Rise Time of the Amplitudes of Time Harmonic Fields in Multicell Cavities *

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Abstract

Wall losses can cause a coupling between eigenmodes in a cavity. The magnitude of the effect can be determined by means of eigenmode expansion. The influence on rise time of forced oscillations is calculated. Results for a brick resonator and a six-cell iris structure are presented.

I. INTRODUCTION

The operation of superconducting and conventional linear colliders under multibunch conditions requires the recovery of the accelerating field and damping of wake fields being completed before the arrival of the next bunch in the train. In either case the study of time behaviour of the accelerating resp. wakefields is essential. For example, for TESLA [1] a train of 800 bunches, following each other in 1µs distance, is foreseen. For TESLA accelerator sections there have been experiments and calculations based on lumped circuit theory showing good agreement between measurement and calculations [2].

In order to investigate the time behaviour of generator or beam driven cavities we decided to use a more general approach.

II. GENERAL THEORY

A. Basic Equations

We consider a driven cavity and want to express the solutions of the time dependent Maxwell equations (1) in terms of cavity eigenmodes.

$$\nabla \times \mathbf{H} = \varepsilon_0 \partial_t \mathbf{E} + \mathbf{J} \ , \ \nabla \times \mathbf{E} = -\mu_0 \partial_t \mathbf{H} \tag{1}$$

The eigenmodes satisfy the following set of equations [3]:

$$\nabla \times \mathbf{H}_j = i\omega_j \varepsilon_0 \mathbf{E}_j \ , \ \nabla \times \mathbf{E}_j = -i\omega_j \mu_0 \mathbf{H}_j \tag{2}$$

The solutions of the time dependent equations (1) may be expanded as

$$\mathbf{E}(\mathbf{r},t) = \sum_j a_j(t)\mathbf{E}_j(\mathbf{r}) \ , \ \mathbf{H}(\mathbf{r},t) = \sum_j b_j(t)\mathbf{H}_j(\mathbf{r}) \tag{3}$$

if the driving term can be expressed in the same way.

$$\mathbf{J}(\mathbf{r},t) = \sum_j c_j(t)\mathbf{E}_j(\mathbf{r}) \tag{4}$$

As shown in (5) the eigenmodes are normalized to unity.

$$\left\{ \begin{matrix} \mu_0 \\ \varepsilon_0 \end{matrix} \right\} \iiint_V \left\{ \begin{matrix} \mathbf{H}_j^* \cdot \mathbf{H}_k \\ \mathbf{E}_j^* \cdot \mathbf{E}_k \end{matrix} \right\} dV = \delta_{jk} \tag{5}$$

Wall losses are taken into account by assuming the following boundary condition for the parallel electric field on the surface, R_a being the surface impedance [3].

$$\mathbf{E}_{tan} = (1+i)R_a \mathbf{H}_{tan} \times \mathbf{n} \tag{6}$$

We multiply equations (1) with $\mathbf{E}_j^*$, $\mathbf{H}_j^*$ resp., use (3), (4), and (5), integrate both equations over the cavity volume and apply Gauss' integral identity. The appearing integral of the function $\mathbf{E} \times \mathbf{H}_j^*$ can be evaluated (using (6)) to a sum of $b_k(t)$ with coefficients depending only on the magnetic eigenfields. These interaction terms are denoted by A_{jk}.

$$\oint_{\partial V} \left(\mathbf{E} \times \mathbf{H}_j^*\right) \cdot \mathbf{n}\, ds = (1+i)R_a \sum_k b_k(t) \oint_{\partial V} \mathbf{H}_j^* \cdot \mathbf{H}_k\, ds$$

$$=: (1+i)R_a \sum_k A_{jk}\, b_k(t) \tag{7}$$

Now we are able to set up a first order system of linear differential equations describing the behaviour of the coefficients for the evaluation of the fields. The dimension is twice the number of modes under consideration.

$$\begin{matrix} \dot{a}_j(t) & -i\omega_j b_j(t) & = -\frac{1}{\varepsilon_0} c_j(t) \\ \dot{b}_j(t) & -i\omega_j a_j(t) + (1+i)R_a \sum_k (A_{jk} b_k(t)) & = 0 \end{matrix} \tag{8}$$

This is equivalent to a second order system:

$$\ddot{b}_j(t) + (1+i)R_a \sum_k \left(A_{jk} \dot{b}_k(t)\right) + \omega_j^2 b_j(t) = -\frac{\omega_j}{\varepsilon_0} c_j(t) \tag{9}$$

One can observe the driven harmonic oscillator characteristic which is modified by the mode interaction in the first order time derivative terms.

