

Gunther Jäger; Department of Statistics, Rhodes University, 6140 Grahamstown, South Africa. Email: g.jager@ru.ac.za

Lattice-valued convergence spaces and regularity

Abstract: We define a regularity axiom for lattice-valued convergence spaces where the lattice is a complete Heyting algebra. To this end, we generalize the characterization of regularity by a "dual form" of a diagonal condition. We show that our axiom ensures that a regular T1-space is separated and that regularity is preserved under initial constructions. Further we present an extension theorem for a continuous mapping from a subspace to a regular space. A characterization in the restricted case that the lattice is a complete Boolean algebra in terms of the closure of an L-filter is given.

Keywords: L-fuzzy convergence, L-topology, L-filter, L-diagonal filter, L-convergence space, pretopological space, diagonal condition, regularity, T1-axiom, T2-axiom, dense subset, continuous extension.

1. Introduction

In [16] and [17] we introduced certain diagonal axioms for lattice-valued convergence spaces (where the lattice, L , is a complete Heyting algebra). These axioms ensure that the lattice-valued convergence is — under additional conditions like being a stratified L -pretopology — stemming from a stratified L -topology. The "dual form" of one of these conditions characterizes, in the classical case, $L = \{0, 1\}$, regularity. This was the definition of regularity for convergence spaces given by Cook and Fischer [6]. The equivalence of this "dual" axiom with the requirement

$$\lim \mathcal{F} \subset \lim \overline{\mathcal{F}}$$

for all filters \mathcal{F} on X was established by Biesterfeld [2]. Here, the filter $\overline{\mathcal{F}}$ is generated by the filter base $\{\overline{F} : F \in \mathcal{F}\}$. Classically, regularity for convergence spaces is mostly defined by this latter requirement. In the Heyting algebra-valued case — lacking an order reversing involution which could induce a complement $-$, however, we do not have a suitable notion of closure for L -sets at our disposal. So we rather choose to go the way of using the "dual form" of a diagonal condition. This is the purpose of this paper. After a preliminary section, where we collect all basic results about lattices, L -sets and L -filters that we will need in the sequel,

we define several categories of lattice-valued convergence spaces [15]. Then we introduce the lower separation axioms $T1$ and $T2$. We show that these are the correct definitions in the sense that they coincide with "good" definitions in the case that the spaces are stratified L -topological spaces. The fourth section is devoted to the definition of our new regularity axiom. We show that under the assumption of regularity, $T1$ implies $T2$ and that regularity is preserved under initial constructions. This gives rise to the reflective subcategory of regular lattice-valued convergence spaces. In order to show that our theory actually works, we present in Section 5 an extension theorem for continuous mappings from a dense subspace to a regular space. To this end we generalize an approach by Cook [5] to the Heyting algebra-valued case. In Section 6, we characterize our axiom in the restricted lattice context of complete Boolean algebras. This restriction seems necessary because we consider stratified L -filters only. The characterization we obtain uses a lattice-valued form of the closure of an L -filter. A similar construction can be found e.g. in Gähler's theory of monadic convergence spaces [9]. Finally we draw some conclusions.

2. Preliminaries

Let L be a complete lattice where finite meets distribute over arbitrary joins: for all $\alpha, \beta_\iota \in L$ ($\iota \in J$) we have

$$\alpha \wedge \bigvee_{\iota \in J} \beta_\iota = \bigvee_{\iota \in J} (\alpha \wedge \beta_\iota).$$

(We will always assume that the largest element, $\top = \bigvee L$, of L is different from the smallest element, $\perp = \bigwedge L$.) Such lattices are called *frames* or *complete Heyting algebras* [10]. They allow the definition of a *residual implication*

$$\alpha \rightarrow \beta = \bigvee \{ \lambda \in L : \alpha \wedge \lambda \leq \beta \}$$

which can be characterized by

$$\delta \leq \alpha \rightarrow \beta \iff \alpha \wedge \delta \leq \beta.$$

The residual implication is order preserving in the second place and order reversing in the first place. For further properties of this operation we refer to [11] and [13].

For notions from category theory we refer to [1].

Given a set X , we can extend the lattice operations pointwise from L to $L^X = \{a : X \longrightarrow L\}$, the set of all L -sets on X . For $A \subset X$ we especially denote by $\top_A : X \longrightarrow L$ the characteristic

function of A , i.e. $\top_A(x) = \top$ if $x \in A$ and $\top_A(x) = \perp$ for $x \notin A$. The constant L -set with value $\alpha \in L$ is denoted by α_X .

A *stratified L -filter on X* is a mapping $\mathcal{F} : L^X \longrightarrow L$ with the following properties [12],[13]:

$$(F1) \quad \mathcal{F}(\top_X) = \top \text{ and } \mathcal{F}(\perp_X) = \perp;$$

$$(F2) \quad a \leq b \text{ implies } \mathcal{F}(a) \leq \mathcal{F}(b);$$

$$(F3) \quad \mathcal{F}(a) \wedge \mathcal{F}(b) \leq \mathcal{F}(a \wedge b);$$

$$(Fs) \quad \mathcal{F}(\alpha_X) \geq \alpha \text{ for all } \alpha \in L.$$

Here, $a, b \in L^X$ are arbitrary L -sets. The set of all stratified L -filters on X is denoted by $\mathcal{F}_L^s(X)$. We further call $\mathcal{F} \in \mathcal{F}_L^s(X)$ *tight* if $\mathcal{F}(\alpha_X) = \alpha$ for all $\alpha \in L$. Tight L -filters appear e.g. in Gähler [8].

An example of a stratified L -filter is the *point L -filter*, $[x]$, defined by $[x](a) = a(x)$ ($a \in L^X$). This stratified L -filter is also tight.

On the set $\mathcal{F}_L^s(X)$ of all stratified L -filters on X we define an order by $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{F}(a) \leq \mathcal{G}(a)$ for all $a \in L^X$. The set $\mathcal{F}_L^s(X)$ then contains maximal elements which are called *stratified L -ultrafilters* [12]. These can be characterized by [12]

$$\mathcal{U} \in \mathcal{F}_L^s(X) \text{ is ultra} \iff \forall a \in L^X : \mathcal{U}(a) = \mathcal{U}(a \rightarrow \perp_X) \rightarrow \perp.$$

For a non-empty family $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ of stratified L -filters the *meet*, $\bigwedge_{\lambda \in \Lambda} \mathcal{F}_\lambda$, can be calculated as

$$\left(\bigwedge_{\lambda \in \Lambda} \mathcal{F}_\lambda \right)(a) = \bigwedge_{\lambda \in \Lambda} (\mathcal{F}_\lambda(a)) \quad (a \in L^X).$$

Clearly, $\bigwedge_{\lambda \in \Lambda} \mathcal{F}_\lambda \in \mathcal{F}_L^s(X)$. In contrast, for two $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, their join need not exist (as a stratified L -filter). Only if $a \wedge b = \perp_X$ implies $\mathcal{F}(a) \wedge \mathcal{G}(b) = \perp$, then this join is in $\mathcal{F}_L^s(X)$ (see [12]). In this case we say that " \mathcal{F} and \mathcal{G} have a join, $\mathcal{F} \vee \mathcal{G}$, in $\mathcal{F}_L^s(X)$ ". Clearly, if for $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathcal{F}_L^s(X)$ we have $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{G} \leq \mathcal{H}$, then \mathcal{F} and \mathcal{G} have a join in $\mathcal{F}_L^s(X)$.

Let $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $f : X \longrightarrow Y$ be a mapping. The *image of \mathcal{F} under f* , $f(\mathcal{F}) \in \mathcal{F}_L^s(Y)$, is defined by $f(\mathcal{F})(b) = \mathcal{F}(f^\leftarrow(b))$ (where $b \in L^Y$ and $f^\leftarrow(b) = b \circ f$) (see [12]). If $\mathcal{F} \in \mathcal{F}_L^s(Y)$, then its *inverse image*, $f^\leftarrow(\mathcal{F})$, is defined by [12]

$$f^\leftarrow(\mathcal{F})(a) = \bigvee \{ \mathcal{F}(b) : f^\leftarrow(b) \leq a \} \quad (a \in L^X).$$

This mapping is not always a stratified L -filter on X . Only if $f^\leftarrow(b) = \perp_X$ implies $\mathcal{F}(b) = \perp$ we have $f^\leftarrow(\mathcal{F}) \in \mathcal{F}_L^s(X)$, see [14].

Example [14]: The stratified L -filter induced on A .

