[x] \times [x] = [x]_x = [(x,x)]$, for all $x \in X$, and so we see how the new definition is a generalisation of the previous one. In addition, the **(LUC5)** given in [16] stated $\Lambda(\mathcal{F}) \wedge \Lambda(\mathcal{G}) \leq \Lambda(\mathcal{F} \circ \mathcal{G})$. For the Heyting algebra case, $* = \wedge$, and so the above definition is thus a useful generalisation as it includes the specific case that was investigated in that work.

Definition 4.2 [16] Let (X, Λ) and (Y, Σ) be stratified L-uniform convergence spaces. A mapping $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ is *uniformly continuous* if for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, we have:

$$\Lambda(\mathcal{F}) \leq \Sigma\big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F})\big).$$

Proposition 4.3 [16] *Let $(X, \Lambda), (Y, \Sigma)$ and (Z, Γ) be stratified L-uniform convergence spaces. Then:*

(i) *The mapping $id_X : (X, \Lambda) \longrightarrow (X, \Lambda)$ is uniformly continuous.*

(ii) *If $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ and $\psi : (Y, \Sigma) \longrightarrow (Z, \Gamma)$ are uniformly continuous, then $\psi \circ \varphi : (X, \Lambda) \longrightarrow (Z, \Gamma)$ is uniformly continuous.*

Result: We have the concrete category *SL-UCS*, where the objects are stratified L-uniform convergence spaces, and the morphisms are the uniformly continuous mappings.

If we have two different stratified L-uniform convergence structures, Λ and Λ', on a set X, we can order them in the following manner:
$(X, \Lambda) \leq (X, \Lambda')$ if and only if, for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, $\Lambda'(\mathcal{F}) \leq \Lambda(\mathcal{F})$. In this case we say $\Lambda \leq \Lambda'$.

Example 4.4 ([16] for L a complete Heyting algebra) The *indiscrete stratified L-uniform convergence structure* Λ_i is defined:

$$\Lambda_i(\mathcal{F}) = \top \quad \text{for all } \mathcal{F} \in \mathcal{F}_L^S(X \times X).$$

Example 4.5 ([16] for L a complete Heyting algebra) For $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, we define the *discrete stratified L-uniform convergence structure* Λ_d by:

$$\Lambda_d(\mathcal{F}) = \begin{cases} \top & \text{if } \mathcal{F} \geq \bigwedge\limits_{x \in E} [(x,x)] \text{ for some finite } E \subset X \\ \bot & \text{else.} \end{cases}$$

Cook and Fischer [2] showed that the classical uniform convergence spaces form a topological category, and Jäger and Burton [16] showed the analogous result for Heyting algebra-valued uniform convergence spaces. Now we show this same result for the case where L is a pseudo-bisymmetric enriched *cl*-premonoid.

Proposition 4.6 [16] *The category SL-UCS is a topological category* (in the sense of Preuss [22]).

PROOF: First we will show the existence of initial structures. Consider a family $\{\varphi_i \,:\, i \in I\}$ such that $X \xrightarrow{\varphi_i} (X_i, \Lambda_i)$ for all $i \in I$.

For $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, define:

$$\Lambda(\mathcal{F}) = \bigwedge_{i \in I} \Lambda_i\Big((\varphi_i \times \varphi_i)^{\rightarrow}(\mathcal{F})\Big).$$

We show that $(X, \Lambda) \in |SL\text{-}UCS|$, but only present here **(LUC1)** and **(LUC5)** as the proofs of the other axioms are similar to those in [16].

LUC1: From the definition of the image of a stratified L-filter, we can see that $(\varphi \times \varphi)^{\rightarrow}\big([(x,x)]\big) = \big[\big(\varphi(x), \varphi(x)\big)\big]$. Now,

$$\begin{aligned} \Lambda\big([(x,x)]\big) &= \bigwedge_{i \in I} \Lambda_i\Big(\big(\varphi_i \times \varphi_i\big)^{\rightarrow}\big([(x,x)]\big)\Big) \\ &= \bigwedge_{i \in I} \Lambda_i\big[\big(\varphi_i(x), \varphi_i(x)\big)\big] \\ &= \top. \end{aligned}$$

LUC5: From Proposition 3.16 we see that

$$(\varphi_i \times \varphi_i)^{\rightarrow}(\mathcal{F} \circ \mathcal{G}) \geq (\varphi_i \times \varphi_i)^{\rightarrow}(\mathcal{F}) \circ (\varphi_i \times \varphi_i)^{\rightarrow}(\mathcal{G}).$$

Now we use that result to show

$$\begin{aligned}\Lambda(\mathcal{F}\circ\mathcal{G}) &= \bigwedge_{i\in I}\Lambda_i\big((\varphi_i\times\varphi_i)^{\rightarrow}(\mathcal{F}\circ\mathcal{G})\big)\\ &\geq \bigwedge_{i\in I}\Lambda_i\Big((\varphi_i\times\varphi_i)^{\rightarrow}(\mathcal{F})\circ(\varphi_i\times\varphi_i)^{\rightarrow}(\mathcal{G})\Big)\\ &\geq \bigwedge_{i\in I}\Big(\Lambda_i\big((\varphi_i\times\varphi_i)^{\rightarrow}(\mathcal{F})\big)*\Lambda_i\big((\varphi_i\times\varphi_i)^{\rightarrow}(\mathcal{G})\big)\Big)\\ &\geq \bigwedge_{i\in I}\Lambda_i\big((\varphi_i\times\varphi_i)^{\rightarrow}(\mathcal{F})\big)*\bigwedge_{i\in I}\Lambda_i\big((\varphi_i\times\varphi_i)^{\rightarrow}(\mathcal{G})\big)\\ &= \Lambda(\mathcal{F})*\Lambda(\mathcal{G}).\end{aligned}$$

Therefore we have that $(X,\Lambda)\in|SL\text{-}UCS|$.

Let $(Y,\Sigma)\in|SL\text{-}UCS|$ and let $\psi:Y\longrightarrow X$ such that for all $i\in I$, $\varphi_i\circ\psi:(Y,\Sigma)\longrightarrow(X_i,\Lambda_i)$ is uniformly continuous. Thus for all $i\in I$ we have

$$\Sigma(\mathcal{F})\leq\Lambda_i\Big(\big((\varphi_i\circ\psi)\times(\varphi_i\circ\psi)\big)^{\rightarrow}(\mathcal{F})\Big)=\Lambda_i\big(((\varphi_i\times\varphi_i)\circ(\psi\times\psi))^{\rightarrow}(\mathcal{F})\big),$$

and consequently

$$\Sigma(\mathcal{F})\leq\bigwedge_{i\in I}\Lambda_i\big((\varphi_i\times\varphi_i)^{\rightarrow}((\psi\times\psi)^{\rightarrow}(\mathcal{F}))\big)=\Lambda\big((\psi\times\psi)^{\rightarrow}(\mathcal{F})\big).$$

Therefore $\psi:Y\longrightarrow X$ is uniformly continuous. Now we have that $\psi:(Y,\Sigma)\longrightarrow(X,\Lambda)$ is uniformly continuous if and only if $\varphi_i\circ\psi:(Y,\Sigma)\longrightarrow(X,\Lambda_i)$ is uniformly continuous. Further, we show the *fibre-smallness* of $SL\text{-}UCS$. Since each stratified L-uniform convergence structure is a mapping $\Lambda:\mathcal{F}_L^S(X\times X)\longrightarrow L$ we have that the class of all possible stratified L-uniform convergence structures on a set X, is a subset of $\{0,1\}^{L^{\mathcal{F}_L^S(X\times X)}}$ and so it is a set.

Lastly, in order to show the *terminal separator* property, consider $X=\{x\}$ and hence $X\times X=\{(x,x)\}$. Since there is only one element in $X\times X$, the only L-sets that exist (i.e. elements of $L^{X\times X}$) are the constant maps $\alpha_{X\times X}(x,x)=\alpha$ for each $\alpha\in L$. By the stratification property **(LFS)** we have $\mathcal{F}\geq[(x,x)]$ for all $\mathcal{F}\in\mathcal{F}_L^S(X\times X)$. Now $\Lambda\big([(x,x)]\big)=\top$, but $\Lambda(\mathcal{F})\geq\Lambda\big([(x,x)]\big)$ and so $\Lambda(\mathcal{F})=\top$ for all $\mathcal{F}\in\mathcal{F}_L^S(X\times X)$.

■

Example 4.7 Let $(X,\Lambda),(Y,\Sigma)\in|SL\text{-}UCS|$ and consider the projections mappings P_1 and P_2. Then the *product L-uniform convergence structure*, $\Lambda\times\Sigma$ on $X\times Y$ is defined by using the initial uniform convergence structure for the projection mappings. That is, for a stratified L-filter $\mathcal{F}\in\mathcal{F}_L^S\big((X\times Y)\times(X\times Y)\big)$:

$$(\Lambda\times\Sigma)(\mathcal{F})=\Lambda\big((P_1\times P_1)^{\rightarrow}(\mathcal{F})\big)\wedge\Sigma\big((P_2\times P_2)^{\rightarrow}(\mathcal{F})\big).$$

In the case of L a complete Heyting algebra, the category SL-UCS has canonical function spaces.

Proposition 4.8 [16] *Let L be a complete Heyting algebra. Then SL-UCS is cartesian closed.*

Remark 4.9 From the stratified L-uniform convergence structures described earlier, it is possible to generate stratified L-limit structures. The stratified L-limit spaces are then sets equipped with a map $\lim(\Lambda) : \mathcal{F}_L^S(X) \longrightarrow L^X$. That is, for each $x \in X$, $\lim(\Lambda)\mathcal{F}(x)$ is the degree to which $\mathcal{F}$ converges to x.

