

Analytical relationship for the cranking inertia

Dorin N. Poenaru^{1,2}, Radu A. Gherghescu¹, and Walter Greiner²

¹*Horia Hulubei National Institute of Physics and Nuclear Engineering, PO Box MG-6, 077125 Bucharest-Magurele, Romania*

²*Frankfurt Institute for Advanced Studies, J W Goethe University, Pf. 111932, D-60054 Frankfurt am Main, Germany*

Abstract

The wave function of a spheroidal harmonic oscillator without spin-orbit interaction is expressed in terms of associated Laguerre and Hermite polynomials. The pairing gap and Fermi energy are found by solving the BCS system of two equations. Analytical relationships for the matrix elements of inertia are obtained function of the main quantum numbers and potential derivative. They may be used to test complex computer codes one should develop in a realistic approach of the fission dynamics. The results given for the ²⁴⁰Pu nucleus are compared with a hydrodynamical model. The importance of taking into account the correction term due to the variation of the occupation number is stressed.

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1 Introduction

By studying fission dynamics [1] one can estimate the value of the disintegration constant λ of the exponential decay law expressing the variation in time of the number of decaying nuclei. The partial decay half-life T is given by $T = \tau \ln 2 = 0.693147/\lambda$. The potential energy surface in a multi-dimensional hyperspace of deformation parameters $\beta_1, \beta_2, \dots, \beta_n$ gives the generalized forces acting on the nucleus. Information concerning how the system reacts to these forces is contained in a tensor of inertial coefficients, or the effective mass parameters $\{B_{ij}\}$. Unlike the potential energy $E = E(\beta)$ which depends on the nuclear shape, the kinetic energy is determined by the contribution of the shape change expressed by

$$E_k = \frac{1}{2} \sum_{i,j=1}^n B_{ij}(\beta) \frac{d\beta_i}{dt} \frac{d\beta_j}{dt} \quad (1)$$

where B_{ij} is the inertia tensor. In a phenomenological approach based on incompressible irrotational flow, the value of an effective mass B^{irr} is usually close to the reduced mass $\mu = (A_1 A_2 / A) M$ in the exit channel of the binary system. Here M is the nucleon mass. One may use the Werner–Wheeler approximation [2].

The microscopic (cranking) model introduced by Inglis [3] leads to much larger values of the inertia. By assuming the adiabatic approximation the shape variations are slower than the single-particle motion. According to the cranking model, after including the BCS pairing correlations [4, 5], the inertia tensor is given by [6, 7]:

$$B_{ij} = 2\hbar^2 \sum_{\nu\mu} \frac{\langle \nu | \partial H / \partial \beta_i | \mu \rangle \langle \mu | \partial H / \partial \beta_j | \nu \rangle}{(E_\nu + E_\mu)^3} (u_\nu v_\mu + u_\mu v_\nu)^2 + P_{ij} \quad (2)$$

where H is the single-particle Hamiltonian allowing to determine the energy levels and the wave functions $|\nu\rangle$, u_ν , v_ν are the BCS occupation probabilities, E_ν is the quasiparticle energy, and P_{ij} gives the contribution of the occupation number variation when the deformation is changed (terms including variation of the gap parameter, Δ , and Fermi energy, λ , $\partial\Delta/\partial\beta_i$ and $\partial\lambda/\partial\beta_i$):

$$\begin{aligned}
P_{ij} = & \frac{\hbar^2}{4} \sum_{\nu} \frac{1}{E_{\nu}^5} \left[\Delta^2 \frac{\partial\lambda}{\partial\beta_i} \frac{\partial\lambda}{\partial\beta_j} + \right. \\
& (\epsilon_{\nu} - \lambda)^2 \frac{\partial\Delta}{\partial\beta_i} \frac{\partial\Delta}{\partial\beta_j} + \Delta(\epsilon_{\nu} - \lambda) \left(\frac{\partial\lambda}{\partial\beta_i} \frac{\partial\Delta}{\partial\beta_j} + \frac{\partial\lambda}{\partial\beta_j} \frac{\partial\Delta}{\partial\beta_i} \right) \\
& - \Delta^2 \left(\frac{\partial\lambda}{\partial\beta_i} \langle \nu | \partial H / \partial \beta_j | \nu \rangle + \frac{\partial\lambda}{\partial\beta_j} \langle \nu | \partial H / \partial \beta_i | \nu \rangle \right) \\
& \left. - \Delta(\epsilon_{\nu} - \lambda) \left(\frac{\partial\Delta}{\partial\beta_i} \langle \nu | \partial H / \partial \beta_j | \nu \rangle + \frac{\partial\Delta}{\partial\beta_j} \langle \nu | \partial H / \partial \beta_i | \nu \rangle \right) \right]
\end{aligned}$$

Similar to the shell correction energy, the total inertia is the sum of contributions given by protons and neutrons $B = B_p + B_n$. The denominator in equation (2) is minimum for the levels in the neighbourhood of the Fermi energy. A large value of inertia is the result of a large density of levels at the Fermi surface. As a result, in a similar way to the shell corrections, one can observe large fluctuations of B_{ii} when the deformation or the number of particles are changed.

