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Compactification of lattice-valued convergence spaces

Abstract: We define compactness for stratified lattice-valued convergence spaces and show that a Tychonoff theorem is true. Further a generalization of the classical Richardson compactification is given. This compactification has a universal property.

Keywords: L -fuzzy convergence, L -topology, L -filter, L -convergence space, compactness, compactification.

1. Introduction

Compactness is certainly one of the most important properties of topological spaces. For convergence spaces, where axiom schemes based on convergence of filters are used, we can use the nice characterization of compactness by convergence of ultrafilters as definition of compactness [3]. Based on such a definition, Richardson [15] constructed a famous compactification of a convergence space which has a universal property. This compactification has many interesting applications, see e.g. [2, 13].

In the case of stratified lattice-valued topological spaces, Höhle [7] used a similar definition based on the convergence of stratified L -ultrafilters. He showed the equivalence of this definition with a covering condition à la Heine-Borel. In this paper, for a complete Heyting algebra as underlying lattice, we use Höhle's definition in the case of lattice-valued convergence spaces. We show the suitability of this approach in two ways: Firstly, we prove the important Tychonoff Theorem. Secondly, as the central point of this paper, a generalization of the classical Richardson compactification to the lattice-valued case is given. It is shown that, as in the classical case, a continuous mapping from a non-compact space to a compact, regular T2-space can be uniquely extended to a mapping from the Richardson compactification. In the restricted case that the lattice is a complete Boolean algebra, the spaces which have regular Richardson compactification are characterized by a condition on the non-convergent L -ultrafilters.

For results on L -topological spaces and L -filters, we refer to [7, 8] and [9]. For a comprehensive discussion of compactification for lattice-valued topological spaces we refer to [16]. For notions of Category Theory we refer to [1] and for a survey on the classical theory of convergence spaces

we recommend the paper [14].

2. Preliminaries

In this paper, let (L, \wedge, \vee) be a *complete Heyting algebra*, i.e. a complete lattice where finite meets distribute over arbitrary joins, i.e. $\alpha \wedge \bigvee_{i \in I} \beta_i = \bigvee_{i \in I} (\alpha \wedge \beta_i)$ holds for all α, β_i ($i \in I$). The bottom and top elements of L are denoted by \perp and \top , respectively. We can then define an *implication* by $\alpha \rightarrow \beta = \bigvee \{\lambda \in L : \alpha \wedge \lambda \leq \beta\}$. Then $\alpha \rightarrow _$ is the right adjoint for $\alpha \wedge _$, i.e. we have $\delta \leq \alpha \rightarrow \beta \iff \delta \wedge \alpha \leq \beta$. This implies that $\alpha \rightarrow \beta$ is order-preserving in the second place and order-reversing in the first place. For further properties of this operation see e.g. [6, 9].

Let X be a set. We extend the operations $\wedge, \vee, \rightarrow$ and the order relation \leq from L to L -sets $a, b, c, \dots \in L^X$ by $(a * b)(x) = a(x) * b(x)$, $(* \in \{\wedge, \vee, \rightarrow\}, x \in X)$ and by the product order ($a \leq b$ if $a(x) \leq b(x)$ for every $x \in X$). With these definitions, also L^X is a complete Heyting algebra and again $a \rightarrow _$ is right adjoint for $a \wedge _$. For a subset $A \subset X$ and $\alpha \in L$ we denote by $\alpha_A : X \rightarrow L, x \mapsto \alpha$ if $x \in A$ and $x \mapsto \perp$ if $x \notin A$. Especially, \top_A is the *characteristic function* of A .

A mapping $\mathcal{F} : L^X \rightarrow L$ subject to the conditions

$$(F1) \quad \mathcal{F}(\top_X) = \top, \quad \mathcal{F}(\perp_X) = \perp$$

$$(F2) \quad f \leq g \implies \mathcal{F}(f) \leq \mathcal{F}(g)$$

$$(F3) \quad \mathcal{F}(f) \wedge \mathcal{F}(g) \leq \mathcal{F}(f \wedge g)$$

$$(Fs) \quad \alpha \wedge \mathcal{F}(f) \leq \mathcal{F}(\alpha_X \wedge f)$$

for all $\alpha \in L, f, g \in L^X$, is called a *stratified L-filter* on X [7, 9].

Example 2.1: For a point $x \in X$ the mapping $[x] : L^X \rightarrow L$ defined by $[x](a) = a(x)$ is a stratified L-filter on X , the *point L-filter* $[x]$ of x .

The set $\mathcal{F}_L^s(X)$ of all stratified L-filters on X is ordered by: $\mathcal{F} \leq \mathcal{G}$ if for all $a \in L^X$ it holds that $\mathcal{F}(a) \leq \mathcal{G}(a)$. The *meet* of $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ can then be calculated by $(\mathcal{F} \wedge \mathcal{G})(a) = \mathcal{F}(a) \wedge \mathcal{G}(a)$. It is obviously again a stratified L-filter on X [7].

For two stratified L-filters on X , \mathcal{F}, \mathcal{G} , we have an upper bound, $\mathcal{H} \in \mathcal{F}_L^s(X)$, $\mathcal{F}, \mathcal{G} \leq \mathcal{H}$, if $\mathcal{F}(a) \wedge \mathcal{G}(b) = \perp$ whenever $a \wedge b = \perp_X$ [7, 9]. In the case of "existence" of an upper bound for $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, the least upper bound, $\mathcal{F} \vee \mathcal{G} \in \mathcal{F}_L^s(X)$, is determined by $\mathcal{F} \vee \mathcal{G}(a) = \bigvee \{\mathcal{F}(f) \wedge \mathcal{G}(g) \mid f \wedge g \leq a\}$. It is shown in [7] that the set $(\mathcal{F}_L^s(X), \leq)$ has maximal elements

which are called *stratified L-ultrafilters*.

