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Integrated Asset Liability Modelling for Property Casuality Insurance: A Portfolio Theoretical Approach

Abstract

In this paper we have developed a financial model of the non-life insurer to provide assistance for the management of the insurance company in making decisions on product, investment and reinsurance mix. The model is based on portfolio theory and recognizes the stochastic nature of and the interaction between the underwriting and investment income of the insurance business. In the context of an empirical application we illustrate how a portfolio optimisation approach can be used for asset-liability management.

Keywords: *Asset Liability Management, Portfolio Optimization, Insurance*

JEL-Classification: *C10, G12, G31, G33*

1. Introduction

Insurance companies can be viewed as levered financial institutions holding financial assets to back up liabilities which are raised by issuing insurance contracts. In this sense the insurance firm is holding two major portfolios: a portfolio of insurance contracts resulting in underwriting profits and a portfolio of financial assets resulting in investment income. The profits of the two portfolios are neither certain nor independent. The uncertainty of the underwriting profits results from the stochastic nature of the insurance business while the uncertainty of the investment income is due to the fact that the returns of most financial assets are, in general, random. The dependencies of underwriting and investment profits are due to (i) non-zero correlations between underwriting profits of different insurance lines, the investment returns of different financial investments and (ii) the reservoir of investable funds which is raised by issuing insurance policies in the different insurance lines.
In this study we are applying modern financial theory to provide assistance for the management of a non-life insurer in making simultaneous decisions on the underwriting and investment activities.
The model presented is based on a portfolio theoretic approach considering the stochastic nature of the insurance business and the dependencies between the underwriting and investment income of the insurance business.[1)] It is assumed that the management of the insurer company seeks to maximize the expected return on shareholders' equity, given a certain level of risk. To accomplish this objective, the management of the insurance company has to determine the optimal value of four sets of decision variables simultaneously: (i) the premium volume written in each insurance line, (ii) the asset allocation of the investable funds, (iii) the degree of reinsurance coverage and (iv) the level of equity capital. In addition a set of constraints reflecting specific features of the insurance business has to be taken into consideration.

2. Construction of a Portfolio Model for Property/Casuality Insurance Companies

2.1 The Basic Model

Consider the following simple financial (one-period) model of the insurance firm: An insurance company with an initial equity capital of C is selling insurance contracts in i = 1,.., n business lines. This leads in each branch to premium proceeds (minus operating costs) amounting to π_i and uncertain aggregated claim costs amounting to S_i, i.e. the underwriting profit in each line is equal to $\pi_i - S_i$. Furthermore the insurance company has a total budget A for financial investments in j = 1,..., m asset classes, each of them receiving a net (after operating investment costs) rate of return of IR_j. Let A_j ($\sum A_j = A$) denote that part of the total investment budget that is invested in asset class j, then the net asset proceeds of the insurance company is given by $\sum A_j\, IR_j$. The total rate of return on stockholders equity ROC is calculated as follows:

$$ROC = \sum_{i=1}^{n} \frac{\pi_i}{C} \cdot PR_i + \sum_{j=1}^{m} \frac{A_j}{C} \cdot IR_j \qquad (1)$$

where PR_i =: 1 - S_i/π_i stands for the net premium return (i.e. one minus the combined ratio) in insurance line i. This equation shows that the company's return on equity can be split into two

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1) See among others *Lambert/Hofflander* 1966, *Krouse* 1970, *Haugen* 1971, *Markle* 1973, *Kahane/Nye* 1975, *Markle/Hofflander* 1975, *Kahane* 1977, *McCabe/Witt* 1980, *Cummins/Nye* 1981, *Loubergé* 1981, *Loubergé* 1983, *Smies-Lok* 1984, *Albrecht* 1986a, *Albrecht* 1986b, *Albrecht/ Zimmermann* 1992 and *Corell* 1998.

components, the result of the portfolio of insurance policies $\sum(\pi_i/C)PR_i$ on the one hand and the result of the portfolio of financial investments $\sum(A_j/C)IR_i$ on the other hand. In general, the underwriting as well as the investment result are considered as stochastic quantities. Even though the insurer writes policies with a negative premium return he makes money as long as the investment result is high enough to compensate the negative underwriting return. Therefore it is important to study the relationship between writing insurance policy and the investable fund of the insurance firm.

