Signatures of a minimal length scale in high precision experiments

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We discuss modifications of the gyromagnetic moment of electrons and muons due to a minimal length scale combined with a modified fundamental scale M_f. First-order deviations from the theoretical standard model value for $g-2$ due to these String Theory-motivated effects are derived. Constraints for the new fundamental scale M_f are given.

I. MOTIVATION AND INTRODUCTION

Although the standard model is a powerful tool to explain the physics of the very basic constituents of matter, it is far from being an exhaustive description of our world. Many questions remain unanswered: What causes the existence of three particle generations? Where do the various quark and lepton masses and coupling constants come from? How to unite gravity and quantum theory? Why is gravity so weak compared to the other forces? Theories such as M-Theory and Superstrings try to give a hint on these questions, but they do not (yet) provide us with measurable quantities. Nevertheless, there are some general features that seem to go hand in hand with all promising candidates for a theory of quantum gravity:

- the need for a higher dimensional space-time and
- the existence of a minimal length scale.

In this paper, we study implications of these extensions in the Dirac equation without the aim to derive them from a first principle theory. Instead we will analyse possible observable modifications

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that may arise by combining the main features of both extra dimensions and a minimal length scale in a simplified model.

II. LARGE EXTRA DIMENSIONS

The idea of Large eXtra Dimensions (LXDs) which was recently proposed in [1, 2, 3, 4, 5] might allow to study first effects of unification or quantum gravity in near future experiments. In these models, only gravitons can propagate into the d compactified LXDs. The standard model particles are bound to our (3+1)-dimensional sub-manifold, often called our 3-brane. This results in a decrease of the Planck scale to a new fundamental scale M_f and gives rise to the exciting possibility of TeV scale GUTs [6]. Therefore, not only the notion of further dimensions of space-time is incorporated, but also the hierarchy-problem is solved, although one might claim it is only shifted to the geometrical sector.

In [1], the following relation between the four-dimensional Planck mass m_p and the higher dimensional fundamental scale M_f is derived:

$$m_p^2 = R^d M_f^{d+2} \quad , \tag{1}$$

where R is the radius of the LXDs. This is a consequence of Gauss' law in $3+d$ spatial dimensions: Two test masses m_1, m_2 within a distance below the compactification radius will feel the gravitational potential

$$\frac{V(r)}{m_1} \sim \frac{1}{M_f^{d+2} r^d} \frac{m_2}{r} \quad , (r \ll R).$$

At distances above the compactification radius, the gravitational flux lines are not further dissolved into the extra dimensions, and one has to regain the usual potential in three spatial dimensions

$$\frac{1}{M_f^{d+2} R^d} \frac{m_2}{r} \stackrel{r \gg R}{\sim} \frac{1}{m_p^2} \frac{m_2}{r} \quad ,$$

which directly yields (1).

This lowered fundamental scale leads to a vast number of observable phenomena of quantum gravity at energies in the range of M_f. In fact, the non-observation of these predicted features in past experiments gives first constraints on the parameters of the model, the number of extra dimensions d and the fundamental scale M_f [7, 8]. This scenario has major consequences:

- Cosmology and astrophysics: Modification of inflation in the early universe and enhanced supernova-cooling due to graviton emission [3, 9, 10, 11, 12].
- Additional processes are expected in high-energetic lepton and hadron interactions [13, 14]: production of real and virtual gravitons [15, 16, 17, 18, 19] and the creation of black holes at energies that can be achieved at colliders in the near future [20, 21, 22, 23, 24, 25, 26] and in ultra high energetic cosmic rays [27].
- One also has to expect the influence of the extra dimensions on high precision measurements; the most obvious being the modification of Newton's law at small distances [28, 29, 30].
- Of highest interest are also modifications of the gyromagnetic moment of Dirac particles which promises new insight into non-standard model couplings and effects [31, 32, 33, 34, 35, 36].

Thus, new phenomena might either be encountered in high energy or high precision experiments.