Lemma 2.2 [7]: $\mathcal{F} \in \mathcal{F}_L^s(X)$ is a stratified L-ultrafilter iff $\mathcal{F}(a) = \mathcal{F}(a \rightarrow \perp_X) \rightarrow \perp$ for all $a \in L^X$.

As a consequence, for a stratified L-ultrafilter $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $a \in L^X$ we have $\mathcal{F}(a) = (\mathcal{F}(a) \rightarrow \perp) \rightarrow \perp$ and $\mathcal{F}(a) \rightarrow \perp = \mathcal{F}(a \rightarrow \perp_X)$ (cf. [7]).

For $\mathcal{F} \in \mathcal{F}_L^s(X)$ and for a mapping $\varphi : X \rightarrow Y$, we define $\varphi(\mathcal{F}) : L^Y \rightarrow L$ by $\varphi(\mathcal{F})(b) = \mathcal{F}(\varphi^\leftarrow(b))$, $b \in L^Y$. Then $\varphi(\mathcal{F})$ is a stratified L-filter on Y [7].

We then have $\varphi([x]) = [\varphi(x)]$ and $\varphi(\mathcal{F} \wedge \mathcal{G}) = \varphi(\mathcal{F}) \wedge \varphi(\mathcal{G})$. From Lemma 2.2 we find moreover that for a stratified L-ultrafilter \mathcal{F} on X , also $\varphi(\mathcal{F})$ is a stratified L-ultrafilter on Y [7].

Lemma 2.3 [10]: Let $\mathcal{F} \in \mathcal{F}_L^s(Y)$ and $\varphi : X \rightarrow Y$ be a mapping. The following is equivalent.

- (i) $\varphi^\leftarrow(\mathcal{F}) \in \mathcal{F}_L^s(X)$, where $\varphi^\leftarrow(\mathcal{F})(a) = \bigvee \{\mathcal{F}(f) \mid \varphi^\leftarrow(f) \leq a\}$.
- (ii) $\mathcal{F}(b) = \perp$ whenever $\varphi^\leftarrow(b) = \perp_X$.

Example 2.4 [10]: For $A \subset X$ we consider the inclusion mapping $\iota_A : A \rightarrow X$. Then $\mathcal{F}_A = \iota_A^\leftarrow(\mathcal{F}) \in \mathcal{F}_L^s(A)$ if and only if $b|_A = \perp_A$ implies $\mathcal{F}(b) = \perp$. In this case we call \mathcal{F}_A the *trace of \mathcal{F} on A* . Note that $\mathcal{F}_A(a) = \bigvee \{\mathcal{F}(b) \mid b|_A \leq a\}$. Also, if $\mathcal{G} \in \mathcal{F}_L^s(A)$, then $[\mathcal{G}] = \iota_A(\mathcal{G}) \in \mathcal{F}_L^s(X)$. Here, $[\mathcal{G}](a) = \mathcal{G}(a|_A)$. We then have $\mathcal{F} \leq [\mathcal{F}_A]$ and if $\mathcal{F}(\top_A) = \top$, then $\mathcal{F} = [\mathcal{F}_A]$.

Example 2.5 [12]: Let J be a set, $\mathcal{G} \in \mathcal{F}_L^s(J)$ and let for $j \in J$, $\mathcal{F}_j \in \mathcal{F}_L^s(X)$. We define for $a \in L^X$

$$\mathcal{F}_{(\cdot)}(a) : \begin{cases} J & \longrightarrow & L \\ j & \longmapsto & \mathcal{F}_j(a) \end{cases},$$

and with this the mapping $\mathcal{G}(\mathcal{F}_{(\cdot)})$ by $\mathcal{G}(\mathcal{F}_{(\cdot)})(a) = \mathcal{G}(\mathcal{F}_{(\cdot)}(a))$ ($a \in L^X$). Then $\mathcal{G}(\mathcal{F}_{(\cdot)}) \in \mathcal{F}_L^s(X)$ is called the *stratified L-diagonal filter*.

3. Lattice-valued convergence spaces

Let X be a set and L be a complete Heyting algebra. A mapping

$$\lim : \mathcal{F}_L^s(X) \longrightarrow L^X, \quad \mathcal{F} \longmapsto \lim \mathcal{F}$$

subject to the conditions

$$(L1) \quad \limx = \top \quad \forall x \in X$$

$$(L2) \quad \mathcal{F} \leq \mathcal{G} \quad \implies \quad \lim \mathcal{F} \leq \lim \mathcal{G}$$

is called a *stratified L -generalized convergence on X* , the pair (X, \lim) a *stratified L -generalized convergence space* [10, 11]. A function $\varphi : X \longrightarrow Y$ between two stratified L -generalized convergence spaces, $(X, \lim^X), (Y, \lim^Y)$, is called *continuous* (w.r.t. \lim^X, \lim^Y) iff

$$\lim^X \mathcal{F}(x) \leq \lim^Y \varphi(\mathcal{F})(\varphi(x)) \quad \forall \mathcal{F} \in \mathcal{F}_L^s(X), x \in X.$$

We showed in [10, 11] that the category *SL-GCS* of stratified L -generalized convergence spaces with objects all stratified L -generalized convergence spaces and continuous mappings as morphisms is a well-fibred topological category which can be identified in case $L = \{\perp, \top\}$ with the category of *generalized convergence spaces* in the sense of Preuss [14]. Moreover, *SL-GCS* is cartesian closed [10], which is for a well-fibred topological category equivalent to the existence of "natural" function space structures [1]. *SL-GCS* contains *SL-TOP*, the category of stratified L -topological spaces [9], as a reflective subcategory [10, 11]. In [17] it is further shown, that also *SL-FTOP*, the category of enriched L -fuzzy topological spaces [9], is a reflective subcategory of *SL-GCS*.

