

Graviton n -point functions

for UV-complete theories in Anti-de Sitter space

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Abstract

We calculate graviton n -point functions in an anti-de Sitter black brane background for effective gravity theories whose linearized equations of motion have at most two time derivatives. We compare the n -point functions in Einstein gravity to those in theories whose leading correction is quadratic in the Riemann tensor. The comparison is made for any number of gravitons and for all physical graviton modes in a kinematic region for which the leading correction can significantly modify the Einstein result. We find that the n -point functions of Einstein gravity depend on at most a single angle, whereas those of the corrected theories may depend on two angles. For the four-point functions, Einstein gravity exhibits linear dependence on the Mandelstam variable s versus a quadratic dependence on s for the corrected theory.

arXiv:1202.2221v1 [hep-th] 10 Feb 2012

1 Introduction

1.1 Motivation and Objectives

The gauge-gravity duality [1, 2, 3, 4] can be used to relate properties of a strongly coupled fluid to those of a weakly coupled theory of anti-de Sitter (AdS) gravity [5] (and references therein). A vast literature is devoted to using graviton and other two-point functions as a means for calculating the two-point correlations of various operators in the gauge theory; see, however, [6, 7]. In particular, the ratio of the shear viscosity to the entropy density η/s has been a focal point of attention [8, 9, 10].

The present treatment broadens the scope to graviton n -point functions for arbitrary n . This is meant as preparation for using the corresponding multi-point correlation functions of the gauge-theory stress tensor as a probe of the gravitational dual of the quark-gluon plasma. This plasma is produced in heavy-ion collisions and, so, of direct observational relevance [11].

The key idea is a recent observation [12] that the effective theory describing gravitational perturbations about a background solution is, itself, highly constrained irrespective of the exact details of the UV-complete theory. The argument is based on considerations of unitarity, which follows naturally from the property of UV-completeness on both sides of the gauge-gravity correspondence.

The argument in [12] is that we should only consider theories whose linearized equation of motion for the gravitons has, at most, two time derivatives. The non-linear interactions of such theories are constrained only by general covariance, which can be contrasted with Lovelock's original construc-

tion [13]. The latter further constrains the form of the interaction terms and limits them to a small finite number for each spacetime dimensionality. This implies that the effective theory of perturbations is of the “Lovelock class” of gravitational models, as defined in detail below. This class contains the Einstein (two-derivative) and Gauss–Bonnet (four-derivative) terms, plus a series of terms with ever-increasing numbers of derivatives. In spite of the higher-derivative extensions, all Lovelock class theories satisfy, by construction, the two-derivative constraint on the equation of motion [13].

Part of the motivation for the current work is the prospect of an experimental test of the multi-particle correlations in heavy-ion collisions. The purpose is to initiate this task, which is accomplished as follows:

We assume an AdS black brane background geometry and calculate the graviton n -point functions for both relevant theories. Considerations are limited to a kinematic regime of a “high momentum”, which is defined further on. Otherwise, we determine all the physically relevant n -point functions for any number of gravitons.

The restriction to the high-momentum kinematic region is chosen with two reasons in mind. First, this kinematic region allows for the suppressed Gauss–Bonnet corrections to compete in the best way with the leading-order Einstein results. Second, this region manifestly reveals how the two theories are fundamentally distinct: Because Einstein gravity is polarization independent, its n -point functions depend on at most a single scattering angle, whereas the Gauss–Bonnet theory is polarization dependent and its n -point functions typically depend on two angles. For the 4-point functions, the distinction is expressed through a quadratic dependence on the Mandelstam

variable s for the Gauss–Bonnet theory versus a linear dependence on s for Einstein’s.

1.2 The meaning of “Lovelock class” theories

We wish to explain in more detail how the condition of having at most two time derivatives in the linearized equations of motion limits the possible class of gravity theories. As is well known and will be evident from Subsection 4.2, the only term in the Gauss–Bonnet Lagrangian which is physically significant is the Riemann-squared term. The other two terms are, essentially, “along for the ride” so as to assure that the equations of motion contain no more than two (time) derivatives. But even this statement can be deceiving, as the two extra terms can be viewed as an artifact of a particular choice of metric variables [14, 15] (and, again, §4.2). And so it is more accurate to say: “the four-derivative unitary extension of Einstein gravity is defined by adding a Riemann-tensor-squared term to the Einstein-Hilbert action, supplemented by boundary conditions that ensure a two-derivative (linearized) field equation.” This, of course, implies that sources for the exorcized modes are not allowed either.

A similar statement should apply to a unitary extension of Einstein gravity to arbitrary order in the number of derivatives. For this reason, six- and higher-derivative corrections can still play a role in a five-dimensional spacetime despite the fact that the Lovelock series terminates at the Gauss–Bonnet extension.¹ However, such corrections are suppressed by factors of momentum divided by the cutoff scale of the gravity theory or, equivalently,

¹This point was missed by us previously.

by inverse powers of the 't Hooft coupling of the gauge theory. For this reason, we will limit the current considerations to Einstein gravity and its leading-order (four-derivative) correction.

Nevertheless, we will, to avoid confusion and clutter, continue to adhere with the standard nomenclature such as Gauss-Bonnet, Lovelock, *etcetera*.

