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Lattice-valued continuous convergence is induced by a lattice-valued uniform convergence structure

1. Introduction

The category SL-GCS of stratified L-generalized convergence spaces [6],[7] is topological over
SET and contains the category SL-TOP of stratified L-topological spaces as reflective sub-
category [6]. The main advantage of SL-GCS over SL-TOP is, from a structural point of
view, the existence of function space structures [6]. This makes SL-GCS cartesian closed. The
corresponding function space structure on the set $C(X, Y)$ of morphisms is c-lim, the structure
of L-continuous convergence.

Another category, SL-UCS, of stratified L-uniform convergence spaces [8] generalizes in a simi-
lar way the category of stratified L-uniform spaces [3], SL-$UNIF$. Also SL-UCS is topological
over SET, cartesian closed and contains SL-$UNIF$ as reflective subcategory. Each object
$(X, \Lambda) \in |SL$-$UCS|$ induces an object $(X, \lim(\Lambda)) \in |SL$-$\lim|$ (see [8]). In this paper we will
show that in the case that $(Y, \Lambda) \in |SL$-$UCS|$ then $(C(X, Y), c$-$\lim) \in |SL$-$UCS|$. To this end,
we generalize a classical result of Cook and Fischer [2] to the lattice-valued case. It follows from
this that the subcategory of all L-limit uniformizable spaces is cartesian closed.

2. Preliminaries

We consider in this paper complete lattices L where finite meets distribute over arbitrary joins.
This means that for all α, β_ι $(\iota \in I)$ we have $\alpha \wedge \bigvee_{\iota \in I} \beta_\iota = \bigvee_{\iota \in I} (\alpha \wedge \beta_\iota)$. Such lattices are called
frames or *complete Heyting algebras* [9]. The bottom (resp. top) element of L is denoted by $\perp$
(resp. $\top$). It is then possible to define an *implication* $\alpha \to \_$ as the right adjoint to $\alpha \wedge \_$ by

$$\alpha \to \beta = \bigvee \{\lambda \in L \ : \ \alpha \wedge \lambda \le \beta\}.$$

So we have $\delta \le \alpha \to \beta$ if and only if $\alpha \wedge \delta \le \beta$. Moreover, this implication, considered as
mapping $L \times L \longrightarrow L$, is order reversing in the first argument and order preserving in the
second argument. For further basic properties of this operator we refer to our earlier papers

[6],[7],[8] as well as to the references given there. The lattice operations are extended pointwise from L to $L^X = \{a : X \longrightarrow L\}$, the set of all L-sets on X. We especially denote the constant L-set on X with value $\alpha \in L$ by α_X.

A *stratified L-filter* $\mathcal{F}$ on X [4],[5] is a mapping $\mathcal{F} : L^X \longrightarrow L$ with the properties

(F1) $\mathcal{F}(\top_X) = \top, \qquad \mathcal{F}(\bot_X) = \bot$

(F2) $f \leq g \qquad \Longrightarrow \qquad \mathcal{F}(f) \leq \mathcal{F}(g)$

(F3) $\mathcal{F}(f) \wedge \mathcal{F}(g) \leq \mathcal{F}(f \wedge g)$

(Fs) $\alpha \wedge \mathcal{F}(f) \leq \mathcal{F}(\alpha_X \wedge f)$,

for all $f, g \in L^X$, $x \in X$, $\alpha \in L$. The set of all stratified L-filters on X is denoted by $\mathcal{F}_L^s(X)$. An example of a stratified L-filter is the *point L-filter* $[x]$ defined by $[x](a) = a(x)$ (see e.g. [5]). An order on $\mathcal{F}_L^s(X)$ is defined pointwise by $\mathcal{F} \leq \mathcal{G}$ if for all $a \in L^X$ we have $\mathcal{F}(a) \leq \mathcal{G}(a)$ (see [5]). For a mapping $f : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$ we define $f(\mathcal{F}) \in \mathcal{F}_L^s(Y)$ by $f(\mathcal{F})(b) = \mathcal{F}(f^{\leftarrow}(b))$ (see [5]) where $f^{\leftarrow}(b) = b \circ f$. The *meet* $\mathcal{F} \wedge \mathcal{G}$ of two L-filters $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$ is defined by $(\mathcal{F} \wedge \mathcal{G})(a) = \mathcal{F}(a) \wedge \mathcal{G}(a)$. Obviously $\mathcal{F} \wedge \mathcal{G} \in \mathcal{F}_L^s(X)$ and it holds $f(\mathcal{F} \wedge \mathcal{G}) = f(\mathcal{F}) \wedge f(\mathcal{G})$. Also we have $f([x]) = [f(x)]$.