In *SL-GCS* initial structures can be easily described. Let $(X \xrightarrow{\varphi_\lambda} (X_\lambda, \lim_\lambda))_{\lambda \in \Lambda}$ be a source. Then

$$\lim \mathcal{F} = \bigwedge_{\lambda \in \Lambda} \varphi_\lambda^\leftarrow(\lim_\lambda \varphi_\lambda(\mathcal{F})) \quad (\mathcal{F} \in \mathcal{F}_L^s(X))$$

is the initial stratified L -generalized convergence on X ([10, 11]. Especially, if $X = \prod_{i \in J} X_i$ and $p_j : X \longrightarrow X_j$ are the projections onto X_j , then we denote

$$\pi\text{-}\lim \mathcal{F}((x_i)) = \bigwedge_{i \in J} \lim_i p_i(\mathcal{F})(x_i)$$

the initial structure and call $(\prod_{i \in J} X_i, \pi\text{-}\lim)$ the *product space*. In case $(A \xrightarrow{\iota_A} X)$ we call the initial construction $(A, \lim|_A)$ a subspace of (X, \lim) . Note that for $\mathcal{F} \in \mathcal{F}_L^s(A)$ and $x \in A$ we have $\lim|_A \mathcal{F}(x) = \lim[\mathcal{F}](x)$.

A stratified L -generalized convergence space (X, \lim) is called a *T2-space* [12] if from $\lim \mathcal{F}(x) = \top = \lim \mathcal{F}(y)$ it always follows that $x = y$. We shall call $\mathcal{F} \in \mathcal{F}_L^s(X)$ *convergent* if there is an $x \in X$ such that $\lim \mathcal{F}(x) = \top$. Otherwise we call \mathcal{F} *non-convergent*. Consequently, a T2-space is characterized by convergent stratified L -filters having unique limit points.

A space $(X, \lim) \in |SL-GCS|$ is called *regular* [12] if it satisfies the following axiom (LR):

$$(LR) \quad \forall J, \forall \psi : J \longrightarrow X, \quad \forall \mathcal{G} \in \mathcal{F}_L^s(J), \quad \forall \mathcal{F}_i \in \mathcal{F}_L^s(X) \ (i \in J), \forall x \in X : \\ \bigwedge_{j \in J} \lim \mathcal{F}_j(\psi(j)) \wedge \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \leq \lim \psi(\mathcal{G})(x).$$

If L is a complete Boolean algebra, i.e. if the *law of double negation*, $(\alpha \rightarrow \perp) \rightarrow \perp = \alpha$ for all $\alpha \in L$, is true [5], then we can characterize regularity of (X, \lim) by the following axiom (LR') (cf. [12]):

$$(LR') \quad \forall \alpha, \beta \in L : \lim \mathcal{F}(x) \geq \alpha \implies \lim \overline{\mathcal{F}}^\beta(x) \geq \alpha \wedge \beta.$$

Here, the β -closure of $\mathcal{F} \in \mathcal{F}_L^s(X)$ is defined by $(a \in L^X)$

$$\overline{\mathcal{F}}^\beta(a) = \bigvee \{ \mathcal{F}(f) \mid f \in L^X \text{ such that } \mathcal{G} \in \mathcal{F}_L^s(X) \text{ with } \lim \mathcal{G}(x) \geq \beta \text{ implies } \mathcal{G}(f) \leq a(x) \}.$$

It is shown in [12] that for all $\beta \in L$, $\overline{\mathcal{F}}^\beta \in \mathcal{F}_L^s(X)$.

We further call $A \subset X$, $(X, \lim) \in |SL-GCS|$, *dense in* (X, \lim) ([12]) if for every $x \in X$

$$H_A^\top(x) = \{ \mathcal{F} \in \mathcal{F}_L^s(X) \mid \mathcal{F}_A \in \mathcal{F}_L^s(A) \text{ and } \lim \mathcal{F}(x) = \top \} \neq \emptyset.$$

4. Compactness

A space $(X, \lim) \in |SL-GCS|$ is called *compact* if every stratified L -ultrafilter converges, i.e. if for all $\mathcal{U} \in \mathcal{F}_L^s(X)$ ultra, there is an $x \in X$ such that $\lim \mathcal{U}(x) = \top$.

This definition goes back to [7], where it was stated for stratified L -topological spaces.

Lemma 4.1: *Let $(X, \lim) \in |SL-GCS|$ be compact and $\varphi : (X, \lim) \longrightarrow (X', \lim')$ be continuous and surjective. Then also (X', \lim') is compact.*

Proof: Let $\mathcal{F}' \in \mathcal{F}_L^s(X')$ be a stratified L -ultrafilter. From the surjectivity of φ , we see that $\varphi^{\leftarrow}(\mathcal{F}') \in \mathcal{F}_L^s(X)$. We choose a stratified L -ultrafilter \mathcal{U} such that $\varphi^{\leftarrow}(\mathcal{F}') \leq \mathcal{U}$. Then $\mathcal{F}' = \varphi(\mathcal{U})$, as \mathcal{F}' is maximal. From the compactness of (X, \lim) there is $x \in X$ such that $\lim \mathcal{U}(x) = \top$. Hence, by continuity of φ , we have

$$\lim' \mathcal{F}'(\varphi(x)) = \lim' \varphi(\mathcal{U})(\varphi(x)) \geq \lim \mathcal{U}(x) = \top.$$

