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The Valuation of Employee Stock Options – How Good is the Standard?*

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Abstract

This study contributes to the valuation of employee stock options (ESO) in two ways: First, a new pricing model is presented, admitting a major part of calculations to be solved in closed form. Designed with a focus on good replication of empirics, the model fits with publicly observable exercise characteristics better than earlier models. In particular, it is able to account for the correlation of the time of exercise and the stock price at exercise, suspected of being crucial for the option value. The impact of correlation is weak, however, whereas cancellations play a central role. The second contribution of this paper is an examination to what extent the ESO pricing method of SFAS 123 is subject to discretion of the accountant. Given my model were true, the SFAS price would be a good proxy. Yet, outside shareholders usually cannot observe one of the SFAS input parameters. On behalf of an example I show that there is wide latitude left to the accountant.

JEL classification: G13; J33; M41; M52

Keywords: Employee stock options; Executive stock options; Barrier options; Exercise Behavior; Fair value accounting

1 Introduction

Firms use employee stock options (ESOs) in order to align the interests of employees to the long-term interests of shareholders. The question of how good stock options perform in this discipline is a true challenge for theorists. Ignoring one side of the coin – incentives, the more difficult side – I focus on the cost of ESOs to shareholders. When shareholders grant ESOs or similar incentive instruments to their employees, they want to know how much they have to pay for incentive. The costs may become substantial in practice; for instance, a sample of 239 German IPOs shows 43 firms with a ratio of outstanding ESOs to outstanding shares above 0.1.[1]

While the assumption that shareholders are unrestricted in trading stocks and treasury notes seems reasonable, one cannot ignore that employees must neither sell nor hedge

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[1] Private sample, unpublished.

ESOs they hold. It may be a consequence of these restrictions that most grantees exercise ESOs considerably earlier than standard option pricing theory predicts for unrestricted holders. Possibly, risk-averse employees decide to forego a part of an option's time value in favor of secure cash obtained by early exercise.

No matter what the reason is, all valuation models have to pay regard for early exercise either way. Literature presents two main types of models. *Rational* models try to explain why an option holder might exercise options early. In contrast, *heuristic* models focus on a good description of exercise behavior. Formally, heuristic models specify the joint distribution of price process and time of exercise. When shareholders are only interested in the cost of stock options, a heuristic model is sufficient.

Since there are no market prices for ESOs, some other observable characteristics of a sample of option exercises must be utilized to see to what degree a model is in line with empirics. Rational models may give a deeper insight than heuristic ones. The latter, in contrast, can be designed easier to fit well with real exercise patterns while keeping things simple. Setting up a heuristic pricing model basically boils down to the following: 1. Choose some characteristics of exercise behavior (like the mean time of exercise), serving as empirical benchmark of a model's fit. 2. Estimate the characteristics. 3. Model the exercise behavior. 5. Fit the model under the physical probability measure. 6. Compute prices under a corresponding risk-neutral measure.

The first contribution of this paper is a heuristic pricing model for plain call options with a vesting period. The simple structure allows to solve essential parts of the formula in closed form. The model adapts to a set of empirical characteristics of exercise better than other models known from the literature. Conform to the results of Carpenter [Car98], the average rate of option cancellations (one of the characteristics) has by far the strongest price impact. The second contribution is to give an example to what extent the accounting method for ESOs in SFAS 123 is subject to discretion and potential misspecification.

1.1 Previous Research

Let me begin with explaining some terms. By *termination* I mean the end of the option contract by any reason. *Exercise* denotes termination with a payoff greater than zero, while *cancellation* is a termination with zero payoff, with no regard to the reasons. I use *forfeiture* as a synonym for cancellation. *Premature* means "not at expiry", no matter what is optimal.

The value of an ESO grant obviously decreases if some options are forfeited, possibly since the holder leaves the firm before vesting or while the option is out of the money. The value also declines when they exercise options at stock prices different from the optimal killing price.[2] The current accounting method for ESOs, SFAS 123, reflects non-optimal exercise as follows: The dividend-adjusted Black/Scholes option price is calculated with a maturity equal to the expected lifetime[3] of the option, given that it vests. In order to correct for cancellations, the result is multiplied by the probability of the option being vested.

By these adjustments, the SFAS method picks a certain exercise strategy with some arbitrariness: Ignoring the (weak) concavity of the Black/Scholes price in time, the SFAS price is correct if options are terminated at some independent random time – be it exercisable or not[4]. The accounting valuation of an ESO therefore maps all probability

[2] Confer Barone-Adesi and Whaley [BAW87].

[3] Lifetime includes the time from grant to exercise as well as to cancellation.

[4] Given that the unhedgeable remaining risk is not priced; cf. Sect. 2.3.

laws of termination time and payoff with the same expected lifetime (given vesting) and probability of vesting to a single price that closely corresponds with independent stopping.

On the one hand, independency is rather implausible for a number of reasons, as Rubinstein [Rub95] argues. On the other hand, the relation between price path and exercise decision can have a large impact on the value of options. For illustration, I compare an option holder who randomly terminates options, independently of stock price movements, with a utility-maximizing risk-neutral option holder, to be completely free of external shocks or any restrictions. In a representative environment[5], the risk-neutral owner would exercise a ten-year ESO – at the optimal killing price of standard theory – after 7.5 years on average. If the other option holder "tossed coins" such that the average lifetime were 7.5 years, too, the SFAS method would assign the same ESO value to both. The SFAS value is close to the correct price when the holder "tossed coins", whereas the correct value for the risk-neutral holder's option is 14% higher. Evidently, a world of risk-neutral, unbiased option holders is far from reality. But at the level of information requested by SFAS 123, such a world is no less substantiated than that of employees tossing coins.

Several authors have modeled the rationales behind exercise decision by utility-maximizing option holders and some exogenous risk they are exposed to. For instance, Kulatilaka and Marcus [KM94], Huddart [Hud94], Rubinstein [Rub95], or Hall and Murphy [HM02] assume that a representative risk-averse option holder decides between holding the option or exercising it and investing the proceeds in the riskless asset.

Of course, rational models are indispensable when incentives to employees are examined. Yet, for the sole purpose of valuation, the less-demanding heuristic approach is justifiable, too: The exercise behavior is not thoroughly explained, but instead some joint probability law of termination time and payoff is supposed that accounts well for empirical observations. The SFAS 123 method obviously follows this "reduced-form" approach. Jennergren and Näslund ([JN93] and [JN95]) incorporate early exercise by an external, independent stopping time as a proxy for option holders resigning or getting fired. If the option is stopped, it is liquidated at its current inner value. If not stopped, it is treated like a European option in [JN95], whereas the American counterpart is considered in [JN93]. The European version is the prototype of the concept of independent termination and allows for a nearly closed formula. The barrier model presented in this paper adopts independent stopping from Jennergren and Näslund, yet the part of "free" decisions differs from that model.

Rubinstein [Rub95] notes that it is difficult to estimate relevant input factors reliably. He suggests a method for accounting that probably underestimates the option value but is based on few reliable factors. Such simple estimates are easier to be compared between firms.

Carpenter [Car98] compares the heuristic model of Jennergren and Näslund [JN93], called *extended American model*, with a three-parameter[6] rational model. As my study refers to this article in several respects, I sketch the content briefly: In a first step, she calibrates both models in order to reproduce a number of statistical values on exercise patterns. Information on stock price paths and option exercises is obtained from a sample of ESO grants in 40 firms. The following benchmarks are used for the goodness of fit: 1. the mean lifetime of an option (conditional on exercise); 2. the normalized mean stock

[5] For the stock related parameters, see Sect. 3.1. Forfeiture before vesting is excluded in this example.

[6] The option holder maximizes utility with CRRA $\gamma = 2$. Her initial non-option wealth is x. The employee is offered a fixed premium y in each period with probability q, to be paid off if she decides to leave the firm. The triplet (x, y, q) is subject to calibration.

price at exercise; 3. the mean cancellation rate as a mix of forfeitures before vesting, after vesting and expirations out of the money. Several parameters form a set of conditions under which the exercise characteristics should be reproduced by the model: the length of vesting period, the mean stock return under the real-world probability measure, volatility, dividend rate and the normalized mean stock price at expiry. In a second step, the best-fitting parametrizations of the models are used to forecast the exercise characteristics on behalf of each firm's specific stock price parameters. In either step of investigation the extended American model appears to perform as well as the utility-maximizing one. The extended American model gives prices strikingly similar to that of the SFAS approach, thus supporting the appropriateness of the SFAS statement 123.

1.2 Adapting to Correlation

At this point I pick up the thread. Although the characteristics of exercise considered by Carpenter [Car98] are certainly relevant, the correlation between exercise time and stock price performance at exercise is well worth a look.

To motivate the focus on correlation, suppose again that all ESO holders behave like unrestricted rational investors, thus exercising if the stock price hits the downward-sloping curve of killing prices. If we had a sample of exercises by such option holders, time and stock price of exercise would be in a strictly negative relation. The correlation coefficient is greater than -1 only by the non-linearity of the killing price function. The representative risk-averse employee in the rational model of Hall and Murphy [HM02] also exercises options along downward-sloping lines in the price vs time graph. Real-world samples of exercises, however, show a positive correlation. In the Carpenter dataset, it amounts to 0.14, while S. Huddart reports a correlation of about 0.2 for a sample of over 50,000 ESO holders from seven companies.[7] It seems that the price vs time scatter plot of exercises has a very different shape.

From the viewpoint of an unrestricted rational option writer, this means cost reduction. A switch from negative to positive correlation, leaving other characteristics constant, should reduce the option value since a higher level of late payoffs (at high discounts) will not totally offset the diminishment of early payoffs (at low discounts). Since runtimes of ESOs are quite long, the effect might be strong.

Correlation may also play a role in the following problem: Recall that, instead of maturity, the SFAS method enters the mean lifetime of an option (given it vests) into the Black/Scholes formula. The time is estimated under real-world probabilities, but used to compute a risk-neutral expectation. While the change of measure leaves an independent stopping time unaffected, it will alter the distribution of exercise time when correlated with stock price at exercise. Suppose that the correlation is strongly positive [negative]. The change of measure will diminish the expected performance at exercise. Given that the correlation is about the same under both measures, the expected time of exercise plausibly should decrease [increase] accordingly. In such settings, the SFAS model uses an overestimated [underestimated] expected exercise time, which typically results in higher [lower] prices.

[7] Private correspondence; for a description of the sample, see Huddart and Lang [HL96] or Heath et al. [HHL99]. The correlation of stock price at exercise and time from exercise to expiration (as opposed to time from grant to exercise) is -0.21. The time from exercise to expiration is roughly, but not precisely a constant minus exercise time since the time to maturity is not constant.

1.2.1 Exercise at a Barrier

I modify the model of Jennergren and Näslund [JN93] as follows. Two independent events will trigger a termination. First, and just like in the earlier models, independent random stopping may enforce that the option is paid off at the inner value, which is zero if the option is unvested or out of the money. Second, an employee is supposed to exercise her (vested) options if the stock price hits some deterministic, constant or exponentially growing target. Formally, it is nothing but the trigger of a barrier option, giving reason to call the whole setting a *barrier model*. Its exercise-related parameters are the barrier's height and growth rate plus the constant intensity of stopping. The simple structure of exercise decisions allows to solve parts of the formula analytically. A double integral is left, to be treated with standard numerical methods. Hull and White [HW03] present a similar model with a constant barrier in a binomial set-up, using backward iteration for valuation. Aside from the discrete setting, it is a special case of the model presented here.

Assuming a deterministic barrier to be the dominant exercise strategy may seem totally arbitrary at first glance, yet there is some evidence that individuals might think in terms of barriers. To construct an extreme case first of all, note that "price targets" and "price potential" are common buzzwords in analysts' forecasts.[8] Given that a stock owner or option holder strongly believed in such a target, completely ignoring the stock price risk, she would sell/exercise if the price target were attained. The stock would "have no further potential" then, provided that the investor has not updated her belief meanwhile. (But in the perceived absence of risk, there is no need for updating.) Every deviation of the stock price from the target would then be perceived just as mispricing. If, moreover, the holder believed in constant proportional growth of the firm, the target would grow exponentially as well.

Investors who do not totally ignore, but underestimate, the stock price risk might behave similar to that extreme case. Possibly, they sell stock (exercise ESOs) if the price is somewhat above their subjective "price target".

