Mach shocks induced by partonic jets in expanding quark–gluon plasma

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Abstract

We study Mach shocks generated by fast partonic jets propagating through a deconfined strongly–interacting matter. Our main goal is to take into account different types of collective motion during the formation and evolution of this matter. We predict a significant deformation of Mach shocks in central Au+Au collisions at RHIC and LHC energies as compared to the case of jet propagation in a static medium. The observed broadening of the near–side two–particle correlations in pseudorapidity space is explained by the Bjorken–like longitudinal expansion. Three–particle correlation measurements are proposed for a more detailed study of the Mach shock waves.

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I. INTRODUCTION

Sideward peaks have been recently observed [1, 2, 3, 4] in azimuthal distributions of secondaries associated with the high p_T hadrons in central Au+Au collisions at $\sqrt{s_{NN}} = 200\,\mathrm{GeV}$. In Ref. [5] such peaks were predicted as a signature of Mach shocks created by partonic jets propagating through a quark–gluon plasma (QGP) formed in a heavy–ion collision. Analogous Mach shock waves were studied previously in a cold hadronic matter [6, 7, 8, 9, 10] as well as in nuclear Fermi liquids [11, 12]. The Mach–shock induced electron emission from metal surfaces have been predicted [13] and then observed experimentally in Ref. [14]. Recently, Mach shocks from jets in the QGP have been studied in Ref. [15] by using a linearized fluid–dynamical approach. It has been argued in Refs. [5, 16] that Mach–like motions of quark–gluon matter can appear via the excitation of collective plasmon waves by the moving color charge associated with the leading jet.

Most of these studies were dealing with the idealized case of homogeneous, static matter. On the other hand, collective expansion effects are known to be strong in relativistic collisions of heavy nuclei [17]. For example, thermal fits of RHIC data give for most central events the radial flow velocities $v_f \sim 0.6c$. The flow effects accompanying jet propagation through the expanding QGP have been considered within different approaches in Refs. [18, 19, 20]. Numerical solutions of fluid–dynamical equations with an additional external source have been studied in Ref. [21].

It is well known [22] that a point–like perturbation (a small body, a hadron or parton etc.) moving with a supersonic speed in the spatially homogeneous ideal fluid produces the so–called Mach region of the perturbed matter. In the fluid rest frame (FRF) the Mach region has a conical shape with an opening angle with respect to the direction of particle propagation given by the expression [27]

$$\widetilde{\theta}_M = \sin^{-1}\left(\frac{c_s}{\widetilde{v}}\right)\,, \tag{1}$$

where c_s denotes the sound velocity of the unperturbed (upstream) fluid and $\widetilde{\boldsymbol{v}}$ is the particle velocity with respect to the fluid. In the FRF, trajectories of fluid elements (perpendicular to the surface of the Mach cone) are inclined at the angle $\Delta\theta = \pi/2 - \widetilde{\theta}_M$

with respect to $\widetilde{\boldsymbol{v}}$. Strictly speaking, formula (1) is applicable only for weak, sound–like perturbations. It is certainly not valid for space–time regions close to a leading particle. Nevertheless, we shall use this simple expression for a qualitative analysis of flow effects. Following Refs. [5, 15] one can estimate the angle of preferential emission of secondaries associated with a fast jet in the QGP. Substituting [28] $\widetilde{v} = 1, c_s = 1/\sqrt{3}$ into Eq. (1) gives the value $\Delta\theta \simeq 0.96$. This agrees well with positions of maxima of the away–side two–particle distributions observed in central Au+Au collisions at RHIC energies.

In this paper we study influence of flow effects on the properties of Mach shocks created by a high–energy parton in expanding QGP. In Sect. II we apply simple kinematic considerations to study weak Mach shocks propagating in transverse and collinear direction with respect to the flow velocity. In Sect. III we use the method of characteristics to calculate the width of the perturbed region in the rapidity space. Finally, in Sect. IV we summarize our results and give outlook for future studies.