The pair $(X, \lim)$ is a *stratified L-limit space* [14] if $\lim : \mathcal{F}_L^S(X) \longrightarrow L^X$ satisfies the following axioms:

(L1) for all $x \in X$, $\lim[x](x) = \top$,

(L2) $\mathcal{F} \leq \mathcal{G} \Longrightarrow \lim \mathcal{F} \leq \lim \mathcal{G}$,

(L3) $\lim \mathcal{F} \wedge \lim \mathcal{G} \leq \lim(\mathcal{F} \wedge \mathcal{G})$.

If $(X, \lim_X)$ and $(Y, \lim_Y)$ are stratified L-limit spaces, then $\varphi : X \longrightarrow Y$ is *continuous* [14] if for all $x \in X$ and for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$, $\lim \mathcal{F}(x) \leq \lim_Y \varphi^{\rightarrow}(\mathcal{F})\big(\varphi(x)\big)$.

Result: SL-LIM is a concrete category, where the objects are stratified L-limit spaces and the morphisms are the continuous mappings defined above. For $(X, \Lambda) \in |SL\text{-}UCS|$ we can define a stratified L-limit structure $\lim(\Lambda) : \mathcal{F}_L^S(X) \longrightarrow L^X$ by:

$$\lim(\Lambda)\mathcal{F}(x) = \Lambda(\mathcal{F}_x).$$

Using Lemma 3.12 (i) and (ii), the proof of the axioms is straightforward. Further, if we have $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ uniformly continuous, then $\varphi : \big(X, \lim(\Lambda)\big) \longrightarrow \big(Y, \lim(\Sigma)\big)$ is continuous. This proof will follow from Lemma 3.12(iii).

Result: We can define a forgetful functor

$$F : \begin{cases} SL\text{-}UCS \longrightarrow SL\text{-}LIM \\ (X, \Lambda) \longmapsto \big(X, \lim(\Lambda)\big) \\ \varphi \longmapsto \varphi \end{cases}$$

Let X be a set and for all $i \in I$, let $(X_i, \lim_i) \in |SL\text{-}LIM|$. For an SL-LIM source $\varphi_i : X \longrightarrow (X_i, \lim_i), i \in I$, the *initial stratified L-limit structure* on X for $\mathcal{F} \in \mathcal{F}_L^S(X)$ is defined [14] by:

$$\lim \mathcal{F}(x) = \bigwedge_{i \in I} \lim_i \varphi_i^{\rightarrow}(\mathcal{F})\big(\varphi_i(x)\big).$$

With this definition, it has been shown [16] that if $\varphi_i : X \longrightarrow (X_i, \Lambda_i)$ is a source in $SL\text{-}UCS$ and Λ is the initial $SL\text{-}UCS$ structure on X, then $\lim(\Lambda)$ is the initial $SL\text{-}LIM$ structure with respect to the source $\varphi_i : X \longrightarrow \big(X_i, \lim(\Lambda_i)\big), i \in I$.

Result: The forgetful functor F preserves initial structures.

5 Example: Lattice-Valued Uniform Spaces

The diagonal of a product $X \times X$ is defined as $\Delta = \{(x,x) \mid x \in X\}$. From this we can define the diagonal L-filter on $X \times X$. Let $a \in L^{X\times X}$:

$$[\Delta](a) = \bigwedge_{x\in X} a(x,x) = \bigwedge_{x\in X} [(x,x)](a), \text{ so } \big[\Delta\big] = \bigwedge_{x\in X} [(x,x)]$$

Definition 5.1 [6], [7] Let X be a non-empty set and $\mathcal{U}$ a stratified L-filter on $X \times X$. If $\mathcal{U}$ satisfies the properties below it is called a *stratified L-uniformity* on X.

(LU1) $\mathcal{U} \leq [\Delta]$,

(LU2) $\mathcal{U} \leq \mathcal{U}^{-1}$,

(LU3) $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$.

The pair $(X,\mathcal{U})$ is called a *stratified L-uniform space*.

Definition 5.2 [6] Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be stratified L-uniform spaces and $\varphi : X \longrightarrow Y$. Then φ is *uniformly continuous* if $(\varphi \times \varphi)^{\rightarrow}(\mathcal{U}) \geq \mathcal{V}$.

Result: We have the category $SL\text{-}UNIF$ [6], where the objects are stratified L-uniform spaces, and the morphisms are the uniformly continuous maps.

Here we will show that $SL\text{-}UNIF$ is a reflective subcategory of $SL\text{-}UCS$. In order to do this we will first introduce the category of principal stratified L-uniform convergence spaces ($SL\text{-}PUCS$), a subcategory of $SL\text{-}UCS$. Then we will proceed by showing that $SL\text{-}UNIF$ is categorically isomorphic to $SL\text{-}PUCS$.

Definition 5.3 [16] The pair $(X,\Lambda) \in |SL\text{-}UCS|$ is a *principal stratified L-uniform convergence space* if there exists a stratified L-filter $\mathcal{U} \in \mathcal{F}_L^S(X \times X)$ such that:

(LUCP) $\Lambda(\mathcal{F}) = \bigwedge\limits_{a\in L^{X\times X}} \big(\mathcal{U}(a) \rightarrow \mathcal{F}(a)\big) \quad$ for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$.

The following lemma shows that from any principal stratified L-uniform convergence space, we can get a stratified L-uniform space.

Lemma 5.4 [16] *Let $(X, \Lambda) \in |SL\text{-}PUCS|$ where:*

$$\Lambda(\mathcal{F}) = \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big).$$

Then $(X, \mathcal{U}) \in |SL\text{-}UNIF|$.

Proof: This can easily be adapted from that given in [16].

Below we show that from any stratified L-uniform space we can generate a principal stratified L-uniform convergence space.

Lemma 5.5 [16] *Let $(X, \mathcal{U}) \in |SL\text{-}UNIF|$, and define:*

$$\Lambda^{\mathcal{U}}(\mathcal{F}) = \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big).$$

Then $(X, \Lambda^{\mathcal{U}}) \in |SL\text{-}PUCS|$.

Proof: Again we show only **(LUC1)** and **(LUC5)**.

LUC1: From **(LU1)** we know that for all $x \in X$ and for all $a \in L^{X \times X}$, $\mathcal{U}(a) \leq a(x, x)$. Therefore

$$\begin{aligned}
\Lambda^{\mathcal{U}}\big([(x, x)]\big) &= \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to [(x, x)](a)\big) \\
&= \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to a(x, x)\big) \\
&= \top.
\end{aligned}$$

LUC5: By definition we have that

$$\Lambda^{\mathcal{U}}(\mathcal{F} \circ \mathcal{G}) = \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F} \circ \mathcal{G}(a)\big).$$

Now since $(X, \mathcal{U}) \in |SL\text{-}UNIF|$ we use **(LU3)** to give

$$\begin{aligned}
\Lambda^{\mathcal{U}}(\mathcal{F} \circ \mathcal{G}) &\geq \bigwedge_{a \in L^{X \times X}} \bigg(\Big(\bigvee_{d_1 \circ d_2 \leq a} \big(\mathcal{U}(d_1) * \mathcal{U}(d_2)\big) \Big) \to \mathcal{F} \circ \mathcal{G}(a) \bigg) \\
&= \bigwedge_{a \in L^{X \times X}} \bigwedge_{d_1 \circ d_2 \leq a} \Big(\mathcal{U}(d_1) * \mathcal{U}(d_2) \to \mathcal{F} \circ \mathcal{G}(a) \Big),
\end{aligned}$$

using property **(iii)** of Lemma 2.1 in the equality. From the definition of the stratified L-filter $\mathcal{F} \circ \mathcal{G}$ we have

$$\begin{aligned}
&\bigwedge_{a \in L^{X \times X}} \bigwedge_{d_1 \circ d_2 \leq a} \Big(\mathcal{U}(d_1) * \mathcal{U}(d_2) \to \mathcal{F} \circ \mathcal{G}(a) \Big) \\
&\geq \bigwedge_{a \in L^{X \times X}} \bigwedge_{d_1 \circ d_2 \leq a} \Big(\mathcal{U}(d_1) * \mathcal{U}(d_2) \to \mathcal{F}(d_1) * \mathcal{G}(d_2) \Big) = Q.
\end{aligned}$$

Here we use Lemma 2.1 (x) in the first step, and properties of infima thereafter, to get

$$\begin{aligned} Q &\geq \bigwedge_{a\in L^{X\times X}} \bigwedge_{d_1\circ d_2\leq a} \Big(\big(\mathcal{U}(d_1)\to\mathcal{F}(d_1)\big) * \big(\mathcal{U}(d_2)\to\mathcal{G}(d_2)\big)\Big) \\ &\geq \bigwedge_{d_1\in L^{X\times X}} \Bigg(\bigwedge_{d_2\in L^{X\times X}} \Big(\big(\mathcal{U}(d_1)\to\mathcal{F}(d_1)\big) * \big(\mathcal{U}(d_2)\to\mathcal{G}(d_2)\big)\Big)\Bigg) \\ &\geq \bigwedge_{d_1\in L^{X\times X}} \big(\mathcal{U}(d_1)\to\mathcal{F}(d_1)\big) * \bigwedge_{d_2\in L^{X\times X}} \big(\mathcal{U}(d_2)\to\mathcal{G}(d_2)\big) \\ &= \Lambda^{\mathcal{U}}(\mathcal{F}) * \Lambda^{\mathcal{U}}(\mathcal{G}). \end{aligned}$$

That is, we have $\Lambda^{\mathcal{U}}(\mathcal{F}\circ\mathcal{G}) \geq \Lambda^{\mathcal{U}}(\mathcal{F}) * \Lambda^{\mathcal{U}}(\mathcal{G})$.