Indeed, employees consider their employer's stock safer than it is. Driscoll et al. [DMSS95] conducted a survey of people whose employer offered a 401(k) plan with the option to invest in products out of a whole range. They find that "participants consider the employer's stock safer than a domestic stock fund."[9] Benartzi [Ben01] analyzes discretionary contributions to 401(k) retirement savings plans. He finds that the decision to invest in company stock is mainly driven by past returns, whereas the impact of volatility and Beta is insignificant in nearly all cases, suggesting that employees do not worry much about their employer's stock price risk. A survey, described in the same article, reveals that "only 16.4 percent of the respondents believe that company stock is riskier than the overall stock market, as indicated by the likelihood of losing half its value over the next five years."[10] Benartzi also reports on similar results of a survey conducted by John Hancock Financial Services [Joh99]. Huberman [Hub01] concludes from the behavior of investors in stock of Regional Bell Operating Companies: "It seems [. . .] that people look favorably upon stocks with which they are familiar and think of them as more likely to deliver higher returns, at lower stock-specific risks."[11]

There is also more direct evidence that barriers play a role for ESO holders. Heath

[8] Google™ responds 4,890 hits to the request *CFA "price target" OR CFA "price targets"*. Even *CFA "three-year price target" OR CFA "3-year price target"*, signalling high self-confidence, yields 15 hits.

[9] See Huberman [Hub01], p.664.

[10] p. 1760

[11] p. 677

et al. [HHL99] analyze the exercises of ESO grants to over 50,000 employees in 7 firms. They regress the weekly rate of option exercises on a range of variables related to the stock price history. It turns out that, if the current stock price is above the one-year maximum of prices, the exercise frequency nearly doubles. Moving maxima over two years and some shorter distances are significant, too. Although this looks like (and can be) a dynamic exercise strategy, exercise at a fixed barrier can also explain the significance of moving maxima, since under typical conditions (non-decreasing barrier; moving time frame starts after vesting), a hit of the barrier can only take place if the moving maximum increases. A period of growing moving maximum has therefore a higher chance to be the first-hitting time.

Furthermore, Heath et al. find that the stock price return in the observation week and those through three preceding weeks are positively related to exercise frequency. The relation is consistent with exercise at barriers, as some of the short-term returns that precede a first hit must be positive.[12]

To return to my model, it seems merely restrictive to assume a single barrier to be representative for all option holders. Instead, one could imagine a portfolio of ESOs held by a group of employees who define different barriers. Yet, numerical examples show that it is negligible whether the barrier is diversified or not.[13] I will therefore stick to a single barrier.

To sum up, I do not claim that employees do behave as my model supposes. It is rather designed to combine analytical tractability with a good adaptation to empirical findings in such a manner that it still provides economic intuition.

1.2.2 Comparison

I check the model with Carpenter's empirical values from [Car98]. Compared with the extended American model, the barrier model better suits not only the three characteristics used there, but allows to adapt for the correlation of exercise time and stock price. As the extended American model cannot produce a positive correlation, it only makes sense to compare prices when correlation is not adapted to. The input characteristics from [Car98] then imply a barrier model price 9% lower than that of the extended American model.

In order to find out what characteristic should be measured with particular diligence, I conduct comparative statics. I change each observed characteristic by a small proportion of its standard deviation, seeing how much the newly-adapted price alters. In this sense, the average cancellation rate is by far the most important parameter, followed by the mean stock price at exercise, and the exercise time. The correlation has the weakest impact, though strong enough that it should not be ignored.

The calibration results in a mean stopping rate of about 9 %. It is, in contrast, impossible to get a good fit without independent stopping. In other words, a pure strategy of exercising at an exponential barrier is unable to explain the exercise pattern, which conforms to Carpenter's result that idiosyncratic stopping events play an important role. It seems, furthermore, that only one half or less of the stopping events is caused by

[12] The result was supplemented by a simple simulation study. First, I generated stock price paths according to the parameters in [HHL99]. Second, options were exercised if a given exponential barrier was hit. This provided me with observations, consisting of the 4 most recent weekly returns, a dummy taking the value one if the recent stock price exceeded the price maximum over the last year, and the exercise decision. Third, I regressed the exercise dummy on returns and maximum dummy. All coefficients are significant and in the same range as those found by Heath et al. Detailed simulation results are available upon request.

[13] See Appendix, 5.5.

staff turnover: Bravely inferring from top executives to all ESO grantees, about 3% p.a. of option holders leave their company by reasons that should trigger an immediate termination.[14]

1.3 Discretion in Implementing SFAS 123

Providing a simple yet flexible pricing model is one purpose of this study. In addition, I use the model to investigate how prices according to the SFAS method could relate to a true value. Encouraged by the flexibility of the barrier model, I venture to take it for the truth, looking at "errors" of SFAS 123 prices with regard to prices of the "true" barrier model.

The pricing method of SFAS 123 integrates, besides market-based parameters, two exercise related inputs: the probability that an option vests (input one), and the mean lifetime of an option under condition that it vests (input two). I choose various parameter sets for the barrier model, compute the corresponding SFAS inputs and compare the resulting SFAS price with the "true" one. For a wide range of barrier model parameters, the error is rather small. The strong link relies, however, on the assumption that the probability of vesting can be measured properly, which is doubtful. The typical outside shareholder must rest on public filings, which do not enable to distinguish whether an option was cancelled before vesting or after, or simply because the option expired out of the money. Public filings report only the aggregated number of options held, so that cancellations usually cannot be addressed to grants uniquely. Only an average of the mean cancellation rate over the whole lifetime of an option can be estimated reliably.[15] SFAS input two, the option's mean lifetime, given vesting, is typically unobservable for the same reason, as cancellations are one kind an option's "life" is terminated.

What I mean by discretion is to what extent an accountant who prices an ESO under SFAS 123 is able to manipulate the published ESO value, while keeping it in line with some publicly available data. To get a picture of that extent, a certain parametrization of the barrier model plays the role of truth again. The accountant has the freedom to choose some other parametrization of the barrier model in the role of her (claimed) perception of truth, yet she feels bound to choose one that implies the correct mean cancellation rate and/or mean exercise time. So, these characteristics are the criteria of the accountant's fidelity. I select right them because the mean cancellation rate is conceptually closest to the probability of vesting (SFAS input one), and mean exercise time is close to mean lifetime (SFAS input two).

The "truth" and "the accountant's belief" do not need to coincide, resulting in price discrepancies between -22% and $+10\%$.

Although I think that the barrier model provides a more precise picture of the truth, the advantage of simplicity of the SFAS method is not to be neglected. Under the barrier model, the SFAS 123 method could be regarded as a reliable proxy if the inputs were defined with more precision. The lack of specification how the probability of vesting should be estimated is a loophole, which can be closed at low effort. The simplest way would be to specify how the probability of vesting should be obtained from publicly available figures.

[14] Evidence on management turnover is collected in Appendix 5.6.

[15] For details, see Carpenter [Car98, Section 5].

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2 The Barrier Model

2.1 Assumptions

Suppose the stock price S follows the stochastic differential equation

$$dS_t = S_t \mu \, dt + S_t \sigma \, dW_t \tag{1}$$

with a standard Brownian motion on a complete probability space $[\Omega, \mathcal{F}, \mathbf{P}]$. Let μ and σ be constant. The stock pays dividends at a constant rate of δ. These values are assumed not to be under control of the option holder. The money market account pays interest at a constant rate r. Shareholders, who authorize ESO grants, are assumed to be unrestricted both in holding stock and investing in the money market account. Moreover, they can trade continuously.

The employee holds a plain, non-dividend-protected[16] call on S with strike price K, starting at time 0 and expiring in T. It is not exercisable until V, $0 \leq V \leq T$, and fully vested afterwards. Other exercise constraints like period around earnings announcements are neglected. Equation (1) excludes that any dilution of the stock price takes place through the lifetime of the option. I presume that the price has been adjusted before grant date.

Independent stopping is excluded for the moment and considered in Section 2.3. I specify the exercise behavior as follows: Every option holder is supposed to have chosen some target of stock price performance in advance. If the target is hit, she exercises all options at once. The target, or barrier, is a function of type

$$b(t) = B \exp\{\alpha (t - V)\}, \quad V \leq t \leq T, \tag{2}$$

with constants $B > K$, $\alpha \geq 0$. The option holder immediately exercises all options in t if $S_t \geq b(t)$. I will refer to b as the *barrier.* Because the holder has to wait until vesting, the option is exercised in V if $S_V \geq b(V)$. Given the price has never hit the barrier between V and T, the option matures like a European call.

In the sequel, I will consider *one single* barrier function. Such a single line is flawed as the joint distribution of exercise time and stock price is degenerate onto a zigzag line in $[0, T] \times [0, \infty)$, which seems unrealistic (see Figure 1). In Appendix 5.5, I will replace the unique barrier by a bundle of barriers. The generalization is skipped, however, because implications are negligible.

2.2 Pricing without Independent Stopping

It is convenient to cut the option payoff into three options, each paying out only along one part of the zigzag line in Figure 1. Let be

$$\tau^* := \inf\{s \geq V : S_s \geq b(s)\} \quad \text{or} \quad \tau^* := \infty \text{ if never hit.} \tag{3}$$

Consider the following events:

$$\begin{aligned}
\mathcal{V} \ &: \ = \{S_V \geq b(V)\} \quad \text{(exercise at vesting)} \\
\mathcal{B} \ &: \ = \{S_V < b(V), \tau^* \leq T\} \quad \text{(barrier is hit)} \\
\mathcal{E} \ &: \ = \{S_V < b(V), \tau^* > T\} \quad \text{(termination at expiry).}
\end{aligned}$$

[16] Protection against dividends is desirable from a perspective on agency conflicts but rarely observed.

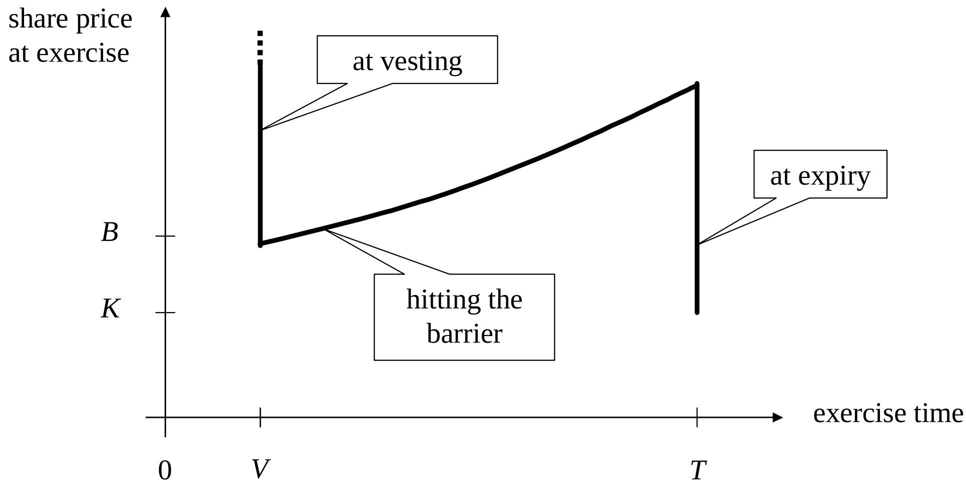

Figure 1: Support of the joint distribution of exercise time and the price at exercise; the vertical line on the left consists of exercises immediately after vesting; the flat line growing from B at time V until T represents hits of the barrier, leading to an option exercise; the vertical line on the right are exercises in the money at expiry.

Then the random option lifetime τ can be written as

$$\tau = \begin{cases} V & : \ \omega \in \mathcal{V} \\ \tau^* & : \ \omega \in \mathcal{B} \\ T & : \ \omega \in \mathcal{E} \end{cases}$$

Like τ^*, the random value τ is a stopping time of the augmented filtration $\mathbb{F}^S := \left(\mathcal{F}_t^S\right)_{t\geq 0}$ generated by S.[17] It meets the prerequisites for the definition of American contingent claims in the sense of Musiela and Rutkowski [MR98, Ch. 8.1]. In the absence of arbitrage, the price of such contingent claims is the expectation of the discounted payoff under the unique equivalent martingale measure $\mathbf{Q}$. In the present setting, $\mathbf{Q}$ is the measure under which $\ln S$ is an arithmetic Brownian motion with volatility σ and constant drift $r - \delta - \sigma^2/2$.