1.3 Difference between Einstein and Lovelock theories

The extra pair of derivatives of Gauss-Bonnet gravity is directly responsible for one of its two physical distinctions with Einstein's theory; namely, the structure of the higher- (than two) point functions. The other physical difference is that Gauss-Bonnet theories disobey Einstein's equivalence principle. This first distinction is essential for the following reason: The defining feature of any Lovelock theory is that the linearized field equation is at most quadratic in derivatives. Hence, the two-point functions of Lovelock tend to all look rather the same, at least when compared at a fixed choice of polarization. Conversely, the higher-point functions can and will be substantially different.

This difference in the higher-point functions becomes apparent when these are re-expressed in terms of scattering angles. As will be made clear, Einstein is unique among gravity theories in that any of its n -point functions depend on at most a single independent angle. This outcome can be viewed as a consequence of a redundancy that was already alluded to by Hofman [7]. He demonstrated that, for a strictly two-derivative theory, the higher-point functions carry what is redundant information about the propagator. We see this quite literally in the current work, inasmuch as any Einstein $2n$ -point

function could be obtained directly from a two-point function, using only simple combinatorial arguments.

The simple nature of the Einstein angular dependence, when compared to Gauss-Bonnet and other higher-derivative theories, is already well understood from the work of Hofman and Maldacena [6]. There, however, the more complicated angular dependence of such “non-Einstein” models is viewed from the field-theory perspective and attributed to a discrepancy in the central charges of the gauge-theory dual. This discrepancy is absent for gauge theories with Einstein duals but generally is not. Our main point here is that this distinction could already be deduced from the bulk point of view without detailed knowledge about the gauge theory.

As an aside, let us point out that the same logic that underlies this distinction between Einstein and Gauss-Bonnet gravity can be extended to the purpose of comparing Lovelock models of arbitrary order. Just as the four- and higher-point functions are redundant for Einstein, the same must be true for the six- and higher-point functions of Gauss-Bonnet. Then, with each additional inclusion of a term from the Lovelock series, the order that this redundancy sets in will increase accordingly. So that, in scenarios where higher-order Lovelock extensions could be relevant, there is an in-principle means of distinguishing the different models by looking at $2n$ -point functions with an increasingly larger value of n .

1.4 Contents

The rest of the paper proceeds as follows. The next section describes the basic set-up and strategies, introduces some important formulas and fixes

conventions. Sections 3 and 4 are dedicated to calculating the graviton $2n$ -point functions (a function with an odd number of gravitons vanishes trivially) for the Einstein and Gauss–Bonnet cases, respectively. In Section 5, we elaborate on and substantiate the statements about angular dependence, as well as make the connection to the gauge theory. Section 6 summarizes our conclusions. One of the supporting calculations is deferred to an appendix.

2 The basic framework

Our starting point is a gauge field theory and its presumed AdS gravitational dual. The premise is to learn about the strongly coupled properties of the former from the weakly coupled limit of the latter. We assume that the AdS bulk spacetime is described by string theory, which is a unitary and UV-complete theory. It is also assumed that the full UV completion can be approximated by a gravitational action that includes the Einstein term along with higher-derivative corrections. Lastly, we assume that the action’s equations of motion support a stationary black brane solution, as this geometry will serve as the background.

2.1 Formalism and conventions

A D -dimensional (asymptotically) AdS black brane can be described by the following background metric:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + \frac{r^2}{L^2}dx_i^2 . \quad (1)$$

The index denotes the transverse space dimensions $i = 1, \dots, D - 2$, L is the AdS radius of curvature and r is the radial coordinate (orthogonal to the

brane). The functions f and g are constrained to asymptote to r^2/L^2 at the AdS boundary ($r \rightarrow \infty$) and vanish on the horizon ($r = r_h$); meaning that all AdS brane solutions look exactly the same on these two surfaces (the latter because of “no-hair” theorems).

Irrespective of the higher-derivative terms and other matter fields in the string theory, we can expect f and g to agree with their Einstein forms, $f, g = \frac{r^2}{L^2} \left(1 - \frac{r_h^{D-1}}{r^{D-1}}\right)$, up to perturbatively small corrections. These corrections are, on general grounds, of the order l_p^2/L^2 with l_P being the Planck length. From now on, we set $L = 1$ unless stated otherwise.

Small metric perturbations about this background solution, $g_{ab} \rightarrow g_{ab} + h_{ab}$, should have a description in terms of an effective field theory. The effective model will naturally inherit higher-derivative corrections; however, as explained in [12] and commented above, unitarity constrains these corrections to be organized into Lovelock extensions of Einstein’s theory.

The immediate aim is to calculate the graviton n -point functions for arbitrary n . As discussed in [16], these functions can be viewed as a measure of the gravitational coupling between n interacting gravitons and, so, can be determined by expanding out the Lagrangian density $\sqrt{-g}\mathcal{L}$ to the relevant perturbative order. To this end, it is useful to define the tensor

$$\mathcal{X}^{abcd} \equiv \frac{\partial \mathcal{L}}{\partial \mathcal{R}_{abcd}}. \quad (2)$$