■

Lemma 5.6 [16] *Let $(X,\mathcal{U}) \in |SL\text{-}UNIF|$. Then for $a \in L^{X\times X}$,*

$$\mathcal{U}(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda^{\mathcal{U}}(\mathcal{F}) \to \mathcal{F}(a)\big).$$

We have shown that there is a one-to-one relationship between the class of objects of $SL\text{-}PUCS$ and $SL\text{-}UNIF$, and now do the same for the class of morphisms.

Lemma 5.7 [16] *Let $(X,\mathcal{U}),(Y,\mathcal{V}) \in |SL\text{-}UNIF|$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then the following are equivalent:*

(i) *$\varphi : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ is uniformly continuous,*

(ii) *$\varphi : (X,\Lambda^{\mathcal{U}}) \longrightarrow (Y,\Lambda^{\mathcal{V}})$ is uniformly continuous.*

Corollary 5.8 [16] *The categories $SL\text{-}PUCS$ and $SL\text{-}UNIF$ are categorically isomorphic.*

Having shown the above result, we now proceed to show that $SL\text{-}PUCS$ is a reflective subcategory of $SL\text{-}UCS$. This will then give us the corollary that $SL\text{-}UNIF$ is isomorphic to a reflective subcategory of $SL\text{-}UCS$.

In order to prove that $SL\text{-}PUCS$ is a reflective subcategory of $SL\text{-}UCS$ we follow a suggestion of J. Gutiérrez García and consider the subset of L:

$$L^{\otimes} = \big\{\alpha\in L \,:\, \alpha \leq \alpha\otimes\alpha\big\}.$$

Clearly, for $\otimes = \wedge$ and $\otimes = \circledast$ we have $L^{\otimes} = L$. The result where we must consider this subset is the one below, where we will show that from any stratified L-uniform convergence space, we can define a stratified L-filter. This will then in turn be used to generate a principal stratified L-uniform convergence space. This fact will then give us that the stratified L-filter defined below is in fact a stratified L-uniformity.

Lemma 5.9 *Let $(L, \leq, \otimes, *)$ be a pseudo-bisymmetric enriched cl-premonoid and let $(X, \Lambda) \in |SL\text{-}UCS|$ such that $\Lambda\big(\mathcal{F}_L^S(X \times X)\big) \subseteq L^{\otimes}$. We define the mapping $\mathcal{U}_\Lambda : L^{X\times X} \longrightarrow L$ for $a \in L^{X\times X}$ by:*

$$\mathcal{U}_\Lambda(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big).$$

Then $\mathcal{U}_\Lambda$ is a stratified L-filter on $X \times X$.

Proof:

LF0: $\mathcal{U}_\Lambda(\perp_{X\times X}) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \mathcal{F}(\perp_{X\times X})$. Now we can take $x \in X$ and consider the stratified L-filter $[(x,x)]$, where $\Lambda\big([(x,x)]\big) = \top$. In this case, $\Lambda\big([(x,x)]\big) \to [(x,x)](\perp_{X\times X}) = \top \to \perp = \perp$, and so we have $\mathcal{U}_\Lambda(\perp_{X\times X}) = \perp$.

Now consider $\mathcal{U}_\Lambda(\top_{X\times X}) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Lambda(\mathcal{F}) \to \mathcal{F}(\top_{X\times X})$. Clearly, for all $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ we have $\mathcal{F}(\top_{X\times X}) = \top$. Now, for any $\mathcal{F} \in \mathcal{F}_L^S(X \times X)$ we have that $\Lambda(\mathcal{F}) \to \top = \top$ and therefore $\mathcal{U}_\Lambda(\top_{X\times X}) = \top$.

LF1: straightforward.

LF2: Let $a, b \in L^{X\times X}$.

$$\begin{aligned}\mathcal{U}_\Lambda(a) \otimes \mathcal{U}_\Lambda(b) &= \Big(\bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big)\Big) \otimes \Big(\bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(b)\big)\Big) \\ &\leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X\times X)} \Big(\big(\Lambda(\mathcal{G}) \to \mathcal{G}(a)\big) \otimes \big(\Lambda(\mathcal{G}) \to \mathcal{G}(b)\big)\Big).\end{aligned}$$

Now with Lemma 2.3 we have that

$$\mathcal{U}_\Lambda(a) \otimes \mathcal{U}_\Lambda(b) \leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X\times X)} \Big(\Lambda(\mathcal{G}) \to \big(\mathcal{G}(a) \otimes \mathcal{G}(b)\big)\Big).$$

We can then use **(LF2)** of the stratified L-filters to get

$$\begin{aligned}\mathcal{U}_\Lambda(a) \otimes \mathcal{U}_\Lambda(b) &\leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{G}) \to \mathcal{G}(a \otimes b)\big) \\ &= \mathcal{U}_\Lambda(a \otimes b).\end{aligned}$$

LFS: Let $\alpha \in L$ and $a \in L^{X\times X}$. Then using Lemma 2.1(xi) and **(LFS)** of each $\mathcal{F} \in \mathcal{F}_L^S(X\times X)$, we get

$$\begin{aligned}
\alpha * \mathcal{U}_\Lambda(a) &= \alpha * \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big) \\
&= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\alpha * \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big)\Big) \\
&\le \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\Lambda(\mathcal{F}) \to \big(\alpha * \mathcal{F}(a)\big)\Big) \\
&\le \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \Big(\Lambda(\mathcal{F}) \to \mathcal{F}(\alpha_{X\times X} * a)\Big) \\
&= \mathcal{U}_\Lambda(\alpha_{X\times X} * a).
\end{aligned}$$

■

Now that we have shown that $\mathcal{U}_\Lambda$ is a stratified L-filter on $X \times X$, we can proceed with the main result.

Lemma 5.10 *Let $(L, \le, \otimes, *)$ be an enriched cl-premonoid and let $(X, \Lambda) \in |SL\text{-}UCS|$ such that $\Lambda\big(\mathcal{F}_L^S(X \times X)\big) \subseteq L^{\otimes}$. Then SL-PUCS is a reflective subcategory of SL-UCS.*

Proof: For any $(X, \Lambda) \in |SL\text{-}PUCS|$ it is clear that $(X, \Lambda) \in |SL\text{-}UCS|$. Therefore we can consider the embedding functor

$$E : \begin{cases} SL\text{-}PUCS \longrightarrow SL\text{-}UCS \\ (X, \Lambda) \longmapsto (X, \Lambda) \\ \varphi \longmapsto \varphi \end{cases}.$$

Now we let $(X, \Lambda) \in |SL\text{-}UCS|$ and define:

$$\mathcal{U}_\Lambda(a) = \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big).$$

From the previous lemma we have that $\mathcal{U}_\Lambda$ is a stratified L-filter on $X \times X$. Moreover, for $x \in X$ we have $\mathcal{U}_\Lambda(a) \le \Lambda([(x,x)]) \to [(x,x)](a) = [(x,x)](a)$ and so we find $\mathcal{U}_\Lambda \le [\Delta]$. Also

$$\begin{aligned}
\mathcal{U}_\Lambda(a) \quad &\le \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} (\Lambda(\mathcal{F}^{-1}) \to \mathcal{F}^{-1}(a)) \\
&= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^S(X\times X)} (\Lambda(\mathcal{F}) \to \mathcal{F}(a^{-1})) \\
&= \mathcal{U}_\Lambda(a^{-1}) \ = \ \mathcal{U}_\Lambda^{-1}(a),
\end{aligned}$$

showing that $\mathcal{U}_\Lambda \le \mathcal{U}_\Lambda^{-1}$. Lastly, using Lemma 3.16, we get $\mathcal{U}_\Lambda \circ \mathcal{U}_\Lambda \in \mathcal{F}_L^S(X \times X)$, since for any $x \in X$ we have $\mathcal{U}_\Lambda \circ \mathcal{U}_\Lambda \le [(x,x)] \circ [(x,x)] \in \mathcal{F}_L^S(X \times X)$.

We define now

$$I_\Lambda = \{\mathcal{G} \in \mathcal{F}_L^S(X \times X) \,:\, \mathcal{G} \leq \mathcal{U}_\Lambda,\ \mathcal{G} \leq \mathcal{G} \circ \mathcal{G}\}.$$

Then $I_\Lambda \neq \emptyset$, as the coarsest stratified L-filter on $X \times X$, $[X \times X] = \bigwedge_{(x,y) \in X \times X} [(x,y)]$ is in I_Λ. We further define

$$\mathcal{U}_\Lambda^* = \bigvee_{\mathcal{G} \in I_\Lambda} \mathcal{G}.$$

Clearly, $\mathcal{U}_\Lambda^* \in \mathcal{F}_L^S(X \times X)$ and $\mathcal{U}_\Lambda^* \leq \mathcal{U}_\Lambda$. Moreover

$$\mathcal{G} \leq \mathcal{G} \circ \mathcal{G} \leq \mathcal{U}_\Lambda^* \circ \mathcal{U}_\Lambda^*$$

for any $\mathcal{G} \in I_\Lambda$ and therefore also $\mathcal{U}_\Lambda^* \leq \mathcal{U}_\Lambda^* \circ \mathcal{U}_\Lambda^*$. Hence $\mathcal{U}_\Lambda^*$ satisfies **(LU3)**. Also, from $\mathcal{U}_\Lambda^* \leq \mathcal{U}_\Lambda \leq [\Delta]$ we see that **(LU1)** is satisfied. For $\mathcal{G} \in I_\Lambda$ we have

$$\mathcal{G}^{-1} \leq \mathcal{U}_\Lambda^{-1} = \mathcal{U}_\Lambda$$

and

$$\mathcal{G}^{-1} \leq (\mathcal{G} \circ \mathcal{G})^{-1} = \mathcal{G}^{-1} \circ \mathcal{G}^{-1}.$$

