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Pretopological and topological lattice-valued convergence spaces

Abstract: We show that the classical axiom which characterizes pretopological convergence spaces splits into two axioms in the general Heyting algebra-valued case. Furthermore we present a generalization of Kowalski's diagonal condition to the lattice-valued case.

Keywords: L-fuzzy convergence, L-topology, L-filter, L-convergence space, limit space, pretopological space, diagonal condition.

1. Introduction

In the paper [4] Flores et al. define a nice category of lattice valued convergence spaces. Their category is a supercategory of the category $SL\text{-}GCS$ of stratified L-generalized convergence spaces [9],[10]. In the special case of left continuity Flores et al.'s category and $SL\text{-}GCS$ are isomorphic [4]. Flores and his colleagues further sketch the definitions of stratified L-pretopological spaces and claim that their definition is "more natural" than the definition of principal lattice valued convergence spaces, as defined as a subcategory of $SL\text{-}GCS$ [9],[10]. In this paper, we have a twofold purpose. First we shall demonstrate that the category $SL\text{-}PCS$ of stratified L-principal convergence spaces (which are also called — similar to the classical case — *stratified L-pretopological spaces*) is isomorphic to the category $SL\text{-}INT$ of spaces with not necessarily idempotent interior operators. Keeping in mind that we do not have the notion of closure at our disposal and therefore have to use the dual notion of interior, the latter spaces seem to be the correct generalization of Čech's closure spaces [2]. For $L = \{0, 1\}$ Čech's closure spaces are isomorphic to principal convergence spaces. We conclude therefore that the category $SL\text{-}PCS$ is the correct generalization of the category of principal convergence spaces to the Heyting algebra-valued case. This shows on the one hand that a generalization of principal convergence spaces in Flores et al.'s sense must require the property of left-continuity. On the other hand, imposing this condition, the question remains if the axiom stated in [4] is then equivalent to the axiom (Lp) of principal stratified L-convergence spaces. We show, secondly, that this is not always the case. The axiom (Lp) splits into two axioms, one of which is Flores et al.'s axiom and the other one is a kind of relaxation of the (Lp) condition. We show with two examples, that the two axioms are in general independent of each other.

A third topic of this paper is a generalization of Kowalski's diagonal condition [12]. As in the classical theory of convergence spaces the validity of the diagonal condition ensures that a principal stratified L-convergence space is topological. The generalization we obtain is on the one hand, in case of $L = \{0, 1\}$, equivalent to Kowalski's condition. On the other hand it is of a remarkable simplicity, which also in the classical case allows a neat formulation of Kowalski's diagonal condition.

2. Preliminaries

We consider in this paper complete lattices L where finite meets distribute over arbitrary joins, i.e. $\alpha \wedge \bigvee_{\iota \in I} \beta_\iota = \bigvee_{\iota \in I} (\alpha \wedge \beta_\iota)$ holds for all α, β_ι ($\iota \in I$). These lattices are called *complete Heyting algebras*. The bottom (resp. top) element of L is denoted by $\perp$ (resp. $\top$). We can then define a *residual implication* by

$$\alpha \rightarrow \beta = \bigvee \{\lambda \in L \,:\, \alpha \wedge \lambda \leq \beta\}.$$

We will often use, without explicitely mentioning, the following properties of the residual implication.

Lemma 2.1 [7]: *Let L be a complete Heyting algebra. The following holds:*

(i) $\alpha \leq \beta \rightarrow \gamma \iff \alpha \wedge \beta \leq \gamma$

(ii) $\alpha \wedge (\alpha \rightarrow \beta) \leq \beta$

(iii) $(\alpha \rightarrow \beta) \rightarrow \beta \geq \alpha$

(iv) $\alpha \leq \beta \implies \alpha \rightarrow \gamma \geq \beta \rightarrow \gamma$ *and* $\gamma \rightarrow \alpha \leq \gamma \rightarrow \beta$

(v) $\alpha \rightarrow (\beta \wedge \gamma) = (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma)$

(vi) $(\alpha \vee \beta) \rightarrow \gamma = (\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)$

(vii) $\alpha \rightarrow \top = \top$ *and* $\top \rightarrow \alpha = \alpha$.