B. Treatment of the Exchange Terms A_{jk}

The $A_{jj} \cdot R_a$ are proportional to the wall losses in the mode j. The single-mode Q is given by:

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$$Q_j = \frac{\omega_j}{A_{jj} R_a} \tag{10}$$

The A_{jk} describe power exchange between modes. From (7) it is apparent that:

$$A_{jk} = A^*_{kj} \tag{11}$$

Further it can be shown with aid of the sentence of Bunjakowski-Schwarz [4] that there is an upper limit for the value of the A_{jk}.

$$|A_{jk}| \le \sqrt{A_{jj} A_{kk}} \tag{12}$$

For some simple geometries like brick or pillbox cavities there are analytical solutions for the A_{jk}. In general a numerical determination of fields has to be done, e.g. use of MAFIA [5] or similar codes.

III. NUMERICAL AND ANALYTICAL EXAMPLE

Starting with (8) one first seeks the solution of the homogenous system. For simplicity, in the following we restrict ourselves to two modes. This is no limitation of the procedure.

$$\begin{pmatrix} \dot{a}_1 \\ \dot{b}_1 \\ \dot{a}_2 \\ \dot{b}_2 \end{pmatrix} = \begin{pmatrix} 0 & i\omega_1 & 0 & 0 \\ i\omega_1 & -(1+i)R_a A_{11} & 0 & -(1+i)R_a A_{12} \\ 0 & 0 & 0 & i\omega_2 \\ 0 & -(1+i)R_a A^*_{12} & i\omega_2 & -(1+i)R_a A_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} \tag{13}$$

The general solution of the homogenous system can be written as:

$$f(t) = u_1 V_1 e^{\lambda_1 t} + \;....\; + u_4 V_4 e^{\lambda_4 t} \tag{14}$$

where $f^T = (a_1, b_1, a_2, b_2)$, λ_j and V_j are the eigenvalues and eigenvectors of the system matrix, and u_j are arbitrary constants. To solve the inhomogenous system variation of constants u_j is used. With the assumption of the same harmonic time dependence of both c_1, c_2 (they may differ in phase and amplitude) we get for the inhomogenous part of (8):

$$-\frac{1}{\varepsilon_0}\begin{pmatrix} c_1(t) \\ 0 \\ c_2(t) \\ 0 \end{pmatrix} = \kappa e^{i\omega_0 t} s(t) = \begin{pmatrix} \kappa_1 \\ 0 \\ \kappa_2 \\ 0 \end{pmatrix} e^{i\omega_0 t} s(t) \tag{15}$$

where s(t) is an arbitrary function controlling the complex amplitude of the excitation. The solution is, e.g. $u_1(t)$:

$$u_1(t) = \frac{\det(\kappa, V_2, V_3, V_4)}{\det(V_1, V_2, V_3, V_4)} \int_0^t \frac{e^{i\omega_0 \tau} s(\tau)}{e^{\lambda_1 \tau}} d\tau \tag{16}$$

Inserting into (14) gives the result.

The figures 1.-4. show the envelope of the values $|a_1|$, $|a_2|$ of the forced $\exp(i\omega_0 t)$-oscillations. The eigenvectors and