For later use, \mathcal{X}^{abcd} inherits all of the (anti-) symmetry properties of the Riemann tensor \mathcal{R}^{abcd} and, for Lovelock theories in particular, must satisfy the identity [13]

$$\nabla_a \mathcal{X}^{abcd} = 0. \quad (3)$$

As is standard procedure in the analysis of AdS brane models, we impose the radial gauge on the gravitons or $h_{ra} = 0$ for any choice of a . This gauge allows us to separate the gravitons into three sectors: tensor, vector and scalar, which are sometimes also called transverse/traceless, shear and sound [17]. With z denoting the direction of graviton propagation parallel to the brane and x, y , any pair of transverse brane directions that are orthogonal to z , these sectors can be classified respectively as

$$h_2 = \{h_{xy}\} \tag{4}$$

$$h_1 = \{h_{zx}, h_{tx}\} \tag{5}$$

$$h_0 = \{h_{tt}, h_{zz}, h_{zt}, h_{x_i}^{x_i}\} . \tag{6}$$

Only special combinations of the modes have physical gauge-invariant meaning [18]. Respectively, these are

$$h_{xy} , \quad \nabla_t h_{zx} - \nabla_z h_{tx} , \quad \square h_a^a .$$

The scalar-mode interactions involve at least two derivatives and, as explained in Appendix A, any occurrence of a $\nabla_a \nabla_b h_{cd}$ can be eliminated via the equations of motion. Hence, the scalar mode decouples on-shell. A physical scalar mode requires an external source in addition to the background brane.

Thus, we are left to consider the tensor and vector modes. Vector interactions involve at least one derivative per mode, and so their maximum number in a $2n$ -point function is set by the highest derivative term of the gravity theory. To understand why, let us consider the coordinate transformation $x^a \rightarrow x^a + \xi^{x_i} \delta_{x_i}^a$ such that $\nabla_t \xi_{x_i} = -h_{tx_i}$. Then h_{tx_i} is set to 0 but, as readily verified, $\nabla_z h_{tx} - \nabla_t h_{zx}$ does not change.

Physically, this can be understood by the vector modes having an effective description as components of an electromagnetic vector potential [9]. For instance, the compactification of x reduces $D = 5$ Einstein gravity to a $D = 4$ Einstein–Maxwell theory such that $A_0 = h_{tx}$ and $A_z = h_{zx}$. We will, therefore, sometimes write the vector modes as

$$F_{tz}^{(j)} \equiv \omega_j h_{zx}^{(j)} + k_j h_{tx}^{(j)}. \quad (7)$$

The choice of notation emphasizes that, as far as these modes are concerned, the field-strength tensor is the only physical quantity. So that, from this point of view, a polarization-dependent theory in 5D is equivalent to polarization-independent gravity in 4D coupled to a $U(1)$ field strength.

The $2n$ -point functions are further simplified by restricting to a kinematic region of “high momentum”; meaning that we intend to take only the terms with the highest power of ω , $k \equiv |\vec{k}|$ in a given $2n$ -point function. Here, we have introduced the convention $h_{ab} \propto \phi(r)e^{i(\omega t - \vec{k} \cdot \vec{x})}$. This high-momentum region still falls within the hydrodynamic paradigm, where the frequency ω and transverse momentum k are considered to be parametrically lower than the temperature [8, 10].

A subtle point is that, even for this high-momentum regime, radial derivatives, whether acting on gravitons or the background, cannot be immediately disregarded. This is because $g^{rr}\nabla_r\nabla_r \sim \frac{r^2}{r^2} \sim 1$, whereas (*e.g.*) $g^{tt}\nabla_t\nabla_t \sim \frac{\omega^2}{r^2} \propto r^{-2}$. Hence, the radial derivatives seem to dominate at the AdS boundary for any finite values of ω and k . However, the process of holographic renormalization [19, 20, 21] for bulk quantities requires negative powers of r [22] to survive. Then, since our ultimate interest is the gauge theory, a derivative will always implicitly mean either ∇_t or $\nabla_z \equiv \vec{k} \cdot \vec{\nabla}$.

2.2 Expanding the Lagrangian

Let us close this section with some useful comments about perturbatively expanding the metric and our general strategy for expanding $\sqrt{-g}\mathcal{L}$. We adopt the 't Hooft–Veltman [23] convention, whereby the expansion of any covariant metric (or metric with both indices down) stops at linear order. That is,

$$g_{ab} = \bar{g}_{ab} + h_{ab} \quad (8)$$

is exact to all orders. Note that an overlined quantity signifies the background and indices on a graviton are always raised by a background (contravariant) metric.

One then finds that

$$g^{ab} = \bar{g}^{ab} - h^{ab} + h^a_c h^{cb} + \mathcal{O}[h^3], \quad (9)$$

$$\sqrt{-g} = \sqrt{-\bar{g}} \left[1 + \frac{1}{2} h^a_a - \frac{1}{4} h^a_c h^c_a + \frac{1}{8} (h^a_a)^2 \right] + \mathcal{O}[h^3]. \quad (10)$$

When carrying out the calculations, we arrange that calculation so that only covariant gravitons are acted on by derivatives. Then, since there can be at most one derivative per graviton (see Appendix A), the following exact expression suffices to handle all appearances of a differentiated graviton:

$$\Gamma_{abc} = \bar{\Gamma}_{abc} + \frac{1}{2} [\bar{\nabla}_a h_{bc} + \bar{\nabla}_b h_{ac} - \bar{\nabla}_c h_{ab}]. \quad (11)$$

The following second-order expansion also proves to be useful:

$$\delta\mathcal{R}_{abcd}[h^2] = [\nabla_c - \bar{\nabla}_c] \Gamma_{bda}(h) - \{c \leftrightarrow d\}, \quad (12)$$

where we have used $\nabla - \bar{\nabla} \sim \Gamma(h)$.