The lattice operations are extended pointwise from L to $L^X = \{a : X \longrightarrow L\}$, the set of all L-sets on X. We denote especially for $A \subset X$ the characteristic function by $1_A : X \longrightarrow L$, $1_A(x) = \top$ if $x \in A$ and $= \perp$ otherwise. For notions from category theory we refer to the textbook [1].

A *stratified L-filter* $\mathcal{F}$ on X [8] is a mapping $\mathcal{F} : L^X \longrightarrow L$ with the properties

(F1) $\mathcal{F}(1_X) = \top, \qquad \mathcal{F}(1_\emptyset) = \bot,$

(F2) $a \leq b \qquad \Longrightarrow \qquad \mathcal{F}(a) \leq \mathcal{F}(b),$

(F3) $\mathcal{F}(a) \wedge \mathcal{F}(b) \leq \mathcal{F}(a \wedge b)$ and

(Fs) $\alpha \wedge \mathcal{F}(a) \leq \mathcal{F}(\alpha 1_X \wedge a) \qquad \forall \alpha \in L,\ f \in L^X$

(where $a, b \in L^X$). The set of all stratified L-filters on X is denoted by $\mathcal{F}_L^s(X)$. An example of a stratified L-filter is the *point L-filter* $[x]$ defined by $[x](a) = a(x)$ (see e.g. [8]). An order on $\mathcal{F}_L^s(X)$ can be defined by $\mathcal{F} \leq \mathcal{G}$ iff for all $a \in L^X$, $\mathcal{F}(a) \leq \mathcal{G}(a)$. For a mapping $f : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$ we define $f(\mathcal{F}) \in \mathcal{F}_L^s(Y)$ by $f(\mathcal{F})(b) = \mathcal{F}(f^{\leftarrow}(b))$ where $f^{\leftarrow}(b) = b \circ f$ (see e.g. [8]). The *meet* $\bigwedge_{\iota \in I} \mathcal{F}_\iota$ of a family L-filters $\{\mathcal{F}_\iota \in \mathcal{F}_L^s(X) \mid \iota \in I\}$ is defined by $(\bigwedge_{\iota \in I} \mathcal{F}_\iota)(a) = \bigwedge_{\iota \in I}(\mathcal{F}_\iota(a))$. Obviously $\bigwedge_{\iota \in I} \mathcal{F}_\iota \in \mathcal{F}_L^s(X)$. We especially denote the coarsest stratified L-filter on X by $\mathcal{F}_0 = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \mathcal{F}$.

3. Stratified lattice-valued principal convergence spaces

A *stratified L-generalized convergence space* $(X, \lim)$ [9],[10] is a set X together with a limit map $\lim : \mathcal{F}_L^s(X) \longrightarrow L^X$ satisfying the axioms

(L1) $\forall\, x \in X, \quad \lim[x](x) = \top$

(L2) $\mathcal{F} \leq \mathcal{G} \qquad \Longrightarrow \qquad \lim\mathcal{F} \leq \lim\mathcal{G}$

A mapping $f : (X, \lim) \longrightarrow (X', \lim')$ between two stratified L-generalized convergence spaces $(X, \lim), (X', \lim')$ is called *continuous* if for all $x \in X$ and for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have

$$\lim \mathcal{F}(x) \leq \lim{}' f(\mathcal{F})(f(x)).$$

The category *SL-GCS* has as objects the stratified L-generalized convergence spaces and as morphisms the continuous mappings. This category is well-fibred, topological over SET and cartesian closed [9], i.e. it has very nice structural properties.