In the present work we consider a single-particle model of a spheroidal harmonic oscillator without spin-orbit interaction for which the cranking approach allows to obtain analytical relationships of the nuclear inertia. Despite of the limited interest of this simple single-particle model, the result of the present work may be used to test complex computer codes developed in a realistic treatment of the fission dynamics based on the deformed two center shell model [8]. The results illustrated for ^{240}Pu nucleus are compared with a hydrodynamical model.

2 Nuclear shape parametrization

The shape of a spheroid with semiaxes a, c (c is the semiaxis along the symmetry) expressed in units of the spherical radius $R_0 = r_0 A^{1/3}$ may be determined by a single deformation coordinate which can be one of the following quantities: the semiaxes ratio c/a ; the eccentricity $e = (1 - a^2/c^2)$; the deformation $\delta = 1.5(c^2 - a^2)/(2c^2 + a^2)$, or the quadrupolar deformation[9] $\varepsilon = 3(c - a)/(2c + a)$ which will be used in the following, and according to which the two oscillator frequencies are expressed as:

$$\omega_{\perp}(\varepsilon) = \omega_0 \left(1 + \frac{\varepsilon}{3} \right) \quad (3)$$

$$\omega_z(\varepsilon) = \omega_0 \left(1 - \frac{2\varepsilon}{3} \right) \quad (4)$$

and by taking into account the condition of the volume conservation $\omega_{\perp}^2 \omega_z = (\omega_0^0)^3$ where $\hbar\omega_0^0 = 41A^{-1/3}$ MeV, one has

$$\omega_0 = \omega_0^0 \left[1 - \varepsilon^2 \left(\frac{1}{3} + \frac{2\varepsilon}{27} \right) \right]^{-1/3} \quad (5)$$

A particularly interesting value is $\varepsilon = 0.6$ of a superdeformed spheroid with the ratio $c/a = 2$.

3 Spheroidal harmonic oscillator

The eigenvalues [1] in units of $\hbar\omega_0$ are given by

$$E = \hbar\omega_{\perp}(n_{\perp} + 1) + \hbar\omega_z(n_z + 1/2) = \hbar\omega_0[N + 3/2 + \varepsilon(n_{\perp} - 2N/3)] \quad (6)$$

where the quantum numbers n_{\perp} and n_z are nonnegative integers. Their summation gives the main quantum number $N = n_{\perp} + n_z$. In units of $\hbar\omega_0$ one has a linear variation of the energy levels in function of deformation ε . By including the variation of ω_0 with ε , one obtains the analytical relationship

$$\epsilon_i = E_i/\hbar\omega_0^0 = [N + 3/2 + \varepsilon(n_{\perp} - 2N/3)][1 - \varepsilon^2(1/3 + 2\varepsilon/27)]^{-1/3} \quad (7)$$

in units of $\hbar\omega_0^0$. Due to the Pauli principle, each energy level ϵ_i , with quantum numbers n_{\perp} and N , can accommodate $g = 2n_{\perp} + 2$ nucleons. One has a number of $(N + 1)(N + 2)$ nucleons in a completely occupied shell characterized by the main quantum number N , and the total number of states for the lowest $N + 1$ shells is $\sum_{N=0}^N (N + 1)(N + 2) = (N + 1)(N + 2)(N + 3)/3$. For each value of N there are $N + 1$ levels with $n_{\perp} = 0, 1, 2, \dots, N$. When the deformation $\varepsilon > 0$ increases, a level with $n_{\perp} = 0$ decreases in energy and the one with $n_{\perp} = N$ increases. For some particular values of the deformation parameter there is a crossing of several levels in the same point leading to a degeneracy followed by an empty gap. If no spin-orbit coupling is considered for the vanishing deformation parameter, $\varepsilon = 0$, one has the following sequence of magic numbers: 2, 8, 20, 40, 70, 112, 168, 240, ... If now $\varepsilon = 0.6$ they become 2, 4, 10, 16, 28, 40, 60, 80, 110, 140, 182, ... The known experimental values can be obtained only with a spin-orbit coupling included.