Let $A \subset X$ and $\iota_A : A \longrightarrow X$ be the inclusion mapping. If $A \neq X$ then ι_A is not surjective

and for $\mathcal{F} \in \mathcal{F}_L^s(X)$ the inverse image $\iota_A^\leftarrow(\mathcal{F}) \in \mathcal{F}_L^s(A)$ iff for all $b \in L^X$, $\iota_A^\leftarrow(b) = b|_A = \perp_A$ implies $\mathcal{F}(b) = \perp$. In this case we call \mathcal{F}_A defined by $\mathcal{F}_A(a) := \iota_A^\leftarrow(\mathcal{F})(a) = \bigvee \{\mathcal{F}(b) \mid b|_A \leq a\}$ the *stratified L -filter on A induced by \mathcal{F}* . We then also say that \mathcal{F} has a trace, \mathcal{F}_A , on A . If moreover $\mathcal{G} \in \mathcal{F}_L^s(A)$ then $[\mathcal{G}] := \iota_A(\mathcal{G}) \in \mathcal{F}_L^s(X)$. We have, for $a \in L^X$, $[\mathcal{G}](a) = \mathcal{G}(a|_A)$. Clearly for $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have $\mathcal{F} \leq [\mathcal{F}_A]$ and if $\mathcal{F}(\top_A) = 1$ then equality holds. Also it is easily checked that if $A \subset B \subset X$ and for $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have $\mathcal{F}_A \in \mathcal{F}_L^s(A)$, then also $\mathcal{F}_B \in \mathcal{F}_L^s(B)$ and $(\mathcal{F}_B)_A = \mathcal{F}_A$. For later use we state the following lemma.

Lemma 2.1: *Let $A \subset B \subset X$ and let $f : A \longrightarrow Y$ be a mapping. If for $\mathcal{G} \in \mathcal{F}_L^s(B)$ we have that $\mathcal{G}_A \in \mathcal{F}_L^s(A)$, then $f(\mathcal{G}_A) = f([\mathcal{G}]_A)$.*

Proof: Let $b \in L^Y$. Then for $x \in A$ trivially $f^\leftarrow(b)|_A(x) = b(f(x))$ and therefore

$$\begin{aligned} f([\mathcal{G}]_A)(b) &= [\mathcal{G}]_A(f^\leftarrow(b)) \\ &= \bigvee \{[\mathcal{G}](c) : c \in L^X, c|_A \leq f^\leftarrow(b)\} \\ &= \bigvee \{\mathcal{G}(c|_B) : c \in L^X, c|_A \leq f^\leftarrow(b)\} \\ &= \bigvee \{\mathcal{G}(c) : c \in L^B, c|_A \leq f^\leftarrow(b)\} \\ &= \mathcal{G}_A(f^\leftarrow(b)) = f(\mathcal{G}_A)(b). \end{aligned}$$

□

In [17] we defined a so-called *stratified L -diagonal filter*. This L -filter will play a crucial role later.

Lemma and Definition 2.2 [17]: *Let J be a set, $\mathcal{G} \in \mathcal{F}_L^s(J)$ and let for all $i \in J$, $\mathcal{F}_i \in \mathcal{F}_L^s(X)$. We define for $a \in L^X$*

$$\mathcal{F}_{(\cdot)}(a) : \begin{cases} J & \longrightarrow & L \\ i & \longmapsto & \mathcal{F}_i(a) \end{cases}$$

i.e. $\mathcal{F}_{(\cdot)}(a) \in L^J$. Then the mapping $\mathcal{G}(\mathcal{F}_{(\cdot)})$, defined by

$$\mathcal{G}(\mathcal{F}_{(\cdot)})(a) = \mathcal{G}(\mathcal{F}_{(\cdot)}(a)) \quad (a \in L^X),$$

is a stratified L -filter on X . It is called the stratified L -diagonal filter of $(\mathcal{G}, (\mathcal{F}_i)_{i \in J})$.

We collect the properties of this stratified L -diagonal filter in the following Lemma.

Lemma 2.3 [17]: *Let J be a set, $\mathcal{G}, \mathcal{H} \in \mathcal{F}_L^s(J)$ and let for all $i \in J$, $\mathcal{F}_i, \mathcal{K}_i \in \mathcal{F}_L^s(X)$.*

- (1) *If for all $i \in J$, $\mathcal{F}_i \leq \mathcal{K}_i$ and if $\mathcal{G} \leq \mathcal{H}$ then $\mathcal{G}(\mathcal{F}_{(\cdot)}) \leq \mathcal{H}(\mathcal{K}_{(\cdot)})$.*
- (2) *$(\mathcal{G} \wedge \mathcal{H})(\mathcal{F}_{(\cdot)}) = \mathcal{G}(\mathcal{F}_{(\cdot)}) \wedge \mathcal{H}(\mathcal{F}_{(\cdot)})$.*
- (3) *If $f : X \longrightarrow Y$ is a mapping then $\mathcal{G}(f(\mathcal{F}_{(\cdot)})) = f(\mathcal{G}(\mathcal{F}_{(\cdot)}))$.*

Remark: The case $L = \{0, 1\}$.

We denote the set of all ordinary filters on X by $\mathcal{F}(X)$. For a family of (ordinary) filters $\mathcal{F}_i \in \mathcal{F}(X)$ ($i \in J$) and a filter $\mathcal{G} \in \mathcal{F}(J)$, the *diagonal filter* is defined by

$$\kappa(\mathcal{G}, (\mathcal{F}_i)_{i \in J}) = \bigvee_{F \in \mathcal{G}} \bigwedge_{i \in F} \mathcal{F}_i$$

(see [18], [19]). If we identify an ordinary filter $\mathcal{F} \in \mathcal{F}(Z)$ with the stratified $\{0, 1\}$ -filter

$$1_{\mathcal{F}} : \begin{cases} \{0, 1\}^Z & \longrightarrow & \{0, 1\} \\ 1_A & \longmapsto & \begin{cases} 1 & \text{if } A \in \mathcal{F} \\ 0 & \text{if } A \notin \mathcal{F} \end{cases} \end{cases}$$

then we have

$$1 = (1_{\mathcal{F}_{(\cdot)}})(1_A)(i) = 1_{\mathcal{F}_i}(1_A) \iff A \in \mathcal{F}_i,$$

i.e. we can identify the crisp set $\{i \in J \mid A \in \mathcal{F}_i\}$ with the $\{0, 1\}$ -set $(1_{\mathcal{F}_{(\cdot)}})(1_A)$. Thus the $\{0, 1\}$ -diagonal filter satisfies

$$1_{\mathcal{G}}((1_{\mathcal{F}_{(\cdot)}}))(1_A) = 1 \iff \{i \in J \mid A \in \mathcal{F}_i\} \in \mathcal{G}.$$

In the sense of this identification, we showed in [17] that the stratified $\{0, 1\}$ -diagonal filter is nothing else than the classical diagonal filter i.e.

$$1_{\kappa(\mathcal{G}, (\mathcal{F}_i)_{i \in J})} = 1_{\mathcal{G}}((1_{\mathcal{F}_{(\cdot)}})).$$

3. Lattice-valued convergence spaces

Let X be a set. We consider a mapping

$$\lim : \mathcal{F}_L^s(X) \longrightarrow L^X$$

which satisfies the following axioms.

$$(L1) \quad \forall x \in X : \limx = \top;$$

$$(L2) \quad \mathcal{F} \leq \mathcal{G} \implies \lim \mathcal{F} \leq \lim \mathcal{G}.$$

The pair (X, \lim) is then called a *stratified L -generalized convergence space* [14],[15]. A mapping $f : X \longrightarrow X'$ between two stratified L -generalized convergence spaces (X, \lim) , (X', \lim') is called *continuous* [14] if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and all $x \in X$ we have

$$\lim \mathcal{F}(x) \leq \lim' f(\mathcal{F})(f(x)).$$

The category $SL-GCS$ has as objects all stratified L -generalized convergence spaces and as morphisms the continuous mappings. This category is topological over SET [14]. For a given source $(f_\lambda : X \longrightarrow (X, \lim_\lambda))_{\lambda \in \Lambda}$ the *initial structure*, $\lim = \text{init}(\lim_\lambda)$ on X is given by [14]

$$\lim \mathcal{F}(x) = \bigwedge_{\lambda \in \Lambda} \lim_\lambda f_\lambda(\mathcal{F})(f_\lambda(x)) \quad (\mathcal{F} \in \mathcal{F}_L^s(X), x \in X).$$

Example: Subspaces

Let $A \subset X$, $(X, \lim) \in |SL-GCS|$, and consider the inclusion mapping

$$\iota_A : \begin{cases} A & \longrightarrow & X \\ x & \longmapsto & x \end{cases}.$$

If we denote the initial structure on A with respect to ι_A by $\lim|_A$, then we find for $\mathcal{F} \in \mathcal{F}_L^s(A)$ and $x \in A$

$$\lim|_A \mathcal{F}(x) = \lim \iota_A(\mathcal{F})(\iota_A(x)) = \lim[\mathcal{F}](x).$$

Further examples for initial constructions are product spaces with $f_\mu : \prod_{\lambda \in \Lambda} X_\lambda \longrightarrow X_\mu$ the projection mappings.

Moreover, $SL-GCS$ is cartesian closed. The corresponding function space structure on $C(X, X')$, the set of all morphisms from X to X' is given by *continuous convergence* [14]:

$$c - \lim \mathcal{F}(f) = \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} \bigwedge_{x \in X} (\lim \mathcal{G}(x) \rightarrow \lim' ev(\mathcal{F} \times \mathcal{G})(f(x))).$$

Here, $\mathcal{F} \in \mathcal{F}_L^s(C(X, X'))$, $f \in C(X, X')$ and $ev : (f, x) \longmapsto f(x)$ is the evaluation mapping restricted on $C(X, X') \times X$ (see [14] for details).

