SOME ASPECTS

OF THE

THEORY, APPLICATION, AND COMPUTATION

OF

GENERALISED INVERSES OF MATRICES

PATRICIA C. CRETCHLEY

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C O N T E N T S

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I N T R O D U C T I O N

The idea of generalising the classical notion of the inverse of a non-singular matrix arose as far back as in 1920, but it was not until the late fifties that the development of the theory gained any impetus. Since then, as is the case in the development of many new concepts, work done in parallel in various parts of the world has resulted in a great deal of untidiness in the literature : confusion over terminology, and even duplication of theory.

More recently, however, some attempts have been made to bring together people active in the field of generalised inverses, in order to reach consensus on some aspects of definition and terminology, and to publish more general works on the subject. Towards this purpose, a symposium on the theory and application of generalised inverses of matrices was held in Lubbock, Texas, and its proceedings published in 1968 (see [25]). A few other works of this nature (see [4], [19a]) have appeared, but the bulk of the literature still comprises numerous diverse papers offering further ideas on the theoretical properties which these matrices have, and drawing attention to their application in areas of statistics, numerical analysis, filtering, modern control and estimation theory, pattern recognition and many others.

This essay offers a look at generalised inverses in the following way: firstly a broad basis and background is established in the first three chapters to provide greater understanding of the motivation for the remaining chapters, where the approach then changes to become far more detailed. Within this general framework, Chapter 1 offers a brief glimpse of the history and development of work in the field.

In Chapter 2 some of the most significant properties of these inverses are described, while in Chapters 3 and 4 and 5 attention is given to interesting and remarkable computational algorithms relating to generalised inverses (some well suited to machine processing). The material of Chapters 4 and 5 is largely due to Decell, Stallings and Boullion, and Tanabe, in [6], [24] and [27], respectively, while the source of material for the first three chapters is the literature generally, with Penrose's two papers providing a rough framework for Chapters 1 and 2 (see [17]).

CHAPTER 1

THE HISTORY AND DEVELOPMENT OF GENERALISED INVERSES OF MATRICES

Way back in 1920, E.H. Moore first established the existence, uniqueness, and even form, of what he called the "general reciprocal" of an arbitrary matrix, while working within the context of integral and differential operators (see [14] 1920). He developed this work further in 1935 (see [14] 1935), while in 1936, van Neumann provided an algebraic basis for this work in his own work in regular rings (see [32]). No major contribution to this concept was made until the fifties. In 1955 Roger Penrose, who worked quite independently of Moore's results, gave in an algebraic setting a definition different but equivalent to Moore's for a generalised inverse of an arbitrary matrix (see [17]).

In 1956, in [17], Penrose noted the remarkable "least square" property of his generalised inverse when considering the solution of inconsistent systems of equations. A little earlier, in 1951, Bjerhammar quite unaware of Moore's work, had found exactly this "least square" property of "reciprocal matrices" while working with geodetic calculations (see [2]).

Since these significant works, many different definitions of generalised inverses have been given and the merits of each expounded. A great number of these, however, are found to be weakened variations of Penrose's basic definition, which was based on the following four equations:

(1) $AXA = A$

(2) $XAX = X$

(3) $(AX)^* = AX$

(4) $(XA)^* = XA$

(where M^* denotes the conjugate transpose of M).

He proved that for an arbitrary complex matrix A, equations (1) to (4) have a unique solution which he called the "generalised inverse", A^+, of A. It has since been found, however, that different combinations of these equations define matrices which have some of the properties which one would hope a generalisation of the inverse of a matrix might have. This has resulted in a wide acceptance of these "weaker" definitions, and the use of the term "generalised inverse" to describe any matrix which satisfies (1), the term largely due to Rao.

Rohde (see [21]) gathered many of these definitions together, establishing a hierarchy of generalised inverses. Although there is still no universal acceptance of terminology, we summarise two of the more widely used notation schemes below. The first given is that suggested by Boullion and Odell in [4] , while the second is that described by Pringle in [18] .

Defining equations	Name and symbol	Alternative Name and Symbol
(1)	generalised inverse A^g	g_1 - inverse X^{g_1}
(1) and (2)	reflexive g.i. A^r	g_2 - inverse X^{g_2}
(1),(2) and (3)	right weak g.i. A^n	g_3 - inverse X^{g_3}
(1),(2) and (4)	left weak g.i. A^w	———
(1),(2),(3) and (4)	pseudo-inverse A^+	the g-inverse X^g

Goldman and Zelen (see [22], p.19) used the term "weak" generalised inverse. While Urquhart called a matrix satisfying (1), (2) and (4) a "normalized" g.i., Rohde used the term "normalized" for conditions (1), (2) and (3), and since both he and Urquhart have worked extensively in the field, the need for some standardisation of terminology can be seen.

Other possible combinations of equations (1) to (4) have been used (e.g. by Chipman and Khatri; refer [18]) but the above seem to draw most attention in the literature. Other formulations of definitions have also been made, some of which can be proved to be equivalent to those above.

In this work, the notation and terminology of Boullion and Odell (i.e. that of the middle column of the table) will be adhered to. Thus in particular, the term "pseudo-inverse" will be used for the Moore - Penrose inverse, A^+.

The existence of all the inverses in this hierarchy was guaranteed when Penrose proved that his pseudo-inverse A^+, existed and was unique for any matrix A. (We shall prove this in Chapter 2. Refer [17]). It was easy to show, however, that the weaker definitions yield non-unique inverses (e.g. see [22], Ch. 1). This highlighted the desirability of the pseudo-inverse, while at the same time offered families of alternatives to the latter, depending on the circumstances, since the weaker inverses preserve some of the properties of the pseudo-inverse and are easier to find since they satisfy less stringent conditions.

Some of the variations which fall a little away from the hierarchy given here, but yield particularly interesting inverses, include that of a "weighted pseudoinverse". The definition here retains equations (1) and (2), but brings in a weighting effect in equations (3) and (4). This concept has been worked on by Chipman and Meicler (see [4] p.4). Another of these variations, yielding an inverse with spectral properties, has been formulated by Greville (see [25] pp.26-46), while the Drazin inverse also has spectral properties. Some work on the differentiation of this latter inverse has recently (1976) been published by S.L. Campbell (see [5]). The bulk

of the literature, however, is devoted to studies and applications of generalised inverses within our hierarchy, with the pseudo-inverse capturing a great deal of interest on the strength of its uniqueness.

Tracing the work done analogously on linear operators, we find that in the fifties, Tseng extended Moore's results to generalised inverses of closed linear operators on a Hilbert space (see [29]). The term "Moore - Tseng" inverse is often used here for the analog of the pseudo-inverse, of "Moore - Penrose" inverse. In [1] in 1963, Ben-Israel and Charnes reviewed work done in this context and in that of Euclidean spaces. Robinson's paper [20] presents some of these analogous results for an arbitrary linear transformation, in a manner suitable for use as classroom notes.

The most recent works show a tendency to investigate applications of generalised inverses in a wealth of different spheres and, as these uses become better known, to look for widely varying methods for their computation. The most dramatic results in the spectrum of applications, however, are still those expounded in Penrose's two papers, for it was here that the significance of the generalised inverse (in particular, the pseudo-inverse) was shown in the solution of systems of equations, both consistent and inconsistent. (Refer Chapter 2).

Works like [4], [18], [22] and [25] indicate some of the ways in which statisticians have used generalised inverses while other fields of application include the algebra of networks, estimation theory, etc. Further comments on their computation will follow in Chapters 3, 4 and 5.

CHAPTER 2.

FUNDAMENTAL PROPERTIES OF GENERALISED INVERSES.

We look at some of the properties of generalised inverses, proving in detail those most significant in later chapters.

We prove first Penrose's theorem, referred to earlier, which establishes at once the existence of all the generalised inverses of our tabled hierarchy, and also yields the uniqueness of the pseudo-inverse. His proof is lengthy, but remarkably simple, calling only upon the well-known Cayley-Hamilton theorem, and the following two lemmas:

Lemma 1:

$A^*A = 0 \Rightarrow A = 0$
(similarly $AA^* = 0 \Rightarrow A = 0$)

Proof:

Each diagonal element of A^*A is the sum of the squares of the absolute values of the elements of the column of A, and equals zero.

Hence the elements of columns of A are zero.

Hence $A = 0$.

Lemma 2:

(a) $BAA^* = CAA^* \Rightarrow BA = CA$
(b) $BA^*A = CA^*A \Rightarrow BA^* = CA^*$

Proof:

$(BAA^* - CAA^*)(B - C)^* = 0$ if $BAA^* = CAA^*$
$\therefore BAA^*B^* - CAA^*B^* - BAA^*C^* + CAA^*C^* = 0$
$\therefore (BA - CA)(BA)^* - (BA - CA)(CA)^* = 0$
$\therefore (BA - CA)(BA - CA)^* = 0$

So by Lemma 1, $BA = CA$
(b) is proved similarly.

Theorem 2.1: The 4 equations

(1) $AXA = A$

(2) $XAX = X$

(3) $(AX)^* = AX$

(4) $(XA)^* = XA$

have a unique solution for any matrix A.

Proof:

Equations (2) and (3) are equivalent to

$$XX^*A^* = X \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

for certainly (3) substituted into (2) yields (5).
On the other hand, from (5) we get

$$AXX^*A^* = AX$$

from which we can establish $(AX)^* = AX$, i.e. (3)
Then (3) substituted into (5) yields (2).

Similarly, equations (1) and (4) are equivalent to

$$XAA^* = A^* \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

So we seek, instead, to establish the existence of a solution, X, to (5) and (6).

If there exists B satisfying

$$BA^* AA^* = A$$

then certainly $BA^* = X$ will satisfy (6)
Also (6) $\Longrightarrow$ (1), so $BA^* = X$ will satisfy

$$A^*X^*A^* = A^*$$

$$\therefore \quad BA^*X^*A^* = BA^*$$

so BA^* satisfies (5)

So we must show the existence of B satisfying

$$BA^* AA^* = A^*$$

Now by the Cayley-Hamilton theorem (every matrix satisfies the matrix equivalent of its own charateristic equation) it can be concluded that there exist scalars $b_1, \ldots, b_k$, not all zero, so that

$b_1 A^*A + b_2 (A^*A)^2 + \ldots + b_k (A^*A)^k = 0$

Let b_r be the first non-zero b.

Then $b_r (A^*A)^r = -(b_{r+1}(A^*A)^{r+1} + \ldots + b_k(A^*A)^k)$

Let $B = -\frac{1}{b_r}(b_{r+1} + b_{r+2} A^*A + \ldots + B_k(A^*A)^{k-r-1})$

Then $(A^*A)^r = B (A^*A)^{r+1}$

Now using Lemma 2(a) and (b) repeatedly, we obtain $A^* = BA^*AA^*$ as required.