Given the call option is terminated at τ, the payoff equals $\pi(\tau) := [S_\tau - K]^+$. Its value in $t = 0$ is denoted by P and fulfils

$$P = \mathbf{E_Q} e^{-r\tau} \pi(\tau) = \mathbf{E_Q} e^{-r\tau} [S_\tau - K]^+$$

The option is now written as a portfolio of contingent claims paying only in $\mathcal{V}$, $\mathcal{B}$ and $\mathcal{E}$, respectively:

$$\begin{aligned} \pi(\tau) &= \pi_1(\tau) + \pi_2(\tau) + \pi_3(\tau) \quad \text{with} && (4) \\ \pi_1(\tau) &:= [S_\tau - K]^+ I_{\mathcal{V}} = [S_V - K]^+ I_{\mathcal{V}} \text{ (exercise at vesting)} && (5) \\ \pi_2(\tau) &:= [S_\tau - K]^+ I_{\mathcal{B}} = [b(\tau^*) - K]^+ I_{\mathcal{B}} \quad \text{(barrier is hit)} && \\ \pi_3(\tau) &:= [S_\tau - K]^+ I_{\mathcal{E}} = [S_T - K]^+ I_{\mathcal{E}} \quad \text{(exercise at expiry)} && (6) \end{aligned}$$

Accordingly, if P_i denotes the price of $\pi_i(\tau)$ at $t = 0$, the total price is obtained with $P = P_1 + P_2 + P_3$. Part $\pi_1(\tau)$ is nothing but a European call option maturing at V,

[17]See Karatzas and Shreve [KS88, Ch. 1.2.]

with an additional hurdle at the height of $B = b(V)$. Its price has a well-known closed solution. Part $\pi_2(\tau)$ is similar to a barrier option, but vesting must be taken into account. It is possible to solve a part of the integral by conditioning on $\mathcal{F}_V^S$, leaving a one-dimensional integral to be solved numerically. Part $\pi_3(\tau)$ looks, at first glance, like a capped European call, however the probability law of S_T under $\mathcal{E}$ is not that of the geometric Brownian motion since all paths having hit the barrier are filtered out. A one-dimensional numerical integral is left for P_3, too. A detailed analysis and pricing formulae are found in Appendix 5.1–5.3.

2.3 Independent Stopping

This step expands the model by independent events terminating the option contract. They are intended to subsume things like liquidity shocks, dismissal, or sudden disability. Following Jennergren and Näslund [JN93], I assume that such events are idiosyncratic to the option holder. Formally, there is a random time $\varphi \geq 0$, to be independent of $\mathcal{F}_T^S$, and exponentially distributed with constant intensity λ, called *stopping rate*. If $\varphi \leq \tau$, the option is immediately paid off if exercisable, or forfeited otherwise. I set

$$\pi_{\text{stop}}(t) := \begin{cases} 0 & : \; t < V \\ [S_t - K]^+ & : \; \text{else} \end{cases} \quad , \; t \leq T \; ,$$

and define $\pi_{\text{stop}}(\tau \wedge \varphi)$ to be the option's payoff with independent stopping. Note that, by independence, a change of measure on $\mathcal{F}_T^S$ has no influence on the distribution of φ. As a preparation for the next, consider $\pi_{\text{stop}}(\tau \wedge \varphi)$ under the measure $\mathbf{P}(\cdot \,|\, \varphi)$. The law of S and τ is not affected by this condition, which means that $\mathbf{Q}(\cdot \,|\, \varphi)$ is the equivalent martingale measure of $\mathbf{P}(\cdot \,|\, \varphi)$. The payoff $\pi_{\text{stop}}(\tau \wedge \varphi)$ for a fixed φ is the same as the non-stopped $\pi(\tau)$ from Section 2.2 if T is replaced by $T \wedge \varphi$. In order to give emphasis to the impact of T on the payoff $\pi(\tau)$, I henceforth write $\pi(\tau, T)$ and $P(T)$ accordingly. It follows that under the condition $\{\varphi = t\}$

$$\text{price}_{\varphi=t}(\pi_{\text{stop}}(\tau \wedge \varphi)) = \mathbf{E}_{\mathbf{Q}} e^{-r(\tau \wedge t)} \pi(\tau, T \wedge t) = P(T \wedge t) \; . \tag{7}$$

Returning to the unconditional measure, the contingent claim $\pi_{\text{stop}}(\tau \wedge \varphi)$ cannot be perfectly hedged by investing in stock and the riskless asset, as $\tau \wedge \varphi$ is obviously not a stopping time of $\mathbb{F}^S$, which destroys the integral representability of $\pi_{\text{stop}}(\tau \wedge \varphi)$. Clearly, a sole arbitrage argument is unable to derive a unique price. I assume, as the preceding authors, that there is no premium for the additional risk arising from independent stopping. Appendix 5.4 gives a formal justification based on a diversification argument. The contingent claim $\pi_{\text{stop}}(\tau \wedge \varphi)$ is therefore priced at its expected present value, just like a perfectly hedgeable option. This immediately leads to the price formula. Let $\lambda e^{\lambda t}$ be the density of φ. With $\mathbf{Q}(\varphi \geq T) = e^{-\lambda T}$, I get from (7)

$$\begin{aligned} P_{\text{stop}} &= \mathbf{E}_{\mathbf{Q}} \text{price}_{\varphi}(\pi_{\text{stop}}(\tau \wedge \varphi)) \\ &= \mathbf{E}_{\mathbf{Q}} e^{-r(\tau \wedge \varphi)} \pi(\tau, T \wedge \varphi) \\ &= \mathbf{E}_{\mathbf{Q}} e^{-r\tau} \pi(\tau, T) I_{\{\varphi \geq T\}} + \mathbf{E}_{\mathbf{Q}} e^{-r(\tau \wedge \varphi)} \pi(\tau, \varphi) I_{\{\varphi < T\}} \\ &= e^{-\lambda T} P(T) + \int_V^T P(t) \lambda e^{\lambda t} \, dt \end{aligned} \tag{8}$$

The integral must be computed numerically since $P(t)$ is determined by numeric integration either. The calculations altogether lead to a two-dimensional integral, which needs (potentially) much time to be computed. However, smooth integrands make the algorithm converge fast[18].

3 Comparison of the Models

3.1 Calibration

For lack of market prices, the question how an ESO pricing model should be calibrated is somewhat delicate. Following the approach of Carpenter [Car98] and previous authors, I try to reconcile certain characteristics (or moments) of the modeled distribution of S, τ and φ with their empirical counterparts. In order to account for, say, an atypical market environment, it may be useful not to apply the physical measure but a conditional measure $\mathbf{M}$ (to be discussed later). The following model characteristics are selected because they are publicly observable:

- the mean lifetime of an option, conditional on exercise , with $\kappa := \tau \wedge \varphi$ defined as

$$\widehat{\kappa} := \mathbf{E}_{\mathbf{M}} \left[\kappa \,|\, \text{exercise} \right] \; ;$$

- the mean stock price at the time of exercise, normalized by the strike price – likewise under condition of exercise:

$$\widehat{S_\kappa} := \mathbf{E}_{\mathbf{M}} \left[S_\kappa \,|\, \text{exercise} \right] \; ;$$

- the average cancellation rate, i.e., the average over $[0, T]$ of the expected cancellation rate, given the option was not terminated before. So it is defined as an average hazard rate[19]. Note that holders can forfeit their options by stopping before vesting, by stopping of an underwater option or by expiration out of the money. The formal continuous-time definition is

$$\widehat{c} := \int_0^T \frac{\mathbf{M}\left(\kappa \in dt \, , \, \text{cancellation at } t\right)}{\mathbf{M}\left(\kappa \geq t\right)} \; ;$$

- the correlation of exercise time and stock price at exercise, conditional on exercise:

$$\widehat{\rho} := \operatorname{corr}_{\mathbf{M}(\cdot \,|\, \text{exercise})} \left(S_\kappa, \kappa \right) \; .$$

Carpenter [Car98] uses $\widehat{\kappa}$, $\widehat{S_\kappa}$, and $\widehat{c}$ to calibrate the models. For reasons explained in Section 1.2, I add $\widehat{\rho}$, which requires no additional data. Carpenter [Car98] analyses a sample of ESO exercises from 40 firms, indicated here by i. All contracts have been running over 10 years. The above exercise/forfeiture characteristics plus firm-specific parameters like volatility σ_i, dividend rate δ_i, mean length of the vesting period V_i and the mean stock price return S_{10}/S_0 from grant to expiration of the ESO have been calculated as averages over grants for each firm separately. The firm-specific averages of κ, S_κ, and c form the final sample. I will refer to this sample, parts of which are at my disposal by courtesy of J. Carpenter. For comprehensive descriptive statistics, see Carpenter [Car98, Table 1]. An excerpt is found in Table 3.1. I will refer to these values as a benchmark,

characteristic (firm-specific)	average	standard deviation
mean time of exercise	$\widetilde{\kappa} = 5.83$	2.25
mean normalized stock price at exercise	$\widetilde{S_\kappa} = 275$	142
mean cancellation rate	$\widetilde{c} = 7.3\,\%$	$7.1\,\%$
correlation of κ and S_κ	$\widetilde{\rho} = 0.14$	0.14[20]
volatility σ_i	$\sigma = 31\,\%$	$10\,\%$
dividend rate δ_i	$\rho = 3\,\%$	$2\,\%$
vesting period V_i	$V = 1.96$	1.03
mean normalized stock price at expiry $S_{10,i}$	$S_{10} = 327$	225

Table 1: Summary statistics of ESO exercises, cancellations and stock price movement from the sample in Carpenter [Car98].

trying to adapt the model to the overall averages of κ, S_κ, c and to correlation. These are denoted by $\widetilde{\kappa}, \widetilde{S_\kappa}, \widetilde{c}$, and $\widetilde{\rho}$.

The rest of input parameters is taken from the same source: a mean annual stock return $\mu = 15.5\ \%$, a riskless return rate $r = 7\,\%$ and a time to expiration $T = 10$. I assume $S_0 = K = 100$, as most options are granted at the money.

Since, on the one hand, every sample is biased by random and, on the other hand, several of the observed values are correlated, it may be a good idea to account for anomalies by an appropriate choice of $\mathbf{M}$ instead of $\mathbf{P}$. The most sophisticated way would be to condition every exercise decision on the stock price path the option holder really witnessed. In this case, $\mathbf{M}$ should be close to the discrete measure that assigns probability N^{-1} to each of N observed stock price paths and zero to the rest.[21] This approach exploits a maximum of information (at high computational effort) and must be based on more detailed data. The simplest way would be not to account for anomalies with $\mathbf{M} = \mathbf{P}$.

Carpenter takes a way "in between": Her approach accounts for atypical stock returns by conditioning the physical measure on the assumption that every stock price path ends in the sample mean value. Formally, she sets $\mathbf{M} = \mathbf{P}\,(\,\cdot\,|S_{10} = 327)$, turning the log of stock price path into a Brownian bridge.[22] As the option always expires in the money under this measure, all probability of cancellation must go back to premature terminations of underwater or unvested options – in contrast to the sample, where 15 % of the firms have a negative mean return.[23] The impact of underwater expiration is considerable. For example, I apply the extended American Model from Carpenter [Car98, Section 3.3] to the setting of Table 3.1. Carpenter reports that – given the Brownian bridge – an annual stopping rate $\lambda = 11\,\%$ produces a cancellation rate $\widehat{c} = 7\,\%$. When $\mathbf{M} = \mathbf{P}$, in contrast, the same cancellation rate is achieved with $\lambda = 5.6\,\%$. The different stopping rates

[18] Routines are written in C++. An option valuation takes 2 seconds on a 400 MHz personal computer.

[19] Hazard rates (intensities) are frequently used in mortality models such as in reliability theory, credit risk or life insurance. Confer Barlow and Proschan [BP75], for example.

[20] Obtained from a bootstrap algorithm.

[21] In a continuous-time model, $\mathbf{M}$ must be absolutely continuous with regard to $\mathbf{P}$ in order to enable inference from the model onto the model characteristics. As discrete measures are not integrable under the Wiener measure, an absolutely continuous proxy of the counting measure must be used.

[22] See Karatzas and Shreve [KS88].

[23] Note that 15% here is the proportion of *firms* with paths expiring underwater on average. I expect the proportion to be even higher in the disaggregated sample of *option grants*.

correspond to "extended American" prices of 26.64[24] ($\lambda = 11\,\%$) and 32.14 ($\lambda = 5.6\,\%$), which is a difference of 21 %.