Therefore $\mathcal{G}^{-1} \in I_\Lambda$ and hence

$$\mathcal{U}_\Lambda^* \geq \bigvee_{\mathcal{G} \in I_\Lambda} \mathcal{G}^{-1} = (\mathcal{U}_\Lambda^*)^{-1}.$$

From this it follows $(\mathcal{U}_\Lambda^*)^{-1} \geq ((\mathcal{U}_\Lambda^*)^{-1})^{-1} = \mathcal{U}_\Lambda^*$ and also **(LU2)** holds for $\mathcal{U}_\Lambda^*$. We define now

$$\Lambda^*(\mathcal{F}) = \bigwedge_{a \in L^{X \times X}} (\mathcal{U}_\Lambda^*(a) \to \mathcal{F}(a)).$$

Then, by Lemma 5.5, $(X, \Lambda^*) \in |SL\text{-}PUCS|$. Moreover

$$\begin{aligned}
\Lambda^*(\mathcal{F}) &= \bigwedge_{a \in L^{X \times X}} (\mathcal{U}_\Lambda^*(a) \to \mathcal{F}(a)) \\
&\geq \bigwedge_{a \in L^{X \times X}} (\mathcal{U}_\Lambda(a) \to \mathcal{F}(a)) \\
&\geq \bigwedge_{a \in L^{X \times X}} ((\Lambda(\mathcal{F}) \to \mathcal{F}(a)) \to \mathcal{F}(a)) \\
&\geq \Lambda(\mathcal{F}).
\end{aligned}$$

Therefore $\Lambda^* \leq \Lambda$. Let now $\widetilde{\Lambda} \leq \Lambda$ satisfy

$$\widetilde{\Lambda}(\mathcal{F}) = \bigwedge_{a \in L^{X \times X}} \big(\mathcal{U}(a) \to \mathcal{F}(a)\big)$$

with some stratified L-filter $\mathcal{U}$ such that $(X, \widetilde{\Lambda}) \in |SL\text{-}PUCS|$. Then

$$\mathcal{U}(a) = \bigwedge_{a \in L^{X \times X}} \big(\widetilde{\Lambda}(\mathcal{F}) \to \mathcal{F}(a)\big) \leq \bigwedge_{a \in L^{X \times X}} \big(\Lambda(\mathcal{F}) \to \mathcal{F}(a)\big) = \mathcal{U}_\Lambda(a).$$

By the maximality of $\mathcal{U}^*_\Lambda$ we then conclude that $\mathcal{U} \le \mathcal{U}^*_\Lambda$ and hence

$$\widetilde{\Lambda}(\mathcal{F}) \ge \bigwedge_{a \in L^{X \times X}} (\mathcal{U}^*_\Lambda(a) \to \mathcal{F}(a)) = \Lambda^*(\mathcal{F}).$$

In other words, $\widetilde{\Lambda} \le \Lambda^*$ and therefore Λ^* is the finest SL-$PUCS$-structure on X which is coarser than Λ. Now, let $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ be a morphism in SL-UCS. We want to show that $\varphi : (X, \Lambda^*) \longrightarrow (Y, \Sigma^*)$ is a morphism in SL-$PUCS$. To this end it is sufficient to show $\mathcal{U}^*_\Sigma \le (\varphi \times \varphi)^\to(\mathcal{U}^*_\Lambda)$.

Let $\mathcal{G} \le \mathcal{U}_\Sigma$ such that $\mathcal{G} \le \mathcal{G} \circ \mathcal{G}$. We know from Lemma 5.7 and the continuity of $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ that $(\varphi \times \varphi)^\to(\mathcal{U}_\Lambda) \ge \mathcal{U}_\Sigma$. Hence also $\mathcal{G} \le (\varphi \times \varphi)^\to(\mathcal{U}_\Lambda)$. By Lemma 3.4(iii) then $(\varphi \times \varphi)^\leftarrow(\mathcal{G})$ exists and, by Lemma 3.4(i), we get

$$(\varphi \times \varphi)^\leftarrow(\mathcal{G}) \le (\varphi \times \varphi)^\leftarrow((\varphi \times \varphi)(\mathcal{U}_\Lambda)) \le \mathcal{U}_\Lambda.$$

Further, by Lemma 3.16(iii),

$$(\varphi \times \varphi)^\leftarrow(\mathcal{G}) \le (\varphi \times \varphi)^\leftarrow(\mathcal{G} \circ \mathcal{G}) \le (\varphi \times \varphi)^\leftarrow(\mathcal{G}) \circ (\varphi \times \varphi)^\leftarrow(\mathcal{G})$$

and hence $(\varphi \times \varphi)^\leftarrow(\mathcal{G}) \in I_\Lambda$. Thus $(\varphi \times \varphi)^\leftarrow(\mathcal{G}) \le \mathcal{U}^*_\Lambda$ and by Lemma 3.4 (ii) we finally get

$$\mathcal{G} \le (\varphi \times \varphi)^\to((\varphi \times \varphi)^\leftarrow(\mathcal{G})) \le (\varphi \times \varphi)^\to(\mathcal{U}^*_\Lambda).$$

Hence, by arbitrariness of $\mathcal{G} \in I_\Sigma$ we conclude $\mathcal{U}^*_\Sigma \le (\varphi \times \varphi)^\to(\mathcal{U}^*_\Lambda)$, and $\varphi : (X, \Lambda^*) \longrightarrow (Y, \Sigma^*)$ is continuous. Now that we know that the uniformly continuous mappings will remain morphisms, we can define a functor:

$$K : \begin{cases} SL\text{-}UCS \longrightarrow SL\text{-}PUCS \\ (X, \Lambda) \longmapsto (X, \Lambda^*) \\ \varphi \longmapsto \varphi \end{cases}$$

For $(X, \Lambda) \in |SL\text{-}PUCS|$ we have $K\big(E(X, \Lambda)\big) = (X, \Lambda)$ and since $\Lambda^* \le \Lambda$ we know that $E\big(K(X, \Lambda)\big) \le (X, \Lambda)$ for $(X, \Lambda) \in |SL\text{-}UCS|$.

This in turn means that $id_X : (X, \Lambda) \longrightarrow E(X, \Lambda^*)$ is continuous. Therefore, for $(X, \Lambda) \in |SL\text{-}UCS|$ we propose our E-universal map to be $\big(id_X, (X, \Lambda^*)\big)$.

We show now that this is an E-universal map for (X, Λ).

Let $(Z, \Gamma) \in |SL\text{-}PUCS|$ and $\psi : (X, \Lambda) \longrightarrow E\big((Z, \Gamma)\big)$. We require a unique SL-$PUCS$ morphism $\phi : (X, \Lambda^*) \longrightarrow (Z, \Gamma)$ such that the following diagram commutes:

$$
\begin{tikzcd}
(X,\Lambda) \arrow[r, "\psi"] \arrow[d, "id_X"'] & E\big((Z,\Gamma)\big) & (Z,\Gamma) \\
E\big((X,\Lambda^*)\big) \arrow[ur, "E(\phi)"'] & & (X,\Lambda^*) \arrow[u, "\phi"']
\end{tikzcd}
$$

It is clear that the mapping ϕ will be none other than $K(\psi) = \psi$.

■

In [16], a similar result to the one above was stated for the case where L is a complete Heyting algebra. The proof presented in [16] is unfortunately not correct. However, the result of Lemma 5.10 does then show that the result in [16] is in fact still valid. We are grateful to one of the reviewers for pointing out this inaccuracy.

Corollary 5.11 *Let $(L, \leq, \otimes, *)$ be a pseudo-bisymmetric enriched cl-premonoid and let $(X, \Lambda) \in |SL\text{-}UCS|$ such that $\Lambda\big(\mathcal{F}_L^S(X \times X)\big) \subseteq L^{\otimes}$. Then $SL\text{-}UNIF$ is isomorphic to a reflective subcategory of $SL\text{-}UCS$.*

Remark 5.12 Now we show that for $(X,\mathcal{U}) \in |SL\text{-}UNIF|$ there are two ways of inducing a convergence function. What is remarkable is that these two pathways produce identical convergence structures.

It is shown in [6] that from a stratified L-uniformity we can define a stratified L-neighbourhood system for each $x \in X$:

$$\mathcal{N}_{\mathcal{U}}^x(a) = \bigvee \{\mathcal{U}(d) \mid d(\cdot, x) \leq a\}.$$

The stratified L-neighbourhood space $\big(X, (\mathcal{N}_{\mathcal{U}}^x)_{x\in X}\big)$ is equivalent to a stratified L-topological space. From this stratified L-topological space it is shown in [11] that we can induce a stratified L-limit space by:

$$\lim(\mathcal{U})\mathcal{F}(x) = \bigwedge_{a\in L^X} \big(\mathcal{N}_{\mathcal{U}}^x(a) \to \mathcal{F}(a)\big).$$

We can also consider the stratified L-uniform space $(X,\mathcal{U})$ as a principal stratified L-uniform convergence space (X, Λ) where

$$\Lambda^{\mathcal{U}}(\mathcal{F}) = \bigwedge_{d\in L^{X\times X}} \big(\mathcal{U}(d) \to \mathcal{F}(d)\big).$$

From here we can consider the induced stratified L-limit structure:

$$\lim(\Lambda)\mathcal{F}(x) = \Lambda(\mathcal{F}_x) = \bigwedge_{d\in L^{X\times X}} \big(\mathcal{U}(d) \to (\mathcal{F}_x)(d)\big).$$

The result below was proved in [16] for the case of L a complete Heyting algebra. Our proof for the case of L a pseudo-bisymmetric enriched *cl*-premonoid uses the same procedure, except that when considering the induced L-limit space, we make use of the filter $\mathcal{F}_x$ on the product space.