□

Theorem 4.2 (Tychonoff): *Let $(X_i, \lim_i) \in |SL-GCS|$ for $i \in J$ and let $(\prod_{i \in J} X_i, \pi\text{-}\lim)$ be the product space. Then*

$$\left(\prod_{i \in J} X_i, \pi\text{-}\lim \right) \text{ is compact} \iff \forall i \in J : (X_i, \lim_i) \text{ is compact}.$$

Proof: As the projections $p_j : (\prod_{i \in J} X_i, \pi\text{-}\lim) \longrightarrow (X_j, \lim_j)$ are continuous and surjective, from the compactness of (X, \lim) we obtain the compactness of the (X_i, \lim_i) . Let, conversely,

all (X_i, \lim_i) be compact and let $\mathcal{U} \in \mathcal{F}_L^s(X)$ be a stratified L -ultrafilter. Then for each $i \in J$ we have that $p_i(\mathcal{U}) \in \mathcal{F}_L^s(X_i)$ is a stratified L -ultrafilter. Thus for each $i \in J$ there is an $x_i \in X_i$ such that $\lim_i p_i(\mathcal{U})(x_i) = \top$. With $x = (x_i) \in \prod_{i \in J} X_i$ we then get

$$\pi\text{-}\lim \mathcal{U}(x) = \bigwedge_{i \in J} \lim_i p_i(\mathcal{U})(x_i) = \top.$$

Consequently, (X, \lim) is compact. \square

5. Richardson compactification

We consider in this section a non-compact stratified L -generalized convergence space (X, \lim) . This means that there is at least one non-convergent stratified L -ultrafilter $\mathcal{U} \in \mathcal{F}_L^s(X)$. We denote $X^* = \{[x] \mid x \in X\} \cup \{\mathcal{V} \in \mathcal{F}_L^s(X) \mid \mathcal{V} \text{ non-convergent stratified } L\text{-ultrafilter}\}$. For $a \in L^X$ we attach an L -set, a^* , on X^* by $a^*(\mathcal{V}) = \mathcal{V}(a)$ ($\mathcal{V} \in X^*$). We denote the stratified L -filters on X^* by Φ, Ψ, \dots in order to distinguish them from the stratified L -filters, $\mathcal{F}, \mathcal{G}, \dots$, on X . For $\Phi \in \mathcal{F}_L^s(X^*)$ we define $\tilde{\Phi}$ by $\tilde{\Phi}(a) = \Phi(a^*)$. Note that we assume (X, \lim) to be non-compact. Of course, if (X, \lim) is compact, then $X^* = \{[x] \mid x \in X\}$ can be identified with X and, similarly, a^* with a and $\tilde{\Phi}$ with Φ in a natural way. The resulting compactification that we are going to construct will then coincide with (X, \lim) .

Lemma 5.1: *If $\Phi \in \mathcal{F}_L^s(X^*)$, then $\tilde{\Phi} \in \mathcal{F}_L^s(X)$. If moreover Φ is a stratified L -ultrafilter, then so is $\tilde{\Phi}$.*

Proof: It is readily checked that for L -sets $a, b \in L^X$ we have $(\top_X)^* = \top_{X^*}$, $(\perp_X)^* = \perp_{X^*}$, $a \leq b$ implies $a^* \leq b^*$ and $(a \wedge b)^* = a^* \wedge b^*$. Moreover, for $\alpha \in L$, we have $\alpha_{X^*} \wedge a^* \leq (\alpha_X \wedge a)^*$. From these relations, the first claim follows directly. To show that together with Φ also $\tilde{\Phi}$ is maximal, we make use of $(a \rightarrow \perp_X)^* = a^* \rightarrow \perp_{X^*}$. This follows as for a stratified L -ultrafilter, $\mathcal{V} \in \mathcal{F}_L^s(X)$, we have, according to the remark after Lemma 2.2,

$$(a \rightarrow \perp_X)^*(\mathcal{V}) = \mathcal{V}(a \rightarrow \perp_X) = \mathcal{V}(a) \rightarrow \perp = a * (\mathcal{V}) \rightarrow \perp = (a^* \rightarrow \perp_{X^*})(\mathcal{V}).$$

Hence

$$\tilde{\Phi}(a) = \Phi(a^*) = \Phi(a^* \rightarrow \perp_{X^*}) \rightarrow \perp = \Phi((a \rightarrow \perp_X)^*) \rightarrow \perp = \tilde{\Phi}(a \rightarrow \perp_X) \rightarrow \perp.$$

and $\tilde{\Phi}$ is a stratified L -ultrafilter by Lemma 2.2. \square

We define now the following mapping. For $\Phi \in \mathcal{F}_L^s(X^*)$ and $\mathcal{V} \in X^*$ we define

$$\begin{aligned} \lim^* \Phi(\mathcal{V}) &= \lim \tilde{\Phi}(x) && \text{if } \mathcal{V} = [x] \\ \lim^* \Phi(\mathcal{V}) &= \begin{cases} \top & \text{if } \mathcal{V} = \tilde{\Phi} \\ \perp & \text{else} \end{cases} && \text{if } \mathcal{V} \text{ is a non-convergent } L\text{-ultrafilter on } X. \end{aligned}$$

Lemma 5.2: *If $(X, \lim) \in |SL-GCS|$ is non-compact, then $(X^*, \lim^*) \in |SL-GCS|$.*

Proof: (L1). Let first $\Phi = [[x]]$ for an $x \in X$. Then $\widetilde{[[x]]}(a) = [[x]](a^*) = a^*([x]) = [x](a)$ and hence

$$\lim^* [[x]]([x]) = \lim x = \top.$$

For $\Phi = [\mathcal{V}]$ with a non-convergent stratified L -ultrafilter \mathcal{V} we have $\widetilde{[\mathcal{V}]}(a) = [\mathcal{V}](a^*) = a^*(\mathcal{V}) = \mathcal{V}(a)$ and hence $\widetilde{[\mathcal{V}]} = \mathcal{V}$. Consequently, $\lim^* \mathcal{V} = \top$.