A slight generalization of *SL-GCS* was studied by Flores et al. A *stratified L-convergence space* [4] is a pair $(X, \overline{q})$, where $\overline{q} = (q_\alpha)_{\alpha \in L}$ satisifes the conditions

(a) $[x] \overset{q_\alpha}{\to} x, \mathcal{F}_0 \overset{q_\bot}{\to} x$ for each $x \in X$

(b) $\mathcal{G} \geq \mathcal{F} \overset{q_\alpha}{\to} x$ implies $\mathcal{G} \overset{q_\alpha}{\to} x$

(c) $\mathcal{F} \overset{q_\alpha}{\to} x$ implies $\mathcal{F} \overset{q_\beta}{\to} x$ whenever $\beta \leq \alpha$.

$\mathcal{F} \overset{q_\alpha}{\to} x$ is given the interpretation that $\mathcal{F}$ converges to x with probability at least α. A mapping $f : (X, \overline{q}) \longrightarrow (Y\overline{p})$ is called *continuous* provided $\mathcal{F} \overset{q_\alpha}{\to} x$ implies $f(\mathcal{F}) \overset{p_\alpha}{\to} f(x)$ for all $\mathcal{F} \in \mathcal{F}_L^s(X)$, $x \in X$ and $\alpha \in L$. The category *SL-CS* has as objects all stratified L-convergence spaces and as morphisms the continuous mappings. The category *SL-CS* is well-fibred, topological over SET, cartesian closed and extensional [4]. The subcategory of left-continuous stratified L-convergence spaces, SL-LC-CS, is isomorphic to the category SL-GCS [4]. Here, a stratified L-convergence space is called *left-continuous* provided that $\mathcal{F} \stackrel{q_\alpha}{\to} x$ for each $\alpha \in A$ and $\beta = \bigvee A$ implies $\mathcal{F} \stackrel{q_\beta}{\to} x$ [4]. An isomorphism is given by

$$\Phi : \begin{cases} SL\text{-}GCS & \longrightarrow & SL\text{-}LC\text{-}CS \\ (X, \lim) & \longmapsto & (X, \overline{q}) \\ f & \longmapsto & f \end{cases}$$

with $\mathcal{F} \stackrel{q_\alpha}{\to} x$ iff $\lim \mathcal{F}(x) \geq \alpha$ (see [4]).

Remark: Similar to the proof in [4] that the category SL-CS is extensional, one can show that the category SL-GCS also possesses this property.

An important reflective subcategory of SL-GCS arises by adding an additional axiom:

$$\text{(Lp)} \qquad \lim \mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a)) \qquad \forall\, \mathcal{F} \in \mathcal{F}_L^s(X),\, x \in X.$$

Here, the stratified L-neighbourhood filter, $\mathcal{U}^x$, of $x \in X$ is defined by

$$\mathcal{U}^x(a) = \bigwedge_{\mathcal{G} \in \mathcal{F}_L^s(X)} (\lim \mathcal{G}(x) \to \mathcal{G}(a)).$$

The definition of $\lim \mathcal{F}(x)$ is an L-valued interpretation of "a filter converges to x iff it is finer than the neighbourhood filter of x". Similarly, the definition of $\mathcal{U}^x$ is an L-valued interpretation of "U is a neighbourhood of x iff U belongs to every filter converging to x".

A space satisfying (L1), (L2) and (Lp) is called a *stratified L-principal convergence space* or a *stratified L-pretopological space* [9],[10]. The category with these spaces as objects and continuous mappings as morphisms is denoted by SL-PCS. The next example shows that the category of non-idemtpotent interior spaces is isomorphic to SL-PCS.

4. Example: The category SL-INT

We introduce a category of stratified L-interior spaces. These spaces are similar to spaces with stratified L-interior operator as defined in [8], only the idempotency is dropped. In the case $L = \{0, 1\}$ it is well-known that these spaces are isomorphic to Čech's closure spaces [2].