We have thus established the existence of X satisfying (1) to (4).
It remains to prove that X is unique:

Just as it was proved that (5) and (6)are equivalent to (1) to (4), it can be proved (by using pairs of equations (2) and (4), then (1) and (3)) that Y satisfies

$$Y = A^*Y^*Y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

and $$A^* = A^*AY \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

if and only if Y satisfies (1) to (4) (with X replaced by Y).

Suppose X satisfies (5) and (6), and Y satisfies (7) and (8). Then $Y = A^*Y^*Y = XAA^*Y^*Y = XAY = XX^*A^*AY = XX^*A^* = X$
Hence X is unique.

On the basis of (1) to (4), Penrose also easily established some significant fundamental properties of A^+, including the following:

(for proof of these see [17], (1955)).

(9) A nonsingular $\Rightarrow$ $A^{-1} = A^{+}$
(justifying the conviction that A^{+} serves as a generalisation of the classical inverse of a nonsingular matrix, if the behaviour of A^{+} in (1) to (4) does not adequately do so).

(10) $(A^{+})^{+} = A$
(generalising a further property of the classical inverse).

(11) $(\lambda A)^{+} = \lambda^{-1} A^{+}$ (or even $\lambda^{+}A^{+}$) $\lambda \neq 0$

(12) $(A^{*})^{+} = (a^{+})^{*}$ and $(A^{*}A)^{+} = A^{+}A^{+*}$

(13) if A is normal, i.e. $A^{*}A = AA^{*}$, then $A^{+}A = AA^{+}$ and $(a^{n})^{+} = (A^{+})^{n}$
(n a positive integer)

(14) A, A^{+}, $A^{*}A$ and $A^{+}A$ all have rank equal to trace $A^{+}A$.

Some of these properties, and many of similar nature have been generalised to A^{g}, A^{r}, A^{w} and A^{n} and an excellent summary of these is reported in [4] pp. 7 - 9. We note, however, that these leave gaps. Disappointingly there is no clear analogous result to that for nonsingular A and B, yielding $B^{-1} A^{-1}$ as the inverse of AB. Rohde and Greville (see [22] p.28) have found conditional results, the latter in the case of the pseudo-inverse, but less strict conditions should exist for generalised inverses lower down in the hierarchy. Also note that in the case of the pseudo-inverse, Penrose himself found normality was a condition yielding $(A^{n})^{+} = (A^{+})^{n}$ (see (13) above). Clearly it would be interesting to find conditions for the generalised inverses of A^{n} in terms of the generalised inverses of A.

Many further algebraic properties have been studied, these often resulting from work in some field of application. In the sixties, names like Bose, Reid and Cline, hitherto unmentioned in this work, occured frequently in the literature.

More recently (1974, 1975) new names like those of the Russians Morovoz, Sokovnin, Žukovskiĭ and Lipcer have appeared, their work showing particular interest in the pseudo-inverse (e.g. see [33]). Morovoz's work, and that of Kammerer and Nashed who took up the investigation of the pseudo-inverse in Hilbert spaces (publishing during 1968 to 1972 - see [13], [15]), provided the background for Sokovnin's work in this field. For an extensive bibliography here, refer to [34].

We consider now, the properties of these matrices with a view to their application. One theorem stands possibly well ahead of all others. For historical reasons we state and prove this as Penrose did in [17], 1955.

Theorem 2.2:

A necessary and sufficient condition for the equation $AXB = C$ to have a solution is

$$AA^{+}CB^{+}B = C$$

in which case the general solution is

$$X = A^{+}CB^{+} + Y - A^{+}AYBB^{+}$$

where Y is arbitrary to within dimension properties.

Proof:

Suppose X is a solution of $AXB = C$
Then since $AA^{+}A = A$ and $BB^{+}B = B$, we have

$$C = AXB = AA^{+}AXBB^{+}B = AA^{+}CB^{+}B$$
$$\therefore \quad C = AA^{+}CB^{+}B$$

Conversely, if $C = AA^{+}CB^{+}B$, then clearly $A^{+}CB^{+}$ is a solution of $AXB = C$.

We have proved that $C = AA^{+}CB^{+}B$ is a necessary and sufficient condition for consistency, and we have found $A^{+}CB^{+}$ is a particular solution of $AXB = C$.

We consider $AXB = 0$, now.

If $X = Y - A^{+}AYBB^{+}$, Y arbitrary, then
since $AA^{+}A = A$, $BB^{+}B = B$, we have that
X is a solution of $AXB = 0$.
Conversely if X is a solution of $AXB = 0$,
then $X = X - A^{+}AXBB^{+}$
So $Y - A^{+}AYBB^{+}$ is the general solution of $AXB = 0$.
Hence $A^{+}CB^{+} + Y - A^{+}AYBB^{+}$ is the general solution of $AXB = C$.

We note, immediately, as did Penrose, that the only property of the pseudo-inverse which is required in the proof, is (1), so that the theorem holds, in fact, for any generalised inverse (further motivation for this terminology).

The significance of this theorem in the solution of systems of linear equations need not be spelled out - hence its usefulness to statisticians in their work, for example, with linear models. We note, too, that it is this property which provides the backbone of the further chapters of this essay. In Chapter 4 it is used to obtain an algorithm for the computation of the pseudo-inverse of any arbitrary matrix A, while in Chapter 5 it is used in an algorithm for solving a system of linear equations, which also yields a process for computing a generalised inverse.

Contributing further to the statistical problem of finding "best" approximate solutions to inconsistent systems of linear equations, Penrose established yet another significant result for his pseudo-inverse in a second paper [17], 1956.

He defined a matrix norm $\|.\|$ as follows:-
if $A = (a_{ij})$, let $\|A\| = \sum_{ij} |a_{ij}|^2$.

Considering the matrix equation $AX = B$, he defined X_o as the "best approximate" solution if for all X, either

(i) $\|AX - B\| > \|AX_o - B\|$, or else

(ii) $\|AX - B\| = \|AX_o - B\|$ and $\|X\| \geq \|X_o\|$

We note that $\|A\| = (\sum_{ij} |a_{ij}|^2)^{\frac{1}{2}}$ is more common. We also note that by his definition, X_o is a "least squares" solution of $AX = B$, but is also one with minimum norm amongst all "least squares" solutions. (Odell conceived the idea of solutions that minimize different abstract norms).

Penrose proved the following:

Theorem 2.3:

A^+B is the unique best approximate solution of $AX = B$.

Proof:

We note first that $\|A\|$ = trace A^*A, for all A.
Also $\|A\| > 0$ unless $A = 0$.

Now recall that $A^*AA^+ = A^*$ (equation (8) for the pseudo-inverse, arising in Theorem 2.1).

$$\therefore \{AP + (I-AA^+)Q\}^* \{AP + (I - AA^+)Q\}$$
$$= (AP)^* AP + \{(I - AA^+)Q\}^* AP + (AP)^* (I - AA^+)Q + \{(I - AA^+)Q\}^* (I - AA^+)Q.$$
$$= (AP)^* AP + 0 + 0 + \{(I - AA^+)Q\}^* (I - AA^+)Q$$

(by expanding and applying $A^*AA^+ = A^*$)

By considering the trace of these matrices, we find

$$\|AP + (I - AA^+)Q\| = \|AP\| + \|(I - AA^+)Q\|. \qquad (15)$$

Since P and Q are arbitrary here, to within dimension, we have, in particular,

$$\| AX - B \| = \| A (X - A^+B) + (I - AA^+) (-B) \|$$
$$= \| AX - AA^+B \| + \| AA^+B - B \|$$

So $\| AX - B \| \geq \| AA^+B - B \|$ and in fact Ⓐ

$\| AX - B \| = \| AA^+B - B \|$ only when

$AX = AA^+B$ Ⓐ*

Equation (15) holds similarly with A^+ replacing A, so we have in particular (since $A^{++} = A$)

$$\| A^+B + (I - A^+A) X \| = \| A^+B \| + \| (I - A^+A) X \|$$

So if $AX = AA^+B$, then we have

$$\| A^+B + (X - A^+AA^+B) \| = \| A^+B \| + \| X - A^+AA^+B \|$$

Now $A^+AA^+ = A^+$ by equation (2) for A^+, so

$$\| X \| = \| A^+B \| + \| X - A^+B \|$$

Hence if $AX = AA^+B$, then

$$\| A^+B \| \leq \| X \|$$

So if $\| AX - B \| = \| A(A^+B) - B \|$, then

$$\| A^+B \| \leq \| X \|$$ Ⓑ

By Ⓐ and Ⓑ, A^+B is the best approximate solution of $AX = B$.

This least square character was, as noted before, also observed and used by Bjerhammar in his geodetic applications: adjusting observations which gave rise to singular or ill-conditioned matrices.

As an example of more recent applications of this last theorem, in 1972 Tewarson (see [28]) observed that if Theorem 2.2 was applied to equation Ⓐ* of Theorem 2.3, the result was that the family

$$x = A^+b + (I - A^+A)y \qquad \text{(y arbitrary)}$$

comprises all the least square solutions of the system $Ax = b$. He used this result to reformulate an algorithm, based on Stiefel's ascent algorithm, for the computation of the Chebyshev (minimax) solutions of an inconsistent system of linear equations.

Pringle, in his work on linear least squares estimation (see [18]) makes use of a result related to the above theorem and proved by Rao in [19] :

the vector $x = Xb$ is a least squares solution of $Ax = b$ if an only if X satisfies equations (1) and (3) for generalised inverses.

For the purposes of this essay, the above indication of where the major general applications lie must suffice, as it would be impossible to refer even briefly to all the numerous more specialised works. Boullion and Odell have singled out a few of these for inclusion in [4] though they observe that the choice was made with a certain amount of diffidence. Applications described by Chipman, Ben-Israel and Rohde are presented in [25] . The great majority of such works are, however, not yet to be found collectively and lie scattered in the literature.

CHAPTER 3.

COMPUTATION METHODS

Algorithms for the computation of generalised inverses have received attention at every stage of the development of the theory. This work cannot, therefore, include the details of each of these. In this chapter, the literature is thus surveyed as a whole. Chapters 4 and 5 single out two algorithms for close examination.