Notation The vector $\left(\widehat{\kappa}, \widehat{S_\kappa}, \widehat{c}, \widehat{\rho}\right)$ is denoted by $\widehat{\theta}$, and its empirical counterpart by $\widetilde{\theta}$. Subscripts like 1101 correspond to sub-vectors, with elements eliminated that correspond to zero.

For lack of grant-specific exercise data, I will set $\mathbf{M} = \mathbf{P}$, which means that empirical characteristics are compared to model characteristics under the physical measure. In order to obtain a good fit of statistical and model characteristics, I seek to minimize a quadratic distance. The least value of

$$
\begin{aligned}
\mathrm{dist}_{1111} \quad &: \quad = C_1\left(\widetilde{\kappa} - \widehat{\kappa}\right)^2 + C_2\left(\widetilde{S_\kappa} - \widehat{S_\kappa}\right)^2 + C_3\left(\widetilde{c} - \widehat{c}\right)^2 + C_4\left(\widetilde{\rho} - \widehat{\rho}\right)^2, \qquad (9) \\
\mathrm{dist}_{1110} \quad &: \quad = C_1\left(\widetilde{\kappa} - \widehat{\kappa}\right)^2 + C_2\left(\widetilde{S_\kappa} - \widehat{S_\kappa}\right)^2 + C_3\left(\widetilde{c} - \widehat{c}\right)^2, \\
\mathrm{dist}_{1101} \quad &: \quad = C_1\left(\widetilde{\kappa} - \widehat{\kappa}\right)^2 + C_2\left(\widetilde{S_\kappa} - \widehat{S_\kappa}\right)^2 + C_4\left(\widetilde{\rho} - \widehat{\rho}\right)^2,
\end{aligned}
$$

is searched by variation of B, α and λ. The coefficients C_1 to C_4 are set equal to one over the empirical variance of the underlying characteristic. In doing so, I assign equal "importance" to each of them.

3.2 Results

I compare the prices of the extended American model and the barrier model under different specifications. One aspect concerns the characteristics to be relevant for a good fit, which is controlled by the type of distance: either the full term (dist_{1111}), without correlation (dist_{1110}), or without cancellation (dist_{1101}). Another aspect refers to the freedom of choice for the model parameters B, α and λ. The parameters B and α are always free to be optimized. The cancellation rate is either fixed at zero (attempt 1), at 3% (attempt 2), or free for optimization like B and α (attempt 3). On behalf of attempt 1, I check whether independent stopping is essential for a good fit. Attempt 2 was motivated by a practitioner's rule-of-thumb, claiming that high-level employees fluctuate at a mean rate of 3 %.[25] The stopping rate in the extended American model, as the only parameter, is always optimized.

A comparison of model prices with those under SFAS 123 is not possible in the same way, as the exercise related SFAS inputs are only loosely connected with $\widehat{\theta}$, as described in Section 1.3. A *unique* SFAS price therefore cannot be derived from a given $\widetilde{\theta}$. Instead, I compute SFAS prices *as if the model was true*: Given values for B, α and λ, the SFAS inputs are derived from the barrier model. The same procedure is repeated for the extended American model.

Following Carpenter, I interpret the "expected option life" as the expected time until termination (including cancellation) given that the option vests, i.e. $\mathbf{E_P}\left[\kappa \,|\, \kappa \geq V\right]$. It is computed under the barrier model and inserted into the Black/Scholes formula as maturity. The Black/Scholes price is multiplied then by the probability of survival over the vesting period, which equals $e^{-\lambda V}$ in the barrier model.

Table 3.2 presents prices for different types of distance and different specifications of λ. Every block of rows summarizes models that are fitted for a common type of distance

[24]Despite thorough tests, I cannot resolve a contradiction between a price of 26.6 from my own computations and that of 29 reported by Carpenter [Car98].

[25]See Appendix 5.6 for empirical evidence on management turnover.

but with different specifications of λ. P denotes the model price, whereas P_{SFAS} is the corresponding price under SFAS 123, given the model is true. Figure 2 gives a summary of prices.

First, I look how well the model can be fitted. The extended American model seems to produce only negative correlations[26]. When dist_{1111} or dist_{1101} are applied, both including correlation, the positive target value $\widetilde{\rho} = 0.14$ forces the extended American model to make unrealistic "compromises" in other characteristics, which results in extreme prices. Correlation should therefore not be a criterion for fit of the extended American model. Under dist_{1110}, the optimal stopping rate $\lambda = 8.1\,\%$ is significantly lower than that of $11\,\%$ reported by Carpenter. I attribute the difference to the choice of $\mathbf{M}$, depending on whether $\mathbf{P}$ is conditioned on the stock price at expiration or not (confer Section 3.1).

Next, I check whether the concept of a barrier option alone is flexible enough to achieve a good fit, or, in other words, whether independent stopping is negligible. When λ is fixed at 0, the distance is worst among all specifications. The fit of correlation is good, whereas a cancellation rate of $\widehat{c} \approx 4\,\%$ (now caused by underwater expirations only) is much too low. The prices under condition $\lambda = 0$ are the highest of all, coming close to the standard optimal American call price[27], which amounts to 39.2.

Next, the barrier model is "freely" adapted, i.e. B, α and λ are subject to optimization. It is no surprise that the highest number of free parameters gives the best fit under all distances, but the improvement is substantial. Remarkably, under dist_{1110}, which ignores correlation, the barrier model produces a *positive* correlation of $30.2\,\%$. With due care, I judge the fact that the barrier model "spontaneously" takes the correct sign of correlation as a signal that it captures some aspects of real-world exercise patterns well.

The parametrization with fixed $\lambda = 3\,\%$ gives a fit in between. It provides no further insight, except that it probably would be a doubtful practice to estimate the stopping rate by turnover rates of employees. The calibrated stopping rates of the barrier model, $\lambda = 9.8\,\%$ (dist_{1111}) or $\lambda = 9.6\,\%$ (dist_{1110}), are similar to that of the extended American model. Its high value – compared with typical rates of staff fluctuation[28] – confirms the conjecture from the bad fit of the setting with $\lambda = 0$: Externally driven terminations seem to play an important part in exercise patterns.

The *prices* of the barrier model (see also Figure 2) suggest robustness regarding the choice of distance as long as the cancellation rate is involved. The range of prices coming from different distances is below $3.8\,\%$ for $\lambda = 0$ and $\lambda = 3\,\%$, and equals $2.5\,\%$ for free λ (without dist_{1101}). The exceptional low price under dist_{1101} signals a lack of stability when $\widetilde{c}$ is ignored. Therefore, I will drop dist_{1101} in the further.

Under each distance, the barrier model yields a considerable decrease in prices through an increase in λ (see also Section 3.2.2). The overall price level of the barrier model for free λ is below that of the extended American model under dist_{1110}, which amounts to 29.76. The prices deviate by $-6.6\,\%$ (dist_{1111}) and $-8.9\,\%$ (dist_{1110}). I attribute the loss of value to the "inefficiency" coupled with positive correlation, as mentioned in Section 1.2. Let me illustrate the impact by a reference model with negative correlation: The last row of Table 3.2 shows a fit of the barrier model with the original empirical benchmarks for $\widehat{\kappa}$, $\widehat{S_\kappa}$, and $\widehat{c}$, but with $\widetilde{\rho} = -29.8\,\%$, which is the outcome of $\widehat{\rho}$ for the extended American model under dist_{1110}. The barrier model then yields a price of 30.84, now even higher than that of the extended American model.[29] Hence, the lower correlation significantly

[26] Checked for $\lambda \in [0, 20\,\%]$, keeping everything else constant.

[27] The price reflects the vesting period, however the impact is weak.

[28] See appendix, Sect. 5.6.

[29] At first glance, an "inefficient" exercise strategy, like that of the barrier model, cannot give a higher

model + **parametrization**	**type of distance**	**fitted parameters**			**prices**		**characteristics**				**distance**
		B	α	λ	P	P_{SFAS}	$\widehat{\kappa}$	$\widehat{S_\kappa}$	$\widehat{c}$	$\widehat{\rho}$	dist_{iiii}
barrier mod., $\lambda = 0$	1111	1.77	16.1 %		36.87	33.38	5.91	237	3.9 %	13.8 %	.305
barrier mod., $\lambda = 0.03$	1111	1.87	16.8 %		33.79	31.14	5.94	242	5.2 %	13.9 %	.148
barrier mod., λ free	1111	2.29	16.6 %	9.8 %	27.48	26.54	6.05	252	7.5 %	14.1 %	.037
extended Am. model	1111			19.9 %	19.60	19.20	4.92	224	13.7 %	−4.9 %	.928
barrier mod., $\lambda = 0$	1110	3.67	−12.2 %		38.28	33.62	6.10	235	6.8 %	−98.2 %	.100
barrier mod., $\lambda = 0.03$	1110	3.53	−8.5 %		34.91	31.37	6.11	241	7.3 %	−73.0 %	.074
barrier mod., λ free	1110	1.63	66.1 %	9.6 %	26.82	26.47	5.92	264	7.3 %	30.2 %	.008
extended Am. model	1110			8.1 %	29.44	27.84	6.23	253	8.2 %	−29.8 %	.072
barrier mod., $\lambda = 0$	1101	1.79	17.2 %		37.05	33.64	6.10	243	3.8 %	14.1 %	.065
barrier mod., $\lambda = 0.03$	1101	1.90	17.4 %		33.88	31.31	6.08	246	5.1 %	14.1 %	.053
barrier mod., λ free	1101	4.89	0.0 %	16.7 %	21.68	22.02	5.93	259	10.3 %	14.1 %	.015
extended Am. model	1101			20.0 %	19.52	19.12	4.91	224	13.8 %	−4.8 %	2.102
empirical target of optimization							**5.83**	**275**	**7.3 %**	**14 %**	
modified target for comparison with extended American model (dist_{1110})							*5.83*	*275*	*7.3 %*	*-29.8 %*	
barrier mod., λ free	*1111*	*290*	*0.9 %*	*6.7 %*	*.3084*	*.2865*	*6.08*	*245*	*7.5 %*	*-29.7 %*	*.058*

Table 2: The extended American Model and three parametrizations of the barrier model. Either model is calibrated in order to fit best with given observable characteristics of exercise: $\widehat{\kappa}$, the mean time of exercise, given vesting; $\widehat{S_\kappa}$, the mean stock price performance at exercise, given vesting; $\widehat{c}$, the mean cancellation rate; $\widehat{\rho}$, the correlation between exercise time and performance at exercise. The empirical target characteristics are found in row 3 from below. The corresponding characteristics achieved by the models are listed above.
Each block of rows summarizes models fitted under one type of distance between empirics and model. "1111" includes the fit of $\widehat{\kappa}$, $\widehat{S_\kappa}$, $\widehat{c}$, and $\widehat{\rho}$. "1110" ignores $\widehat{\rho}$, and "1101" ignores $\widehat{c}$. In the barrier model under "$\lambda = 0$" and "$\lambda = 0.03$", only the starting level of the barrier $B = b(V)$ and its growth rate α are subject to optimization, under "free λ" the stopping rate is optimized as well. The price P is computed under the model with fitted parameters, P_{SFAS} is the price according to SFAS 123, given the model is true: the European Black/Scholes price with a maturity equal to the expected lifetime of the option, given it vests (computed under the model), adjusted for the probability of cancellations before vesting $(1 - \exp\{-\lambda V\})$. The last column contains the remaining distance to the target characteristics after fitting. Under "1101", a numerical restriction $\lambda \leq 20\%$ has become binding for the extended American model. The last row shows the outcome of a fit with the characteristics above, except correlation, now equal to the model outcome of the extended American model under dist_{1110}.

