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A COMBINATORIAL ANALYSIS OF BARRED PREFERENTIAL ARRANGEMENTS

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Dedicated to my mother **Nkonkobe Noritise**.

Abstract

For a non-negative integer n an ordered partition of a set X_n with n distinct elements is called a preferential arrangement (PA). A barred preferential arrangement (BPA) is a preferential arrangement with bars in between the blocks of the partition. An integer sequence a_n associated with the counting PA's of X_n has been intensely studied over a century and a half in many different contexts. In this thesis we develop a unified combinatorial framework to study the enumeration of BPAs and a special subclass of BPAs. The results of the study lead to a positive settlement of an open problem and a conjecture by Nelsen. We derive few important identities pertaining to the number of BPAs and restricted BPAs of an n element set using generatingfunctionology. Later we show that the number of restricted BPAs of X_n are intricately related to well-known numbers such as Eulerian numbers, Bell numbers, Poly-Bernoulli numbers and the number of equivalence classes of fuzzy subsets of X_n under some equivalent relation.

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Chapter 1

Introduction.

The integer sequence 1,1,3,13,75,541... with exponential generating function $\frac{1}{2-e^x}$ has been a subject of study for more than a century now. The sequence seem to first have been proposed by Arthur Cayley in [Cay59], an 1859 paper in connection with analytical forms called trees. The sequence has since then attracted much attention in combinatorics, being given interpretations in various contexts for instance in [VC95, Gro62, Men82, Slo, Mur06a, Mur06b, Mac90]. The sequence is A000670 on the On-line encyclopedia of integer sequences [Slo]. MacMahon in [Mac90] has interpreted the sequence in a graph theoretic context using Yoke-chains and multipartite decompositions.

Mendelson in [Men82] has interpreted the sequence as giving the number of outcomes in a race in which ties are allowed. Murali in [Mur06a, Mur06b] has interpreted the sequence as giving the number of preferential fuzzy subsets of a set X_n. In [Gro62] Gross has interpreted the sequence as giving the number of preferential arrangements of a set X_n. The term preferential arrangement is due to Gross [Gro62].

We now give two examples of preferential arrangements of the set $X_4 = \{1, 2, 3, 4\}$.

a) 2 13 4

b) 3 1 4 2

The preferential arrangement in a) has three blocks. The first block of the preferential arrangement contains the element 2. The second block of the preferential arrangement contains two elements: the elements 1 and 3. The third block of the preferential arrangement contains the element 4. The spacing between elements indicates which elements are on the same block and which elements are not. The preferential arrangement in b) contains four blocks of which each of the blocks contains a singleton element.

Barred Preferential Arrangements.

The introduction of bars in between blocks of preferential arrangements results in the notion of barred preferential arrangements [CAP13]. Barred preferential arrangements with a single bar seem to first appear in [Pip10], a 2010 paper by Nicholas Pippenger. The idea was later generalised by Ahlbach et al in [CAP13] to multiple bars, a 2013 paper. We now give two examples of barred preferential arrangements of X_4, the first one having three bars and the second one having two bars.

a) $|4 \quad 13| \quad |2$

b) $3 \quad 4 \quad 2 \quad | \quad |1$

The barred preferential arrangement in a) has four sections. The first section being the empty section to left of the first bar (from left to right). The second section being the section in between the first and second bar i.e the section having the two blocks, where the first block contains the element 4 and the second block contains the two elements 1 and 3. The third section is the empty section in between the second and the third bar. The fourth section is the section containing the element 2.

The barred preferential arrangement in b) has three sections. The first section contains three blocks. The second section is empty. The third section contains the element 1. In general $r \in \mathbb{N}_0 = \{0, 1, 2, \ldots, n\}$ bars separate a barred preferential arrangement into $r + 1$ sections [CAP13]. We denote by J_n^r the number of barred preferential arrangements of X_n having r bars.

In Chapter 2 and [NM15a] we propose identities and a property satisfied by the number of barred preferential arrangements.

Roger Nelsen and Harvey Schmidt in [NJ91] have proposed the family of generating functions $\frac{e^{rx}}{2-e^x}$, where $r \in \mathbb{N}_0$. The generating function for $r = 0$ is known to be the generating function for the number of preferential arrangements [Gro62]. In [NJ91] Nelsen and Schmidt proposed that for $r = 2$, the generating function is that of the number of chains in the power set of X_n. They then asked the following question.

Question.

Could there be other combinatorial structures that can be associated with either the set X_n or the power set of X_n whose integer sequences are generated members of the family $\frac{e^{rx}}{2-e^x}$ for other values of r other than for the two values? Benjamin in [Ben96] proposed an answer to a campus security problem, his solution to the campus security problem also answers the question of Nelsen and Schmidt problem for $r = 1$. In [Mur06b] the author has proposed an answer to the question of Nelsen and Schmidt for the value of $r = 0, 1, 2, 3$ using the idea of preferential fuzzy subsets of a set X_n. The main aim of this study is to propose an answer to this question of Nelsen and Schmidt for all values of r in non-negative integers. We do this by putting a restriction on barred preferential arrangements of X_n that some sections to have a maximum of one block. This we do in Chapter 3 and [NM17]. In the same chapter we show how the generating function of Nelsen and Schmidt can be interpreted for negative values of r as well.

Chapter 4 is a literature survey chapter. In the chapter we show how the number of restricted barred preferential arrangements are related to some well known integer sequences. Which are Bell numbers, Eulerian numbers, the number of chains in power set of X_n and the number of equivalence classes of fuzzy subsets of X_n. In Chapter 5 and [NM15b] we show relations between number of restricted barred preferential arrangements and poly-Bernoulli numbers. We also discuss some common properties between poly-Bernoulli numbers and number of restricted barred preferential arrangements.

In this study we will refer to generating functions of the form $\frac{e^{rx}}{2-e^x}$ where $r \in \mathbb{N}_0$ a family of generating functions of Nelsen-Schmidt type since the original problem regarding the generating functions first proposed by Nelsen and Schmidt in [NJ91]. The entries in the tables presented in this study were generated using a computer program code in python which the author developed in his thesis [Nko14]. In this study when referring to a generating function of a sequence we will be referring to its exponential generating function unless stated otherwise.

Preliminaries.

We denote by Q_n^r the set of all barred preferential arrangements of X_n having $(r \in \mathbb{N}_0)$ bars, and the number of these barred preferential arrangements by J_n^r. We end this chapter by stating a closed form, recurrence relation and a generating function for the numbers J_n^r.

Theorem 1.1. *[CAP13] For all $n, r \in \mathbb{N}_0$,*

$$J_n^r = \sum_{s=0}^{n} \left\{ {n \atop s} \right\} s! \binom{r+s}{s},$$

where $\left\{ {n \atop s} \right\}$ are the Stirling numbers of the second kind.

Theorem 1.2. *[CAP13] For $r \in \mathbb{N}_0$,*

$$J_n^r = \sum_{s=0}^{n} \binom{n}{s} J_s^0 J_{n-s}^{r-1} \quad \textit{where } J_0^r = 1,$$

for all $n \in \mathbb{N}_0$.

Theorem 1.3. *[CAP13] For $r \in \mathbb{N}_0$,*

$$q^r(x) = \sum_{n=0}^{\infty} \frac{J_n^r x^n}{n!} = \frac{1}{(2-e^x)^{r+1}} \, .$$

Remark. Whether we view a barred preferential arrangement as a result of the introduction of bars in between blocks of a preferential arrangement

Or viewing it as a result of first placing and then preferentially arranging elements on the sections, the resulting number of barred preferential arrangements is the same [Pip10].

Note. A list of symbols and notations used in this study is given in Appendix B (see Page 83). The set X_n throughout this study is assumed to have elements which are all distinct.

Chapter 2

Barred Preferential Arrangements.

In this chapter we discuss properties of barred preferential arrangements and prove combinatorial identities satisfied by the number J_n^r of barred preferential arrangements of X_n. In the process we highlight some important consequences of the identities.

2.1 Properties of Barred Preferential Arrangements.

A table for the number of barred preferential arrangements.

Table 2.1: Table for the number of barred preferential arrangements [Nko14].

$n\backslash r$	0	1	2	3	4
1	1	2	3	4	5
2	3	8	15	24	35
3	13	44	99	184	305
4	75	308	807	1704	3155
5	541	2612	7803	18424	37625
6	4683	25988	87135	227304	507035
7	47293	296564	1102419	3147064	7608305
8	545835	3816548	15575127	48278184	125687555
9	7087261	54667412	242943723	812387704	2265230825
10	102247563	862440068	4145495055	14872295784	44210200235
11	1622632573	14857100084	76797289539	294192418744	928594230305
12	28091567595	277474957988	1534762643847	6251984167464	20880079975955

We observe that the last digit of entries in the first column of Table 2.1 has a four cycle 1,3,3,5. On the second column the four cycle of the last digit is 2,8,4,8. We also observe that the other columns have four cycles.

We state the result as a lemma.

Lemma 2.1. *[Nko14] For fixed $r \in \mathbb{N}_0$ the last digit of the number J_n^r has a four cycle for $n \in \mathbb{N} = \{1, 2, \ldots\}$.*

We will prove the property presented in the lemma in a generalised form in Chapter 4 as Theorem 4.9.

A Duality Property of Preferential Arrangements.

We recall that J_n^r denotes the number of barred preferential arrangements of X_n having r bars, where the set of these barred preferential arrangements is denoted by Q_n^r.

Definition 2.1. *A* ***dual*** *of a preferential arrangement of X_n is the same preferential arrangement only now being read from right to left instead of left to right.*

Given the preferential arrangement 3 1 4 2 of X_4, its dual is 2 4 1 3.

Definition 2.2. *The* ***trivial preferential arrangement*** *of X_n is the preferential arrangement which has only one block.*

So on the set,

$$Q_2^0 = \{12; 1 \quad 2; 2 \quad 1\},$$

which is the set of all preferential arrangements of X_2. The preferential arrangement 12 is the trivial preferential arrangement. The preferential arrangements 1 2 and 2 1 are duals of each other.

Illustration. In this illustration we generate the set Q_3^0 of all preferential arrangements of the set X_3 using the duality property. From Table 2.1 on Page 15 we observe that $|Q_3^0| = 13$.

We collect as below a random list of 6 non-trivial preferential arrangements of X_3 whose dual are all not in the collected list to form a set C_6.

3 12

2 1   3

3 2   1

1 3   2

23 1

13 2

We generate duals of the above preferential arrangements by reading the preferential arrangements from right to left to form a set C_6^*.

$$
\begin{array}{c}
12 \quad 3 \\
3 \quad 1 \quad\quad 2 \\
1 \quad 2 \quad\quad 3 \\
2 \quad 3 \quad\quad 1 \\
1 \quad 23 \\
2 \quad 13
\end{array}
$$

Combining the sets C_6, C_6^* and the trivial preferential arrangement 123, we obtain the set Q_3^0.

Indeed $|C_6|+|C_6^*| + 1 = 6 + 6 + 1 = 13 = |Q_3^0|$.

Conjecture 2.1. *We conjecture that in general the set Q_n^0 of all preferential arrangements of X_n, for $n \geq 2$ can be partitioned into three subsets. Where the first subset C_n contain all non-trivial preferential arrangements of X_n such that each preferential arrangement of C_n does not have its dual in C_n. The second subset being the collection C_n^* of all those preferential arrangements of X_n and whose duals are in C_n. The third subset being composed of the trivial preferential arrangement. The two sets C_n^* and C_n are each of* *cardinality* $\frac{J_n^0 - 1}{2}$.