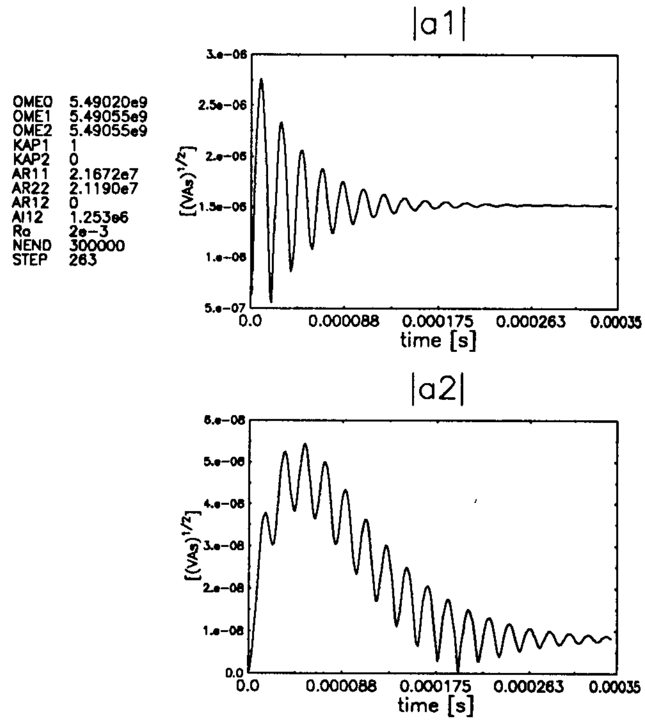

Figure 1. Brick resonator driven slightly below resonance of both degenerated modes (TM_{111}, TE_{111}). In the parameter block the generator (OME0), the two angular eigenfrequencies (OME1, OME2), direct coupling constants according to (15) (KAP1, KAP2), the A_{jk} and the wall impedance are printed.

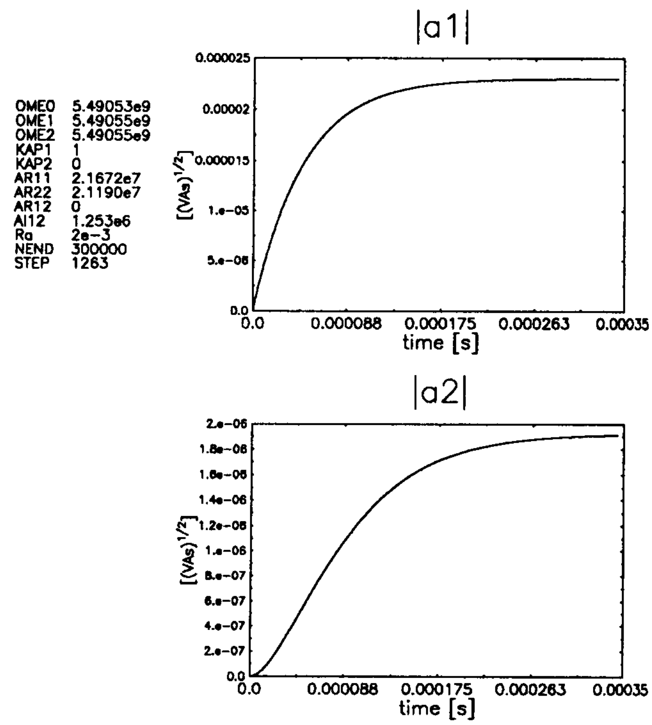

Figure 2. Brick resonator driven very close to resonance. $|a_2|$ reaches about 10% of $|a_1|$.

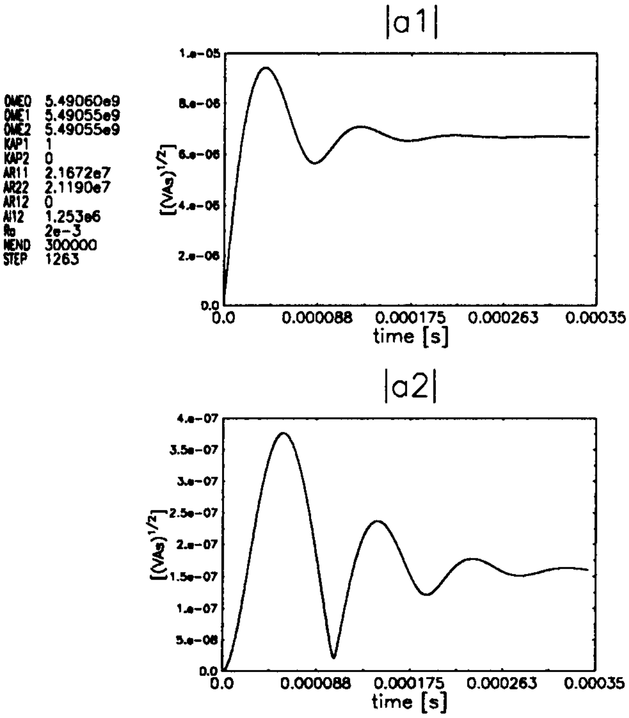

Figure 3. Brick resonator driven above resonance. Stabilization of second mode takes twice the time of the first.