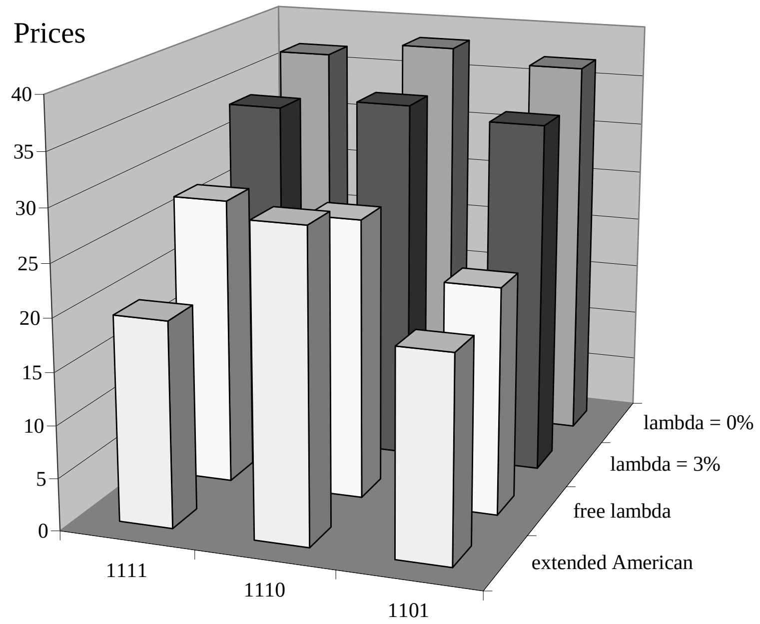

Figure 2: Prices of the extended American model and three parametrizations of the barrier model. Each model is calibrated in order to fit best with given observable characteristics of exercise. Different notions of distance between modeled and empirical characteristics form the x-axis. Here, "1111" includes the mean values of exercise time, stock price performance at exercise, cancellation rate, and correlation; "1110" excludes correlation; "1101" excludes the cancellation rate. Under "$\lambda = 0\%$" and "$\lambda = 3\%$", only level and growth rate of the barrier are subject to a minimization of distance, whereas under "free λ" the stopping rate is also adapted.

increases the price.

3.2.1 SFAS Prices and Discretion

Figure 3 compares the model prices with their corresponding SFAS prices. The input characteristics of exercise for SFAS 123 (the probability of vesting p_{vest}, and the mean lifetime given vesting $\widetilde{\kappa}_{\text{vest}}$) are hard to be observed. Like Carpenter [Car98], I obtain the inputs from the fitted models, implicitly assuming the models were true. All SFAS prices are below their model counterparts (with one exception in case "free λ under dist_{1110}", a non-robust case). While the discount is around 10 % for $\lambda = 0$, it reduces to 3 % for the free λ, and to 5.6 % for the extended American model. Above I suspected the real-world mean exercise time of being too high to be a realistic (risk-neutral) maturity for the Black/Scholes formula. An overstated time could lead to higher prices. Since I observe *lower* SFAS prices in most cases, my conjecture is not confirmed.

A more elaborate check compares barrier model prices and SFAS prices for parameters on a grid G over the ranges $B \in [1.1, 5.0]$, $\alpha \in [0.0, 0.4]$, and $\lambda \in [0.0, 0.2]$. It turns out that the proportional "error" lies between -10% and $+3\%$.

value than that of the optimal policy, to be applied in the extended American model. Note that, however, λ is much lower in the barrier model's parametrization since the barrier already shortens the average option lifetime considerably. An equal λ in both models clearly leads to lower prices for the barrier model.

It may be reasonable to tolerate an error of 10% (a hypothetical one, given the model is true) in favor of keeping things simple. However, it is doubtful whether the SFAS method could be fed with correct inputs – even if holders did behave in accordance with the barrier model. People who must rely on public data cannot estimate the probability of vesting p_{vest} directly because they will not be able to separate cancellations before vesting from those after. Some model or assumption is needed to infer p_{vest} and $\widetilde{\kappa}_{\mathrm{vest}}$ from other characteristics. As there is no rule how the SFAS inputs should be obtained, there is room for arbitrariness. To what extent could an accountant influence the SFAS price? Computing the SFAS price *under the assumption that the barrier model is true* means that (by fitting the barrier model) four characteristics of exercise are processed in order to get p_{vest} and $\widetilde{\kappa}_{\mathrm{vest}}$. To my knowledge, such precision is not common.

The next paragraphs develop a scenario of a "somewhat dishonest" accountant who is interested in manipulating the reported SFAS option price in whatever direction and a mistrustful outside shareholder who is able to check the accountant's statement to some degree. Both the accountant and the shareholder refer to one sample of option exercises, of which the shareholder can observe the cancellation rate $\widetilde{c}$ and the mean time of exercise $\widetilde{\kappa}$ only. The accountant might or might not be able to observe the SFAS inputs – she seeks to report a low p_{vest} and $\widetilde{\kappa}_{\mathrm{vest}}$ to derive a low SFAS price, or high values for high prices. Although these values cannot be checked by the shareholder, it is clear that not every value of p_{vest} can plausibly be paired with every $\widetilde{c}$ and not every $\widetilde{\kappa}_{\mathrm{vest}}$ with every $\widetilde{\kappa}$. What of relations between these four characteristics is plausible to the shareholder is modeled with parametrizations (B, α, λ) of the barrier model. Each of them establishes a quadruple $(p_{\mathrm{vest}}, \widetilde{c}, \widetilde{\kappa}_{\mathrm{vest}}, \widetilde{\kappa})$. Since the shareholder observes $\widetilde{c}$ and $\widetilde{\kappa}$, she accepts the accountant's statement about p_{vest} and $\widetilde{\kappa}_{\mathrm{vest}}$ only if there is a barrier model (B, α, λ) that produces both $(\widetilde{c}, \widetilde{\kappa})$ and $(p_{\mathrm{vest}}, \widetilde{\kappa}_{\mathrm{vest}})$.

In particular, I suppose that option holders behave in accordance with a certain but unknown parametrization of the barrier model, leading to an observation of $\widetilde{c} = 0.073$ and $\widetilde{\kappa} = 5.83$.[30] The accountant now reveals her "estimates" of p_{vest} and $\widetilde{\kappa}_{\mathrm{vest}}$ in accordance with $\widetilde{c}$ and $\widetilde{\kappa}$, which means that there is an "explaining" barrier model with $(B, \alpha, \lambda) \in G$. The accountant forms, based on the observation of $\widetilde{c}$, a "belief" about the true exercise behavior in the shape of some (possibly different) parametrization of the barrier model. She picks a barrier model that produces a correct $\widetilde{c} = 0.073 \pm 0.001$, and obtains from it the implicit $p_{\mathrm{vest}} = \exp\{-\lambda V\}$. Yet, the belief does not need to reproduce $\widetilde{\kappa}$ likewise. This way, I demand informativeness at a level of widely accepted ad-hoc statements like "The probability of cancellation is assumed to spread evenly over the option's runtime" (leading to $p_{\mathrm{vest}} := (T - V)/T\, \mathbf{P}\,(\text{cancellation})$) or "...to be distributed with a constant hazard rate" (leading to $p_{\mathrm{vest}} := \exp\{-V\widetilde{c}\}$).

With p_{vest} and $\widetilde{\kappa}_{\mathrm{vest}}$, the accountant is ready to calculate the SFAS price. Let both, truth and the accountant's belief, be points in the grid G introduced above. Since truth and belief do not necessarily coincide, any possible true parametrization yielding $\widetilde{c} \approx 0.073$ and $\widetilde{\kappa} \approx 5.83$ may face some arbitrary belief yielding the correct $\widetilde{c}$ as well. Given $\widetilde{c}$, $\widetilde{\kappa}$, and the grid, the true model prices range from 27.5 to 32.6, whereas the SFAS prices corresponding to the accountant's belief take values in [25.3, 30.1]. Assuming every combination of truth and belief to be admissible yields proportional discrepancies between -22% and $+10\%$. Since it is not clear whether all points in G are sufficiently realistic, a more restrictive scenario pairs the beliefs with the (only) truth of the parametrization $B = 2.29$, $\alpha = 16.6\,\%$, and $\lambda = 9.8\%$, which is the model's free fit under dist_{1111}. Then,

[30] Selected from Table 3.1.

mispricing lies between -8% and 9.5%. Note that the example is restricted to a universe of barrier models, suggesting that the accountant is given a wider latitude in reality.

The degree of discretion seems to be serious, leading me to assert that some precision should be added to the accounting standard. When p_{vest}, the rate of options being vested, is intransparent (a common situation, as I see it), a rule should specify how to estimate p_{vest} from the average cancellation rate $\widetilde{c}$ – from a value that outsiders can verify. One obvious way of implementation is a rule-of-thumb like setting

$$p_{\text{vest}} := \exp\{-\widetilde{c}V\}. \tag{10}$$

Even if such a rule is systematically biased, at least the comparative power of SFAS option values should improve. Under the barrier model, for instance, where $\lambda > \widehat{c}$, the rule was even not so bad since it increases the SFAS price relative to the "precise" SFAS price, which uses $\exp\{-\lambda V\}$. This way, rule-of-thumb (10) would rebalance parts of the undervaluation of the SFAS method.

Alternatively, the procedure chosen to link between $\widetilde{c}$ and p_{vest} could remain at discretion of the accountant, but in this case evidence should be requested whether the procedure conforms with characteristics beyond $\widetilde{c}$ as well. In terms of the above example, the accountant would be ordered to verify if her belief on the barrier model produces the correct $\widetilde{\kappa}$. Yet, the valuation process became more complicated, giving away the main advantage of the SFAS method – simplicity.

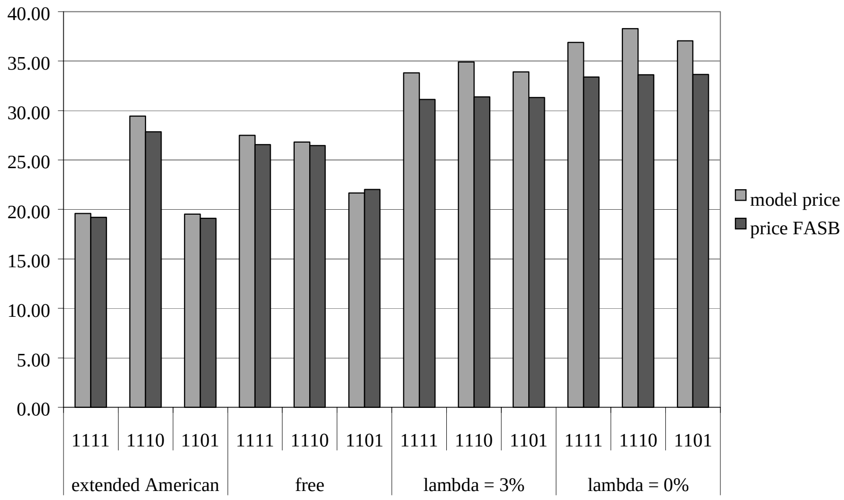

Figure 3: Model prices vs prices according to SFAS 123 under the assumption that the models were true: The implicit probability of vesting and the mean stopping time, given that the option vests, are taken from the models and used to compute the SFAS price.

3.2.2 Value Drivers

The sensitivity of the price to model parameters is investigated graphically, looking at the price as a function of B, λ, or α, each with some representative values for remaining parameters.

The price depends nonlinearly on the starting level B of the barrier at time V (Figure 4). While the curves show strong and monotonous growth roughly up to $B = 200$, the price may even decrease beyond this value – presumably, since an exponential barrier crudely substitutes the optimal killing price.

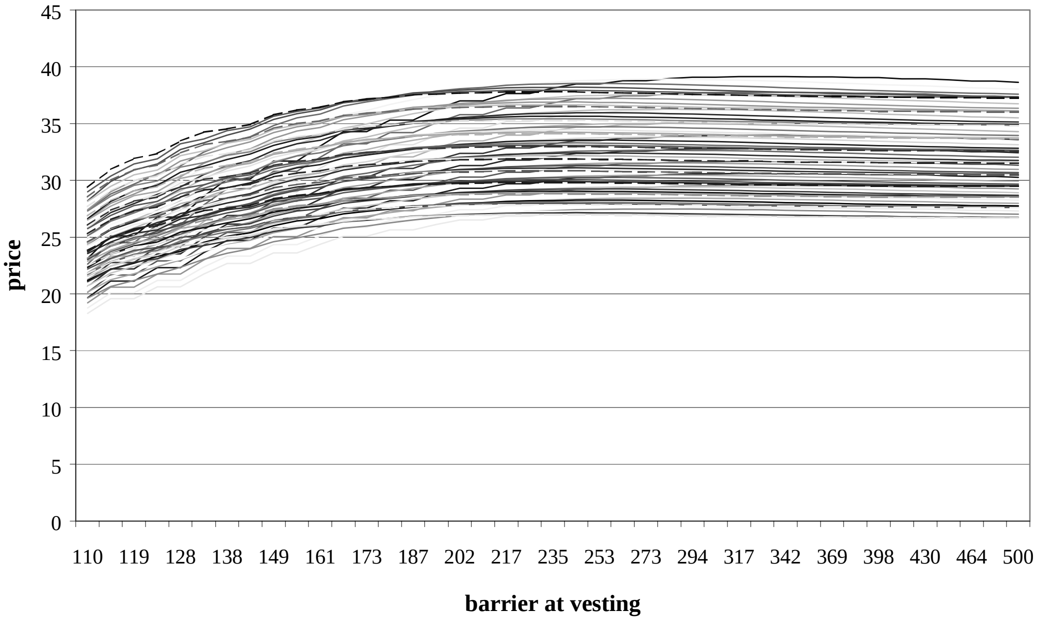

Figure 4: Prices of the barrier model as a function of B, the barrier level immediately after vesting, for selected pairs $(\alpha, \lambda) \in [0, 0.4] \times [0, 0.1]$, where α is the growth rate of the barrier and λ the continuous stopping rate. The abscissa is logarithmic.