(L2). If $\Phi \leq \Psi$, then we have $\tilde{\Phi} \leq \tilde{\Psi}$. Hence for $\mathcal{V} = [x]$ we obtain

$$\lim^* \Phi([x]) = \lim \tilde{\Phi}(x) \leq \lim \tilde{\Psi}(x) = \lim^* \Psi([x]).$$

For a non-convergent stratified L -ultrafilter $\mathcal{V} \in \mathcal{F}_L^s(X)$ we have that if $\lim^* \Phi(\mathcal{V}) = \top$, then $\mathcal{V} \leq \tilde{\Phi} \leq \tilde{\Psi}$, and hence $\lim^* \Psi(\mathcal{V}) = \top$. \square

Lemma 5.3: *If $(X, \lim) \in |SL-GCS|$ is non-compact, then (X^*, \lim^*) is compact.*

Proof: Let $\Phi \in \mathcal{F}_L^s(X^*)$ be a stratified L -ultrafilter. Then $\tilde{\Phi}$ is a stratified L -ultrafilter on X . If there is an $x \in X$ such that $\lim \tilde{\Phi}(x) = \top$, then $\lim^* \Phi([x]) = \top$. If there is no such $x \in X$, then $\tilde{\Phi} \in X^*$ and hence $\lim^* \Phi(\tilde{\Phi}) = \top$. Hence every stratified L -ultrafilter on X^* converges and (X^*, \lim^*) is compact. \square

Lemma 5.4: *If $(X, \lim) \in |SL-GCS|$ is a non-compact T2-space, then (X^*, \lim^*) is a T2-space.*

Proof: Let $\mathcal{U}, \mathcal{V} \in X^*$ and assume $\lim^* \Phi(\mathcal{U}) = \lim^* \Phi(\mathcal{V}) = \top$. We distinguish three cases.

Case 1: $\mathcal{U} = [x]$ and $\mathcal{V} = [y]$ with $x, y \in X$. Then $\top = \lim \tilde{\Phi}(x) = \lim \tilde{\Phi}(y)$ and hence by (T2) we conclude $x = y$, i.e. $\mathcal{U} = \mathcal{V}$.

Case 2: \mathcal{U}, \mathcal{V} non-convergent stratified L -ultrafilters on X . Then $\mathcal{U} = \tilde{\Phi} = \mathcal{V}$ by definition of \lim^* .

Case 3: $\mathcal{U} = [x]$ with $x \in X$ and \mathcal{V} a non-convergent stratified L -ultrafilter on X . Then $\mathcal{V} = \tilde{\Phi}$ and from $\lim^* \Phi([x]) = \top$ we conclude $\lim \mathcal{V}(x) = \lim \tilde{\Phi}(x) = \top$. Thus this case cannot occur.

\square

We consider in the sequel the following embedding of X into X^* , $\iota : \begin{cases} X & \longrightarrow & X^* \\ x & \longmapsto & [x] \end{cases}$.

Lemma 5.5: $\iota : (X, \text{lim}) \longrightarrow (X^*, \text{lim}^*)$ is an embedding.

Proof: Clearly the mapping $\iota : x \longmapsto [x]$ is one-to-one. We have to show that $\text{lim}^* \iota(\mathcal{F})(\iota(x)) = \text{lim} \mathcal{F}(x)$ for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and all $x \in X$. By definition of lim^* we have $\text{lim}^* \iota(\mathcal{F})(\iota(x)) = \text{lim}^* \iota(\widetilde{\mathcal{F}})(x)$. So it is sufficient to show that $\widetilde{\iota(\mathcal{F})} = \mathcal{F}$ for $\mathcal{F} \in \mathcal{F}_L^s(X)$. Let $a \in L^X$. We first note that for $x \in X$

$$\iota^\leftarrow(a^*)(x) = a^*(\iota(x)) = a^*([x]) = [x](a) = a(x).$$

Hence $\iota^\leftarrow(a^*) = a$ and we obtain

$$\iota(\widetilde{\mathcal{F}})(a) = \iota(\mathcal{F})(a^*) = \mathcal{F}(\iota^\leftarrow(a^*)) = \mathcal{F}(a),$$

and the proof is complete. \square

Lemma 5.6: $\iota(X)$ is dense in (X^*, lim^*) .