Earliest methods include those of Penrose, Hestenes and Greville for computing A^+. Penrose's Theorem 2.1, discussed in Chapter 2, incorporated a rather lengthy computational method relevant to the method singled out in Chapter 4. In his second paper he described a partition method and a repetitive method, the latter based on Frame's method for computing classical inverses. It was found, similarly, that Hestenes' biorthogonalization method of computing inverses of non-singular matrices, could also be adapted for the computation of A^+ (by adding orthogonal rows to rectangular A). Greville (see [8]) described an early recursive algorithm which starts with a column vector, and computes A^+ by obtaining further columns successively.

During the sixties, a variety of methods direct and indirect were proposed. In 1971, a work by Shinokazi, Sibuya and Tanabe, [23], was published in which direct methods for computing A^+ published up till then, were surveyed and compared.

The following classification emerged:

A: methods based on matrix decomposition
B: methods using bordered matrices
C: others (including Greville's method referred to above, and Pyle's gradient projection method).

On the basis of some test matrices (chosen to satisfy clearly stated conditions) they concluded that for machine computation, methods of type B (proposed by Hestenes and Gemain-Bonne) as well as those of type A which involve decomposition into two matrices of full rank, seemed generally preferable to the others.

This survey, however, had an omission: that of Decell's algorithm, subject of Chapter 4 of this essay. The omission of this work from the reasonably comprehensive bibliography of the survey may well be explained by the observation that Decell's method, though it seems to have been obtained independently of Penrose's Theorem 2.1, can be obtained exactly analogously to the process described in the latter, yielding BA^* as A^+. Decell merely employs a slight modification resulting in a simpler computational algorithm.

On the other hand, the paper [31] of Vioth, Vogt and Mickle, published in 1972, includes a comparison with Decell's algorithm, though its main purpose is to describe another interesting direct method. Classical minimization theory is applied to finding the best approximate solution (in the sense of Penrose) to $AX = I$ (i.e. A^+). A^+ arises out of the construction of two sequences of related matrices, the equations yielding their elements being simple in form for machine computation. Methods tested, using a computer, included Penrose's partitioning method (referred to earlier), Decell's algorithm, and Penrose's related algorithm (the relationship between the last two was not noted). Just two small matrices, one real, one complex, were used, with no particular properties being indicated. The conclusion drawn rated the classical minimization algorithm and Decell's algorithm as more useful and efficient than the others if no information is easily available about A, e.g. rank. It is noted that both these algorithms do, however, involve many operations. In Chapter 4 we observe one advantage of Decell's algorithm over the other.

The seventies have brought further ideas on the subject. A paper of Hallum and Pore, [9], published in 1975 defines a recursive partitioning method useful for machine computation of A^+ when A is large, since it avoids excessive storage. A Russian paper ([33] of 1975), offers a recurrent algorithm. Hestenes, still working on pseudo-inverses noted in [10] of 1975, that a "conjugate-gradient" algorithm may be extended to compute A^+. Another convergent iterative method, for the approximate calculation of the pseudo-inverse of a normally solvable bounded linear operator, between Hilbert spaces, is described by Sokovnin in his work referred to earlier.

A certain preoccupation with the pseudo-inverse rather than the weaker inverses may be observed in the above review. The reason for this is that it is the pseudo-inverse which presents the greatest difficulty for computation, there being a great number of easy means of finding an A^g in the family of generalised inverses (e.g. Searle [22] and others - see further details in paragraph 5.1 of Chapter 5 of this work) while Urquhart has described methods for finding A^r, A^n and A^w from A^g. A clear discussion of this lies in Chapter 1 of [4].

The criteria used to judge any method of computation must be affected by the accuracy of result required for one's purposes. For example,

I - one may wish to use Theorem 2.2, to test, with complete certainty, the consistency of a large system of linear equations. Clearly A^+ should exact.

II - one may seek the "best" solution of a system proved inconsistent, using Theorem 2.3. Here a certain degree of inaccuracy may be tolerated.

III - the work required to solve a consistent system exactly may not be worthwhile. Here an iterative method, or an exact method prone to rounding errors, may be considered.

With a view to I above, any method presupposing properties of A, or yielding rounding errors, must be rejected. In Chapter 4 an exact method is described, yielding A^+ for a given matrix A, making use of the computer in accordance with modern needs.

In Chapter 5, considering II and III, an iterative method is described, which solves a system of linear equations approximately, while also yielding a generalised inverse if wanted.

The particular merits of these two algorithms singled out for detailed examination, are given in the relevant Chapters, while Chapter 5 also considers in a little more detail, the computation of weaker generalised inverses.

CHAPTER 4.

A METHOD FOR EXACT COMPUTATION OF PSEUDO-INVERSES.

4.1. INTRODUCTION

Rounding errors present a problem in almost all machine calculable algorithms for A^+.

In 1966/1967, papers [16] , [26] and [3] published by Newman, Szabo and Tanaka, Borosh and Frankel, respectively, proposed the use of residue arithmetic to avoid this problem. The idea was to work only with integers so that computations are performed exactly by the machine (provided overflow is avoided). In 1969/1970, Jo Ann Howell and Robert T. Gregory published more detailed computational procedures, using this technique to solve systems of linear equations exactly. (see [11] .)

A study of these works leads one to conclude that residue arithmetic can be effective in avoiding rounding errors under the following conditions :

> suppose an algorithm for the computation of vector/matrix X from integral vector/matrix Y can be described, yielding
>
> $$X = f(g(Y))$$
>
> where g involves operations with an analog in residue arithmetic, g(Y) is integral, and f is a simple function easy to perform possibly without computer aid.

Then the analogous residue arithmetic is used to compute the residue of g(Y) modulo some integer, m, starting with the residue form of Y. It is found that if m has been chosen wisely, the true integral value of g(Y) can be restored from its residue. Lastly, the simple calculation of X from g(Y), through f, is calculated exactly if possible.

The papers referred to above illustrate how this procedure works in application. But, clearly, the success of such a method in yielding A^+ exactly, is firstly dependent on the existence of such an explicit form for A^+, comprising a large integral part which need only be manipulated simply to yield A^+. If this last stage involves a process whereby rounding errors may occur, it should not be performed by the computer : hence the necessity for it to be simple, if the method is to be worthwhile.

Decell's paper [6] yields exactly such a form for A^+, in the following theorem:

4.2. DECELL'S ALGORITHM.

Theorem 4.2.1.

Let A be any n x m complex matrix and let

$$f(\lambda) = (-1)^n (a_o\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n).$$

(with $a_o = 1$) be the characteristic polynomial of AA^*.

If $k \neq o$ is the largest integer such that $a_k \neq o$, then the pseudo-inverse of A is given by

$$A^+ = -\frac{1}{a_k} A^* \left[(AA^*)^{k-1} + a_1(AA^*)^{k-2} + \ldots + a_{k-1}I \right]$$

If $k = o$ is the largest integer such that $a_k \neq 0$, then $A^+ = 0$.

Decell proved this theorem by applying Theorem 2.2 proved earlier, to a matrix equation for AA^*, obtained from the Cayley - Hamilton theorem. His careful use of (Penrose's) Theorem 2.2 here, indicates that he may not have been aware that Penrose's existence and uniqueness theorem for A^+, Theorem 2.1 of this treatise, incorporated a proof of this expression for A^+ (an observation we made in Chapter 3). For this reason, Decell's proof is not given here. We offer the equivalent result via Penrose's Theorem 2.1.

The proof may be outlined as follows:

Penrose proved that the equations

(5) $XX^*A^* = X$ and (6) $XAA^* = A^*$

are equivalent to the system (1) to (4).

He also proved that if B exists satisfying

$$BA^*AA^* = A^*$$

then BA^* is a solution to (5) and (6)
so $BA^* = A^+$.

He finally proved the existence of B by using Cayley-Hamilton applied to A^*A, showing that

$$B = -\frac{1}{b_r} (b_{r+1} I + b_{r+2} A^*A + \ldots + b_k (A^*A)^{k-r-1})$$

(where the b_i are the coefficients in the characteristic equation for A^*A)

has the desired property.

Consider the exactly analogous argument for the equations (equivalent to (1) - (4))

(7) $Y = A^*Y^*Y$ and (8) $A^* = A^*AY$

Suppose C satisfies $CAA^*A = A$

Then $(CA)^*$ is easily shown to satisfy (7) and (8)

So $A^+ = (CA)^*$

Applying Cayley - Hamilton to AA^* this time, we get, in the notation of Decell

$$a_o (AA^*)^n + a_1 (AA^*)^{n-1} + \ldots + a_{n-1} AA^* + a_n I = 0$$

(with $a_o = 1$).

Then if a_k is defined as in Decell's theorem ($k \neq 0$), we find that

$$C = -\frac{1}{a_k}\left[(AA^*)^{k-1} + a_1(AA^*)^{k-2} + \ldots + a_{k-1}I \right]$$

has the desired property.

Hence $A^+ = A^*C^*$

$$\therefore A^+ = -\frac{1}{a_k} A^* \left[(AA^*)^{k-1} + a_1(AA^*)^{k-2} + \ldots + a_{k-1}I \right]$$

Lastly: if $k = 0$, then $(AA^*)^n = 0$

$$\therefore A = 0$$

$$\therefore A^+ = 0$$

Decell's paper, however, offers a modification due to Faddeev, which yields the following remarkably convenient computing algorithm :

the matrices A_i and B_i are constructed as follows:-

$$A_0 = 0 \qquad q_0 = -1 \qquad B_0 = I$$

$$A_1 = AA^*B_0 \qquad q_1 = \frac{\text{trace } A_1}{1} \qquad B_1 = A_1 - q_1 I$$

$$A_2 = AA^*B_1 \qquad q_2 = \frac{\text{trace } A_2}{2} \qquad B_2 = A_2 - q_2 I$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$A_{k-1} = AA^*B_{k-2} \qquad q_{k-1} = \frac{\text{trace } A_{k-1}}{k-1} \qquad B_{k-1} = A_{k-1} - q_{k-1}I$$

$$A_k = AA^*B_{k-1} \qquad q_k = \frac{\text{trace } A_k}{k} \qquad B_k = A_k - q_k I$$

Using Newton's formulae which relate the a_i and eigenvalues, Faddeev proved (see [7]) $q_i = -a_i$ for $i = 1,2,\ldots,k$.

So we have, using Theorem 4.2.1

$$A^+ = -\frac{1}{a_k} A^* B_{k-1} \quad , \text{ with } a_k = -q_k .$$

(We add that k may well not be known in advance, but that this algorithm, if taken to the (k+1)th step, will yield the first zero value for q_k, identifying k immediately).

4.3. RESIDUE ARITHMETIC IN DECELL'S ALGORITHM.

Stallings and Boullion (see [24]) noticed that the above equation has exactly the form suited to the technique of residue arithmetic, if A is integral, for then, A^*, AA^* and hence a_i, i = 1,2,..., k are also integral (though A^+ may not be).