Definition 4.1: A pair (X, int) of a set X and a mapping $int : L^X \longrightarrow L^X$ is called a *stratified L-interior space* iff

$$\begin{array}{ll} (I1) & int(1_X) = 1_X \\ (I2) & int(a) \leq int(b) \text{ whenever } a \leq b \\ (I3) & int(a) \wedge int(b) \leq int(a \wedge b) \\ (I4) & int(a) \leq a \\ (Is) & \alpha 1_X \leq int(\alpha 1_X) \quad \forall \alpha \in L \end{array}$$

A mapping $f : (X, int) \longrightarrow (X', int')$ is called *continuous* if $int'(b)(f(x)) \leq int(f^{\leftarrow}(b))(x)$ for all $b \in L^{X'}$ and all $x \in X$. The category $SL\text{-}INT$ has as objects the stratified L-interior spaces and as morphisms the continuous mappings.

We define two functors. The functor $\Phi : SL\text{-}INT \longrightarrow SL\text{-}PCS$ is defined by

$$\Phi : \left\{ \begin{array}{ccc} (X, int) & \longmapsto & (X, \lim(int)) \\ f & \longmapsto & f \end{array} \right. .$$

Here, the limit function, $\lim(int)$, is defined by

$$\lim(int)\mathcal{F}(x) = \bigwedge_{b \in L^X} (int(b)(x) \to \mathcal{F}(b)).$$

From (I4) we immediately obtain

$$\lim(int)[x](x) = \bigwedge_{b \in L^X} (int(b)(x) \to [x](b)) = \top.$$

Furthermore, it is obvious that for $\mathcal{F} \leq \mathcal{G}$ then $\lim(int)\mathcal{F} \leq \lim(int)\mathcal{G}$. For the axiom (Lp) we show that $\mathcal{U}^x_{\lim(int)}(a) = int(a)(x)$. We have on the one hand

$$\begin{aligned} \mathcal{U}^x_{\lim(int)}(a) &= \bigwedge_{\mathcal{F} \in \mathcal{F}^s_L(X)} (\lim(int)\mathcal{F}(x) \to \mathcal{F}(a)) \\ &\geq \bigwedge_{\mathcal{F} \in \mathcal{F}^s_L(X)} ((int(a)(x) \to \mathcal{F}(a)) \to \mathcal{F}(a)) \\ &\geq int(a)(x). \end{aligned}$$

On the other hand, it is clear that

$$\mathcal{U}^x_{\lim(int)}(a) \leq \lim(int)\mathcal{U}^x(x) \to \mathcal{U}^x(a),$$

with the stratified L-filter $\mathcal{U}^x$ defined by $\mathcal{U}^x(a) = int(a)(x)$. As apparently $\lim(int)\mathcal{U}^x(x) = \top$ we conclude therefore that also $\mathcal{U}^x_{\lim(int)}(a) \leq int(a)(x)$. Hence

$$\lim(int)\mathcal{F}(x) = \bigwedge_{b \in L^X} \left(\mathcal{U}^x_{\lim(int)}(b) \to \mathcal{F}(b) \right)$$

and the axiom (Lp) holds. If $f : (X, int_X) \longrightarrow (Y, int_Y)$ is continuous, then for $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have

$$\begin{aligned}
\lim(int_X)\mathcal{F}(x) &= \bigwedge_{a\in L^X} (int_X(a)(x) \to \mathcal{F}(a)) \\
&\le \bigwedge_{b\in L^Y} (int_X(f^{\leftarrow}(b))(x) \to \mathcal{F}(f^{\leftarrow}(b))) \\
&\le \bigwedge_{b\in L^Y} (int_Y(b)(f(x)) \to f(\mathcal{F})(b)) \\
&= \lim(int_Y)f(\mathcal{F})(f(x)).
\end{aligned}$$

So $f : (X, \lim(int_X)) \longrightarrow (Y, \lim(int_Y))$ is also continuous. Hence Φ is a functor.