Proposition 5.13 *Let* $(X,\mathcal{U}) \in |SL\text{-}UNIF|$. *Then* $\big(X, \lim(\mathcal{U})\big) = \big(X, \lim(\Lambda^{\mathcal{U}})\big)$.

Using the two routes of obtaining a stratified L-limit space that are described above, we can show the following diagram commutes when L is a pseudo-bisymmetric enriched cl-premonoid:

$$\begin{tikzcd}
(X,\mathcal{U}) \arrow[r, "\text{(forget)}"] \arrow[d, "\text{(embed)}"] & \big(X, \tau(\mathcal{U})\big) \arrow[d, "\text{(embed)}"] \\
(X, \Lambda^{\mathcal{U}}) \arrow[r, "\text{(forget)}"] & \big(X, \lim(\Lambda^{\mathcal{U}})\big) = \big(X, \lim(\mathcal{U})\big)
\end{tikzcd}$$

6 Example: Probabilistic Uniform Limit Spaces

In this section we shall consider the enriched cl-premonoid $\big([0,1], \wedge, *\big)$ where $*$ is a left-continuous t-norm. We denote by $\mathbb{F}(X)$ the set of all classical filters on a set X. For $\Psi, \eta \in \mathbb{F}(X \times X)$ consider the filter $\Psi^{-1} = \{F^{-1} : F \in \Psi\}$ where $F^{-1} = \{(y,x) : (x,y) \in F\}$. Now for $F, G \subset X \times X$, we have $F \circ G = \{(x,y) | \exists z \in X$ such that $(x,z) \in F, (z,y) \in G\}$. The collection $\Psi \circ \eta = \big[\{F \circ G : F \in \Psi, G \in \eta\}\big]$ is a filter if and only if $F \circ G \neq \emptyset$, for all $F \in \Psi, G \in \eta$. We also consider the filter $\dot{x}$, generated by an element of X, where $\dot{x} = \{F \subset X : x \in F\}$.

Definition 6.1 [20], [21] For any $\alpha \in [0,1]$, consider $\mathcal{L}^\alpha$, a non-empty collection of filters on $X \times X$. The collection $(\mathcal{L}^\alpha)_{\alpha \in [0,1]}$ is a *probabilistic uniform limit structure* if it satisfies the following axioms

(UC1) for all $x \in X$, $\alpha \in [0,1]$, $\dot{x} \times \dot{x} \in \mathcal{L}^\alpha$,

(UC2) $\Psi \in \mathcal{L}^\alpha$, $\Psi \leq \eta \Rightarrow \eta \in \mathcal{L}^\alpha$,

(UC3) $\Psi \in \mathcal{L}^\alpha \Rightarrow \Psi^{-1} \in \mathcal{L}^\alpha$,

(UC4) $\Psi, \eta \in \mathcal{L}^\alpha \Rightarrow \Psi \wedge \eta \in \mathcal{L}^\alpha$,

(P1) $\alpha \leq \beta \Rightarrow \mathcal{L}^\beta \subset \mathcal{L}^\alpha$,

(P2) $\mathcal{L}^0 = \mathbb{F}(X \times X)$,

(PULIM) $\Psi \in \mathcal{L}^\alpha, \eta \in \mathcal{L}^\beta \Rightarrow \Psi \circ \eta \in \mathcal{L}^{\alpha * \beta}$.

The pair $\big(X, (\mathcal{L}^\alpha)\big)$ is called a *probabilistic uniform limit space*

Definition 6.2 [20], [21] A mapping φ, from one probabilistic uniform limit space to another, $\varphi : \big(X, (\mathcal{L}^\alpha)\big) \longrightarrow \big(Y, (\mathcal{K}^\alpha)\big)$ is said to be *uniformly continuous* if for all $\alpha \in [0,1]$, $(\varphi \times \varphi)^{\rightarrow}(\mathcal{L}^\alpha) \subset \mathcal{K}^\alpha$.

Definition 6.3 A probabilistic uniform limit structure, $(\mathcal{L}^\alpha)_{\alpha \in [0,1]}$, is *left-continuous* if whenever we have $\Psi \in \mathcal{L}^\alpha$ for all $\alpha \in A$, then $\Psi \in \mathcal{L}^{\bigvee A}$.

Note that if $\Psi \notin \mathcal{L}^\alpha$ when $(\mathcal{L}^\alpha)_{\alpha\in[0,1]}$ is left-continuous, this implies that there exists $\epsilon > 0$ such that $\Psi \notin \mathcal{L}^{\alpha-\epsilon}$. The left-continuous probabilistic uniform limit spaces with the uniformly continuous mappings form a category $PULIM^*$.

In order to show that probabilistic uniform limit spaces are an example of our stratified L-uniform convergence spaces, we must define a method for moving between L-filters and classical filters. The following two constructions are special cases of definitions given in [5]. For $\mathcal{F} \in \mathcal{F}^S_{[0,1]}(X\times X)$ define

$$\Phi_\mathcal{F} = \big\{A \subset X\times X \;:\; \mathcal{F}(\top_A) = \top\big\}.$$

Then $\Phi_\mathcal{F} \in \mathbb{F}(X\times X)$ [3].

Lemma 6.4 *Let $\mathcal{F},\mathcal{G} \in \mathcal{F}^S_{[0,1]}(X\times X)$ and consider the associated filters $\Phi_\mathcal{F}$ and $\Phi_\mathcal{G}$. Further, let $\varphi : X \longrightarrow Y$. Then*

(i) $\Phi_{[(x,x)]} = \dot{x}\times\dot{x}$,

(ii) $\mathcal{F} \le \mathcal{G} \Longrightarrow \Phi_\mathcal{F} \le \Phi_\mathcal{G}$,

(iii) $\Phi_{(\mathcal{F}^{-1})} = \big(\Phi_\mathcal{F}\big)^{-1}$,

(iv) $\Phi_{\mathcal{F}\wedge\mathcal{G}} = \Phi_\mathcal{F} \wedge \Phi_\mathcal{G}$,

(v) *if $\mathcal{F}\circ\mathcal{G} \in \mathcal{F}^S_{[0,1]}(X\times X)$, then $\Phi_\mathcal{F}\circ\Phi_\mathcal{G} \in \mathbb{F}(X\times X)$ and $\Phi_{\mathcal{F}\circ\mathcal{G}} \ge \Phi_\mathcal{F}\circ\Phi_\mathcal{G}$,*

(vi) $(\varphi\times\varphi)^\rightarrow(\Phi_\mathcal{F}) = \Phi_{(\varphi\times\varphi)^\rightarrow(\mathcal{F})}$.

Proof: Results (i) to (iv) follow easily from the definitions. For (v) it must first be shown that for $F, G \subset X\times X$ we have $1_F \circ 1_G = 1_{F\circ G}$. The reverse inequality appears only to be true for $L = \{\bot,\top\}$. For (vi) we use the fact that for $A \subset X\times X$, $(\varphi\times\varphi)^\rightarrow(1_A) = 1_{(\varphi\times\varphi)^\rightarrow(A)}$.

■

Conversely, if we let $\Psi \in \mathbb{F}(X\times X)$, we can define for $a \in L^{X\times X}$:

$$\mathcal{F}_\Psi(a) = \bigvee\big\{\alpha \in L \;:\; [a\ge\alpha] \in \Psi\big\} \text{ where } [a\ge\alpha] = \big\{(x,y) \;:\; a(x,y)\ge\alpha\big\}.$$

Lemma 6.5 *Let $\Psi,\eta \in \mathbb{F}(X\times X)$ and consider the associated stratified L-filters $\mathcal{F}_\Psi$ and $\mathcal{F}_\eta$. Further let $\varphi : X \longrightarrow Y$ be a mapping. Then*

(i) $\mathcal{F}_{\dot{x}\times\dot{x}} = [(x,x)]$,

(ii) $\Psi \le \eta \Longrightarrow \mathcal{F}_\Psi \le \mathcal{F}_\eta$,

(iii) $\mathcal{F}_{\Psi^{-1}} = (\mathcal{F}_\Psi)^{-1}$,

(iv) $\mathcal{F}_\Psi \wedge \mathcal{F}_\eta \leq \mathcal{F}_{\Psi\wedge\eta}$,

(v) *if* $\Psi \circ \eta \in \mathbb{F}(X \times X)$, *then* $\mathcal{F}_\Psi \circ \mathcal{F}_\eta \in \mathcal{F}^S_{[0,1]}(X \times X)$ *and* $\mathcal{F}_\Psi \circ \mathcal{F}_\eta \leq \mathcal{F}_{\Psi\circ\eta}$.

(vi) $(\varphi \times \varphi)^{\rightarrow}(\mathcal{F}_\Psi) = \mathcal{F}_{(\varphi\times\varphi)^{\rightarrow}(\Psi)}$.