Moreover, the category $SL-TOP$ of stratified L -topological spaces [13] is isomorphic to a reflective subcategory, $SL-TCS$, of $SL-GCS$, defined below [14],[15]. Here, a stratified L -topological space is a set X together with a set $\Delta \subset L^X$ which is closed under finite infima and arbitrary suprema and contains all constants α_X , see e.g. [13]. The embedding of $SL-TOP$ into $SL-GCS$ is given by defining a stratified L -neighbourhood filter for a stratified L -topological space (X, Δ) by (Höhle and Sostak [13])

$$\mathcal{U}^x(a) = \underline{a}(x) = \bigvee_{g \in \Delta, g \leq a} g(x)$$

and defining

$$\lim(\Delta)\mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{F}(a)).$$

One can also show that $SL-GCS$ is extensional, following the proof of the corresponding property in Flores et al's category $SL-CS$ [7]. (We do not want to go into detail here about this

category but only remark that Flores et al's category is a tower construction (see e.g. [9], Section 12) and is slightly more general than our category $SL-GCS$. However, in order to embed $SL-TOP$ into this category it seems necessary to impose an axiom (Lpw1) stated below. This axiom at present has not been expressed in terms of the "level convergence structures" which are used to construct the objects of $SL-CS$, see [16].) This all shows that the category $SL-GCS$ has very nice categorical properties.

Important reflective subcategories of $SL-GCS$ arise if we add suitable axioms. In this paper we will consider the following:

$$\begin{aligned} (Lp) \quad & \forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall x \in X : \lim \mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{F}(a)); \\ (Lt) \quad & \forall x \in X : \mathcal{U}^x \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)}). \end{aligned}$$

If (X, \lim) satisfies (Lp) then it is called a *stratified L -pretopological space*. Here, the stratified L -neighbourhood filter is defined by [14]

$$\mathcal{U}^x(a) = \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} (\lim \mathcal{G}(x) \rightarrow \mathcal{G}(a)) \quad (a \in L^X).$$

The category of stratified L -pretopological spaces, $SL-PCS$, generalizes the category of Čech's closure spaces [4] to the Heyting algebra-valued setting [16]. A space satisfying (Lp) and (Lt) is called a stratified L -topological convergence space. The category of these spaces, $SL-TCS$, is isomorphic to $SL-TOP$ [15].

We showed in [16] that the axiom (Lp) splits into the following two axioms which are independent of each other.

$$\begin{aligned} (Lpw1) \quad & \forall \alpha \in L, \forall x \in X : [\mathcal{U}^x \wedge \alpha] = \mathcal{U}_\alpha^x; \\ (Lpw2) \quad & \forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall x \in X : \lim \mathcal{F}(x) = \bigvee \{ \alpha \in L : \mathcal{F} \geq \mathcal{U}_\alpha^x \}. \end{aligned}$$

Here the *stratified α -level neighbourhood L -filter* is defined by [16]

$$\mathcal{U}_\alpha^x = \bigwedge_{\lim \mathcal{F}(x) \geq \alpha} \mathcal{F}$$

and the left-hand side of (Lpw1) has the meaning

$$[\mathcal{U}^x \wedge \alpha] = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X), \mathcal{F}(a) \geq \mathcal{U}^x(a) \wedge \alpha \forall a \in L^X} \mathcal{F}.$$

Restating (Lpw1) in the following form

$$(Lpw1) \quad \forall \alpha \in L, \forall x \in X :$$

$$\bigwedge \{ \mathcal{F} \in \mathcal{F}_L^s(X) : \alpha \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{F}(a)) \} = \bigwedge \{ \mathcal{F} \in \mathcal{F}_L^s(X) : \alpha \leq \lim \mathcal{F}(x) \}$$

shows how "close" (Lpw1) and (Lp) are. Interestingly, the axiom (Lpw2) is equivalent to the following requirement [16].

$$(Lpw2') \quad \forall \mathcal{F}_\iota \in \mathcal{F}_L^s(X) \quad (\iota \in J) : \lim \left(\bigwedge_{\iota \in J} \mathcal{F}_\iota \right) = \bigwedge_{\iota \in J} (\lim \mathcal{F}_\iota),$$

which is in the case $L = \{0, 1\}$ equivalent to (Lp). Another equivalent form of this axiom which we will use later is [16]

$$(Lpw2'') \quad \forall \alpha \in L, \forall x \in X \quad \forall \mathcal{F} \in \mathcal{F}_L^s(X) : \lim \mathcal{F}(x) \geq \alpha \iff \mathcal{F} \geq \mathcal{U}_\alpha^x.$$

In [16] we showed that, for a space satisfying (Lp), the "topological" axiom (Lt) (which ensures the idempotency of the interior operator $\underline{a}(x) = \mathcal{U}^x(a)$) is equivalent to the following diagonal condition

$$(LK) \quad \forall \mathcal{G} \in \mathcal{F}_L^s(X), \forall \mathcal{F}_y \in \mathcal{F}_L^s(X) \quad (y \in X), \forall x \in X : \\ \bigwedge_{y \in X} \lim \mathcal{F}_y(y) \wedge \lim \mathcal{G}(x) \leq \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

This diagonal condition generalizes a corresponding classical condition due to Kowalski [19]. So a stratified L -pretopological space which satisfies (LK) is L -topological (in the sense that it is a stratified L -topological convergence space). In the classical theory of convergence spaces, there is a stronger axiom, (F) (named after H.R. Fischer, first published in [6]), which ensures that a convergence space satisfying this axiom (F) is topological. The reason for this is that this axiom implies Kowalski's diagonal axiom and that the convergence space is pretopological. We generalized this axiom to the Heyting algebra-valued case in [17].

$$(LF) \quad \forall J, \forall \psi : J \longrightarrow X, \quad \forall \mathcal{G} \in \mathcal{F}_L^s(J), \forall \mathcal{F}_i \in \mathcal{F}_L^s(X) \quad (i \in J), \forall x \in X :$$

$$\bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \wedge \lim \psi(\mathcal{G})(x) \leq \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

Apparently for the case $L = \{0, 1\}$, our axiom is the same as the axiom (F). However, it turned out that this axiom (LF) does not imply (Lp) but it only implies (Lpw2), see [17]. So in order that a stratified L -generalized convergences space is L -topological, we have to demand both the axioms (LF) and (Lpw1) (the latter being classically, $L = \{0, 1\}$, always true).

We showed in [17] that the axiom (LF) is preserved under initial constructions, i.e. that the subcategory of all stratified L -convergence spaces satisfying (LF) is reflective in $SL\text{-}GCS$ and topological over SET . For the axiom (LK) this is not the case. We only have the following result.

Lemma 3.1: *If all $(X_\lambda, \lim_\lambda) \in |SL\text{-}GCS|$ satisfy the axiom (LK) and if all $f_\lambda : X \rightarrow (X_\lambda, \lim_\lambda)$ are injective ($\lambda \in \Lambda$) and $\lim = \text{init}(\lim_\lambda)$ then (X, \lim) satisfies (LK).*

Proof: Let $\mathcal{G} \in \mathcal{F}_L^s(X)$ and for each $y \in X$ let $\mathcal{F}_y \in \mathcal{F}_L^s(X)$. Then

$$\begin{aligned} \lim \mathcal{G}(x) \wedge \bigwedge_{y \in X} \lim \mathcal{F}_y(y) &= \bigwedge_{\lambda \in \Lambda} \lim_\lambda f_\lambda(\mathcal{G})(f_\lambda(x)) \wedge \bigwedge_{y \in X} \bigwedge_{\lambda \in \Lambda} \lim_\lambda f_\lambda(\mathcal{F}_y)(f_\lambda(y)) \\ &\leq \bigwedge_{\lambda \in \Lambda} \left(\lim_\lambda f_\lambda(\mathcal{G})(f_\lambda(x)) \wedge \bigwedge_{y \in X} \lim_\lambda f_\lambda(\mathcal{F}_y)(f_\lambda(y)) \right). \end{aligned}$$

We define now for $y_\lambda \in f_\lambda(X)$ the stratified L -filter $\mathcal{H}_{y_\lambda} = f_\lambda(\mathcal{F}_y)$ with the unique $y = f_\lambda^{-1}(y_\lambda)$, and for $y_\lambda \notin f_\lambda(X)$ we define $\mathcal{H}_{y_\lambda} = [y_\lambda]$. Then by (LK) we obtain

$$\begin{aligned} \lim \mathcal{G}(x) \wedge \bigwedge_{y \in X} \lim \mathcal{F}_y(y) &= \bigwedge_{\lambda \in \Lambda} \left(\lim_\lambda f_\lambda(\mathcal{G})(f_\lambda(x)) \wedge \bigwedge_{y_\lambda \in X_\lambda} \lim_\lambda \mathcal{H}_{y_\lambda}(y_\lambda) \right) \\ &\leq \bigwedge_{\lambda \in \Lambda} \lim_\lambda f_\lambda(\mathcal{G})(\mathcal{H}_{(\cdot)})(f_\lambda(x)). \end{aligned}$$