Proof: We have to show that for every $\mathcal{V} \in X^*$,

$$H_{\iota(X)}^\top(\mathcal{V}) = \{\Phi \in \mathcal{F}_L^s(X^*) \mid \Phi_{\iota(X)} \in \mathcal{F}_L^s(X), \text{ and } \text{lim}^* \Phi(\mathcal{V}) = \top\} \neq \emptyset.$$

If $\mathcal{V} = [x]$ with $x \in X$, then clearly $[[x]] \in \mathcal{F}_L^s(X^*)$ and $\text{lim}^* [[x]]([x]) = \top$. Further it is an easy exercise to check that $[[x]]_{\iota(X)} \in \mathcal{F}_L^s(X)$. If $\mathcal{V} \in \mathcal{F}_L^s(X)$ is a non-convergent stratified L -ultrafilter, then $\iota(\mathcal{V}) \in \mathcal{F}_L^s(X^*)$. We have seen above that $\widetilde{\iota(\mathcal{V})} = \mathcal{V}$. Hence $\text{lim}^* \iota(\mathcal{V})(\mathcal{V}) = \top$ and it only remains to show that $\iota(\mathcal{V})_{\iota(X)} \in \mathcal{F}_L^s(X)$. Let $b \in L^{X^*}$ such that $b|_{\iota(X)} = \perp_{\iota(X)}$. Then $\iota(\mathcal{V})(b) = \mathcal{V}(\iota^\leftarrow(b)) = \mathcal{V}(\perp_X) = \perp$ and the proof is complete. \square

We call a pair $((X^+, \text{lim}^+), k)$ of a compact space (X^+, lim^+) and a dense embedding $k : (X, \text{lim}) \longrightarrow (X^+, \text{lim}^+)$, a *compactification* of (X, lim) . Hence $((X^*, \text{lim}^*), \iota)$ is a compactification of (X, lim) . In case (X, lim) is a T2-space then also the compactification (X^*, lim^*) is a T2-space. We call (X^*, lim^*) the *lattice-valued Richardson compactification* of (X, lim) , as it coincides in case $L = \{\perp, \top\}$ with the classical Richardson compactification of a convergence space [15]. A nice feature of this compactification is the following property.

Lemma 5.7: If $(X, \text{lim}) \in |SL\text{-}GCS|$ is a non-compact T2-space and $(\overline{X}, \overline{\text{lim}}) \in |SL\text{-}GCS|$ is a compact and regular T2-space and $\varphi : (X, \text{lim}) \longrightarrow (\overline{X}, \overline{\text{lim}})$ is continuous, then there is a unique continuous mapping $\varphi^* : (X^*, \text{lim}^*) \longrightarrow (\overline{X}, \overline{\text{lim}})$ such that $\varphi = \varphi^* \circ \iota$.

Proof: We define the mapping φ^* as follows. For $\mathcal{V} \in X^*$ we know that $\varphi(\mathcal{V})$ is a stratified L -ultrafilter on \overline{X} . From the compactness and the (T2) axiom of $(\overline{X}, \overline{\text{lim}})$ there is a unique $\overline{x} \in \overline{X}$ such that $\overline{\text{lim}} \varphi(\mathcal{V})(\overline{x}) = \top$. So we can define $\varphi^*(\mathcal{V}) = \overline{x}$. Clearly, $\varphi^* \circ \iota(x) = \varphi^*([x]) = \varphi(x)$ as

$\overline{\lim} \varphi([x])(\varphi(x)) = \overline{\lim} \varphi(x) = \top$. In order to show the continuity of φ^* , we provide the following lemma.

Lemma 5.8: *In the situation of Lemma 5.7, let $\Phi \in \mathcal{F}_L^s(X^*)$. Then $\overline{\lim} \varphi(\tilde{\Phi}) \leq \overline{\lim} \varphi^*(\Phi)$.*

Proof of Lemma 5.8: We use the notation of the regularity axiom (LR). Choose $J = X^*$ and $\psi = \varphi^* : J \longrightarrow \overline{X}$. For $\mathcal{V} \in J$ we put $\mathcal{F}_{\mathcal{V}} = \varphi(\mathcal{V}) \in \mathcal{F}_L^s(X)$. Then for $c \in L^{\overline{X}}$ we find

$$\mathcal{F}_{(\cdot)}(c)(\mathcal{V}) = \varphi(\mathcal{V})(c) = \mathcal{V}(\varphi^{\leftarrow}(c)) = (\varphi^{\leftarrow}(c))^*(\mathcal{V}).$$

Hence, for $\Phi \in \mathcal{F}_L^s(X^*)$,

$$\Phi(\mathcal{F}_{(\cdot)})(c) = \Phi(\mathcal{F}_{(\cdot)}(c)) = \Phi((\varphi^{\leftarrow}(c))^*) = \tilde{\Phi}(\varphi^{\leftarrow}(c)) = \varphi(\tilde{\Phi})(c).$$

So we have $\Phi(\mathcal{F}_{(\cdot)}) = \varphi(\tilde{\Phi})$. Noting further that, by definition of φ^* , for $\mathcal{V} \in X^*$ we have $\overline{\lim} \varphi(\mathcal{V})(\varphi^*(\mathcal{V})) = \top$, the regularity axiom (LR) finally yields

$$\overline{\lim} \varphi(\tilde{\Phi})(\overline{x}) = \bigwedge_{\mathcal{V} \in X^*} \overline{\lim} \varphi(\mathcal{V})(\varphi^*(\mathcal{V})) \wedge \overline{\lim} \varphi(\tilde{\Phi})(\overline{x}) \leq \overline{\lim} \varphi^*(\Phi)(\overline{x}).$$

□

We return back to the proof of Lemma 5.7. Let $\Phi \in \mathcal{F}_L^s(X^*)$ and $\mathcal{V} \in X^*$. If $\mathcal{V} = [x]$, then

$$\lim^* \Phi([x]) = \lim \tilde{\Phi}(x) \leq \overline{\lim} \varphi(\tilde{\Phi})(\varphi(x)) \leq \overline{\lim} \varphi^*(\Phi)(\varphi^*([x])).$$

If \mathcal{V} is a non-convergent L -ultrafilter, then if $\top = \lim^* \Phi(\mathcal{V})$ we have $\mathcal{V} = \tilde{\Phi}$. But then

$$\top = \overline{\lim} \varphi(\mathcal{V})(\varphi^*(\mathcal{V})) = \overline{\lim} \varphi(\tilde{\Phi})(\varphi^*(\mathcal{V})) \leq \overline{\lim} \varphi^*(\Phi)(\varphi^*(\mathcal{V})).$$