Proof: For (i) to (iii) the proofs are straightforward and hence left to the reader. For (iv), let $a \in L^{X\times X}$, then

$$\begin{aligned}
\mathcal{F}_\Psi(a) \wedge \mathcal{F}_\eta(a) &= \bigvee \{\alpha \,:\, [a \geq \alpha] \in \Psi\} \wedge \bigvee \{\beta \,:\, [a \geq \beta] \in \eta\} \\
&= \bigvee \{\alpha \wedge \beta \,:\, [a \geq \alpha] \in \Psi, [a \geq \beta] \in \eta\} \\
&\leq \bigvee \{\alpha \wedge \beta \,:\, [a \geq \alpha \wedge \beta] \in \Psi, [a \geq \alpha \wedge \beta] \in \eta\} \\
&\leq \bigvee \{\gamma \,:\, [a \geq \gamma] \in \Psi, [a \geq \gamma] \in \eta\} \\
&\leq \bigvee \{\gamma \,:\, [a \geq \gamma] \in \Psi \wedge \eta\}.
\end{aligned}$$

Now for (v) we show that $\mathcal{F}_\Psi \circ \mathcal{F}_\eta \leq \mathcal{F}_{\Psi\circ\eta}$. From this the existence of $\mathcal{F}_\Psi \circ \mathcal{F}_\eta$ will follow. For $c \circ d \leq a$ we have

$$\mathcal{F}_\Psi(c) * \mathcal{F}_\eta(d) = \bigvee \{\alpha * \beta \,:\, [c \geq \alpha] \in \Psi, [d \geq \beta] \in \eta\} \leq \bigvee \{\alpha * \beta \,:\, [c \geq \alpha] \circ [d \geq \beta] \in \Psi \circ \eta\}.$$

For $(x, y) \in [c \geq \alpha] \circ [d \geq \beta]$ we have $z \in X$ such that $c(x, z) \geq \alpha$ and $d(z, y) \geq \beta$ and hence $c \circ d(x, y) \geq \alpha * \beta$. Consequently $(x, y) \in [c \circ d \geq \alpha * \beta]$. Therefore

$$\mathcal{F}_\Psi(c) \circ \mathcal{F}_\eta(d) \leq \bigvee \{\alpha * \beta \,:\, [c \circ d \geq \alpha * \beta] \in \Psi \circ \eta\} \leq \mathcal{F}_{\Psi\circ\eta}(c \circ d) \leq \mathcal{F}_{\Psi\circ\eta}(a).$$

In order to show (vi) we use the fact that $[(\varphi \times \varphi)^{-1}(a) \geq \alpha] = (\varphi \times \varphi)^{\leftarrow}[a \geq \alpha]$ and also that $(\varphi \times \varphi)^{\leftarrow}[a \geq \alpha] \in \Psi$ if and only if $[a \geq \alpha] \in (\varphi \times \varphi)^{\rightarrow}(\Psi)$.

■

Lemma 6.6 *Suppose* $\mathcal{F} \in \mathcal{F}^S_{[0,1]}(X \times X)$ *and* $\Psi \in \mathbb{F}(X \times X)$. *Then*

(i) $\mathcal{F}_{\Phi_\mathcal{F}} \leq \mathcal{F}$,

(ii) $\Phi_{\mathcal{F}_\Psi} = \Psi$.

Proof: To show (i), we let $a \in L^{X\times X}$ and see

$$\begin{aligned}
\mathcal{F}_{\Phi_\mathcal{F}}(a) &= \bigvee \{\alpha \,:\, [a \geq \alpha] \in \Phi_\mathcal{F}\} \\
&= \bigvee \{\alpha \,:\, \mathcal{F}(1_{[a\geq\alpha]}) = 1\} \\
&\leq \bigvee \{\alpha \,:\, \alpha * \mathcal{F}(1_{[a\geq\alpha]}) = \alpha\}.
\end{aligned}$$

Then, from the stratification of $\mathcal{F}$, we see

$$\begin{aligned}\mathcal{F}_{\Phi_\mathcal{F}}(a) &\leq \bigvee \big\{\alpha \,:\, \mathcal{F}(\underbrace{\alpha * 1_{[a \geq \alpha]}}_{\leq a}) \geq \alpha\big\} \\ &\leq \bigvee \big\{\alpha \,:\, \mathcal{F}(a) \geq \alpha\big\} = \mathcal{F}(a).\end{aligned}$$

For (ii), it is simple to show that

$$A \in \Phi_{\mathcal{F}_\Psi} \Longleftrightarrow \mathcal{F}_\Psi(1_A) = 1 \Longleftrightarrow \bigvee \big\{\alpha \,:\, A = [1_A \geq \alpha] \in \Psi\big\} = 1 \Longleftrightarrow A \in \Psi.$$

■

For $\big(X, (\mathcal{L}^\alpha)_{\alpha \in [0,1]}\big) \in |PULIM^*|$ we define

$$\Lambda_\mathcal{L} : \mathcal{F}^S_{[0,1]}(X \times X) \longrightarrow [0,1]$$

for $\mathcal{F} \in \mathcal{F}^S_{[0,1]}(X \times X)$ by

$$\Lambda_\mathcal{L}(\mathcal{F}) = \bigvee \Big\{\alpha \in [0,1] \,:\, \Phi_\mathcal{F} \in \mathcal{L}^\alpha\Big\}.$$

Proposition 6.7 *The mapping $\Lambda_\mathcal{L}$ is a stratified $[0,1]$-uniform convergence structure.*

Proof:

LUC1: The filter $\Phi_{[(x,x)]} = \dot{x} \times \dot{x} \in \mathcal{L}^1$. Therefore $\Lambda_\mathcal{L}\big([(x,x)]\big) = 1$.

LUC2: If $\mathcal{F} \leq \mathcal{G}$ by Lemma 6.4(ii) we have that $\Phi_\mathcal{F} \leq \Phi_\mathcal{G}$. Now

$$\begin{aligned}\Lambda_\mathcal{L}(\mathcal{F}) &= \bigvee \big\{\alpha \,:\, \Phi_\mathcal{F} \in \mathcal{L}^\alpha\big\} \\ &\leq \bigvee \big\{\alpha \,:\, \Phi_\mathcal{G} \in \mathcal{L}^\alpha\big\} \\ &= \Lambda_\mathcal{L}(\mathcal{G}).\end{aligned}$$

LUC3: If $\Phi_\mathcal{F} \in \mathcal{L}^\alpha$, then by **(UC3)** and Lemma 6.4(iii), $(\Phi_\mathcal{F})^{-1} = \Phi_{(\mathcal{F}^{-1})} \in \mathcal{L}^\alpha$. Now

$$\begin{aligned}\Lambda_\mathcal{L}(\mathcal{F}) &= \bigvee \big\{\alpha \,:\, \Phi_\mathcal{F} \in \mathcal{L}^\alpha\big\} \\ &\leq \bigvee \big\{\alpha \,:\, \Phi_{(\mathcal{F}^{-1})} \in \mathcal{L}^\alpha\big\} \\ &= \Lambda_\mathcal{L}(\mathcal{F}^{-1}).\end{aligned}$$

LUC4: Here we have

$$\begin{aligned}\Lambda_\mathcal{L}(\mathcal{F}) \wedge \Lambda_\mathcal{L}(\mathcal{G}) &= \bigvee \big\{\alpha \,:\, \Phi_\mathcal{F} \in \mathcal{L}^\alpha\big\} \wedge \bigvee \big\{\beta \,:\, \Phi_\mathcal{G} \in \mathcal{L}^\beta\big\} \\ &= \bigvee \big\{\alpha \wedge \beta \,:\, \Phi_\mathcal{F} \in \mathcal{L}^\alpha, \Phi_\mathcal{G} \in \mathcal{L}^\beta\big\}.\end{aligned}$$

From **(P1)** we will then get

$$\begin{aligned}\Lambda_{\mathcal{L}}(\mathcal{F}) \wedge \Lambda_{\mathcal{L}}(\mathcal{G}) &\leq \bigvee \left\{\alpha \wedge \beta \,:\, \Phi_{\mathcal{F}} \in \mathcal{L}^{\alpha\wedge\beta}, \Phi_{\mathcal{G}} \in \mathcal{L}^{\alpha\wedge\beta}\right\} \\ &\leq \bigvee \left\{\gamma \,:\, \Phi_{\mathcal{F}} \in \mathcal{L}^{\gamma}, \Phi_{\mathcal{G}} \in \mathcal{L}^{\gamma}\right\}.\end{aligned}$$

Now using **(UC4)** and Lemma 6.4 (iv) we see

$$\begin{aligned}\Lambda_{\mathcal{L}}(\mathcal{F}) \wedge \Lambda_{\mathcal{L}}(\mathcal{G}) &\leq \bigvee \left\{\gamma \,:\, \Phi_{\mathcal{F}} \cap \Phi_{\mathcal{G}} = \Phi_{\mathcal{F}\wedge\mathcal{G}} \in \mathcal{L}^{\gamma}\right\} \\ &= \Lambda_{\mathcal{L}}(\mathcal{F} \wedge \mathcal{G}).\end{aligned}$$

LUC5: Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}^S_{[0,1]}(X \times X)$ and suppose that $\mathcal{F} \circ \mathcal{G}$ exists. By Lemma 6.4(v) we have that $\Phi_{\mathcal{F}} \circ \Phi_{\mathcal{G}}$ exists and that $\Phi_{\mathcal{F}\circ\mathcal{G}} \geq \Phi_{\mathcal{F}} \circ \Phi_{\mathcal{G}}$. By definition we have that

$$\Lambda_{\mathcal{L}}(\mathcal{F}) * \Lambda_{\mathcal{L}}(\mathcal{G} = \bigvee \left\{\epsilon \,:\, \Phi_{\mathcal{F}} \in \mathcal{L}^{\epsilon}\right\} * \bigvee \left\{\delta \,:\, \Phi_{\mathcal{G}} \in \mathcal{L}^{\delta}\right\} = \bigvee \left\{\epsilon * \delta \,:\, \Phi_{\mathcal{F}} \in \mathcal{L}^{\epsilon}, \Phi_{\mathcal{G}} \in \mathcal{L}^{\delta}\right\}.$$

Using **(PULIM)** and then **(UC2)** we then see that

$$\begin{aligned}\Lambda_{\mathcal{L}}(\mathcal{F}) * \Lambda_{\mathcal{L}}(\mathcal{G}) &\leq \bigvee \left\{\epsilon * \delta \,:\, \Phi_{\mathcal{F}} \circ \Phi_{\mathcal{G}} \in \mathcal{L}^{\epsilon*\delta}\right\} \\ &\leq \bigvee \left\{\gamma \,:\, \Phi_{\mathcal{F}} \circ \Phi_{\mathcal{G}} \in \mathcal{L}^{\gamma}\right\} \\ &\leq \bigvee \left\{\gamma \,:\, \Phi_{\mathcal{F}\circ\mathcal{G}} \in \mathcal{L}^{\gamma}\right\} \\ &= \Lambda_{\mathcal{L}}(\mathcal{F} \circ \mathcal{G}).\end{aligned}$$

■

Proposition 6.8 *The map* $\delta : PULIM^* \longrightarrow S[0,1]\text{-}UCS, \big(X, (\mathcal{L}^{\alpha})\big) \longmapsto (X, \Lambda_{\mathcal{L}})$ *is injective on objects.*

Proof: Consider $\big(X, (\mathcal{L}^{\alpha})_{\alpha\in[0,1]}\big)$ and $\big(X, (\mathcal{K}^{\alpha})_{\alpha\in[0,1]}\big)$ such that $(\mathcal{L}^{\alpha})_{\alpha\in[0,1]} \neq (\mathcal{K}^{\alpha})_{\alpha\in[0,1]}$. We must then have an α_0 such that $\mathcal{L}^{\alpha_0} \neq \mathcal{K}^{\alpha_0}$ and therefore there must exist Ψ such that $\Psi \in \mathcal{L}^{\alpha_0}$ but $\Psi \notin \mathcal{K}^{\alpha_0}$.