The total fund disposal for financial investments $A = \sum A_i$ is derived from shareholder-supplied capital and from policyholder-supplied funds, which are referred to as liability reserves.[2)] Issuing insurance policies generates investable funds, because there is a time lag between collecting the premiums and paying the losses. While the premiums are in general paid at the beginning of the insurance period, claim payments for loss events occur during and/or -because of administrative and legal delays- also after the insurance period. To bridge this time lag between premium receipts and claim payments the insurance company has to build up liability reserves (i.e unearned premium and loss reserves). The assets backing these liabilities constitute the investable funds obtained by writing insurance policies. Based on an idea from *Mc Cabe/Witt* (1980) and elaborated in more detail by *Cummins/Nye* (1981, p. 421) and *Albrecht* (1990, pp. 132-133) the following approximation for the total investment budget is reasonable:

$$A = \alpha \cdot C + \sum_{i=1}^{n} h_i \cdot \pi_i \qquad (2)$$

Therein $0 < \gamma < 1$ denotes that part of the equity capital that is not bound in (non-earning) operating assets. Accordingly, $\gamma \cdot C$ of stockholders equity capital can be invested into financial assets. The variable h_i is called the "funds generating coefficient[2)]" and approximates the average amount of liability reserves available for financial investment, which is generated by writing one unit of premium in the *i*th insurance line. Because of different settlement horizons, funds generating coefficients differ among insurances lines. For example, in short-tailed lines such as auto physical damage, losses are settled relatively quickly, which results in small loss reserves. On the other hand, in long-tailed lines such as general liability insurance there are substantial time lags between the occurrence and the settlement of losses resulting in relatively high loss reserves. One method for approximating funds generating coefficients is to divide the sum of current outstanding loss reserves and unearned premium reserves by premiums written in each insurance line. Applying this method for short-tail lines, the ratio is typically between zero and one, and in excess of one for long-tailed lines.

Let a_j ($\sum a_j = 1$) denote the fraction of the total investment budget A invested in asset class j, and by substituting equation (2) in (1) one obtains the following expression for the return on equity of the insurance firm:

$$\begin{aligned} ROC &= \sum_{i=1}^{n} \frac{\pi_i}{C} \cdot PR_i \quad + \quad \sum_{j=1}^{m} \frac{a_j}{C} (€ \cdot C + \sum_{i=1}^{n} h_i \pi_i) \cdot IR_j \\ &= \sum_{i=1}^{n} x_i \cdot PR_i + \sum_{j=1}^{m} y_j \cdot IR_j \end{aligned} \qquad (3)$$

Here $x_{\mathrm{i}} = \pi_i/C$ stands for the premium-to-surplus ratio in i-th insurance line and $y_j = (a_j/C)(\chi C +$

2) See *Fairley* 1979 and *MacCabe/Witt* 1980.

3) See *Kahane* 1978, p. 69, *Cummins/Nye* 1981, p. 420, *Albrecht* 1986, p. 117 and *Cummins* 1991, p. 284.

$\sum h_i \pi_i$) for the asset-to-surplus ratio in the asset class j. The sum $\sum x_i$ is also denominated as insurance leverage[3)], i.e. the insurer can be viewed as a levered corporation which raises debt by issuing insurance contracts. However, raising debt by issuing insurance policies is quite different from conventional debt instruments such as bonds. While bonds have, in general, fixed coupon payments at fixed maturity dates, the payment time and amount of insurance policies are stochastic.[5)] Therefore, insurance leverage is not equivalent to financial leverage.[6)]
The insurance company's management has to find the optimal product mix of underwriting activities and investments in financial assets. Assuming that the portfolio decision has no effect on the probability distribution of individual premium and asset returns, it is possible to use x_i and y_j as decision variables. Given a fixed amount of equity capital C the decision on $x_i = x_i\,(\pi_i)$ means that the management of the insurance firm decides on the premium exposure π_i in each insurance line, and the decision on $y_j = y_j(a_j)$ determines the asset allocation (i.e. the relative investment weights a_j) of the total investable fund A. Note that if the management of the insurance company has the possibility to increase or decrease a given level of equity capital, the asset-to-surplus and the premium-to-surplus ratios are also influenced by C.
To be able to evaluate the different investment and insurance strategies (i.e. the probability distributions of the return on stockholders' equity capital) determined by the vector of the premium-to-surplus ratio x_i and asset to surplus ratios y_i in a quantitative framework, it is necessary to introduce a formal criterion for decision making under uncertainty. In this paper we make the standard assumption of a risk-averse management of an insurance firm who uses variance or standard deviation (sometimes referred to as volatility) of returns as the measure of risk and applies the mean-variance rule introduced by *Markowitz* to evaluate the different portfolio strategies. This means that a higher expected value and a lower variance of return on equity is more desirable for the firm.
Returning to equation (3) the expected return on equity can be specified in terms of the decision variables x_i and y_j by:

$$E(ROC) = \sum_{i=1}^{n} x_i \cdot E(PR_i) + \sum_{j=1}^{m} y_j \cdot E(IR_j) \tag{4}$$