Of special interest for us are stratified L-filters on products $X \times Y$. The first examples of such L-filters are *products of L-filters* $\mathcal{F} \in \mathcal{F}_L^s(X)$, $\mathcal{G} \in \mathcal{F}_L^s(Y)$ defined by

$$\mathcal{F} \times \mathcal{G}(a) = \bigvee \{\mathcal{F}(a_1) \wedge \mathcal{G}(a_2) \mid a_1 \times a_2 \leq a\}$$

for $a \in L^{X \times Y}$ (see [6]), where for $a_1 \in L^X$ and $a_2 \in L^Y$ it is defined $a_1 \times a_2(x, y) = a_1(x) \wedge a_2(y)$. If we denote the projections from $X \times Y$ onto X, resp. onto Y, by π_X, resp. π_Y, then for $\mathcal{H} \in \mathcal{F}_L^s(X \times Y)$ we have $\pi_X(\mathcal{H}) \times \pi_Y(\mathcal{H}) \leq \mathcal{H}$ and for $\mathcal{F} \in \mathcal{F}_L^s(X)$, $\mathcal{G} \in \mathcal{F}_L^s(Y)$ we have $\mathcal{F} \leq \pi_X(\mathcal{F} \times \mathcal{G})$ and $\mathcal{G} \leq \pi_Y(\mathcal{F} \times \mathcal{G})$ (see [6]). In other words: the mapping $\_ \times \_ : \mathcal{F}_L^s(X) \times \mathcal{F}_L^s(Y) \longrightarrow \mathcal{F}_L^s(X \times Y)$ is left adjoint to the mapping $\pi = (\pi_X, \pi_Y) : \mathcal{F}_L^s(X \times Y) \longrightarrow \mathcal{F}_L^s(X) \times \mathcal{F}_L^s(Y)$. Further we have

$$\mathcal{F} \times (\mathcal{G} \wedge \mathcal{H}) = (\mathcal{F} \times \mathcal{G}) \wedge (\mathcal{F} \times \mathcal{H})$$

(see [8]).

Lemma 2.1: *For $x \in X$ and $y \in Y$ we have $[(x, y)] = [x] \times [y]$.*

Proof: The inequality $[x] \times [y] \leq [(x, y)]$ follows from $\pi_X([(x, y)]) = [\pi_X(x, y)] = [x]$ and, similarly, $\pi_Y([(x, y)]) = [y]$ as $\_ \times \_$ is left adjoint to $\pi = (\pi_X, \pi_Y)$. On the other hand, for $(x, y) \in X \times Y$ and $a \in L^{X \times Y}$ we define the L-sets

$$a_x : \begin{cases} Y & \longrightarrow & L \\ z & \longmapsto & a(x, z) \end{cases} \qquad \text{and} \qquad a_y : \begin{cases} X & \longrightarrow & L \\ z & \longmapsto & a(z, y) \end{cases} .$$

Then by definition $[x] \times [y](a) \geq [x](a_y) \wedge [y](a_x) = a(x,y) = [(x,y)](a)$ and the proof is completed. $\square$

If Y^X denotes the set of mappings from X to Y, the evaluation mapping is defined as usual by

$$ev : Y^X \times X \longrightarrow Y, \ (f, x) \longmapsto f(x).$$

We obtain as a Corollary of Lemma 2.1:

Corollary: We have for $f : X \longrightarrow Y$ and $x \in X$ that $ev([f] \times [x]) = [f(x)]$.

It is further a simple exercise to prove the following lemma.

Lemma 2.2: Let $\mathcal{F} \in \mathcal{F}_L^s(Y^X)$, $\mathcal{G} \in \mathcal{F}_L^s(X)$, $x \in X$ and $f : X \longrightarrow Y$. Then

$$ev(\mathcal{F} \times (\mathcal{G} \wedge [x])) \times ev([f] \times (\mathcal{G} \wedge [x]))$$
$$= \ (ev(\mathcal{F} \times \mathcal{G}) \times ev([f] \times \mathcal{G})) \wedge (ev(\mathcal{F} \times \mathcal{G}) \times ev([f]) \times [x])) \wedge$$
$$\wedge (ev(\mathcal{F} \times [x]) \times ev([f] \times \mathcal{G})) \wedge (ev(\mathcal{F} \times [x]) \times ev([f] \times [x])).$$

The following bijection

$$\eta : \begin{cases} (X \times X) \times (Y \times Y) & \longrightarrow & (X \times Y) \times (X \times Y) \\ ((x_1, x_2), (y_1, y_2)) & \longmapsto & ((x_1, y_1), (x_2, y_2)) \end{cases}$$

plays a prominent role later. The proofs of the next two lemmas are straightforward and are therefore again left to the reader.