The spin Σ contributes with positive or negative values (up or down) for every state with quantum numbers $n_z, n_r = 0, 1, 2, \dots, n_{\perp}$ and $m = n_{\perp} - 2n_r$, hence in a system of cylindrical coordinates (ρ, φ, z) the wave function [10, 11] can be written as

$$\Psi = |n_r m n_z \Sigma\rangle = \psi_{n_r}^m(\rho)\Phi_m(\varphi)\psi_{n_z}(z)\chi(\Sigma) \quad (8)$$

Few examples of the quantum numbers $n_r m n_z$ belonging to the lowest levels with $N = 0, 1, 2$ are given in the table 1.

The eigenfunctions are given by

$$\psi_{n_r}^m(\rho) = \frac{\sqrt{2}}{\alpha_{\perp}} N_{n_r}^m \eta^{\frac{|m|}{2}} e^{-\frac{\eta}{2}} L_{n_r}^{|m|}(\eta) = \frac{\sqrt{2}}{\alpha_{\perp}} \psi_{n_r}^m(\eta) \quad (9)$$

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \quad (10)$$

$$\psi_{n_z}(z) = \frac{1}{\sqrt{\alpha_z}} N_{n_z} e^{-\frac{\xi^2}{2}} H_{n_z}(\xi) = \frac{1}{\sqrt{\alpha_z}} \psi_{n_z}(\xi) \quad (11)$$

where $L_{n_r}^{|m|}$ are the associated (or generalized) Laguerre polynomials and H_{n_z} are the Hermite polynomials. The new dimension-less variables η and ξ are defined by

$$\eta = \frac{\rho^2}{\alpha_{\perp}^2} \quad (12)$$

Table 1: Quantum numbers of the lowest states of a spheroidal harmonic oscillator

N	n_{\perp}	n_r	$n_z = N - n_{\perp}$	$m = n_{\perp} - 2n_r$	ϵ for $\epsilon = 0$
0	0	0	0	0	1.5
1	0	0	1	0	2.5
	1	0	0	1	
	1	1	0	-1	
2	0	0	2	0	3.5
	1	0	1	1	
	1	1	1	-1	
	2	0	0	2	
	2	1	0	0	
	2	2	0	-2	

$$\xi = \frac{z}{\alpha_z} \quad (13)$$

where the quantities

$$\alpha_{\perp} = \sqrt{\frac{\hbar}{M\omega_{\perp}}} \approx A^{1/6} \sqrt{\frac{\omega_0^0}{\omega_{\perp}}} \quad \alpha_z = \sqrt{\frac{\hbar}{M\omega_z}} \approx A^{1/6} \sqrt{\frac{\omega_0^0}{\omega_z}} \quad (14)$$

which depend on the nucleon mass, M , possess a dimension of a length. Their numerical values, in fm, can be estimated by knowing that $\hbar^2/M \approx 41.5 \text{ MeV}\cdot\text{fm}^2$ and $\hbar\omega_0^0 = 41A^{-1/3} \text{ MeV}$. The normalization constants

$$(N_{n_r}^m)^2 = \frac{n_r!}{(n_r + |m|)!} \quad (15)$$

$$(N_{n_z})^2 = \frac{1}{\sqrt{\pi} 2^{n_z} n_z!} \quad (16)$$

are obtained from the orthonormalization conditions

$$\int_0^{\infty} \psi_{n_r'}^m(\rho) \psi_{n_r}^m(\rho) \rho d\rho = \delta_{n_r' n_r} \quad (17)$$

$$\int_{-\infty}^{\infty} \psi_{n_z'}(z) \psi_{n_z}(z) dz = \delta_{n_z' n_z} \quad (18)$$

$$\int_0^{2\pi} \Phi_{m'}^*(\varphi) \Phi_m(\varphi) d\varphi = \delta_{m' m} \quad (19)$$

One should take into account that the factorial $0! = 1! = 1$. We shall substitute the wave functions $\psi_{n_r}^m(\eta)$ and $\psi_{n_z}(\xi)$ in the equation (2) of the nuclear inertia.

4 Nuclear inertia

By ignoring the spin-orbit coupling the Hamiltonian of the harmonic spheroidal oscillator contains the kinetic energy and the potential energy term, V :

$$V(\eta, \xi; \varepsilon) = \frac{1}{2}M(\omega_{\perp}^2\rho^2 + \omega_z^2z^2) = \frac{1}{2}\hbar\omega_{\perp}\eta + \frac{1}{2}\hbar\omega_z\xi^2 = \frac{\hbar\omega_0^0[(3+\varepsilon)\eta + (3-2\varepsilon)\xi^2]}{2[27-\varepsilon^2(9+2\varepsilon)]^{1/3}} \quad (20)$$

Now we are making some changes in the equation (2), first of all replacing the deformation β by ε .

One may assume [6, 7, 10] that only the leading term of the Hamiltonian, namely the potential written above, contributes essentially to the derivative,

$$\frac{dH}{d\varepsilon} \simeq \frac{dV}{d\varepsilon} \quad (21)$$

The contribution of P_{ij} , denoted by P_{ε} for a system with one deformation coordinate, sometimes assumed to be negligible small, will be discussed in the last section.