What is left is to expand out of the contravariant metrics and the determinant. As these gravitons are undifferentiated, they must be tensors. So, the task simplifies. The relevant expressions are now

$$g^{xx} = \bar{g}^{xx} + h^x_y h^{yx} + (h^x_y h^{yx})^2 + \dots + (h^x_y h^{yx})^p + \dots, \quad (13)$$

$$\begin{aligned} \sqrt{-g} = \sqrt{-\bar{g}} & \left[1 - \frac{1}{2} h^x_y h^{yx} - \frac{1}{4 \cdot 2!} (h^x_y h^{yx})^2 - \frac{3}{2^3 \cdot 3!} (h^x_y h^{yx})^3 \dots \right. \\ & \left. - \Theta(p) (h^x_y h^{yx})^p - \dots \right], \quad (14) \end{aligned}$$

such that

$$\Theta(p) \equiv \frac{\Gamma[p - \frac{1}{2}]}{2\sqrt{\pi}p!}, \quad p = 0, 1, 2, \dots \quad (15)$$

In Eqs. (13-15) we have made the physically motivated choice of $D = 5$, which is the case from now on. None of our conclusions would change for larger values of D .

3 The Einstein n-point functions

We first recall the Einstein Lagrangian $\mathcal{L}_E = (1/16\pi G_5)\mathcal{R}$ and its variation with respect to the Riemann tensor,

$$\mathcal{X}_E^{abcd} = \frac{1}{32\pi G_5} [g^{ac} g^{bd} - g^{ad} g^{bc}], \quad (16)$$

where G_5 is the five-dimensional Newton's constant.

3.1 Two-point functions

Let us begin here with the two-point functions. Because of the high-momentum restriction, we only take into account terms in which every available derivative acts on a graviton. As the scalar modes have been deemed irrelevant,

the only possibilities are two differentiated tensor modes or two differentiated vector modes. A “mixed combination” of a tensor and a vector cannot contribute since general covariance requires any term to have an even number of both x and y indices, and $\nabla_x = \nabla_y = 0$.

By way of Eqs. (11,12,16) and some simplification, the case of two tensor modes work can be worked out. Using the notation $h_{ab}^{(j)} \propto \exp[i\omega_j t - k_j z]$, we find

$$\langle h_2 h_2 \rangle_E = -\frac{1}{32\pi G_5} \sqrt{-g} g^{xx} g^{yy} [h_{xy}^{(1)} (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2) h_{xy}^{(2)}] . \quad (17)$$

Here and throughout, the large-momentum regime is implied.

When there are, rather, two vector modes, the result is then

$$\begin{aligned} \langle h_1 h_1 \rangle_E &= \frac{1}{16\pi G_5} \sqrt{-g} g^{xx} g^{zz} (-g^{tt}) [\omega_1 h_{zx}^{(1)} + k_1 h_{tx}^{(1)}] [\omega_2 h_{zx}^{(2)} + k_2 h_{tx}^{(2)}] \\ &= \frac{1}{16\pi G_5} \sqrt{-g} g^{xx} g^{zz} (-g^{tt}) F_{tz}^{(1)} F_{tz}^{(2)} . \end{aligned} \quad (18)$$

All expressions should be understood as symmetrized with respect to x and y , so that (*e.g.*) $g^{xx} h_{zx} h_{tx}$ really means $\frac{1}{2} [g^{xx} h_{zx} h_{tx} + g^{yy} h_{zy} h_{ty}]$.

3.2 Higher-point functions

Because vector modes appear only through F and so must be differentiated, and because undifferentiated tensor modes can only be added in pairs (*cf.* Eqs. (13,14)), the n -point functions with odd numbers of gravitons vanish. So let us next consider the $2n$ -point functions with $n \geq 2$. These could either be worked out by a brute force expansion or deduced from the two-point functions by way of simple combinatorial arguments.

The coefficients of the contravariant-metric expansion are given in Eq. (13) and those of the determinant, in Eq. (14), which leads us to

$$\begin{aligned} \langle (h_2)^{2n} \rangle_E &= \binom{2n}{2} \sum_{p=0}^{n-1} (n-p) \Theta(p) \frac{\sqrt{-g} g^{xx} g^{yy}}{32\pi G_5} \left[\prod_{j=2}^n (h^x_y)^{(2j-1)} (h^y_x)^{(2j)} \right] \\ &\times [h_{xy}^{(1)} (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2) h_{xy}^{(2)}] . \end{aligned} \quad (19)$$

In the previous equation, the binomial factor in front of the sum accounts for the number of ways of drawing two (differentiated) tensor modes out of the $2n$ available, the summation index counts the number of pairs of modes in the expansion of the determinant and the factor of $(n-p)$ is the number of ways of drawing the remaining $n-1-p$ pairs out of two contravariant metrics. Here, we have used that the number of ways of drawing q identical objects from m distinct “boxes” is $\binom{q+m-1}{m-1}$. The summation can be done explicitly,

$$\begin{aligned} \langle (h_2)^{2n} \rangle_E &= - \binom{2n}{2} \frac{\Gamma [n + \frac{1}{2}]}{\sqrt{\pi} \Gamma [n]} \frac{\sqrt{-g} g^{xx} g^{yy}}{16\pi G_5} \left[\prod_{j=2}^n (h^x_y)^{(2j-1)} (h^y_x)^{(2j)} \right] \\ &\times [h_{xy}^{(1)} (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2) h_{xy}^{(2)}] \\ &= - \frac{(2n-1) \Gamma [n + \frac{1}{2}]}{\sqrt{\pi} \Gamma [n-1]} \frac{\sqrt{-g} g^{xx} g^{yy}}{16\pi G_5} \left[\prod_{j=2}^n (h^x_y)^{(2j-1)} (h^y_x)^{(2j)} \right] \\ &\times [h_{xy}^{(1)} (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2) h_{xy}^{(2)}] . \end{aligned} \quad (20)$$