Lemma 2.3: Let $a, b \in L^X$, $c, d \in L^Y$. Then

$$\eta((a \times b) \times (c \times d)) = (a \times c) \times (b \times d).$$

We deduce from this

Lemma 2.4: Let $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $\mathcal{H}, \mathcal{K} \in \mathcal{F}_L^s(Y)$. Then

$$\eta((\mathcal{F} \times \mathcal{G}) \times (\mathcal{H} \times \mathcal{K})) = (\mathcal{F} \times \mathcal{H}) \times (\mathcal{G} \times \mathcal{K}).$$

From stratified L-filters on $X \times X$ we construct new ones with the following two constructions [8]. Let $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$. Then $\mathcal{F}^{-1} \in \mathcal{F}_L^s(X \times X)$ is defined by

$$\mathcal{F}^{-1}(a) = \mathcal{F}(a^{-1}).$$

Here, for $a \in L^{X \times X}$, it is defined $a^{-1}(x, y) = a(y, x)$. Then $(\mathcal{F} \times \mathcal{G})^{-1} = \mathcal{G} \times \mathcal{F}$ (see [8]). If $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X \times X)$ we define further $\mathcal{F} \circ \mathcal{G}$ by

$$\mathcal{F} \circ \mathcal{G}(a) = \bigvee \{\mathcal{F}(f) \wedge \mathcal{G}(g) \mid f \circ g \leq a\} \qquad (a \in L^{X \times X}).$$

Here, for $f, g \in L^{X \times X}$, it is defined $f \circ g(x, y) = \bigvee \{f(x, z) \wedge g(z, y) \mid z \in X\}$. Then $\mathcal{F} \circ \mathcal{G} \in \mathcal{F}_L^s(X \times X)$ if and only if $f \circ g = \perp_{X \times X}$ implies $\mathcal{F}(f) \wedge \mathcal{G}(g) = \perp$ (see [8]).

Lemma 2.5: Let $\mathcal{G} \in \mathcal{F}_L^s(X)$. Then $(\mathcal{G} \times \mathcal{G}) \circ (\mathcal{G} \times \mathcal{G}) \in \mathcal{F}_L^s(X \times X)$ and $(\mathcal{G} \times \mathcal{G}) \circ (\mathcal{G} \times \mathcal{G}) \geq \mathcal{G} \times \mathcal{G}$.

Proof: That $(\mathcal{G} \times \mathcal{G}) \circ (\mathcal{G} \times \mathcal{G}) \in \mathcal{F}_L^s(X \times X)$ follows from the the following observation. Let $a, b \in L^{X \times X}$. If $a \circ b = \perp_{X \times X}$, then for all $x, y \in X$ we have

$$\perp = \bigvee_{z \in X} a(x, z) \wedge b(z, y) = \bigvee_{z \in X} a^{-1}(z, x) \wedge b(z, y).$$

Taking $x = y$ we conclude

$$\perp = \bigvee_{z \in X} (a^{-1} \wedge b)(z, x).$$

Hence for all $x, z \in X$ we have $a^{-1} \wedge b(z, x) = \perp$ and this means $a^{-1} \wedge b = \perp_{X \times X}$. Hence

$$\begin{aligned}
(\mathcal{G} \times \mathcal{G})(a) \wedge (\mathcal{G} \times \mathcal{G})(b) &= (\mathcal{G} \times \mathcal{G})^{-1}(a) \wedge (\mathcal{G} \times \mathcal{G})(b) \\
&= (\mathcal{G} \times \mathcal{G})(a^{-1}) \wedge (\mathcal{G} \times \mathcal{G})(b) \\
&\leq (\mathcal{G} \times \mathcal{G})(a^{-1} \wedge b) = \perp,
\end{aligned}$$

as $\mathcal{G} \times \mathcal{G}$ is a stratified L-filter. For the inequality we remark that trivially we have for $f_1, f_2, g_1, g_2 \in L^X$

$$(f_1 \times f_2) \circ (g_1 \times g_2) \leq f_1 \times g_2.$$

Hence

$$\begin{aligned}
(\mathcal{G} \times \mathcal{G}) \circ (\mathcal{G} \times \mathcal{G})(a) &= \bigvee_{(f_1 \times f_2) \circ (g_1 \times g_2) \leq a} \mathcal{G}(f_1) \wedge \mathcal{G}(f_2) \wedge \mathcal{G}(g_1) \wedge \mathcal{G}(g_2) \\
&\geq \bigvee_{f_1, g_2 \text{ s.t. } f_1 \times g_2 = (f_1 \times 1_X) \circ (1_X \times g_2) \leq a} \mathcal{G}(f_1) \wedge \mathcal{G}(g_2) \\
&= \mathcal{G} \times \mathcal{G}(a). \qquad \square
\end{aligned}$$

A somewhat related result is the following.

Lemma 2.6: Let $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X$. Then $(\mathcal{F} \times [x]) \circ ([x] \times \mathcal{F}) \leq \mathcal{F} \times \mathcal{F}$ and consequently $(\mathcal{F} \times [x]) \circ ([x] \times \mathcal{F}) \in \mathcal{F}_L^s(X \times X)$.