II. DEFORMATION OF MACH SHOCKS DUE TO RADIAL FLOW

Let us consider first the case when the away–side jet propagates with velocity $\boldsymbol{v}$ parallel to the matter flow velocity $\boldsymbol{u}$. Assuming that $\boldsymbol{u}$ does not change with space and time, and performing the Lorentz boost to the FRF, one can see that a weak Mach shock has a conical shape with the axis along $\boldsymbol{v}$. In this reference frame, the shock front angle $\widetilde{\theta}_M$ is given by Eq. (1). Transformation from the FRF to the c.m. frame (CMF) shows that the Mach region remains conical, but the Mach angle becomes smaller in the CMF:

$$\tan\theta_M = \frac{1}{\gamma_u}\tan\widetilde{\theta}_M\,, \tag{2}$$

where $\gamma_u \equiv (1-u^2)^{-1/2}$ is the Lorentz factor corresponding to the flow velocity $\mathbf{u}$. Eqs. (1)–(2) give the resulting expression for the Mach angle in the CMF

$$\theta_M = \tan^{-1}\left(c_s\sqrt{\frac{1-u^2}{\widetilde{v}^2-c_s^2}}\right)\,, \tag{3}$$

where

$$\widetilde{v} = \frac{v\mp u}{1\mp vu}\,, \tag{4}$$

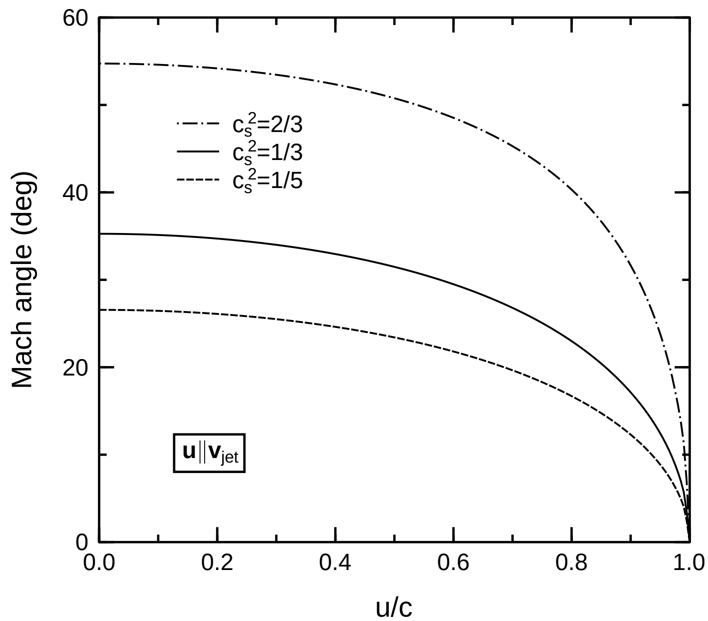

FIG. 1: Mach cone angles for jet propagating collinearly to the matter flow as a function of fluid velocity u. Different curves correspond to different values of sound velocity c_s.

and upper (lower) sign corresponds to the jet's motion in (or opposite to) the direction of collective flow. For ultrarelativistic jets ($v \to 1$) one can take $\widetilde{v} \simeq 1$ which leads to a simpler expression

$$\theta_M \simeq \tan^{-1}\left(\frac{c_s\gamma_s}{\gamma_u}\right) = \sin^{-1}\left(c_s\sqrt{\frac{1-u^2}{1-u^2\,c_s^2}}\right) , \qquad (5)$$

where $\gamma_s = (1-c_s^2)^{-1/2}$. According to Eq. (5), in the ultrarelativistic limit θ_M does not depend on the direction of flow with respect to the jet. The Mach cone becomes more narrow as compared to jet propagation in static matter. This narrowing effect has a purely relativistic origin. Indeed, the difference between θ_M from Eq. (5) and the Mach angle in absence of flow ($\lim\limits_{u\to 0}\theta_M = \sin^{-1} c_s$) is of the second order in the collective velocity u. The Mach angle calculated from Eq. (5) is shown in Fig. 1 as a function of u for different sound velocities c_s. Following Ref. [15], the value $c_s^2 = 1/5$ is identified with the hadronic matter and $c_s^2 = 1/3$ with ideal QGP composed of massless quarks and gluons. The value

$c_s^2 = 2/3$ is chosen to represent a strongly coupled QGP [23].