The functor $\Psi : SL\text{-}PCS \longrightarrow SL\text{-}INT$ is defined by

$$\Psi : \begin{cases} (X, \lim) & \longmapsto \quad (X, int(\lim)) \\ \quad f & \longmapsto \qquad f \end{cases}$$

Here, $int(\lim)(a)(x) = \mathcal{U}^x(a)$, with the neighborhood L-filter $\mathcal{U}^x$ of $(X, \lim)$. We leave the straightforward check that $(X, int(\lim)) \in |SL\text{-}INT|$ to the reader. The properties of the residual implication of Lemma 2.1 are used. If $f : (X, \lim_X) \to (Y, \lim_Y)$ is a continuous mapping, then

$$\begin{aligned}
int(\lim{}_Y)(b)(f(x)) &= \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(Y)} (\lim{}_Y\mathcal{F}(f(x)) \to \mathcal{F}(b)) \\
&\le \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X)} (\lim{}_Y f(\mathcal{G})(f(x)) \to f(\mathcal{G})(b)) \\
&\le \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(X)} (\lim{}_X\mathcal{G}(x) \to \mathcal{G}(f^{\leftarrow}(b))) \\
&= int(\lim{}_X)(f^{\rightarrow}(b))(x).
\end{aligned}$$

Hence $f : (X, int(\lim{}_X)) \longrightarrow (Y, int(\lim{}_Y))$ is also continous and Ψ is a functor.

We next show that $\Phi \circ \Psi = id_{SL\text{-}PCS}$. To this end, let $(X, \lim) \in |SL\text{-}PCS|$. Then for $\mathcal{F} \in \mathcal{F}_L^s(X)$ we have

$$\begin{aligned}
\lim(int(\lim))\mathcal{F}(x) &= \bigwedge_{b\in L^X} (int(\lim)(b)(x) \to \mathcal{F}(b)) \\
&= \bigwedge_{b\in L^X} (\mathcal{U}^x(b) \to \mathcal{F}(b)) \\
&= \lim\mathcal{F}(x).
\end{aligned}$$

Finally we show that $\Psi \circ \Phi = id_{SL\text{-}INT}$. Let $(X, int) \in |SL\text{-}INT|$. Then for $a \in L^X$ we have

$$\begin{aligned} int(\lim(int))(a) &= \bigwedge_{\mathcal{F} \in \mathcal{F}^s_L(X)} (\lim(int)\mathcal{F}(x) \to \mathcal{F}(a)) \\ &\geq \bigwedge_{\mathcal{F} \in \mathcal{F}^s_L(X)} ((int(a)(x) \to \mathcal{F}(a)) \to \mathcal{F}(a)) \\ &\geq int(a)(x). \end{aligned}$$

On the other hand, with the stratified L filter $\mathcal{U}^x$ defined by $\mathcal{U}^x(a) = int(a)(x)$, we see:

$$int(\lim(int))(a) \leq \left(\bigwedge_{b \in L^X} (int(b)(x) \to \mathcal{U}^x(b)) \right) \to \mathcal{U}^x(a) = \top \to \mathcal{U}^x(a) = \mathcal{U}^x(a) = int(a)(x).$$

We collect our findings in the following Lemma.

Lemma 4.2: *The category SL-INT is isomorphic to the category SL-PCS.*

5. A characterization of the (Lp) axiom

We adapt a definition of Flores et al. [4] to our setting. For $\alpha \in L$ we denote

$$\mathcal{U}^x_\alpha = \bigwedge_{\mathcal{F} : \lim \mathcal{F}(x) \geq \alpha} \mathcal{F},$$

the *stratified α-level L-neighbourhood filter of x*. We can characterize the stratified L-neighbourhood filter of x, $\mathcal{U}^x$, by the levels.