The residue arithmetic analog of Decell's computing algorithm can thus be used to find the residues of a_k and B_{k-1}, modulo m (provided the latter is well chosen). Proper choice of m again ensures that the true integer values of a_k and B_{k-1} may be restored, and finally, A^+ can be computed as a simple product.

We note that if this can be done to find A^+ when A is integral, then it can also be done to compute B^+ when B has rational elements, since $(\lambda A)^+ = \lambda^{-1} A^+$

and B can be scaled to become λA, with A integral. We note too that any fixed-word-length computer can only store rational numbers, so that we need look for no further generalisations if we accept the need to use a computer.

Stallings and Boullion also assume A is n × p with $n \leq p$, but show that this poses no problem since if B is n × p with n > p, then B^* is p × n with p < n, so that $(B^*)^+$ may be found

Then $(B^+)^* = (B^*)^+$

so $B^+ = (B^{*+})^*$

Proof that the claims made for residue arithmetic do indeed hold, necessitates a look at the fundamental definitions, and the well known and easily proved properties of residues, a tedious task to do in detail. Stallings and Boullion simply summarised those needed - the policy adopted here. A fuller presentation is given in [11], while any reasonably comprehensive number theory text will yield the details.

Definition 4.3.1.

Given integers a,b and $m \neq o$,

$a \equiv b \pmod m$ if an only if m divides $b - a$.
If $a \equiv b \pmod m$ and $-\frac{m}{2} < b \leq \frac{m}{2}$

then b is called the residue of a mod m, and we write
$b = |a|_m$.

Definition 4.3.2.

Given a matrix $A = (a_{ij})$ and $m \neq 0$, then matrix $R = (r_{ij})$ is called the residue of A mod m

if $r_{ij} = |a_{ij}|_m$ for all i and j.
We write $R = |A|_m$.

Nothing is lost by assuming $m > 1$ here.

Theorem 4.3.3.

(i) For any integer a, $|a|_m$ is unique.

(ii) $|ab|_m = ||a|_m . b|_m = |a.|b|_m|_m = ||a|_m.|b|_m|_m$

(iii) $|a \pm b|_m = ||a|_m \pm b|_m = |a \pm |b|_m|_m = ||a|_m \pm |b|_m|_m$

Theorem 4.3.4.

(i) If $(a, m) = 1$, there exists integer b so that

$$|ab|_m = 1$$

If $-\frac{m}{2} < b \leq \frac{m}{2}$, then b is unique.

We write $b = a^{-1}(m)$.

(ii) If $a^{-1}(m)$ exists then $|ax|_m = |b|_m$ has a unique solution, $|x|_m = |b.a^{-1}(m)|_m$

Theorem 4.3.5.

Theorem 4.3.3. remains valid if a and b are replaced by matrices A and B (with suitably compatible dimensions).

Theorem 4.3.6.

Given an integer d, if m is chosen so that

(i) $m > 2|d|$

and d' is formed from $|d|_m$, satisfying

(ii) $|d'|_m = |d|_m$ and

(iii) $|d'| < \frac{m}{2}$

then $d' = d$

Theorem 4.3.7.

Given $A = (a_{ij})$, if m is chosen so that

(i) $m > 2 \max_{ij} |a_{ij}|$

and if B is formed from $|A|_m$ satisfying

(ii) $|B|_m = |A|_m$ and

(iii) $\max_{ij} |b_{ij}| < \frac{m}{2}$

then $B = A$

Examining Decell's algorithm, it is easily seen that all the operations to calculate B_{k-1} and q_k can be performed in modulo m residue arithmetic provided that "division by $r = 1,2,\ldots,k$ can be accomplished. From Theorem 4.3.4. it can be concluded that m must satisfy

(A) $(m,r) = 1$, $r = 1,2,\ldots,k$.

The next stage, restoration of the exact values of q_k (= $-a_k$) and B_{k-1}, is easily accomplished via Theorems 4.3.6 and 4.3.7 if m is chosen so that

(B) $$m > 2 \max \{|a_k| , \max_{ij} |b_{ij}|\}$$

for then the residues of a_k and B_{k-1} modulo m, are in fact their true values.

Lastly, A^+ is obtained exactly if the product $-\frac{1}{a_k} A^* B_{k-1}$ is calculated exactly.

4.4. CRITERIA FOR SELECTION OF m.

The success of this method is clearly dependent on the careful selection of m. Most papers suggesting the use of residue arithmetic, devote considerable space to selection criteria. That of Stallings and Boullion is no exception.

The chances of m satisfying (A) and (B) are increased if m is chosen as a large prime. For a particular computer, however, there is an upper bound for m (twice the square root of the largest integer which can be stored), if multiplication of the residues of any 2 integers is to be achieved without overflow. In general, the larger the modulus, the slower the computation. This suggests that m should be kept as low as possible, and that calculation of a lower bound for m would be worthwhile.

Stallings and Boullion arrive at an alternative to (B) :

(B') $$m > 2 \max \{M^n ; n(n-k+1) M^{n-1}\}$$

(where $M = \min \{ \text{trace } AA^* ; \|AA^*\| \}$).

Such an alternative is necessary if little is known about values of a_k and B_{k-1} at the outset, so that (B) is meaningless. (B') still includes k which may also not be known, but setting k = 0 offers a conservative alternative to (B'). Clearly any additional information (like rank A, or eigenvalues of AA^*) is extremely useful in lowering the bound, and should be used for this purpose.

(B') is justified as follows : some results are needed from matrix algebra ;

Lemma 4.4.1.

Suppose $AA^* = (a_{ij})$ for any n × p matrix A.

Let $a = \max_{ij} |a_{ij}|$. Then

(i) $a_{ii} \geq 0$ for all i

(ii) there exists an i so that $a = a_{ii}$.

Proof:

$a_{ij} = (A_i, A_j)$ for all $i = 1,2,\ldots,n$
$j = 1,2,\ldots,n$

where A_i is the i'th row of A. Fix i and j.

Suppose $|A_i| \leq |A_j|$ (where $|A_i|$ denotes the (Euclidean) length of vector A_i).

Then

(i) $a_{ii} = (A_i, A_i) \geq 0$

(ii) $|a_{ij}| = |(A_i\ A_j)| \leq |A_i|\ |A_j| \leq |A_j|^2 = a_{jj}$
by Schwarz's inequality.

Hence the diagonal element has the largest absolute value in the i'th row, for each i.

Hence value a is assumed by some diagonal element.

The following well known results are not proved here :

Lemma 4.4.2.

(i) For any A, AA^* is symmetric and positive semi-definite : hence the eigenvalues of AA^* are real and non-negative.

(ii) AA^* symmetric implies there exists orthogonal P so that

$$PAA^*P^* = D = \text{diag}\ [\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0]$$

where $\lambda_1, \ldots, \lambda_k$ are the non-zero eigenvalues of AA^* (k in number, by definition of a_k earlier.)

Proof of (B'):

Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the eigenvalues of $-\frac{1}{a_k} B_{k-1}$, and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be those for B_{k-1}.

Then $P\left(-\frac{1}{a_k} B_{k-1}\right) P^* = -\frac{1}{a_k}\left(D^{k-1} + \ldots + a_{k-2}D + a_{k-1}I\right)$

$$= \text{diag}\ [\gamma_1, \gamma_2, \ldots, \gamma_n]$$

$$= G$$

Now $A^+ = -\frac{1}{a_k} A^* B_{k-1}$

$\therefore\ AA^+AA^* = -\frac{1}{a_k} AA^* B_{k-1} AA^*$

$\therefore\ AA^* = -\frac{1}{a_k} AA^* B_{k-1} AA^*$ (by equation (6) of Ch. 2)

Now P orthogonal implies $PP^* = P^*P = I$

$\therefore\ DGD = -\frac{1}{a_k} (PAA^*P^*)\ (P B_{k-1} P^*)\ (P AA^*P^*)$

$$= D$$

Hence $\gamma_i = \frac{1}{\lambda_i}$ for $i = 1, 2, \ldots, k$

and $\gamma_i = \frac{-a_{k-1}}{a_k}$ for $i = k+1, \ldots, n$

⊛ It is well known that the $(-1)^i a_i$ are the elementary symmetric polynomials in λ_i, $i=1,\ldots,k$.

Hence a_{k-1} is non-zero and opposite in sign to a_k

Thus $\gamma_i > 0$ for $i = 1,2,\ldots,n$ since

$\lambda_i > 0$ for $i = 1,2,\ldots,k$.

Now $\frac{-1}{a_k} B_{k-1}$ is real and symmetric and thus now also positive definite. So there exists non-singular Q so that $QQ^* = \frac{-1}{a_k} B_{k-1}$.

Applying lemma 4.4.1, $\frac{-1}{a_k} B_{k-1}$ has non-negative diagonal elements, of which one yields the greatest absolute value of all elements of $\frac{-1}{a_k} B_{k-1}$.

let $B_{k-1} = (b_{ij})$. Then $\max_{ij} |b_{ij}| \leq |\text{ trace } B_{k-1}|$

Note that by definition γ_i , σ_i , we have

$$\sigma_i = -a_k \gamma_i , \qquad i = 1,\ldots,n.$$

So $$\sigma_i = \frac{-a_k}{\lambda_i} , \qquad i = 1,2,\ldots,k$$

and $$\sigma_i = a_{k-1} , \qquad i = k+1,\ldots,n.$$

$$|\text{Trace } B_{k-1}| = |\sum_{i=1}^{n} \sigma_i|$$

$$= |-a_k \sum_{i=1}^{k} \frac{1}{\lambda_i} + (n-k)\, a_{k-1}|$$

$$= |-a_k \frac{(-1)^{k-1} a_{k-1}}{(-1)^k a_k} + (n-k)\, a_{k-1}|$$

(by making use of ⊛)

$$= |(n-k+1)\, a_{k-1}|$$

Hence $$\max_{ij} |b_{ij}| \leq (n-k+1) |a_{k-1}|$$

From this we get an alternative to B :

(B") choose $m > 2 \max \{ |a_k| ; (n-k+1) |a_{k-1}| \}$

which is useful if rank A = k, is known, as well as the eigenvalues of AA^*. If not, we define

$$\lambda = \text{spectrum } AA^*$$

then $0 < \lambda_i \leq \lambda$ for $i = 1,\ldots,k$.