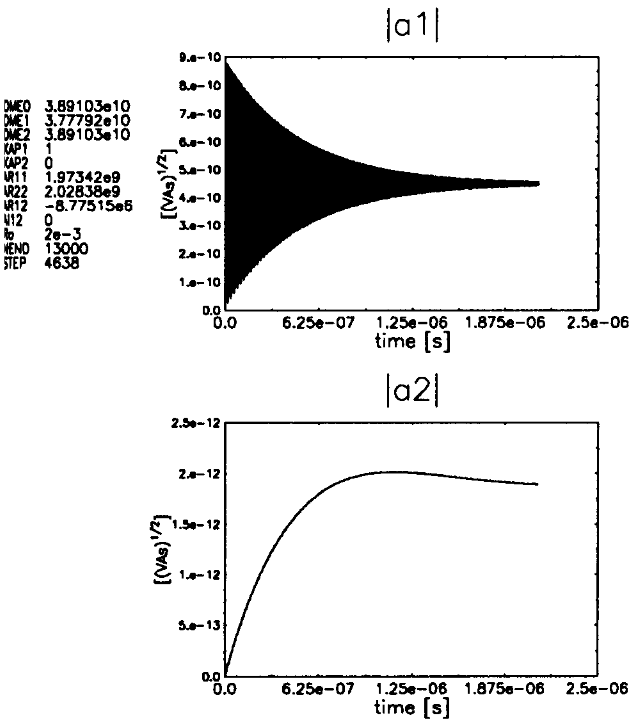

Figure 4. Six-cell iris structure. First mode is $\pi/6$, second mode is $\pi/3$. Excitation at eigenfrequency of second mode. The relatively far distance to ω_1 causes a fast oscillation of $|a_1|$. $|a_2|$ reaches about 0.5% of $|a_1|$.

eigenvalues as well as the $u_j(t)$ were calculated numerically. The function s(t) has been chosen

$$s(\tau) = \begin{cases} 0.5\left[1-\cos(\frac{\tau}{T_{st}}\pi)\right] & \tau \le T_{st} \\ 1 & \tau > T_{st} \end{cases} \tag{17}$$

that analytical time integration is possible.

For the brick resonator ideal degeneration of modes is possible. Therefore we investigated the interaction between the TM_{111} and the TE_{111} mode. The A_{jk} werde determined analytically.

As an example of a multicell structure we chose a six-cell iris cavity. The TM_{010}-$\pi/6$ and the neighbouring $\pi/3$ mode were calculated by means of MAFIA, then the magnetic surface fields had to be extracted from the result file in order to compute the A_{jk}.

IV. CONCLUSIONS

There is a coupling between modes due to wall losses. The effect depends on the distance of frequencies of the involved modes, the value of wall impedance, and the geometrically determined interaction terms A_{jk}. The coupling strength is limited according to (12). In most cases there is no need to take care of the effect. But it can be of some importance for degenerated modes or multicell accelerator structures with low coupling between cells, equivalent to narrow passbands. A similar coupling mechanism is to be expected for HOM-damped structures.

V. ACKNOWLEDGEMENTS

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VI. REFERENCES

[1] A Proposal to Construct and Test Prototype Superconducting R.F. Structures for Linear Colliders, The TESLA Collaboration, Hamburg, February 1992

[2] D. Proch, Private Communication, November 1992

[3] H.-G. Unger, Elektromagnetische Theorie für die Hochfrequenztechnik, Teil 2, pp. 275, Heidelberg 1981

[4] Bronstein-Semendjajew, Taschenbuch der Mathematik, p. 124, Thun und Frankfurt/Main 1985

[5] T. Weiland, Solving Maxwell's Equations in 2D and 3D by Means of MAFIA, Proc. Conf. on Computer Codes and the Linear Accelerator Community, pp.3, Los Alamos 1990