The statement of the conjecture is supported by Lemma 2.1 above. From the conjecture we deduce that $J_n^0 = |C_n| + |C_n^*| + 1 = 2r + 1$ (where $r \in \mathbb{N}$) i.e J_n^0 is odd. From Lemma 2.1 we deduce that the last digit of the sequence J_n^0 has the four cycle 1-3-3-5, meaning J_n^0 is odd for all integer $n \geq 2$.

2.2 Identities on Barred Preferential Arrangements.

Lemma 2.2. *For* $r \in \mathbb{N}$,

$$J_n^r = J_n^{r-1} + \sum_{s=0}^{n-1} \binom{n}{s} J_s^{r-1} J_{n-s}^0 \,,$$

for all $n \in \mathbb{N}$.

The lemma will be proved in Chapter 3 in a more general form as Theorem 3.6.

Theorem 2.1. *For $r \in \mathbb{N}$,*

$$J_n^r = J_n^{r-1} + \sum_{s=0}^{n-1} \binom{n}{s} J_s^r \ ,$$

for all $n \in \mathbb{N}$.

Proof. We recall that J_n^r is the number of barred preferential arrangements of X_n having r bars, where the set of these barred preferential arrangements is denoted by Q_n^r. On each barred preferential arrangement in Q_n^r either the first section is empty or non-empty.

When the first section is empty, it means that all the elements of X_n are preferentially arranged on the r other section. Hence the number of elements of Q_n^r having this property is J_n^{r-1}.

In the second scenario the first section being non-empty means that the first section will always have at least one block. From left to right there will always be a first block on the first section in this case. The maximum number of elements which are not to form part of the first block of the first section is $n-1$ and the minimum number is 0. Lets says there are s elements which are not to form part of the first block of the first section. The s can be chosen in $\binom{n}{s}$ ways. The s elements can be preferentially arranged on the $r+1$ sections in J_s^r ways. The remaining $n-s$ elements attach them on the first section as the first block. Taking the product and summing over s we obtain $\sum_{s=0}^{n-1}\binom{n}{s}J_s^r \times 1^{n-s}$. $\square$

Lemma 2.3. *[Pip10]For $n \in \mathbb{N}_0$*

$$J_{n+1}^0 = \sum_{s=0}^{n}\binom{n}{s}J_s^1.$$

We end this chapter with the following theorem which is a generalization of Lemma 2.3.

Theorem 2.2. *For $r \in \mathbb{N}_0$,*

$$J_{n+1}^r = (r+1)\sum_{s=0}^{n}\binom{n}{s}J_s^{r+1} \quad \textit{where} \quad J_0^r = 1,$$

for all $n \in \mathbb{N}_0$.

Proof. Considering the set X_{n+1} we base our argument on one element of X_{n+1} say the element x_i. In all barred preferential arrangements of X_{n+1} having $r \in \mathbb{N}_0$ bars we determine the position of x_i using two entities; the section in which the element is in and the block within that section to which the element belongs. On barred preferential arrangements of X_{n+1} having r bars the element x_i can be in anyone of the $r+1$ sections, say the element is in the t^{st} section. On all barred preferential arrangements of X_{n+1} the maximum number of elements which are not in the same block as x_i is n and the minimum number is 0. Lets say there are s elements which are not in the same block as x_i in the t^{st} section. Treating the marked element as a bar, those elements in the same block with x_i will be interpreted as the first block to the right of x_i. We now have $r+1$ bars, consequently $r+2$ sections. There are $\binom{n}{s}$ ways of selecting the s elements. The s elements can be preferentially arranged in the $r+2$ sections in J_s^{r+1} ways. Hence the number of barred preferential arrangements is $(r+1)\sum_{s=0}^{n}\binom{n}{s}J_s^{r+1}$.

□

Chapter 3

Generating functions of Nelsen-Schmidt type.

Roger Nelsen and Harvey Schmidt in [NJ91] proposed the family of generating functions $\frac{e^{rx}}{2-e^x}$ where $r \in \mathbb{N}_0$, which for $r = 0$ and $r = 2$ are known to be the generating functions for the number of preferential arrangements of X_n and the number of chains in the power set of X_n respectively. They then asked, "could there be other combinatorial structures that can be associated with either the set X_n or the power set of X_n whose integer sequences are generated by members of the family $\frac{e^{rx}}{2-e^x}$ for other values of r in $\mathbb{N}_0$ other than the two values?" In this chapter we proposed an answer to this open problem of Nelsen and Schmidt. Here we propose an answer to the question for all values of r in $\mathbb{N}_0$. We further propose a much more general family of generating functions $\left(\frac{e^{rx}}{(2-e^x)^j} \text{ for } r, j \in \mathbb{N}_0\right)$ of which the family of Nelsen and Schmidt is a subset.

3.1 A family of Nelsen-Schmidt type.

In finding the total number of barred preferential arrangements of X_n having $r \in \mathbb{N}_0$ bars, here we view a barred preferential arrangement as a result of bars having been placed first and then elements being distributed into the resulting sections between the bars. We generalise Equation 24 of [Pip10] from one bar to r bars using the same kind of argument. We argue as follows: r bars result in $r+1$ sections. We assume there are w_i elements on the i^{th} section, where $0 \leq i \leq r+1$. There are $C(n; w_1, w_2, \ldots, w_{r+1}) = \frac{n!}{\prod_{i=1}^{r+1} w_i!}$ number of ways of distributing the n elements of X_n into $r+1$ sections ($C(n; w_1, w_2, \ldots, w_{r+1})$ is the multinomial coefficient). There are $J_{w_i}^0$ ways of preferentially arranging elements in the i^{th} section for $0 \leq i \leq r+1$. Hence the total number J_n^r of barred preferential arrangements of X_n having r bars is given by the following theorem.

Theorem 3.1. *For $r, n \in \mathbb{N}_0$,*

$$J_n^r = \sum_{w_1+\cdots+w_i+\cdots+w_{r+1}=n} C(n; w_1, \cdots, w_i, \cdots, w_{r+1}) \prod_{i=1}^{r+1} J_{w_i}^0, \tag{3.1}$$

where the summation is taken over all solutions of the equation $\sum_{i=1}^{r+1} w_i = n$ in non-negative integers.

Equation 3.1 is a dual of the closed form in Theorem 1.1 on Page 12 above. Theorem 1.1 is proved in [CAP13] using an argument where barred preferential arrangements are viewed as being a result of elements first being preferentially arranged into blocks, and then bars being introduced in between the blocks of preferential arrangements.

Definition 3.1. *[NM17] A* `free section` *of a barred preferential arrangement is a section which contains elements that are preferentially arranged into any number of blocks.*

What Definition 3.1 means is that, elements on a free section can be preferentially arranged into any possible number of blocks. For w_j elements on a free section, the number of possible ways of preferentially arranging the elements on the free section is $J^0_{w_j}$.

Definition 3.2. *[NM17] "A* ***restricted section*** *of a barred preferential arrangement is a section containing elements arranged into a single block."*

What Definition 3.2 means is that, for any number of elements on a restricted section, there is one way of preferentially arranging them.

Note 3.1. *Either kind of sections above, free or restricted may be empty.*

Definition 3.3. *A* ***restricted barred preferential arrangement*** *is a barred preferential arrangement which has some of its sections restricted and others free.*

We denote by $p^r_1(n)$ the number of restricted barred preferential arrangements of X_n having $r \in \mathbb{N}_0$ restricted sections and one fixed predefined free section. We choose the free section to be the $(r+1)^{st}$ section without loss of generality. In the case $r = 0$ we have $p^0_1(n) = J^0_n$. The $(r+1)^{st}$ section being a free section means that elements which go into the section can be preferentially arranged in any possible number of blocks. The r other sections being restricted sections means that each of the sections can have at most a single block. In getting the total number $p^r_1(n)$ of such restricted barred preferential arrangements we argue as follows. There are $C(n; w_1, \cdots, w_i, \cdots, w_{r+1})$ number of ways of distributing the n elements of X_n into $r+1$ sections. There is one way of preferentially arranging elements on each of the r restricted sections. There are $J^0_{w_{r+1}}$ ways of preferentially arranging elements on the $(r+1)^{st}$ section (since the section is a free section). Hence the total number $p^r_1(n)$ is given by,

$$p^r_1(n) = \sum_{w_1+\cdots+w_i+\cdots+w_{r+1}=n} C(n; w_1, \cdots, w_i, \ldots, w_{r+1}) \left[\prod_{i=1}^{r}(1)\right] J^0_{w_{r+1}}, \quad (3.2)$$

where the summation is taken over all solutions of the equation $\sum_{i=1}^{r+1} w_i = n$ in $\mathbb{N}_0$.

We denote by $G^r_1(n)$ the set of these restricted barred preferential arrangements. So $p^r_1(n) = |G^r_1(n)|$. The set $G^r_1(n)$ is a subset of the set Q^r_n of all barred preferential arrangements of X_n without any restrictions discussed in Chapter 2.

We now ask the following question.

Question.

What sort of a family of exponential generating functions do the numbers $p_1^r(n)$ (denoting the generating function by $P_1^r(x)$) have and is it related to the family of Nelsen-Schmidt type?

By (3.2), we deduce that $p_1^r(n)$ is the coefficient of $\frac{x^n}{n!}$ in the following convolution of exponential generating functions (for a discussion on convolution of generating functions see Appendix A on Page 80),

$$\left[\prod_{i=1}^{r}\left(\sum_{n_i=0}^{\infty}\frac{x^{n_i}}{n_i!}\right)\right]\left(\sum_{n_{r+1}=0}^{\infty}\frac{J^0_{n_{r+1}}x^{n_{r+1}}}{n_{r+1}!}\right)=\sum_{n=0}^{\infty}\frac{p_1^r(n)x^n}{n!}=P_1^r(x). \tag{3.3}$$

By Theorem 1.3 on Page 12 and the fact that $\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}=e^x$, Equation 3.3 becomes

$$\left[\prod_{i=1}^{r}e^x\right]\frac{1}{2-e^x}=\frac{e^{rx}}{2-e^x}=\sum_{n=0}^{\infty}\frac{p_1^r(n)\,x^n}{n!}=P_1^r(x). \tag{3.4}$$

We recognise the generating function in (3.4) as being the same as the one proposed by Nelsen and Schmidt in [NJ91], introduced in this chapter. Hence the Nelsen-Schmidt generating function for an arbitrary fixed $r \in \mathbb{N}_0$ is that of the number of restricted barred preferential arrangements of X_n, having r restricted sections and one free section. Thus in answering the question of Nelsen and Schmidt, a structure that can associate with the set X_n whose integer sequences are generated by members of the family $\frac{e^{rx}}{2-e^x}$ for all values of $r \in \mathbb{N}_0$, is restricted barred preferential arrangements of X_n having r restricted sections and one free section. This settles the open question as posed by Nelsen and Schmidt.

We now propose theorems, lemmas and conjectures on the numbers $p_1^r(n)$.

Theorem 3.2. *For $r \in \mathbb{N}$,*

$$p_1^r(n) = \sum_{s=0}^{n} \binom{n}{s} p_1^0(s) r^{n-s} \quad \textit{where} \;\; p_1^0(0) = 1,$$

for all $n \in \mathbb{N}_0$.