The derivative is written as

$$\frac{1}{\hbar\omega_0^0} \frac{dV}{d\varepsilon} = \frac{3}{2} [f_1(\varepsilon)\eta + f_2(\varepsilon)\xi^2] \quad (22)$$

in which

$$f_1 = \frac{\varepsilon(\varepsilon+6)+9}{[27-\varepsilon^2(9+2\varepsilon)]^{4/3}} \quad (23)$$

$$f_2 = 2 \frac{\varepsilon(2\varepsilon+3)-9}{[27-\varepsilon^2(9+2\varepsilon)]^{4/3}} \quad (24)$$

For a single deformation parameter the inertia tensor becomes a scalar B_{ε} which a summation in eq. 2 performed for all states ν, μ taken into consideration in the pairing interaction [12].

4.1 Pairing interaction

We consider a set of doubly degenerate energy levels $\{\epsilon_i\}$ expressed in units of $\hbar\omega_0^0$. Calculations for neutrons are similar with those for protons, hence for the moment we shall consider only protons. In the absence of a pairing field, the first $Z/2$ levels are occupied, from a total number of n_t levels available. Only few levels below (n) and above (n') the Fermi energy are contributing to the pairing correlations. Usually $n' = n$. If \tilde{g}_s is the density of states at Fermi energy obtained from the shell correction calculation $\tilde{g}_s = dZ/d\epsilon$, expressed in number of levels per $\hbar\omega_0^0$ spacing, the level density is half of this quantity: $\tilde{g}_n = \tilde{g}_s/2$.

We can choose as computing parameter, the cut-off energy (in units of $\hbar\omega_0^0$), $\Omega \simeq 1 \gg \tilde{\Delta}$. Let us take the integer part of the following expression

$$\Omega\tilde{g}_s/2 = n = n' \quad (25)$$

When from calculation we get $n > Z/2$ we shall take $n = Z/2$ and similarly if $n' > n_t - Z/2$ we consider $n' = n_t - Z/2$.

The gap parameter $\Delta = |G| \sum_k u_k v_k$ and the Fermi energy with pairing correlations λ (both in units of $\hbar\omega_0^0$) are obtained as solutions of a nonlinear system of two BCS equations

$$n' - n = \sum_{k=k_i}^{k_f} \frac{\epsilon_k - \lambda}{\sqrt{(\epsilon_k - \lambda)^2 + \Delta^2}} \quad (26)$$

$$\frac{2}{G} = \sum_{k=k_i}^{k_f} \frac{1}{\sqrt{(\epsilon_k - \lambda)^2 + \Delta^2}} \quad (27)$$

where $k_i = Z/2 - n + 1$; $k_f = Z/2 + n'$.

The pairing interaction G is calculated from a continuous distribution of levels

$$\frac{2}{G} = \int_{\tilde{\lambda}-\Omega}^{\tilde{\lambda}+\Omega} \frac{\tilde{g}(\epsilon)d\epsilon}{\sqrt{(\epsilon - \tilde{\lambda})^2 + \tilde{\Delta}^2}} \quad (28)$$

where $\tilde{\lambda}$ is the Fermi energy deduced from the shell correction calculations and $\tilde{\Delta}$ is the gap parameter, obtained from a fit to experimental data, usually taken as $\tilde{\Delta} = 12/\sqrt{A}\hbar\omega_0^0$. Both Δ_p and Δ_n decrease with increasing asymmetry $(N - Z)/A$. From the above integral we get

$$\frac{2}{G} \simeq 2\tilde{g}(\tilde{\lambda}) \ln\left(\frac{2\Omega}{\tilde{\Delta}}\right) \quad (29)$$

Real positive solutions of BCS equations are allowed if

$$\frac{G}{2} \sum_k \frac{1}{|\epsilon_k - \lambda|} > 1 \quad (30)$$

i.e. for a pairing force (G-parameter) large enough at a given distribution of levels. The system can be solved numerically by Newton-Raphson method refining an initial guess

$$\begin{aligned} \lambda_0 &= (n_s \epsilon_d + n_d \epsilon_s)/(n_s + n_d) + G(n_s - n_d)/2 \\ \Delta_0^2 &= n_s n_d G^2 - (\epsilon_d - \epsilon_s)/4 \end{aligned} \quad (31)$$

where ϵ_s, n_s are the energy and degeneracy of the last occupied level and ϵ_d, n_d are the same quantities for the next level. Solutions around magic numbers, when $\Delta \rightarrow 0$, have been derived by Kumar et al. [14].