Proof: By definition for $a \in L^{X \times X}$

$$(\mathcal{F} \times [x]) \circ ([x] \times \mathcal{F})(a) = \bigvee_{a_1 \circ a_2 \leq a} ((\mathcal{F} \times [x])(a_1) \wedge (([x] \times \mathcal{F})(a_2)).$$

Now if $a_1 \circ a_2 \leq a$ it follows, similar to the proof of Lemma 2.5, that for all $y, z \in X$

$$a_1(y, x) \wedge a_2^{-1}(z, x) \leq a(y, z)$$

i.e. that $a_1(\cdot, x) \times a_2^{-1}(\cdot, x) \leq a$. Further we find

$$(\mathcal{F} \times [x])(a_1) = \bigvee_{b_1 \times c_1 \leq a_1} \mathcal{F}(b_1) \wedge c_1(x).$$

As $\mathcal{F}$ is a stratified L-filter this is

$$\leq \bigvee_{b_1 \times c_1 \leq a_1} \mathcal{F}(b_1 \wedge c_1(x)) \leq \mathcal{F}(a_1(\cdot, x)).$$

Similarly we find $([x] \times \mathcal{F})(a_2) \leq \mathcal{F}(a_2^{-1}(\cdot, x))$. Hence

$$\begin{aligned}
(\mathcal{F} \times [x]) \circ ([x] \times \mathcal{F})(a) &\leq \bigvee_{a_1(\cdot, x) \times a_2^{-1}(\cdot, x) \leq a} \mathcal{F}(a_1(\cdot, x)) \wedge \mathcal{F}(a_2^{-1}(\cdot, x)) \\
&\leq \bigvee_{d_1 \times d_2 \leq a} \mathcal{F}(d_1) \wedge \mathcal{F}(d_2) = (\mathcal{F} \times \mathcal{F})(a).
\end{aligned}$$

From this inequality it also follows that $(\mathcal{F} \times [x]) \circ ([x] \times \mathcal{F}) \in \mathcal{F}_L^s(X \times X)$, as $\mathcal{F} \times \mathcal{F}$ is a stratified L-filter. $\square$

3. Stratified lattice-valued limit spaces and stratified lattice-valued uniform convergence spaces

A *stratified L-limit space* $(X, \lim)$ [7] is a set X together with a limit map $\lim : \mathcal{F}_L^s(X) \longrightarrow L^X$ satisfying the axioms

(L1) $\lim[x](x) = \top$

(L2) $\mathcal{F} \leq \mathcal{G} \quad \Longrightarrow \quad \lim \mathcal{F} \leq \lim \mathcal{G}$

(L3) $\lim \mathcal{F} \wedge \lim \mathcal{G} \leq \lim(\mathcal{F} \wedge \mathcal{G})$

for all $x \in X$ and all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$. A limit preserving mapping $f : (X, \lim) \longrightarrow (X', \lim')$ between two stratified L-limit spaces $(X, \lim), (X', \lim')$ is called *continuous* [7]. Here, "limit preserving" means that for all $x \in X$ and for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have

$$\lim \mathcal{F}(x) \leq \lim' f(\mathcal{F})(f(x)).$$

The category SL-LIM has as objects the stratified L-limit spaces and as morphisms the continuous mappings. It is a reflective subcategory of SL-GCS, the category of L-generalized convergence spaces mentioned in the introduction ([6],[7]). Like this category, SL-LIM is well-fibred and topological over SET. The initial structures are defined as follows. Given a source

$$(f_\iota : X \longrightarrow (X_\iota, \lim_\iota))_{\iota_i n I}$$

the unique limit map on X making all the f_ι continuous is given by

$$\lim \mathcal{F}(x) = \bigwedge_{\iota \in I} \lim_\iota f_\iota(\mathcal{F})(f_\iota(x)),$$

for $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X$ (see [6],[7]). Especially, SL-LIM has (finite) products (taking as $X = \prod X_\iota$ the cartesian product of the X_ι and as the $f_\iota = \pi_\iota$ the projection mappings). Moreover, SL-LIM is cartesian closed [7]. A category $\mathbf{A}$ is called *cartesian closed* if it has finite products and if for each pair of objects (A, B) there exists a *power object* B^A and an *evaluation morphism* $ev : B^A \times A \longrightarrow B$ with the property, that for each morphisms $f : C \times A \longrightarrow B$ there is a unique *exponential morphism* $\hat{f} : C \longrightarrow B^A$ such that $ev \circ (\hat{f} \times id_A) = f$ [1]. In our case, being cartesian closed is equivalent to having function spaces [1]: Given $(X, \lim), (X', \lim') \in |SL$-$LIM|$ we can choose as power object the set $C(X, X') = \{f : (X, \lim) \longrightarrow (X', \lim') \mid f \text{ continuous}\}$ of all continuous mappings from X to X' with the function space structure on $C(X, X')$ defined by

$$c\text{-}\lim \mathcal{F}(f) = \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} \left(\lim \mathcal{G}(x) \to \lim' ev(\mathcal{F} \times \mathcal{G})(f(x)) \right).$$

Here $\mathcal{F} \in \mathcal{F}_L^s(C(X, X'))$, $f \in C(X, X')$ and $ev : (f, x) \longmapsto f(x)$ is the evaluation mapping restricted on $C(X, Y) \times X$ (see [7]).