From the left continuity of $(\mathcal{K}^{\alpha})_{\alpha\in[0,1]}$ we have therefore that $\Psi \notin \mathcal{K}^{\alpha_0-\epsilon}$ for some $\epsilon > 0$.

For Ψ and $a \in L^{X\times X}$ we have $\mathcal{F}_{\Psi}(a) = \bigvee\limits_{[a\geq\alpha]\in\Psi} \alpha$ and $\mathcal{F}_{\Psi} \in \mathcal{F}^S_{[0,1]}(X \times X)$. Further, $\Phi_{(\mathcal{F}_{\Psi})} = \Psi$.

Now $\Lambda_{\mathcal{L}}(\mathcal{F}_{\Psi}) = \bigvee \left\{\alpha \,:\, \Phi_{(\mathcal{F}_{\Psi})} = \Psi \in \mathcal{L}^{\alpha}\right\} \geq \alpha_0$ and $\Lambda_{\mathcal{K}}(\mathcal{F}_{\Psi}) = \bigvee \left\{\alpha \,:\, \Phi_{(\mathcal{F}_{\Psi})} = \Psi \in \mathcal{K}^{\alpha}\right\} < \alpha_0$. This second inequality is as a result of the fact that from **(P1)** we have that for $\beta \geq \alpha_0$ that $\Psi \notin \mathcal{K}^{\beta}$. Therefore we have shown that $(X, \Lambda_{\mathcal{L}}) \neq (X, \Lambda_{\mathcal{K}})$.

■

Proposition 6.9 *The mapping* δ *preserves morphisms.*

PROOF: Let $\varphi : (X, \mathcal{L}^\alpha) \longrightarrow (Y, \mathcal{K}^\alpha)$ be uniformly continuous. That is, for all $\alpha \in [0,1]$ we have $(\varphi \times \varphi)^{\rightarrow}(\mathcal{L}^\alpha) \subset \mathcal{K}^\alpha$. In order to show that φ is a morphism in $S[0,1]$-UCS we must show that for all $\mathcal{F} \in \mathcal{F}^S_{[0,1]}(X \times X)$ we have $\Lambda_{\mathcal{L}}(\mathcal{F}) \le \Lambda_{\mathcal{K}}\big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F})\big)$.

We know that $\Lambda_{\mathcal{L}}(\mathcal{F}) = \bigvee \big\{\alpha : \Phi_{\mathcal{F}} \in \mathcal{L}^\alpha\big\}$. If $\Phi \in \mathcal{L}^\alpha$ then by Lemma 6.4(vi) we get $(\varphi \times \varphi)^{\rightarrow}(\Phi_{\mathcal{F}}) = \Phi_{(\varphi\times\varphi)^{\rightarrow}(\mathcal{F})} \in \mathcal{K}^\alpha$. Hence we can see that $\big\{\alpha : \Phi_{\mathcal{F}} \in \mathcal{L}^\alpha\big\} \subset \big\{\alpha : \Phi_{(\varphi\times\varphi)^{\rightarrow}(\mathcal{F})} \in \mathcal{K}^\alpha\big\}$ and therefore $\Lambda_{\mathcal{L}}(\mathcal{F}) \le \Lambda_{\mathcal{K}}\big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F})\big)$.

■

Using the above propositions, we define an embedding functor for $L = [0,1]$:

$$\delta : \begin{cases} PULIM^* \longrightarrow S[0,1]\text{-}UCS \\ \begin{cases} \big(X, (\mathcal{L}^\alpha)_{\alpha\in[0,1]}\big) \longmapsto (X, \Lambda_{\mathcal{L}}) \\ \varphi \longmapsto \varphi \end{cases} \end{cases}$$

Proposition 6.10 *The functor δ is full. That is,*
$mor\Big(\big(X, (\mathcal{L}^\alpha)_{\alpha\in[0,1]}\big), \big(Y, (\mathcal{K}^\alpha)_{\alpha\in[0,1]}\big)\Big) = mor\big((X, \Lambda_{\mathcal{L}}), (Y, \Lambda_{\mathcal{K}})\big)$.

PROOF: Let $\varphi : (X, \Lambda_{\mathcal{L}}) \longrightarrow (Y, \Lambda_{\mathcal{K}})$ be uniformly continuous. That is, for all $\mathcal{F} \in \mathcal{F}^S_{[0,1]}(X \times X)$ we have $\Lambda_{\mathcal{L}}(\mathcal{F}) \le \Lambda_{\mathcal{K}}\big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F})\big)$. Now if we let $\Psi \in \mathbb{F}(Y \times Y)$, $\alpha_0 \in [0,1]$ and $\Psi \in (\varphi \times \varphi)^{\rightarrow}(\mathcal{L}^{\alpha_0})$ then there must exist some $\eta \in \mathcal{L}^{\alpha_0}$ such that $(\varphi \times \varphi)^{\rightarrow}(\eta) \le \Psi$.

For $\mathcal{F}_\eta$ we have $\Phi_{(\mathcal{F}_\eta)} = \eta$ and therefore $\Lambda_{\mathcal{L}}(\mathcal{F}_\eta) = \bigvee \big\{\alpha : \Phi_{(\mathcal{F}_\eta)} = \eta \in \mathcal{L}^\alpha\big\} \ge \alpha_0$. Hence $\Lambda_{\mathcal{K}}\big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F}_\eta)\big) \ge \alpha_0$.

We have that

$$\Lambda_{\mathcal{K}}\big((\varphi \times \varphi)^{\rightarrow}(\mathcal{F}_\eta)\big) = \bigvee \big\{\beta : \Phi_{\big((\varphi\times\varphi)^{\rightarrow}(\mathcal{F}_\eta)\big)} = (\varphi \times \varphi)^{\rightarrow}\big(\Phi_{(\mathcal{F}_\eta)}\big) = (\varphi \times \varphi)^{\rightarrow}(\eta) \in \mathcal{K}^\beta\big\}.$$

This then gives us $\bigvee \big\{\beta : (\varphi\times\varphi)^{\rightarrow}(\eta) \in \mathcal{K}^\beta\big\} \ge \alpha_0$. Now if we consider $A = \big\{\beta : (\varphi\times\varphi)^{\rightarrow}(\eta) \in \mathcal{K}^\beta\big\}$, then $(\varphi\times\varphi)^{\rightarrow}(\eta) \in \mathcal{K}^\beta$ for all $\beta \in A$. By left-continuity we then get $(\varphi\times\varphi)^{\rightarrow}(\eta) \in \mathcal{K}^{\bigvee A}$. Since $\bigvee A \ge \alpha_0$ we have that $\mathcal{K}^{\bigvee A} \subset \mathcal{K}^{\alpha_0}$ and hence $(\varphi \times \varphi)^{\rightarrow}(\eta) \in \mathcal{K}^{\alpha_0}$.

■

In order to show that we can also generate a probabilistic uniform limit space from a stratified L-uniform convergence space for $\Psi \in \mathbb{F}(X \times X)$ and $\alpha \in [0,1]$, we define

$$\Psi \in \mathcal{L}^\alpha_\Lambda \Longleftrightarrow \Lambda(\mathcal{F}_\Psi) \ge \alpha.$$

Proposition 6.11 *If $(X, \Lambda) \in |S[0,1]\text{-}UCS|$, then $\big(X, (\mathcal{L}^\alpha_\Lambda)_{\alpha\in[0,1]}\big)$ is a probabilistic uniform limit space.*

UC1: By Lemma 6.5(i), $\mathcal{F}_{\dot{x}\times\dot{x}} = [(x,x)]$ and $\Lambda\big([(x,x)]\big) = 1$, we have $\dot{x}\times\dot{x} \in \mathcal{L}^{\alpha}_{\Lambda}$ for all $\alpha \in [0,1]$.

UC2: Suppose $\Psi \le \eta$ and $\Psi \in \mathcal{L}^{\alpha}_{\Lambda}$. Then clearly $\Lambda(\mathcal{F}_{\psi}) \ge \alpha$. From Lemma 6.5(ii) and **(LUC2)** $\Lambda(\mathcal{F}_{\eta}) \ge \Lambda(\mathcal{F}_{\Psi}) \ge \alpha$ and thus we see $\eta \in \mathcal{L}^{\alpha}_{\Lambda}$.