Here $\mathrm{E}(PR_i)$ stands for the expected underwriting return of the ith insurance line and $\mathrm{E}(IR_i)$ for the expected return of asset class j. The variance is given by

$$\begin{aligned} \mathrm{Var}(ROC) = & \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \cdot x_j \cdot \mathrm{Cov}(PR_i, PR_j) \\ & + \sum_{i=1}^{n}\sum_{j=i}^{m} x_i \cdot y_j \cdot \mathrm{Cov}(PR_i, IR_j) \\ & + \sum_{i=1}^{m}\sum_{j=1}^{m} y_i \cdot y_j \cdot \mathrm{Cov}(IR_i, IR_j) \end{aligned} \tag{5}$$

where $\mathrm{Cov}(PR_i, PR_j)$ is the covariance between the premium returns in the ith and jth insurance line, $\mathrm{Cov}(IR_i, IR_j)$ is the covariance between the ith and the jth asset class and $\mathrm{Cov}(PR_i, IR_j)$ stands for the relationship between asset returns and the premium returns in the different insurance lines. Note, that both types of returns could be correlated through general economic

4) See *Cummins/Nye* 1981, S. 415.

5) See *Cummins* 1990, p. 149.
6) See *McCabe/Witt* 1980, p. 620 and *Albrecht* 1986b.

activity.[7)]

The usual next step in portfolio selection is to determine, for a given menu of risky assets, the set of portfolios that minimises the risk for given levels of expected return, i.e. the mean-variance efficient frontier. However, with respect to specific institutional features of the insurance business it is necessary to impose some constraints which reduces the set of admissible premium-to-surplus and investment-to-surplus ratios.

Balance sheet identity: At the beginning of the accounting period the sum of non earning operating assets and financial assets has to be equal to the sum of stockholders' equity capital and insurance company liability reserves. Using the relationship between liabilities and premiums according to equation (2) this can formally be expressed by $(1-\gamma)C + A = C + \sum h_i \pi_i$. Dividing by C and rearranging terms the following equation

$$\sum_{j=1}^{m} y_j - \sum_{i=1}^{n} h_i \cdot x_i = \gamma, \tag{6}$$

ensures this accounting identity.

Insurance market constraints: It is not unproblematic to assume a perfect independence between the premium-to-surplus ratio and the premium returns for the different insurance lines. Instead it is more realistic that (in the short run) the insurance company can vary the premium volume within a certain bandwidth $\pi_i^{\min} \le \pi_i \le \pi_i^{\max}$. Using $x_i = \pi_i / C$ and imposing the constraints

$$x_i^{\min} \le x_i \le x_i^{\max} \tag{7}$$

it is possible to model the variation of the premium-to-surplus ratios in a realistic way. The insurance line specific minimum $x_i^{min} \ge 0$ and the maximum limit $x_i^{max} > x_i^{min}$ reflect demand compounds and high market entry respectively exit costs. Note that if stockholders equity capital can vary within a certain interval $[C^{\min}; C^{\max}]$, the upper and lower bound for the premium-to-surplus is given by $x_i^{\min} = \pi_i^{\min} / C^{\max}$ and $x_i^{\max} = \pi_i^{\max} / C^{\min}$. If product complementary, i.e. cross-selling- respectively cross-cancellation-effects between different insurance lines, should also be taken into consideration. This can formally be expressed in the following way:

$$x_i \ge \beta_{ij}\, x_j \tag{8}$$

The factor β_{ij} determines the relationship between the premium volume written in insurance i and j. For example, if $\beta_{ij} = 0.3$ and the insurer writes 100 € of premiums in line j the insurer has to write at least 30 € in line j.[8)]

Constraints from insurance regulation: Insurance companies are in business to provide financial protection, i.e. to reimburse the individual in case the insured event occurs. Thus, the individual transfers the insured risk to the company. However, because the results of underwriting and investment activities are stochastic in nature the company may become insolvent and therefore be unable to pay. *Kahneman/Tversky* 1979 introduced the term probabilistic insurance to point out that most insurance is, in fact, only pseudo-certain. The centrepiece of insurance regulation is to bound this default risk by controlling the financial stability of an insurance company. State regulation of German insurers imposes at least two important constraints considering financial ratios: solvency requirements and restrictions on

7) For an empirical examination of this point see *Cummins/Nye* 1980 for US- and *Maurer* 2000, p. 221-254 for German property-liability insurance companies. Both study reported statistically significant correlations between the yearly returns on different asset classes and between the yearly underwriting returns of different insurance lines. However, in both studies the correlations between asset and underwriting returns are in general not statistically different from zero.