Now we observe that for $a \in L^{X_\lambda}$ and $z \in X$ we have

$$f_\lambda^{\leftarrow}(\mathcal{H}_{(\cdot)}(a))(z) = \mathcal{H}_{(\cdot)}(a)(f_\lambda(z)) = \mathcal{H}_{f_\lambda(z)}(a) = f_\lambda(\mathcal{F}_z)(a) = [f_\lambda(\mathcal{F}_{(\cdot)})(a)](z).$$

Therefore

$$\begin{aligned} f_\lambda(\mathcal{G})(\mathcal{H}_{(\cdot)})(a) &= f_\lambda(\mathcal{G})(\mathcal{H}_{(\cdot)}(a)) = \mathcal{G}(f_\lambda^{\leftarrow}(\mathcal{H}_{(\cdot)}(a))) \\ &= \mathcal{G}(f_\lambda(\mathcal{F}_{(\cdot)})(a)) = \mathcal{G}(\mathcal{F}_{(\cdot)})(f_\lambda^{\leftarrow}(a)) \\ &= \mathcal{G}(\mathcal{F}_{(\cdot)})(f_\lambda^{\leftarrow}(a)) = f_\lambda(\mathcal{G}(\mathcal{F}_{(\cdot)}))(a). \end{aligned}$$

Hence we obtain finally

$$\lim \mathcal{G}(x) \wedge \bigwedge_{y \in X} \lim \mathcal{F}_y(y) \leq \bigwedge_{\lambda \in \Lambda} \lim_\lambda f_\lambda(\mathcal{G}(\mathcal{F}_{(\cdot)}))(f_\lambda(x)) = \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

□

Kent and Richardson showed in [18] that for the case $L = \{0, 1\}$ we cannot omit the requirement of injectivity for the mappings f_λ . Interesting for us is the following corollary.

Corollary 3.2: *Let $(X, \lim) \in |SL\text{-}GCS|$ satisfy (LK) and let $A \subset X$. Then the subspace $(A, \lim|_A)$ satisfies (LK).*

4. Lower separation axioms

We call $(X, \lim) \in |SL-GCS|$ a *T1-space* if it satisfies the axiom

$$(T1) \quad \forall x, y \in X : \lim[y](x) = \top \Rightarrow x = y,$$

and a *T2-space* if it satisfies the axiom

$$(T2) \quad \forall \mathcal{F} \in \mathcal{F}_L^s(X) : \lim \mathcal{F}(x) = \lim \mathcal{F}(y) = \top \Rightarrow x = y.$$

Lemma 4.1: *A T2-space (X, \lim) satisfies T1.*

Proof: If $\lim[y](x) = \top$ then, as by (L1) also $\limy = \top$, (T2) implies $x = y$. \square

The separation axioms (T1) and (T2) are productive and inherited by subspaces:

Lemma 4.2: *If all $(X_\lambda, \lim_\lambda)$ are T1-spaces (resp. T2 spaces) $(\lambda \in \Lambda)$ and if the family of mappings $(f_\lambda : X \rightarrow X_\lambda)_{\lambda \in \Lambda}$ separates points (i.e. for $x \neq y$ there is $\lambda \in \Lambda$ such that $f_\lambda(x) \neq f_\lambda(y)$), then $(X, \text{init}(\lim_\lambda))$ is a T1-space (resp. a T2-space).*

Proof: For (T1), we make use of $f_\lambda([y]) = [f_\lambda(y)]$ (see [14]) and argue as follows. If

$$\top = \lim[y](x) = \bigwedge_{\lambda \in \Lambda} \lim_\lambda f_\lambda([y])(f_\lambda(x)) = \bigwedge_{\lambda \in \Lambda} \lim_\lambda [f_\lambda(y)](f_\lambda(x))$$

then for every $\lambda \in \Lambda$ we have $\top = \lim_\lambda [f_\lambda(y)](f_\lambda(x))$. Hence for all $\lambda \in \Lambda$ we conclude with (T1) that $f_\lambda(y) = f_\lambda(x)$ from which, by point-separatedness, $x = y$ follows. The proof for (T2) is very similar and we leave it to the reader. \square

Corollary 4.3: *Subspaces and products of T1-spaces (resp. T2-spaces) satisfy T1 (resp. T2).*

We are now going to characterize these separation axioms in some subcategories of *SL-GCS*.

Lemma 4.4: *Let $(X, \lim) \in SL-PCS$. Then*

$$(T1) \quad \Longleftrightarrow \quad \forall x, y \in X : \mathcal{U}^x \leq [y] \Rightarrow x = y;$$

$$(T2) \quad \Longleftrightarrow \quad \forall x, y \in X : \mathcal{U}^x \vee \mathcal{U}^y \in \mathcal{F}_L^s(X) \Rightarrow x = y.$$

Proof: Let first (X, \lim) satisfy (T1). If $\mathcal{U}^x \leq [y]$ then by (Lp)

$$\lim[y](x) = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow [y](a)) = \top$$

and hence $x = y$. On the other hand it follows from $\top = \lim[y](x)$ and (Lp) that $\mathcal{U}^x \leq [y]$ and therefore $x = y$ by the condition stated in the lemma.

Let now (X, \lim) satisfy the T2-axiom and let $\mathcal{U}^x \vee \mathcal{U}^y \in \mathcal{F}_L^s(X)$. Then by (Lp)

$$\top = \lim \mathcal{U}^x(x) \leq \lim \mathcal{U}^x \vee \mathcal{U}^y(x)$$

and

$$\top = \lim \mathcal{U}^y(y) \leq \lim \mathcal{U}^x \vee \mathcal{U}^y(y).$$

Therefore, by (T2), $x = y$. On the other hand it follows with (Lp) from $\top = \lim \mathcal{F}(x) = \lim \mathcal{F}(y)$ that $\mathcal{U}^x \leq \mathcal{F}$ and $\mathcal{U}^y \leq \mathcal{F}$. Therefore $\mathcal{U}^x \vee \mathcal{U}^y \in \mathcal{F}_L^s(X)$ and by the condition of the lemma we get $x = y$. \square

We identify in the sequel a stratified L -topological convergence space with a stratified L -topological space by defining the stratified L -topology by $\Delta = \{a \in L^X : a \leq \underline{a}\}$, where $\underline{a}(x) = \mathcal{U}^x(a)$ for $x \in X$.

Lemma 4.5: *Let $(X, \lim) \in SL-TCS$. Then*

$$(T1) \iff (\forall x, y \in X : g(x) \leq g(y) \text{ for all } g \in \Delta \implies x = y);$$

$$(T2) \iff (\forall x, y \in X : x \neq y \implies \exists g, h \in \Delta \text{ such that } g(x) \wedge h(y) \neq \perp \text{ and } g \wedge h = \perp_X).$$

Proof: Let first (X, \lim) satisfy (T1). If $g(x) \leq g(y)$ for all $g \in \Delta$, then, as $\underline{a} \in \Delta$ for all $a \in L^X$, we find

$$\mathcal{U}^x(a) = \underline{a}(x) \leq \underline{a}(y) \leq a(y) = [y](a).$$

Hence $\mathcal{U}^x \leq [y]$ and therefore $x = y$ by Lemma 4.4. If on the other hand $\mathcal{U}^x \leq [y]$ then for all $g \in \Delta$

$$g(x) = \mathcal{U}^x(g) \leq [y](g) = g(y)$$

and therefore $x = y$.

Let now (X, \lim) satisfy (T2). Then for $x \neq y$, $\mathcal{U}^x \vee \mathcal{U}^y \notin \mathcal{F}_L^s(X)$. Therefore there are $g, h \in L^X$ such that $g \wedge h = \perp_X$ and $\mathcal{U}^x(g) \wedge \mathcal{U}^y(h) \neq \perp$. It follows that $\underline{g}, \underline{h} \in \Delta$ and $\underline{g} \wedge \underline{h} = \perp_X$ and $\underline{g}(x) \wedge \underline{h}(y) \neq \perp$. On the other hand let the condition of the lemma be true. Then $\mathcal{U}^x \vee \mathcal{U}^y \notin \mathcal{F}_L^s(X)$ for $x \neq y$, which means that whenever $\mathcal{U}^x \vee \mathcal{U}^y \in \mathcal{F}_L^s(X)$ we must have that $x = y$, i.e. the axiom (T2) holds by Lemma 4.4. \square

This shows that our T1- and T2-axioms are generalizations of known definitions in *SL-TOP* (e.g. Höhle and Šostak [13]).