Lemma 5.1: *For $(X, \lim) \in |SL\text{-}GCS|$ we have*

$$\bigwedge_{\alpha \in L} (\alpha \to \mathcal{U}^x_\alpha(a)) \leq \mathcal{U}^x(a) \quad \forall a \in L^X.$$

If $(X, \lim)$ satisfies the axiom (Lp), then equality holds.

Proof: We have by definition

$$\mathcal{U}^x(a) = \bigwedge_{\mathcal{F} \in \mathcal{F}^s_L(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)).$$

With $\alpha_{\mathcal{F}} = \lim \mathcal{F}(x)$ we have $\mathcal{U}^x_{\alpha_{\mathcal{F}}} \leq \mathcal{F}$. Hence

$$\mathcal{U}^x(a) \geq \bigwedge_{\mathcal{F} \in \mathcal{F}^s_L(X)} \left(\alpha_{\mathcal{F}} \to \mathcal{U}^x_{\alpha_{\mathcal{F}}}(a)\right) \geq \bigwedge_{\alpha \in L} (\alpha \to \mathcal{U}^x_\alpha(a)).$$

If $(X, \lim)$ is a stratified L-principal convergence space, then we have by definition of $\mathcal{U}^x_\alpha$:

$$\begin{aligned}
\lim \mathcal{U}^x_\alpha(x) &= \bigwedge_{a \in L^X} \left(\mathcal{U}^x(a) \to \bigwedge_{\mathcal{F}:\lim \mathcal{F}(x) \geq \alpha} \mathcal{F}(a) \right) \\
&= \bigwedge_{a \in L^X} \bigwedge_{\mathcal{F}:\lim \mathcal{F}(x) \geq \alpha} (\mathcal{U}^x(a) \to \mathcal{F}(a)) \\
&= \bigwedge_{\mathcal{F}:\lim \mathcal{F}(x) \geq \alpha} \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a)) \\
&= \bigwedge_{\mathcal{F}:\lim \mathcal{F}(x) \geq \alpha} \lim \mathcal{F}(x) \ \geq \ \alpha.
\end{aligned}$$

With this we obtain

$$\mathcal{U}^x(a) \leq \bigwedge_{\alpha \in L} (\lim \mathcal{U}^x_\alpha(x) \to \mathcal{U}^x_\alpha(a)) \leq \bigwedge_{\alpha \in L} (\alpha \to \mathcal{U}^x_\alpha(a)) \,.$$

□

Next we look at the limit function.

Lemma 5.2: *For $(X, \lim) \in |SL\text{-}GCS|$ we have*

$$\lim \mathcal{F}(x) \leq \bigvee \{\alpha \in L \ : \ \mathcal{F} \geq \mathcal{U}^x_\alpha\}.$$

If $(X, \lim)$ satisfies the axiom (Lp), then equality holds.

Proof: By definition, $\lim \mathcal{F}(x) \geq \alpha$ implies $\mathcal{U}^x_\alpha \leq \mathcal{F}$. Hence $\alpha \in \{\beta \in L \ : \ \mathcal{F} \geq \mathcal{U}^x_\beta\}$. Taking $\alpha = \lim \mathcal{F}(x)$ we obtain the first claim. Now let $(X, \lim)$ staisfy the axiom (Lp). If $\mathcal{U}^x_\alpha \leq \mathcal{F}$ we conclude

$$\mathcal{U}^x(a) \leq \alpha \to \mathcal{U}^x_\alpha(a) \leq \alpha \to \mathcal{F}(a).$$

Therefore

$$\mathcal{U}^x(a) \to \mathcal{F}(a) \geq (\alpha \to \mathcal{F}(a)) \to \mathcal{F}(a) \geq \alpha.$$

This holds for any $a \in L^X$ and from (Lp) we conclude $\lim \mathcal{F}(x) \geq \alpha$. From this the second claim follows. □

We will next address the question of wether the axiom (Lp) is equivalent to the requirement

$$\lim \mathcal{F}(x) = \bigvee \{\alpha \in L \ : \ \mathcal{F} \geq \mathcal{U}^x_\alpha\}.$$

To this end, we note the following trivial result.