8) See *Cummins/Nye* 1981, S. 423.

financial investments. The centrepiece of the solvency requirements is to limit the exposure of the underwriting risk with respect to a certain level of equity (solvency) capital.[9)] In our model this can be expressed by an upper bound on the insurance leverage, i.e. the sum of premium-to-surplus ratios over all insurance lines:

$$\sum_{i=1}^{n} x_i \leq \chi. \tag{9}$$

A reasonable rule of thumb[10)] is that $\chi = 1/0.18$, if reinsurance coverage is neglected. Important restrictions regarding the composition of the insurance company asset allocation are the exclusion of short sales and maximum investment weights for certain risky assets (§ 54a II no. 1 –13 VAG) such as stocks and real estate. More formally, this can be modelled by

$$0 \leq y_j \leq a_j^{\max} \sum_{j=1}^{m} y_j \,. \tag{10}$$

Here $0 < a_j^{max} \leq 1$ stands for the maximum possible fraction of the total investable fund which can be invested in asset class j. For example, according to § 54a VAG the maximum weight for stocks is 30% and for real estate 25%. However, these numbers are based on accounting data, i.e. the book value of stocks should not exceed 30% of the book value of the part of the investable fund which is backing the liability reserves of the insurance company. Hence, the maximal investment weight for stocks with respect to the market value of the total investable fund can be much higher. Of course, the management of the insurance company can impose additional constraints on investment weights, e.g. to guarantee a well diversified investment portfolio or to control estimation risk of asset manager.[11)]

Probability of insolvency: Empirical studies about consumer behaviour of *Wakker* et al. (1997) and *Albrecht/Maurer* (2001) reported evidence that people's willingness to pay for insurance products is dramatically reduced if the default risk of an insurance company exceed a certain level. These results about people's reluctance to purchase probabilistic insurance contracts have practical implications for insurance companies. Their products can only be attractive compared to competitors if they employ a safety-first strategy in their business operations to keep their default risk as low as possible. Therefore, it could be reasonable (e.g. to get a certain rating level) to incorporate a explicit constraint on the risk of an eventual insolvency. Following traditional actuarial risk theory we chose the ruin probability

$$\text{Prob}(ROC < -\gamma) \leq \varepsilon \tag{11}$$

as the measure of risk Equation (11) says that the probability that a negative return on equity exhausts the insurers solvency capital[12] is bound by a maximal small value $\varepsilon > 0$. To implement this stability criterion it is necessary to make a reasonable assumption about the probability distribution of the return on equity. While the normal distribution is the natural approach, empirical studies, such as *Cummins/Nye* 1980 for the US and *Maurer* 2000 for the German insurance market, show that the distribution of underwriting returns is substantially skewed. Therefore, to avoid a possible underestimation of the ruin probability it is necessary to use an approach which allows for reflecting the skewness. Following *Surnick/Grandisson* (1999) we

9) In reality the solvency capital of an insurance company is neither equal to the book nor to the market value of the equity capital. See for this point *Schradin* 1995, 209-220.

10) This is due to the so called Beitragsindex codified in §§ 1, 2 KapitalausstattungsVO and Rundschreiben des BAV R 3/88, VerBAV, 1988, pp. 195 ff., c.f. *Maurer* 2000, p. 215.

11) See *Grauer/Shen* 2000.

12) Here we model the solvency capital as the part of the equity capital which is not invested in operating assets.

represented the return of equity by fitting a modified lognormal distribution.[13)] To generate this distribution, a lognormal distribution is flipped so that the tail is on the negative side and then it is shifted by a constant to the right, which represents its maximum value.[14)] Such a distribution allows for continuous outcomes, is skewed to the left and assigns more probability weight to extreme negative results (which are in the core of actuarial ruin-theory) than to the normal distribution.

2.2 Reinsurance coverage

Reinsurance is a financial arrangement between a reinsurance and an insurance company, whereby the reinsurer agrees, against the payment of a certain amount of money (the reinsurance premium), to reimburse a part of the uncertain claims for losses that the ceding insurer is called upon to pay the original policyholders.[15)] In this sense, reinsurance may be defined as the direct insurer's insurance. From an economic point of view, the rational of writing reinsurance is to (hopefully) improve the probability distribution of the uncertain return on stockholders' equity in conjunction with a sufficient level solvency of the ceding insurance company.

Reinsurance contracts can be divided into two main groups: facultative agreement and treaty binding both parties. In the first case each arrangement refers to a specific insurance contract written by the direct insurer, which has to be separately negotiated between the reinsurer and the ceding insurer for each contract. In contrast to these case-by-case reinsurance trades, a treaty concerns a whole set of insurance contracts written by the direct insurer typically in a particular insurance line (fire, homeowners) during a specific period of time. The primary writer has to cede and the reinsurance company is obligated to accept all contracts for which the treaty has been signed. While historically reinsurance was signed first on a facultative basis, today reinsurance coverage occurs mostly on a treaty basis.

Basically, one can distinguish between proportional and non-proportional reinsurance treaties. In the non-proportional form, a so-called priority is arranged. If the loss for an individual contract (stop-loss-treaty) or the losses for a set of contracts (excess-of-loss-treaty) incurred by the direct insurer on the reinsured contract set is lower than this priority, the reinsurer has no obligation to pay. This means, the reinsurer has to bear the risk above the priority. Hence, because the intervention of the reinsurer is contingent upon the severity of losses suffered by the direct insurer and the reinsurance premium is fix, profits and losses are not shared proportionally between both parties. In contrast to this, in the case of proportional treaties, all profits and losses incurred by the primary writer in the reinsured population of contracts are shared by the reinsurer according to a defined percentage. In the case the most important proportional treaties are the quota share and the surplus reinsurance.