For the case of two vector modes, there is no need for a leading binomial factor and only one contravariant metric is expanded; and similar methods yield

$$\begin{aligned} \langle (h_1)^2 (h_2)^{2n-2} \rangle_E &= \sum_{p=0}^{n-1} \Theta(p) \frac{\sqrt{-g} g^{xx} g^{zz} g^{tt}}{16\pi G_5} F_{tz}^{(1)} F_{tz}^{(2)} \prod_{j=2}^n (h^x_y)^{(2j-1)} (h^y_x)^{(2j)} \\ &= - \frac{\Gamma [n - \frac{1}{2}]}{\sqrt{\pi} \Gamma [n]} \frac{\sqrt{-g} g^{xx} g^{zz} g^{tt}}{16\pi G_5} F_{tz}^{(1)} F_{tz}^{(2)} \prod_{j=2}^n (h^x_y)^{(2j-1)} (h^y_x)^{(2j)}. \end{aligned} \quad (21)$$

Equations (20) and (21) exhaust all possible n -point functions.

4 The Gauss–Bonnet n -point functions

4.1 Initial considerations

We view the various Gauss–Bonnet expressions as extensions to the leading Einstein term. So, the Lagrangian for this theory is (with L momentarily restored)

$$\frac{1}{16\pi G_5} \mathcal{R} + \frac{1}{L} \mathcal{L}_{GB} = \frac{1}{G_5} \left[\frac{\mathcal{R}}{16\pi} + \frac{l_p^2}{L^2} L^2 \mathcal{L}_{GB} \right], \quad (22)$$

where we have used $l_p = \sqrt{G_4} \sim \sqrt{G_5/L}$. This makes it clear that the relative strength of the Gauss-Bonnet extension goes as l_p^2/L^2 , which is parametrically smaller than unity.

Let us now recall the Gauss–Bonnet Lagrangian and its variation,

$$\mathcal{L}_{GB} = \lambda [\mathcal{R}^{abcd} R_{abcd} - 4\mathcal{R}^{ab} \mathcal{R}_{ab} + \mathcal{R}^2], \quad (23)$$

$$\begin{aligned} \mathcal{X}_{GB}^{abcd} = \lambda [& \mathcal{R}^{abcd} - \mathcal{R}^{abdc} - 2g^{ac} \mathcal{R}^{bd} - 2g^{bd} \mathcal{R}^{ac} \\ & + 2g^{ad} \mathcal{R}^{bc} + 2g^{bc} \mathcal{R}^{ad} + \mathcal{R} g^{ac} g^{bd} - \mathcal{R} g^{ad} g^{bc}], \end{aligned} \quad (24)$$

where λ is a dimensionless number of order unity.

We will first look at the four-point functions. In light of previous considerations, there are only three viable ways of selecting the four gravitons: four tensor modes, four vector modes or two of each. We will, however, proceed to argue that only the first of these choices can have any physical relevance and, even for this one, the calculation is much simpler than it might appear.

4.2 Simplifying the Gauss–Bonnet calculations

Let us begin here with the case of four tensor modes. It is only necessary to include the contribution from the Riemann-tensor-squared term of \mathcal{L}_{GB} since it already contains all physical information about the scattering of four tensor modes [24] (and references therein). This claim can be readily understood from the perspective of field redefinitions [14, 15].

To clarify the above argument, suppose that we start with the following term in the two-point function, $h_{ab}h^{ab}$. Now, redefine the tensor modes $h_{ab} \rightarrow h_{ab} + \delta^{(1)}\mathcal{R}_{ab} + \delta^{(2)}\mathcal{R}_{ab} + \dots$. One of the products of this transformation goes as $\delta^{(2)}\mathcal{R}^{ab}\delta^{(2)}\mathcal{R}_{ab}$; that is, precisely the fourth-order contribution from Ricci-tensor-squared term. Similarly, we can reproduce the fourth-order contribution from the Ricci-scalar-squared term with the redefinition $h_{ab} \rightarrow h_{ab} + \bar{g}_{ab}\delta^{(1)}\mathcal{R} + \bar{g}_{ab}\delta^{(2)}\mathcal{R} + \dots$. We are, of course, free to combine these (retaining only the relevant parts): $h_{ab} \rightarrow h_{ab} + a_1\delta^{(2)}\mathcal{R}_{ab} + a_2\bar{g}_{ab}\delta^{(2)}\mathcal{R}$. Now, if one wants to do away with the fourth-order contributions from the Ricci-tensor-squared and the Ricci-scalar-squared terms, it becomes the matter of appropriately choosing the numerical coefficients a_1 and a_2 .

There is, however, no such field redefinition that can produce the fourth-order term from the Riemann-tensor-squared term. Each graviton has two symmetric indices, and so a contraction like $h_{ab}h^{ab}$ cannot reproduce the requisite four-index structure of the Riemann-tensor-squared expansion.

To sum up, of the three Gauss–Bonnet terms, only the Riemann-tensor-squared term is of physical relevance.