Lemma 5.3: *The axiom (Lp) is equivalent to*

$$\forall \alpha \in L: \ \{\mathcal{F} \in \mathcal{F}^s_L(X) \ : \ \alpha \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a))\} = \{\mathcal{F} \in \mathcal{F}^s_L(X) \ : \ \alpha \leq \lim \mathcal{F}(x)\}.$$

Weakening this condition we arrive at the following axiom

$$\text{(Lpw1)} \qquad \forall \alpha \in L : \bigwedge \{\mathcal{F} \in \mathcal{F}^s_L(X) \ : \ \alpha \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{F}(a))\} = \bigwedge \{\mathcal{F} \in \mathcal{F}^s_L(X) \ : \ \alpha \leq \lim \mathcal{F}(x)\}.$$

Clearly (Lp) impies (Lpw1). Note that on the right side of (Lpw1) we see the stratified α-level neighbourhood L-filter, $\mathcal{U}^x_\alpha$. For the left side we find

$$\begin{aligned}
\bigwedge \{\mathcal{F} \in \mathcal{F}^s_L(X) \ : \ \alpha \leq \bigwedge_{a \in L^X} (\mathcal{U}^x(a) \rightarrow \mathcal{F}(a))\} &= \bigwedge \{\mathcal{F} \in \mathcal{F}^s_L(X) \ : \ \alpha \leq \mathcal{U}^x(a) \rightarrow \mathcal{F}(a) \quad \forall a \in L^X\} \\
&= \bigwedge \{\mathcal{F} \in \mathcal{F}^s_L(X) \ : \ \alpha \wedge \mathcal{U}^x(a) \leq \mathcal{F}(a) \quad \forall a \in L^X\}.
\end{aligned}$$

If we denote the last expression by

$$[\alpha \wedge \mathcal{U}^x] = \bigwedge \{\mathcal{F} \in \mathcal{F}^s_L(X) \ : \ \alpha \wedge \mathcal{U}^x(a) \leq \mathcal{F}(a) \quad \forall a \in L^X\}$$

then we can write the axiom (Lpw1) in the succinct form

$$\text{(Lpw1)} \qquad \forall \alpha \in L : \quad [\alpha \wedge \mathcal{U}^x] = \mathcal{U}^x_\alpha$$

In this way, we can characterize the stratified α-level neighbourhood filters by the L-neighborhood filter. Note that especially for $\alpha = \top$

$$\mathcal{U}^x = [\top \wedge \mathcal{U}^x] = \mathcal{U}^x_\top$$

for an (Lpw1)-space.

We further define the following axiom.

$$\text{(Lpw2)} \qquad \lim \mathcal{F}(x) = \bigvee \{\alpha \in L \ : \ \mathcal{F} \geq \mathcal{U}^x_\alpha\}$$

By Lemma 5.2 we know that (Lp) implies (Lpw2). We can characterize the axiom (Lpw2).

Lemma 5.4: *For $(X, \lim) \in |SL\text{-}GCS|$ the following are equivalent.*

(1) *(Lpw2)*

(2) $\forall \alpha \in L : \ (\lim \mathcal{F}(x) \geq \alpha \iff \mathcal{U}^x_\alpha \leq \mathcal{F})$

(3) $\forall \alpha \in L : \ \lim \mathcal{U}^x_\alpha(x) \geq \alpha$

(4) $\forall \mathcal{F}_i \in \mathcal{F}^s_L(X) \ (i \in I) : \ \lim \left(\bigwedge_{i \in I} \mathcal{F}_i\right)(x) = \bigwedge_{i \in I} \lim \mathcal{F}_i(x)$

Proof:

(1)⇒(2): If $\lim \mathcal{F}(x) \geq \alpha$, then always $\mathcal{U}^x_\alpha \leq \mathcal{F}$. On the other hand let $\mathcal{U}^x_\alpha \leq \mathcal{F}$. Then

$\alpha \in \{\beta \in L \,:\, \mathcal{F} \geq \mathcal{U}^x_\beta\}$ and hence by (Lpw2) $\lim \mathcal{F}(x) \geq \alpha$.

