Is Jump Risk Priced? — What We Can (and Cannot) Learn From Option Hedging Errors

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Abstract

When options are traded, one can use their prices and price changes to draw inference about the set of risk factors and their risk premia. We analyze tests for the existence and the sign of the market prices of jump risk that are based on option hedging errors.

We derive a closed-form solution for the option hedging error and its expectation in a stochastic jump model under continuous trading and correct model specification. Jump risk is structurally different from, e.g., stochastic volatility: there is one market price of risk for each jump size (and not just *the* market price of jump risk). Thus, the expected hedging error cannot identify the exact structure of the compensation for jump risk.

Furthermore, we derive closed form solutions for the expected option hedging error under discrete trading and model mis-specification. Compared to the ideal case, the sign of the expected hedging error can change, so that empirical tests based on simplifying assumptions about trading frequency and the model may lead to incorrect conclusions.

JEL: G12, G13

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1 Introduction and Motivation

It is the goal of asset pricing theory to explain expected excess returns on financial assets. The two key issues are the identification of the relevant risk factors, and the determination of the associated market prices of risk. Given this information the expected excess return can be computed for an arbitrary contingent claim. In derivative pricing researchers are particularly interested in stochastic volatility (SV) and stochastic jumps (SJ), which represent the main risk factors, besides the price risk of the underlying, discussed in the option pricing literature. An important example is the model developed by Bakshi, Cao, and Chen (1997). SV and SJ are distinctly different, as shown by Das and Sundaram (1999) and by Carr and Wu (2003) who discuss the implications of SV and SJ on the pricing of derivative securities. Further differences will be discussed in this paper.

The focus of this paper is on jump risk. We analyze the theoretical properties of tests that try to identify the pricing of jump risk from the properties of option hedging errors. The main idea of these tests is that the expected return of an option depends on the compensation for stock price risk and some other risk factors. Once stock price risk is eliminated via a standard delta hedge the remaining hedging error is due to these other risk factors and can thus be used to learn something about the associated market prices of risk. Note that our goal is not to find the risk-minimizing hedge with the smallest possible error, but to use the hedging error to identify risk factors and to learn about the characteristics of the associated risk premia.

When a researcher tries to apply this approach in an empirical study he or she is faced with (at least) two problems. Although the theory is usually developed in continuous time, there will be only discrete observations in real data. Furthermore, in an empirical application the true model is not known. The additional question is thus whether we can still identify the jump risk premium in case of discrete trading and model mis-specification. Stated differently, we investigate the robustness of tests developed in continuous-time models when these tests are performed in an economy with discrete trading and a true model which is unknown to the researcher.

Recently, there has been growing research interest in the question whether there are additional risk factors besides stock price risk, and whether these risk factors are priced. Based on the result that the pricing kernel can be spanned by two assets if and only if there are no additional risk factors besides price risk of the underlying, Buraschi and Jackwerth (2001) perform an empirical test and find that due to the presence of additional risk factors deterministic volatility models appear to be misspecified. Coval and Shumway (2001) empirically show that option returns cannot be explained by the risk-free interest rate and stock returns, and Bakshi and Kapadia (2003) perform a hedging test with the main result that options cannot be hedged by the stock and the money market account. These papers suggest the presence of additional risk factors on options markets. Branger and Schlag (2003) analyze the properties of standard option hedging tests for the market price of volatility risk. Their main finding is that discretization error and model mis-specification will in many cases destroy the relationship between the sign of the expected hedging error and the sign of the volatility risk premium, which holds under the ideal scenario of continuous trading and correct model specification.

The contribution of this paper is an analysis of hedging based tests to infer the market prices of risk for stochastic jumps. We derive closed-form solutions for the expected hedging error (EHE) not only under ideal conditions, but also for the empirically relevant situations of discrete trading and for model mis-specification. Although the analysis is very formal in nature we also focus on the economic interpretation of the results derived in our propositions and corollaries. We show how to use the EHE to learn about the market prices of jump risk, and we discuss limitations due to the specialness of jump risk is special (compared to SV), and due to discretization and model mis-specification.

Jump risk is more difficult to hedge than volatility risk, since one has to set up a hedge against every possible jump size. The number of hedge instruments is thus equal to the number of possible jump sizes, and each jump size can be interpreted as one risk-factor with an associated market price of risk. The identification problem of all these risk premia is therefore much more demanding than in the case of SV. As an aside, we show that the fact that there is one market price of risk for each jump size also implies a more complicated general structure for the risk premia of assets. For example, the jump risk premia on a call option and on its underlying asset may have different signs, despite the fact that the stock and the call react to a jump in the same direction.

In the course of the analysis of option hedging errors under ideal conditions we show that the risk of the option can be decomposed into an exposure to stock price risk and an additional exposure to jump risk that is not captured by the option delta. The EHE is shown to depend on this additional exposure to jump risk and the associated premium. Under the additional restriction that only jump intensity (but not jump size) is priced, the EHE identifies the sign of market price of jump intensity risk. The EHE is highest for short term at-the-money (ATM) options, so these options are basically well-suited for an empirical identification of the market prices of jump risk.

We then analyze the impact of model mis-specification and of discretization on the EHE. Concerning the impact of model mis-specification we find that the difference between the EHE under this scenario and the ideal EHE depends on the slope of the volatility smile and on the risk premium of the stock. For a positive premium and a downward sloping volatility smile, the EHE is higher than in the ideal case. In case of discretization error the difference to the ideal EHE is caused by a stale hedge ratio. It has two components, a diffusion component depending on the gamma of the claim and a jump component, which is related to the discrete changes in delta caused by jumps. Our numerical examples show that the differences between the EHEs under discrete trading and model mis-specification and the ideal EHE are highest for short term ATM options. This is an important result, since it shows that ATM options are not as useful empirically for the identification of jump risk premia as they might appear from a theoretical perspective.

The remainder of the paper is organized as follows. The main idea of a hedging-based test for the sign of the jump risk premium as well as some definitions related to hedging errors are presented in Section 2. Section 3 introduces the model setup with SJ and provides a discussion of expected returns on the stock and on associated contingent claims in the presence of jumps. Closed-form solutions for the EHE under ideal conditions as well as under discretization and model mis-specification are presented in Section 4, where we also provide some numerical examples. Section 5 gives the conclusion and makes some suggestions for further research.

2 Hedging Based Tests

2.1 Idea

The return on a contingent claim can be explained by the short rate and the risk factors it is exposed to, like, e.g., stock price risk, SV, or SJ. When we hedge the contingent claim using the stock and the money market account, we control for the impact of stock price risk, so that the hedging error can be attributed to non-traded risk factors. The key idea of the hedging based identification of risk premia represents some kind of re-engineering in using this hedging error to learn something about the market prices of risk of these risk factors. In a first step one may simply be interested in the sign of the premium, as are Bakshi and Kapadia (2003) or Branger and Schlag (2003). We discuss whether and how this idea can be used to identify the market prices of jump risk.

The implicit assumption of such a test is that we can indeed eliminate stock price risk completely. However, in empirical studies, there are two problems, the consequences of which will also be analyzed in this paper. First, trading does not take place continuously. Second, the true data-generating process is not known, so that the researcher is likely to use a mis-specified model to set up the hedge.

It should be noted explicitly that this paper is not on risk management, and our objective is not to minimize the residual risk of the hedge portfolio, but to identify the market prices of risk. However, in practical applications, simple delta hedging strategies and the use of the Black-Scholes (BS) model to set up hedges are highly relevant, and since we discuss the properties of the resulting hedging error both for continuous and for discrete trading, our results are also of interest to practitioners.

2.2 Hedging Strategy and Hedging Error

A hedge portfolio is characterized by its initial value Π_0 and by a trading strategy. At time t the number of shares of the stock in the portfolio is given by H_t, and the investment in the money market account (earning a deterministic interest rate r) is determined such that the portfolio is self-financing. This yields a value of the hedge portfolio at time t given by

$$\Pi_t = e^{rt}\left(\Pi_0 + \int_0^t e^{-ru} H_u (dS_u - rS_u du)\right)$$

where S_t is the price of the underlying at time t.

The hedging error at time t, D_t, is the difference between the price C_t of the contingent claim and the value of the hedge portfolio, H_t, i.e.

$$D_t = C_t - \Pi_t.$$

Analogously, the hedging error over the interval from t to $t+\tau$ is defined as

$$D(t, t+\tau) = (C_{t+\tau} - \Pi_{t+\tau}) - e^{r\tau}(C_t - \Pi_t).$$

In the following, we concentrate on the hedging errors over a single period with discrete length. In case of continuous hedging, the hedge ratio can change at every point in time. The continuous hedging error over the interval $[t, t+\tau]$ is denoted by $D^c(t, t+\tau)$, and it is given by

$$\begin{aligned} D^c(t, t+\tau) &= (C_{t+\tau} - \Pi_{t+\tau}) - e^{r\tau}(C_t - \Pi_t) \\ &= e^{r(t+\tau)}\left(\int_t^{t+\tau} e^{-ru}(dC_u - rC_u du) - \int_t^{t+\tau} e^{-ru} H_u (dS_u - rS_u du)\right). \end{aligned} \quad (1)$$

In case of discrete hedging, the hedge portfolio is only rebalanced at the discrete trading dates $0 = t_0 < t_1 < \ldots < t_n$, where $t_n \equiv T$ is the maturity date of the derivative contract. The hedge ratio over the interval $[t_i, t_{i+1}]$ is constant and equal to H_{t_i}, i.e. to the value at the beginning of the period. The discrete hedging error over the interval $[t_i, t_{i+1}]$ is denoted by $D^d(t_i, t_{i+1})$ with

$$D^d(t_i, t_{i+1}) = C_{t_{i+1}} - C_{t_i} e^{r(t_{i+1}-t_i)} - H_{t_i}\left(S_{t_{i+1}} - S_{t_i} e^{r(t_{i+1}-t_i)}\right). \quad (2)$$

3 Jump-Diffusion Model

3.1 Model Setup

We consider a jump-diffusion model. The stock price process under the physical measure P is represented by the stochastic differential equation

$$dS_t \;=\; \mu S_{t-}dt + \sigma_S S_{t-}dW_t^S + S_{t-}\left(X_t dN_t - h^P \mathrm{E}^P[X]dt\right), \tag{3}$$

where N is a Poisson process with an intensity under P denoted by h^P. The sizes X of the different jumps are assumed to be independent and identically distributed. The mean jump size under P is equal to $\mathrm{E}^P[X]$ and -1 is a lower bound on the jump size. For ease of notation, we assume that the jump intensity, the distribution of the jump size, the drift μ, and the volatility σ_S neither depend on calendar time t nor on the current stock price S_t.