By similar reasoning, one can argue that *any* (four-derivative) four-point function with vector modes is devoid of physical meaning, irrespective of

the interactions. This is because, as previously discussed, vector interactions can be represented in terms of field-strength tensors (*cf.* Eq. (7)) which do not involve x or y indices. As a consequence, a fourth-order term containing vector modes must be one of the four simple forms — $F_{ab}F_{cd}F^{ad}F^{bc}$, $F_{ab}F^{ab}F_{cd}F^{cd}$, $F_{ab}F^{ab}\nabla_c h_{de}\nabla^c h^{de}$, or $F_{ac}F_b^c\nabla^a h_{de}\nabla^b h^{de}$ — any of which can be attained by suitably redefining a graviton. For instance, $\langle FF \rangle$ and $F_{ab} \rightarrow F_{ab} + F_{ac}F_b^c$ leads to the first form, $\langle FF \rangle$ and $\bar{g}_{ab} \rightarrow \bar{g}_{ab} + h_{ab} \rightarrow \bar{g}_{ab} + h_{ab} + F_{ac}F_b^c$ yields the second (via the determinant); whereas $\langle h_2 h_2 \rangle$ and the preceding transformation gives us the latter pair (respectively by way of the determinant and a contravariant metric).

There is yet another argument that allows us to reach the same conclusion about the vector-mode amplitudes. Gauss–Bonnet gravity leads to equations of motion that are at most quadratic in derivatives. So a fourth-order expansion of its Lagrangian in what are field-strength tensors had better give back either the fourth-order term in the Born–Infeld Lagrangian, since Born–Infeld’s theory [25] is the electromagnetic analogue of Lovelock gravity [26], or nothing at all. Our actual calculations of $\langle h_1 h_1 h_1 h_1 \rangle_{GB}$ do indeed lead to the latter result. Meanwhile, the “mixed” four-point function $\langle h_1 h_1 h_2 h_2 \rangle_{GB}$ is constrained (and found) to vanish by similar reasoning, as the possible form of term in the Lagrangian producing such an amplitude is $\mathcal{R}_{abcd}F^{ad}F^{bc}$, and this would lead to equations of motion with higher than two derivatives.

4.3 The results

And so the four-point functions amount to a single calculation, expanding Riemann-tensor-squared to fourth order in tensors. We have performed this

expansion and obtained

$$\begin{aligned} \langle h_2 h_2 h_2 h_2 \rangle_{GB} &= \frac{3}{4} \lambda \sqrt{-g} (g^{xx})^2 (g^{yy})^2 [h_{xy}^{(1)} (\omega_1 g^{tt} \omega_2 + k_1 g^{zz} k_2) h_{xy}^{(2)}] \\ &\times [h_{xy}^{(3)} (\omega_3 g^{tt} \omega_4 + k_3 g^{zz} k_4) h_{xy}^{(4)}] . \end{aligned} \quad (25)$$

To keep the calculation tractable, it is better to keep all linearized Γ 's in their covariant form as in Eq. (11) and apply the Riemann (anti-) symmetry properties only at the end of the calculation.

Let us now move on to the $2n$ -point functions with $n \geq 4$. We need, of course, only consider the prospect of having all tensors, as adding additional pairs of undifferentiated gravitons cannot invalidate the previous arguments.

Again calling upon simple combinatorics, we find that

$$\begin{aligned} \langle (h_2)^{2n} \rangle_{GB} &= -\frac{3}{4} \lambda \binom{2n}{4} \sum_{p=0}^{n-2} \binom{n-p+1}{3} \Theta(p) \sqrt{-g} (g^{xx} g^{yy})^2 \\ &\times \left[\prod_{j=2}^n (h^x_y)^{(2j-1)} (h^y_x)^{(2j)} \right] \prod_{l=1}^2 [h_{xy}^{(2l-1)} (\omega_{2l-1} g^{tt} \omega_{2l} + k_{2l-1} g^{zz} k_{2l}) h_{xy}^{(2l)}] \\ &= \frac{2}{5} \lambda \binom{2n}{4} \frac{\Gamma[n + \frac{3}{2}]}{\sqrt{\pi} \Gamma[n-1]} \sqrt{-g} (g^{xx} g^{yy})^2 \times \left[\prod_{j=2}^n (h^x_y)^{(2j-1)} (h^y_x)^{(2j)} \right] \\ &\times \prod_{l=1}^2 [h_{xy}^{(2l-1)} (\omega_{2l-1} g^{tt} \omega_{2l} + k_{2l-1} g^{zz} k_{2l}) h_{xy}^{(2l)}] . \end{aligned} \quad (26)$$

In the top line, the left-most binomial factor is the number of ways of drawing four differentiated tensors modes from the $2n$ available, the summation index is again counting the pairs of modes that are drawn out of the determinant and the right-most binomial factor accounts for the number of ways of extracting the $n - 2 - p$ remaining pairs from the four contravariant metrics.

This exhausts the possible n -point functions in the high-momentum regime.

5 Comparing Einstein and Gauss-Bonnet

5.1 Angular dependence of the n -point functions

We can use the results of Sections 3 and 4, to express the statements about scattering angles in a precise way. We work at $r \rightarrow \infty$, as appropriate for making contact with the gauge theory, although a different choice of r would be inconsequential.²

Let us begin with the Einstein $2n$ -point functions and assume, for the moment, only tensor modes. Then, for $n = 1$,

$$\lim_{r \rightarrow \infty} \langle h_2 h_2 \rangle_E \sim h_{xy}^{(1)} \left[\omega_1 \omega_2 - \vec{k}_1 \cdot \vec{k}_2 \right] h_{xy}^{(2)}, \quad (27)$$

where the arbitrariness of the propagation direction has now been made explicit and the \sim indicates some normalization factors that are not essential to our discussion. Here, it becomes the simple matter of applying momentum conservation. That is, $\vec{k}_2 = -\vec{k}_1$, and so we obtain the angular-independent form

$$\lim_{r \rightarrow \infty} \langle h_2 h_2 \rangle_E \sim k_1^2 h_{xy}^{(1)} h_{xy}^{(2)}. \quad (28)$$

Although not yet crucial, the on-shell condition $\omega_j = k_j \equiv |\vec{k}_j|$ has also been imposed.