In a similar way as stratified L-limit spaces generalize stratified L-topological spaces, stratified L-uniform convergence spaces generalize stratified L-uniform spaces ([3],[8]). They are defined as follows. A *stratified L-uniform convergence space* (X, Λ) [8] is a set X together with a mapping $\Lambda : \mathcal{F}_L^s(X \times X) \longrightarrow L$ satisfying the axioms

(UC1) $\Lambda([x] \times [x]) = \top$

(UC2) $\mathcal{F} \leq \mathcal{G} \quad \implies \quad \Lambda(\mathcal{F}) \leq \Lambda(\mathcal{G})$

(UC3) $\Lambda(\mathcal{F}) \leq \Lambda(\mathcal{F}^{-1})$

(UC4) $\Lambda(\mathcal{F}) \wedge \Lambda(\mathcal{G}) \leq \Lambda(\mathcal{F} \wedge \mathcal{G})$

(UC5) $\Lambda(\mathcal{F}) \wedge \Lambda(\mathcal{G}) \leq \Lambda(\mathcal{F} \circ \mathcal{G})$ whenever $\mathcal{F} \circ \mathcal{G} \in \mathcal{F}_L^s(X \times X)$,

for all $x \in X$ and for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X \times X)$. A mapping $f : (X, \Lambda) \longrightarrow (X', \Lambda')$ between two stratified L-uniform convergence spaces $(X, \Lambda), (X', \Lambda')$ is called *uniformly continuous* [8] if for all $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$ we have

$$\Lambda(\mathcal{F}) \leq \Lambda'((f \times f)(\mathcal{F})).$$

The category $SL\text{-}UCS$ has as objects the stratified L-uniform convergence spaces and as morphisms the uniformly continuous mappings [8]. $SL\text{-}UCS$ is topological over SET. Initial structures are defined as follows [8]. Given a source

$$(f_\iota : X \longrightarrow (X_\iota, \Lambda_\iota))_{\iota \in I}$$

the unique structure, Λ, on X making all the f_ι uniformly continuous is given by

$$\Lambda(\mathcal{F}) = \bigwedge_{\iota \in I} \Lambda_\iota((f_\iota \times f_\iota)(\mathcal{F})).$$

Especially also $SL\text{-}UCS$ has (finite) products. Also this category is cartesian closed [8]. The function space structure on $UC = UC(X, X') = \{f : (X, \Lambda) \longrightarrow (X', \Lambda') \mid f \text{ uniformly continuous}\}$ is defined by

$$\Lambda_{uc}(\Phi) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X \times X)} \left(\Lambda(\mathcal{F}) \to \Lambda'((ev \times ev)(\eta(\Phi \times \mathcal{F}))) \right).$$

Here $\Phi \in \mathcal{F}_L^s(UC \times UC)$, ev is the evaluation mapping restricted on $UC(X, Y) \times X$ and η is the bijection

$$\eta : \begin{cases} (UC \times UC) \times (X \times X) & \longrightarrow & (UC \times X) \times (UC \times X) \\ ((f,g),(x,y)) & \longmapsto & ((f,x),(g,y)) \end{cases}.$$

Every stratified L-uniform convergence space (X, Λ) induces a stratified L-limit space $(X, \lim(\Lambda))$ by defining the limit map [8]

$$\lim(\Lambda)\mathcal{F}(x) = \Lambda(\mathcal{F} \times [x]).$$

A uniformly continuous map $f : (X, \Lambda) \longrightarrow (X', \Lambda')$ then induces a continuous map $f : (X, \lim(\Lambda)) \longrightarrow (X', \lim(\Lambda'))$ (see [8]). Hence we can define a forgetful functor

$$\Phi : \begin{cases} SL\text{-}UCS & \longrightarrow & SL\text{-}LIM \\ (X, \Lambda) & \longmapsto & (X, \lim(\Lambda)) \\ f & \longmapsto & f \end{cases}.$$

Lemma 3.1: *The forgetful functor* $\Phi : SL\text{-}UCS \longrightarrow SL\text{-}LIM$ *preserves initial structures.*

Proof: Let

$$(f_\iota : X \longrightarrow (X_\iota, \Lambda_\iota))_{\iota \in I}$$

be a source. By definition of the initial structure Λ on X we have for $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $x \in X$

$$\begin{aligned} \lim(\Lambda)\mathcal{F}(x) &= \Lambda(\mathcal{F} \times [x]) \\ &= \bigwedge_{\iota \in I} \Lambda_\iota((f_\iota \times f_\iota)(\mathcal{F} \times [x])) \end{aligned}$$

$$= \bigwedge_{\iota \in I} \Lambda_\iota(f_\iota(\mathcal{F}) \times [f_\iota(x)])$$

$$= \bigwedge_{\iota \in I} \lim(\Lambda_\iota) f_\iota(\mathcal{F})(f_\iota(x)).$$