The sensitivity of the price to changing λ, the continuous stopping rate (Figure 5), is strong, decreasing and weakly concave.[31] Furthermore, the steepness of the function roughly corresponds to a linear function of the absolute level of prices at some fixed λ.

Compared to the other parameters, α is a weak value driver. The sensitivity is still strongest for low B (Figure 6).

To sum up, the price is a smooth function of the parameters (B, α, λ). The stopping rate and the barrier level are more relevant for the price than α.

Since four characteristics of exercise are taken into account under dist_{1111}, whereas only three parameters can be calibrated, I clearly cannot control a single characteristic, leaving the other ones unchanged. It is therefore not obvious how a fitted model, seen as a map $\left(\widetilde{\kappa}, \widetilde{S_\kappa}, \widetilde{c}, \widetilde{\rho}\right) = \widetilde{\theta} \mapsto (B, \alpha, \lambda)$, responds to changes in $\widetilde{\theta}$. The question should ideally be treated with the help of a representative sample of characteristics. For lack of such data, I will present comparative statics. Taking the optimal parametrization from "free λ with dist_{1111}" as a reference point, the price change is measured when each component of $\widetilde{\theta}$ alters. Price changes are expressed in units of each characteristic's standard deviation σ_j from Table 3.1, in order to see which factors most drive the option value "in practice". I move each characteristic j in steps of size $0.2\,\sigma_j$ from $-\,0.6\,\sigma_j$ to $+0.6\,\sigma_j$. The range is narrow, first, to avoid problems with nonlinearity, second, because the characteristics of $\widetilde{\theta}$ will be correlated[32], whereas isolated variations of characteristics could lead to unrealistic

[31] The relation admits quadratic interpolation. A parabola, pinned at prices for $\lambda \in \{0;\, 5\,\%;\, 10\,\%\}$, yields proportional errors less than 10^{-3} within $\lambda \in [0, 10\,\%]$. The error is around 10^{-2} with linear approximation.

[32] See Carpenter [Car98, Table 1].

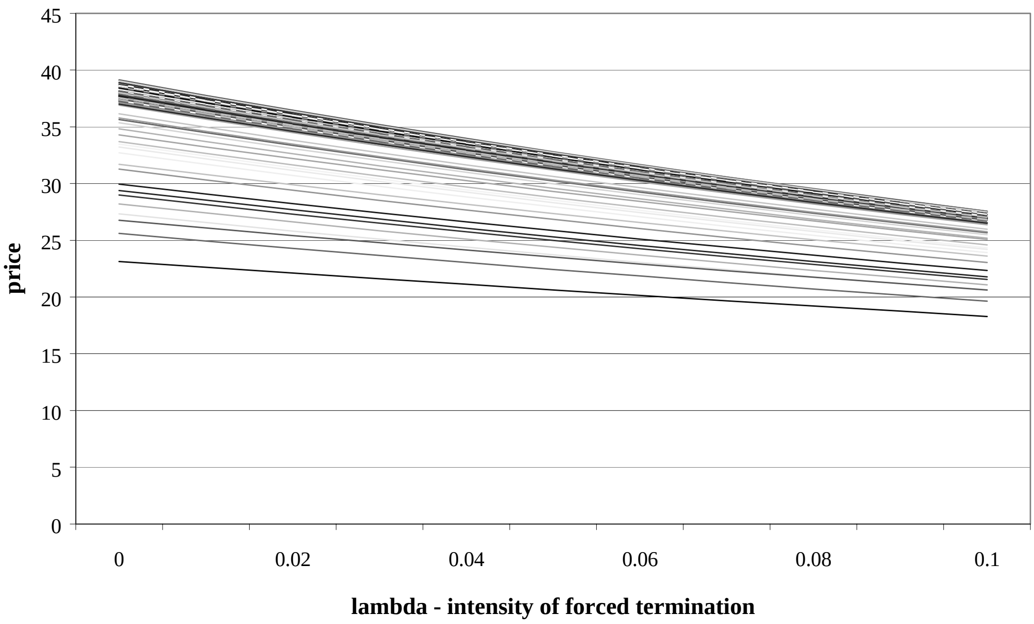

Figure 5: Prices of the barrier model as a function of the stopping rate λ for selected pairs $(B, \alpha) \in [110, 500] \times [0, 0.4]$. B is the height of the barrier at time V, the growth rate of the barrier is denoted by α.

combinations if outer quantiles of marginal distributions are taken on.

The following table presents the sensitivity of price to an increase of each $\widetilde{\theta}_j$ by one standard deviation. For a detailed summary, see Table 5.6 in the appendix.

characteristic $\widetilde{\theta}_j$	$\Delta\ \widetilde{\theta}_j$ **(one stand. deviation)**	$\Delta P/P$
mean time of exercise $\widetilde{\kappa}$	2.25	$-6.6\,\%$
mean stock price perf. at exercise $\widetilde{S_\kappa}$	1.42	$-14.6\,\%$
mean cancellation rate $\widetilde{c}$	$7.1\,\%$	$-50.4\,\%$
correlation $\widetilde{\rho}$ of κ and S_κ	0.14	$-3.2\,\%$

Cancellations have by far the strongest impact, followed by the stock price performance at exercise and exercise time. Note that the price *de*creases when the time of exercise rises – as opposed to the Black/Scholes model used in SFAS 123. Deferring the average time of exercise does not mean realizing a larger part of time value in general. Here, the postponement of exercise is achieved by raising the barrier, accompanied by more intensive independent stopping to keep $\widetilde{S_\kappa}$ down. A portion of profitable payoffs from stopping at the barrier is therefore replaced by payoffs at independent stops, leading a considerable amount of options to be cancelled out of the money.

The influence of correlation is weak, yet it should be noticed that it is one objective of this paper to clarify whether correlation is an important issue *at all*. The answer is that a model of similar flexibility as the barrier model should not *ignore* correlation, whereas a crude estimate of $\widetilde{\rho}$ seems sufficient. Moreover, it is remarkable that $\widetilde{S_\kappa}$ – not relevant in SFAS 123 – has a price impact two times stronger than $\widetilde{\kappa}$, which is close to the second SFAS input $\widetilde{\kappa}_{\mathrm{vest}}$, the mean lifetime given vesting. The significance of $\widetilde{S_\kappa}$, however, might be weaker when the characteristics are computed under a measure $\mathbf{M}$ that accounts for a bullish/bearish market (cf. Sect. 3.1).

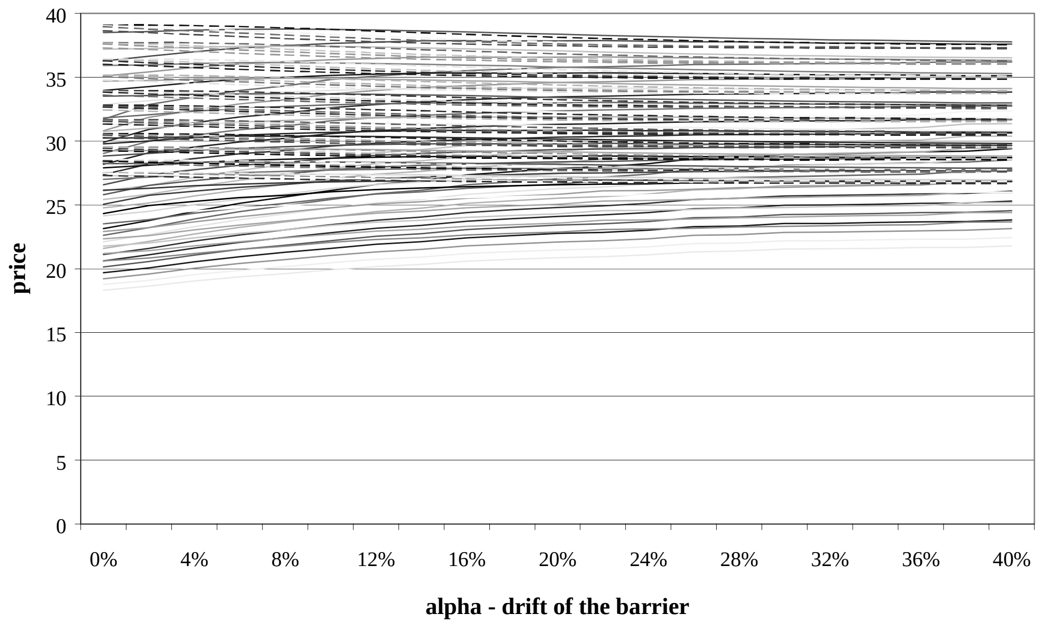

Figure 6: Prices of the barrier model as a function of the barrier's growth rate α for selected pairs $(B, \lambda) \in [110, 500] \times [0, 0.1]$. B is the height of the barrier at time V, λ is the continuous stopping rate. The lower B, the steeper the curves are.

4 Conclusion

This paper deals with the valuation of employee stock options from an external perspective such as that of shareholders or analysts. Unlike shareholders, who are assumed to be able to freely take any position in shares or debt, option holders are not allowed to hedge ESOs. This leads to option exercise patterns that substantially deviate from that of standard theory. To account for this fact, I follow the approach of Carpenter [Car98] in general. Observing certain characteristics of an exercise pattern, I specify a model's parameters such that it best reproduces the observations. The model incorporates vesting periods as well as forced termination of the option. A grantee is supposed to exercise her option if the stock price passes a deterministic threshold, which may grow exponentially. Another source of forced terminations of the option life is some independent, exponentially distributed random time. Parts of the calculation are solved in closed form, leaving a smooth function to be integrated in two dimensions.

The model is specified by the intensity of independent stopping and level plus drift of the barrier that triggers exercise. It adapts better to a representative set of empirical characteristics than the extended American model from Carpenter [Car98]. Although higher flexibility puts the model's robustness at risk, the model *is* robust provided that the annual cancellation rate of options is part of the characteristics fitted to. Hence, the barrier model is applicable both to internal purposes and external reporting, as it is based on publicly observable characteristics.

Besides practical application, a more theoretical contribution of this paper is the investigation of the influence of the correlation between exercise time and stock price at exercise. Correlation is interesting because it largely deviates from theoretical values in practice. As the model is able to incorporate a given correlation, it provides an opportunity to study the impact of correlation, given the model was true. The effect of correlation exists, but it is weak. In general, the model prices are slightly lower than those of the extended American model. Comparative statics show that the annual cancellation rate is the most important value driver, followed by the stock price performance at exercise.

Since cancellations are such important for the option value, I use my model to investigate how precisely the standard valuation approach accounts for options being canceled. Supposed that the barrier model were true and all inputs of the SFAS method were measured reliably, SFAS prices would be rather stable and slightly lower than barrier model prices. There is no evidence for suspecting SFAS 123 to be an unreliable proxy – at such a favorable level of data provision.

The SFAS approach uses the probability of forfeiture before vesting as an input, whereas an aggregated cancellation rate over different grants is often the only publicly available information about cancellations. Using the barrier model in a double role as the "truth" and "the accountant's belief", I observe a wide latitude of discretion left to the accountant.

5 Appendix

In order to value the option when independent stopping is excluded ($\lambda = 0$), the payoff $\pi(\tau)$ is decomposed into $\pi_1(\tau) := [S_\tau - K]^+ I_{\mathcal{V}}$ (exercise at vesting), $\pi_2(\tau) := [S_\tau - K]^+ I_{\mathcal{B}}$ (barrier is hit), and $\pi_3(\tau) := [S_\tau - K]^+ I_{\mathcal{E}}$ (option expires). Each payoff is valued separately.