5. The regularity axiom

We call a stratified L -generalized convergence space (X, \lim) *regular* if it satisfies the following axiom (LR):

$$(LR) \quad \forall J, \forall \psi : J \longrightarrow X, \quad \forall \mathcal{G} \in \mathcal{F}_L^s(J), \forall \mathcal{F}_i \in \mathcal{F}_L^s(X) \ (i \in J), \forall x \in X : \\ \bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \wedge \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \leq \lim \psi(\mathcal{G})(x).$$

Using the pointwise evaluation of the residual implication, we can state this axiom also in the following form:

$$(LR') \quad \forall J, \forall \psi : J \longrightarrow X, \quad \forall \mathcal{G} \in \mathcal{F}_L^s(J), \forall \mathcal{F}_i \in \mathcal{F}_L^s(X) \ (i \in J) : \\ \bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \leq \lim \mathcal{G}(\mathcal{F}_{(\cdot)}) \rightarrow \lim \psi(\mathcal{G}).$$

Similarly we can restate the axiom (LF):

$$(LF') \quad \forall J, \forall \psi : J \longrightarrow X, \quad \forall \mathcal{G} \in \mathcal{F}_L^s(J), \forall \mathcal{F}_i \in \mathcal{F}_L^s(X) \ (i \in J) : \\ \bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \leq \lim \psi(\mathcal{G}) \rightarrow \lim \mathcal{G}(\mathcal{F}_{(\cdot)}).$$

This form shows in which sense (LR) and the diagonal axiom (LF) are "dual" to each other: The arguments in the residual implication appear reversed.

Lemma 5.1: *Let $(X, \lim) \in |SL-GCS|$ be regular and satisfy (T1). Then it also satisfies (T2).*

Proof: Let $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $\lim \mathcal{F}(x) = \lim \mathcal{F}(y) = \top$. We define $J = \{\mathcal{G} \in \mathcal{F}_L^s(X) : \lim \mathcal{G}(y) = \top\}$ and for each $\mathcal{G} \in J$ we define $\mathcal{F}_{\mathcal{G}} = \mathcal{G}$. Further we consider the constant mapping

$$\psi : \begin{cases} J & \longrightarrow & X \\ \mathcal{G} & \longmapsto & y \end{cases}.$$

From $\lim \mathcal{F}(y) = \top$ we see that $\mathcal{F} \in J$ and hence $[\mathcal{F}] \in \mathcal{F}_L^s(J)$ (with $[\mathcal{F}](a) = a(\mathcal{F})$ for any $a \in L^J$). Further we have for $a \in L^X$:

$$\psi([\mathcal{F}](a)) = [\mathcal{F}](\psi^{\leftarrow}(a)) = \psi^{\leftarrow}(a)(\mathcal{F}) = a(\psi(\mathcal{F})) = a(y) = [y](a),$$

and therefore $\psi([\mathcal{F}]) = [y]$. Moreover we find for $a \in L^X$

$$[\mathcal{F}](\mathcal{F}_{(\cdot)})(a) = [\mathcal{F}](\mathcal{F}_{(\cdot)}(a)) = \mathcal{F}_{\mathcal{F}}(a) = \mathcal{F}(a),$$

i.e. $[\mathcal{F}](\mathcal{F}_{(\cdot)}) = \mathcal{F}$. From regularity we thus obtain

$$\top = \bigwedge_{\mathcal{G} \in J} \lim \mathcal{G}(y) \wedge \lim \mathcal{F}(x) = \bigwedge_{\mathcal{G} \in J} \lim \mathcal{F}_{\mathcal{G}}(\psi(\mathcal{G})) \wedge \lim [\mathcal{F}](\mathcal{F}_{(\cdot)})(x) \leq \lim \psi([\mathcal{F}](x)) = \lim [y](x).$$

By (T1) this yields $x = y$ and therefore (T2) holds. \square

The next lemma shows that regularity is preserved under the formation of subspaces and product spaces.

Lemma 5.2: *Let all $(X_\lambda, \lim_\lambda) \in |SL-GCS|$ be regular ($\lambda \in \Lambda$) and let $(f_\lambda : X \longrightarrow X_\lambda)_{\lambda \in \Lambda}$ be a source. Then also $(X, \text{init}(\lim_\lambda))$ is regular.*

Proof: Let J be a set, $\psi : J \longrightarrow X$ be a mapping, $\mathcal{G} \in \mathcal{F}_L^s(J)$ and for every $i \in J$ let $\mathcal{F}_i \in \mathcal{F}_L^s(X)$.

Then for $x \in X$ we have

$$\begin{aligned} \bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \wedge \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) &= \bigwedge_{i \in J} \bigwedge_{\lambda \in \Lambda} \lim_\lambda f_\lambda(\mathcal{F}_i)(f_\lambda(\psi(i))) \wedge \bigwedge_{\lambda \in \Lambda} \lim_\lambda f_\lambda(\mathcal{G}(\mathcal{F}_{(\cdot)}))(f_\lambda(x)) \\ &\leq \bigwedge_{\lambda \in \Lambda} \left(\bigwedge_{i \in J} \lim_\lambda f_\lambda(\mathcal{F}_i)(f_\lambda(\psi(i))) \wedge \lim_\lambda f_\lambda(\mathcal{G}(\mathcal{F}_{(\cdot)}))(f_\lambda(x)) \right). \end{aligned}$$

With $\psi_\lambda = f_\lambda \circ \psi$ and $\mathcal{F}_i^\lambda = f_\lambda(\mathcal{F}_i) \in \mathcal{F}_L^s(X_\lambda)$ for all $i \in J$ and (LR) for each $(X_\lambda, \lim_\lambda)$ we obtain

$$\begin{aligned} &\bigwedge_{i \in J} \lim \mathcal{F}_i(\psi(i)) \wedge \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \\ &\leq \bigwedge_{\lambda \in \Lambda} \left(\bigwedge_{i \in J} \lim_\lambda \mathcal{F}_i^\lambda(\psi_\lambda(i)) \wedge \lim_\lambda \mathcal{G}(\mathcal{F}_{(\cdot)}^\lambda)(f_\lambda(x)) \right) \\ &\leq \bigwedge_{\lambda \in \Lambda} \lim_\lambda \psi_\lambda(\mathcal{G})(f_\lambda(x)) \\ &= \bigwedge_{\lambda \in \Lambda} \lim_\lambda f_\lambda(\psi(\mathcal{G}))(f_\lambda(x)) \\ &= \lim \psi(\mathcal{G})(x). \end{aligned}$$

\square

We can consider the subcategory $SL-RGCS$ of $SL-GCS$ with objects the regular stratified L -generalized convergence spaces and morphisms the continuous mappings. Lemma 5.2 then shows that this subcategory is topological over SET and concretely reflective in $SL-GCS$ (see [1]).

We finally present a trivial characterization which shows how to define regularity in Flores et al's category $SL-CS$ [7]. See in this respect also the definition of regularity for probabilistic convergence spaces [3],[20].

Lemma 5.3: For $(X, \lim) \in |SL-GCS|$ it is equivalent

- (1) (LR)
- (2) $\forall \alpha, \beta \in L, \forall J, \forall \psi : J \longrightarrow X, \forall \mathcal{G} \in \mathcal{F}_L^s(J), \forall \mathcal{F}_i \in \mathcal{F}_L^s(X) (i \in J), \forall x \in X :$
 $\alpha \leq \lim \mathcal{F}_i(\psi(i)) \ \forall i \in J, \quad \beta \leq \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) \quad \Rightarrow \quad \alpha \wedge \beta \leq \lim \psi(\mathcal{G})(x).$

6. A theorem on continuous extensions

We consider in this section the following situation. For stratified L -generalized convergence spaces (X, \lim) and (Y, \lim') and $A \subset X$ and a mapping $f : A \longrightarrow Y$, we ask if it is possible to extend f to the whole of X . Naturally, we will demand that f and its extension are continuous. For $\alpha \in L$ we define

$$\begin{aligned} H^\alpha(x) &= H_A^\alpha(x) = \{\mathcal{F} \in \mathcal{F}_L^s(X) : \mathcal{F}_A \in \mathcal{F}_L^s(A) \text{ and } \lim \mathcal{F}(x) \geq \alpha\} \\ F^\alpha(x) &= F_A^\alpha(x) = \{y \in Y : \lim' f(\mathcal{F}_A)(y) \geq \alpha \text{ for all } \mathcal{F} \in H_A^\alpha(x)\} \end{aligned}$$

We note that for $\alpha \leq \beta$ we have $H^\beta(x) \subset H^\alpha(x)$ and that therefore the requirement $H^\top(x) \neq \emptyset$ assures the non-emptiness of all $H^\alpha(x)$. We further define

$$X_0 = \{x \in X : H^\top(x) \neq \emptyset \text{ and } \bigcap_{\alpha \in L} F^\alpha(x) \neq \emptyset\}.$$

Note that if $f : (A, \lim|_A) \longrightarrow (Y, \lim')$ is continuous, then $A \subset X_0$: If $x \in A$, then $[x]_A \in \mathcal{F}_L^s(A)$ and by (L1) $\limx = \top$. Hence $H^\top(x) \neq \emptyset$. Further, if $\mathcal{F} \in H^\alpha(x)$, then $\lim|_A \mathcal{F}_A(x) = \lim[\mathcal{F}_A](x) \geq \lim \mathcal{F}(x) \geq \alpha$ and therefore, by continuity, $\lim' f(\mathcal{F}_A)(f(x)) \geq \alpha$. Hence $f(x) \in F^\alpha(x)$. This is true for any $\alpha \in L$, so that $f(x) \in \bigcap_{\alpha \in L} F^\alpha(x)$.