But $\text{trace } (AA^*) = \sum_{i=1}^{k} \lambda_i$, and so

$$\lambda \leq \text{trace } (AA^*)$$

It is well known that $\lambda \leq \|AA^*\|$ for any norm of AA^*.
Choose $M = \min \{\text{trace } (AA^*) ; \|AA^*\|\}$

then $\lambda \leq M$

Using ⊛ again, $|a_k| \leq \lambda^k \leq \lambda^n \leq M^n$, more useful if k is unknown as we have supposed.
Similarly $|a_{k-1}| \leq k \lambda^{k-1} \leq n\lambda^{n-1} \leq n M^{n-1}$

Using these bounds on $|a_k|$ and $|a_{k-1}|$ in (B") we get

(B') choose $m > 2 \max \{ M^n ; n(n-k+1) M^{n-1} \}$

4.5. MULTI-MODULUS METHODS

In practice, conservative bounds for m, like B', may still yield unfortunately large numbers, and the work required to tighten the bounds may be lengthy. It can be observed, though, that the necessity for condition B only arises after the modulo arithmetic is completed. Szabo and Tanaka noted this,

and suggested that the computation be carried out with respect to each modulus m_i in a system, $m_1, m_2, \ldots, m_s$

Restoration of the values which have a representation as an s-tuple of residues modulo $m_1,\ldots,m_s$, is then via the Chinese Remainder Theorem, favoured historically, or, preferably, via a conversion of the residue representation to a "mixed radix" representation from which an easy computational process yields the original value. (See [26]).

The advantage of this multi-modulus alternative is that the residue computation may be performed with moduli m_i satisfying far less stringent conditions than B so that they need not be so large. In the case of Decell's algorithm, though, it is found that new conditions arise which must be satisfied by the m_i if multi-modulus methods are applied. These are not difficult to meet, however, and it is only the product $M = m_1 \ldots m_s$ which must satisfy B. All the conditions are as follows :

$$(m_i , r) = 1 \qquad i = 1,\ldots,s \qquad r = 1,\ldots,k$$

$$|a_k|_{m_i} \neq 0 \qquad i = 1,\ldots,s$$

$$(m_i, m_j) = 1 \qquad \text{for all } i \neq j$$

$$M = m_1 \ldots m_s \qquad \text{satisfies B.}$$

The details of multi-modulus residue arithmetic, and the theorems yielding methods of restoration, can be found in [26] and [11]. They are lengthy to state but easy to follow. The major advantage of mixed radix, over Chinese Rem. Th. restoration, is that the former reduces the amount of multilength arithmetic (very slow, computationally) required.

Finally we note that comparable direct methods, like Voith's classical minimization one, do not yield the form necessary for A^+, if residue methods are to be applied.

CHAPTER 5

AN ITERATIVE METHOD FOR SOLVING SYSTEMS OF LINEAR EQUATIONS AND FOR COMPUTING GENERALISED INVERSES.

5.1. THE PLACE OF TANABE'S PAPER [27] IN THE LITERATURE.

In Chapter 3 it was claimed that there exist many ways of computing the weaker generalised inverses of any matrix A; for example, in [18] , Chapter 4, methods for each of the inverses A^g, A^r, A^n (and A^+) are discussed. Similarly Chapter 11 of [19a] reviews a few such methods for A^g, A^r, (and A^+). Chapters of this essay have, however, been directed mainly towards the computation of A^+. In the area of solving linear equations, though, the weaker generalised inverses have significance (see Theorem 2.2) and it is in this context that Tanabe's paper [27] proves remarkable.

Firstly it describes an iterative method which converges for any system of linear equations for which the coefficient matrix has non-zero rows; secondly, it can be used to test for consistency as described in Th. 2.2, and it generates all solutions for a consistent system via the general solution described there. Since it is constructed, then, around some generalised inverse of A (if the system is Ax=b), it can also be used to yield this generalised inverse if required. Clearly this approximate generalised inverse may be obtained via computation more lengthy and tedious here than if the other method referred to were used. But this method holds as well for a matrix about which little is known, matrix size is no stumbling block, affecting only the speed of computation, and the method is well suited to computer calculation.

For the reasons just stated, this paper helped fill a gap in the literature, which Pringle and Rayner refer to in Ch. 4 of [18]. Most papers on computation of generalised inverses are theoretical in nature, yielding alternative forms for the various types without considering the usefulness in practice of such forms. Certainly it would be tedious to consider all aspects in a comparison of existing methods, and the value of doing so may be questioned, since a great deal of the usefulness of generalised inverses is theoretical rather than practical. There should exist, however, some practical methods of computing them for possibly large matrices without particular characteristics. Tanabe's paper incorporated exactly such a "safe" method at the time (1969 - 1971) when little else was offered. Recently, Hallum and Pore ([9], 1975) have looked at the computational aspects of such generalised matrix inversion by the computer, showing recognition of the need for research in this direction.

Tanabe's is an iterative method, and it is important to note that Ben-Israel's iterative method [0], a favourite historically, has the advantage that it yields A^+, whereas Tanabe's yields an inverse somewhere between A^W and A^+ in the hierarchy. (It is easy to show, though, that $A^+ = A^W A A^n$ and one wonders whether as a result of the symmetries between the definitions of A^W and A^n, there might be an argument parallel to Tanabe's, yielding A^n. Such a method used for obtaining A^+ would lose some of its value by being very long). Under certain circumstances, Tanabe's algorithm does yield A^+ but the generality is lost and the method must then come in line for comparison with others with preconditions placed on A; e.g. Sokovnin, referred to earlier, describes such an iterative method. It is also difficult, if not impossible, to establish criteria for comparison between methods which rely on widely varying properties of A.

No such comparison of the iterative methods has been seen by the writer, not even one for methods which hold for any A, and there is certainly place for the latter. The paper [23] (discussed in Chapter 3), served exactly this purpose for direct methods published before 1971. Note that Tanabe was a co-author of that paper.

In summary, Tanabe's algorithm may be comparatively long, computationally, and it does not yield A^+ in general (though some modification might). It yields, for any A with non-zero rows, an inverse which satisfies the three conditions of A^W, as well as a weakened forth condition. The method arises as part of his more general algorithm for solving a system of linear equations Ax = b, by an iterative method which is unusual and significant in that it converges for any A with non-zero rows, even when the system is singular and inconsistent.

The following discussion of this algorithm is thus given in full awareness of its limitations, but recognising where its merits lie in respect of recent literature on generalised inverses and solution of linear equations.

5.2. BACKGROUND : OPERATOR THEORY.

The following are well-known results to be found in many elementary linear algebra, or functional analysis, texts:

Definition 5.2.1: If U and V are subspaces of C^n (the space of n-dimensional vectors with complex elements) with $U + V = C^n$ and $U \wedge V = 0$ then we write $C^n = U \oplus V$ and say C^n is the direct sum of U and V.

Property 5.2.2: If $C^n = U \oplus V$, then for every vector w in C^n, there exists unique x in U, y in V so that $w = x + y$.

Property 5.2.3: If U is any closed subspace of C^n, then $C^n = U \oplus U^{\perp}$ (where $U^{\perp}$ is the orthogonal complement of U).

Definition 5.2.4: If $C^n = U \oplus V$, then the linear operator P defined as follows, is called the projection on U along V:

for any w in C^n, suppose $w = x+y$, x in U, y in V, then define $Pw = x$.
P is called the orthogonal projection on U, if V is the orthogonal complement of U in C^n.

Property 5.2.5: A linear operator is a projection on some subspace if and only if it is idempotent. It is an orthogonal projection if and only if it is both hermitian and idempotent.

Notation:

Im A	will denote the range of the mapping defined by A
Ker A	will denote the nullspace of A.
$A/_T$	will denote the restriction of map A to subspace T of C^n.
P_T	represents the orthogonal projection onto linear subspace T of C^n.
(x,y)	denotes the usual inner product on C^n. i.e. $(x,y) = y^*x$ (the matrix product of x and the conjugate transpose of y, where x and y are n-dimensional column vectors in C^n).
$\|x\|_n$	$= \{(x,x)_n\}^{\frac{1}{2}}$ (where x is in C^n)
$\|A\|$	$= \sup_{x \text{ in } C^n} \frac{\|Ax\|_m}{\|x\|_n}$, ($\|x\| \neq 0$, and A an m × n complex matrix)

Property 5.2.6: If P is an orthogonal projection, $\|P\| = 1$.

Property 5.2.7: A linear operator P is a projection if and only if I - P is a projection. Moreover, if P is a projection on U along V, then I - P is a projection on V along U.

5.3. THE PROJECTION METHOD FOR SOLVING A SYSTEM OF LINEAR EQUATIONS.

We first note that this is basically the well known method devised by Kaczmarz (see [12]) for the solution of a system of linearly independent equations, but that it is shown to hold as well for singular systems.

Suppose A is an $m \times n$ complex matrix, with non-zero rows, and b is an m-dimensional column vector with complex elements. We wish to solve the system $Ax = b$. let a_i be the i-th column vector of A^*. Then $Ax = b$ can be expressed as the system of equations

$$a_i^* x = b_i \ , \qquad i = 1,\ldots,m.$$

since a_i^* is the i-th row of A. Equivalently,

$$(x, a_i) = b_i \ , \qquad i = 1,\ldots,m.$$

Now A has non-zero rows, so $a_i^* \neq 0$.

Thus $a_i \neq 0$, and $(a_i, a_i) \neq 0$, $\qquad i = 1,\ldots,m.$

We define $d_i = (a_i, a_i)$ and have d_i positive for all $i = 1,\ldots,m$.

Let $f_i(x) = x - d_i^{-1} \ ((x,a_i) - b_i) \ a_i$, so that f_i maps C^n into C^n.

Let $F(b,x) = f_1 \circ F_2 \circ \ldots \circ f_m (x)$ so that F maps C^{m+n} into C^n.

Now suppose $x_\circ$ is arbitrary.

Let $x_1 = F(b,x_\circ)$ and in fact

let $x_{i+1} = F(b,x_i)$ for $i = 0,1,2,\ldots$

so generating a sequence of vectors $x_\circ, x_1, x_2, \ldots$ which defines the iterative method.

Then $f_i(x) = x - d_i^{-1}(x,a_i)\,a_i + d_i^{-1} b_i a_i$

$= x - d_i^{-1} a_i a_i^* x + d_i^{-1} b_i a_i$

$\therefore\ f_i(x) = P_i x + d_i^{-1} b_i a_i\ , \quad i = 1,\ldots,m$

where $P_i = I - d_i^{-1} a_i a_i^*$

$= (p_{k\ell})$

where $p_{k\ell} = \delta_{k\ell} - d_i^{-1} a_{ik}\,\overline{a_{i\ell}}$

(i) It is easy to check that $\overline{p_{\ell k}} = p_{k\ell}$, so that P_i is hermitian (self-adjoint).