As a consequence of the pairing correlation, the levels situated below the Fermi energy are only partially filled, while those above the Fermi energy are partially empty; there is a given probability for each level to be occupied by a quasiparticle

$$v_k^2 = \frac{1}{2} \left[1 - \frac{\epsilon_k - \lambda}{\sqrt{(\epsilon_k - \lambda)^2 + \Delta^2}} \right] \quad (32)$$

or a hole

$$u_k^2 = 1 - v_k^2 \quad (33)$$

Only the levels in the near vicinity of the Fermi energy (in a range of the order of Δ around it) are influenced by the pairing correlations. For this reason, it is sufficient for the value of the cut-off parameter to exceed a given limit $\Omega \gg \tilde{\Delta}$, the value in itself having no significance.

4.2 Variation with deformation

The following relationship allows to calculate the effective mass in units of $\hbar^2/(\hbar\omega_0^0)$

$$\frac{\hbar\omega_0^0}{\hbar^2}B_\varepsilon = \frac{9}{2} \sum_{\nu\mu} \frac{\langle\nu|f_1\eta + f_2\xi^2|\mu\rangle\langle\mu|f_1\eta + f_2\xi^2|\nu\rangle}{(E_\nu + E_\mu)^3} (u_\nu v_\mu + u_\mu v_\nu)^2 \quad (34)$$

Matrix elements are calculated by performing some integrals

$$\begin{aligned} \langle n'_z n'_r m' | f_1(\varepsilon)\eta + f_2(\varepsilon)\xi^2 | n_z n_r m \rangle &= \delta_{m'm} N_{n'_r}^m N_{n_r}^m N_{n'_z} N_{n_z} \cdot [\\ & f_1 \int_0^\infty d\eta \eta^{|m|+1} e^{-\eta} L_{n'_r}^{|m|}(\eta) L_{n_r}^{|m|}(\eta) \int_{-\infty}^\infty d\xi e^{-\xi^2} H_{n'_z}(\xi) H_{n_z}(\xi) + \\ & f_2 \int_0^\infty d\eta \eta^{|m|} e^{-\eta} L_{n'_r}^{|m|}(\eta) L_{n_r}^{|m|}(\eta) \int_{-\infty}^\infty d\xi \xi^2 e^{-\xi^2} H_{n'_z}(\xi) H_{n_z}(\xi)] \end{aligned}$$

where

$$N_{n'_z} N_{n_z} \int_{-\infty}^\infty d\xi e^{-\xi^2} H_{n'_z}(\xi) H_{n_z}(\xi) = \delta_{n'_z n_z} \quad (35)$$

$$N_{n'_r}^m N_{n_r}^m \int_0^\infty d\eta \eta^{|m|} e^{-\eta} L_{n'_r}^{|m|}(\eta) L_{n_r}^{|m|}(\eta) = \delta_{n'_r n_r} \quad (36)$$

so that

$$\begin{aligned} \langle n'_z n'_r m' | f_1(\varepsilon)\eta + f_2(\varepsilon)\xi^2 | n_z n_r m \rangle &= \delta_{m'm} N_{n'_r}^m N_{n_r}^m N_{n'_z} N_{n_z} \cdot \\ & \left[\delta_{n'_z n_z} f_1 \int_0^\infty d\eta \eta^{|m|+1} e^{-\eta} L_{n'_r}^{|m|}(\eta) L_{n_r}^{|m|}(\eta) + \delta_{n'_r n_r} f_2 \int_{-\infty}^\infty d\xi \xi^2 e^{-\xi^2} H_{n'_z}(\xi) H_{n_z}(\xi) \right] \end{aligned}$$

Next we can use the relationships [16]:

$$\int_0^\infty d\eta \eta^{|m|+1} e^{-\eta} L_{n'_r}^{|m|}(\eta) L_{n_r}^{|m|}(\eta) = \delta_{n'_r n_r} (2n_r + |m| + 1) \frac{(n_r + |m|)!}{n_r!} \quad (37)$$

$$\int_{-\infty}^\infty d\xi \xi^2 e^{-\xi^2} H_{n'_z}(\xi) H_{n_z}(\xi) = \sqrt{\pi} n_z! 2^{n_z} \left[\delta_{n'_z n_z} (n_z + \frac{1}{2}) + \delta_{n'_z n_z + 2} (n_z + 1)(n_z + 2) + \delta_{n'_z n_z - 2} \frac{1}{4} \right] \quad (38)$$

which were obtained by using the recurrence relations and orthonormalization conditions for Hermite polynomials and associated Laguerre polynomials.

$$L_n^k(x) = L_n^{k+1}(x) - L_{n-1}^{k+1}(x) ; \quad (n+1)L_{n+1}^k(x) = [(2n+k+1) - x]L_n^k(x) - (n+k)L_{n-1}^k(x) \quad (39)$$

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \quad (40)$$

with particular values $L_0^k(x) = 1$, $L_1^k(x) = 1 + k - x$, $H_0(x) = 1$, $H_1(x) = 2x$.