We can thus choose for $x \notin A$, $x \in X_0$, a fixed value $y_x \in \bigcap_{\alpha \in L} F^\alpha(x)$ and define a function $\bar{f} : X_0 \longrightarrow Y$ by putting

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ y_x & \text{if } x \in X_0 - A \end{cases}.$$

Clearly, $\bar{f}|_A = f$. The next Lemma gives conditions which guarantee that \bar{f} is continuous.

Lemma 6.1: Let $(X, \lim) \in |SL-GCS|$ satisfy the axiom (LK) and let $(Y, \lim') \in |SL-GCS|$ be regular and $A \subset X$. If $f : (A, \lim|_A) \longrightarrow (Y, \lim')$ is continuous then $\bar{f} : (X_0, \lim|_{X_0}) \longrightarrow (Y, \lim')$ is continuous.

Proof: We remark first that the subspace $(X_0, \lim|_{X_0})$ satisfies the diagonal axiom (LK). Let now $\mathcal{G} \in \mathcal{F}_L^s(X_0)$ and let $x_0 \in X_0$ and

$$\alpha = \lim|_{X_0} \mathcal{G}(x_0) = \lim[\mathcal{G}](x_0).$$

For each $x \in X_0$ we choose $\mathcal{F}_x \in H^\top(x)$. As $(\mathcal{F}_x)_A \in \mathcal{F}_L^s(A)$ and $A \subset X_0$, also $(\mathcal{F}_x)_{X_0} \in \mathcal{F}_L^s(X_0)$ and

$$\lim|_{X_0}(\mathcal{F}_x)_{X_0}(x) = \lim[(\mathcal{F}_x)_{X_0}](x) \geq \lim \mathcal{F}_x(x) = \top.$$

So if we define $\mathcal{H}_x = (\mathcal{F}_x)_{X_0} \in \mathcal{F}_L^s(X_0)$ we have $\lim|_{X_0} \mathcal{H}_x(x) = \top$ for all $x \in X_0$. From the axiom (LK) it therefore follows that

$$\alpha = \alpha \wedge \top = \lim|_{X_0} \mathcal{G}(x_0) \wedge \bigwedge_{x \in X_0} \lim|_{X_0} \mathcal{H}_x(x) \leq \lim|_{X_0} \mathcal{G}(\mathcal{H}_{(\cdot)})(x_0).$$

We note that $\mathcal{G}(\mathcal{H}_{(\cdot)})$ has a trace on A : Let $b|_A = \perp_A$ for $b \in L^{X_0}$. Then for $x \in X_0$ we have

$$\mathcal{H}_x(b) = (\mathcal{F}_x)_{X_0}(b) = \bigvee \{\mathcal{F}_x(c) : c|_{X_0} \leq b\}.$$

If $c|_{X_0} \leq b$ then $c|_A \leq b|_A = \perp_A$. As \mathcal{F}_x has a trace on A we therefore conclude $\mathcal{F}_x(c) = \perp$. Therefore $\mathcal{H}_x(b) = \perp$. From this we conclude that $\mathcal{H}_{(\cdot)}(b) = \perp_{X_0}$ and therefore

$$\mathcal{G}(\mathcal{H}_{(\cdot)})(b) = \mathcal{G}(\perp_{X_0}) = \perp.$$

As a consequence we have that $[\mathcal{G}(\mathcal{H}_{(\cdot)})] \in H^\alpha(x_0)$ and therefore

$$\lim' f([\mathcal{G}(\mathcal{H}_{(\cdot)})]_A)(\bar{f}(x_0)) = \lim' f(\mathcal{G}(\mathcal{H}_{(\cdot)})_A)(\bar{f}(x_0)) \geq \alpha.$$

Clearly, $(\mathcal{H}_x)_A = ((\mathcal{F}_x)_{X_0})_A = (\mathcal{F}_x)_A \in \mathcal{F}_L^s(A)$ and we can define for $x \in X_0$

$$\mathcal{K}_x = f((\mathcal{H}_x)_A) \in \mathcal{F}_L^s(Y).$$

For $b \in L^Y$ we find

$$\begin{aligned} \mathcal{G}(\mathcal{K}_{(\cdot)})(b) &= \mathcal{G}(\mathcal{K}_{(\cdot)}(b)) = \mathcal{G}(f((\mathcal{H}_{(\cdot)})_A(b))) \\ &= \mathcal{G}((\mathcal{H}_{(\cdot)})_A(f^\leftarrow(b))) = \mathcal{G}((\mathcal{H}_{(\cdot)})_A)(f^\leftarrow(b)) = f(\mathcal{G}((\mathcal{H}_{(\cdot)})_A))(b). \end{aligned}$$

By this we see that $f(\mathcal{G}((\mathcal{H}_{(\cdot)})_A)) = \mathcal{G}(\mathcal{K}_{(\cdot)})$ and therefore

$$\lim' \mathcal{G}(\mathcal{K}_{(\cdot)})(\bar{f}(x_0)) \geq \alpha.$$

Further we find $[\mathcal{H}_x]_A = [(\mathcal{F}_x)_{X_0}]_A = (\mathcal{F}_x)_A \in \mathcal{F}_L^s(A)$ and $\lim[\mathcal{H}_x](x) \geq \lim \mathcal{F}_x(x) = \top$. Hence for every $x \in X_0$ we have $[\mathcal{H}_x] \in H^\top(x)$ and therefore $\top = \lim' f([\mathcal{H}_x]_A)(\bar{f}(x)) = \lim' f((\mathcal{H}_x)_A)(\bar{f}(x))$. We conclude therefore

$$\top = \lim' f((\mathcal{H}_x)_A)(\bar{f}(x)) = \lim' \mathcal{K}_x(\bar{f}(x)).$$

Using the regularity of (Y, \lim') and $J = X_0$, $\psi = \bar{f}$ we thus obtain

$$\alpha \leq \bigwedge_{x \in X_0} \lim' \mathcal{K}_x(\bar{f}(x)) \wedge \lim' \mathcal{G}(\mathcal{K}_{(\cdot)})(\bar{f}(x_0)) \leq \lim' \bar{f}(\mathcal{G})(\bar{f}(x_0))$$

and \bar{f} is continuous. \square

We note that if (Y, \lim') is a T2-space, then the extension \bar{f} will be unique: if $y_1, y_2 \in \bigcap_{\alpha \in L} F^\alpha(x)$, then especially $y_1, y_2 \in F^\top(x)$ and for any $\mathcal{F} \in H^\top(x)$ then $\lim' f(\mathcal{F}_A)(y_1) = \top$ and $\lim' f(\mathcal{F}_A)(y_2) = \top$ and hence by (T2), $y_1 = y_2$.

Let us further call a subset $A \subset X$ *dense* in (X, \lim) (or simply *dense* if the space (X, \lim) is clear) if from $x \in X$ it follows that $H_A^\top(x) \neq \emptyset$. We can characterize this concept in stratified L -pretopological spaces as follows.

Lemma 6.2: *Let $(X, \lim) \in |SL-PCS|$ and $A \subset X$. Then A is dense if and only if for every $x \in X$, \mathcal{U}^x has a trace on A .*

Proof: Let first A be dense and $x \in X$. Then there is $\mathcal{F} \in H_A^\top(x)$, i.e. $\mathcal{F}_A \in \mathcal{F}_L^s(A)$ and $\lim \mathcal{F}(x) = \top$. From this we see that $\mathcal{F} \geq \mathcal{U}^x$ and therefore $\mathcal{U}_A^x \in \mathcal{F}_L^s(A)$. The converse follows from $[\mathcal{U}_A^x] \geq \mathcal{U}^x$ and $[\mathcal{U}_A^x]_A = \mathcal{U}_A^x$. \square

For stratified L -topological spaces, this concept coincides with the definition of denseness in Hohle and Šostak [13].

Lemma 6.3: *Let $(X, \lim) \in |SL-TCS|$ and $A \subset X$. Then A is dense if and only if for every $a \in \Delta$, $a|_A = \perp_A$ implies $a = \perp_X$.*

Proof: Let first A be dense and let $a \in \Delta$ and $a|_A = \perp_A$. For any $x \in X$ we know that \mathcal{U}_A^x exists. Therefore $\mathcal{U}^x(a) = a(x) = \perp$. Conversely, we see from the condition of the Lemma that \mathcal{U}^x has a trace on A for every $x \in X$: Let $a|_A = \perp_A$. Then also $\underline{a}|_A = \perp_A$ and $\underline{a} \in \Delta$. Therefore for every $x \in X$ we find $\perp = \underline{a}(x) = \mathcal{U}^x(a)$. \square

We can now state our main theorem.