Under the risk-neutral measure Q the dynamics of the stock price are

$$dS_t \;=\; rS_{t-}dt + \sigma_S S_{t-}d\widetilde{W}_t^S + S_{t-}\left(X_t dN_t - h^Q \mathrm{E}^Q[X]dt\right). \tag{4}$$

When changing the measure from P to Q we introduce a new standard Brownian motion $d\widetilde{W}_t^S$ according to

$$d\widetilde{W}_t^S = dW_t^S + \lambda^W dt$$

with λ^W as the market price of W-risk. For the jump risk component, the intensity changes from h^P to h^Q, and the distribution of the jump size changes from $P(dx)$ under the physical measure to $Q(dx)$ under the risk-neutral measure. An interpretation of these changes in terms of risk premia will be given in Section 3.2. Technical details on the change of measure for the jump process can be found in the appendix.

3.2 Expected Return on the Stock

In our model the expected return on the stock is given by

$$\begin{aligned} \mu &= r + \sigma_S \lambda^W + E^P[X]h^P - E^Q[X]h^Q \\ &= r + \sigma_S \lambda^W + \int_{-1}^{\infty} x \left[h^P P(dx) - h^Q Q(dx) \right]. \end{aligned} \tag{5}$$

Note that the stock is exposed to diffusion risk and to jump risk, which is different from the case of SV, where the stock as a linear contract is not exposed to SV. It is important to be clear about the terminology. In the following discussion the term 'diffusion risk' relates to W, 'jump risk' relates to X and N, and 'stock price risk' sums up diffusion risk and jump risk. The exposure to diffusion risk is measured by the volatility coefficient σ_S. To be hedged against this type of risk, we need one instrument with non-zero diffusion sensitivity, and consequently, there is one market price λ^W of diffusion risk.

On the other hand, there is one market price of risk for each possible jump size. This follows from the fact that to be hedged against jump risk, we need one instrument for each possible jump size. To see this consider the following simple example where the stock is only exposed to jump risk. There are two possible jump sizes x_1 and x_2, and no diffusion risk. The change in the stock price is given by

$$dS_t = S_{t-}(\mu - h^P \mathrm{E}^P[X])dt + S_{t-}x_1 dN_t^{(1)} + S_{t-}x_2 dN_t^{(2)},$$

where $dN_t^{(i)} = 1$ $(i = 1, 2)$ if a jump of size x_i occurs, and zero otherwise. The price of a derivative contract C is a function of the stock price, and the change in its value is given by

$$dC_t = C_{t-}\mu_C dt + C_{t-}f^C(x_1)dN_t^{(1)} + C_{t-}f^C(x_2)dN_t^{(2)},$$

where $f^C(x)$ is the percentage change in the claim price if the stock jumps by x, and μ_C is the drift of the claim price. At the moment, we are not concerned with the exact functional form of f^C and μ_C. To create a risk-free portfolio, we form a portfolio of the claim and the hedge instruments which is no longer exposed to any of the risk factors. With two risk factors, namely a jump of size x_1 and a jump of size x_2, the situation is similar to a trinomial model (where we implicitly assume that the two Poisson processes do not jump simultaneously), and we need in general two instruments exposed to these risk factors to eliminate jump risk. The example can be easily generalized to more than two jump sizes. We need one hedge instrument for every possible jump size, and thus, there is not one market price of jump risk, but there is one market price of jump risk for each possible jump size x. Consequently, for a continuous jump size distribution there are infinitely many market prices of jump risk.

According to equation (5) the total jump premium of the stock can be represented as the integral over the compensations for each individual jump size x. Obviously, the exposure of the stock to a jump of size x is just equal to x, and its contribution to the jump risk premium is equal to this exposure times the difference $h^P P(dx) - h^Q Q(dx)$ between the 'intensity of a jump of size x' under the physical and under the risk-neutral measure. This difference thus includes all the information about the pricing of jump of size x.

If the intensities of a jump of size x are the same under P and Q, a jump of size x in the stock is not priced. To see what happens if a jump is priced, consider a negative jump size $x < 0$ first. If the risk neutral intensity is greater than the physical intensity, the difference $h^P P(dx) - h^Q Q(dx)$ is negative, and the contribution to the total risk premium is positive. Intuitively, when it comes to pricing, the investor 'over'-estimates the probability of this negative jump, that is he demands a compensation for this jump. For a positive jump the argument goes just the other way around. If its risk-neutral intensity is larger than the physical intensity, the investor is willing to pay a compensation for this jump and its contribution to the total jump risk premium is negative.

3.3 Expected Return on a Contingent Claim

We now consider the expected return on a contingent claim written on the stock. Assume that the change in the claim price C is

$$dC = \left(\mu_C - h^P \mathrm{E}^P[f^C(X)]\right) C dt + \sigma_C C dW_t^S + f^C(X) C dN_t,$$

where μ_C is the expected return, σ_C is the volatility of the claim, and the function $f^C(x)$ represents the impact of a jump of size x on the claim price (the problem of determining these sensitivities will be discussed later on). For ease of notation, we have suppressed any possible time dependence. Analogously to the expected return on the stock, the expected return on the claim is given by

$$\mu_C = r + \sigma_C \lambda^W + \int_{-1}^{\infty} f^C(x) \left[h^P P(dx) - h^Q Q(dx) \right].$$

To determine this expected return we obviously need to know the market prices of diffusion and of jump risk. In the following, we consider empirical procedures (which will be called 'tests') that try to identify these market prices of risk on the basis of option hedging errors.

For a continuous jump size distribution, there are infinitely many market prices of jump risk. This not only complicates their identification, but it generally makes jump risk more difficult to deal with than volatility risk in an SV model. To give a flavor of the complexity of jump risk, we compare the jump risk premium on a stock to that on a call written on that stock. In our setup, the call price is an increasing function of the stock price, so it seems obvious at first sight that the jump risk premia on the stock and on the call must have the same sign. However, there are situations where this conjecture is not true. Assume $h^Q > h^P$ and $Q(dx) = P(dx)$. Then the jump risk premium of a contingent claim C is $\mathrm{E}^P \left[f^C(X) \right] (h^P - h^Q)$, and it is positive if and only if the mean exposure $\mathrm{E}^P \left[f^C(X) \right]$ to jumps is negative. For a stock, the mean exposure to jump risk is $\mathrm{E}^P[X]$, since in this case $f^C(X) = X$. For a call option, the mean exposure to jump risk is given by

$$\mathrm{E}^P \left[c^{SJ}(t, S_{t-} + S_{t-}X_t) - c^{SJ}(t, S_{t-}) | \mathcal{F}_{t-} \right].$$

Here $c^{SJ}(t, S_t)$ is the call price function $c^{SJ}(t, s)$ evaluated at time t for the current stock price $s = S_t$. For $\mathrm{E}^P[X] = 0$, the mean exposure of the call is positive, since the call price is convex in the stock price, and the mean exposure of the stock is zero. Thus, the jump risk premium of the call is negative, whereas the jump risk premium of the stock is zero. The intuition behind this result is that the investor pays a compensation for positive jumps and requires a compensation for negative jumps. Under the assumption $\mathrm{E}^P[X] = 0$, the negative and positive compensations just cancel out for the stock. The call, on the other hand, is a convex function of the stock, so that the price increase due to positive jumps is greater in absolute terms than the price decrease due to negative jumps. The compensation paid for positive jumps thus exceeds the compensation required for negative jumps, and the total jump risk premium becomes negative. We can even find distributions for the jump size $P(dx)$, for which the jump risk premium of the stock is positive (with $\mathrm{E}^P[X] < 0$), while the jump risk premium of an ATM call is negative (since the mean exposure of the call is still positive). Furthermore, the jump risk premium for a call far in the money converges towards the jumps risk premium of the stock and is positive, so that the sign of the jump risk premium of the call even depends on its moneyness. Aagain, these results highlight the special character of jump risk, since they do not hold, e.g., for the SV case.

4 Analysis of Expected Hedging Errors

To identify the market prices of jump risk, we use hedging based tests as described in Section 2. These tests are based on the hypothesis that there is some relation between the properties of the hedging error, in particular its mean, and the market prices of risk. We now analyze the EHE to see the actual form of this relationship and to see whether and how the EHE can be used to infer information about the market prices of jump risk.

We consider different delta-hedging strategies for the call, all of which only use the stock and the money market account. These strategies differ with respect to whether they are adjusted continuously or only at discrete points in time, and with respect to the hedge ratio used. This hedge ratio will be given either by the partial derivative of the call price from the SJ model or from the (in this case mis-specified) BS model.

The call price C_t^{SJ} is written as $C_t^{SJ} = c^{SJ}(t, S_t)$. The hedging error over the interval from t to $t+\tau$ is

$$D(t,t+\tau) = \int_t^{t+\tau} e^{r(t+\tau-u)} \left[dC_u^{SJ} - rC_{u-}^{SJ}du - H_u(dS_u - rS_{u-}du)\right].$$

The following proposition gives a general representation of the EHE in the SJ model.