The case of a jet propagating at nonzero angle with respect to the flow velocity is more complicated. As will be shown later, Mach shocks become nonconical for non–collinear flows. For simplicity, below we study only the case when the jet and flow velocities are orthogonal to each other, $\boldsymbol{v} \perp \boldsymbol{u}$. Let axes OX and OY be directed along $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. As before, we first make transition to the FRF by performing the Lorentz boost along the OX axis. The components of the jet velocity $\widetilde{\boldsymbol{v}}$ in the new reference frame are equal to

$$\widetilde{v}_x = -u\,, \qquad \widetilde{v}_y = v\sqrt{1-u^2}\,. \tag{6}$$

The angle $\widetilde{\varphi}$ between vectors $\widetilde{\boldsymbol{v}}$ and $\boldsymbol{v}$ (see Fig. 2) can be found from the relations

$$\tan\widetilde{\varphi} = -\frac{\widetilde{v}_x}{\widetilde{v}_y} = \frac{\gamma_u u}{v} \underset{v\to 1}{\longrightarrow} \gamma_u u\,. \tag{7}$$

Let us now consider the situation when the jet propagates along the path $OA = \widetilde{v}\widetilde{t}$ during the time interval $\widetilde{t}$ in the FRF, as illustrated in Fig. 2. At the same time the wave front from a point–like perturbation (created at point O) reaches a spherical surface with radius $OB = OC = c_s\widetilde{t}$. Two tangent lines AB and AC show boundaries of the Mach region [29] with the symmetry axis OA. This region has a conical shape with opening angles $\widetilde{\theta}$ determined by the expressions (cf. Eq. (1))

$$\sin\widetilde{\theta} = \frac{OC}{OA} = \frac{c_s}{\widetilde{v}} \simeq c_s\,. \tag{8}$$

Performing inverse transformation from FRF to CMF, it is easy to show that the Mach region is modified in two ways. First, it is no longer symmetrical with respect to the jet trajectory in the CMF. The insert in Fig. 2 shows that the boundaries of Mach wave have different angles, $\theta_+ \neq \theta_-$, with respect to $\boldsymbol{v}$ in this reference frame. One can interpret this effect as a consequence of transverse flow which acts like a "wind" deforming the Mach cone along the direction OX. On the other hand, the angles of the Mach front with respect to the beam (OZ) axis are not changed under the transformation to CMF. We conclude, that due to effects of transverse flow, the Mach region in the CMF should have a shape of a deformed cone with an elliptic base.

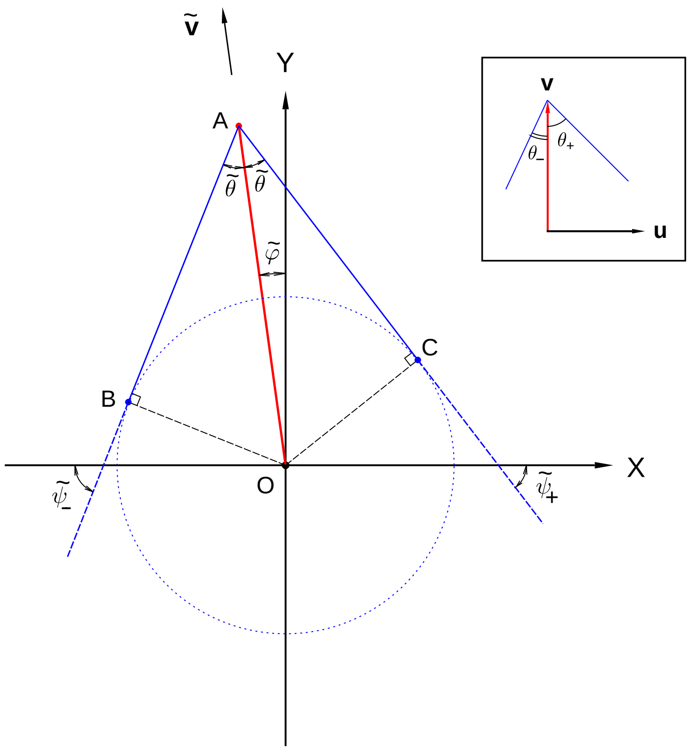

FIG. 2: Mach region created by jet moving with velocity $\boldsymbol{v}$ orthogonal to the fluid velocity $\boldsymbol{u}$. Main plot and insert correspond to FRF and CMF, respectively. It is assumed that jet moves from O to A in FRF. Dotted circle represents the front of sound wave generated at point O.

To find the Mach angles $\theta_\pm$ it is useful to introduce the angles $\widetilde{\psi}_\pm$ of the boundary lines AB and AC with respect to the OX axis in the FRF (see Fig. 2). Under the Lorentz boost to CMF, $\widetilde{\psi}_\pm$ are transformed to angles $\psi_\pm$ which can be found by using the relations

$$\cot\psi_\pm = \gamma_u \cot\widetilde{\psi}_\pm = \gamma_u \tan(\widetilde{\theta} \pm \widetilde{\varphi})\,. \tag{9}$$

The final expressions for Mach angles $\theta_\pm = \pi/2 - \psi_\pm$ take the form

$$\tan\theta_\pm = \cot\psi_\pm \simeq \gamma_u \frac{\gamma_s c_s \pm \gamma_u u}{1 \mp \gamma_s c_s \gamma_u u}\,. \tag{10}$$

The last equality gives the approximate formula in the ultrarelativistic case $v \simeq 1$[30]. According to Eq. (10), θ_- becomes negative for supersonic flows, i.e. at $u > c_s$. For such strong flows, the jet trajectory lies outside the Mach region. One can see that the difference of the Mach angles $\theta_\pm$ in moving and static matter is approximately linear in u.