Hence $\lim(\Lambda)$ is the initial structure on X with respect to the source

$$(f_\iota : X \longrightarrow (X_\iota, \lim(\Lambda_{(\iota)})))_{\iota \in I}. \qquad \square$$

4. A stratified L-uniform convergence structure which induces c-lim

We consider in this section a stratified L-limit space $(X, \lim)$ and a stratified L-uniform convergence space (Y, Λ). The set

$$C(X, Y) = \{f : (X, \lim) \longrightarrow (Y, \lim(\Lambda)) \mid f \text{ continuous }\}$$

can be endowed with the stratified L-limit structure of continuous convergence c-lim. We will now define a stratified L-uniform convergence structure on $C(X, Y)$ as follows. Let $\Phi \in \mathcal{F}_L^s(C(X, Y) \times C(X, Y))$ and define

$$\Lambda_c(\Phi) = \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} \left(\lim\mathcal{G}(x) \to \Lambda((ev \times ev)(\eta(\Phi \times (\mathcal{G} \times \mathcal{G})))) \right).$$

Lemma 4.1: $(C(X, Y), \Lambda_c) \in |SL\text{-}UCS|$.

Proof: We check the axioms.
(UC1): Let $f \in C(X, Y)$. Then for all $\mathcal{G} \in \mathcal{F}_L^s(X)$ and for all $x \in X$ we have

$$\lim\mathcal{G}(x) \le \lim(\Lambda)f(\mathcal{G})(f(x)) = \Lambda(f(\mathcal{G}) \times [f(x)]).$$

From [6], Lemma 8.2 we know $f(\mathcal{G}) \le ev([f] \times \mathcal{G})$ and therefore we conclude

$$\lim\mathcal{G}(x) \le \Lambda(ev([f] \times \mathcal{G}) \times [f(x)]).$$

(UC5), Lemma 2.6 and (UC3) then yield

$$\begin{aligned}
\lim\mathcal{G}(x) &\le \Lambda((ev([f] \times \mathcal{G}) \times [f(x)]) \circ ([f(x)] \times ev([f] \times \mathcal{G}))) \\
&\le \Lambda(ev([f] \times \mathcal{G}) \times ev([f] \times \mathcal{G})) \\
&= \Lambda(ev \times ev(\eta(([f] \times [f]) \times (\mathcal{G} \times \mathcal{G})))).
\end{aligned}$$

Therefore $\Lambda_c([f] \times [f]) = \top$.
(UC2) is obvious.

(UC3): We have

$$\Lambda_c(\Phi^{-1}) = \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X), x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev \times ev(\eta(\Phi^{-1} \times (\mathcal{G} \times \mathcal{G}))))).$$

From [8], Lemma 3.8 we conclude that

$$\eta(\Phi^{-1} \times (\mathcal{G} \times \mathcal{G})) = (\eta(\Phi \times (\mathcal{G} \times \mathcal{G})^{-1}))^{-1} = (\eta(\Phi \times (\mathcal{G} \times \mathcal{G})))^{-1}.$$

Moreover it is generally true that $(f \times f)(\Phi^{-1}) = ((f \times f)(\Phi))^{-1}$, as the reader may readily verify. Thus we conclude

$$
\begin{aligned}
\Lambda_c(\Phi^{-1}) &= \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X), x\in X} (\lim\mathcal{G}(x) \to \Lambda((ev \times ev)(\eta(\Phi \times (\mathcal{G} \times \mathcal{G})))^{-1})) \\
&\geq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X), x\in X} (\lim\mathcal{G}(x) \to \Lambda((ev \times ev)(\eta(\Phi \times (\mathcal{G} \times \mathcal{G}))))) \\
&= \Lambda_c(\Phi).
\end{aligned}
$$

(UC4) follows with $\alpha \to (\beta \wedge \gamma) = (\alpha \to \beta) \wedge (\alpha \to \gamma)$ directly from (UC4) for (Y, Λ).

For (UC5) we use [8], Lemma D. Let $\Phi \circ \Psi \in \mathcal{F}_L^s(C(X, Y) \times C(X, Y))$. Then

$$
\begin{aligned}
\Lambda_c(\Phi \circ \Psi) &= \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X), x\in X} (\lim\mathcal{G}(x) \to \Lambda((ev \times ev)(\eta((\Phi \circ \Psi) \times (\mathcal{G} \times \mathcal{G}))))) \\
&\geq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(((ev \times ev)(\eta(\Phi \times (\mathcal{G} \times \mathcal{G})))) \circ \\
&\qquad\qquad \circ((ev \times ev)(\eta(\Psi \times ((\mathcal{G} \times \mathcal{G})^{-1} \circ (\mathcal{G} \times \mathcal{G}))))))).
\end{aligned}
$$

Lemma 2.5 shows that the last expression is

$$\geq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda((ev \times ev)(\eta(\Phi \times (\mathcal{G} \times \mathcal{G}))) \circ (ev \times ev)(\eta(\Psi \times (\mathcal{G} \times \mathcal{G}))))).$$

From (UC5) for (Y, Λ) then finally $\Lambda_c(\Phi \circ \Psi) \geq \Lambda_c(\Phi) \wedge \Lambda_c(\Psi)$ follows. $\square$

Next we present a useful description of the induced stratified L-convergence of Λ_c.