Proposition 1 (SJ: EHE for arbitrary hedge ratio) *The true model is the SJ model given by (3). Then the EHE over the interval* $[t, t+\tau]$ *with a hedge ratio of* H_t *is*

$$\begin{aligned} &E^P[D(t,t+\tau)\,|\,\mathcal{F}_t] \\ &= \int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[\int_{-1}^{\infty} f^{SJ}(u,S_{u-},x)\left[h^P P(dx) - h^Q Q(dx)\right] \,\Big|\, \mathcal{F}_t\right] du \\ &\quad + (\mu - r)\int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[S_u\left(\frac{\partial c^{SJ}}{\partial s}(u,S_{u-}) - H_u\right) \,\Big|\, \mathcal{F}_t\right] du, \end{aligned} \tag{6}$$

where

$$f^{SJ}(u,s,x) \;=\; c^{SJ}(u,s+sx) - c^{SJ}(u,s) - \frac{\partial c^{SJ}}{\partial s}(u,s)\,sx. \tag{7}$$

Proof: See the appendix. □

The expression for the EHE in Equation (6) will form the basis for the analysis of the EHE under ideal conditions as well as for the impact of the discretization error and model mis-specification.

4.1 EHE under Ideal Conditions

4.1.1 Delta hedge

We consider a delta-hedge. The hedge ratio is equal to the partial derivative of the call price with respect to the stock price, i.e.

$$H_t = \frac{\partial c^{SJ}}{\partial s}(t, S_{t-}).$$

By plugging this expression into Equation (6) we immediately obtain the following corollary:

Corollary 1 (SJ: Expected hedging error) *The true model is the SJ model given by (3). The hedge portfolio is rebalanced continuously, the hedge ratio is the partial derivative of the claim price with respect to the stock price. Then, the EHE over the interval* $[t, t+\tau]$ *is*

$$\begin{aligned} & E^P\left[D^c(t,t+\tau) \mid \mathcal{F}_t\right] \\ = & \int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[\int_{-1}^{\infty} f^{SJ}(u, S_{u-}, x)\left[h^P P(dx) - h^Q Q(dx)\right] \mid \mathcal{F}_t\right] du. \end{aligned}$$

Note that this corollary corrects a mistake in a formula given in Bakshi and Kapadia (2003). For each jump size, the function f is multiplied by the difference between the jump intensities under the true and under the risk-neutral measure, whereas in their formula, the delta of the claim times the jump size (which is last part of the function f given in (7)) is only multiplied by the jump intensity under the risk-neutral measure.

The expression given in our corollary is the 'ideal' EHE for continuous trading and a correctly specified hedging model. To use this ideal EHE for the purpose of identifying the market prices of jump risk, we proceed in two steps. First, we consider the structure of the realized hedging error and explain the EHE by the market prices of jump risk, i.e. we analyze the relationship between the EHE and the market prices of risk. Secondly, we try to extract the market prices of risk from the EHE. As stated above there is no such thing as *the* market price of jump risk, but rather a market price of jump risk for each jump size x. Thus, we cannot expect to identify all these risk premia from just one option hedging error. However, the question of what we can actually learn from an EHE still remains.

4.1.2 Explaining the ideal EHE

To analyze the ideal EHE in more detail, we consider the realized change of the call price dC_t^{SJ}. It follows from applying Ito's lemma to the call price, inserting the fundamental partial differential equation and using equations (3) and (5) for the change in the stock price and the expected return on the stock:

$$
\begin{aligned}
dC_t^{SJ} &= rC_{t-}^{SJ}dt + \frac{\partial c^{SJ}}{\partial s}(t, S_{t-})\,\sigma_S S_{t-}\left(\lambda^W dt + dW_t\right) \\
&\quad + \left[c^{SJ}(t, S_{t-} + S_{t-}X_t) - c^{SJ}(t, S_{t-})\right] dN_t \\
&\quad - E^Q\left[c^{SJ}(t, S_{t-} + S_{t-}X_t) - c^{SJ}(t, S_{t-}) \,|\, \mathcal{F}_{t-}\right] h^Q dt.
\end{aligned}
$$

The call is obviously exposed to both diffusion and jump risk. Its exposure to diffusion risk is measured by the call delta and is a linear function of the diffusion risk exposure of the stock. The exposure to a jump of size x is given by the difference between the call value after a jump of size x and the value before, i.e. by

$$
c^{SJ}(t, S_{t-} + S_{t-}x) - c^{SJ}(t, S_{t-}).
$$

This expression can be decomposed into two parts, namely the exposure to the price risk of the stock, which is itself exposed to jump risk, and an additional exposure to jump risk. The first part is again given by the delta:

$$
\frac{\partial c^{SJ}}{\partial s}(t, S_{t-})S_{t-}x. \tag{8}
$$

For a non-linear contract like a call, there is also some additional exposure to jump risk which cannot be explained by the delta of the claim. This additional exposure to a jump of size x is given by the function f^{SJ} from Equation (7) in Proposition 1. For a convex claim like the call, the additional exposure is positive, irrespective of the sign of the jump in the stock price.

When we set up a delta hedge, we control for stock price risk, and the remaining hedging error is explained by the non-hedgeable risks. Diffusion risk will be hedged, just like the indirect jump risk exposure (8). However, the additional jump risk exposure from Equation (7) cannot be hedged:

Proposition 2 (SJ: Realized hedging error) *The true model is the SJ model given by (3). The hedge portfolio is rebalanced continuously, the hedge ratio is given by the delta* *of the claim. Then, the realized hedging error over the interval* $[t, t+\tau]$ *is*

$$\begin{aligned} D^c(t, t+\tau) &= \int_t^{t+\tau} e^{r(t+\tau-u)} f^{SJ}(u, S_{u-}, X_u) dN_u \\ &\quad - \int_t^{t+\tau} e^{r(t+\tau-u)} E^Q[f^{SJ}(u, S_{u-}, X_u)|\mathcal{F}_{u-}] h^Q du. \end{aligned}$$

For a convex claim the function f^{SJ} is positive. If no jumps occur ($dN_u \equiv 0$), the realized hedging error for a call will thus be negative. For each jump a positive term is added. If the stock price jumps often enough and if the jumps are high enough, the realized hedging error ultimately becomes positive. Note that this result holds irrespective of the jump directions and irrespective of the market prices of jump risk.

From Proposition 2 we can conclude that the EHE depends on the probability distribution for the jumps. The higher the true probabilities for positive hedging errors in case of jumps, the higher the EHE. If jump risk is not priced, that is if the jump size has the same distribution under P and Q and if $h^P = h^Q$, the EHE is zero. It is positive if the physical distribution puts more weight on the jumps, and it is negative if the physical distribution puts less weight on the jumps than the risk-neutral measure.

The EHE can also be explained by the remaining risk exposure. The exposure of the hedge portfolio to a stock price jump of size x is given by the additional jump risk exposure $f^{SJ}(t, S_{t-}, x)$ from equation (7), and the compensation is

$$f^{SJ}(t, S_{t-}, x) \left[h^P P(dx) - h^Q Q(dx) \right].$$

The ideal EHE depends on the compensation for additional jump risk of all possible jump sizes $-1 < x < \infty$, i.e. on the integral

$$\int_{-1}^{\infty} f^{SJ}(t, S_{t-}, x_t) \left[h^P P(dx) - h^Q Q(dx) \right],$$

as can also be seen in Proposition 1. In the next section, we discuss how this compensation for additional jump risk depends on the change in the jump intensity and on the change in the jump size distribution when we move from P to Q.

4.1.3 Linking the Ideal EHE to the Market Prices of Jump Risk

We are now going to discuss what we can learn from the EHE about the market prices of jump risk. For a continuous jump size distribution, there are infinitely many market prices of jump risk. It is therefore not possible to identify all market prices of risk even from more than one EHE without further restrictions. We consider three types of such restrictions. First, only the jump intensity changes when we switch from the physical measure P to the risk-neutral measure Q. Second, only the distribution of jump size changes, and third, both the intensity and the size distribution change. Since the first case is the one which allows the most detailed analysis, we will mainly focus on this scenario and consider the other two cases only briefly.

In all three cases, we provide numerical examples and analyze how the EHE depends on the moneyness and on the maturity of the call option. For these examples, we assume that jumps are lognormally distributed, which results in the Merton (1976) jump-diffusion model. Although this is a simplifying assumption, it seems justifiable in the context of our analysis, since we are only interested in the fundamental properties of SJ models. We simplify the analysis in that we do not evaluate the expectations explicitly, but approximate the EHE by multiplying the integrand at time t by the length of the hedge interval. If one was interested in a more detailed analysis of the numerical characteristics of the EHE, one would have to resort to techniques like Monte-Carlo simulation. In the examples in this section, we fix the risk-neutral measure. Under Q, we set the jump intensity to 1.0, and the mean jump size to -0.2. The volatility of the jump size and the volatility of the stock are both set equal to 0.2. Furthermore, we assume that the market price of diffusion risk is equal to 0.5 so that for the stock, the diffusion risk premium is 0.1.

In the case of different jump intensities, but equal jump size distributions under P

and Q the ideal EHE is

$$
\begin{aligned}
&E^P\left[D^c(t,t+\tau)\;\middle|\;\mathcal{F}_t\right]\\
&= \int_t^{t+\tau} e^{r(t+\tau-u)}\mathrm{E}^P\left[\int_{-1}^{\infty} f^{SJ}(u,S_{u-},x)\left[h^P P(dx)-h^Q Q(dx)\right]du\;\middle|\;\mathcal{F}_t\right]\\
&= \int_t^{t+\tau} e^{r(t+\tau-u)}\mathrm{E}^P\left[\mathrm{E}^Q\left[f^{SJ}(u,S_{u-},X_u)|\mathcal{F}_{u-}\right]du|\mathcal{F}_t\right]\,(h^P-h^Q).
\end{aligned}
$$

It is proportional to the mean additional exposure to jumps which is identical under P and Q in this case. Once we know the sign of this mean additional exposure, we can infer the sign of the difference between the intensities under the physical and under the risk-neutral measure from the EHE. For a convex claim like a call or put option, $f^{SJ}\geq 0$, and so the mean additional exposure to jumps is also positive. Note that this holds irrespective of the sign of $\mathrm{E}^P[X]$. So, for convex claims sign(EHE) = sign(h^P-h^Q).