(ii) Also $P_i P_i = (I - d_i^{-1} a_i a_i^*)\,(I - d_i^{-1} a_i a_i^*)$

which yields after expansion

$= I - d_i^{-1} a_i a_i^* = P_i$

So P_i is idempotent.

From (i) and (ii), P_i is an orthogonal projection.

Define $Q_0 = I$

$Q_i = P_1 P_2 \ldots P_i \qquad i = 1,\ldots,m$

and R as the $n \times m$ matrix with i-th column vector $d_i^{-1} Q_{i-1} a_i\ , \qquad i = 1,\ldots,m.$

Then $Rb = \sum_{i=1}^{m} d_i^{-1} b_i Q_{i-1} a_i$

and $F(b,x) = f_1 \circ f_2 \circ \ldots \circ f_m(x)$

$= P_1 \ldots P_m x + P_1 \cdots P_{m-1} d_m^{-1} b_m a_m$

$+ \ldots + P_1 d_2^{-1} b_2 a_2 + d_1^{-1} b_1 a_1$

$= Qx + \sum_{i=1}^{m} d_i^{-1} Q_i b_i a_i$

Comparing with Rb given above, we conclude

$$F(b,x) = Qx + Rb$$

this being the defining equation of the iterative method (where Q and R are dependent only on A).

Theorem 5.3.1.

$$Q + RA = I$$

Proof:

i-th row of A is a_i^* (by definition)

i-th column of R is $d_i^{-1} Q_{i-1} a_i$ (by definition)

$$\therefore RA = d_1^{-1} Q_0 a_1 a_1^* + d_2^{-1} Q_1 a_2 a_2^* + \ldots + d_m^{-1} Q_{m-1} a_m a_m^*$$

But $I - P_i = d_i^{-1} a_i a_i^*$

$$\therefore RA = Q_0 (I-P_1) + Q_1 (I-P_2) + \ldots + Q_{m-1} (I-P_m)$$

$$= Q_0 - Q_0 P_1 + Q_1 - Q_1 P_2 + \ldots + Q_{m-1} - Q_{m-1} P_m$$

But $Q_i = P_1 \ldots P_i = Q_{i-1} P_i$

$$\therefore RA = I - Q_1 + Q_1 - Q_2 + \ldots + Q_{m-1} - Q$$

$$\therefore RA = I - Q$$

So $F(b,x) = Qx + Rb$ is a linear stationary iterative method of first degree (it is also consistent with system $Ax = b$ - see texts on Num. Anal.).

It is not difficult to show Ker A is a closed subspace of C^n, so by property 5.2.3

$$C^n = \text{Ker } A \oplus (\text{Ker } A)^{\perp}$$

Claim A:

$$C^n = \text{Ker } A \oplus \text{Im } A^*$$

Proof:

This will follow if $\text{Im } A^* = (\text{Ker } A)^\perp$.

Suppose x is in $\text{Im } A^*$

then there exists y so that $A^*y = x$.
Let z be any element of Ker A, then $Az = 0$

$$(z,x) = (z,A^*y) = (Az,y) = (0,y) = 0$$

Hence x is in $(\text{Ker } A)^\perp$
So $\text{Im } A^* \subseteq (\text{Ker } A)^\perp$

We must prove $(\text{Ker } A)^\perp \subseteq \text{Im } A^*$
(We prove instead $\text{Ker } A \supseteq (\text{Im } A^*)^\perp$, then
use the facts that $(G^\perp)^\perp = G$ and

$$G \subset H \implies G^\perp \supset H^\perp$$

to conclude $\text{Im } A^* \supseteq (\text{Ker } A)^\perp$.)

Suppose x is in $(\text{Im } A^*)^\perp$

then $(x,y) = 0$ for any y in $\text{Im } A^*$.

But a_i is in $\text{Im } A^*$, since if e_i is the one-vector (column) with all elements 0, except the i-th which is a 1, $A^*e_i = a_i$.

So $(x,a_i) = 0$ for $i = 1,\ldots,m$.
$\therefore$ $Ax = 0$
$\therefore$ x is in Ker A

So $(\text{Im } A^*)^\perp \subseteq \text{Ker } A$
So $(\text{Ker } A)^\perp \subseteq \text{Im } A^*$ and the claim follows, by Prop. 5.2.3 since Ker A is a closed subgroup of C^n.

Claim B:

$$\text{Ker } A = \bigcap_{i=1}^{m} \left\{ x \text{ in } C^n : P_i x = x \right\}$$

Proof:

Suppose x is in Ker A, so $Ax = 0$

$$\therefore (x, a_i) = 0, \quad i = 1,\ldots,m$$

$$P_i(x) = (I - d_i^{-1} a_i a_i^*)x$$

$$= x - d_i^{-1} a_i (x, a_i)$$

$$= x - d_i^{-1} a_i \, 0 .$$

$$\therefore P_i(x) = x \qquad \text{for } i = 1,\ldots,m$$

So $\text{Ker } A \subseteq \bigcap_{i=1}^{m} \{ x \text{ in } C^n : P_i x = x \}$

Now suppose $P_i x = x$, for $i = 1,\ldots,m$

$\therefore x - d_i^{-1} a_i a_i^* x = x$ for $i = 1,\ldots,m$

$\therefore a_i a_i^* x = 0$ for $i = 1,\ldots,m$

$\therefore (x, a_i)\, a_i = 0$ for $i = 1,\ldots,m$

$\therefore (x, a_i) = 0$ for $i = 1,\ldots,m$

since a_i is not the null-vector (given)

$\therefore Ax = 0$

so x is in Ker A

So $\bigcap_{i=1}^{m} \{ x \text{ in } C^n : P_i x = x \} \subseteq \text{Ker } A$

The Claim follows.

Claim C:

$$\text{Im } A^* = \bigcup_{i=1}^{m} \{ x \text{ in } C^n : P_i x = 0 \}$$

Proof:

Suppose $P_{i_\circ} = 0$ for some $i_\circ$ in $\{1,2,\ldots,m\}$

then $(I - d_{i_\circ}^{-1} a_{i_\circ} a_{i_\circ}^*) x = 0$

$\therefore d_{i_\circ}^{-1} a_{i_\circ} a_{i_\circ}^* x = x$

$\therefore d_{i_\circ}^{-1} (x,a_{i_\circ}) a_{i_\circ} = x$

Let y be any element of Ker A

then $(y,a_i) = 0$ for $i = 1,\ldots,m$

so $(y,x) = (y, d_{i_\circ}^{-1} (x,a_{i_\circ}) a_{i_\circ})$

$= 0$ since $(y,a_{i_\circ}) = 0$

so x is in $(\text{Ker } A)^\perp = \text{Im } A^*$ by Claim A

So $\bigcup_{i=1}^{m} \{ x \text{ in } C^n : P_i x = 0 \} \subseteq \text{Im } A^*$

We must prove $\text{Im } A^* \subseteq \bigcup_{i=1}^{m} \{x \in C^n : P_i x = 0\}$

We prove instead

$$\text{Ker } A \supseteq \left(\bigcup_{i=1}^{m} \{ x \in C^n : P_i x = 0 \} \right)^\perp$$

Suppose x is in this last set

then if y is such that $P_j y = 0$ for some j in $\{1,2,\ldots,m\}$, then

$(x,y) = 0$

$$\text{Now } P_i a_i = a_i - d_i^{-1} a_i a_i^* a_i$$
$$= a_i - d_i^{-1} (a_i, a_i)\, a_i$$
$$= a_i - d_i^{-1} d_i a_i$$
$$= 0$$

$\therefore (x, a_i) = 0$ by the previous observation.

So $(x, a_i) = 0$ for all $i = 1, \ldots, m$

$\therefore Ax = 0$

$\therefore x$ is in Ker A

$$\therefore \text{Im } A^* = (\text{Ker } A)^{\perp} \subseteq \bigcup_{i=1}^{m} \{ x \text{ in } C^n : P_i x = 0 \}$$

and Claim C follows.

Lemma 5.3.2

$\|Qx\| = \|x\|$ if and only if x is in Ker A.

Proof:

(necessity) : Suppose x is not in Ker A.
Then by Claim B, $P_i x \neq x$ for some $i \leq m$.
Let i_o be the largest such i,

then $\| P_{i_o} P_{i_o+1} \cdots P_{m-1} P_m x \| = \| P_{i_o} x \|$

By definition $\|P_{i_o}\|$, $\|P_{i_o}\| \geq \dfrac{\|P_{i_o} x\|}{\|x\|}$

so $\| P_{i_o} x \| \leq \|P_{i_o}\| \, \|x\|$

$\therefore \| P_{i_o} x \| \leq \|x\|$ since $\|P_{i_o}\| = 1$.

Suppose $\|P_{i_o} x\| = \|x\|$

then $(P_{i_o} x, P_{i_o} x) = (x,x)$

$\therefore (P_{i_o} x, x) = (x,x)$ since P_{i_o} is an orthogonal projection.

similarly $(x, P_{i_o} x) = (P_{i_o} x, P_{i_o} x)$

adding: $(P_{i_o} x, x) + (x, P_{i_o} x) = (x,x) + (P_{i_o} x, P_{i_o} x)$

or $(P_{i_o} x, P_{i_o} x) - (x, P_{i_o} x) - (P_{i_o} x, x) - (x,x) = 0$

or $(P_{i_o} x - x, P_{i_o} x - x) = 0$

$\therefore P_{i_o} x = x$. Contradiction .

Hence $\| P_{i_o} x \| < \|x\|$

Also $\|Q_i\| \leq \|P_1\| \|P_2\| \cdots \|P_i\| = 1, \quad i=1,\ldots,m$

So $\|Q_i\| \leq 1$ for all $i = 1,\ldots,m$ *

$$\|Q_m x\| = \|P_1 \cdots P_{i_o-1} \cdot P_{i_o} \cdot P_{i_o+1} \cdots P_m x\|$$

$$= \|Q_{i_o-1} \cdot P_{i_o} \cdot P_{i_o+1} \cdots P_m x\|$$

$$\leq \|Q_{i_o-1}\| \|P_{i_o} x\|$$

$$\|Q_m x\| \leq \|x\|$$

So x not in Ker A implies $\|Q x\| \neq \|x\|$

(sufficiency) : suppose x is in Ker A.

By Claim B, $P_i x = x$ for $i = 1,\ldots,m$

$$Qx = P_1 P_2 \cdots P_m x = x$$

$\therefore$ x in Ker A implies $\|Q x\| = \|x\|$

Corollary 5.3.3.