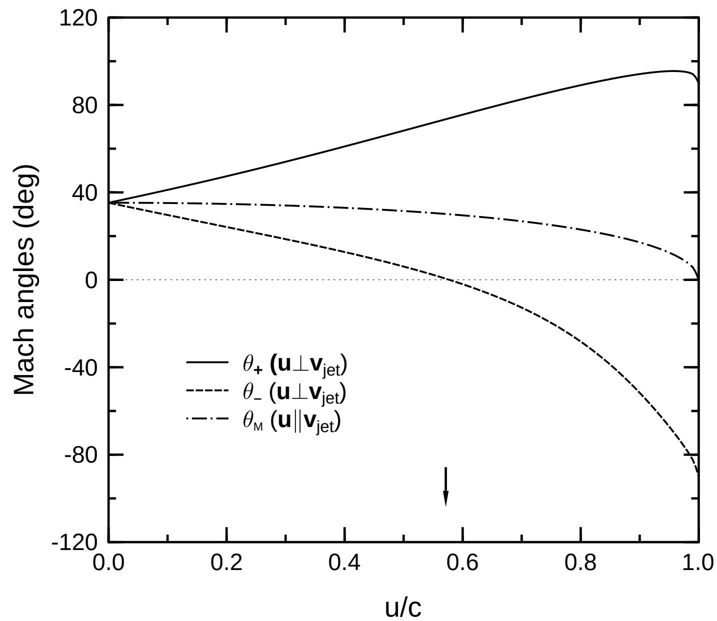

FIG. 3: Angles of Mach region created by a jet moving transversely (solid and dashed curves) and collinearly (dashed–dotted line) to the fluid velocity $\boldsymbol{u}$ in the CMF. All curves correspond to the case $c_s^2 = 1/3$. Arrow marks the value $u = c_s$.

Figure 3 shows the numerical values of the Mach angles for an ultrarelativistic jet moving through the QGP transversely or collinearly to its flow velocity. The solid and dashed curves are calculated by using Eq. (10) with $c_s = 1/\sqrt{3}$. We point out a much stronger sensitivity of the Mach angles $\theta_\pm$ to the transverse flow velocity as compared with the collinear flow.

To discuss possible observable effects, in Fig. 4 we schematically show events with different di–jet axes $A_iB_i\,(i = 1, 2, 3)$ with respect to the center of a fireball [31]. In the $2 - 2'$ event, the away–side jet '2' propagates along the diameter A_2B_2, i.e. collinearly with respect to the collective flow. In the two other cases, the di–jet axes are oriented along the chords, A_1B_1 and A_3B_3, respectively. In such events, the fluid velocity has both transverse and collinear components with respect to the jet axis. In Fig. 4 we also show how the Mach fronts will be deformed in expanding matter. It is easy to see that

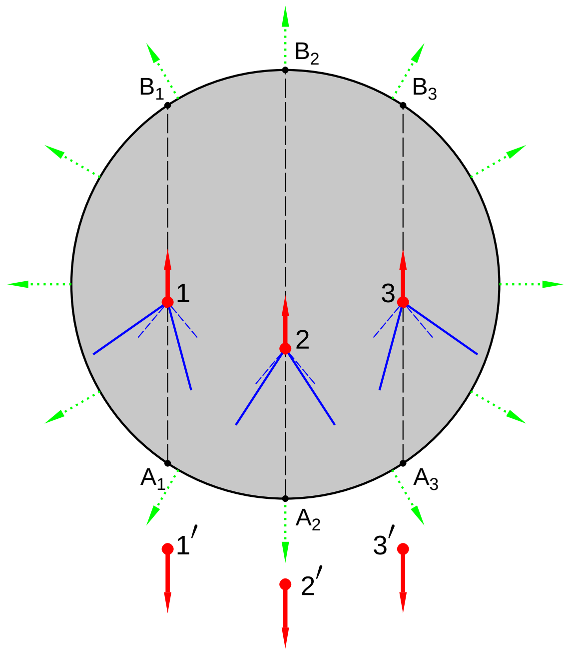

FIG. 4: Schematic picture of Mach shocks from jets $1, 2, 3$ propagating through the fireball matter (shaded circle) created in a central heavy–ion collision. Dotted arrows represent local velocities of fireball expansion. Thick downward arrows show associated trigger jets. The Mach shock boundaries are shown by solid lines. Short–dashed lines give positions of shock fronts in the case of static fireball.