A graphical sensitivity analysis of the ideal EHE with respect to strike price and time to maturity is provided in Figure 1. The jump intensity under the true measure is set to 0.5 and is thus lower than the jump intensity under the risk-neutral measure. In this case, the ideal EHE from Proposition 1 is negative. It is highest in absolute terms for ATM options, which exhibit the highest gamma and therefore the most pronounced non-linearity in their jump size exposure. Concerning the impact of time to maturity, we have to distinguish between ATM options, where the ideal EHE becomes less negative with increasing time to maturity, and in-the-money (ITM) and out-of-the-money (OTM) options, where the ideal EHE is more or less constant with respect to maturity.

When the two jump intensities are equal, but the distributions of the jump size differ under P and Q the ideal EHE becomes

$$
\begin{aligned}
&E^P\left[D^c(t,t+\tau)\;\middle|\;\mathcal{F}_t\right]\\
&= \int_t^{t+\tau} e^{r(t+\tau-u)}E^P\left[E^P\left[f^{SJ}(u,S_{u-},X_u)\,|\,\mathcal{F}_{u-}\right]-E^Q\left[f^{SJ}(u,S_{u-},X_u)\,|\,\mathcal{F}_{u-}\right]|\mathcal{F}_t\right]du\,h^Q.
\end{aligned}
$$

It depends on the difference between the mean exposure under P and under Q. The behavior of the EHE is again analyzed numerically, where we impose the additional restriction that only the mean jump size differs between the measures P and Q. The graphs in Figure

2 are based on a mean jump size under the physical measure of -0.1, 0, and 0.1, which imply a jump risk premium for the stock of 0.1, 0.2 and 0.3. The graphs show that for most options the sign of the EHE is negative. However, for calls slightly out of the money, the EHE becomes positive, and this effect is the more pronounced with a higher mean jump size under the physical measure and thus with a larger jump risk premium.

When both the intensity and the distribution of the jump size are different under the two measures, the additional jump risk premium is given by

$$\begin{aligned} & E^P\left[D^c(t,t+\tau) \mid \mathcal{F}_t\right] \\ &= \frac{h^P}{h^Q}\int_t^{t+\tau} e^{r(t+\tau-u)}E^P\left[E^P\left[f^{SJ}(u,S_{u-},X_u)\,|\,\mathcal{F}_{u-}\right]-E^Q\left[f^{SJ}(u,S_{u-},X_u)\,|\,\mathcal{F}_{u-}\right]|\mathcal{F}_t\right]du\,h^Q \\ &\quad + \int_t^{t+\tau} e^{r(t+\tau-u)}E^P\left[E^Q\left[f^{SJ}(u,S_{u-},X_u)\,|\,\mathcal{F}_{u-}\right]|\mathcal{F}_t\right]du\,(h^P-h^Q). \end{aligned}$$

It is equal to the sum of the EHE when only the jump size distribution changes and the EHE when only the jump intensity changes, where the former is multiplied by the ratio of the jump intensity under the physical and under the risk-neutral measure.

Finally, we consider the EHE conditional on no jumps. It is given by

$$\begin{aligned} & E^P\left[D^c(t,t+\tau) \mid \mathcal{F}_t, N_{t+\tau}-N_t=0\right] \\ &= -\frac{E^P\left[\int_t^{t+\tau} e^{r(t+\tau-u)}E^Q[f^{SJ}(u,S_{u-},X_u)|\mathcal{F}_{u-}]h^Q du \mid \mathcal{F}_t, N_{t+\tau}-N_t=0\right]}{1-e^{-h^P\cdot\tau}}. \end{aligned}$$

For small τ this EHE is approximately equal to the negative of the mean exposure times the intensity, both calculated under the risk-neutral measure. An important implication of this result is that the EHE conditional on no jumps can be used to learn something about the term

$$E^Q[f^{SJ}(t,S_{t-},X_t)|\mathcal{F}_{t-}]h^Q$$

by observing the process under the *true* measure.

Summing up, our results show that it is not possible to identify all market prices of jump risk just from the ideal EHE for a set of options. Nevertheless, the EHE contains some useful information about the pricing of jump risk. Under the assumption that only jump intensity is priced and that jump size risk is not priced, the sign of the ideal EHE can be used to infer whether the risk-neutral jump intensity is larger or smaller than the jump intensity under P.

4.2 Hedging in Continuous Time Under Model Mis-Specification

Model mis-specification denotes a situation in which a wrong hedge model is used. In practical applications this is an almost unavoidable risk, so it is of interest to see how this problem affects the properties of the EHE. In our setup, we capture model mis-specification by wrongly using the BS delta as the hedging coefficient instead of the delta from the true model. The volatility is set equal to the implied BS volatility of the call to be hedged, i.e.

$$H_t = \frac{\partial c^{BS}}{\partial s}(t, S_{t-})$$

with $c^{BS}(u,s)$ as the call price in the BS model. The EHE is given in the next proposition:

Proposition 3 (SJ: EHE under model mis-specification) *The true model is the SJ model given by (3). The hedge portfolio is rebalanced continuously, the hedge ratio is the BS delta based on the implied volatility. Then, the EHE over the interval* $[t, t+\tau]$ *is*

$$\begin{aligned} & E^P\left[D^c(t,t+\tau) \,|\, \mathcal{F}_t\right] \\ &= \int_t^{t+\tau} e^{r(t+\tau-u)} E^P \left[\left[\int_{-1}^{\infty} f^{SJ}(u, S_{u-}, x) \left[h^P P(dx) - h^Q Q(dx) \right] \,\Big|\, \mathcal{F}_t \right] du \\ &\quad - (\mu - r) \int_t^{t+\tau} e^{r(t+\tau-u)} E^P \left[\frac{\partial c^{BS}}{\partial \sigma}(u, S_{u-}) \frac{\partial \sigma^{BS}}{\partial m}(u, M_{u-}) M_{u-} \,|\, \mathcal{F}_t \right] du \qquad (9) \end{aligned}$$

where $M = \frac{K}{S}$ *is the moneyness of the option, and* σ^{BS} *is its implied volatility which depends on time* t *and on moneyness* M_t*.*

Proof: See the appendix.

The first term in Equation (9) is the ideal EHE if the partial derivative from the true jump-diffusion model is used (see Corollary 1). The second term arises due to the use of the BS delta instead of the partial derivative from the SJ model, so that it can be attributed to model mis-specification. The hedge portfolio is now still exposed to stock price risk, and the remaining exposure to stock price risk depends on the difference between the two deltas. Equation (16) in the appendix shows that the difference between the deltas depends on the slope of the implied volatility function. This difference is multiplied by the premium $\mu - r$ for stock risk exposure, which depends on both the diffusion risk premium and on the premium for jump risk.

Most empirical studies find a positive equity risk premium and an implied volatility which is decreasing in the moneyness of the options. Under these conditions, the second term in (9) is positive, so that the EHE under model-misspecification is greater than the ideal EHE. Figures 3 provides some numerical examples for the case where only jump intensity risk is priced. We fix the equity risk premium at 0.2 and vary the proportions of the diffusion risk premium and the jump risk premium. The higher the jump risk premium, the more negativ is the EHE. Due to model mis-specification, the EHE increases, and if the jump risk premium is low compared to the diffusion risk premium, than the sign of the EHE changes.

4.3 Hedging Error for Discrete Trading

Up to now, the EHE was calculated under the ideal condition of continuous trading. We now turn to the analysis of the EHE when the hedge is rebalanced discretely. First, we derive a general formula for the expected hedging error. In a second step, we take a closer look at the hedging error when the hedge ratio is given by the delta of the claim.

Proposition 4 (SJ: Expected hedging error for discrete trading) *The true model is the SJ model given by (3). Trading is discrete, the hedge ratio over the interval* $[t, t+\tau]$

is constant and equal to H_t. Then, the EHE over the interval $[t, t+\tau]$ is

$$
\begin{aligned}
& E^P[D^d(t,t+\tau) \,|\, \mathcal{F}_t] \\
&= \int_t^{t+\tau} e^{r(t+\tau-u)} E^P \left[\int_{-1}^{\infty} f^{SJ}(u, S_{u-}, x) \left[h^P P(dx) - h^Q Q(dx) \right] \,\Big|\, \mathcal{F}_t \right] du \\
&\quad + (\mu - r) \int_t^{t+\tau} e^{r(t+\tau-u)} E^P \left[S_{u-} \int_t^u \frac{\partial^2 c^{SJ}}{\partial s^2}(v, S_{v-})\, \lambda^W \sigma_S S_{v-} dv \,|\, \mathcal{F}_t \right] du \\
&\quad + (\mu - r) \int_t^{t+\tau} e^{r(t+\tau-u)} \\
&\qquad\qquad E^P \left[S_{u-} \int_t^u \int_{-1}^{\infty} g(v, S_{v-}, x)(1+x) \left[h^P P(dx) - h^Q Q(dx) \right] dv \,\Big|\, \mathcal{F}_t \right] du \\
&\quad + (\mu - r) \left(\frac{\partial c^{SJ}}{\partial s}(t, S_{t-}) - H_t \right) \int_t^{t+\tau} e^{r(t+\tau-u)} E^P[S_u \,|\, \mathcal{F}_t]\, du
\end{aligned}
$$

where

$$
g(t, s, x) \;=\; \frac{\partial c^{SJ}}{\partial s}(t, s + sx) - \frac{\partial c^{SJ}}{\partial s}(t, s).
$$

Proof: See the appendix.

4.3.1 Discretization error

To analyze the pure discretization error, we set the hedge ratio equal to the delta of the claim in the SJ model.