III. THE MINIMAL SCALE

As discussed above, String theory suggests the existence of a minimal length scale. In perturbative string theory [37, 38], the feature of a fundamental minimal length scale arises from the fact that strings can not probe distances smaller than the string scale. If the energy of a string reaches the Planck mass m_p, excitations of the string can occur and increase the extension [39]. Due to this, uncertainty in position measurement can never become smaller than $l_p = \hbar/m_p$. For a detailed review, the reader is referred to Refs. [40, 41].

However, in the present model with LXDs, this fact grows important for collider physics at high energies or for high precision measurements at low energies due to the lowered fundamental scale M_f, which results in a new fundamental length scale $L_f = \hbar/M_f$.

Naturally, this minimum length uncertainty is related to a modification of the standard commutation relations between position and momentum [42, 43]. Application of this is of high interest for quantum fluctuations in the early universe and inflation [44, 45, 46, 47, 48, 49, 50, 51, 52]. We will follow the propositions made in [53, 54].

IV. INCORPORATION INTO QUANTUM THEORY

In order to implement the notion of a minimal length L_f, let us now suppose that one increases the momentum p of a particle arbitrarily, but that the wave number k has an upper bound. This effect leads to pronounced deviations from the linear dependence when p approaches the scale M_f. The physical interpretation of this is that particles can not possess arbitrarily small Compton wavelengths $\lambda = 2\pi/k$ so that arbitrarily small scales cannot be resolved anymore.

To incorporate this behaviour, we assume a relation $k = k(p)$ between p and k which is an uneven function (because of parity) and which asymptotically approaches $1/L_f$. Furthermore, we demand the functional relation between the energy E and the frequency ω to be the same as that between the wave vector k and the momentum p. A possible choice for the relations is

$$L_f k(p) = \tanh^{1/\gamma}\left[\left(\frac{p}{M_f}\right)^{\gamma}\right] \quad , \tag{2}$$

$$L_f \omega(E) = \tanh^{1/\gamma}\left[\left(\frac{E}{M_f}\right)^{\gamma}\right] \quad , \tag{3}$$

with a real, positive constant γ.

In the following, we restrict our study to the low momentum approximation, namely the regime of first effects including the orders $(p/M_f)^3$. For this purpose, we expand the function in a Taylor series for small arguments.

Because the exact functional dependence is unknown, we assume an arbitrary factor α in front of the order $(p/M_f)^3$-term. Therefore the relations for $k(p)$ and $\omega(E)$ which are used in the following are

$$L_f k(p) \approx \frac{p}{M_f} - \alpha\left(\frac{p}{M_f}\right)^3 \quad , \tag{4}$$

$$L_f \omega(E) \approx \frac{E}{M_f} - \alpha\left(\frac{E}{M_f}\right)^3 \quad , \tag{5}$$

$$\frac{1}{M_f} p(k) \approx k L_f + \alpha\,(k L_f)^3 \quad , \tag{6}$$

$$\frac{1}{M_f} E(\omega) \approx \omega L_f + \alpha\,(\omega L_f)^3 \quad , \tag{7}$$

with α being of order one (e.g. $\alpha = 1/3$ for $\gamma = 1$), but in general negative values of α can not be excluded.

This yields to 3$^{\text{rd}}$ order

$$\frac{1}{\hbar}\frac{\partial p}{\partial k} \approx 1 + 3\alpha\left(\frac{p}{M_f}\right)^2 \quad . \tag{8}$$

The quantisation of these relations is straight forward. The commutators between $\hat{k}$ and $\hat{x}$ remain in the standard form:

$$[\hat{x}_i, \hat{k}_j] = \mathrm{i}\delta_{ij} \quad . \tag{9}$$

Inserting the functional relation between the wave vector and the momentum then yields the modified commutator for the momentum. With the commutator relation

$$[\,\hat{x}, \hat{A}(k)] = +\mathrm{i}\frac{\partial A}{\partial k} \quad , \tag{10}$$

the modified commutator algebra now reads

$$[\,\hat{x}, \hat{p}] = +\mathrm{i}\frac{\partial p}{\partial k} \quad . \tag{11}$$