the radial expansion of the fireball should cause broadening of the sideward peaks in the $\Delta\phi$–distributions of associated hadrons. Due to the radial expansion, the peaks will acquire an additional width of the order of $<\theta_+ - \theta_->$. Here $\theta_\pm$ are local values of the Mach angles in individual events. The angular brackets mean averaging over the jet trajectory in a given event as well as over all events with different positions of di–jet axes. Assuming that particle emission is perpendicular to the surface of Mach cone and taking $<u>\sim 0.4, c_s \simeq 1/\sqrt{3}$, we estimate the angular spread of emitted hadrons in the range $30^\circ - 50^\circ$. This is comparable with the half distance between the away–side peaks of the $\Delta\phi$ distribution observed by the STAR and PHENIX Collaborations [1, 2, 3, 4].

On the basis of this analysis we conclude that in individual events the sideward maxima should be asymmetric and more narrow than in an ensemble of different events. Due to a stronger absorption of particles emitted from the inner part of the shock (events 1, 3 in Fig. 4), the two peaks may have different amplitudes. We think that these effects can be observed by measuring three–particle correlations (see next section).

There is one more reason for broadening of the $\Delta\phi$–distributions which one should keep in mind when comparing with experimental data: due to the momentum spread of the initial parton distributions, $\Delta p_* \lesssim 1\,\mathrm{GeV}$, the di–jet system has a nonzero total momentum with respect to the global c.m. frame. As a consequence, the angle θ_* between the trigger– and the away–side jet is generally–speaking not equal to π, as was assumed above. Taking typical momenta of initial partons as p_0, with $p_0 > 4 - 6\,\mathrm{GeV}$ [1, 2, 3, 4], we estimate the angular spread as $|\pi - \theta_*| \sim \Delta p_*/p_0 \lesssim 0.1$. Therefore, the considered broadening should be much less than the typical shift of the Mach angles due to the collective flow.

III. INFLUENCE OF LONGITUDINAL EXPANSION OF QGP

In the preceding section, only effects of transverse flow have been considered. On the other hand, experimental data on the jet–induced pseudorapidity correlations in central Au+Au collisions at RHIC energies have appeared recently [25]. According to these data, "near–side" hadrons (emitted in the forward hemisphere, $\Delta\phi < \pi/2$, with respect to the trigger jet) exhibit much broader distributions in the relative pseudorapidity, $\Delta\eta$, as compared with pp and d+Au interactions. The widths of the $\Delta\eta$ distributions increase with decreasing p_T of secondary hadrons, reaching values of $\Delta\eta \sim 1$ for $p_T \to 0$.

This broadening can be naturally explained as a consequence of the longitudinal expansion of fireball matter created at early stages of a heavy–ion collision. Let us consider a trigger jet emitted at the initial stage of the reaction ($t \simeq 0$) with the c.m. pseudorapidity $\eta \simeq 0$, i.e. in the transverse plane $z \simeq 0$, where z is the beam axis. Below we consider the Bjorken–type scenario of nuclear collisions [24] and assuming that a cylindrical, longitudinally expanding volume of QGP with radius R is formed at proper time

$\tau \equiv \sqrt{t^2 - z^2} = \tau_0$, where $\tau_0 \simeq 0.4 - 0.6\,\mathrm{fm/c}$. In the following discussion we disregard the radial expansion of the fireball, assuming that the near–side correlations are formed during a comparatively short time, $\tau - \tau_0 < R/c_s$,which is needed for the trigger jet to leave the fireball.