Hence $\varphi^* : (X^*, \lim^*) \longrightarrow (\overline{X}, \overline{\lim})$ is continuous. Assume finally a further continuous mapping $\tilde{\varphi} : X^* \longrightarrow \overline{X}$ with $\varphi = \tilde{\varphi} \circ \iota$. Then for $x \in X$ we have $\tilde{\varphi}([x]) = \varphi(x) = \varphi^*([x])$. For a non-convergent stratified L -ultrafilter, $\mathcal{V} \in X^*$, we have that $\lim^* \iota(\mathcal{V})(\mathcal{V}) = \top$. Hence both

$$\overline{\lim} \varphi(\mathcal{V})(\varphi^*(\mathcal{V})) = \overline{\lim} \varphi^*(\iota(\mathcal{V}))(\varphi^*(\mathcal{V})) = \top = \overline{\lim} \tilde{\varphi}(\iota(\mathcal{V}))(\tilde{\varphi}(\mathcal{V})) = \overline{\lim} \varphi(\mathcal{V})(\tilde{\varphi}(\mathcal{V})).$$

Hence, $(\overline{X}, \overline{\lim})$ being a T2-space, $\tilde{\varphi}(\mathcal{V}) = \varphi^*(\mathcal{V})$ and φ^* is unique. □

Having this "universal property", it is a natural question, if we restrict to non-compact, regular T2-spaces, if the Richardson compactification is also a categorical compactification in the sense of [16]. Unfortunately the answer to this question is negative, as in general (X^*, \lim^*) is not regular even if (X, \lim) is so (see e.g. [4] for the case $L = \{\perp, \top\}$). We give a precise criterion for the case that L is a complete Boolean algebra in the next section. This criterion generalizes a classical result of Gazik [4].

6. The Boolean case: Regularity of the Richardson compactification

In this section, we will always assume that L is a complete Boolean algebra. Further, $(X, \lim) \in |SL-GCS|$ is a non-compact T2-space and (X^*, \lim^*) the Richardson compactification of (X, \lim) , and $\iota : x \mapsto [x]$ the embedding of X into X^* .

Lemma 6.1: *Let $\mathcal{V} \in \mathcal{F}_L^s(X)$ be a non-convergent stratified L -ultrafilter. If (X^*, \lim^*) is regular, then for every $\beta \in L \setminus \{\perp\}$ we have $\mathcal{V} \leq \overline{\mathcal{V}}^\beta$.*

Proof: Let \mathcal{V} be a non-convergent stratified L -ultrafilter on X and $\beta \in L \setminus \{\perp\}$. Because $\widetilde{\iota(\mathcal{V})} = \mathcal{V}$ we have $\lim^* \iota(\mathcal{V})(\mathcal{V}) = \top$. Hence by regularity also $\lim^* \overline{\iota(\mathcal{V})}^\beta(\mathcal{V}) \geq \beta$ and as $\beta \neq \perp$ even $\lim^* \overline{\iota(\mathcal{V})}^\beta(\mathcal{V}) = \top$. This implies $\mathcal{V} = \overline{\iota(\mathcal{V})}^\beta$. Hence for $a \in L^X$

$$\begin{aligned} \mathcal{V}(a) &= \overline{\iota(\mathcal{V})}^\beta(a) = \overline{\iota(\mathcal{V})}^\beta(a^*) \\ &= \bigvee \{ \iota(\mathcal{V})(c) \mid \lim^* \Phi(x^*) \geq \beta \implies \Phi(c) \leq a^*(x^*) \} \\ &= \bigvee \{ \mathcal{V}(\iota^\leftarrow(c)) \mid \lim^* \Phi(x^*) \geq \beta \implies \Phi(c) \leq a^*(x^*) \}. \end{aligned}$$

Let now $\mathcal{G} \in \mathcal{F}_L^s(X)$ such that $\lim \mathcal{G}(x) \geq \beta$. If $c \in \{d \mid \lim^* \Phi(x^*) \geq \beta \implies \Phi(d) \leq a^*(x^*)\}$, then with $\Phi = \iota(\mathcal{G})$ we have $\lim^* \iota(\mathcal{G})([x]) = \lim \widetilde{\iota(\mathcal{G})}(x) = \lim \mathcal{G}(x) \geq \beta$. Hence $\mathcal{G}(\iota^\leftarrow(c)) = \iota(\mathcal{G})(c) = \Phi(c) \leq a^*([x]) = a(x)$. Therefore

$$\begin{aligned} \mathcal{V}(a) &\leq \bigvee \{ \mathcal{V}(\iota^\leftarrow(c)) \mid \lim \mathcal{G}(x) \geq \beta \implies \mathcal{G}(\iota^\leftarrow(c)) \leq a(x) \} \\ &\leq \bigvee \{ \mathcal{V}(g) \mid \lim \mathcal{G}(x) \geq \beta \implies \mathcal{G}(g) \leq a(x) \} \\ &= \overline{\mathcal{V}}^\beta(a). \end{aligned}$$

□

Note that, as \mathcal{V} is maximal, we even have $\mathcal{V} = \overline{\mathcal{V}}^\beta$ for all $\beta \in L \setminus \{\perp\}$. Note further that with (X^*, \lim^*) also (X, \lim) is regular, see [12].