Continuing to larger values of n , we find that

$$\begin{aligned} \lim_{r \rightarrow \infty} \langle h_2 \dots h_2 \rangle_E &\sim h_{xy}^{(1)} [k_1 k_2 (1 - \cos \theta)] h_{xy}^{(2)} \\ &\sim s h_{xy}^{(1)} h_{xy}^{(2)}, \end{aligned} \quad (29)$$

²An exception is at the brane horizon, where the divergence of g^{tt} effectively wipes out all information about the transverse momenta. So that, for a hypothetical scattering experiment on the horizon, the n -point functions would appear angular independent for any theory and for any n .

for any even number of gravitons greater than two. In the second equality of Eq. (29), we have introduced the Mandelstam variable $s = k_1^\mu k_{2\mu}$ such that $k_i^\mu = (\omega_i, \vec{k}_i)$. Recall that, for massless particles, the sum of the three Mandelstam variables vanishes, $s + t + u = 0$ ($t = -k_1^\mu k_{3\mu}$, $u = -k_1^\mu k_{4\mu}$).

The situation is even simpler for a $2n$ -point function that contains two necessarily differentiated vector modes. One can deduce that there is never any angular dependence for any value of n by simply recognizing that the explicit frequencies and momenta are already included in the definition of the physical modes $F_{tz}^{(1,2)}$ and there are no other derivatives available to introduce additional angular dependence.

Let us now find the corresponding n -point functions for the Gauss–Bonnet theory, as these can then be compared to the Einstein expressions. Given our interest in the high-momentum regime, the simplest case is the four-point function. When symmetrized with respect to the four gravitons, this goes as

$$\begin{aligned} \lim_{r \rightarrow \infty} \langle h_2 h_2 h_2 h_2 \rangle_{GB} &\sim h_{xy}^{(1)} \left[\omega_{(1} \omega_2 - \vec{k}_{(1} \cdot \vec{k}_2 \right] h_{xy}^{(2)} h_{xy}^{(3)} \left[\omega_3 \omega_4 - \vec{k}_3 \cdot \vec{k}_4 \right] h_{xy}^{(4)} \\ &\sim h_{xy}^{(1)} \left[k_{(1}^\mu \cdot k_{2\mu} \right] h_{xy}^{(2)} h_{xy}^{(3)} \left[k_{\mu 3} \cdot k_{4}^\mu \right] h_{xy}^{(4)} , \end{aligned} \quad (30)$$

which can, when on-shell, be simplified in terms of the Mandelstam variables,

$$\begin{aligned} \lim_{r \rightarrow \infty} \langle h_2 h_2 h_2 h_2 \rangle_{GB} &\sim h_{xy}^{(1)} h_{xy}^{(2)} [-s(t + u)] h_{xy}^{(3)} h_{xy}^{(4)} \\ &\sim s^2 h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)} . \end{aligned} \quad (31)$$

That leaves us to look at the Gauss–Bonnet $2n$ -point functions with $n \geq 3$. As for the analogous Einstein calculation, the condition of momentum conservation $\sum_{j=1}^{2n} \vec{k}_j = 0$ is no longer useful; meaning that two angles now require specification. A simple way to account for this new angle is

to introduce a “generalized Mandelstam variable” $v = -k_1^\mu \sum_{j=5}^{2n} k_{j\mu}$, for which it is readily confirmed that $s + t + u = -v$. Then, with the on-shell condition imposed,

$$\begin{aligned} \lim_{r \rightarrow \infty} \langle h_2 h_2 h_2 h_2 \rangle_{GB} &\sim h_{xy}^{(1)} h_{xy}^{(2)} [-s(t+u)] h_{xy}^{(3)} h_{xy}^{(4)} \\ &\sim s(s+v) h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)}. \end{aligned} \quad (32)$$

5.2 The gauge-theory perspective

So as to connect with experiment, our ultimate interest is in the corresponding n -point stress-tensor correlators for the gauge-theory dual. To this end, the standard prescription is to send a bulk quantity toward the AdS boundary and then apply standard subtraction techniques before taking the final $r \rightarrow \infty$ limit [19, 20, 21]. And so it should, with some effort, be possible to translate our results into statements about the gauge theory [27].

The stress-energy tensor correlators of the gauge theory can be expected to inherit the angular dependence of the graviton n -point functions. This is because the subtraction process is equivalent to a process of matching and then stripping off the divergent bulk and boundary conformal factors [2, 28, 29, 30]. Such a process would not change the angular dependence of the correlators because the metric components g^{tt} and g^{zz} are dispersed democratically and exhibit the same radial dependence at the boundary.

6 Conclusion

Beginning with the premise of a UV-complete gauge theory and its UV-complete gravitational dual, we applied our argument [12] that an effective

theory describing gravitational perturbations about a background solution must be organized into Einstein gravity plus terms of the Lovelock class. The leading-order effective description must then either be Einstein gravity or a Gauss–Bonnet extension thereof.

Given an AdS black brane background solution and a kinematic regime of high-momentum, we have calculated, for any number of gravitons, all of the physical n -point functions. This was done for both of the proposed effective theories, with all the results having been expressed in terms of gauge-invariant gravitational modes.