Eventually from the sum of equation (34) one is left with an important diagonal contribution and two nondiagonal terms

$$\frac{\hbar\omega_0^0}{\hbar^2}B_{\varepsilon 1} = \frac{9}{4} \delta_{n'_r n_r} \delta_{m' m} \sum_{\nu=k_i}^{k_f} [f_1(2n_r + |m| + 1) + f_2(n_z + 1/2)]^2 \frac{(u_\nu v_\nu)^2}{E_\nu^3} \delta_{n'_z n_z} \quad (41)$$

$$\frac{\hbar\omega_0^0}{\hbar^2}B_{\varepsilon 2} = \frac{9}{4} \delta_{n'_r n_r} \delta_{m' m} \sum_{\nu \neq \mu} \frac{f_2^2}{2} (n_z + 1)(n_z + 2) \frac{(u_\nu v_\mu + u_\mu v_\nu)^2}{(E_\nu + E_\mu)^3} \delta_{n'_z n_z + 2} \quad (42)$$

$$\frac{\hbar\omega_0^0}{\hbar^2}B_{\varepsilon 3} = \frac{9}{4}\delta_{n'_r n_r}\delta_{m' m}\sum_{\nu \neq \mu} \frac{f_2^2}{2}(n_z - 1)n_z \frac{(u_\nu v_\mu + u_\mu v_\nu)^2}{(E_\nu + E_\mu)^3}\delta_{n'_z n_z - 2} \quad (43)$$

where k_i and k_f have been defined above. In order to perform the summations of the nondiagonal terms for a state with a certain ν (specified quantum numbers $n_z n_r m$) one has to consider only the states with $\mu \neq \nu$ and $n'_r = n_r$; $m' = m$ for which $n'_z = n_z + 2$ or $n'_z = n_z - 2$ respectively. Finally one arrives at the nuclear inertia in units of \hbar^2/MeV by adding the three terms and dividing by $\hbar\omega_0^0$.

4.3 Hydrodynamical formulae

There are several hydrodynamical formulae [15] of the mass parameters. For a spherical liquid drop with a radius $R_0 = 1.2249A^{1/3}$ fm one has

$$B^{irr}(0) = \frac{2}{15}MAR_0^2 = \frac{2}{15} \frac{\hbar^2}{41.5\text{MeV}\cdot\text{fm}^2} 1.2249^2 A^{5/3}\text{fm}^2 = 0.0048205A^{5/3} \frac{\hbar^2}{\text{MeV}} \quad (44)$$

When the spheroidal deformation is switched on it becomes

$$B_\varepsilon^{irr}(\varepsilon) = B^{irr}(0) \frac{81}{[27 - \varepsilon^2(9 + 2\varepsilon)]^{4/3}} \frac{9 + 2\varepsilon^2}{(3 - 2\varepsilon)^2} \quad (45)$$

Good results for the fission halflives of actinides have been obtained by using an inertia larger by about an order of magnitude.

One can also employ a formula with a fit parameter k

$$B_r^{ph}(r) = \mu + k(B_r^{irr}(r) - \mu) \quad (46)$$

where k shows how much deviates the flow of nuclear matter from an irrotational one. By substituting the above expression of the irrotational term, one obtains

$$B_r^{ph}(r) = \mu \left[1 + k \frac{17}{15} \exp\left(-\frac{r - 3R_0/4}{d}\right) \right] \quad (47)$$

with parameter values of $d = d_0 = R_0/2.452$ and $k = 11.5$, or $d = 2d_0$ and $k = 6.5$.

5 Results

The main result of the present study is represented by the equations (41–43), which could be used to test complex computer codes developed for realistic single-particle levels, for which it is not possible to obtain analytical relationships. Nuclear inertia of ^{240}Pu calculated with the equation (44) for a spherical liquid drop and with eq. (45) for spheroidal shapes is illustrated in figure 1. One can see how $B^{irr}(0)$ increases when the mass number of the nucleus is increased. The irrotational value $B_\varepsilon^{irr}(\varepsilon)$ monotonously increases with the spheroidal deformation parameter ε . Due to the fact that in this single center model the nucleus only became longer without developing a neck and never arriving to a scission configuration when the deformation is increased, the reduced mass is not reached as it should be in a two center model [2].

The cranking inertia of the spheroidal harmonic oscillator calculated by using the analytical relationships (41–43) and the correction given in the next section shows very pronounced fluctuations which are correlated to the shell corrections (calculated with the macroscopic-microscopic method [13]) plotted at the bottom of the figure 1.