This results in the generalised uncertainty relation

$$\Delta p \Delta x \geq \frac{1}{2}\left| \left\langle \frac{\partial p}{\partial k} \right\rangle \right| \quad . \tag{12}$$

With the approximations (4)-(7), the results of Ref. [44] are reproduced up to the factor α:

$$[\hat{x}, \hat{p}] \approx \mathrm{i}\hbar \left(1 + 3\alpha \frac{\hat{p}^2}{M_f^2} \right) \tag{13}$$

giving the generalised uncertainty relation

$$\Delta p \Delta x \geq \frac{1}{2}\hbar \left(1 + 3\alpha \frac{\langle \hat{p}^2 \rangle}{M_f^2} \right) \quad . \tag{14}$$

We give the operators in the position representation which is suited best for this purpose:

$$\begin{aligned} \hat{x} &= x \quad , \quad \hat{k} = -\mathrm{i}\partial_x \\ \hat{p} &= \hat{p}(\hat{k}) \quad , \end{aligned} \tag{15}$$

yielding the new momentum operator

$$\hat{p}(\hat{k}) \approx -\mathrm{i}\hbar \left(1 - \alpha L_f^2 \partial_x^2 \right) \partial_x \quad . \tag{16}$$

In ordinary relativistic quantum mechanics the Hamiltonian of the Dirac Particle is [1]

$$\hat{H} = \mathrm{i}\hbar\partial_0 = \gamma^0 \left(\mathrm{i}\hbar\gamma^i \partial_i + m \right). \tag{17}$$

[1] Greek indices run from 0 to 3, roman indices run from 1 to 3.

This leads to the Dirac Equation

$$(\not{p} - m)\psi = 0 \quad , \tag{18}$$

with the following standard abbreviation $\gamma^\nu A_\nu := \not{A}$ and $p_\nu = \mathrm{i}\hbar\partial_\nu$. To include the modifications due to the generalised uncertainty principle, we start with the relation

$$\hat{E}(\omega) = \gamma^0 \left(\gamma^i \hat{p}_i(k) + m\right) . \tag{19}$$

Including the altered momentum wave vector relation $\hat{p}(\hat{k})$ from Eq. (16), this yields again Eq. (18) with the modified momentum operator

$$(\not{p}(\hat{k}) - m)\psi = 0 \quad . \tag{20}$$

This equation is Lorentz invariant by construction. It contains in position representation 3rd order derivatives in space coordinates and 3rd order time-derivatives. In our approximation, we can solve the equation for a single order time derivative by using the energy condition $E^2 = p^2 + m^2$. This leads effectively to a replacement of time derivatives by space derivatives:

$$\hbar\hat{\omega} \approx \hat{E} - \alpha\hat{E}^3/M_f^2 = \hat{E}\left(1 - \alpha\frac{\hat{p}^i\hat{p}_i + m^2}{M_f^2}\right) \quad . \tag{21}$$

Inserting the modified $\hat{E}(\omega)$ and $\hat{p}(k)$ and keeping only terms up to 3rd order, we obtain the following expression of the Dirac Equation:

$$\omega|\psi\rangle \approx \gamma^0 \left(\gamma^i \hat{k}_i + \frac{m}{\hbar}\right)\left(1 - \alpha\frac{\hbar^2\hat{k}^i\hat{k}_i + m^2}{M_f^2}\right)|\psi\rangle \ . \tag{22}$$

V. THE GYROMAGNETIC MOMENT

The task is now to derive the modifications of the anomalous gyromagnetic moment due to the existence of a minimal length. Therefore we assume as usual the particle is placed inside a homogeneous and static magnetic field B. Regarding the energy levels of an electron the magnetic field leads to a splitting of the energetic degenerated values which is proportional to the magnetic field B and the gyromagnetic moment g. Since the energy of the particle in the field is not modified (see (18)) there is no modification of the splitting as one might have expected from the fact that the particles spin is responsible for the anomaly.