Proof. We recall that $p_1^r(n) = |G_1^r(n)|$, is the number of restricted barred preferential arrangements of X_n having the first r sections restricted and the $(r+1)^{st}$ section free, where $G_1^r(n)$ is the set of these conditional barred preferential arrangements. We base the proof of the theorem on the number of elements that are to go into $(r+1)^{st}$ section. We argue as follows; the minimum number of elements which are to go to the $(r+1)^{st}$ section on each $\mathfrak{B} \in G_1^r(n)$ is 0 and the maximum number is n. Lets say there are s elements which are to go into the $(r+1)^{st}$ section. There are $\binom{n}{s}$ of selecting the s elements. There are $p_1^0(s)$ ways of preferentially arranging the s elements within the section. The remaining $n-s$ elements can be preferentially arranged among the r restricted sections in r^{n-s} ways. Taking the product and summing over s we obtain the result of the theorem. □

Lemma 3.1. *For* $r \in \mathbb{N}_0$,

$$p_1^{r+1}(n) = \sum_{s=0}^{n} \binom{n}{s} p_1^r(s) \quad \textit{where} \quad p_1^r(0) = 1,$$

for all $n \in \mathbb{N}_0$.

Proof. The proof of the lemma is similar to that of Theorem 3.2. In this lemma we are dealing with $r+2$ sections, this is from the definition of $p_1^{r+1}(n)$. In proving the lemma we base our argument on a single fixed restricted section, for argument sake say its the i^{st} section. We recall that $p_1^{r+1}(n) = |G_n^{r+1}|$. We argue as follows. The maximum number of elements of X_n that can go into the i^{th} section is n and the minimum number is 0. For s elements on the i^{st} section (where $0 \leq s \leq n$), there are $\binom{n}{s}$ ways of selecting the s elements. There are $p_1^r(n)$ ways of preferentially arranging the s elements among the other $r+1$ sections other than the i^{st} section (by definition of $p_1^{r+1}(n)$). The remaining $n-s$ elements can be preferentially arranged on the i^{st} section in 1 way (since the i^{st} section is a restricted section). Taking the product and summing over s we obtain the result of the lemma. □

We will prove the result of Lemma 3.1 above in a more general form in Section 3.2 as Theorem 3.5.

Lemma 3.2. *For $r \in \mathbb{N}$,*

$$p_1^r(n) = r^n + \sum_{s=0}^{n-1} \binom{n}{s} r^s p_1^o(n-s),$$

for all $n \in \mathbb{N}$.

Proof. In proving the result of the theorem we consider two cases. Both cases will be based on the free section on elements of $G_1^r(n)$. The elements of $G_1^r(n)$ can be partitioned into two disjoint subsets. Where one subset contains those elements of $G_1^r(n)$ whose free section is empty, and the other subset containing those elements of $G_1^r(n)$ whose free section is non-empty.

In the first case, the free section being empty on $\mathfrak{T} \in G_1^r(n)$ means that the n elements are distributed on the r restricted sections. The number of ways of preferentially arranging the n elements among these r restricted sections is r^n.

In the second, case the free section being non-empty on $\mathfrak{T} \in G_1^r(n)$ means that the maximum number of elements which are not in the free section on $\mathfrak{T}$ in this case is $n-1$. Lets say there are s elements which are not in the free section. There are $\binom{n}{s}$ ways of selecting the s which are not to go to the free section. There are r^s ways of preferentially arranging the s elements among the r restricted sections. The remaining $n-s$ elements can be preferentially arranged on the free section in $p_1^0(n-s)$ ways. Taking the product and summing over s we obtain the number of elements in this case as $\sum_{s=0}^{n-1} r^s p_1^0(n-s)$. Combining the two cases we obtain the result of the lemma. □

Lemma 3.3. *For* $r \in \mathbb{N}$,

$$p_1^r(n) = p_1^{r-1}(n) + \sum_{s=0}^{n-1} \binom{n}{s} p_1^{r-1}(s) \quad \textit{where } p_1^r(0) = 1,$$

for all $n \in \mathbb{N}$.

We will generalise Lemma 3.3 in Section 3.2 as Theorem 3.7 and give a proof of the theorem.

3.2 Generalised family of Nelsen-Schmidt type.

Apart from proposing an answer to the open question of Nelsen and Schmidt, we further ask the following..

Question.

In constructing restricted barred preferential arrangements X_n having r bars, instead of the requirement we made in Section 3.1 that one fixed predefined section be a free section and the remaining r sections be restricted sections, what if we allow more that one section to be free? What sort of generating functions would such restricted barred preferential arrangements have and how would the family of such generating functions relate to that of Nelsen and Schmidt considered in Section 3.1 above?

Answering the question we argue in the following way. Constructing restricted barred preferential arrangements of X_n having $m \in \mathbb{N}_0$ bars here we require without loss of generality the first $r \in \mathbb{N}_0$ of the $m+1$ sections be restricted sections and the remaining j fixed predefined sections be free (where $j = m + 1 - r$). We denote such restricted barred preferential arrangements of $p_j^r(n)$ and the set of these arrangements by $G_j^r(n)$. The number $p_j^r(n)$ using a similar argument to that used to obtain (3.2) of Section 3.1 is,

$$p_j^r(n) = \sum_{\Omega} C(n; w_1, \cdots, w_r, , w_{r+1}, \cdots, w_{r+j}) \left[\prod_{i=1}^{r}(1)\right] \left[\prod_{s=r+1}^{r+j} J_{w_s}^0\right], \quad (3.5)$$

where the summation is taken over all solutions Ω in non-negative integers of the equation $w_1 + \cdots + w_i + \cdots + w_r + w_{r+1} + \cdots + w_s + \cdots + w_{r+j} = n$.

Using a similar argument to that used in (3.4) of Section 3.1 the generating function (denoting it by $P_j^r(x)$) for the numbers $p_j^r(n)$ is given by the following theorem.

Theorem 3.3. *For $j \in \mathbb{N}_0$ the generating function for the number $p_j^r(n)$ of restricted barred preferential arrangements of X_n having $r \in \mathbb{N}_0$ restricted sections and $j \in \mathbb{N}_0$ free sections is,*

$$P_j^r(x) = \sum_{n=0}^{\infty} \frac{p_j^r(n)\, x^n}{n!} = \frac{e^{rx}}{(2-e^x)^j}\,.$$

We observe that the family of generating functions $\frac{e^{rx}}{2-e^x}$ proposed by Nelsen and Schmidt in Section 3.1 is a special case of the family of generating functions in Theorem 3.3, this occurs when $j = 1$ on the generating function in Theorem 3.3. Hence Theorem 3.3 offers a generalised answer to the open problem of Nelsen and Schmidt.

We now propose theorems and conjectures on the numbers $p_j^r(n)$.

Theorem 3.4. *For $j \in \mathbb{N}_0$ and $r \in \mathbb{N}$,*

$$p_j^r(n) = \sum_{s=0}^{n} \binom{n}{s} r^s p_j^0(n-s) \quad \textit{where} \quad p_0^0(0) = 1,$$

for all $r \in \mathbb{N}$.

Proof. We recall that $p_j^r(n)$ is the number of restricted barred preferential arrangements of X_n having $r \in \mathbb{N}_0$ restricted sections and $j \in \mathbb{N}_0$ free sections, and the set of these restricted barred preferential arrangements being $G_j^r(n)$. In proving the theorem we view the r restricted sections as a single unit and the j free sections also as a single unit. We argue as follows, on an element $\mathfrak{Q} \in G_j^r(n)$ we assume there are s elements which are distributed within the r restricted sections (where $0 \leq s \leq n$). There are $\binom{n}{s}$ ways of selecting the s elements. The s elements can be preferentially arranging among the r sections in r^s ways. The remaining $n-s$ elements can be preferentially arrangement within the j free sections in $p_j^0(n-s)$ ways. Taking the product and summing over s we obtain the result of the theorem. □

We now generalise Lemma 3.1 of Section 3.1.

Theorem 3.5. *For $r, j \in \mathbb{N}_0$,*

$$p_j^{r+1}(n) = \sum_{s=0}^{n} \binom{n}{s} p_j^r(n-s) \qquad \textit{where} \quad p_0^r(0) = 1,$$

for all $n \in \mathbb{N}$.

Proof. In proving the result we base our argument on a fixed predefined restricted section. For argument sake we say this fixed restricted section is the i^{th} section. We assume, out of $G_j^{r+1}(n)$ elements, there are s elements on the i^{th} section (where $0 \leq s \leq n$). There are $\binom{n}{s}$ ways of choosing the s elements which are to go into the i^{th} section. There is one way of preferentially arranging the s elements on the i^{th} (since the i^{th} section is a restricted section). There are $p_j^r(n-s)$ ways of preferentially arranging the remaining $n-s$ elements among the other $r+j$ sections, of which r of them are restricted and j of them are free sections. Taking the product and summing over s we obtain the result of the theorem. □

The following theorem is a generalisation of both Lemma 2.2 of Chapter 2 and Lemma 3.2 of Section 3.1.

Theorem 3.6. *For $r \in \mathbb{N}_0$ and $j \in \mathbb{N}$,*

$$p_j^r(n) = p_{j-1}^r(n) + \sum_{s=0}^{n-1} \binom{n}{s} p_{j-1}^r(s) p_1^0(n-s) \quad \textit{where} \quad p_0^0(0) = 1,$$

for all $n \in \mathbb{N}$.

Proof. In obtaining the number $p_j^r(n)$ we base our argument on a fixed free predefined section. For argument sake we say the fixed free section is the i^{st} section. We consider two scenarios, the first one being elements of $G_j^r(n)$ having their i^{st} section empty and the second one being elements of $G_j^r(n)$ having their i^{st} section being non-empty. In the first scenario on an element $\mathfrak{P} \in G_j^r(n)$ the i^{st} section being empty means that the n elements of X_n are preferentially arranged among the other $r+j-1$ sections of $\mathfrak{P}$, of which $j-1$ of them are free sections and r of them are restricted sections. The number of such restricted barred preferential arrangements is $p^r_{j-1}(n)$ (by definition of $p^r_{j-1}(n)$).

In the second scenario the i^{st} section having at least one element means that on $\mathfrak{P} \in G^r_j(n)$ the maximum number of elements which are not on the i^{st} section is $n-1$. The total number of such elements of $G^r_j(n)$ whose i^{st} section has at least one element is given as follows. Suppose there are s elements which are not to go into the i^{st} section on elements of $G^r_j(n)$. There are $\binom{n}{s}$ ways of selecting the s. The s elements can be preferentially arranged on the $r+j-1$ other sections in $p^r_{j-1}(s)$ ways. The remaining $n-s$ elements can be preferentially arranged in the i^{st} section in $p^0_1(n-s)$ ways. Taking the product and summing over s we obtain the number $\sum_{s=0}^{n-1} \binom{n}{s} p^r_{j-1}(s) p^0_1(n-s)$. □

We now generalise Lemma 3.3 of Section 3.1 to the following theorem.

Theorem 3.7. *For $j \in \mathbb{N}_0$ and $r \in \mathbb{N}$,*

$$p^r_j(n) = p^{r-1}_j(n) + \sum_{s=0}^{n-1} \binom{n}{s} p^{r-1}_j(s) \quad \textit{where} \quad p^0_0(0) = 1,$$

for all $n \in \mathbb{N}$.

Proof. We prove the theorem in a similar way to Theorem 3.6. Here instead of basing our argument on a fixed free section we base our argument on a fixed restricted section. For a fixed predefined restricted section say the i^{st} section, we argue as follows in obtaining the number $p_j^r(n)$. We have two cases, the first one being the i^{st} section being empty and the second one being the i^{st} section being non-empty. In the first case the i^{st} section being empty on a $\mathfrak{J} \in G_j^r(n)$ means that the n elements of X_n are preferentially arranged on the other $r + j - 1$ sections, $r - 1$ of which are restricted and j are free. The number of elements of $G_j^r(n)$ having this property is $p_j^{r-1}(n)$.