Theorem 6.4: *Let $(X, \lim) \in |SL-GCS|$ satisfy the axiom (LK) and let $(Y, \lim') \in |SL-GCS|$ be regular and T2 and let $A \subset X$ be dense in (X, \lim) . Then a continuous mapping $f : (A, \lim|_A) \rightarrow (Y, \lim')$ has a unique continuous extension $\bar{f} : (X, \lim) \rightarrow (Y, \lim')$ if and only if for every $x \in X$, $\bigcap_{\alpha \in L} F^\alpha(x) \neq \emptyset$.*

Proof: If f has a continuous extension, \bar{f} , then for every $\mathcal{G} \in \mathcal{F}_L^s(X)$ we have

$$\lim' \bar{f}(\mathcal{G})(\bar{f}(x)) \geq \lim \mathcal{G}(x).$$

If $\mathcal{F} \in H^\alpha(x)$, then $\mathcal{F}_A \in \mathcal{F}_L^s(A)$ and $\lim \mathcal{F}(x) \geq \alpha$. Hence

$$\lim' \overline{f}([\mathcal{F}_A])(\overline{f}(x)) \geq \lim' \overline{f}(\mathcal{F})(\overline{f}(x)) \geq \lim \mathcal{F}(x) \geq \alpha.$$

We will now show that $\overline{f}([\mathcal{F}_A]) = f(\mathcal{F}_A)$: Let $b \in L^Y$. Then

$$\overline{f}([\mathcal{F}_A])(b) = [\mathcal{F}_A](\overline{f}^\leftarrow(b)) = \mathcal{F}(\overline{f}^\leftarrow(b)|_A).$$

For $x \in A$ we have

$$\overline{f}^\leftarrow(b)|_A(x) = b(\overline{f}(x)) = b(f(x)) = f^\leftarrow(b)(x).$$

Therefore

$$\mathcal{F}_A(\overline{f}^\leftarrow(b)|_A) = \mathcal{F}_A(f^\leftarrow(b)) = f(\mathcal{F}_A)(b).$$

So, $\overline{f}([\mathcal{F}_A]) = f(\mathcal{F}_A)$ and therefore $\overline{f}(x) \in F^\alpha(x)$ and the condition is satisfied. For the converse we note that under the condition and the assumptions of the Lemma, $X_0 = X$ and therefore the existence of the continuous extension follows from Lemma 6.1. The uniqueness follows from the assumption of (T2) for (Y, \lim') as shown above. \square

7. Characterization of regularity in the Boolean case

In this section we characterize regularity using a mapping $\overline{\mathcal{F}}^\beta : L^X \longrightarrow L$ for a given stratified L -filter $\mathcal{F} \in \mathcal{F}_L^s(X)$. A close inspection of the proofs shows that we need to demand that all L -filters are tight to ensure that $\overline{\mathcal{F}}^\beta$ is stratified (the other L -filter axioms are always satisfied). One way to go is to drop the requirement of stratification *for all L -filters* in our theory (see in this respect a similar construction in Gähler [9]). The other way to go is to restrict the lattice context. For a stratified L -filter $\mathcal{F} \in \mathcal{F}_L^s(X)$ it holds for any $a \in L^X$

$$\bigwedge_{x \in X} a(x) \leq \mathcal{F}(a) \leq ((\bigvee_{x \in X} a(x)) \rightarrow \perp) \rightarrow \perp$$

(see [12]). From this we see that if in L the *law of double negation*,

$$(\alpha \rightarrow \perp) \rightarrow \perp = \alpha \quad \forall \alpha \in L,$$

holds, then all stratified L -filters will be tight. On the other hand, we cannot find a weaker condition: if all stratified L -filters are tight, then for a stratified L -ultrafilter \mathcal{U} it follows

$$\alpha = \mathcal{U}(\alpha_X) = \mathcal{U}(\alpha_X \rightarrow \perp) \rightarrow \perp = (\alpha \rightarrow \perp) \rightarrow \perp.$$

So in order that the theory of this section works we have to impose the law of double negation, which is equivalent to L being a complete Boolean algebra (cf. e.g. [10]).

In the sequel let (X, lim) be a stratified L -generalized convergence space. For $\mathcal{F} \in \mathcal{F}_L^s(X)$, $\beta \in L$ and $a \in L^X$ we define

$$\overline{\mathcal{F}}^\beta(a) = \bigvee \{ \mathcal{F}(f) : f \in L^X \text{ such that for all } \mathcal{G} \in \mathcal{F}_L^s(X) \text{ with } \text{lim } \mathcal{G}(x) \geq \beta \text{ we have } \mathcal{G}(f) \leq a(x) \}.$$

Lemma 7.1: *If L is a complete Boolean algebra then $\overline{\mathcal{F}}^\beta \in \mathcal{F}_L^s(X)$.*

Proof: Denote $J = \{(\mathcal{G}, y) : \text{lim } \mathcal{G}(y) \geq \beta\}$. We can then write

$$\overline{\mathcal{F}}^\beta(a) = \bigvee_{f : \mathcal{G}(f) \leq a(y) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f).$$

We check the axioms.

(F1) $\overline{\mathcal{F}}^\beta(\top_X) \geq \mathcal{F}(\top_X) = \top$. Further, if $\mathcal{G}(f) = \perp$ for all $(\mathcal{G}, y) \in J$, then especially for all $y \in X$ we have $f(y) = [y](f) = \perp$, i.e. $f = \perp_X$. Hence $\overline{\mathcal{F}}^\beta(\perp_X) = \mathcal{F}(\perp_X) = \perp$.

(F2) Let $a \leq b$. If $\mathcal{G}(f) \leq a(y)$, then also $\mathcal{G}(f) \leq b(y)$ and hence $\overline{\mathcal{F}}^\beta(a) \leq \overline{\mathcal{F}}^\beta(b)$.

(F3) If $\mathcal{G}(f) \leq a(y)$ for all $(\mathcal{G}, y) \in J$ and $\mathcal{G}(g) \leq b(y)$ for all $(\mathcal{G}, y) \in J$, then also $\mathcal{G}(a \wedge b) \leq a \wedge b(y)$ for all $(\mathcal{G}, y) \in J$. Therefore

$$\begin{aligned} \overline{\mathcal{F}}^\beta(a \wedge b) &= \bigvee_{\mathcal{G}(f) \leq a \wedge b(y) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f) \geq \bigvee_{\mathcal{G}(f \wedge g) \leq a \wedge b(y) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f \wedge g) \\ &\geq \bigvee_{\substack{f : \mathcal{G}(f) \leq a \wedge b(y) \ \forall (\mathcal{G}, y) \in J \\ g : \mathcal{G}(g) \leq a \wedge b(y) \ \forall (\mathcal{G}, y) \in J}} \mathcal{F}(f) \wedge \mathcal{F}(g) \\ &= \bigvee_{\mathcal{G}(f) \leq a(y) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f) \wedge \bigvee_{\mathcal{G}(g) \leq b(y) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(g) = \overline{\mathcal{F}}^\beta(a) \wedge \overline{\mathcal{F}}^\beta(b). \end{aligned}$$

(Fs) This is the place where we make use of the tightness of the L -filters, i.e. we use $\mathcal{G}(\alpha_X) \leq \alpha$ for all $\alpha \in L$ and all $\mathcal{G} \in \mathcal{F}_L^s(X)$. We have

$$\overline{\mathcal{F}}^\beta(\alpha_X) = \bigvee_{\mathcal{G}(f) \leq \alpha \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f) \geq \mathcal{F}(\alpha_X) \geq \alpha.$$

□

We call $\overline{\mathcal{F}}^\beta$ the β -closure of \mathcal{F} . It is not difficult to see that in the case $L = \{0, 1\}$ the 1-closure $\overline{\mathcal{F}}^1$ can be identified with the closure $\overline{\mathcal{F}} = [\{\overline{F} : F \in \mathcal{F}\}]$, where \overline{F} is the closure operator in (X, lim) (see e.g. [6] for the definition of the closure operator).