However, if we look at the precession of a dipole in a magnetic field without minimal length and compare its precession frequency to that of the spin 1/2 particle under investigation, again the factor g occurs. Without minimal length the frequency from quantum mechanics is two times the classical one. In that case a further modification from the minimal length is expected, as has been under investigation in an alternative approach in [55]. In our model, this modification results from the new relation between energy and frequency.

Equation (22) with minimally coupled electromagnetic fields reads:

$$\omega|\psi\rangle \approx \gamma^0 \left(\gamma^i \hat{K}_i + \frac{m}{\hbar}\right) \left(1 - \alpha \frac{\hbar^2 \hat{K}^i \hat{K}_i + m^2}{M_f^2}\right) |\psi\rangle \tag{23}$$

where $\hat{K} = \hat{k} + e\hat{A}/\hbar$. Higher derivatives acting on the magnetic potential can be dropped too for a static and uniform field. In addition, the constant electric potential can be set to zero. In the non-relativistic approximation we can simplify this equation in the Coulomb gauge to:

$$\left(E + m\hat{F}\right) |\chi\rangle = \left(\frac{(\hbar\hat{K})^2}{2m}\hat{F} + \frac{e\hbar}{2m}\sigma\hat{B}\hat{F}\right) |\chi\rangle \tag{24}$$

with

$$\hat{F} = \left(1 - \alpha \frac{\hbar^2 \hat{K}^i \hat{K}_i + m^2}{M_f^2}\right) \quad , \quad |\psi\rangle = \left|\begin{matrix}\chi \\ \phi\end{matrix}\right\rangle \quad . \tag{25}$$

Here χ is the upper component of the Dirac spinor and σ denotes the Pauli matrices.

Therefore, the modified expression $\tilde{g}$ for the gyromagnetic moment for $k \to 0$ is:

$$\tilde{g} = g \cdot \left(1 - \alpha \frac{m^2}{M_f^2}\right) \quad . \tag{26}$$

The experimental data concerning the muon gyromagnetic moment are as follows: Davier and collaborates provide two standard model theory results; they differ in the experimental input[2] used to the hadronic contributions [56]. It is convenient to use the quantity $a_\mu = (g-2)/2$ to denote the gyromagnetic factor of the muon:

$$\begin{aligned} a_{\mu,\tau} &= 11659195.6(11.1) \times 10^{-10} \\ a_{\mu,e^+e^-} &= 11659180.9(9.7) \times 10^{-10} \quad . \end{aligned}$$

The experimental 'world average' is [57]:

$$a_\mu = 11659203(8) \times 10^{-10} \quad . \tag{27}$$

[2] The indices indicate the source of the vector spectral functions; they are obtained by either hadronic τ decays or e^+e^--annihilation cross-sections.

The results indicate that modifications to the standard model calculation have to be smaller than 10^{-8}. This leads to the following constraint on the fundamental scale of the theory:

$$M_f/\sqrt{|\alpha|} \geq 1 \text{ TeV} \quad . \tag{28}$$

For the commonly used setting $\gamma = 1$ ($\alpha = 1/3$), a specific limit on the fundamental scale M_f can be obtained from present $g-2$ data: $M_f \geq 577$ GeV.

Note that there might further be corrections due to graviton loops [58, 59]. However, recent calculations show that neither sign nor value of these corrections are predictable due to unknown form-factors and cutoff parameters [60].

VI. SUMMARY

A phenomenological model, which combines both Large Extra Dimensions and the minimal length scale L_f is studied. The existence of a minimal length scale leads to modifications of quantum mechanics. With the recently proposed idea of Large Extra Dimensions, this new scale might be in reach of present day experiments. The modified Dirac equation is used to derive first-order deviations of the gyromagnetic moment of spin 1/2 particles. Our results for the muon $g-2$ value are compared to the values predicted by QED and experiment.

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