Corollary 2 (SJ: Discretization error) *The true model is the SJ model given by (3). Trading is discrete, the hedge ratio over the interval $[t, t+\tau]$ is constant and equal to the partial derivative of the claim price w.r.t. the stock price at time t. Then, the EHE over the interval $[t, t+\tau]$ is*

$$
\begin{aligned}
& E^P[D^d(t,t+\tau) \,|\, \mathcal{F}_t] \qquad (10) \\
&= \int_t^{t+\tau} e^{r(t+\tau-u)} E^P \left[\int_{-1}^{\infty} f^{SJ}(u, S_{u-}, x) \left[h^P P(dx) - h^Q Q(dx) \right] \,\Big|\, \mathcal{F}_t \right] du \\
&\quad + (\mu - r) \int_t^{t+\tau} e^{r(t+\tau-u)} E^P \left[S_{u-} \int_t^u \frac{\partial^2 c^{SJ}}{\partial s^2}(v, S_{v-})\, \lambda^W \sigma_S S_{v-} dv \,|\, \mathcal{F}_t \right] du \\
&\quad + (\mu - r) \int_t^{t+\tau} e^{r(t+\tau-u)} \\
&\qquad\qquad E^P \left[S_{u-} \int_t^u \int_{-1}^{\infty} g(v, S_{v-}, x)(1+x) \left[h^P P(dx) - h^Q Q(dx) \right] dv \,\Big|\, \mathcal{F}_t \right] du
\end{aligned}
$$

Proof: Follows directly from Proposition 4.

The first term in Equation (10) is again the ideal EHE. The second and third term represent the discretization error. They arise since in the case of discrete trading, we use a stale delta, which no longer eliminates stock price risk completely. The importance of this term depends on the error in the delta and on the equity risk premium $\mu - r$.

The second term in (10) depends on the gamma of the claim. It captures the change in delta due to the diffusion term. This term is proportional to the market price of diffusion risk λ^W, and it vanishes if this market price of risk is zero. Analogously, the third term can be attributed to changes in delta caused by jumps in the stock price. The function g describes the change in delta after a jump of size x and can be interpreted as a 'discrete gamma'. This term depends on the market prices of jump risk. It vanishes if these market prices of risk are all equal to zero.

The discretization error, i.e. the sum of the second and third term on the right-hand side of Equation (10), is in most cases positive, and it is approximately proportional to the length of the time interval.

4.3.2 Model Mis-Specification and Discretization Error

On real-world financial markets trading only takes place at discrete points in time. Furthermore, we are exposed to the risk of model mis-specification. Both these problems have an impact on the EHE, and the next corollary shows that their effects are additive.

Corollary 3 (SJ: Model mis-specification and discretization error) *The true model is the SJ model given by (3). Trading is discrete, the hedge ratio is the BS delta where the implied volatility of the claim to be hedged is used. Then, the expected hedging error over* *the interval* $[t, t+\tau]$ *is*

$$
\begin{aligned}
&E^P[D^d(t,t+\tau)\,|\,\mathcal{F}_t] \\
&= \int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[\int_{-1}^{\infty} f^{SJ}(u, S_{u-}, x)\left[h^P P(dx) - h^Q Q(dx)\right] \,\Big|\, \mathcal{F}_t\right] du \\
&\quad + (\mu - r)\int_t^{t+\tau} e^{r(t+\tau-u)} E^P\left[S_{u-}\int_t^u \frac{\partial^2 c^{SJ}}{\partial s^2}(v, S_{v-})\,\lambda^W \sigma_S S_{v-} dv\,|\,\mathcal{F}_t\right] du \\
&\quad + (\mu - r)\int_t^{t+\tau} e^{r(t+\tau-u)} \\
&\qquad E^P\left[S_{u-}\int_t^u \int_{-1}^{\infty} g(v, S_{v-}, x)(1+x)\left[h^P P(dx) - h^Q Q(dx)\right] dv \,\Big|\, \mathcal{F}_t\right] du \\
&\quad - (\mu - r)\left(\frac{\partial c^{BS}}{\partial \sigma}(t, S_{t-})\frac{\partial \sigma^{BS}}{\partial m}(t, M_{t-})\frac{M_{t-}}{S_{t-}}\right)\int_t^{t+\tau} e^{r(t+\tau-u)} E^P[S_u\,|\,\mathcal{F}_t]\, du
\end{aligned}
$$

where $M = \frac{K}{S}$.

Proof: The result follows directly from Proposition 4 and from Equation (16) for the relation between the deltas in the BS and SJ model.

The first three terms are already known from the previous propositions and corollaries. The fourth term represents the impact of model mis-specification. A comparison with Proposition 3 shows that the resulting expressions are quite similar. While the term in Proposition 3 captures the impact of the wrong delta at every point in time, the delta is fixed here due to discrete trading, and we only have to consider the error in delta at time t. Again, this error depend on the slope of the implied volatility function.

Figure 4 visualizes the EHE under model mis-specification and discrete trading as a function of the strike price and of the length of the hedge interval. We assume that only jump intensity, but not jump size is priced and that the ideal EHE is negative. The equity risk premium is fixed at 0.2, and in the upper graph, the jump risk premium of the stock is 0.1, while it is equal to 0.05 in the lower graph. As expected, the impact of the discretization error increases with the length of the hedge interval. The example shows that it depends on the parameter scenario whether the sign of the EHE changes due to model mis-specification and discrete trading. For a jump risk premium of 0.1 (and a diffusion risk premium of 0.1), the sign of the EHE remains negative as in the ideal case, while it is mainly positive for a jump risk premium of 0.05 (and a diffusion risk premium of 0.15). Furthermore, the impact of discretization error and model mis-specification are largest for short term ATM options. Exactly for these options, the ideal EHE is largest in absolute terms also which makes them basically good candidates for the identification of the market prices of jump risk. However, the results show that ultimately they are of only limited use for that purpose.

5 Conclusion

A key insight from the analysis in this paper is that jump risk is structurally different from volatility risk. First, there are infinitely many market prices of risk, so the pricing of jump risk cannot be inferred from the EHE for one option only. Second, jump risk premia may behave counter-intuitively, e.g. when the sign of the jump risk premium for the stock is opposite to the sign of this premium for a call option.

In this paper, we analyze the theoretical behaviour of tests that try to infer the market prices of jump risk from option hedging errors. The closed form expressions show that the EHE depends on what we call the additional exposure of the contingent claim to jump risk and on the market prices of jump risk, but also on further 'risk' factors like discrete trading or model mis-specification. Under ideal conditions, i.e. with continuous trading and under the correct model, and if only jump intensity is priced, the sign of the EHE identifies the sign of the difference between the intensities under the physical and under the risk-neutral measure. Short-term ATM options would be suited best for this identification, since their EHE is highest.

Concerning the empirical applicability of a hedging based test for the identification of the sign of the market price of jump risk, the unavoidable discretization error causes the EHE to have additional components for (non-hedged) diffusion risk and (non-hedged) 'stock exposure to jump risk'. Under model mis-specification the EHE additionally depends on the slope of the implied volatility smile. Again, these additional components are highest for ATM options. This limits the applicability of the test to identify the market prices of jump risk.

Further research could aim at developing tests which are more robust against model mis-specification and discretization error. It might also be of interest to integrate the results for volatility risk and jump risk, and to analyze the EHE concerning its potential to help distinguish between these two sources of risk, probably based on a cross-section of option hedging errors.

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A Appendix

A.1 Ito for Jump Processes

Assume that the jump-diffusion process Y is given by

$$dY_t = \mu_Y dt + \sigma_Y dW_t + X_t dN_t.$$

Let f be a function of t and y with continuous second derivatives. Then, Itos formula for jump processes gives

$$\begin{aligned} df(t, Y_t) &= \frac{\partial f}{\partial t}(t, Y_{t-})dt + \frac{\partial f}{\partial y}(t, Y_{t-})\left(dY_t - X_t dN_t\right) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t, Y_{t-})\sigma_Y^2 dt \\ &\quad + \left[f(t, Y_{t-} + X_t) - f(t, Y_{t-})\right] dN_t. \end{aligned}$$

For $f(y) = e^y$, we get

$$de^{Y_t} = e^{Y_{t-}}\left[\left(\mu_Y + \frac{1}{2}\sigma_Y^2\right) dt + \sigma_Y dW_t\right] + e^{Y_{t-}}\left[e^{X_t} - 1\right] dN_t. \tag{11}$$

A.2 Measure Transform for Jump Processes

Let N be a Poisson process with intensity h_t, and let X_t be the stochastic jump size at time t with measure $P(t, dx)$. To be in line with the jump diffusion model in Section 3, we assume $X_t > -1$. The change of measure from P to Q is given by the Radon-Nikodym derivative

$$\xi_T = \frac{dQ}{dP}, \qquad \xi_t = \mathrm{E}^P[\xi_T|\mathcal{F}_t].$$

This Radon-Nikodym derivative can be represented by

$$\begin{aligned} \xi_t &= \exp\left\{-\int_0^t \int_{-1}^{\infty} \left[\phi(u, x) - 1\right] P(u, dx) h_u du + \int_0^t \ln \phi(u, X_u) dN_u\right\} \\ &= \exp\left\{-\int_0^t E^P[\phi(u, X_u) - 1|\mathcal{F}_{u-}] h_u du + \int_0^t \ln \phi(u, X_u) dN_u\right\}. \end{aligned}$$

where ϕ is some function of time t and of the jump size X that describes the measure transform (similar to the term λ in case of a diffusion process which captures the change in the drift).

The stochastic differential equation of ξ follows from Equation (11):

$$d\xi_t \;=\; \xi_{t-}\left\{-\int_{-1}^{\infty}\left[\phi(t,x)-1\right]P(t,dx)h_t dt+\left[\phi(t,X_t)-1\right]dN_t\right\}.$$

Form this expression for $d\xi_t$, we see that ξ is indeed a P-martingale.