In the second case, the i^{st} section being non-empty means that, there can be a maximum of $n - 1$ elements which are not to go into the i^{st} section. Suppose there are s elements which are not to go into the i^{st} section. Then there are $\binom{n}{s}$ ways of selecting them, there are $p_j^{r-1}(s)$ ways of preferentially arranging them into the $r + j - 1$ other sections. The remaining $n - s$ elements can be preferentially arranged on the i^{st} section in one way. Taking the product and summing over s we obtain $\sum\limits_{s=0}^{n-1} \binom{n}{s} p_j^{r-1}(s)$ as the number of elements of $G_j^r(n)$ whose i^{st} section is non-empty. $\square$

Lemma 3.4. *[NJ91]For $r \in \mathbb{N}_0$,*

$$p_1^{r+1}(n) = 2p_1^r(n) - r^n,$$

for all $n \in \mathbb{N}_0$.

Lemma 3.4 was proposed and proved by Nelsen and Schmidt in [NJ91] using an algebraic argument. We prove the lemma in a generalised form in the following theorem using a combinatorial argument instead.

Theorem 3.8. *For $r \in \mathbb{N}_0$ and $j \in \mathbb{N}$,*

$$p_j^{r+1}(n) = 2p_j^r(n) - p_{j-1}^r(n) \quad \textit{where} \quad p_0^0(0) = 1,$$

for all $n \in \mathbb{N}_0$.

Proof. In proving the theorem we construct the set $G_j^{r+1}(n)$ using elements of the set $G_j^r(n)$. We assume that the last section (from left to right) of each element of $G_j^r(n)$ is a free section. In obtaining the number of elements of the set $G_j^{r+1}(n)$ we argue as follows. On each element of $G_j^r(n)$ we add an extra bar $\overset{*}{|}$ to the far right of each $\mathfrak{O} \in G_j^r(n)$ to form a set $D_j^r(n)$. So the bar $\overset{*}{|}$ is to the right of the last section of $\mathfrak{A} \in D_j^r(n)$. To the left of the bar $\overset{*}{|}$ on each $\mathfrak{A} \in D_j^r(n)$ is a free section and to the right of the bar is an empty section. Also each element of the set $D_j^r(n)$ has $r+j+1$ sections due to the introduction of the extra bar $\overset{*}{|}$. We observe that the addition of the extra bar $\overset{*}{|}$ on elements of $G_j^r(n)$ does not affect counting. So $|D_j^r(n)| = p_j^r(n) = |G_j^r(n)|$. We now form a new set $R_j^r(n) = \{0,1\} \times D_j^r(n)$ which contains the same elements as the set $D_j^r(n)$ where the bar $\overset{*}{|}$ now has an index 1 and separately an index 0 (see [CAP13]). So the set $R_j^r(n)$ has twice the number of elements as the set $D_j^r(n)$, where on half of the elements of $R_j^r(n)$ the bar $\overset{*}{|}$ has the index 0 and on the other half the bar $\overset{*}{|}$ has index 1.

We now make use of elements of the set $R_j^r(n)$ to construct elements of the set $G_j^{r+1}(n)$. We base our argument on the index of the bar $\overset{*}{|}$ of each element of $R_j^r(n)$ which is either 0 or 1. We argue as follows.

I. We collect all those elements of the set $R_j^r(n)$ whose index on the bar $\overset{*}{|}$ is 0 to form a subset L. The number of elements in the set L is $p_j^r(n)$ (this is so since half of elements of $R_j^r(n)$ have the index 0). Elements of L have $r+j+1$ sections of which the $(r+j+1)^{st}$ section is empty. We interpret elements of L as those elements of $G_j^{r+1}(n)$ whose $(r+j+1)^{nd}$ section is empty.

II. On those elements of $R_j^r(n)$ whose index on the bar $\overset{*}{|}$ is 1, we shift the block of the free section closest to the bar $\overset{*}{|}$ to be the only block to the right of the bar $\overset{*}{|}$ to form a set Y (see [CAP13]). There are $p_j^r(n)$ elements on $R_j^r(n)$ whose index on the bar $\overset{*}{|}$ is 1. So $|Y| = p_j^r(n)$ with possibly empty $(r+j+1)^{st}$ sections.

When not considering the indexing on the bar $\overset{*}{|}$ on elements of Y and elements of L we observe that some elements are common elements between the two sets when interpreted as elements of $G_j^{r+1}(n)$. These common elements occur in the construction of the set Y, where the free section closest to the bar $\overset{*}{|}$ is empty. In this case there is no block to shift to the right of the bar $\overset{*}{|}$. This occurs when the n elements of the underlying set X_n are preferentially arranged on the $r+j-1$ other sections on $\Phi \in G_j^r(n)$ other than the $(r+j+1)^{st}$ section. The number of elements of $G_j^r(n)$ with this property is $p_{j-1}^r(n)$ (by definition of $p_{j-1}^r(n)$). Hence $|Y \cap L| = p_{j-1}^r(n)$. Now $|Y \cup L| = |Y| + |L| - |Y \cap L| \Rightarrow |Y \cup L| = p_j^r(n) + p_j^r(n) - p_{j-1}^r(n)$. The set $|Y \cup L|$ is the set of all restricted barred preferential arrangements of X_n having $r+1$ restricted sections and j free sections.. By definition of $G_j^{r+1}(n)$ we have $Y \cup L = G_j^{r+1}(n)$. This completes the proof. $\square$

Conjecture 3.1. *For* $r, j \in \mathbb{N}_0$,

$$p_j^r(n) = \sum_{s=0}^{\infty} \tbinom{j+s-1}{s} \tfrac{(r+s)^n}{2^{j+s}},$$

for all $n \in \mathbb{N}$.

Conjecture 3.2. *For* $r, j \in \mathbb{N}$,

$$jp_{j+1}^{r+1}(n) = p_j^r(n+1) - rp_j^r(n),$$

for all $n \in \mathbb{N}$.

Conjecture 3.3. *For* $r, j \in \mathbb{N}_0$,

$$p_1^r(n+1) = rp_1^r(n) + \sum_{s=0}^{n} \tbinom{n}{s} p_1^r(s) p_1^1(n-s),$$

for all $n \in \mathbb{N}_0$.

It would be interesting to prove the above conjectures using a combinatorial argument.

3.3 A note on the family of generating functions of Nelsen-Schmidt type for negative values.

Question.

We recall the question of Nelsen and Schmidt as discussed in Section 3.1 above. "Can there be combinatorial structures that can be associated with either the set X_n or the power set of X_n, whose integer sequences are generated by members of the family $\frac{e^{rx}}{2-e^x}$ for other values of r other than $r = 0$ and $r = 2$?" Here we ask the same question now instead of asking it for non-negative integers as in Sections 3.1 and 3.2 we now ask for negative values of r, which we denote by $\overset{*}{r}$ (where $\overset{*}{r} \in \mathbb{Z}^- = \{0, -1, -2, -3, \ldots\}$).

We recall the number $p_j^r(n)$ as defined in previous sections as the number of restricted barred preferential arrangements of X_n having r restricted sections and j free sections, where $r, j \in \mathbb{N}_0$. The set of such restricted barred preferential arrangements being denoted by $G_j^r(n)$. We now interpret restricted sections of restricted barred preferential arrangements of X_n as a single unit. Where the number restricted sections is r and the number of free sections is j. We refer to those restricted barred preferential arrangements which are having an even number of elements on the restricted sections combined as even restricted barred preferential arrangements. We denote their set and their number by $G_j^{r,e}(n)$ and $p_j^{r,e}(n)$ respectively. Similarly we refer to those restricted barred preferential arrangements which are having an odd number of elements on the restricted sections combined as odd restricted barred preferential arrangements. We denote their set and their number by $G_j^{r,o}(n)$ and $p_j^{r,o}(n)$ respectively.

Lemma 3.5. *For $r, j \in \mathbb{N}_0$,*

$$p_j^{r,e}(n) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2s} \times r^{2s} \times p_j^0(n-2s),$$

for all $n \in \mathbb{N}_0$.

Proof. In proving the lemma we consider $r+j$ sections of the arrangement, r of which are restricted, and the remaining j are free. For s elements on the restricted sections (where $0 \leq s \leq \lfloor \frac{n}{2} \rfloor$), there are $\binom{n}{2s}$ ways of choosing an even number of elements from the elements of X_n. There are r^{2s} ways of preferentially arranging the $2s$ elements on the restricted sections. The remaining $n-2s$ elements can be preferentially arranged on the j free sections in $p_j^0(n-2s)$ ways. This completes the proof. $\square$

Lemma 3.6. *For* $r, j \in \mathbb{N}_0$,

$$p_j^{r,o}(n) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2s+1} \times r^{2s+1} \times p_j^0(n-2s-1),$$

for all $n \in \mathbb{N}_0$.

Using a similar argument to that of Lemma 3.5 we obtain the result of Lemma 3.6. We end this chapter with the following theorem.

Theorem 3.9. *For fixed* $\overset{*}{r} \in \mathbb{Z}^-$ *and* $j \in \mathbb{N}_0$ *the quantity* $p_j^{\overset{*}{r}}(n) = [\frac{x^n}{n!}]\frac{e^{\overset{*}{r}x}}{(2-e^x)^j}$ *is the difference between the number of restricted barred preferential arrangements of* $G_j^r(n)$ *having an even number of elements on the restricted sections minus the number of those restricted barred preferential arrangements of* $G_j^r(n)$ *having an odd number of elements on their restricted sections.*

Proof. Treating $\frac{e^{\overset{*}{r}x}}{(2-e^x)^j}$ as a convolution, we have

$$p_j^{\overset{*}{r}}(n) = \sum_{s=0}^{n} \binom{n}{s} (-1)^s \times r^s \times p_j^0(n-s) \quad \text{for} \quad r \in \mathbb{N}_0 \tag{3.6}$$

Applying Lemmas 3.5 and 3.6 to (3.6) we obtain the result of the theorem. □

Chapter 4

A Literature Survey of Some Studies Related to the Study of Restricted Barred Preferential Arrangements.

In this chapter we do a literature survey of some studies related to the study of restricted barred preferential arrangements and their associated generating functions.

4.1 A Generalisation of Nelsen's Conjecture and Related Problems.

We now study the generating function $\frac{e^{rx}}{(2-e^x)^j}$ which was discussed in Chapter 3, here for $r, j \in \mathbb{R}$, instead of $r, j \in \mathbb{N}_0$ as in Chapter 3. In [GLW+84] Nelsen conjectured that $\sum_{k=0}^{n} \sum_{s=0}^{k} \binom{k}{s} (-1)^{k-s} (r+s)^n = \frac{1}{2} \sum_{s=0}^{\infty} \frac{(r+s)^n}{2^s}$ (where $r \in \mathbb{R}$). Three alternative algebraic proofs of the conjecture were proposed by Donald Knuth et al in [NKBW87]. Here we propose and prove an alternative identity generalising Nelsen's conjecture. We further show that such a generalisation of Nelsen's conjecture under certain restrictions of r and j, may be interpreted combinatorially using the idea of restricted barred preferential arrangements.

Lemma 4.1. *[GLW+84]**Nelsen's Conjecture.** For $r \in \mathbb{R}$,*

$$\sum_{k=0}^{n} \sum_{s=0}^{k} \binom{k}{s} (-1)^{k-s} (r+s)^n = \frac{1}{2} \sum_{s=0}^{\infty} \frac{(r+s)^n}{2^s},$$

for all $n \geq 0$.

We now generalise Nelsen's conjecture.

We let $u_j^r(n) = [\frac{x^n}{n!}] U_j^r(x)$, where $U_j^r(x) = \frac{e^{rx}}{(2-e^x)^j}$ for $r, j \in \mathbb{R}$.