Lemma 4.2: *Let* $(X, \lim) \in |SL\text{-}LIM|$ *and* $(Y, \Lambda) \in |SL\text{-}UCS|$. *Then*

$$\lim(\Lambda_c)\mathcal{F}(f) = \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X)} \bigwedge_{x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F} \times (\mathcal{G} \wedge [x])) \times ev([f] \times (\mathcal{G} \wedge [x])))).$$

Proof: From Lemma 2.4 we know that

$$\eta((\mathcal{F} \times [f]) \times (\mathcal{G} \times \mathcal{G})) = (\mathcal{F} \times \mathcal{G}) \times ([f] \times \mathcal{G}).$$

Therfore we conclude

$$\lim(\Lambda_c)\mathcal{F}(f) \;\; = \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F}\times\mathcal{G})\times ev([f]\times\mathcal{G})))$$

$$\leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim(\mathcal{G}\wedge[x])(x) \to \Lambda(ev(\mathcal{F}\times(\mathcal{G}\wedge[x]))\times ev([f]\times(\mathcal{G}\wedge[x])))).$$

From (L1), (L2) and (L3) it follows $\lim\mathcal{G}(x) = \lim(\mathcal{G}\wedge[x])(x)$ and hence

$$\lim(\Lambda_c)\mathcal{F}(f) \leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F}\times(\mathcal{G}\wedge[x]))\times ev([f]\times(\mathcal{G}\wedge[x])))).$$

As $\mathcal{G}\wedge[x] \leq \mathcal{G}$ we conclude with (UC2) for (Y,Λ) finally

$$\lim(\Lambda_c)\mathcal{F}(f) \;\; \leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F}\times(\mathcal{G}\wedge[x]))\times ev([f]\times(\mathcal{G}\wedge[x]))))$$

$$\leq \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F}\times\mathcal{G})\times ev([f]\times\mathcal{G})))$$

$$= \;\; \lim(\Lambda_c)\mathcal{F}(f). \qquad \square$$

Lemma 4.2 yields, together with Lemma 2.2, (UC2) and (UC4), the following equality $(*)$

$$\lim(\Lambda_c)\mathcal{F}(f) \;\; = \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F}\times\mathcal{G})\times ev([f]\times\mathcal{G})))$$

$$\wedge \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F}\times\mathcal{G})\times ev([f]\times[x])))$$

$$\wedge \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F}\times[x])\times ev([f]\times\mathcal{G})))$$

$$\wedge \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F}\times[x])\times ev([f]\times[x]))).$$

We will use this equality to compare $\lim(\Lambda_c)$ and c-lim. For the stratified L-limit spaces $(X,\lim)$ and $(Y,\lim(\Lambda))$ the structure of continuous convergence is defined as follows. For $\mathcal{F}\in\mathcal{F}_L^s(C(X,Y))$ and $f\in C(X,Y)$ we have

$$c\text{-}\lim\mathcal{F}(f) \;\; = \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \lim(\Lambda)(ev(\mathcal{F}\times\mathcal{G}))(f(x)))$$

$$= \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X),\, x\in X} (\lim\mathcal{G}(x) \to \Lambda(ev(\mathcal{F}\times\mathcal{G})\times[f(x)])).$$

As, by the Corollary to Lemma 2.1, $ev([f]\times[x]) = [f(x)]$ we immediately see that $\lim(\Lambda_c)\mathcal{F}(f) \leq c\text{-}\lim\mathcal{F}(f)$. We will now argue that even equality holds, showing that the three remaining expressions in $(*)$ are all greater of equal to $c\text{-}\lim\mathcal{F}(f)$.

Firstly, taking $\mathcal{G} = [x]$ we find

$$c\text{-lim}\mathcal{F}(f) \leq \bigwedge_{x \in X} (\top \to \Lambda(ev(\mathcal{F} \times [x]) \times ev([f] \times [x]))).$$

As the residual implication operator is order reversing in the first argument we conclude therefore

$$c\text{-lim}\mathcal{F}(f) \leq \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X),\, x \in X} (\text{lim}\mathcal{G}(x) \to \Lambda(ev(\mathcal{F} \times [x]) \times ev([f] \times [x]))).$$

Secondly, it follows from the continuity of $f \in C(X,Y)$ that

$$\text{lim}\mathcal{G}(x) \leq \text{lim}(\Lambda)f(\mathcal{G})(f(x)) = \Lambda(f(\mathcal{G}) \times [f(x)]).$$

Using again $f(\mathcal{G}) \leq ev([f] \times \mathcal{G})$ ([6], Lemma 8.2), we obtain from this

$$\text{lim}\mathcal{G}(x) \leq \Lambda(ev([f] \times \mathcal{G}) \times [f(x)]).$$