5.1 Part I: Exercise Immediately After Vesting

$\pi_1(\tau)$ is a European call with maturity V, strike K, and an additional exercise hurdle of height B. The hurdle option can be decomposed into a European call with strike B and a digital option paying out $B - K$ iff $S_V \geq B$. Both are well-known, and the price of $\pi_1(\tau)$ amounts to

$$P_1 = e^{-rV}\mathbf{E_Q}\pi_1(\tau) = e^{-\delta V}S_0\Phi(d_1) - e^{-rV}K\Phi(d_2)$$

where $\Phi(\cdot)$ denotes the standard normal distribution function and

$$d_{1,2} := \frac{\ln(S_0/B) + \left(r - \delta \pm \frac{1}{2}\sigma^2\right)V}{\sigma\sqrt{V}}$$

5.2 Part II: Exercise at the Barrier

In a first step, I condition the expected value on $\mathcal{F}_V^S$:

$$P_2 = \mathbf{E_Q}e^{-r\tau}\pi_2(\tau) = \mathbf{E_Q}\left[\mathbf{E_Q}\left[e^{-r\tau^*}(b(\tau^*) - K)I_{\mathcal{B}}\middle|\mathcal{F}_V^S\right]\right]$$

The random time τ^*, restricted to $\mathcal{B}$, is $\sigma(S_t, V \leq t \leq T)$-measurable. On $\mathcal{B}$, it is therefore a stopping time of the augmentation of the filtration $(\sigma(S_s, V \leq s \leq t))_{V \leq t \leq T}$. Since the stock price path has the strong Markov property, I can replace $\mathcal{F}_V^S$ by $\sigma(S_V)$:

$$P_2 = \mathbf{E_Q}\left[\mathbf{E_Q}\left[e^{-r\tau^*}(b(\tau^*) - K)I_{\mathcal{B}}\middle|S_V\right]\right] \tag{11}$$

Consider now the inner conditional expectation at some fixed S_V. The structure is similar to a barrier option[33]. The distribution of τ^* under $\mathbf{Q}(\cdot|S_V)$ is essential for computation.

[33] Rich [Ric94] calls such an option an *up-and-out call* with a *rebate* equal to the barrier minus strike.

Given $\mathbf{Q}\left(\cdot|\, S_V\right)$, the process $Y_t := \ln\left(S_t/B\right)$, $t \geq V$, is a Brownian motion with a constant drift $r - \delta - \sigma^2/\,2$ and a starting point $-\ln\left(B/S_V\right)$. The hitting condition $S_t \geq b\left(t\right)$ from (3) can be rewritten with $C := \ln\left(B/S_V\right)$ and the Brownian motion $Z_t := Y_t - \alpha\left(t - V\right) + C$ to

$$Z_t \geq C \tag{12}$$

Hence, τ^* is the hitting time of Z for a constant barrier C, where Z starts at time V in 0, and runs at a constant drift $\beta := r - \delta - \alpha - \sigma^2/2$. The distribution of τ^* has a well-known density h^*, which amounts to[34]

$$h^*\left(t, C\right)\, dt := \mathbf{Q}\left(\tau^* \in dt|\, C\right) = \frac{C}{\sigma\sqrt{2\pi}\left(t - V\right)^{\frac{3}{2}}} \exp\left\{-\frac{\left(C - \beta\left(t - V\right)\right)^2}{2\sigma^2\left(t - V\right)}\right\} dt$$

I return now to the unconditional measure $\mathbf{Q}$. Taking into account that

$$C \sim N\left(\ln\left(B/S_0\right) - \beta V, \sigma\sqrt{V}\right) ,$$

the density h of τ^* under $\mathbf{Q}$ is determined by integration over C: Set $q := \ln\left(B/S_0\right) - \alpha V$. Then

$$h\left(t\right) = \frac{1}{\sigma\sqrt{2\pi V}} \int_0^\infty \exp\left\{-\frac{\left(x - q\right)^2}{2\sigma^2 V}\right\} h^*\left(t, x\right)\, dx\ .$$

After substitutions, the density can be rewritten to

$$h\left(t\right) = \left(a\left(t\right) + d\left(t\right)\right)\exp\left\{g\left(t\right)\right\} \tag{13}$$

with

$$\begin{aligned} \psi\left(t\right) &:= \frac{q}{\sigma}\sqrt{\frac{t - V}{tV}}\,,\quad g\left(t\right) := -\frac{\left(q - \beta t\right)^2}{2t\sigma^2} \\ a\left(t\right) &:= \Phi\left(\psi\left(t\right)\right)\frac{q}{\sigma\sqrt{2\pi t^3}}\,,\quad d\left(t\right) := \frac{\sqrt{V}}{2\pi t\sqrt{t - V}} \exp\left\{-\frac{\psi^2\left(t\right)}{2}\right\} \end{aligned}$$

For further computation, I introduce

$$H\left(\gamma\right) := \int_V^T \exp\left\{-\gamma t\right\} h\left(t\right)\, dt\,,\ \gamma \in \mathbb{R}\,, \tag{14}$$

which is applied when returning to the price of $\pi_2\left(\tau\right)$:

$$\begin{aligned} P_2 &= \mathbf{E_Q} e^{-r\tau^*}\left(b\left(\tau^*\right) - K\right) I_{\mathcal{B}} = \int_V^T e^{-rt}\left(Be^{\alpha(t-V)} - K\right) h\left(t\right)\, dt \\ &= Be^{-\alpha V} H\left(r - \alpha\right) - KH\left(r\right)\ . \end{aligned}$$

Computation The function H must be evaluated numerically. All parts of the integrand are bounded, except d at $t \downarrow V$. The peak is eliminated by a further substitution: h is split up by resolving the brace in (13) into $a\left(t\right)\exp\left\{g\left(t\right)\right\}$ and $d\left(t\right)\exp\left\{g\left(t\right)\right\}$. The

[34] See Rich [Ric94].

first part is evaluated as before; for the second part I replace t by $s := (t-V)^{1/2}$, arriving at

$$\begin{aligned} I(\gamma) \quad : \quad &= \int_V^T \exp\{g(t) - \gamma t\}\, b(t)\ dt \\ &= \int_0^{\sqrt{T-V}} 2s \exp\left\{g\left(s^2+V\right) - \gamma\left(s^2+V\right)\right\} b\left(s^2+V\right)\ ds \\ &\vdots \\ &= \int_0^{\sqrt{T-V}} \frac{\sqrt{V}}{\pi\left(s^2+V\right)} \exp\left\{-\frac{q^2s^2 + \left(q-\left(s^2+V\right)\right)^2}{2\sigma^2\left(s^2+V\right)} - \gamma\left(s^2+V\right)\right\} ds \end{aligned}$$

which has a bounded and equicontinuous integrand that enables numerical integration. Finally,

$$H(\gamma) = I(\gamma) + \int_V^T \exp\{g(t) - \gamma t\}\, a(t)\ dt$$

which is used in (14) as before.

5.3 Part III: Exercise at expiration

The remaining part $\pi_3(\tau)$ collects cases in which the barrier was not hit before T. It is similar to a European call capped at $b(T)$, but not equivalent since some of the paths of S that would mature within $[K, b(T)]$ do not do so because they hit b before. The distribution is biased downwards. It is more convenient to turn now over from Z to $X_t := \ln(S_t/S_0) - \alpha t$, $t \in [0,T]$, which has the same drift β but starts at zero. Define $M := \sup_{V \le t \le T} X_t$. Recalling $q = \ln(b(0)/S_0)$, the condition of not hitting the barrier turns into

$$X_t < q,\ V \le t \le T, \quad \text{or, equivalently,} \quad M < q.$$

For the integral, the distribution of $S_T I_{\mathcal{E}}$ or that of $X_T I_{\mathcal{E}}$ is needed. I condition the probability on X_V, which is equivalent to S_V with regard to the generated σ-algebras:

$$\mathbf{Q}(M < q, X_T \le z) = \mathbf{E_Q}\,\mathbf{Q}\left[M < q, X_T \le z \mid X_V\right]\ . \tag{15}$$

Note that $\mathcal{E} \subset \{S_V < B\}$, or, equivalently, $\mathcal{E} \subset \{X_V < q\}$. An application of the reflection principle and Girsanov's theorem[35] yields

$$\begin{aligned} & \mathbf{Q}\left[M < q, X_T \le z \mid X_V = x\right],\ x < q \\ = \ & \mathbf{Q}\left[M - x < q - x, X_T - x \le z - x \mid X_V = x\right],\ x < q \\ = \ & \Phi\left(\frac{z - x - \beta(T-V)}{\sigma\sqrt{T-V}}\right) - \exp\left\{\frac{2\beta(q-x)}{\sigma^2}\right\} \Phi\left(\frac{z + x - 2q - \beta(T-V)}{\sigma\sqrt{T-V}}\right) \end{aligned}$$

According to (15), this has to be integrated over $X_V \sim N\left(\beta V, \sigma\sqrt{V}\right)$:

$$\begin{aligned} & \mathbf{Q}(M < q, X_T \le z) \\ = \ & \frac{1}{\sigma\sqrt{2\pi V}} \int_{-\infty}^{q} \mathbf{Q}\left[M < q, X_T \le z \mid X_V = x\right] \exp\left\{-\frac{(x-\beta V)^2}{2\sigma^2 V}\right\} dx\,. \end{aligned} \tag{16}$$

[35] Confer Musiela and Rutkowski [MR98, Sect. B3].

For integration over the payoff $\pi_3(\tau)$, the density $l(z)$ of X_T is needed. I obtain it by differentiation of (16)

$$\begin{aligned}
l(z)\ dz &= \mathbf{Q}(M<q, X_T \in dz) \ = \ dz\frac{d}{dz}\mathbf{Q}(M<q, X_T \le z) \\
&= \frac{dz}{\sigma\sqrt{2\pi V}}\int_{-\infty}^{q}\exp\left\{-\frac{(x-\beta V)^2}{2\sigma^2 V}\right\}\frac{dx}{\sigma\sqrt{2\pi(T-V)}}\times \\
&\quad \times\left[\exp\left\{-\frac{(x+\beta(T-V)-z)^2}{2\sigma^2(T-V)}\right\}-\right. \\
&\quad \left.-\exp\left\{\frac{2\beta(q-x)}{\sigma^2}\right\}\exp\left\{-\frac{(z+x-2q-\beta(T-V))^2}{2\sigma^2(T-V)}\right\}\right] \\
&\ \ \vdots \\
&= \left[f(z)\,\Phi\left(\frac{q-\mu_1}{\sigma_1}\right)-g(z)\,\Phi\left(\frac{q-\mu_2}{\sigma_1}\right)\right]\ dz
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 :=&\quad \frac{V}{T}z\,,\ \ \mu_2 := \frac{V}{T}(2q-z)\ , \\
U :=&\quad T-V\,,\ \ \sigma_1 := \sigma\sqrt{\frac{V}{T}U}\ , \\
f(z) :=&\quad \frac{1}{\sigma\sqrt{2\pi T}}\exp\left\{\frac{1}{2\sigma^2 U}\left(\frac{V}{T}z^2-(\beta U-z)^2-V\beta^2 U\right)\right\}\ , \\
g(z) :=&\quad \frac{1}{\sigma\sqrt{2\pi T}}\exp\left\{\frac{1}{2\sigma^2 U}\left(\frac{V}{T}(z-2q)^2-\beta^2 VU+4\beta qU-(z-2q-\beta U)^2\right)\right\}\ .
\end{aligned}$$

Using $S_T = S_0\exp\{\alpha T + X_T\}$, the price P_3 of the part "exercise in T" finally amounts to

$$\begin{aligned}
P_3 &= \mathbf{E}_{\mathbf{Q}}e^{-rT}[S_T-K]^+ I_{\mathcal{E}} \\
&= e^{-rT}S_0\int_{\ln(K/S_0)-\alpha T}^{q}\left(e^{\alpha T}e^z-\frac{K}{S_0}\right)l(z)\ dz\ ,
\end{aligned}$$

which can be solved numerically in a straightforward manner.