We have, from a novel perspective, explained why the Einstein n -point functions have a simpler angular dependence than those of Gauss–Bonnet gravity and then used our results to quantify the angular dependence of both theories in a precise manner.

The graviton n -point functions have a direct correspondence with stress-energy tensor correlators in the gauge theory. Holography implies that these gauge-theory correlators should inherit the same angular dependence. This means that there should be fundamental and testable distinction between the Einstein and Gauss–Bonnet models. Following ideas from [6, 7], we have proposed that heavy-ion scattering experiments can be used for such purposes.

Acknowledgments

The research of RB was supported by the Israel Science Foundation grant no. 239/10. The research of AJMM received support from Rhodes University and

the Claude Leon Merit Award grant no. CLMA05/2011. AJMM thanks the Arnold Sommerfeld Center for Theoretical Physics at Ludwig-Maximilians-Universität München for their hospitality.

A The case against two derivatives on a graviton

In working within the high-momentum regime, we require that all derivatives (two for Einstein, four for Gauss–Bonnet, *etc.*) act on a graviton. At a first glance, it would appear that there could well be terms with two derivatives acting on the same graviton. However, as we now show, such terms cannot contribute on-shell. This can be verified by explicit calculations but can also be understood through the following simple argument, whose domain of applicability is discussed at the end.

Let us demonstrate that the above claim is true for any Lovelock theory with four derivatives acting on gravitons. The extension to other cases, including the simplest case of two derivatives, is then straightforward.

The linearized field equation for any Lovelock theory can be expressed as

$$\overline{\mathcal{X}}^{abcd} \delta^{(1)} \mathcal{R}_{abcd} = 0, \quad (33)$$

such that

$$\delta^{(1)} \mathcal{R}_{abcd} = \overline{\nabla}_c \Gamma_{bda}(h) - \overline{\nabla}_d \Gamma_{bca}(h). \quad (34)$$

Here, a numerical superscript denotes the number of gravitons, an over-lined quantity signifies the background and, in this discussion, we often neglect the usual (anti-) symmetrization of indices.

The simplicity of the field equation (33) follows from that of a generic theory of gravity [31, 32],

$$2\nabla_b\nabla_a\mathcal{X}^{apqb} - \mathcal{X}^{abcp}\mathcal{R}_{abc}{}^a + \frac{1}{2}g^{pq}\mathcal{L} = 0, \quad (35)$$

along with the Lovelock identity (3).³

Let us next look at the fourth-order expansion of the Lagrangian density of a Lovelock theory. Considering just the terms with exactly four derivatives acting on gravitons, we have (up to inconsequential numerical factors)

$$\frac{\delta^{(4)}(\sqrt{-g}\mathcal{L})}{\sqrt{-\bar{g}}} = [h\delta^{(1)}\mathcal{R}_{pqrs} + \delta^{(2)}\mathcal{R}_{pqrs}] \bar{\mathcal{Y}}^{abcd} [h\delta^{(1)}\mathcal{R}_{abcd} + \delta^{(2)}\mathcal{R}_{abcd}], \quad (36)$$

where \mathcal{Y} is the variation of \mathcal{X} with respect to \mathcal{R} (indices suppressed) and a depicted “ h ” is meant to indicate that a single undifferentiated graviton from the expansion of the determinant or the contravariant metric. Notice that $\delta^{(2)}\mathcal{R} \sim \Gamma(h)\Gamma(h)$; cf, Eq. (12).

The crucial point is that the tensor \mathcal{Y} inherits, just like \mathcal{X} does, all the (anti-) symmetry properties of the Riemann tensor. In fact, this tensor has two sets of four indices, each of which is Riemannian in structure. Hence, its contraction with $h\delta^{(1)}\mathcal{R}$, either from the left or the right, must necessarily give back a form that is proportional to the linearized field equation (however, see below). Meaning that the on-shell form of this fourth-order Lagrangian density is simply

$$\frac{\delta^{(4)}(\sqrt{-g}\mathcal{L})}{\sqrt{-\bar{g}}} = \delta^{(2)}\mathcal{R}_{pqrs} \left(\bar{\mathcal{Y}}^{abcd} \right) \delta^{(2)}\mathcal{R}_{abcd}. \quad (37)$$

³For the relevant modes (vectors and tensors), the third term in the field equation must be a mass term and ends up being absorbed into the mass terms of $\delta^{(1)}\mathcal{R} \sim \nabla\Gamma(h) \sim \square h$.

The same basic argument persists for any number of pairs of derivatives all acting on gravitons. That is, it can be applied to any order of Lovelock theory (and, in particular, Einstein gravity) with always the same outcome: no more than one derivative per graviton.

Strictly speaking, this argument is only rigorous at the AdS boundary and at the horizon, as these are the only the surfaces where the metric and its descendants (\mathcal{R} , \mathcal{X} , \mathcal{Y} , *etc.*) are assured to be insensitive to the polarization.⁴ This being a sufficient (albeit not necessary) condition for $\mathcal{Y} \cdot \delta^{(2)}\mathcal{R} \propto \mathcal{X} \cdot \delta^{(2)}\mathcal{R}$. However, at any radius, the background metric can be regarded as (polarization-independent) Einstein plus $\mathcal{O}[l_p^2/L^2]$ corrections. Hence, any violation of this argument is suppressed by an *additional* factor of l_p^2/L^2 relative to other contributions from the same Lagrangian.

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⁴The horizon, effectively so, because the metric reduces to a 1+1 conformally flat space.

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