Theorem 4.1. *For $r, j \in \mathbb{R}$,*

$$\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}u_{j-1}^{r+s}(n) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{u_{j-1}^{r+k}(n)}{2^k},$$

for all $n \in \mathbb{N}$.

Proof. $U_j^r(x) = \frac{e^{rx}}{(2-e^x)^j} = \frac{1}{2}\frac{e^{rx}}{(2-e^x)^{j-1}}\sum_{k=0}^{\infty}\frac{e^{xk}}{2^k} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{e^{(r+k)x}}{2^k(2-e^x)^{j-1}}$
$\implies [\frac{x^n}{n!}]\frac{e^{rx}}{(2-e^x)^j} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{u_{j-1}^{r+k}(n)}{2^k}$.
Also $U_j^r(x) = \frac{e^{rx}}{(2-e^x)^j} = \frac{e^{rx}}{(2-e^x)^{j-1}}\sum_{k=0}^{\infty}(e^x-1)^k$. Since,
$\frac{d^n}{dx^n}(\frac{e^{rx}}{(2-e^x)^{j-1}}\sum_{k=0}^{\infty}(e^x-1)^k)|_{x=0}$=0 for all $k > n$ then
$\frac{d^n}{dx^n}(\frac{e^{rx}}{(2-e^x)^{j-1}}\sum_{k=0}^{\infty}(e^x-1)^k)|_{x=0} = \frac{d^n}{dx^n}(\frac{e^{rx}}{(2-e^x)^{j-1}}\sum_{k=0}^{n}(e^x-1)^k)|_{x=0}$
$\implies [\frac{x^n}{n!}](\frac{e^{rx}}{(2-e^x)^j}) = \sum_{k=0}^{n}\sum_{s=0}^{k}(-1)^{k-s}u_{j-1}^{r+s}(n)$. The argument used in proving the theorem is the same kind of argument as that used in proving Equations (2) and (4) of [Gro62].

□

Note 4.1. *On the statement of the Theorem4.1, when $j = 1$ we have $u_j^r(n) = (r+s)^n$, which makes Theorem 4.1 corresponding to Nelsen's conjecture.*

When we restrict the parameters r and j to be non-negative integers on the generating function $\frac{e^{rx}}{(2-e^x)^j}$ which we used in proving Theorem 4.1

referring to Chapter 3, the generating functions have combinatorial interpretations of being the generating functions for the numbers $p_j^r(n)$ of restricted barred preferential arrangements. So $p_j^r(n) = u_j^r(n)$ for $r, j \in \mathbb{N}_0$.

Lemma 4.2. *[Gould&Mays[GM87]] The number of chains in the power set of X_n is,*

$$\sum_{k=0}^{n} \sum_{s=0}^{k} \binom{k}{s} (2+s)^n (-1)^{k-s},$$

for all $n \in \mathbb{N}_0$.

Lemma 4.3. *For $r \in \mathbb{N}_0$, the number of restricted barred preferential arrangements of X_n having $r + k$ restricted sections such that the k restricted sections are non-empty for $k = 0, 1, 2, \ldots, n$ is,*

$$\sum_{k=0}^{n} \sum_{s=0}^{k} \binom{k}{s} (r+s)^n (-1)^{k-s},$$

for all $n \in \mathbb{N}_0$.

Theorem 4.2. *For $r \in \mathbb{N}_0$ and $j \in \mathbb{N}$ the number of all restricted barred preferential arrangements of X_n having $r + k$ restricted sections and $j - 1$ free sections such that the k fixed restricted sections are non-empty,*

for $k = 0, 1, 2 \ldots, n$ is,

$$\sum_{k=0}^{n} \sum_{s=0}^{k} \binom{k}{s} p_{j-1}^{r+s}(n)(-1)^{k-s},$$

for all $n \in \mathbb{N}_0$.

Proof. In proving the theorem we use the same kind of argument as that used by Gould and Mays in proving Theorem 1 of [GM87] even though the work is in a different context the argument used is relevant in proving this one. When there are $r+k$ restricted sections and $j-1$ free sections. The number of restricted barred preferential arrangements of X_n such that one of the chosen k sections is empty is $p_{j-1}^{r+k-1}(n)$ and there are $\binom{k}{1}$ ways of choosing one of the k sections. The number of restricted barred preferential arrangements of X_n such that 2 of the k sections are empty is $p_{j-1}^{r+k-1}(n)$ and there are $\binom{n}{2}$ ways of choosing 2 of the k sections,,,,,,,,,,,,, Consequently by the inclusion/exclusion principle the number of restricted barred preferential arrangements of X_n such that the k sections are non-empty is $\sum_{k=0}^{n} \sum_{s=0}^{k} \binom{k}{s} p_{j-1}^{r+k-s}(n)(-1)^{k-s}$.

□

Note 4.2. *On Theorem 4.2 when* $j = 1$*, we have* $\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}(r+s)^n$*, which is one side of Nelsen's conjecture. So the theorem offers a generalised combinatorial interpretation of one side of Nelsen's conjecture.*

Note 4.3. *On the above theorem when* $r = 2$ *and* $j = 1$ *we have* $\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}(2+s)^n$ *which is one side of Gould & May's lemma above (see Lemma 4.2). This suggests that for* $r = 2$ *and* $j = 1$ *there must be a 1-1 correspondence between restricted barred preferential arrangements of* X_n *and the number of chains in the power set of* X_n*.*

Lemma 4.4. *For* $r \in \mathbb{N}$*,*

$$\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}(-1)^s p_0^{k-s+r}(n) = p_1^r(n),$$

for all $n \in \mathbb{N}$*.*

Theorem 4.3. *For* $j \in \mathbb{N}$ *and* $r \in \mathbb{N}_0$*,*

$$\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}(-1)^s p_{j-1}^{r+k-s}(n) = p_j^r(n),$$

for all $n \in \mathbb{N}$*.*

Proof. By Theorem 4.2 we have $\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}(-1)^s p_{j-1}^{r+k-s}(n)$ as the number of restricted barred preferential arrangements of X_n having $r + k$ restricted sections and $j-1$ free sections such that k fixed restricted sections are non-empty. Treating the k restricted sections as a single unit, the distribution of elements on the k restricted sections form a preferential arrangement of a subset of X_n. So the k restricted sections as a single unit, may be interpreted as a single free section. Hence the number of arrangements of X_n having $r+k$ restricted sections and $j-1$ free sections is the same as the number of restricted barred preferential arrangements of X_n having r restricted sections and j free sections. Thus $\sum_{s=0}^{k} \binom{k}{s}(-1)^s p_{j-1}^{r+k-s}(n) = p_j^r(n)$. When summing over k we obtain the result of the theorem.

□

4.2 Relation to Eulerian numbers.

In this section we relate the number $p_j^r(n)$ of restricted barred preferential arrangements to Eulerian numbers. We consider the permutation 6 2 3 4 1 5 of the set $X_6 = \{1, 2, 3, 4, 5, 6\}$. The permutation has three increasing runs, which are 6, 2-3-4 and 1-5. For $n \in \mathbb{N}$ when the set of all permutations of $X_n = \{1, 2, 3, \ldots, n\}$ is partitioned according to number of increasing runs, we obtain numbers called Eulerian numbers (see [VC95]). We denote by

$E(n,s)$ the number of all permutations of X_n having s increasing runs (where $1 \leq s \leq n$). Following is a table for the Eulerian numbers, where the entry on column i of row n is $E(n,i)$.

Table 4.1: [VC95]

n	$E(n,1)$	$E(n,2)$	$E(n,3)$	$E(n,4)$	$E(n,5)$
1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
5	1	26	66	26	1

Lemma 4.5. *[VC95] For $n \in \mathbb{N}$,*

$$p_1^0(n) = \sum_{s=1}^{n} E(n,s)2^{n-s} = \tfrac{1}{2}\sum_{s=0}^{\infty} \tfrac{s^n}{2^s}\ .$$

Theorem 4.4. *For $r \in \mathbb{N}$,*

$$p_1^r(n) = 2^r \sum_{y=1}^{n} E(n,y)2^{n-y} - 2^{r-1}\sum_{y=0}^{r-1} \tfrac{y^n}{2^y}\ ,$$

for all $n \in \mathbb{N}$.

Proof. By Theorem 4.1 and 4.3 of Section 4.1 we have,

$$p_1^r(n) = \frac{1}{2}\sum_{s=0}^{\infty} \frac{p_0^{r+s}(n)}{2^s} = \frac{1}{2}\sum_{s=0}^{\infty} \frac{(r+s)^n}{2^s}. \tag{4.1}$$

Letting $y = r + s$ on (4.1) we obtain,

$$p_1^r(n) = \frac{1}{2}\sum_{s=0}^{\infty}\frac{(r+s)^n}{2^s} = 2^{r-1}\sum_{y=r}^{\infty}\frac{y^n}{2^y} \tag{4.2}$$

We write $\sum\limits_{y=0}^{\infty}\frac{y^n}{2^y} = \sum\limits_{y=0}^{r-1}\frac{y^n}{2^y} + \sum\limits_{y=r}^{\infty}\frac{y^n}{2^y}$. It follows that $\frac{2^r}{2}\sum\limits_{y=0}^{\infty}\frac{y^n}{2^y} = 2^{r-1}\sum\limits_{y=0}^{r-1}\frac{y^n}{2^y} + 2^{r-1}\sum\limits_{y=r}^{\infty}\frac{y^n}{2^y}$. Using (4.2) we have $p_1^r(n) = \frac{2^r}{2}\sum\limits_{y=0}^{\infty}\frac{y^n}{2^y} - 2^{r-1}\sum\limits_{y=0}^{r-1}\frac{y^n}{2^y}$. Applying Lemma 4.5 we have

$$p_1^r(n) = 2^r\sum_{y=1}^{n}E(n,y)2^{n-y} - 2^{r-1}\sum_{y=0}^{r-1}\frac{y^n}{2^y}\,. \tag{4.3}$$

□

4.3 Relation to Bell numbers.

In this section we establish a relation between the number of restricted barred preferential arrangements and Bell numbers (B_n). Combinatorially the Bell numbers give the number of unordered partitions of X_n [Slo]. A definition of the Bell numbers B_n is $\sum\limits_{n=0}^{\infty}\frac{B_n\,x^n}{n!} = e^{e^x-1}$ (see [Wil, Slo]).

We consider the sequence having code A059099 on the oeis (see [Slo]); which is the sequence 1,2,7,33,198,1453,... We denote the n^{th} term of the sequence A059099 by $m(n)$ and the generating function of the sequence by $M(x)$. The generating function $M(x)$ is given by (see [Slo]);

$$M(x) = \frac{e^{e^x - 1}}{2 - e^x} \tag{4.4}$$

We recall as defined in Chapter 3 that the generating function for the number $p_1^r(n)$ of restricted barred preferential arrangements of X_n having $r \in \mathbb{N}_0$ predefined restricted sections and 1 free section is given by,

$$P_1^r(x) = \frac{e^{rx}}{2 - e^x} \,. \tag{4.5}$$

Theorem 4.5. *A bivariate generating function for the numbers $p_1^r(n)$,*

$$P(x,y) = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} p_1^r(n)\frac{x^n}{n!}\frac{y^r}{r!} = \frac{e^{e^x y}}{2-e^x} \,.$$

Proof. $P(x,y) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} p_1^r(n)\frac{x^n}{n!}\frac{y^r}{r!}$ (from definition of $P(x,y)$)

$$= \sum_{r=0}^{\infty}\left(\sum_{n=0}^{\infty} p_1^r(n)\frac{x^n}{n!}\right)\frac{y^r}{r!}$$

$$= \sum_{r=0}^{\infty}\left(\frac{e^{rx}}{2-e^x}\right)\frac{y^r}{r!} \qquad [By\ (4.5)]$$

$$= \frac{1}{2-e^x}\left(\sum_{r=0}^{\infty} e^{rx}\frac{y^r}{r!}\right)$$
$$= \frac{e^{e^x y}}{2-e^x} \qquad \square$$

Theorem 4.6. *For $n \in \mathbb{N}_0$,*

$$m(n) = \frac{1}{e}\sum_{r=0}^{\infty} \frac{p_1^r(n)}{r!}\ .$$

Proof. By Theorem 4.5 we have $P(x, y) = \sum_{r=0}^{\infty} \frac{e^{rx}}{2-e^x}\frac{y^r}{r!}$.
$\implies e^{-1}P(x, y) = \frac{e^{-1}}{2-e^x}\sum_{r=0}^{\infty}\frac{(e^x y)^r}{r!} = \frac{e^{e^x y-1}}{2-e^x}$.