The most frequent proportional treaty is the quota share reinsurance, which is studied here.[16)] In this case the reinsurance company is participating proportionally to the arranged quota $0 \leq q \leq 1$

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13) Another possibility elaborated by *Albrecht/Zimmermann* 1991 is to use the Normal-Power-Approximation for the distribution of the return on equity. However, in this case it is necessary to estimate the covariance-matrix and the co-skewness-matrix for the underwriting return of each insurance line and investment returns of each asset class.

14) More formally, given a constant z we assume that the random variable $(z - ROC) \sim \text{LN}(m, v^2)$ is lognormal distributed with parameters m and v^2. The parameters of the modified lognormal distribution can be obtained from the expected return and the variance of the random return on equity, see *Maurer* 2000, pp. 97-101.

15) See in the following *Loubergé* 1981, 1983, *Waters* 1983 and *Schradin* 1998, chapter two.

16) In the case of a surplus treaty, the reinsurance covers losses only for those contracts in the line for which the value exceeds a certain limit, c.f. *Loubergé* 1983, p. 46.

in the uncertain loss-payments S for all contracts written by the primary writer in the reinsured line. For this transfer of risk the primary writer has to pay a reinsurance premium of $q\cdot \pi$, where π is the original premium received by the direct insurer. It is usual in practice that the reinsurer gives back some part of the received premium as a ceding commission to the primary writer. The ceding commission's function is to let the reinsurer participate in the overhead costs of the insurer. Therefore, the ceding commission is often calculated as a variable part of the reinsurance premium subject to the combined ratio, and this component can be directly modelled in relation to the premiums, respectively the claims. This means that insurer and reinsurer share, proportional to the quota q the total underwriting result of a reinsured business line€By assuming that such a (perfect) proportional splitting takes place, introducing quota-share reinsurance into the direct insurer's portfolio model is straightforward. If q_i represents the percentage reinsured in the i-th insurance line, the return on stockholders equity may by rewritten as follows:

$$\begin{aligned} ROC &= \sum_{i=1}^{n} \frac{\pi_i}{C} \cdot (1-q_i) \cdot PR_i + \sum_{j=1}^{m} \frac{A_j}{C} \cdot IR_j \\ &= \sum_{i=1}^{n} x_i \cdot (1-q_i) \cdot PR_i + \sum_{j=1}^{m} y_j \cdot IR_j . \end{aligned} \tag{12}$$

This equation shows that from the viewpoint of the primary writer, quota-share reinsurance results in a linear reduction of the underwriting return in each business line. The direct insurer's task is to decide simultaneously about the underwriting activities x_i, the financial investments y_j and the reinsurance policies q_i subject to some constraints. The expected return on equity has to be redefined by

$$E(ROC) = \sum_{i=1}^{n} x_i \cdot (1-q_i) E(PR_i) + \sum_{j=1}^{m} y_j \cdot E(IR_j) \tag{13}$$

and the variance by

$$\begin{aligned} \mathrm{Var}(ROC) = &\sum_{i=1}^{n} \sum_{l=1}^{n} x_i \cdot x_l \cdot (1-q_i) \cdot (1-q_l) \cdot \mathrm{Cov}(PR_i, PR_l) \\ &+ \sum_{i=1}^{n} \sum_{j=i}^{m} x_i \cdot y_j \cdot (1-q_i) \cdot \mathrm{Cov}(PR_i, IR_j) \\ &+ \sum_{k=1}^{m} \sum_{j=1}^{m} y_k \cdot y_j \cdot \mathrm{Cov}(IR_k, IR_j). \end{aligned} \tag{14}$$

As can be seen from (13) and (14) transferring some part of the underwriting exposure affects the expected return on equity and variance. By assuming, that different intensities of reinsurance have no effect on the level of non-earning operation assets and that the premiums for reinsurance have to be paid at the beginning of the period, the total reservoir of funds available for financial investments is determined as follows:

$$A = \alpha \cdot C + \sum_{i=1}^{n} (1-q_i) \cdot h_i \cdot \pi_i . \tag{15}$$

Dividing by stockholders equity capital C on both sides and rearranging the balance sheet constraint, including the possibility of reinsurance, is given by:

$$\sum_{j=1}^{m} y_i - \sum_{i=1}^{n} h_i (1-q_i) \cdot x_i = \alpha. \tag{16}$$

Due to the long-term business-connection between the insurer and reinsurer it is feasible that the reinsurance quota q_i cannot be reduced to zero. Likewise, it will be hard to find a reinsurer that will overtake the overall business of an insurance line. Thus, it could be reasonable to restrict the reinsurance quota to a minimum respectively maximum limit:

$$0 \leq q_i^{\min} \leq q_i \leq q_i^{\max} \leq 1. \tag{17}$$

In practice, quota share reinsurance treaties are explicitly or implicitly connecting over different insurance lines, typically with different expected underwriting results. For example, the reinsurer is only willing to cover contracts of a line with a low or negative expected underwriting profit, if the direct insurer at the same time cedes some part of the contracts written in a line with a positive expected underwriting result. More formally, this cross-reinsurance effect can be modelled as follows:

$$q_i \leq b_{ij} q_j \tag{18}$$

whereby the extent to which the reinsurer is willing to cover contracts written in (the low profitable) line i depends on the direct insurer ceding at least $b_{ij}q_i$ contracts ($0 \leq b_{ij} \leq 1$) written in (the more profitable) line j.

The last extension which includes proportional insurance coverage into the decision problem is the constraint due to solvency requirements by state insurance regulation. In general, quota-share arrangements result in a linear reduction of the premium-to-surplus ratio and therefore in an extension of the capacity of the primary writer.[17)] However, the possibility to increase the capacity is restricted. To take the default risk of the reinsurance company, which in general not under insurance regulation, into consideration, in the solvency requirements reinsurance coverage is only taken into consideration up to a limit of 50% of the primary writer premium volume. This leads to the following modified solvency criterion reflecting reinsurance coverage:

$$\max[0.5\sum_{i=1}^{n} x_i ; \sum_{i=1}^{n}(1-q_i)\cdot x_i] \leq \chi \tag{19}$$

In order to determine the set of efficient asset and liability portfolios that minimise risk for given levels of expected return (i.e. the insurance companies mean-variance efficient frontier), the following quadratic optimisation problem with respect to some linear and non-linear constraint should be solved simultaneously for the vector of premium to surplus weights ($x_1, x_2,\ldots, x_n$), the vector of asset-to-surplus weights ($y_1, y_2,\ldots, y_m$) and the vector of reinsurance ratios ($q_1, q_2,\ldots, q_n$):

$$\min Var[ROC(x_i, q_i, y_j)] \tag{20a}$$

for all $i = 1,\ldots, n$, $j = 1, \ldots, m$ and for all admissible values of E(ROC) under the constraints:

17) For insurance capacity see in general *Stone* 1973a, 1973b and *Albrecht/Zimmermann* 1991.

$$\begin{aligned}
&\sum_{j=1}^{m} y_i - \sum_{i=1}^{n} h_i (1-q_i)\cdot x_i = \alpha \\
&0 \le x_i^{\min} \le x_i \le x_i^{\max};\; x_i \ge \beta_{ij}\, x_j \\
&0 \le q_i^{\min} \le q_i \le q_i^{\max} \le 1;\; q_i \le b_{ij} q_j \\
&0 \le y_j \le a_j^{\max} \sum_{j=1}^{m} y_j \\
&\max[0.5\sum_{i=1}^{n} x_i; \sum_{i=1}^{n}(1-q_i)\cdot x_i] \le \chi \\
&\text{Prob}(ROC < -\gamma) \le \varepsilon
\end{aligned} \tag{20b}$$

The optimal investment proportions generally depend on the insurance positions, which themselves are affected by the reinsurance positions and vice versa. As special cases of this simultaneous choice of premium-to-surplus-ratios, asset-to-surplus ratios and reinsurance positions, which can be referred to as mutual fund solution, is by setting all of the reinsurance positions and/or premium-to-surplus-ratios equal to $q_i = 1$ and/or $x_i = 0$, respectively. In the case the insurer has no underwriting exposure investing stockholders' capital in different financial assets.[18)]

3. Empirical Application

3.1 Data Description

The objective of this section is to illustrate how the portfolio approach can be used to assist the management of an individual insurance company in practical decision making regarding the optimal product, reinsurance and investment mix. Therefore we have implemented and solved the optimisation model under several combinations of constraints reflecting the business of a mid-size multi line insurance company that is under German supervision. The company writes insurance contracts in eight lines for which proportional reinsurance coverage is available and invests in six asset classes. The company has a current equity capital of 410 Mio. € where 25%, that means 102,5 Mio €, is invested in non-earning operating assets. This amount was fixed in all examinations and the probability that a negative profit will consume the total equity capital not invested in operating assets is limit to 0.01%. The insurers' management has the possibility to decrease (increase) the equity capital to 210 Mio. € (550 Mio. €). Table 1 shows management judgements about the expected values, the volatilities and the correlations of the underwriting and investment returns of the different insurance lines and asset classes. Furthermore, the table summarizes maxima, minima and product complementary constraints regarding the premium volume, certain cross-reinsurance effects, the constraints on the asset allocation and the funds generating coefficients.