$\|Q\| \leq 1$. If rank A < n, then $\|Q\| = 1$.

Proof:

$\|Q\| = \|Q_m\| \leq 1$ by * of the lemma above.

Suppose rank A < n.

Then there exists non-zero $x_\circ$ so that

$$Ax_\circ = 0$$

So $x_\circ$ is in Ker A, and so $\|Q x_\circ\| = \|x_\circ\|$ by the lemma above.

$$\text{So} \quad \frac{\|Qx_\circ\|}{\|x_\circ\|} = 1$$

$\therefore \quad \|Q\| \geq 1$ by definition $\|Q\|$

But $\|Q\| \leq 1$

$\therefore \quad \|Q\| = 1$ when rank A < n.

Corollary 5.3.4.

$Qx = x$ if and only if x is in Ker A.

Proof:

Suppose $Qx = x$, then $\|Qx\| = \|x\|$

so x is in Ker A (Lemma 5.3.2.)

Suppose x is in Ker A, then $Ax = 0$

$\therefore \quad RAx = 0$

$\therefore \quad (I-RA) = x$

$\therefore \quad Q x = x$ (Theorem 5.3.1)

Theorem 5.3.5.

(i) (a) Ker A and Im A^* are invariant subspaces under Q

(b) $Q/_{\text{Ker A}} = I_{\text{Ker A}}$ (the identity map on Ker A)

(c) hence $Q = P_{\text{Ker A}} \oplus \tilde{Q}$ and $P_{\text{Ker A}} \cdot \tilde{Q} = \tilde{Q} \cdot P_{\text{Ker A}} = 0$

where $\tilde{Q} = Q\, P_{\text{Im } A^*}$

(Note: In subscripts, will abbreviate Ker A to K, ImA^* to I^*).

(ii) $$\| \tilde{Q} \| = \sup_{\substack{x \text{ in Im } A^* \\ \| x \| = 1}} \| Qx \| < 1$$

Proof:

(i) Ker A is invariant under Q, since Lemma 5.3.2 proved x in Ker A implies Qx = x.

So also $Q/_{\text{Ker A}} = I_{\text{Ker A}}$, so (b) is proved.

If y is in Im A^*, and x is any element of Ker A, then

$$\begin{aligned} (x, Qy) &= (Q^* x, y) \\ &= (P_m^* \ldots P_2^* P_1^* x, y) \\ &= (P_m \ldots P_1 x, y) \\ &= (x, y) \end{aligned}$$

since x in Ker A implies $P_i x = x$ for all $i = 1, \ldots, m$ by Claim B.

Now Im $A^* = (\text{Ker } A)^{\perp}$, so $(x,y) = 0$

$\therefore$ $(x, Qy) = 0$ for any x in Ker A

$\therefore$ Qy is in $(\text{Ker } A)^{\perp}$ for any y in Im A^*

$\therefore$ Qy is in Im A^* for any y in Im A^*

So Im A^* is an invariant subspace under Q.

So (a) is proved.

Recall $C^n = \text{Ker } A \oplus \text{Im } A^*$

so we may express Q as follows, using (a) and (b) just proved

$$Q = P_K \oplus Q \cdot P_I^* = P_K \oplus \tilde{Q} \quad \text{and}$$

where $P_K P_I^* = P_I^* P_K = 0$ and $\tilde{Q}P_K = P_K\tilde{Q} = 0$

(this notation simply indicates that if $z = x + y$ where x is in Ker A, y is in Im A^*, uniquely, then

$$Qz = Qx + Qy = x + QIy \quad \text{by (a) and (b)}$$
$$= P_K z + QP_I^* z$$
$$= (P_K + QP_I^*)\, z)$$

So (c) is proved.

(ii) $$\|Q\| = \sup_{x \text{ in } C^n} \frac{\|\tilde{Q}x\|}{\|x\|} \geq \sup_{x \text{ in Im } A^*} \frac{\|\tilde{Q}\, x\|}{\|x\|} \qquad (\|x\| \neq 0)$$

If x is in Ker A, $P_I^* x = 0$ (by definition P_I^*)
also if x is in Im A^*, $P_I^* x = x$ (by definition P_I^*)
So for z in C^n, if $z = x + y$, x in Ker A, y in Im A^*

$$\tilde{Q}z = QP_I^* x + QP_I^* y = Qy$$

$$\therefore \frac{\|Qz\|}{\|z\|} = \frac{\|Qy\|}{\|z\|} \leq \frac{\|Qy\|}{\|y\|} \qquad \text{(since y in Im } A^* = (\text{Ker } A)^{\perp} \text{ yields } \|y\| \leq \|z\|)$$

Hence $$\sup_{z \text{ in } C^n} \frac{\|\tilde{Q}z\|}{\|z\|} \leq \sup_{y \text{ in Im } A^*} \frac{\|Qy\|}{\|y\|}$$

Hence $$\|\tilde{Q}\| = \sup_{y \text{ in Im}A^*} \frac{\|Qy\|}{\|y\|} = \sup_{\substack{y \text{ in Im } A^* \\ \|y\| = 1}} \|Qy\|$$

Note that $\| \tilde{Q} \| = \| Q P_I^* \| \leq \| Q \| \| P_I^* \| \leq 1$

Suppose $\| \tilde{Q} \| = 1$:

Then since $\| Qx \|$ is a continuous function on the compact set $\{ x \text{ in Im } A^* : \| x \| = 1 \}$, $\| Qx \|$ must attain its supremum for some $x_\circ$ in this set.

So there exists $x_\circ$ in Im A^* with $\| X_\circ \| = 1$

so that $\| Qx_\circ \| = 1$

$\therefore \| Qx_\circ \| = \| x_\circ \|$

By Lemma 5.3.2, then $x_\circ$ is in Ker A

$\therefore$ $x_\circ$ is in Im $A^* \cap$ Ker A, so $x_\circ = 0$,

contradicting $\| x_\circ \| = 1$.

So $\| \tilde{Q} \| < 1$ and (ii) follows.

If rank A = n, then Ker A = {0} and $Q = \tilde{Q}$.

But $\| \tilde{Q} \| < 1$, so $\| Q \| < 1$ in this case.

Corollary 5.3.6.

$$\lim_{i \to \infty} Q^i = P_K$$

Proof:

$$\| Q^i - P_K \| = \| (P_K + \tilde{Q})^i - P_K^i \|$$

$$= \| P_K^i + \tilde{Q}^i - P_K^i \|$$

(since all other terms in $(P_K + \tilde{Q})^i$ contain $P_K \tilde{Q} = 0$)

$$= \| \tilde{Q}^i \|$$

$$\leq \| \tilde{Q} \|^i$$ which approaches 0 as i approaches infinity, since $\| \tilde{Q} \| \leq 1$.

$\therefore$ $\lim_{i \to \infty} Q^i = P_K$

Theorem 5.3.7.

(i) $\lim_{i\to\infty} \sum_{j=0}^{i} (Q^j R)$ exists and equals $(I - \tilde{Q})^{-1} R$

(ii) the $n \times m$ matrix $G = (I - Q)^{-1}R$ is a generalised inverse of A satisfying

$AGA = A, \quad GAG = G, \quad GA = P_I{}^*, \quad AG = P$

where P is the projection onto Im A along Ker R.

(Note that $P_I{}^*$ is an orthogonal projection, so that the third condition here incorporates Penrose's condition (4)).

Proof:

The i-th column vector of R is $d_i^{-1} \cdot Q_{i-1} \cdot A_i$.

Now x in Ker A yields $(a_i, x) = 0$ for $i = 1, \ldots, m$.

So $(Q_{i-1} a_i, x) = (a_i, Q_{i-1}^* x)$

$= (a_i, x)$ (by Claim A)

$= 0$ for $i = 1, \ldots, m$

So $Q_{i-1} a_i$ is in $(\text{Ker } A)^{\perp}$ for $i = 1, \ldots, m$

So $d_i^{-1} Q_{i-1} a_i$ is in $\text{Im } A^*$ for $i = 1, \ldots, m$

But $\text{Im } A^*$ is invariant under Q.

So the column vectors of QR are also in $\text{Im } A^*$

$\therefore Q^j R = \tilde{Q}^j R$ for $j = D, 1, 2, \ldots, i$

and $\sum_{j=0}^{i} Q^j R = \sum_{j=0}^{i} \tilde{Q}^j R$

Now $\|Q\| < 1$ yields $\sum_{j=0}^{\infty} \tilde{Q}^j$ convergent

$\therefore \sum_{j=0}^{\infty} \tilde{Q}^j R$ exists, since

$\sum_{j=0}^{i} \tilde{Q}^j R = \left(\sum_{j=0}^{i} \tilde{Q}^j\right) R$ which has a limit as i approached ∞.

So $\sum_{j=0}^{\infty} Q^j R$ also exists and equals $\sum_{j=o}^{\infty} \tilde{Q}^j R$.

It is easy to check that $(I - \tilde{Q})^{-1} = \lim_{i\to\infty} \sum_{j=o}^{i} \tilde{Q}^j$

where $I - \tilde{Q}$ is invertible since $\| \tilde{Q} \| < 1$.

Hence $\sum_{j=0}^{\infty} Q^j R = (I - \tilde{Q})^{-1} R$

(Note that it may well not follow that $\sum_{j=o}^{\infty} Q^j$ exists so that R may not be removed as a factor from this series, in general).

(ii) The column vectors of A^* are in Im A^* (they are the images of the usual one-vectors in C^n).

$$\begin{aligned} \text{So } (I - Q) A^* &= A^* - QA^* \\ &= A^* - (P_K + \tilde{Q}) A^* \\ &= A^* - \tilde{Q}A^* \\ &= (I - \tilde{Q}) A^* \end{aligned}$$

But $I - Q = RA$ by Theorem 5.3.1

So $RAA^* = (I - \tilde{Q}) A^*$

So $(I - \tilde{Q})^{-1} RAA^* = A^*$

So $GAA^* = A^*$

This is simply equation (6) of Theorem 2.1 in Chapter 2 where it was shown to be equivalent to Penrose's equations (1) and (4)

Hence $\underline{AGA = A}$ and $(GA)^* = GA$

Also $(GA)^2 = G(AGA) = GA$

So <u>GA is hermitian and idempotent</u>, hence is an orthogonal projection (See Property 5.2.5)

$$\text{Also} \quad RA\,(I - \tilde{Q})^{-1} = (I - Q)(I - \tilde{Q})^{-1}$$

$$= (I - \tilde{Q} - P_K)(I - \tilde{Q})^{-1}$$

$$= (I - \tilde{Q})(I - \tilde{Q})^{-1} - P_K(I - \tilde{Q})^{-1}$$

$$= I - P_K\,(I - \tilde{Q})^{-1}$$

$$\text{(but} \quad P_K(I - \tilde{Q}) = P_K - P_K\tilde{Q} = P_K$$

$$\text{so} \quad P_K = P_K(I - \tilde{Q})^{-1}\ )$$

$$\therefore\ RA\,(I - \tilde{Q})^{-1}\,R = (I - P_K)\,R$$

$$= R - P_K R$$

$$= R$$

$$\therefore\ (I - \tilde{Q})^{-1}\,RA(I - \tilde{Q})^{-1}R = (I - \tilde{Q})^{-1}\,R$$

or $\underline{GAG = G}$

Also $(AG)^2 = AGAG = AG$

So $\underline{AG \text{ is idempotent}}$, so is a projection (Property 5.2.5)

We have still, however, to establish the subspace onto which these projections map.