Implying that,

$$e^{-1}P(x, 1) = \frac{e^{e^x-1}}{2-e^x}\,. \tag{4.6}$$

It follows that,

$$\frac{d^n}{dx^n}\left(e^{-1}P(x, 1)\right)|_{x=0} = \frac{1}{e}\sum_{r=0}^{\infty}\frac{p_1^r(n)}{r!}\,. \tag{4.7}$$

By (4.4),(4.6) and (4.7) we obtain the result of the theorem. $\square$

Theorem 4.7. *For $n \in \mathbb{N}_0$,*

$$m(n) = \sum_{s=0}^{n}\binom{n}{s}p_1^0(s)B_{n-s}\ .$$

Proof. We construct barred preferential arrangements of X_n having 1 bar in the following way. We require one fixed predefined section to be a free section and elements which are to go to the other section to be partitioned into non-empty unordered subsets in any possible way. In obtaining the total number of such barred preferential arrangements we argue as follows. There are $\binom{n}{s}$ ways of choosing s elements which are to go into the free section (where $0 \leq s \leq n$). There are $p_1^0(s)$ ways of preferentially arranging the s elements within the section. The remaining $n - s$ elements can be arranged on the second section in B_{n-s} ways. Taking the product and summing over s we obtain the number $\sum_{s=0}^{n} \binom{n}{s} p_1^0(s) B_{n-s}$. □

The author has submitted the interpretation of the sequence $m(n)$ given in Theorem 4.7 on the oeis [Slo], as the sequence A059099.

4.4 On Some Properties of the Number of Restricted Barred Preferential Arrangements.

In this section we propose two theorems generalising equations in [Gro62]. We recall from Chapter 3 that for $r, j \in \mathbb{N}_0$ the generating function

$P_j^r(x) = \frac{e^{rx}}{(2-e^x)^j}$ is that of the number $p_j^r(n)$ of restricted barred preferential arrangements of X_n having r fixed predefined sections restricted and j sections free.

Lemma 4.6. *[Gro62] For $n \in \mathbb{N}$,*

$$p_1^0(n) = \sum_{k=0}^{\infty} \nabla^k k^n = \sum_{k=0}^{\infty} \nabla^k k^n \ .$$

In the above lemma, ∇ is the backward difference operator defined as, $\nabla f(x) = f(x+1) - f(x)$ (see [Sch88]).

Theorem 4.8. *For $r \in \mathbb{N}_0$,*

$$p_1^r(n) = \sum_{k=0}^{\infty} E^r \nabla^k k^n = \sum_{k=0}^{\infty} \nabla^k E^r k^n,$$

for all $n \in \mathbb{N}$.

Proof. The following proof of the theorem uses the same kind of argument as that used in obtaining Equation 8 of [Gro62].

$P_1^r(x) = \frac{e^{rx}}{2-e^x} = \frac{e^{rx}}{1-(e^x-1)} = e^{rx} \sum_{k=0}^{\infty} (e^x - 1)^k = e^{rx} \sum_{k=0}^{\infty} \sum_{s=0}^{k} \binom{k}{s} (-1)^s (e^x)^{k-s}$

$\implies p_1^r(n) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \binom{k}{s} (-1)^s (k-s+r)^n.$

$\implies p_1^r(n) = \sum_{k=0}^{\infty} E^r \sum_{s=0}^{k} \binom{k}{s} (-1)^s (k-s)^n.$

The fact that $\nabla^k = [1 - E^{-1}]^k = \sum_{s=0}^{k} \binom{k}{s} (-1)^s E^{-s}$ (see [Sch88]),

$$\Longrightarrow \ p_1^r(n) = \sum_{k=0}^{\infty} E^r \nabla^k k^n = \sum_{k=0}^{\infty} \nabla^k E^r k^n.$$

The second equality is due to the fact that $\nabla E = E \nabla$ (see [Sch88]). □

On Lemma 2.1 of Chapter 2 we have illustrated a four cycle property satisfied by the last digit of the number of barred preferential arrangements, here we prove the result for the number of restricted barred preferential arrangements, which in a certain way can be viewed as a generalisation of barred preferential arrangements.

Lemma 4.7. *[Gro62] For a fixed $s \in \mathbb{N}_0$ the following congruence holds,*

$$s^{n+4} - s^n \ \equiv \ 0 \, mod \, 10 \ ,$$

for all $n \in \mathbb{N}$.

Lemma 4.8. *[Gro62] The last digit of the sequence $p_1^0(n)$ has a four cycle for all $n \in \mathbb{N}$.*

Theorem 4.9. *For fixed $r, j \in \mathbb{N}_0$ the last digit of the sequence $p_j^r(n)$ has a four cycle for all $n \in \mathbb{N}$.*

Proof. The theorem and its proof are generalisations of an equation in [Gro62] using the same kind of argument. Showing that the last digit of $p_j^r(n)$ has a four cycle is equivalent to showing that

$p_j^r(n+4) - p_j^r(n) \equiv 0 \bmod 10$ (see [Gro62]).

We have $P_j^r(x) = \frac{e^{rx}}{(2-e^x)^j} = \frac{1}{2^j}\sum_{s=0}^{\infty} \frac{\binom{-j}{s}(-1)^s e^{(r+s)x}}{2^s}$.

$\implies \frac{d^n}{dx^n}\left(P_j^r(x)\right)|_{x=0} = \frac{1}{2^j}\sum_{s=0}^{\infty} \frac{\binom{-j}{s}(-1)^s (r+s)^n}{2^s} = p_j^r(n)$. We make the change of variable $c = r+s$. So $p_j^r(n) = \frac{1}{2^j}\sum_{c=r}^{\infty} \frac{\binom{-j}{c-r}(-1)^{c-r} c^n}{2^{c-r}}$.

$\implies p_j^r(n+4) = \frac{1}{2^j}\sum_{c=r}^{\infty} \frac{\binom{-j}{c-r}(-1)^{c-r} c^{n+4}}{2^{c-r}}$.

So

$$p_j^r(n+4) - p_j^r(n) = \frac{1}{2^j}\sum_{c=r}^{\infty} \frac{\binom{-j}{c-r}(-1)^{c-r}}{2^{c-r}}[c^{n+4} - c^n] \qquad (4.8)$$

Applying Lemma 4.7 to the expression $c^{n+4} - c^n$ on (4.8) we obtain the result of the theorem. □

We now illustrate the four cycle property in the theorem for some well known integer sequences.

Illustration 1.

We consider the sequence $p_1^2(n)$ for $n \geq 1$ generated by the generating function $\frac{e^{2x}}{2-e^x}$.

The last digit of the sequence has the four cycle 1-3-1-1-3. The sequence for $n \in \mathbb{N}_0$ has the code A007047 in [Slo] (the on-line encyclopedia of integer

Table 4.2

n	$p_1^2(n)$
1	3
2	11
3	51
4	299
5	2163
6	18731
7	189171
8	2183339
9	28349043
10	408990251
$\vdots$	$\vdots$

sequences). The sequence gives the number of chains in the power set of X_n (see [NJ91]).

Illustration 2.

Here we illustrate the four cycle property of the last digit of the sequence $p_1^3(n)$ whose generating function is $\frac{e^{3x}}{2-e^x}$.

Table 4.3

n	$p_1^3(n)$
1	4
2	18
3	94
4	582
5	4294
6	37398
7	378214
8	4366422
9	56697574
10	817979478
$\vdots$	$\vdots$

The sequence has the four cycle property $4-8-4-2$. The author submitted the sequence on the oeis as A259533.

4.5 Relation to Fuzzy Sets.

We use $I = [0, 1]$ the unit interval of real numbers. For a given set $X_n = \{x_1, x_2, \ldots, x_n\}$ a fuzzy subset ω_n of X_n is a mapping $\omega_n : X_n \to [0, 1]$ (see [Zad65]). We denote by I^{X_n} the set of all fuzzy subsets of X_n.

Definition 4.1. *[MM05] Two fuzzy subsets ω_n and β_n of X_n are equivalent if and only if,*

I. $\omega_n(x_i) = 1 \iff \beta_n(x_i) = 1.$

II. $\omega_n(x_i) \geq \omega_n(x_j) \iff \beta_n(x_i) \geq \beta_n(x_j).$

III. $\omega_n(x_i) = 0 \iff \beta_n(x_i) = 0.$

The relation in Definition 4.1 is an equivalence relation [MM05]. We denote by $F(n)$ the number of resultant equivalence classes. We recall from Chapter 3 that $p_j^r(n)$ is the number of restricted barred preferential arrangements of X_n having $r \in \mathbb{N}_0$ restricted sections and $j \in \mathbb{N}_0$ free sections, whose generating function is denote by $P_j^r(x) = \frac{e^{rx}}{(2-e)^j}$. The exponential generating function of $F(n)$ is $\frac{e^{2x}}{2-e^x}$ (see [Mur06a]). This implies that $F(n) = p_1^2(n)$.

Theorem 4.10. *For $j \in \mathbb{N}$,*

$$p_j^r(n) = p_{j-1}^r(n) + \sum_{s=0}^{n-1} \binom{n}{s} p_j^r(s) \ ,$$

for all $n \in \mathbb{N}$.