Lemma 7.2: *Let L be a complete Boolean algebra and let $(X, \text{lim}) \in |\text{SL-GCS}|$. Then (X, lim) is regular if and only if for all $\alpha, \beta \in L$, $\text{lim } \mathcal{F}(x) \geq \alpha$ implies $\text{lim } \overline{\mathcal{F}}^\beta(x) \geq \alpha \wedge \beta$.*

Proof: Let first (X, \lim) be regular, $\alpha, \beta \in L$ and $\lim \mathcal{F}(x) \geq \alpha$. Again we define

$$J = \{(\mathcal{G}, y) : \mathcal{G} \in \mathcal{F}_L^s(X), \lim \mathcal{G}(y) \geq \beta\}.$$

For $i = (\mathcal{G}, y) \in J$ we define $\mathcal{G}_{(\mathcal{G}, y)} = \mathcal{G}$ (the first projection) and the mapping $\psi : J \longrightarrow X$ is defined by $\psi(\mathcal{G}, y) = y$ (the second projection). Then

$$\lim \mathcal{G}_{(\mathcal{G}, y)}(\psi(\mathcal{G}, y)) = \lim \mathcal{G}(y) \geq \beta.$$

We define a stratified L -filter $\mathcal{S} \in \mathcal{F}_L^s(J)$ by

$$\mathcal{S}(a) = \bigvee_{\mathcal{G}(f) \leq a(\mathcal{G}, y) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f) \quad (a \in L^J).$$

(That \mathcal{S} is in fact a stratified L -filter on J can be shown in the same way as for $\overline{\mathcal{F}}^\beta$.) Then $\mathcal{S} \leq \mathcal{S}(\mathcal{G}_{(\cdot)})$. To see this, let $a \in L^X$. Then

$$\mathcal{S}(\mathcal{G}_{(\cdot)})(a) = \mathcal{S}(\mathcal{G}_{(\cdot)}(a)) = \bigvee_{\mathcal{G}(f) \leq \mathcal{G}_{(\cdot)}(a)(\mathcal{G}, y) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f) = \bigvee_{\mathcal{G}(f) \leq \mathcal{G}(a) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f) \geq \mathcal{F}(a).$$

Therefore $\lim \mathcal{S}(\mathcal{G}_{(\cdot)})(x) \geq \alpha$ and by regularity then $\lim \psi(\mathcal{S})(x) \geq \alpha \wedge \beta$. All what remains to show is that $\psi(\mathcal{S}) = \overline{\mathcal{F}}^\beta$. To this end, let $a \in L^X$. Then we note that $\psi^{\leftarrow}(a)(\mathcal{G}, y) = a(y)$ and hence

$$\psi(\mathcal{S})(a) = \mathcal{S}(\psi^{\leftarrow}(a)) = \bigvee_{\mathcal{G}(f) \leq \psi^{\leftarrow}(a)(\mathcal{G}, y) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f) = \bigvee_{\mathcal{G}(f) \leq a(y) \ \forall (\mathcal{G}, y) \in J} \mathcal{F}(f) = \overline{\mathcal{F}}^\beta(a).$$

Hence $\alpha \wedge \beta \leq \overline{\mathcal{F}}^\beta(x)$ and the condition of the Lemma is true.

Conversely, let

$$\begin{aligned} \alpha &\leq \lim \mathcal{G}_i(\psi(i)) \quad \forall i \in J \\ \beta &\leq \lim \mathcal{F}(\mathcal{G}_{(\cdot)})(x). \end{aligned}$$

(with J, ψ, \mathcal{G}_i and \mathcal{F} as in the axiom (LR)). By assumption then

$$\lim \overline{\mathcal{F}(\mathcal{G}_{(\cdot)})}^\alpha(x) \geq \alpha \wedge \beta.$$

If now $f \in L^X$ is given with $\mathcal{G}(f) \leq a(y)$ for all $\mathcal{G} \in \mathcal{F}_L^s(X)$ and $y \in X$ with $\lim \mathcal{G}(y) \geq \alpha$, then also for all \mathcal{G}_i we have $\mathcal{G}_i(f) \leq a(\psi(i)) = \psi^{\leftarrow}(a)(i)$. Therefore

$$\mathcal{F}(\mathcal{G}_{(\cdot)})(f) = \mathcal{F}(\mathcal{G}_{(\cdot)}(f)) \leq \mathcal{F}(\psi^{\leftarrow}(a)) = \psi(\mathcal{F})(a).$$

From this we conclude

$$\overline{\mathcal{F}(\mathcal{G}_{(\cdot)})}^\alpha(a) \leq \psi(\mathcal{F})(a)$$

and hence

$$\lim \psi(\mathcal{F})(x) \geq \alpha \wedge \beta$$

and (X, \lim) is regular. \square

If (X, \lim) satisfies the axiom (Lpw2), then a nice characterization is possible.

Lemma 7.3: *Let L be a complete Boolean algebra and let $(X, \lim) \in |SL-GCS|$ satisfy the axiom (Lpw2). Then (X, \lim) is regular if and only if for all $\alpha, \beta \in L$ and all $x \in X$ we have $\overline{\mathcal{U}_\alpha^x}^\beta \geq \mathcal{U}_{\alpha \wedge \beta}^x$.*

Proof: Let (X, \lim) be regular. We know by (Lpw2) that $\lim \mathcal{U}_\alpha^x(x) \geq \alpha$. Therefore $\lim \overline{\mathcal{U}_\alpha^x}^\beta(x) \geq \alpha \wedge \beta$ which is the same as $\overline{\mathcal{U}_\alpha^x}^\beta \geq \mathcal{U}_{\alpha \wedge \beta}^x$. Conversely let the condition of the Lemma be true and let $\lim \mathcal{F}(x) \geq \alpha$. Then $\mathcal{F} \geq \mathcal{U}_\alpha^x$ and therefore (the β -closure being order preserving) $\overline{\mathcal{F}}^\beta \geq \overline{\mathcal{U}_\alpha^x}^\beta \geq \mathcal{U}_{\alpha \wedge \beta}^x$. By (Lpw2) therefore $\lim \overline{\mathcal{F}}^\beta(x) \geq \alpha \wedge \beta$. \square

8. Conclusions

In this paper we defined a regularity axiom for lattice-valued convergence spaces using a diagonal condition. Our definition generalizes on the one hand a classical definition of regularity for $\{0, 1\}$ -convergence spaces and on the other hand has nice properties: it is preserved under initial constructions (so that subspaces and products of regular stratified L -convergence spaces are again regular) and ensures that a regular T1-space satisfies (T2). We further gave a theorem on continuous extension which demonstrates that the definition really works and characterized it, in the restricted lattice context of complete Boolean algebras, by a requirement on convergence of closures of convergent L -filters.

Interesting, and to date not known, would be a characterization of our regularity in terms of open sets of a stratified L -topology. Is our regularity related to e.g. the regularity (or star-regularity) of Höhle and Šostak [13]? They also prove an extension theorem, however under the stronger assumption of a strongly dense subset A , whereas we only need a dense subset A . This points to the definitions being different.

References

- [1] J. Adamek., H. Herrlich, and G.E. Strecker, *Abstract and Concrete Categories*, Wiley, New York 1989
- [2] H.J. Biesterfeld, Regular convergence spaces, *Indag. Math.* 28 (1966), 605 – 607
- [3] P. Brock and D.C. Kent, Probabilistic convergence spaces and regularity, *Int. J. Math. & Math. Sci.* 20 (1997), 637 – 646
- [4] E. Čech, *Topological Spaces*, Revised edition by M. Frolik and M.Katětov, Wiley, London 1966
- [5] C.H. Cook, On continuous extensions, *Math. Annalen* 176 (1968), 302 – 304
- [6] C.H. Cook and H.R. Fischer, Regular convergence spaces, *Math. Annalen* 174 (1967), 1 – 7
- [7] P.V. Flores, R.N. Mohapatra and G. Richardson, Lattice-valued spaces: fuzzy convergence, *Fuzzy Sets and Systems* 157 (2006), 2706 - 2714
- [8] W. Gähler, Monadic topology — a new concept of generalized topology, in: *Recent Developments of General Topology*, Mathematical Research 67, Berlin 1992, 136 – 149
- [9] W. Gähler, Monadic convergence structures, in: *Topological and Algebraic Structures in Fuzzy Sets* (S.E. Rodabaugh, E.P. Klement, eds.), Kluwer, Dordrecht, Boston, London 2003
- [10] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove and D.S. Scott, *A Compendium of Continuous Lattices*, Springer, Berlin, Heidelberg, New York 1980
- [11] U. Höhle, Commutative, residuated L-monoids, in: *Non-classical Logics and Their Applications to Fuzzy Subsets* (U. Höhle, E.P. Klement, eds.), Kluwer, Dordrecht 1995
- [12] U. Höhle, Locales and L-topologies, in: *Categorical Methods in Algebra and Topology – A collection of papers in honour of Horst Herrlich*, *Mathematik Arbeitspapiere* 48 (1997), 233 – 250
- [13] U. Höhle and A.P. Šostak, Axiomatic foundations of fixed-basis fuzzy topology, in: *Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory* (U.Höhle, S.E. Rodabaugh, eds.), Kluwer, Dordrecht 1999
- [14] G. Jäger, A category of L-fuzzy convergence spaces, *Quaest. Math.* 24 (2001), 501 – 517

- [15] G. Jäger, Subcategories of lattice-valued convergence spaces, *Fuzzy Sets and Systems* 156 (2005), 1 – 24
- [16] G. Jäger, Pretopological and topological lattice-valued convergence spaces, *Fuzzy Sets and Systems* 158 (2006), 424 – 435
- [17] G. Jäger, Fischer’s diagonal condition for lattice-valued convergence spaces, *Quaest. Math.* 31 (2008), to appear
- [18] D.C. Kent and G.D. Richardson, Convergence spaces and diagonal conditions, *Topology and its Applications* 70 (1996), 167 – 174
- [19] H.-J. Kowalski, Limesräume und Kompletttierung, *Math. Nachrichten* 12 (1954), 301 – 340
- [20] G.D. Richardson and D.C. Kent, Probabilistic convergence spaces, *J. Austral. Math. Soc.* 61 (1996), 1 – 21