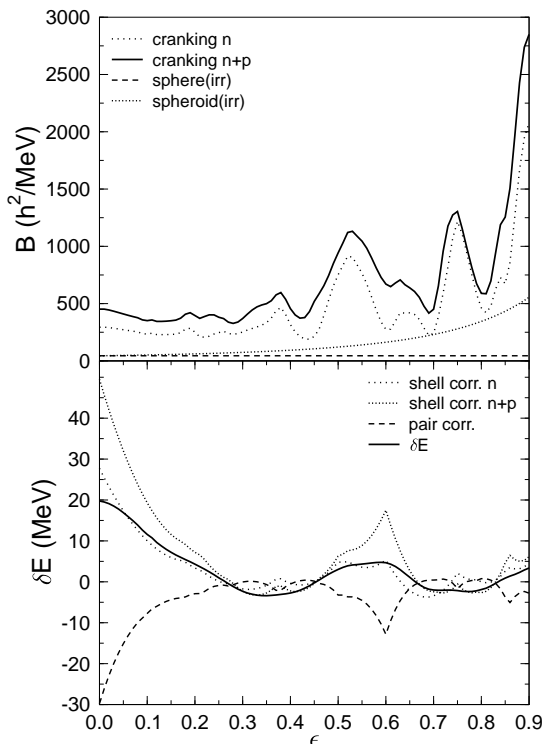


Figure 1: Top: comparison of the effective mass (in units of \hbar^2/MeV) calculated by using the cranking model for the proton plus neutron level schemes, only for neutrons, as well as for the irrotational spheroidal and spherical shapes of ^{240}Pu . Bottom: shell corrections for neutrons and protons, only for neutrons, pairing corrections, and shell plus pairing corrections.

5.1 Variation of the gap parameter and Fermi energy with deformation

Now we can calculate the correction term as

$$\begin{aligned}
P_\epsilon = & \frac{2\hbar^2}{8} \sum_\nu \frac{1}{E_\nu^5} \left[\left(\Delta \frac{d\lambda}{d\epsilon} \right)^2 + \right. \\
& (\epsilon_\nu - \lambda)^2 \left(\frac{d\Delta}{d\epsilon} \right)^2 + 2\Delta(\epsilon_\nu - \lambda) \frac{d\lambda}{d\epsilon} \frac{d\Delta}{d\epsilon} \\
& - 2\Delta^2 \frac{d\lambda}{d\epsilon} \langle \nu | dV/d\epsilon | \nu \rangle \\
& \left. - 2\Delta(\epsilon_\nu - \lambda) \frac{d\Delta}{d\epsilon} \langle \nu | dV/d\epsilon | \nu \rangle \right]
\end{aligned}$$

In figure 2 we plotted the variation with deformation of the solutions of BCS equations for Fermi energy λ (bottom) and the gap parameter Δ (top) of the proton and neutron level schemes for ^{240}Pu nucleus. The dotted line at the value 0.117 corresponds to $\tilde{\Delta}$. Their derivatives with respect to the deformation parameter are given in figure 3. For superdeformed nuclei with $\epsilon > 0.5$ the oscillation amplitudes of $d\lambda_n/d\epsilon$ are approaching their maximum values of about 2 units. In the same range of the deformations the inertia is also larger as a result of the increased density of levels at the Fermi surface.

The result displayed in figure 4 shows the important contribution of the neutron level scheme, $P_{\epsilon n}$ (dotted line), reflecting the larger density of states at the Fermi energy, compared to the

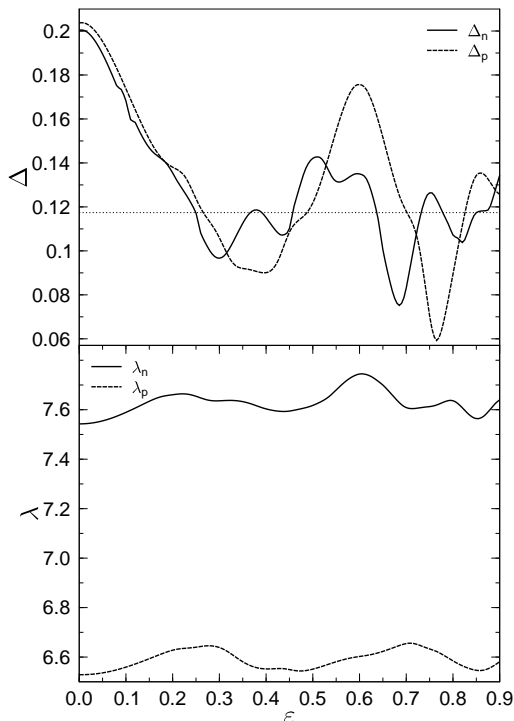


Figure 2: The variation with deformation of the solutions of BCS equations for Fermi energy λ (bottom) and the gap parameter Δ (top) of the proton and neutron level schemes for ^{240}Pu nucleus. The energies are expressed in units of $\hbar\omega_0^0 = 6.597$ MeV. The dotted line in the upper part corresponds to $\tilde{\Delta} = 0.117$.