Proof. The theorem is a generalisation of Equation 3.4 of [Mur06a]. By

Theorem 4.1 and 4.3 of Section 4.1 we have,

$$p_j^r(n) = \frac{1}{2}\sum_{s=0}^{\infty}\frac{p_{j-1}^{r+s}(n)}{2^s}\,. \tag{4.9}$$

This implies that (see [Gro62]);

$$\sum_{m=0}^{n-1}\binom{n}{m}p_j^r(n-m) = \frac{1}{2}\sum_{s=0}^{\infty}\left[\sum_{m=0}^{n-1}\tbinom{n}{m}p_{j-1}^{r+s}(n-m)\right]\frac{1}{2^s}\,.$$

Which implies that (see [Gro62]);

$$\sum_{m=0}^{n-1}\binom{n}{m}p_j^r(n-m) = \frac{1}{2}\sum_{s=0}^{\infty}\left[\sum_{m=0}^{n}\tbinom{n}{m}p_{j-1}^{r+s}(n-m)\times 1^s - 1\right]\frac{1}{2^s}\,. \tag{4.10}$$

On the convolution $P_{j-1}^{r+s}(x)\times P_0^1(x)$ of $P_{j-1}^{r+s}(x) = \frac{e^{rx}}{(2-e^x)^{j-1}}$ and $P_0^1(x) = e^x$ the term $[\frac{x^n}{n!}][P_{j-1}^{r+s}(x)\times P_0^1(x)]$ is (by convolution);

$$p_{j-1}^{r+s+1}(n) = \sum_{m=0}^{n}\binom{n}{m}p_{j-1}^{r+s}(n-m)\times 1^s$$

So (4.10) becomes,

$$\sum_{m=0}^{n-1} \binom{n}{m} p_j^r(n-m) = \frac{1}{2} \sum_{s=0}^{\infty} \left[p_{j-1}^{r+s+1}(n) - 1 \right] \frac{1}{2^s} \, .$$

Hence,

$$\sum_{m=0}^{n-1} \binom{n}{m} p_j^r(n-m) = \frac{1}{2} \sum_{s=0}^{\infty} \frac{p_{j-1}^{r+s+1}(n)}{2^s} - \frac{1}{2} \sum_{s=0}^{\infty} \frac{1}{2^s} \, .$$

It follows that,

$$\sum_{m=0}^{n-1} \binom{n}{m} p_j^r(n-m) = \frac{1}{2} \sum_{s=0}^{\infty} \frac{p_{j-1}^{r+s+1}(n)}{2^s} - 1 \, .$$

We let $s + 1 = p$,

$$\sum_{m=0}^{n-1} \binom{n}{m} p_j^r(n-m) = \sum_{p=1}^{\infty} \frac{p_{j-1}^{r+p}(n)}{2^p} - 1 \, .$$

Which is,

$$\sum_{m=0}^{n-1} \binom{n}{m} p_j^r(n-m) = \sum_{p=0}^{\infty} \frac{p_{j-1}^{r+p}(n)}{2^p} - p_{j-1}^r(n) - 1 \, .$$

By (4.9) we have (see [Gro62]),

$$\sum_{m=0}^{n-1} \binom{n}{m} p_j^r(n-m) = 2p_j^r(n) - p_{j-1}^r(n) - 1 \,.$$

This implies that,

$$p_j^r(n) = \sum_{m=1}^{n} \binom{n}{m} p_j^r(n-m) + p_{j-1}^r(n) \,.$$

It follows that,

$$p_j^r(n) = \sum_{s=0}^{n-1} \binom{n}{s} p_j^r(s) + p_{j-1}^r(n) \,.$$

□

Corollary 4.1. *[Gro62] For $j \in \mathbb{N}$,*

$$p_1^0(n) = \sum_{s=1}^{n-1} \tbinom{n}{s} p_1^0(n-s) + 1$$

for all $n \in \mathbb{N}$.

Corollary 4.2. *[Mur06a] For $j, n \in \mathbb{N}$,*

$$F(n+1) = \sum_{s=0}^{n} \binom{n+1}{s} F(s) + 2^{n+1} ,$$

for all $n \in \mathbb{N}_0$.

Proof. On the theorem when $r = 2$ and $j = 1$ we have,

$$p_1^2(n) = p_0^2(n) + \sum_{s=0}^{n-1} \binom{n}{s} p_1^2(s) .$$

This implies that,

$$p_1^2(n) = 2^n + \sum_{s=0}^{n-1} \binom{n}{s} p_1^2(s) .$$

This is so since the sequence $p_0^2(n)$ has the exponential generating function e^{2x}.

Letting $n+1$ equal to n we have,

$$p_1^2(n+1) = 2^{n+1} + \sum_{s=0}^{n} \binom{n+1}{s} p_1^2(s) . \tag{4.11}$$

Substituting $p_1^2(n)$ by $F(n)$ we obtain the result. □

We end this chapter with three theorems of which the three theorems follow from the property that the generating function of $F(n)$ is $\frac{e^{2x}}{2-e^x}$ (see [Mur06a]).

Theorem 4.11. *For* $n \in \mathbb{N}_0$,

$$F(n) = \sum_{s=0}^{n} \binom{n}{s} p_1^1(s) \ .$$

The theorem follows from convolution of e^x and $\frac{1}{2-e^x}$.

Theorem 4.12. *For* $n \in \mathbb{N}_0$,

$$F(n) = p_1^1(n) + \sum_{s=1}^{n} \binom{n}{s} p_1^1(n-s) \ .$$

The theorem follows from convolution of e^x and $\frac{e^x}{2-e^x}$.

Theorem 4.13. *For* $n \in \mathbb{N}_0$,

$$F(n) = 2^n + \sum_{s=1}^{n} \binom{n}{s} p_1^0(s) 2^{n-s} \ .$$

The theorem follows from convolution of e^{2x} and $\frac{1}{2-e^x}$.

Chapter 5

Poly-Bernoulli Numbers.

In this chapter we study the generating function $\frac{e^{rx}}{(2-e^{j})}$ which was discussed in Chapter 3, generalising that of Nelsen and Schmidt. Here we study the generating function for r and j now both being non-positive integers. We relate the generating function to poly-Bernoulli numbers for non-positive values of r and j. We also interpret poly-Bernoulli numbers as giving the number of restricted barred preferential arrangements.

5.1 Poly-Bernoulli numbers.

Clearly the requirement that r and j be non-positive integers on the generating function $\frac{e^{rx}}{(2-e^x)^j}$ is equivalent to having r and j being non-negative integers on the generating function $\frac{(2-e^x)^j}{e^{rx}}$. We denote, $P_{-j}^{-r}(x) = \frac{(2-e^x)^j}{e^{rx}}$ and $p_{-j}^{-r}(n) = [\frac{x^n}{n!}]P_{-j}^{-r}(x)$, for $r, j \in \mathbb{N}_0$.

Poly-Bernoulli numbers (denoting them by B_n^k, for $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$) seem to first appear in [Kan97], in which the author defined them as,

$\sum\limits_{n=0}^{\infty} B_n^k \frac{x^n}{n!} = \frac{Li_k(1-e^{-x})}{1-e^{-x}}$, where $Li_k(x)$ is a poly-logarithm defined as

$Li_k(x) = \sum\limits_{s=1}^{\infty} \frac{x^s}{s^k}$ (for a fixed $k \in \mathbb{Z}$).

In this section we establish relationships between poly-Bernoulli numbers of index -2 and the number of restricted barred preferential arrangements.

The poly-Bernoulli numbers B_n^{-2} of index -2 have the code A027649 on the oeis [Slo].

A closed form for B_n^{-2} (see [Kam13]),

$$B_n^{-2} = 2 \times 3^n - 2^n \quad where \quad n \in \mathbb{N}_0 \, . \tag{5.1}$$

The formula in (5.1) also appear in [hay09] in a different context without reference to poly-Bernoulli numbers.

Theorem 5.1. *For* $r \in \mathbb{N}_0$*,*

$$p_{-j}^{-r}(n) = (-1)^n B_n^{-2} \ ,$$

for all $n \in \mathbb{N}_0$*.*

Proof. The result follows from the generating function

$P_{-j}^{-r}(x) = \frac{2-e^x}{e^{rx}}$ of the numbers $p_{-j}^{-r}(n)$. □

Theorem 5.2. *For* $j \in \mathbb{N}$ *and natural number* $r \geq 3$*,*

$$p_{j-1}^{r-3}(n) = \sum_{s=0}^{n} \binom{n}{s} (-1)^s B_s^{-2} \times p_j^r(n-s) \ ,$$

for all $n \in \mathbb{N}_0$*.*

Proof. The theorem follows from the convolution of the generating functions

$\frac{2-e^x}{e^{3x}}$ and $\frac{e^{rx}}{(2-e^x)^j}$, where $j \in \mathbb{N}$ and natural number $r \geq 3$. □

5.2 Multi-poly-Bernoulli numbers

Arakawa and Kaneko in [AK99] generalised poly-Bernoulli numbers to multi-poly-Bernoulli numbers (denoting them by $B_n^{(j_1,\ldots,j_b)}$ for $j_1, j_2, \ldots, j_b \in \mathbb{Z}$) in the following way: $\sum_{n=0}^{\infty} B_n^{(j_1,\ldots,j_b)} \frac{x^n}{n!} = \frac{Li_{j_1\ldots j_b}(1-e^{-x})}{(1-e^{-x})}$, where $Li_{j_1\ldots j_b}(x) = \sum_{0<s_1<\cdots<s_b} \frac{x^{s_b}}{s_1{}^{j_1}\cdots s_b{}^{j_b}}$. Here we show how multi-poly-Bernoulli numbers can be related to the number $p_j^r(n)$ of restricted barred preferential arrangements. We then show how a four cycle of last digit property satisfied by the numbers $p_j^r(n)$ as shown in Theorem 4.9 is also satisfied by the multi-poly-Bernoulli numbers.

Theorem 5.3. *[Kam13] A multivariate generating function for multi-poly-Bernoulli numbers,*

$$\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_b=0}^{\infty} \times \sum_{n=0}^{\infty} B_n^{(-j_1,\ldots,-j_b)} \frac{r_1^{j_1}}{j_1!} \frac{r_2^{j_2}}{j_2!} \cdots \frac{r_b^{j_b}}{j_b!} \frac{m^n}{n!}$$
$$= \frac{1}{(e^{-r_1-r_2\cdots-r_b}+e^{-m}-1)(e^{-r_2\cdots-r_b}+e^{-m}-1)\cdots(e^{-r_b}+e^{-m}-1)}\ .$$

Corollary 5.1. *For fixed $b \in \mathbb{N}$,*

$$B_n^{(-j,\overbrace{0,0,\cdots,0}^{b-1})} = \sum_{s=0}^{n} \binom{n}{s} B_s^{(\overbrace{0,0,\cdots,0}^{b-1})} \times B_{n-s}^{-j},$$

for all $\in \mathbb{N}_0$.

Proof. On Theorem 5.3 when we let $r_2 = r_3 = \cdots = r_b = 0$ we obtain,

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} B_n^{(-j,\overbrace{0,0,\cdots,0}^{b-1})} \frac{r^j}{r!}\frac{x^n}{n!} = e^{(b-1)x}\left(\frac{1}{e^{-r}+e^{-x}-1}\right). \tag{5.2}$$

The corollary follows from (5.2) by the convolution property. □

Theorem 5.4. *[Kam13] For a fixed $b \in \mathbb{N}$ and $j_1, j_2, \ldots, j_b \in \mathbb{N}_0$ such that $(j_1, j_2, \ldots, j_b) \neq (0, 0, \ldots, 0)$. Let $j = j_1 + j_2 + \cdots + j_b$. Then the following identity holds*

$$B_n^{(-j_1,\ldots,-j_b)} = \sum_{s=1}^{j} \mu_s^{(j_1,\ldots,j_b)}(s+b)^n$$

Where $\mu_s^{(j_1,\ldots,j_b)}$ are integers recursively defined in the following way.

I. $\mu_s^{(j_1)} = (-1)^{s+j_1} s! \left\{ {j_1 \atop s} \right\}$

II. $\mu_s^{(j_1,\ldots,j_{b-1},0)} = \mu_s^{(j_1,\ldots,j_{b-1})}$

III. $\mu_s^{(j_1,\ldots,j_{b-1},j_b+1)} = (s+b-1)\mu_{s-1}^{(j_1,\ldots,j_{b-1},j_b)} - s \times \mu_s^{(j_1,\ldots,j_{b-1},j_b)}$

$$Where\ \mu_0^{(j_1,\ldots,j_{b-1},j_b)} = \begin{cases} 1 & if\ \ (j_1,\ldots,j_{b-1},j_b) = (0,0,\ldots,0) \\ \\ 0 & otherwise \end{cases}$$

$and\ \mu_s^{(j_1,\ldots,j_{b-1},j_b)} = 0\ for\ all\ s > j.$

By the above theorem we have the following results.

$$B_n^{(-2)} = 2 \times 3^n - 2^n$$

$$B_n^{(-2,0)} = 2 \times 4^n - 3^n$$

$$B_n^{(-2,0,0)} = 2 \times 5^n - 4^n$$

$$B_n^{(-2,0,0,0)} = 2 \times 6^n - 5^n$$

$$B_n^{(-2,0,0,0,0)} = 2 \times 7^n - 6^n$$

Inductively we have,

$$B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})} = 2 \times (3+b)^n - (3+b-1)^n \quad where\ \ b \in \mathbb{N}_0\,. \qquad (5.3)$$

Theorem 5.5. *For* $r, b \in \mathbb{N}_0$

$$p_1^{3+b}(n) = \sum_{s=1}^{n} \tbinom{n}{s} (-1)^{s+1} B_s^{(-2,\overbrace{0,0,\cdots,0}^{b})} \times p_1^{3+b}(n-s),$$

for all $n \in \mathbb{N}_0$.