Given this measure transform, the new intensity of the Poisson process under Q, h^Q, is given by

$$h_t^Q = h_t E^P\left[\phi(t,X_t)|\mathcal{F}_{t-}\right],$$

and the distribution of the jump size under Q is given by

$$Q(t,dx) = \frac{\phi(t,x)}{E^P\left[\phi(t,X_t)|\mathcal{F}_{t-}\right]}\,P(t,dx).$$

A.3 Proof of Proposition 1

We consider a contingent claim with price $C_t^{SJ} = c^{SJ}(t,S_t)$. From Itos formula, we get

$$\begin{aligned} dC_t^{SJ} \;=\;& \frac{\partial c^{SJ}}{\partial t}\,dt+\frac{\partial c^{SJ}}{\partial s}\,dS_t+\frac{1}{2}\,\frac{\partial^2 c^{SJ}}{\partial s^2}\,\sigma_S^2 S^2 dt \\ &+\left(c^{SJ}(t,S+SX)-c^{SJ}(t,S)-\frac{\partial c^{SJ}}{\partial s}\,SX\right)dN. \end{aligned} \tag{12}$$

The fundamental partial differential equation for the claim price is

$$\begin{aligned} &\frac{\partial c^{SJ}}{\partial t}+\frac{\partial c^{SJ}}{\partial s}(r-h^Q\mathrm{E}^Q[X])S+\frac{1}{2}\,\frac{\partial^2 c^{SJ}}{\partial s^2}\,\sigma_S^2 S^2 \\ &\quad +E^Q\left[c^{SJ}(t,S+SX)-c^{SJ}(t,S)\right]h^Q \;=\; rc^{SJ}. \end{aligned} \tag{13}$$

Plugging (13) into (12) we obtain

$$\begin{aligned} dC_t^{SJ} \;=\;& rC^{SJ}dt+\frac{\partial c^{SJ}}{\partial S}\,(dS-rSdt) \\ &+\left(c^{SJ}(t,S+SX)-c^{SJ}(t,S)-\frac{\partial c^{SJ}}{\partial s}\,XS\right)dN \\ &-E^Q\left[c^{SJ}(t,S+SX)-c^{SJ}(t,S)-\frac{\partial c^{SJ}}{\partial s}\,SX\,|\,\mathcal{F}_t\right]h^Q dt. \\ =\;& rC^{SJ}dt+\frac{\partial c^{SJ}}{\partial S}\,(dS-rSdt) \\ &+f^{SJ}(t,S,X)dN-E^Q\left[f^{SJ}(t,S,X)\,|\,\mathcal{F}_t\right]h^Q dt, \end{aligned} \tag{14}$$

where the function $f(t, s, x)$ is given in Equation (7) in Proposition 1. Plugging Equation (14) into the general formula (1) for the hedging error gives

$$\begin{aligned} D^c(t, t+\tau) &= \int_t^{t+\tau} e^{r(t+\tau-u)} \left(\frac{\partial c^{SJ}}{\partial s}(u, S_{u-}) - H_u \right) (dS_u - rS_{u-}du) \\ &+ \int_t^{t+\tau} e^{r(t+\tau-u)} f^{SJ}(u, S_{u-}, X_u) dN_u \\ &- \int_t^{t+\tau} e^{r(t+\tau-u)} E^Q \left[f^{SJ}(u, S_{u-}, X_u) \,|\, \mathcal{F}_u \right] h^Q du \end{aligned}$$

Taking expectations gives the proposition. □

A.4 Proof of Proposition 3

From Proposition 1 we get

$$\begin{aligned} & E^P \left[D^c(t, t+\tau) \;\middle|\; \mathcal{F}_{t_i} \right] \\ &= \int_t^{t+\tau} e^{r(t+\tau-u)} \mathrm{E}^P \left[\int_{-1}^{\infty} f^{SJ}(u, S_{u-}, x) \left[h^P P(dx) - h^Q Q(x) \right] |\mathcal{F}_t \right] du \\ &+ (\mu - r) \int_t^{t+\tau} e^{r(t+\tau-u)} E^P \left[\left(\frac{\partial c^{SJ}}{\partial s} - \frac{\partial c_u^{BS}}{\partial s} \right) S_u \,|\, \mathcal{F}_t \right] du. \end{aligned} \tag{15}$$

To analyze the difference between the deltas in the BS and the SJ model, we start from the definition of the implied volatility $\sigma^{BS}(t, S_t, K)$:

$$c^{BS} \left(t, S_t, \sigma^{BS}(t, S_t, K) \right) = c^{SJ} (t, S_t).$$

Differentiating once w.r.t. the stock price gives

$$\frac{\partial c^{BS}}{\partial s} \left(t, S_t, \sigma^{BS}(t, S_t, K) \right) + \frac{\partial c^{BS}}{\partial \sigma} \left(t, S_t, \sigma^{BS}(t, S_t, K) \right) \cdot \frac{\partial \sigma^{BS}}{\partial s}(t, S_t) = \frac{\partial c^{SJ}}{\partial s}(t, S_t)$$

so that the difference between the two hedge ratios is

$$\frac{\partial c^{SJ}}{\partial s}(t, S_t) - \frac{\partial c^{BS}}{\partial s} \left(t, S_t, \sigma^{BS}(t, S_t, K) \right) = \frac{\partial c^{BS}}{\partial \sigma} \left(t, S_t, \sigma^{BS}(t, S_t, K) \right) \cdot \frac{\partial \sigma^{BS}}{\partial s}(t, S_t).$$

This expression can be simplified by noting that in the BS model the call price is a positive homogenous function of the stock price and of the strike so that

$$\sigma^{BS} (t, S_t, K) = \sigma^{BS} (t, 1, M_t)$$

where the moneyness M_t is defined as $M_t = \frac{K}{S_t}$. This implies

$$\frac{\partial \sigma^{BS}}{\partial S}(t, S_t) \quad = \quad - \frac{\partial \sigma^{BS}}{\partial M_t}(t, 1, M_t) \, \frac{K}{S_t^2}.$$

Finally, we get

$$\begin{aligned} &\frac{\partial c^{SJ}}{\partial s}(t, S_t) - \frac{\partial c^{BS}}{\partial S} \left(t, S_t, \sigma^{BS}(t, S_t, K)\right) \\ &= \; - \frac{\partial c^{BS}}{\partial \sigma} \left(t, S_t, \sigma^{BS}(t, S_t, K)\right) \cdot \frac{\partial \sigma^{BS}}{\partial m}(t, M_t, 1) \, \frac{K}{S_t^2}. \end{aligned} \tag{16}$$

Plugging this into (15) gives the proposition. □

A.5 Proof of Proposition 4

From Proposition 1, we know that the EHE is equal to

$$\begin{aligned} & E^P \left[D^c(t, t+\tau) \,\middle|\, \mathcal{F}_{t_i}\right] \\ = & \int_t^{t+\tau} e^{r(t+\tau-u)} \mathrm{E}^P \left[\int_{-1}^{\infty} f^{SJ}(u, S_{u-}, x) \left[h^P P(dx) - h^Q Q(x)\right] |\mathcal{F}_t\right] du \\ & + (\mu - r) \int_t^{t+\tau} e^{r(t+\tau-u)} E^P \left[\left(\frac{\partial c^{SJ}}{\partial s}(u, S_{u-}) - H_u\right) S_u \,|\, \mathcal{F}_t\right] du. \end{aligned}$$

For $H_u \equiv H_t$, the EHE can be decomposed into

$$\begin{aligned} & \mathrm{E}^P[D(t, t+\tau) \,|\, \mathcal{F}_t] \\ = & \int_t^{t+\tau} e^{r(t+\tau-u)} \mathrm{E}^P \left[\int_{-1}^{\infty} f^{SJ}(u, S_{u-}, x) \left[h^P P(dx) - h^Q Q(x)\right] |\mathcal{F}_t\right] du \\ & + (\mu - r) \left(\frac{\partial c^{SJ}}{\partial s}(t, S_{t-}) - H_t\right) \int_t^{t+\tau} e^{r(t+\tau-u)} \mathrm{E}^P[S_u \,|\, \mathcal{F}_t] \, du \\ & + (\mu - r) \int_t^{t+\tau} e^{r(t+\tau-u)} E^P \left[\left(\frac{\partial c^{SJ}}{\partial s}(u, S_{u-}) - \frac{\partial c^{SJ}}{\partial s}(t, S_{t-})\right) S_u \,|\, \mathcal{F}_t\right] du. \end{aligned} \tag{17}$$

We first derive an expression for the difference

$$\frac{\partial c^{SJ}}{\partial s}(u, S_{u-}) - \frac{\partial c^{SJ}}{\partial s}(t, S_{t-})$$

of the partial derivatives at time u and at time t. Applying Ito's lemma yields the stochastic differential equation for the partial derivative:

$$\begin{aligned} d\frac{\partial c}{\partial s}(t, S_{t-}) \quad = \quad & \frac{\partial^2 c}{\partial s \partial t}(t, S_{t-}) dt + \frac{\partial^2 c}{\partial s^2}(t, S_{t-}) \left(dS_t - X S_{t-} dN_t\right) + \frac{1}{2} \, \frac{\partial^3 c}{\partial s^3}(t, S_{t-}) \sigma_S^2 S_{t-}^2 dt \\ & + \left(\frac{\partial c}{\partial s}(t, S_{t-} + S_{t-} X_t) - \frac{\partial c}{\partial s}(t, S_{t-})\right) dN_t. \end{aligned} \tag{18}$$