proton term $P_{\varepsilon p}$ (dashed line). Their sum is a positive quantity, contributing to an increase of the nuclear inertia. In a dynamical investigation using the quasiclassical WKB approximation, the quantum tunnelling penetrability depends exponentially on the action integral, in which the integral contains a square root of the product of mass parameter and deformation energy. This exponential dependence amplifies very much any variation of the inertia. Consequently, the term P_{ij} should be considered in calculations. A similar conclusion was drawn from a study of a realistic two-center shell model [17].

6 Conclusions

By using the wave functions of the spheroidal harmonic oscillator (the simplest variant of the Nilsson model) without spin-orbit coupling one can obtain analytical relationships for the cranking inertia. Consequently the result may be conveniently used to test complex computer codes developed for realistic two center shell models.

Unfortunately this single center oscillator is not able to describe fission processes reaching the scission configuration or ground states with necked in or diamond shapes. When the deformation parameter is increased the nucleus became longer and longer without developing a neck and reaching the touching point configuration. In this way it is not possible to obtain at the limit a nuclear inertia equal to the reduced mass of the final fragments in a process of fission, alpha decay or cluster radioactivity.

As expected, in agreement with the results obtained by other authors, the cranking inertia is larger

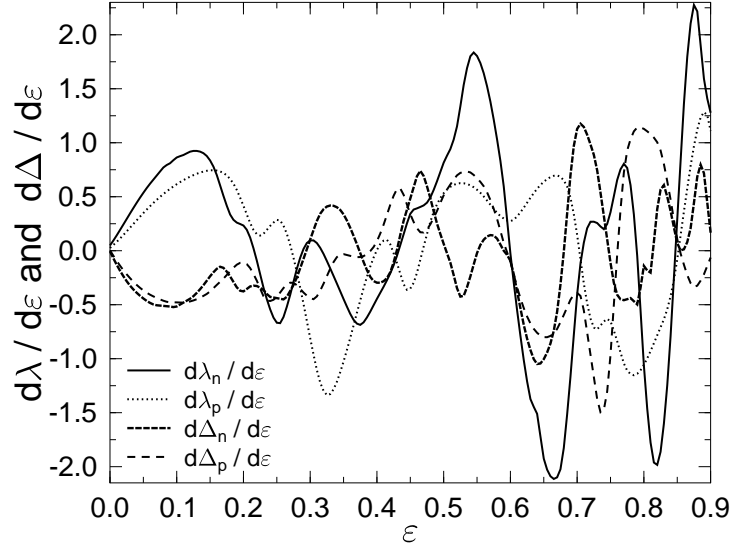


Figure 3: The derivatives with respect to deformation of the solutions of BCS equations for Fermi energy λ and the gap parameter Δ of the proton and neutron level schemes for ^{240}Pu nucleus. The energies are expressed in units of $\hbar\omega_0^0 = 6.597$ MeV.

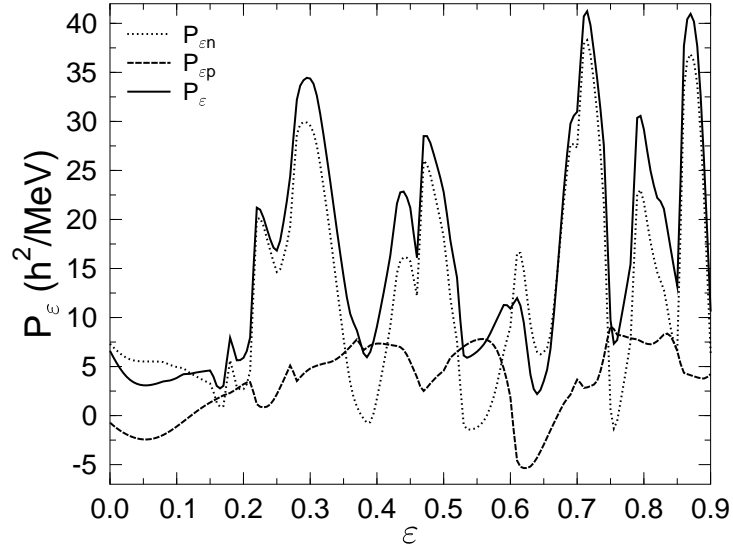


Figure 4: Contribution, P_ε , to the mass parameter of the occupation number variation with deformation for ^{240}Pu nucleus expressed in units of \hbar^2/MeV .

than the hydrodynamical one for a spheroidal shape which is higher than that of a spherical nucleus.

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