Since $AGA = A$, $\quad \text{Im } AG \supset \text{Im } A$ (a)

Also $\quad \text{Ker } AG = \text{Ker } (A\,(I - \tilde{Q})^{-1}\,R)$

so $\quad \text{Ker } AG \supset \text{Ker } R$ (b)

Now $\quad \text{Im } AG \oplus \text{Ker } AG = C^m$

since AG is square, and $\text{Im } AG \cap \text{Ker } AG = \{o\}$.

So $\quad \text{Im } AG \cap \text{Ker } AG = \{o\}$

$\therefore\ \text{Im } A \cap \text{Ker } R = \{0\}$ by (a) and (b)

The i-th column vector of R is

$$d_i^{-1} Q_{i-1} a_i = d_i^{-1} (P_1 \ldots P_{i-2})(P_{i-1} a_i)$$

$$= d_i^{-1} (P_1 \ldots P_{i-2})(I - d_{i-1}^{-1}(a_{i-1} a_{i-1}^*) a_i$$

$$= d_i^{-1} (P_1 \ldots P_{i-2})(a_i - d_{i-1}^{-1}(a_i, a_{i-1}) a_{i-1}$$

and by continuing the expansion

$$= d_i^{-1} \left(a_i - \sum_{j=1}^{i-1} c_j a_j\right)$$

i.e. the columns of R are linear combinations of the columns a_i of A^*

So $\quad \text{Im } R \subseteq \text{Im } A^*$

Similarly $\quad \text{Im } A^* \subseteq \text{Im } R$

So $\quad \text{Im } A^* = \text{Im } R$

$\therefore \dim (\text{Im } A^*) = \dim (\text{Im } R)$

$\therefore \dim (\text{Im } A) = \dim (\text{Im } R^*)$

(but $\text{Im } A \oplus \text{Ker } A^* = C^m$

and $\text{Im } R^* \oplus \text{Ker } R = C^m$, as in Claim A)

$\therefore \dim (\text{Ker } A^*) = \dim (\text{Ker } R)$

But $\dim (\text{Im } A \oplus \text{Ker } A^*) = \dim (C^m)$

Hence $\dim (\text{Im } A + \text{Ker } R) = \dim (C^m)$

but we proved $\text{Im } A \cap \text{Ker } R = \{0\}$

Hence $\quad \text{Im } A \oplus \text{Ker } R = C^m$

So by (a) and (b), AG is the projection onto Im A along Ker R.

Also, since $GA = A^* G^*$, $\text{Im}(GA) \subset \text{Im } A^*$

also $(AGA)^* = A^*$

$\therefore$ $GA\, A^* = A^*$, so $\text{Im } (GA) \supset \text{Im } A^*$

Hence $\quad \text{Im}(GA) = \text{Im } A^*$, i.e. $GA = P_I^*$

Corollary 5.3.8:

For any m × n matrix A with non-zero rows, and any m-dimensional column vector b, the algorithm

$$x_{i+1} = F(b; x_i) \qquad i = 0,1,2,\ldots$$

generates a convergent sequence of vectors such that

$$\lim_{i\to\infty} x_i = P_K x_\circ + Gb$$

where $x_\circ$ is an arbitrary initial vector in C^n.

Proof:

$$F(b, x) = Qx + Rb$$

$$\therefore x_1 = Q x_\circ + Rb$$

$$x_2 = Q (Q x_\circ + Rb) + Rb$$

$$= Q^2 x_\circ + QRb + Rb$$

$$x_3 = Q (Q^2 x_\circ + QRb + Rb) + Rb$$

$$= Q^3 x_\circ + Q^2Rb + QRb + Rb$$

continuing this process, we find

$$x_i = Q^i x_\circ + \sum_{j=\circ}^{i-1} Q^j Rb$$

$$= Q^i x_\circ + \left(\sum_{j=\circ}^{i-1} Q^j R\right) b$$

Now $\lim_{i\to\infty} Q^i x_\circ = P_K x_\circ$ (by Cor 5.3.6)

and $\lim_{i\to\infty} \sum_{j=\circ}^{i-1} Q^j R$ exists (by Th. 5.3.7)

So the sequence of x_i (i = 0,1,2,...) converges to

$$P_K x_\circ + Gb$$

CONCLUSION: (to solve the system $Ax = b$)

By Th. 2.2 our system has a solution if and only if $AGb = b$; and if this condition is satisfied, then $Gb + x_o - GAx_o$ is a solution, for arbitrary x_o.

Now $Gb + (I - GA)\, x_o = Gb + P_K x_o$ (by 5.2.7)
(also, by Th. 2.3, Gb is that solution with minimum norm).

So to solve the system:

(i) calculate Gb by applying the iterative method to x_o the null vector, 0.

(ii) test whether $A(Gb) = b$

(iii) if the system is consistent, calculate P_K by using the method with $b = 0$, and using the (column) one-vectors e_i as starting vectors, $i = 1,2,\ldots,n$. The i-th application of the method will yield the i-th column of P_K.

(iv) $P_K x_o + Gb$ is then the general solution (x_o arbitrary).

5.4. THE METHOD APPLIED TO FINDING THE GENERALISED INVERSE G of A.

Using $x_o = 0$, the null vector, apply the method m times, to solving $Ax = e_i$ $(i = 1,2,\ldots,m)$ (e_i the (column) one-vectors). The i-th application will yield a sequence of vectors converging to $P_K.0 + Ge_i$, i.e. the i-th column of G.

Remark: clearly, if the last condition for G in Th. 5.3.7, (namely, AG is the projection P onto Im A along Ker R) can be strengthened to become

AG is the orthogonal projection onto Im A

then $(AG)^* = AG$, by 5.2.5, and so $G = A^+$.

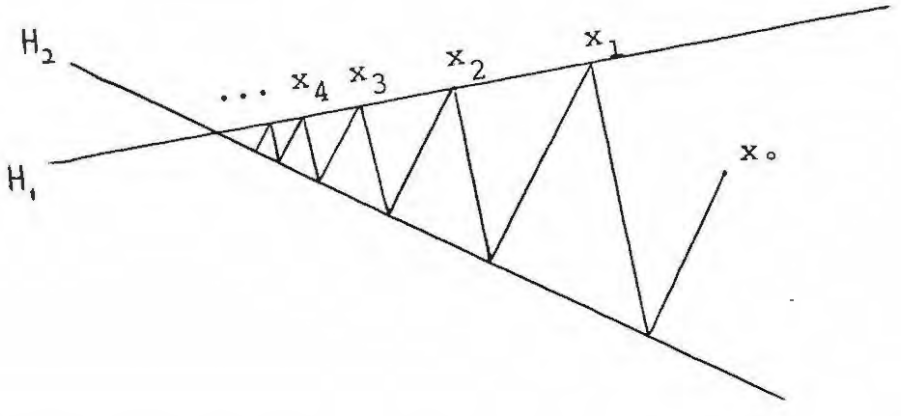
Considering 5.2.4, we conclude that if, for a given matrix A, Ker R = $(\text{Im } A)^{\perp}$, then Tanabe's algorithm will yield A^{+} and not just A^{w}.

5.5. CONCLUDING REMARKS

Tanabe quotes results of six well-chosen examples, used to test this algorithm, and notes that these illustrate an expected phenomenon : convergence of the method is fast if the rows of A are mutually "nearly" orthogonal. That this should be so follows from a consideration of the geometry of the method:

if the equations $(x,a_i) = b_i$ are viewed as defining hyperplanes H_i, $i = 1,2,\ldots,m$, then the maps f_i simply project vectors on which they operate onto the corresponding hyperplanes H_i.
Hence $x_{i+1} = F(b,x_i) = f_1 \circ f_2 \circ \ldots \circ f_m\ (x_i)$ is the vector arising from the projection of x_i onto the H_i in the order $i = m, m-1, \ldots, 1$.

We illustrate this in the case where $m = 2$:



Clearly convergence is fast if H_1 and H_2 are nearly orthogonal. Also notice more generally, that the iterative sequence x_i , $i = 1,2,\ldots$ lies in the hyperplane H_1.

When convergence is slow (e.g. A is "nearly" singular, or $\| \tilde{Q} \|$ close to 1) an acceleration technique is desirable. Tanabe claims that Aitken's Δ^2 - method has the disadvantage that if it is applied to a singular system, it disturbs the dependence (in general) of the limit point on the initial vector (see Cor. 5.3.8). L. Duane Pyle describes on pp. 218 - 235 of [25], a generalised "epsilon - algorithm", which he applies to accelerating the convergence of the "symmetric" Kaczmarz method. The latter is a variation which includes in one stage of the iteration, projection onto each of the hyperplanes $H_1, \ldots, H_n$ in that order, followed by projection back onto $H_{n-1}, \ldots, H_1$ (in that order). The ε_4 - algorithm applied to this example, yields a semi-iterative method (see Varga [30]) which is then useful as an accelerating process.

In comparison to the Gauss - Seidel and Jacobi methods for iteration when A is non-singular, the method of this chapter may be slow (e.g. when A is large and sparse) but has the advantage that it does always converge. A non-singular, but ill-conditioned, 84-dimensional test matrix, was used as A in the system $Ax = b$ solved by Gaussian elimination without iterative or pivoting improvement. The poor results obtained are compared in Tanabe's paper with those at various iterations using the projection method, where the error proves negligible by comparison.

To conclude, it is noted that observation that AA^+ and A^+A may be viewed as orthogonal projections has been used in the search for algorithms which yield A^+, as far back in the literature as 1964. At that time Pyle published one using a "gradient projection" method which Rosen had used frequently in non-linear programming in 1960/1961. (See bibliograph of [19a].)

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