Therefore

$$\bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X),\, x \in X} (\text{lim}\mathcal{G}(x) \to \Lambda(ev([f] \times \mathcal{G}) \times [f(x)])) = \top.$$

Using (UC5), (UC3) and Lemma 2.6 we conclude

$$
\begin{aligned}
c\text{-lim}\mathcal{F}(f) \quad \leq \quad & \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X),\, x \in X} (\text{lim}\mathcal{G}(x) \to \Lambda(ev(\mathcal{F} \times [x]) \times ev([f] \times [x]))) \\
\wedge \quad & \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X),\, x \in X} (\text{lim}\mathcal{G}(x) \to \Lambda([f(x)] \times ev([f] \times \mathcal{G}))) \\
\leq \quad & \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X),\, x \in X} (\text{lim}\mathcal{G}(x) \to \Lambda(ev(\mathcal{F} \times [x]) \times ev([f] \times \mathcal{G}))).
\end{aligned}
$$

Similarly, as $(ev(\mathcal{F} \times \mathcal{G}) \times [f(x)]) \circ ([f(x)] \times ev([f] \times \mathcal{G})) \leq ev(\mathcal{F} \times \mathcal{G}) \times ev([f] \times \mathcal{G})$ we conclude thirdly

$$c\text{-lim}\mathcal{F}(f) \leq \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X),\, x \in X} (\text{lim}\mathcal{G}(x) \to \Lambda(ev(\mathcal{F} \times \mathcal{G}) \times ev([f] \times \mathcal{G}))).$$

Putting everything together, we can state the main theorem of this paper.

Theorem 4.3: *Let $(X, \text{lim}) \in |SL\text{-}LIM|$, $(Y, \Lambda) \in |SL\text{-}UCS|$. Then the stratified L-limit structure of continuous convergence on $C(X,Y)$, c-lim, is induced by the stratified L-uniform convergence structure Λ_c defined by*

$$\Lambda_c(\Phi) = \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} (\text{lim}\mathcal{G}(x) \to \Lambda((ev \times ev)(\eta(\Phi \times (\mathcal{G} \times \mathcal{G}))))).$$

We call $(X, \lim) \in |SL\text{-}LIM|$ *L-limit-uniformizable* if there is a stratified L-uniform convergence structure Λ such that $(X, \Lambda) \in |SL\text{-}UCS|$ and $\lim = \lim(\Lambda)$. The full subcategory of $SL\text{-}LIM$ with objects all L-limit-uniformizable spaces and the usual continuous mappings as morphisms is denoted by $SL\text{-}LIM_{ucs}$. This category is topological over SET as can be seen with Lemma 3.1. By Theorem 4.3 we moreover see that the category $SL\text{-}LIM_{ucs}$ has function spaces in the definition of [1], i.e. that this category is cartesian closed. We collect all this in the following corollary.

Corollary 4.4: *The category $SL\text{-}LIM_{ucs}$ is topological over SET and cartesian closed.*

Remark: For stratified L-uniform convergence spaces $(X, \Lambda), (X', \Lambda')$ we may restrict L-continuous convegence on the set $UC(X, Y)$ of uniformly continuous mappings. That is, we consider c-$\lim$ only for $\mathcal{F} \in \mathcal{F}_L^s(UC(X, Y))$ and $f \in UC(X, Y)$. Then clearly

$$
\begin{aligned}
\lim(\Lambda_{uc})\mathcal{F}(f) &= \bigwedge_{\mathcal{H} \in \mathcal{F}_L^s(X \times X)} (\Lambda(\mathcal{H}) \to \Lambda'((ev \times ev)(\eta((\mathcal{F} \times [f]) \times \mathcal{H})))) \\
&\leq \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X),\, x \in X} (\Lambda(\mathcal{G} \times [x]) \to \Lambda'((ev \times ev)(\eta((\mathcal{F} \times [f]) \times (\mathcal{G} \times [x]))))) \\
&= \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X),\, x \in X} (\lim(\Lambda)\mathcal{G}(x) \to \lim(\Lambda')(ev(\mathcal{F} \times \mathcal{G}))(f(x))) \\
&= c\text{-}\lim \mathcal{F}(f)
\end{aligned}
$$

Note, however, that in general c-$\lim = \lim(\Lambda_c)$ (resp. Λ_c) and $\lim(\Lambda_{uc})$ (resp. Λ_{uc}) are defined on different sets of morphisms.

5. Conclusions

We showed in this paper, by explicitly constructing a function space structure on the set $C(X, Y)$, that the category $SL\text{-}LIM_{ucs}$ of L-limit uniformizable stratified L-limit spaces is cartesian closed. This poses a natural problem: Give necessary and sufficient conditions when a stratified L-limit space is induced by a stratified L-uniform convergence space.

In the classical case of $\{0, 1\}$-limit spaces Keller [10] states such conditions. An extension of this result to the general Heyting-algebra-valued case seems difficult at the moment, as a suitable definition of L-Cauchy filter has not been obtained so far.

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