Lemma 6.2: *Let $\Phi \in \mathcal{F}_L^s(X^*)$. Then for all $\beta \in L \setminus \{\perp\}$ we have $\widetilde{\overline{\Phi}}^\beta \geq \overline{\widetilde{\Phi}}^\beta$.*

Proof: Let $a \in L^X$. Then

$$\begin{aligned} \widetilde{\overline{\Phi}}^\beta(a) &= \bigvee \{ \widetilde{\Phi}(f) \mid \lim \mathcal{G}(x) \geq \beta \implies \mathcal{G}(f) \leq a(x) \} \\ &= \bigvee \{ \Phi(f^*) \mid \lim \mathcal{G}(x) \geq \beta \implies \mathcal{G}(f) \leq a(x) \}. \end{aligned}$$

Let $f \in \{g \in L^X \mid \lim \mathcal{G}(x) \geq \beta \implies \mathcal{G}(g) \leq a(x)\}$. Note that then $f \leq a$. Let now $\Psi \in \mathcal{F}_L^s(X^*)$ such that $\lim^* \Psi(x^*) \geq \beta$. If $x^* = [x]$ with $x \in X$, then $\lim \widetilde{\Psi}(x) \geq \beta$ and hence, by the choice of f , $\Psi(f^*) = \widetilde{\Psi}(f) \leq a(x) = a^*(x^*)$. If $x^* = \mathcal{V}$, with a non-convergent stratified

L -ultrafilter on X , then $\tilde{\Psi} = \mathcal{V}$ and therefore

$$\Psi(f^*) = \tilde{\Psi}(f) = \mathcal{V}(f) \leq \mathcal{V}(a) = a^*(\mathcal{V}) = a^*(x^*).$$

Therefore

$$\begin{aligned} \overline{\Phi}^\beta(a) &\leq \bigvee \{ \Phi(f^*) \mid \lim^* \Psi(x^*) \geq \beta \implies \Psi(f^*) \leq a^*(x^*) \} \\ &\leq \bigvee \{ \Phi(g) \mid \lim^* \Psi(x^*) \geq \beta \implies \Psi(g) \leq a^*(x^*) \} \\ &= \overline{\Phi}^\beta(a^*) = \widetilde{\overline{\Phi}^\beta}(a), \end{aligned}$$

and the proof is complete. \square

Lemma 6.3: *If (X, \lim) is a non-compact regular $T2$ -space and for all non-convergent stratified L -ultrafilters $\mathcal{V} \in \mathcal{F}_L^s(X)$ and all $\beta \in L \setminus \{\perp\}$ we have $\mathcal{V} \leq \overline{\mathcal{V}}^\beta$, then (X^*, \lim^*) is regular.*

Proof: Let $\lim^* \Phi(x^*) \geq \alpha$. We may assume $\alpha, \beta \in L \setminus \{\perp\}$. If $x^* = \mathcal{V}$ with a non-convergent stratified L -ultrafilter on X , then $\lim^* \Phi(\mathcal{V}) = \top$ and hence $\tilde{\Phi} = \mathcal{V}$. But then, with Lemma 6.2,

$$\widetilde{\overline{\Phi}^\beta} \geq \widetilde{\tilde{\Phi}}^\beta = \overline{\mathcal{V}}^\beta \geq \mathcal{V}$$

and therefore $\lim^* \overline{\Phi}^\beta(\mathcal{V}) = \top \geq \alpha \wedge \beta$. If $x^* = [x]$ with $x \in X$, then $\lim \tilde{\Phi}(x) \geq \alpha$ and hence, by regularity of (X, \lim) and Lemma 6.2

$$\lim^* \overline{\Phi}^\beta(x^*) = \lim \widetilde{\overline{\Phi}^\beta}(x) \geq \lim \widetilde{\tilde{\Phi}}^\beta(x) \geq \alpha \wedge \beta.$$

Hence (X^*, \lim^*) is regular. \square

We collect the Lemmas 6.1 and 6.3.

Theorem 6.4: *Let L be a complete Boolean algebra and let (X, \lim) be a non-compact, regular $T2$ -space. Then the following are equivalent:*

- (i) (X^*, \lim^*) is regular.
- (ii) $\mathcal{V} = \overline{\mathcal{V}}^\beta$ for every $\beta \in L \setminus \{\perp\}$ and every non-convergent stratified L -ultrafilter $\mathcal{V} \in \mathcal{V}_L^s(X)$.

7. Conclusions

We defined a notion of compactness for lattice-valued convergence spaces. This notion has good properties and generalizes a definition from [7] for lattice-valued topological spaces. It was shown that a Tychonoff Theorem is valid. Moreover, a compactification for any non-compact lattice-valued convergence space is constructed which has a nice "universal" property. In the sense of

[16], this compactification is of "topological type", i.e. it is a pair consisting of a compact space and a "dense homeomorphic embedding". If we restrict to the subcategory of regular T2-spaces, then our compactification is unfortunately not a "categorical compactification", but we were able to give, at least for the case that L is a complete Boolean algebra, a precise criterion when the regularity of the original space implies the regularity of the compactification. This is in line with the "classical situation" where $L = \{0, 1\}$ (see e.g. [4, 13]). Several questions are still open and will be topics for future research, e.g.:

- When is the Richardson compactification the largest T2-compactification of a given T2-space?
- Construct smallest T2/regular compactifications for a lattice-valued convergence space.
- In case that the space is a lattice-valued topological space, when does the Richardson compactification coincide with a Stone-Čech compactification (see e.g. [16]).
- Replace the T2-axiom by a suitable notion of sobriety (again see [16] for a discussion).
- If (X, \lim) is in some subcategory of $SL-GCS$, will also the Richardson compactification (X^*, \lim^*) be in this subcategory?

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