Proof. We recall the generating function $P_{-1}^{-r}(x) = \frac{2-e^x}{e^{rx}}$ for the numbers $p_{-j}^{-r}(n)$ as defined in Section 5.1. For $r \geq 3$ the generating function $P_{-1}^{-r}(x)$ can be rewritten as follows; $P_{-1}^{-(3+b)}(x) = \frac{2-e^x}{e^{(3+b)x}}$, where $b \in \mathbb{N}_0$.

$\Longrightarrow\ p_{-1}^{-(3+b)}(n) = (-1)^n[2 \times (3+b)^n - (3+b-1)^n]$

So by (5.3)

$$p_{-1}^{-(3+b)}(n) = (-1)^n B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})}\,. \tag{5.4}$$

Since the generating function of $(-1)^n B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})}$ and that of the number $p_1^{3+b}(n)$ of restricted barred preferential arrangements are reciprocals then by the reciprocal property of generating functions (for a discussion on the reciprocal property of generating functions see Appendix A on Page 80) we have,

$$p_1^{3+b}(n) = \sum_{s=1}^{n} \binom{n}{s} (-1)^{s+1} B_s^{(-2,\overbrace{0,0,\cdots,0}^{b})} \times p_1^{3+b}(n-s)\,. \tag{5.5}$$

□

Theorem 5.6. *For fixed $b \in \mathbb{N}_0$, on restricted barred preferential arrangements of X_n having $3+b$ bars where all the sections are restricted. For two fixed sections (i^{st} and j^{st}) the poly-Bernoulli number $B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})}$ is the number of restricted barred preferential arrangements of X_n such that the i^{st} or the j^{st} section is empty.*

Proof. We consider restricted barred preferential arrangements of X_n having $3+b$ bars where all the sections are restricted. We fix two sections (the i^{st} and the j^{st} section). The number of those restricted barred preferential arrangements whose i^{st} section is empty is $(3+b)^n$ and the number of those arrangements whose j^{st} section is empty is also $(3+b)^n$. The number of those restricted barred preferential arrangements whose i^{st} and j^{st} section are empty is $(3+b-1)^n$. Hence the number of those barred preferential arrangements whose i^{st} or j^{st} section are empty is,

$$2 \times (3+b)^n - (3+b-1)^n .$$

Using (5.3) (see Page 75) we obtain the result of theorem. □

Theorem 5.7. *For $n \in \mathbb{N}$ and fixed $b \in \mathbb{N}_0$ the last digit of the sequence* $B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})}$ *has a four cycle.*

Proof. The theorem and its proof are generalisations of an equation in [Gro62] using the same kind of argument. To show that the sequence $B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})}$ has a four cycle we need to show that $B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})}$ and $B_{n+4}^{(-2,\overbrace{0,0,\cdots,0}^{b})}$ have the same last digit. We need to show that $B_{n+4}^{(-2,\overbrace{0,0,\cdots,0}^{b})} - B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})}$ is divisible by 10 (see [Gro62]). From (5.3) $B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})} = 2 \times (3+b)^n - (3+b-1)^n$. Now, $B_{n+4}^{(-2,\overbrace{0,0,\cdots,0}^{b})} - B_n^{(-2,\overbrace{0,0,\cdots,0}^{b})} = 2[(3+b)^{n+4} - (3+b)^n] - [(2+b)^{n+4} - (2+b)^n]$. By Lemma 4.7 of Chapter 4 (see page 60) both $[(3+b)^{n+4} - (3+b)^n]$ and $[(2+b)^{n+4} - (2+b)^n]$ are divisible by 10. This completes the proof ☐

We conclude this chapter and the thesis by discussing some problems that arose.

Firstly the concept of barred preferential arrangements may be generalised by replacing the underlying set with a multi-set (a set with certain elements repeated). The sequences and their combinatorial parameters, their generating functions, closed forms, and recurrence relations of sequences arising from counting barred preferential arrangements of such multi-sets generalising those studied in [NM15a, NM15b, NM17] would prove to be interesting to study.

Secondly as generating functions can be viewed as analytic objects (mappings of the complex plane into itself) it would be interesting to investigate the asymptotic behaviour of the integer sequences arising from counting of restricted barred preferential arrangements. The singularities of the associated generating functions determine their coefficients in an asymptotic way, as in [Gro62, CAP13].

Thirdly MacMahon in [Mac90] has shown a way on how the generating function

$P_1^0(x) = \frac{1}{2-e^x}$ can be interpreted in a graph theoretic context using the idea of yokes and chains. What could be done is to take up the argument to the general generating function $P_1^r(x) = \frac{e^{rx}}{2-e^x}$ of Nelsen and Schmidt. We suspect when you glue r of these yoke-chain graphs, the generating function for the number of yoke-chains is related to the Nelsen-Schmidt generating function $P_1^r(x) = \frac{e^{rx}}{2-e^x}$ and possibly even to the generating function $P_j^r(x) = \frac{e^{rx}}{(2-e^x)^j}$.

Appendix A

Properties of Generating Functions

Convolution of ordinary generating functions.

Some of standard books in combinatorics which include a discussion on convolution of generating functions are Riodarn [Rio03], Stanley [Sta97], Comtet [Com74] and Wilf [Wil].

Below we discuss how a convolution of generating functions is done.

• We consider the three ordinary generating functions; $X_1 = \sum_{n_1=0}^{\infty} b_{n_1} x^{n_1}$, $X_2 = \sum_{n_2=0}^{\infty} b_{n_2} x^{n_2}$ and $X_3 = \sum_{n_3=0}^{\infty} b_{n_3} x^{n_3}$. A convolution of the three generating functions is the generating function $X_1 \times X_2 \times X_3$ defined by

$$X_1 \times X_2 \times X_3 = \left(\sum_{n_1=0}^{\infty} b_{n_1} x^{n_1}\right) \times \left(\sum_{n_2=0}^{\infty} b_{n_2} x^{n_2}\right) \times \left(\sum_{n_3=0}^{\infty} b_{n_3} x^{n_3}\right)$$

$$= \sum_{n_1,n_2,n_3 \geq 0} b_{n_1} \times b_{n_2} \times b_{n_3} \times x^{n_1+n_2+n_3}$$

We let $n_1 + n_2 + n_3 = n$.

$$So\ X_1 \times X_2 \times X_3 = \sum_{n=0}^{\infty} \left(\sum_{n_1+n_2+n_3=n} b_{n_1} \times b_{n_2} \times b_{n_3} \right) x^n ,$$

where the inner summation is taken over all solutions of the equation

$n_1 + n_2 + n_3 = n$ in non-negative integers.

• In general $[x^n]\left(X_1 \times X_2 \times \cdots \times X_r\right) = \sum_{n_1+n_2+\cdots+n_r=n} b_{n_1} \times b_{n_2} \times \cdots \times b_{n_r}$.

Convolution of Exponential Generating Functions.

We consider the three exponential generating functions $X_1 = \sum_{n_1=0}^{\infty} \frac{b_{n_1} x^{n_1}}{n_1!}$, $X_2 = \sum_{n_2=0}^{\infty} \frac{b_{n_2} x^{n_2}}{n_2!}$ and $X_3 = \sum_{n_3=0}^{\infty} \frac{b_{n_3} x^{n_3}}{n_3!}$. Their convolution is the generating function $X_1 \times X_2 \times X_3$ defined as

$$X_1 \times X_2 \times X_3 = \left(\sum_{n_1=0}^{\infty} \frac{b_{n_1} x^{n_1}}{n_1!}\right) \times \left(\sum_{n_2=0}^{\infty} \frac{b_{n_2} x^{n_2}}{n_2!}\right) \times \left(\sum_{n_3=0}^{\infty} \frac{b_{n_3} x^{n_3}}{n_3!}\right),$$

$$= \sum_{n_1,n_2,n_3 \geq 0} \left(\frac{b_{n_1}}{n_1!} \times \frac{b_{n_2}}{n_2} \times \frac{b_{n_3}}{n_3!}\right) x^{n_1+n_2+n_3}.$$

We let $n_1 + n_2 + n_3 = n$.

$$So\ X_1 \times X_2 \times X_3 = \sum_{n=0}^{\infty} \left(\sum_{n_1+n_2+n_3=n} \frac{1}{n_1! \times n_2! \times n_3!} b_{n_1} \times b_{n_2} \times b_{n_3}\right) x^n,$$

$$= \sum_{n=0}^{\infty} \left(\sum_{n_1+n_2+n_3=n} \frac{n!}{n_1! \times n_2! \times n_3!} b_{n_1} \times b_{n_2} \times b_{n_3}\right) \frac{x^n}{n!},$$

where the inner summation is taken over all solutions of the equation

$n_1 + n_2 + n_3 = n$ in non-negative integers.

• In general

$$[\tfrac{x^n}{n!}]\left(X_1 \times X_2 \times \cdots \times X_r\right) = \sum_{n_1+n_2+\cdots+n_r=n} \frac{n!}{n_1! \times n_2! \times \cdots \times n_r!} \times b_{n_1} \times b_{n_2} \times \cdots \times b_{n_r}.$$

Reciprocal of a Generating Function.

Proposition A.1. *[AIKZ14, Wil, Com74] A reciprocal* $\frac{1}{p(x)}$ *of a formal power series* $p(x) = \sum_{n=0}^{\infty} c_n \times x^n$ *exists if* $c_0 \neq 0$*, where* $\frac{1}{p(x)}$ *is defined as* $\frac{1}{p(x)} = \sum_{n=0}^{\infty} c_n^* \times x^n$ *such that* $c_n^* = \frac{-1}{c_0} \sum_{s=1}^{n} c_s c_{n-s}^*$ *with* $c_0^* = \frac{1}{c_0}$.

Appendix B

List of symbols/Abbreviations.

- $\mathbb{N}_0$={0,1,2,3,...}
- $\mathbb{Z}^-$={0,-1,-2,-3,...}
- $\mathbb{R}, \mathbb{N}$ have the usual meaning of being real and natural numbers.
- Q_n^r : the set of all barred preferential arrangements of an n-element set having r bars.
- J_n^r : the number of all barred preferential arrangements of an n-element set having r bars.
- $G_1^r(n)$: the number of barred preferential arrangements of an n - element set having r fixed sections restricted and one section being a free section.

- $G^r_j(n)$: the number of barred preferential arrangements of an n - element set having r fixed sections being restricted sections and j sections being free sections.

- $\left\{ {n \atop s} \right\}$: Stirling numbers of the second kind.

- We denote as $[\frac{x^n}{n!}]f(x)$ the coefficient of $\frac{x^n}{n!}$ on a generating function $f(x)$.

- oeis : the on-line encyclopedia of integer sequences.

- $\mathbb{Z}$: the set of all integers.

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