Weak, sound–like, perturbations of fireball matter propagate in z–direction with velocities $(u \pm c_s)/(1 \pm u c_s)$, where u and c_s are, respectively, the collective longitudinal velocity and the sound speed at a given space–time point (t, z) . To find the trajectory of a sound wave front $z = z_+(t)$, one should solve the equation for the C_+ characteristics [22]:

$$\frac{dz_+}{dt} = \left.\frac{u + c_s}{1 + uc_s}\right|_{z=z_+(t)} . \tag{11}$$

Assuming the scaling law, $u = z/t = \tanh \eta$, for the longitudinal expansion of the fireball, one can solve Eq. (11) analytically, even in the case when c_s is a function of τ . Indeed, changing the variables from (t, z) to (τ, η) , one gets the following equation for the pseudorapidity η_+ of the sound front

$$\tau \frac{d\eta_+}{d\tau} = c_s(\tau) . \tag{12}$$

Assuming that the sound wave is excited at $t = \tau_0$, $z = 0$, we get

$$\eta_+(\tau) = \int\limits_{\tau_0}^{\tau} \frac{d\tau}{\tau} c_s(\tau) . \tag{13}$$

The expression for the C_- characteristics, $\eta = \eta_-(\tau)$, is given by the r.h.s. of (13) with an additional negative sign. From this derivation we conclude that at fixed proper time $\tau > \tau_0$ a primary jet, created at $\tau \simeq 0$, $\eta = 0$, disturbs the fireball matter in the pseudorapidity region $\eta_- < \eta < \eta_+$. In the case $c_s = \mathrm{const}$ one obtains a simple expression

$$\eta_\pm = \pm c_s \log\left(\frac{\tau}{\tau_0}\right) . \tag{14}$$

Within this approach one may expect nonzero correlations of secondaries within the interval of relative pseudorapidities $\eta_- \leqslant \Delta\eta \leqslant \eta_+$. Substituting $\tau_0 = 0.5$ fm/c, $\tau = 5$ fm/c, $c_s = 1/\sqrt{3}$ we get an estimate $|\Delta\eta|_{\max} \simeq 1.3$. This agrees quite well with the experimental data of Refs. [2, 25] (after subtraction of the "background" pp peak from the experimental $\Delta\phi - \Delta\eta$ distribution). As one can see from Eq. (13), the width of the $\Delta\eta$ distribution is sensitive to the QGP formation time τ_0 and, therefore, measurements of these distributions in different p_T intervals can be used as a clock for the thermalization process. More detailed information about the $\Delta\eta - \Delta\phi$ asymmetry can be extracted from 3–particle correlations. In particular, one can use such correlations to select events where the trigger and two associated hadrons have momenta in the axial plane (containing the beam axis). If the pseudorapidity of the trigger jet is small, both associated hadrons will probe the longitudinal flow of the QGP.

In the case of a strong shock we expect that the matter in the inner part of the Mach region will be swept to its surface[32]. This will result in a local depletion of parton density inside the Mach cone. Such a behavior should be observed as a dip in pseudorapidity density of the associated hadrons around $\eta = 0$. It seems that the experimental data on near–side production [2] indeed show such a dip for soft secondaries.

IV. CONCLUSION

In this paper we have investigated properties of Mach shock waves induced by high–energy partons propagating through dense quark–gluon matter created in heavy–ion collisions. By using simple kinematic and hydrodynamic relations we have analyzed deformations of these shocks due to the radial and longitudinal expansion of QGP. Taking typical flow parameters expected in central collisions of nuclei at RHIC and LHC energies we show that the shape and orientation of Mach regions are strongly modified as compared to the case of static (nonexpanding) medium. This may obscure observable signatures of Mach collective waves.

In the future we are going to take into account several additional effects, in particular, bending of the leading parton's trajectory due to the friction force from moving medium. In principle, this bending may also produce isolated peaks in the away–side $\Delta\phi$ distributions. On the contrary, the Mach waves should generate ring–like maxima around the jet axis. We believe that measurements of three–particle correlations with a broad coverage of angular space can be used to differentiate between these two possibilities. We are also planning to investigate properties of strong Mach shocks, in particular, their sensitivity to the equation of state of QGP and to effects of collective expansion.

Acknowledgments

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[27] Here and below the quantities in the FRF are marked by tilde.

[28] Units with $c = 1$ are chosen throughout the paper.

[29] Such region exists only if $\widetilde{v} > c_s$. The latter condition is fulfilled if $v > c_s$ or $u > c_s$ hold.

[30] This formula is not accurate if either u or c_s are close to unity.

[31] For simplicity we consider the case when both trigger (i') and away–side (i) jets have zero pseudorapidities in CMF.

[32] Strong shocks produced by a localized, high energy initial perturbation in the QGP have been studied by using the 3D fluid–dynamical model in Ref. [26]. Clear deviations from Mach cone shape have been predicted.