18) *Cummins* 1990, p. 151.

Table 1:
Constraints and parameter estimates for an insurance company

	Insurance Lines								Asset Classes					
	S1	**S2**	**S3**	**S4**	**S5**	**S6**	**S7**	**S8**	**A1**	**A2**	**A3**	**A4**	**A5**	**A6**
	Underwriting Return (in % p.a.)								**Investment Return (in % p.a.)**					
E(.)	11,0	4,0	0,0	-1,2	4,0	5,0	6,0	-4,4	9,3	9,3	5,4	4,9	5,0	4,2
σ (.)	3,4	5,0	5,0	12,4	9,0	6,0	8,0	16,1	20,0	16,0	4,0	8,0	5,0	1,0
	Premium volume (in Mio. €)								**Investment-Weights**					
Min	40	60	60	30	30	50	50	30	0 %	0 %	0 %	0 %	4%	10 %
Max	70	80	100	60	50	70	70	50	25 % 45 %	25 %	90 %	10 %	10%	-
	Funds-Generating-Coefficients													
h	1,929	1,095	0,346	0,335	0,845	0,845	0,546	0,285	-	-	-	-	-	-
	Correlations													
S1	1,0													
S2	0,6	1,0												
S3	0,7	0,4	1,0											
S4	0,0	-0,3	0,3	1,0										
S5	0,2	0,3	0,3	-0,2	1,0									
S6	0,2	0,2	0,6	0,5	0,3	1,0								
S7	-0,1	0,2	-0,4	-0,2	0,0	-0,3	1,0							
S8	0,0	0,0	0,4	0,7	-0,2	0,7	-0,1	1,0						
A1	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	1,0					
A2	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,5	1,0				
A3	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,2	0,1	1,0			
A4	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,2	0,6	0,2	1,0		
A5	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,1	0,2	0,4	0,2	1,0	
A6	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	-0,1	-0,1	0,0	0,0	-0,1	1,0

Product Complementary: $\pi_4 \geq 0.6\pi_6$
Reinsurance Complementary: $q_3 \leq 0.5q_1$; $q_8 \leq 0.75q_6$; $q_4 \leq 0.6q_6$
Stockholders Equity Capital (in Mio. €): [210 , 550]
Fixed no-earning assets constraint: 102.50 Mio €

The short-cuts for the insurance lines and asset classes is denoted by the following categories:

S1:= General Accident
S2:= General Liability
S3:= Automobile
S4:= Fire
S5:= Household
S6:= Technical
S7:= Transportation
S8:= Business Interruption Insurance

A1 := German Stocks
A2 := International Stocks
A3 := German Bonds
A4 := International Bonds
A5 := German Real Estate
A6 := German Money Market

3.2 Results

The results of the optimisation problem are summarized in table 2.

Table 2:
Efficient Portfolios for an insurance company

E(ROC)	6.4%	8%	10%	12%	14%	16%	18%	20%	22%	24.8%
σ (ROC)	1.46%	1.92%	2.70%	3.47%	4.26%	5.12%	6.23%	8.00%	10.55%	16.31%
π_1	70	70	70	70	70	70	70	70	70	70
(q_1)	14%	0%	0%	0%	0%	0%	0%	0%	0%	0%
π_2	60	60.01	74.12	74.84	74.82	75.37	80	80	80	80
(q_2)	99%	42%	6%	32%	44%	0%	0%	0%	0%	0%
π_3	60	60	60	60	60	60	60	60	68.18	100
(q_3)	7%	0%	0%	0%	0%	0%	0%	0%	0%	0%
π_4	30.12	30.12	30.12	30.12	30.12	30.12	39.34	42.17	42.17	42.17
(q_4)	60%	60%	60%	60%	60%	60%	50%	18%	0%	0%
π_5	30.17	31.49	34.92	37.15	37.66	39.13	50	50	50	50
(q_5)	94%	54%	45%	56%	60%	49%	0%	0%	0%	0%
π_6	50	50	50	50	50	50	65.31	70	70	70
(q_6)	99%	99%	99%	99%	99%	99%	83%	31%	0%	0%
π_7	50.15	63.73	70	70	70	70	70	70	70	70
(q_7)	23%	0%	0%	5%	10%	0%	0%	0%	0%	0%
π_8	30	30	30	30	30	30	30	30	30	30
(q_8)	74%	74%	74%	74%	74%	74%	62%	23%	0%	0%
A_1	2.47	3.37	3.65	2.85	2.42	2.94	4.65	8.47	22.93	53.92
A_2	4.96	8.61	9.47	7.81	7.03	9.06	15.51	28.8	58.1	127.61
A_3	26.33	43.82	40.79	34.03	31.64	41.27	64.65	103.66	121.28	145.37
A_4	0	0	0	0	0	0	0	0	0	0
A_5	24.51	27.35	24.46	17.92	14.57	16.3	18.03	19.62	20.64	21.12
A_6	554.58	600.48	533.2	385.47	308.62	337.92	347.84	329.93	293.06	179.02
C	550	538.68	420.49	282.01	211.31	210	210	210	210	210