To simplify this expression, we differentiate the fundamental partial differential equation (13) for the claim price with respect to s:

$$\begin{aligned}
\frac{\partial^2 c^{SJ}}{\partial t \partial s}(t,s) + \frac{\partial^2 c^{SJ}}{\partial s^2}(t,s)\left(r - h^Q \mathrm{E}^Q[X]\right)s + \frac{\partial c^{SJ}}{\partial s}(t,s)\left(r - h^Q \mathrm{E}^Q[X]\right) \\
+\frac{1}{2}\frac{\partial^3 c^{SJ}}{\partial s^3}(t,s)\sigma_S^2 s^2 + \frac{\partial^2 c^{SJ}}{\partial s^2}(t,s)\sigma_S^2 s \\
+\mathrm{E}^Q\left[\frac{\partial c^{SJ}}{\partial s}(t, s+sX)(1+X) - \frac{\partial c^{SJ}}{\partial s}(t,s)|\mathcal{F}_{t-}\right] h^Q &= r\frac{\partial c^{SJ}}{\partial s}(t,s),
\end{aligned}$$

where we assume that σ_S, h^Q, and $\mathrm{E}^Q[X]$ are constant. After simplifying, we obtain

$$\begin{aligned}
\frac{\partial^2 c^{SJ}}{\partial t \partial s}(t,s) + \frac{\partial^2 c^{SJ}}{\partial s^2}(t,s)\left(r + \sigma_S^2 - h^Q \mathrm{E}^Q[X]\right)s + \frac{1}{2}\frac{\partial^3 c^{SJ}}{\partial s^3}(t,s)\sigma_S^2 s^2 \\
+\mathrm{E}^Q\left[g(t,s,X)(1+X)|\mathcal{F}_{t-}\right] h^Q &= 0 \qquad (19)
\end{aligned}$$

where

$$g(t,s,x) = \frac{\partial c}{\partial s}(t, s+sx) - \frac{\partial c}{\partial s}(t,s).$$

Plugging Equation (19) into (18) yields

$$\begin{aligned}
d\frac{\partial c^{SJ}}{\partial s}(t, S_{t-}) &= \frac{\partial^2 c^{SJ}}{\partial s^2}(t, S_{t-})\left[dS_t - \left(r + \sigma_S^2\right)S_{t-}dt - XS_{t-}dN_t + h^Q \mathrm{E}^Q[X]S_{t-}dt\right] \\
&\quad + g(t, S_{t-}, X_t)dN_t - \mathrm{E}^Q\left[g(t, S_{t-}, X_t)(1+X_t)|\mathcal{F}_{t-}\right] h^Q dt \\
&= \frac{\partial^2 c^{SJ}}{\partial s^2}(t, S_{t-})\, S_{t-}\left[\left(-\sigma_S^2 + \sigma_S \lambda^W\right)dt + \sigma_S dW_t\right] \\
&\quad + g(t, S_{t-}, X_t)dN_t - \mathrm{E}^Q\left[g(t, S_{t-}, X_t)(1+X_t)|\mathcal{F}_{t-}\right] h^Q dt.
\end{aligned}$$

Integrating from t to u gives

$$\begin{aligned}
&\frac{\partial c}{\partial s}(u, S_{u-}) - \frac{\partial c}{\partial s}(t, S_{t-}) \\
&= \int_t^u \frac{\partial^2 c}{\partial s^2}(v, S_{v-})\left[\left(-\sigma_s^2 + \sigma_S \lambda^W\right)S_{v-}dv + \sigma_S S_{v-}dW_v\right] \\
&\quad + \int_t^u g(v, S_{v-}, X_v)dN_v - \int_t^u \mathrm{E}^Q\left[g(v, S_{v-}, X_v)(1+X_v)|\mathcal{F}_{v-}\right] h^Q dv. \qquad (20)
\end{aligned}$$

The term that enters the expected hedging error (17) is

$$\begin{aligned}
&E^P\left[\left(\frac{\partial c^{SJ}}{\partial s}(u, S_{u-}) - \frac{\partial c^{SJ}}{\partial s}(t, S_{t-})\right) S_u \,|\, \mathcal{F}_t\right] \\
&= E^P[S_u|\mathcal{F}_t] E^{P^S}\left[\frac{\partial c^{SJ}}{\partial s}(u, S_{u-}) - \frac{\partial c^{SJ}}{\partial s}(t, S_{t-}) \,|\, \mathcal{F}_t\right], \qquad (21)
\end{aligned}$$

where the measure P^S is defined by

$$\frac{dP^S}{dP} = \frac{S_T}{\mathrm{E}^P[S_T]}.$$

For the second factor on the right-hand side of (21), plugging in (20) gives

$$\begin{aligned} & E^{P^S}\left[\frac{\partial c^{SJ}}{\partial s}(u, S_{u-}) - \frac{\partial c}{\partial s}(t, S_{t-}) \,|\, \mathcal{F}_t\right] \\ = \;& E^{P^S}\left[\int_t^u \frac{\partial^2 c}{\partial s^2}(v, S_{v-}) \left[\left(-\sigma_s^2 + \sigma_S \lambda^W\right) S_{v-} dv + \sigma_S S_{v-} dW_v\right]\right. \\ & \left. + \int_t^u g(v, S_{v-}, X_v) dN_v - \int_t^u \mathrm{E}^Q\left[g(v, S_{v-}, X_v)(1+X) | \mathcal{F}_{v-}\right] h^Q dv \,|\, \mathcal{F}_t\right]. \end{aligned} \quad (22)$$

The Wiener process W^{P^S} is given by

$$dW_u^{P^S} \;=\; dW_u - \sigma_S du.$$

From the formulas for the measure transform given in Appendix A.2, we see that the intensity of the Poisson process under P^S, h^{P^S}, is equal to $h^P E^P[X]$ or equivalently $h^P(1 + \mathrm{E}^P[X])$. The distribution of the jump size has the Radon-Nikodym-derivative

$$\frac{P^S(dx)}{P(dx)} = \frac{x+1}{\mathrm{E}^P[X]+1}.$$

Equation (22) then becomes

$$\begin{aligned} & E^{P^S}\left[\frac{\partial c^{SJ}}{\partial s}(u, S_{u-}) - \frac{\partial c}{\partial s}(t, S_{t-}) \,|\, \mathcal{F}_t\right] \\ = \;& E^{P^S}\left[\int_t^u \frac{\partial^2 c}{\partial s^2}(v, S_{v-}) \lambda^W \sigma_S S_{v-} dv + \int_t^u g(v, S_{v-}, X_v) h^P (1 + \mathrm{E}^P[X]) dv\right. \\ & \left. - \int_t^u \mathrm{E}^Q\left[g(v, S_{v-}, X_v)(1+X_v) | \mathcal{F}_{v-}\right] h^Q dv \;\Big|\; \mathcal{F}_t\right]. \end{aligned} \quad (23)$$

Now, we calculate the expectation of the last two terms conditional on $\mathcal{F}_{v-}$:

$$\begin{aligned} & E^{P^S}\left[g(v, S_{v-}, X_v)(1 + \mathrm{E}^P[X]) | \mathcal{F}_{v-}\right] h^P - E^Q\left[g(v, S_{v-}, X_v)(1+X_v) | \mathcal{F}_{v-}\right] h^Q \\ = \;& E^P\left[\frac{1+X_v}{1+\mathrm{E}^P[X]}\; g(v, S_{v-}, X_v)(1 + \mathrm{E}^P[X]) | \mathcal{F}_{v-}\right] h^P - E^Q\left[g(v, S_{v-}, X_v)(1+X_v) | \mathcal{F}_{v-}\right] h^Q \\ = \;& E^P\left[g(v, S_{v-}, X_v)(1+X_v) | \mathcal{F}_{v-}\right] h^P - E^Q\left[g(v, S_{v-}, X_v)(1+X_v) | \mathcal{F}_{v-}\right] h^Q \end{aligned}$$

Collecting terms, the expected difference (23) of the hedge ratios is

$$\begin{aligned}
& E^{P^S}\left[\left(\frac{\partial c^{SJ}}{\partial s}(u, S_{u-}) - \frac{\partial c^{SJ}}{\partial s}(t, S_{t-})\right) S_u \,|\, \mathcal{F}_t\right] \\
&= E^{P^S}\Bigg[\int_t^u \frac{\partial^2 c}{\partial s^2}(v, S_{v-})\lambda^W \sigma_S S_{v-} dv \\
&\qquad + \int_t^u \int_{-1}^{\infty} g(v, S_{v-}, x)(1+x)\left[h^P P(dx) - h^Q Q(dx)\right] dv \,\Bigg|\, \mathcal{F}_t\Bigg]
\end{aligned}$$

Plugging this into equation (17), we get the expected hedging error:

$$\begin{aligned}
& \mathrm{E}^P[D(t, t+\tau) \,|\, \mathcal{F}_t] \\
&= \int_t^{t+\tau} e^{r(t+\tau-u)} \mathrm{E}^P\left[\int_{-1}^{\infty} f^{SJ}(u, S_{u-}, x)\left[h^P P(dx) - h^Q Q(x)\right] |\mathcal{F}_t\right] du \\
&\quad + (\mu - r)\lambda^W \int_t^{t+\tau} e^{r(t+\tau-u)} \mathrm{E}^P[S_u \,|\, \mathcal{F}_t] E^{P^S}\left[\int_t^u \frac{\partial^2 c}{\partial s^2}(v, S_{v-})\sigma_S S_{v-} dv \,|\, \mathcal{F}_t\right] du \\
&\quad + (\mu - r)\int_t^{t+\tau} e^{r(t+\tau-u)} \mathrm{E}^P[S_u \,|\, \mathcal{F}_t] \\
&\qquad\qquad E^{P^S}\left[\int_t^u \int_{-1}^{\infty} g(v, S_{v-}, x)(1+x)\left[h^P P(dx) - h^Q Q(dx)\right] dv \,\Big|\, \mathcal{F}_t\right] du \\
&\quad + (\mu - r)\left(\frac{\partial c^{SJ}}{\partial s}(t, S_{t-}) - H_t\right)\int_t^{t+\tau} e^{r(t+\tau-u)} \mathrm{E}^P[S_u \,|\, \mathcal{F}_t]\, du
\end{aligned}$$