UC3: If $\Psi \in \mathcal{L}^{\alpha}_{\Lambda}$ then $\Lambda(\mathcal{F}_{\Psi}) \ge \alpha$. By **(LUC3)** and Lemma 6.5(iii) we have $\Lambda(\mathcal{F}_{(\Psi^{-1})}) = \Lambda\big((\mathcal{F}_{\Psi})^{-1}\big) \ge \alpha$ and therefore $\Psi^{-1} \in \mathcal{L}^{\alpha}_{\Lambda}$.

UC4: If $\Psi, \eta \in \mathcal{L}^{\alpha}_{\Lambda}$ then clearly $\Lambda(\mathcal{F}_{\Psi}) \ge \alpha$ and $\Lambda(\mathcal{F}_{\eta}) \ge \alpha$. Using first **(LUC2)** and Lemma 6.5(iv), followed by **(LUC4)** we get

$$\begin{aligned}\Lambda(\mathcal{F}_{\Psi\wedge\eta}) &\ge \Lambda(\mathcal{F}_{\Psi}\wedge\mathcal{F}_{\eta})\\ &\ge \Lambda(\mathcal{F}_{\Psi})\wedge\Lambda(\mathcal{F}_{\eta})\\ &\ge \alpha\wedge\alpha\\ &= \alpha.\end{aligned}$$

Hence $\Psi\wedge\eta \in \mathcal{L}^{\alpha}_{\Lambda}$.

P1: Suppose $\alpha \le \beta$ and $\Psi \in \mathcal{L}$. This gives us $\Lambda(\mathcal{F}_{\Psi}) \ge \beta \ge \alpha$. Therefore $\Psi \in \mathcal{L}^{\alpha}_{\Lambda}$.

P2: If $\Psi \in \mathbb{F}(X\times X)$ then $\Lambda(\mathcal{F}_{\Psi}) \ge 0$ and therefore $\Psi \in \mathcal{L}^{0}_{\Lambda}$. Hence $\mathcal{L}^{0}_{\Lambda} = \mathcal{F}(X\times X)$.

PULIM: Suppose $\Psi \in \mathcal{L}^{\alpha}_{\Lambda}$ and $\eta \in \mathcal{L}^{\beta}_{\Lambda}$, and that $\Psi\circ\eta$ exists. Then by Lemma 6.5(v) $\mathcal{F}_{\Psi}\circ\mathcal{F}_{\eta}$ exists and $\mathcal{F}_{\Psi}\circ\mathcal{F}_{\eta} \le \mathcal{F}_{\Psi\circ\eta}$. Now $\Lambda(\mathcal{F}_{\Psi}) \ge \alpha$ and $\Lambda(\mathcal{F}_{\eta}) \ge \beta$ and therefore by **(LUC5)** we have $\Lambda(\mathcal{F}_{\Psi}\circ\mathcal{F}_{\eta}) \ge \Lambda(\mathcal{F}_{\Psi}) * \Lambda(\mathcal{F}_{\eta}) \ge \alpha * \beta$.

Axiom **(LUC2)** then gives us $\Lambda(\mathcal{F}_{\Psi\circ\eta}) \ge \alpha*\beta$ and hence $\Psi\circ\eta \in \mathcal{L}^{\alpha*\beta}_{\Lambda}$.

■

Proposition 6.12 *If* $\varphi : (X,\Lambda) \longrightarrow (Y,\Sigma)$ *is uniformly continuous, then* $\varphi : \big(X, (\mathcal{L}^{\alpha}_{\Lambda})_{\alpha\in[0,1]}\big) \longrightarrow \big(Y, (\mathcal{L}^{\alpha}_{\Sigma})_{\alpha\in[0,1]}\big)$ *is uniformly continuous.*

Proof: If $\alpha \in [0,1]$ and $\Psi \in \mathcal{L}^{\alpha}_{\Lambda}$ then $\Lambda(\mathcal{F}_{\Psi}) \ge \alpha$. Since φ is uniformly continuous, $\Sigma\big((\varphi\times\varphi)^{\rightarrow}(\mathcal{F}_{\Psi})\big) = \Sigma\big(\mathcal{F}_{(\varphi\times\varphi)^{\rightarrow}(\Psi)}\big) \ge \alpha$, and therefore $(\varphi\times\varphi)^{\rightarrow}(\Psi) \in \mathcal{L}^{\alpha}_{\Sigma}$.

■

As a consequence of the previous two results we can now define a functor:

$$K : \begin{cases} S[0,1]\text{-}UCS \longrightarrow PULIM^* \\ \begin{cases} (X,\Lambda) \longmapsto \big(X, (\mathcal{L}^\alpha_\Lambda)_{\alpha\in[0,1]}\big) \\ \varphi \longmapsto \varphi. \end{cases} \end{cases}$$

Proposition 6.13 *For* $\big(X, (\mathcal{L}^\alpha)_{\alpha\in[0,1]}\big) \in |PULIM^*|$ *and for all* $\alpha \in [0,1]$ *we have* $\mathcal{L}^\alpha = \mathcal{L}^\alpha_{(\Lambda_\mathcal{L})}$. *In other words,* $K\Big(\delta\big(X, (\mathcal{L}^\alpha)_{\alpha\in[0,1]}\big)\Big) = \big(X, (\mathcal{L}^\alpha)_{\alpha\in[0,1]}\big)$.

Proof: Let $\Psi \in \mathcal{L}^\alpha$. Then $\Lambda_\mathcal{L}(\mathcal{F}_\Psi) = \bigvee \big\{\beta : \Phi_{(\mathcal{F}_\Psi)} = \Psi \in \mathcal{L}^\beta\big\} \geq \alpha$ and so $\Psi \in \mathcal{L}^\alpha_{(\Lambda_\mathcal{L})}$.

On the other hand, $\Psi \in \mathcal{L}^\alpha_{(\Lambda_\mathcal{L})}$ if and only if $\Lambda_\mathcal{L}(\mathcal{F}_\Psi) \geq \alpha$. By definition, this will only be the case if $\bigvee \big\{\beta : \Phi_{(\mathcal{F}_\Psi)} = \Psi \in \mathcal{L}^\beta\big\} \geq \alpha$. If we denote $A = \big\{\beta : \Psi \in \mathcal{L}^\beta\big\}$, then we have $\Psi \in \mathcal{L}^\beta$ for all $\beta \in A$ and hence by left continuity $\Psi \in \mathcal{L}^{\bigvee A} \subset \mathcal{L}^\alpha$.

■

Proposition 6.14 *The identity mapping* $id_X : \Big(X, \delta\big(K(X,\Lambda)\big)\Big) \longrightarrow (X,\Lambda)$ *is continuous. In other words,* $(X,\Lambda) \leq \delta\big(K(X,\Lambda)\big)$.

Proof: For $\mathcal{F} \in \mathcal{F}^S_{[0,1]}(X \times X)$,

$$\begin{aligned} \Lambda_{(\mathcal{L}^\alpha_\Lambda)}(\mathcal{F}) &= \bigvee \big\{\alpha : \Phi_\mathcal{F} \in \mathcal{L}^\alpha_\Lambda\big\} \\ &= \bigvee \big\{\alpha : \Lambda\big(\mathcal{F}_{(\Phi_\mathcal{F})}\big) \geq \alpha\big\} \\ &\leq \bigvee \big\{\alpha : \Lambda(\mathcal{F}) \geq \alpha\big\} \\ &= \Lambda(\mathcal{F}). \end{aligned}$$

■

Proposition 6.15 *$PULIM^*$ is a coreflective subcategory of $S[0,1]$-UCS.*

Proof: Let $(X,\Lambda) \in |S[0,1]\text{-}UCS|$. Then $id_X : \Big(X, \delta\big(K(\Lambda)\big)\Big) \longrightarrow (X,\Lambda)$ is a morphism. For a further space $\big(Y, (\mathcal{K}^\alpha)_{\alpha\in[0,1]}\big) \in |PULIM^*|$, and a morphism $\varphi : \big(Y, (\mathcal{K}^\alpha)_{\alpha\in[0,1]}\big) \longrightarrow (X,\Lambda)$ we have that $\bar{\varphi} : \Big(Y, K\big(\delta(\mathcal{K}^\alpha)_{\alpha\in[0,1]}\big)\Big) \longrightarrow \big(X, K(\Lambda)\big)$ is a morphism in $PULIM^*$ since $\Big(Y, K\big(\delta(\mathcal{K}^\alpha)_{\alpha\in[0,1]}\big)\Big) = \big(Y, (\mathcal{K}^\alpha)_{\alpha\in[0,1]}\big)$. Clearly, $id_X \circ \bar{\varphi} = \varphi$.

Now to show the uniqueness of $\bar{\varphi}$, assume that $\hat{\varphi} : \big(Y, (\mathcal{K}^\alpha)\big) \longrightarrow \big(X, K(\Lambda)\big)$ such that $id_X \circ \hat{\varphi} = \varphi$. This would imply that we have $id_X \circ \hat{\varphi}(y) = id_X \circ \bar{\varphi}(y)$ and hence for all $y \in Y$, $\hat{\varphi}(y) = \bar{\varphi}(y)$. Hence $\hat{\varphi} = \bar{\varphi}$.

■

7 Conclusions

We have generalised the category of lattice-valued uniform convergence spaces, previously defined in the restricted case of complete Heyting algebras, to more general enriched lattices. The resulting category, SL-UCS, was shown to be a topological category in the definition of Preuss [22]. Moreover we could show that Gutiérrez García's L-uniform spaces [6] and Nusser's probabilistic uniform limit spaces [20], [21] can be viewed as natural examples of our spaces. This shows that our category is a suitable framework for studying various lattice-valued concepts of uniformities.

A question that remains open is whether or not our category is also cartesian closed. We encountered problems in generalising the function space structure defined in [16] to our more general setting. Also, it would be intersting to know if Nusser's category of probabilistic uniform limit spaces can also be reflectively embedded into our category. We will have a look into these questions in our future work.

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