E(ROC) := Expected return on equity (in %)
σ (ROC) := Standard Deviation of return on equity (in %)
π_i := Premium volume in insurance line $i = 1, \ldots, 8$ (in Mio. €)
A_j := Investment volume in asset class $j = 1, \ldots, 6$ (in Mio. €)
q_i := Proportion of reinsurance coverage in insurance line $i = 1, \ldots, 8$ (in % of π_i)
C := Equity capital at the beginning of the period (in Mio. €)

One can detect several interesting constellations. The analysis of the structure of the Minimum-Variance-Portfolio (MVP), with an expected return on equity of 6.4% and a standard deviation of 1.46% shows that the premium volumes, except of insurance line one, which is, under return-risk aspects, a very attractive insurance line, are all at the possible minimum. The reinsurance quotas q_i are at a relatively high level, e.g. in insurance line two 99% of the underwriting exposure is ceded to the reinsurer. Because of the assumed cross-reinsurance effects only 7% of insurance line three, 60% of insurance line four and 74% of insurance line eight can be ceded through to the reinsurer. The structure of the asset allocation reflects the high degree of risk-aversion of such an insurance company: more than 90% of the investment budget is invested in T-bills but only 1.2% in German and international stocks and 4.3% in German bonds. Moreover, the equity capital is at the maximum of 550 Mio € resulting in a total premium-to-surplus ratio (i.e. the insurance leverage) before (after) reinsurance of 0.69 (0.32). A company with such a business structure has some similarity to a money market investment fund. In contrast to this, if an expected return on equity of 24.8% is required, the insurance leverage is, before and after reinsurance, 2.44. With increasing expected returns on equity, one can detect four effects.

The first effect takes into account the premium volumes in the insurance lines. With an increasing expected return on equity, it is necessary to increase the underwriting volume in the different insurance lines. Because of the cross-selling effect in insurance line four and six it is more advantageous to increase the underwriting volume in line six to the maximum first, when more than a 20% expected return on equity is demanded. This is contrast with the results of lines five and seven, but here no cross selling effects with unattractive insurance lines have to be regarded. To realise the portfolio with the highest return on equity which is in line with the ruin constraint, it is necessary to underwrite, with exception of line four, the maximum volume in each insurance line.

The second effect is that the reinsurance quotas are becoming smaller. Because of the cross-reinsurance effects one can see, again, that one cannot, as in insurance line one for example, reduce the reinsurance quota in the attractive insurance line six without reducing it also in the unattractive insurance line four. If the required return on equity should be 22%, no reinsurance coverage can be observed.

The third effect pertains to the amount of equity capital. Again, with rising expected return on equity it is necessary to reduce the amount of equity capital. This is evident because of the definition of the return on equity. Up from the level of 16% expected return, the equity capital is reduced to the minimum of 210 Mio. €.

Finally, the fourth effect concerns asset allocation. The structure of asset allocation relocates with an increasing expected return on equity from low risk and low return assets like T-bills to high risk and high return assets like stocks and bonds. Looking at the asset allocation for the maximum return on equity portfolio (MVP) about 35% (1.21%) of the investable fund is allocated in German and international stocks, 27.5% (4.3%) in German bonds and only 33% (90.49%) in T-bills. Note the restriction on the maximum investment weight. With given input parameters it is possible to reach (μ, σ)-combinations of up to 26.6% expected return on equity and 20.95% volatility. Nevertheless, expected returns on equity above 24.8% do not fulfil the constraint for a AAA rating (i.e. 0.01% ruin probability) of an insurance company.

Summing up the results, one can point out, that for the realisation of the MVP a high amount of equity capital with the implication of slight premium to surplus ratios is necessary. The reinsurance quotas of the insurance lines are as high as possible and the asset allocation is dominated by money market investments. With rising expected returns on equity four effects ceteribus paribus take place:

- The amount of equity capital is reduced
- Reinsurance quotas are reduced
- Underwriting volumes in insurance lines are increased to the maximum
- Asset allocation is relocated from money market to stocks and bonds

4. Conclusion

This paper approaches mean-variance portfolio theory for a non-life insurance company's business. We developed a model that permits simultaneous determination for the underwriting activities, extent of reinsurance coverage, asset allocation and level of equity capital subject to various constraints reflecting special characteristics of the insurance business. The objective is to extract from a set of admissible business strategies, the efficient frontier regarding the risk and the expected return on shareholders equity. Therefore, the model provides some insight into the management of insurance companies and can be used as a guide by insurance companies in asset liability management.

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