5.4 The Unhedgeable Risk of Independent Stopping

In this section I present a sufficient condition, under which the assumption of Section 2.3 not to price the unhedgeable risk of independent stopping can be justified. Suppose the ESO is not granted to a single person but to a large group of N employees, holding the Nth part each. By assumption, the risk of stopping is idiosyncratic to each of them. Formally, there is a whole number of i.i.d. random times φ_i, the entirety of which is independent of $\mathcal{F}_T^S$. This implies the identity of $\mathbf{P}$ and $\mathbf{Q}$ on $\sigma(\varphi_i, i \in \mathbb{N})$. By independence, holding the portfolio of claims

$$C_{\text{portf}} := \left\{\frac{1}{N}\pi_{\text{stop}}(\tau\wedge\varphi_i)\,,\ \text{due in } \tau\wedge\varphi_i \,\middle|\, i=1,\ldots,N\right\}$$

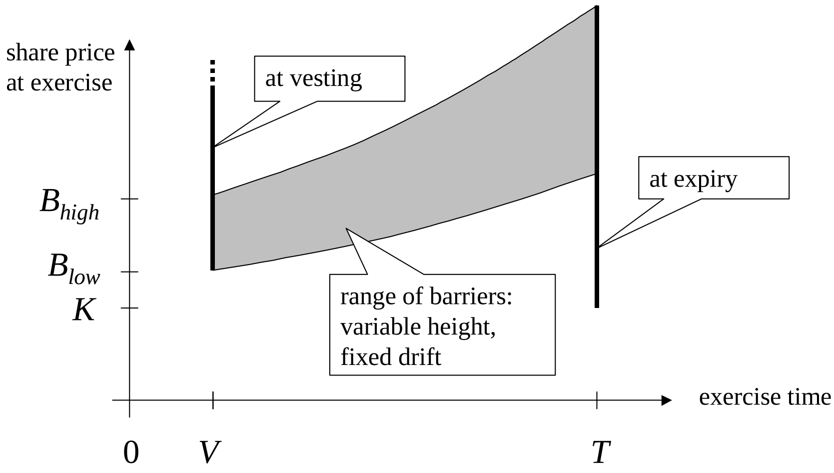

Figure 7: Support of the joint distribution of exercise time and stock price at exercise, given dispersed barriers; the vertical line on the left consists of exercises immediately after vesting; the line on the right of exercises in the money at expiry; the grey area arises from exercises at various barriers, growing at the same proportional rate and starting from a level between B_{high} and B_{low}, to be equally distributed.

is nearly the same for large N as holding a continuum of claims

$$C_{\mathrm{cont}} := \{ \mathbf{P}(\varphi \in dt)\ \pi_{\mathrm{stop}}(\tau \wedge t)\,,\ \text{due in } \tau \wedge t \,|\ t \in [V,T] \}$$

Provided additive prices and infinitely divisible ESOs, the similarity of C_{portf} and C_{cont} is easily proved by the Strong Law of Large Numbers. The price of C_{portf} then nearly equals that of C_{cont}. With (7) and independence of φ and $\mathcal{F}_T^X$, a generalization of (8) is obtained:

$$\lim_{N\to\infty} \mathrm{price}\,(C_{\mathrm{portf}}) = \int_0^\infty \mathrm{price}\,(\pi_{\mathrm{stop}}\,(\tau \wedge t))\ \mathbf{P}\,(\varphi \in dt)\ = \mathbf{E}e^{-r(\tau\wedge\varphi)}\pi\,(\tau, T\wedge\varphi)\ , \quad (17)$$

The same result can be justified by CAPM-like arguments as well.

5.5 Dispersion of the Barrier

Figure 1 shows that the joint distribution of τ and S_τ is degenerate to a zigzag line of Lebesgue measure zero. While it quite frequently happens that holders exercise options immediately after vesting or at expiry, thus causing jumps in the distribution of κ at V and T, a single barrier at the time in between is less plausible. Instead, I follow the idea of a group of option holders, each choosing some individual barrier. For simplicity, only the level B is subject to variation, while the growth rate α remains constant (confer Figure 7). I consider a special setting: B shall be equally diversified on some interval $[B_{\mathrm{low}}, B_{\mathrm{high}}]$ such that $B_{\mathrm{low}} > K$. Let be $\Delta := B_{\mathrm{high}} - B_{\mathrm{low}}$ and $B_{\mathrm{center}} := 1/2\,(B_{\mathrm{high}} + B_{\mathrm{low}})$. The option portfolio is then correctly priced as the average of prices for different barrier functions. Given that $P_{B,\alpha}(T)$ denotes the price for an individual barrier, one obtains

$$P_{\mathrm{dispersed}} = \Delta^{-1} \int_{B_{\mathrm{low}}}^{B_{\mathrm{high}}} P_{B\,,\,\alpha}\,(T)\ dx\,.$$

Keeping B_{center} fixed, I check how much $P_{\mathrm{dispersed}}$ is affected by an increase of Δ. The impact is weak. For illustration, consider the option price as a function of Δ and B_{center}

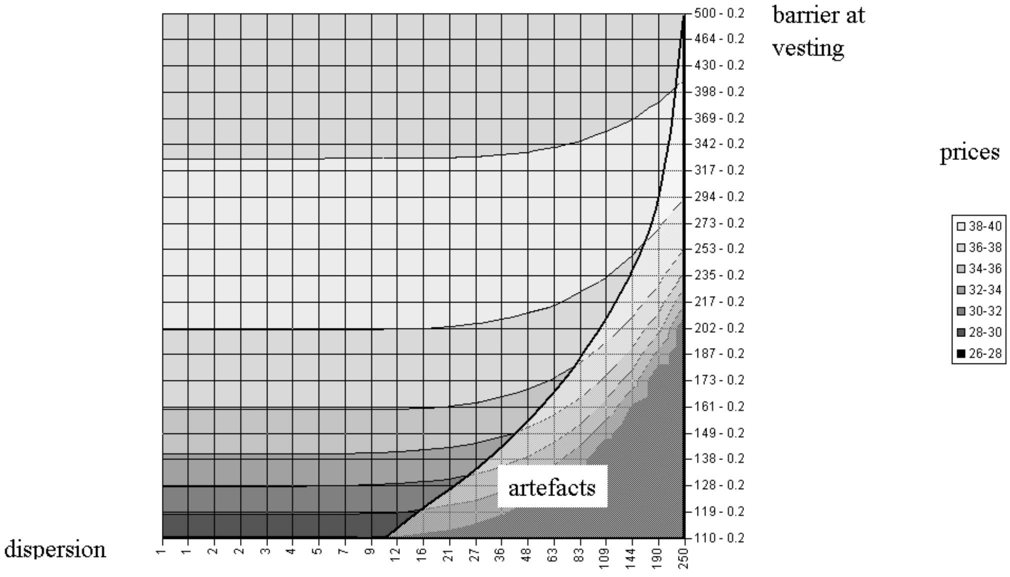

Figure 8: Option prices as a function of dispersion and general level of the barrier. Areas of equal lightness are areas of roughly the same price. Dispersion is measured by the width Δ of the barrier's range $\left[B_{\text{low}}, B_{\text{high}}\right]$. “Barrier at vesting” denotes the midpoint of the interval. The area "artefacts" is irrelevant, as the condition $B_{\text{low}} > K$ is hurt there. Other parameters are fixed at $\alpha = 20\%, \lambda = 3\%$.

for fixed $\alpha = 20\,\%$, $\lambda = 3\,\%$. It is drawn in contour lines in Figure 8. The flat lines on the left half of the field show that the impact of Δ is negligible at a range from 0 to about 50. Note that the range refers to the starting level of b at V. At expiry, this means a range from 0 to 248 when $\alpha = 20\,\%$. Other characteristics show a low sensibility, too. An analysis for $\alpha = 0\,\%$ or $\alpha = 40\,\%$ provides similar results. To sum up, a model with a dispersed barrier looks more aesthetic but gives neither new insight nor essential price differences.

5.6 Some Evidence on Management Turnover

Hadlock and Lumer [HL97] report an annual turnover rate of 3.8% for CEOs from a sample of 259 U.S. firms. Kaplan [Kap94] compares the CEO turnover in large U.S. and Japanese firms, coming up with rates of 2.2% (Japan) and 2.9% (U.S.) if CEOs entering the supervisory board are left out. I assume that they may continue to hold their options. Kang and Shivdasani [KS95] find 3.1% p.a. for Japanese firms when the turnover is corrected for executives who remain on the board. The U.S. sample of Denis et al. [DDS97] yields a weighted mean rate of 7.5%, yet it is not corrected in the above sense. The same holds for the rate of 9.2% from Mikkelson and Partch [MP97], where CEO turnover in unacquired U.S. firms is measured over ten years. Dahya, McConnell and Travlos [DMT02] report forced CEO turnover at rates between 2.7% and 5% from a dataset of 470 industrial firms in the U.K.

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modified target		fitted parameters			prices		achieved characteristics				dis-tance
		B	α	λ	P	P_{SFAS}	$\widehat{\kappa}$	$\widehat{S_\kappa}$	$\widehat{c}$	$\widehat{\rho}$	dist_{1111}
	4.48	175	.144	.072	28.91	26.82	4.95	214	.076	.142	.231
	4.93	189	.155	.081	28.51	26.74	5.33	226	.076	.142	.151
$\widetilde{\kappa}$	*5.83*	*229*	*.167*	*.098*	*27.48*	*26.55*	*6.05*	*252*	*.075*	*.141*	*.037*
	6.28	259	.162	.103	27.04	26.55	6.38	265	.074	.141	.007
	6.73	310	0.143	.106	26.72	26.66	6.69	278	.073	.140	.001
	7.18	396	.105	.105	26.72	27.01	6.98	290	.070	.139	.022
	190	190	.156	.071	29.58	27.66	5.49	230	.070	.138	.105
	218	199	.162	.078	29.03	27.38	5.66	237	.072	.139	.023
	247	211	.167	.088	28.30	26.99	5.85	244	.073	.140	.000
$\widetilde{S_\kappa}$	*275*	*229*	*.167*	*.098*	*27.48*	*26.55*	*6.05*	*252*	*.075*	*.141*	*.037*
	303	259	.157	.110	26.47	26.01	6.26	261	.077	.142	.128
	332	374	.094	.127	24.92	25.14	6.49	273	.081	.143	.269
	360	662	−.014	.128	24.76	25.27	6.68	283	.080	.142	.451
	.030	174	.176	−.007	37.70	34.16	6.13	241	.033	.141	.076
	.045	185	.180	.025	34.33	31.68	6.10	245	.047	.141	.061
	.059	201	.179	.060	3.93	29.15	6.08	248	.061	.141	.048
$\widetilde{c}$	*.073*	*229*	*.167*	*.098*	*27.48*	*26.55*	*6.05*	*252*	*.075*	*.141*	*.037*
	.087	303	.116	.140	23.95	23.87	6.03	256	.089	.141	.026
	.101	501	.000	.166	21.81	22.17	5.93	257	.100	.140	.018
	.116	1816	−.263	.176	21.09	21.50	5.81	256	.111	.137	.021
	.056	244	.120	.093	28.02	26.85	6.06	249	.075	.057	.044
	.084	240	0.134	.096	27.80	26.72	6.07	251	.075	.085	.042
	.112	235	.149	.097	27.64	26.64	6.06	251	.075	.113	.039
$\widetilde{\rho}$	*.140*	*229*	*.167*	*.098*	*27.48*	*26.55*	*6.05*	*252*	*.075*	*.141*	*.037*
	.168	222	.187	.099	27.36	26.50	6.05	253	.075	.169	.034
	.196	216	.212	.100	27.15	26.38	6.04	255	.075	.197	.030
	.224	208	.246	.101	26.97	26.30	6.03	256	.075	.225	.026

Table 3: Comparative statics for the barrier model. The model's optimal fit for free λ under dist_{1111} with empirical target characteristics from Table 3.2 serves as a reference point (emphasized). Either characteristic changes ceteris paribus in steps of 1/5 of its standard deviation from three steps below the reference point up to three above. Given a modified target set of characteristics, the model parameters are now fitted again. Column"achieved characteristic" summarizes the corresponding model characteristics after calibration. P denotes the model price, P_{SFAS} is the price according to SFAS 123, given the model is true. Notation of characteristics: $\widehat{\kappa}$, the mean time of exercise; $\widehat{S_\kappa}$, the mean stock price performance at exercise; $\widehat{c}$, the mean cancellation rate; $\widehat{\rho}$, the correlation of κ and S_κ. Notation of model parameters: B, the value of the barrier at vesting time V; α, the barrier's growth rate; λ, the intensity of independent stopping.