Changing the measure back from $P^{(S)}$ to P gives the proposition. □

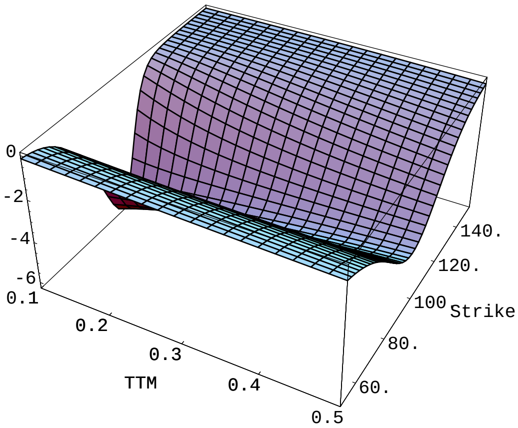


Figure 1: Ideal EHE when jump intensity is priced

The 'instantaneous' EHE for a call option under continuous trading and when the correct model is used is shown as a function of the strike price and the time to maturity. Instead of calculating the EHE over a discrete time interval of length τ, we use the integrand at time $t = 0$ which is the instantaneous EHE per unit of time. The jump risk parameters are $\sigma_X = 0.2$, $\mathrm{E}^Q[X] = -0.2$, $h^Q = 1.0$, $\mathrm{E}^P[X] = -0.2$, $h^P = 0.5$, the other parameters are $S_0 = 100$, $r = 0.0$, $\sigma_S = 0.2$, $\lambda^W = 0.5$. The jump risk premium on the stock is thus equal to $\mathrm{E}^P[X]h^P - \mathrm{E}^Q[X]h^Q = 0.1$, and the diffusion risk premium is $\sigma_S \lambda^W = 0.1$.

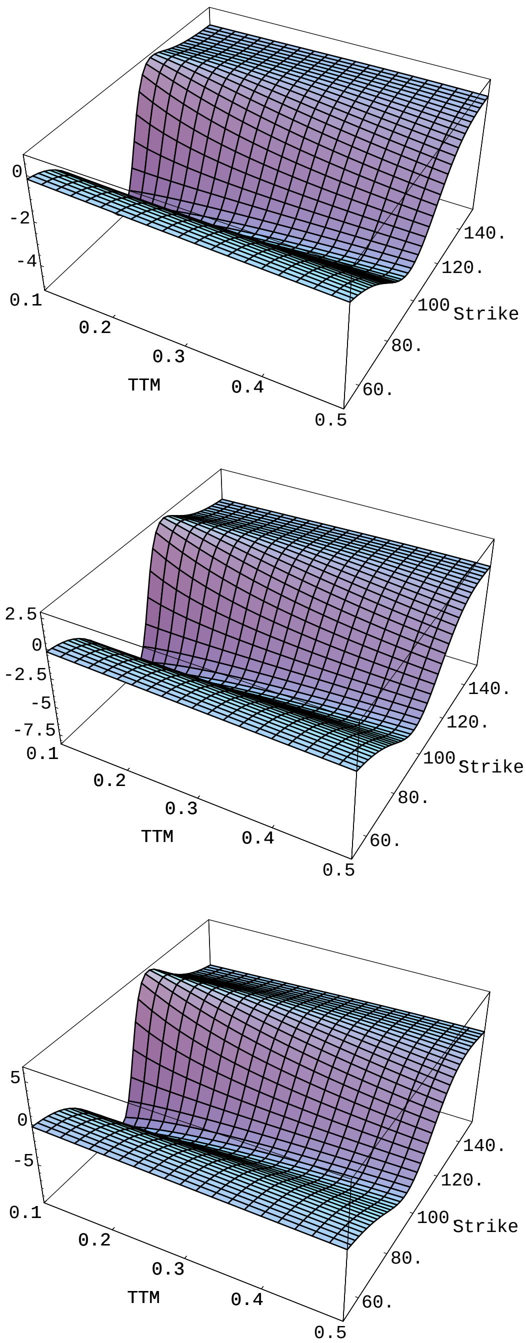


Figure 2: Ideal EHE when jump size is priced

The 'instantaneous' EHE for a call option under continuous trading and when the correct model is used is shown as a function of the strike price and the time to maturity. Instead of calculating the EHE over a discrete time interval of length τ, we use the integrand at time $t = 0$ which is the instantaneous EHE per unit of time. The jump risk parameters are $\sigma_X = 0.2$, $\mathrm{E}^Q[X] = -0.2$, $h^Q = 1.0$, $h^P = 1.0$, and, $\mathrm{E}^P[X] = -0.1, 0.0, 0.1$ (from top to bottom), the other parameters are $S_0 = 100$, $r = 0.0$, $\sigma_S = 0.2$, $\lambda^W = 0.5$. The jump risk premium $\mathrm{E}^P[X]h^P - \mathrm{E}^Q[X]h^Q$ on the stock is thus equal to $0.1, 0.2, 0.3$ (again from top to bottom), and the diffusion risk premium is $\sigma_S \lambda^W = 0.1$.

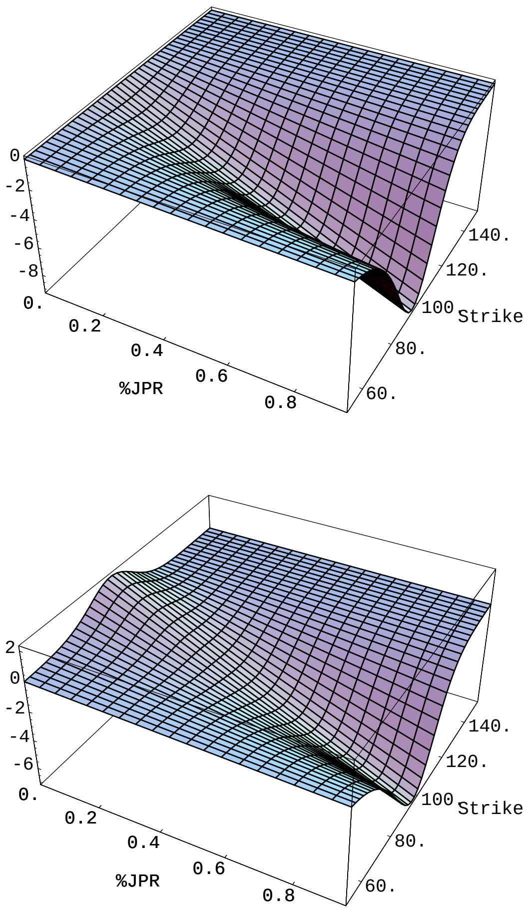
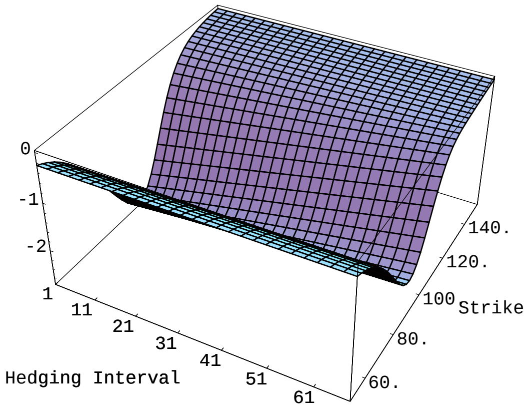


Figure 3: EHE under model mis-specification when jump intensity is priced

The 'instantaneous' EHE for a call option under continuous trading is shown as a function of the strike price and the ratio of the jump risk premium to the total equity risk premium. In the upper graph, the correct model is used, whereas in the lower graph, there is model mis-specification. Instead of calculating the EHE over a discrete time interval of length τ, we use the integrand at time $t = 0$ which is the instantaneous EHE per unit of time. The jump risk parameters are $\sigma_X = 0.2$, $\mathrm{E}^Q[X] = -0.2$, $h^Q = 1.0$, $\mathrm{E}^P[X] = -0.2$, and h^P is chosen such that the jump risk premium $\mathrm{E}^P[X]h^P - \mathrm{E}^Q[X]h^Q$ on the stock varies from 0.0 to 0.2. The other parameters are $S_0 = 100$, $r = 0.0$, $\sigma_S = 0.2$, and λ^W is chosen such that the total risk premium on the stock is constant and equal to 0.2



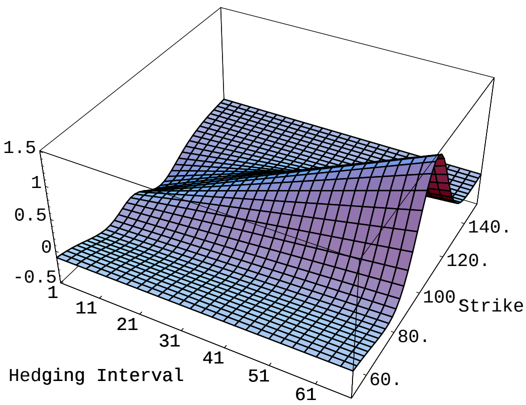


Figure 4: EHE under model mis-specification and discrete trading when jump intensity is priced

The EHE for a call option under discrete trading and model mis-specification is shown as a function of the strike price and the length of the hedging interval. Instead of evaluating the integral over the discrete time interval of length τ, we approximate it by the integrand at time $t = 0$ times the length of the time interval. The jump risk parameters are $\sigma_X = 0.2$, $\mathrm{E}^Q[X] = -0.2$, $h^Q = 1.0$, $\mathrm{E}^P[X] = -0.2$, and $h^P = 0.5, 0.75$ (from top to bottom) so that jump risk premium $\mathrm{E}^P[X]h^P - \mathrm{E}^Q[X]h^Q$ on the stock is 0.1 in the upper graph and 0.05 in the lower graph. The other parameters are $S_0 = 100$, $r = 0.0$, $\sigma_S = 0.2$, and $\lambda^W = 0.5, 0.75$ (again, from top to bottom) so that the diffusion risk premium is 0.1 in the upper graph and 0.15 in the lower graph which gives a total equity risk premium of 0.2 in both graphs.