(2)⇒(3): Obvious.

(3)⇒(4): By (L2) $\lim \bigwedge \mathcal{F}_i(x) \leq \bigwedge \lim \mathcal{F}_i(x)$ always holds. On the other hand, let $\alpha = \bigwedge_{i\in I} \lim \mathcal{F}_i(x) \leq \lim \mathcal{F}_i(x)$ for all $i \in I$. Hence $\mathcal{U}^x_\alpha \leq \mathcal{F}_i$ for all $i \in I$ and therefore also $\mathcal{U}^x_\alpha \leq \bigwedge_{i\in I} \mathcal{F}_i(x)$. From (L2) and (3) we thus conclude $\alpha \leq \lim \bigwedge_{i\in I} \mathcal{F}_i(x)$.

(4)⇒(1): Let $\mathcal{F} \geq \mathcal{U}^x_\beta$. Then $\lim \mathcal{F}(x) \geq \lim \bigwedge_{\lim \mathcal{G}(x)\geq\beta} \mathcal{G} = \bigwedge_{\lim \mathcal{G}(x)\geq\beta} \lim \mathcal{G}(x) \geq \beta$. Hence $\lim \mathcal{F}(x) \geq \bigvee\{\alpha | \mathcal{G} \geq \mathcal{U}^x_\alpha\}$. Since the other inequality is always true (Lemma 5.2), this completes the proof. □

We will next clarify the relation between the axioms (Lp), (Lpw1) and (Lpw2).

Lemma 5.5: *For $(X, \lim) \in |SL\text{-}GCS|$ the following are equivalent.*

(1) *(Lp)*

(2) *(Lpw1) and (Lpw2).*

Proof: (1)⇒(2) was mentioned before.

(2)⇒(1): It is always true that $\lim \mathcal{F}(x) \leq \bigwedge_{a\in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a))$. On the other hand, if we let $\alpha = \bigwedge_{a\in L^X} (\mathcal{U}^x(a) \to \mathcal{F}(a))$, then $\alpha \leq \mathcal{U}^x(a) \to \mathcal{F}(a)$ for all $a \in L^X$ and hence $\alpha \wedge \mathcal{U}^x(a) \leq \mathcal{F}(a)$ for all $a \in L^X$. By (Lpw1) then $\mathcal{U}^x_\alpha \leq \mathcal{F}$ and (Lpw2) thus implies $\lim \mathcal{F}(x) \geq \alpha$. □

Note that for $L = \{0, 1\}$ the axiom (Lpw1) simply states that

$$\begin{aligned} [0] &= \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(X)} \mathcal{F} = [\{X\}], \\ \mathcal{U}^x = [\mathcal{U}^x] &= \bigwedge_{x\in\lim \mathcal{F}} \mathcal{F}. \end{aligned}$$

Therefore in this case we have the equivalence (Lp) $\iff$ (Lpw2).

We give two examples which show that this equivalence is not always true for a Heyting algebra $L \neq \{0, 1\}$.

Example 5.6: We give an example of a (Lpw1)-space which does not satisfy (Lpw2) (and consequently also not (Lp)). Recall that $\mathcal{F}_0 = \bigwedge_{\mathcal{F}\in\mathcal{F}^s_L(X)} \mathcal{F}$ is the coarsest stratified L-filter on X. Then for any $b \in L^X$ we have $\mathcal{F}_0(b) = \bigwedge_{x\in X} b(x)$ [8]. Moreover we find

$$\mathcal{F}_0(b) \leq \bigwedge_{\mathcal{F}\neq\mathcal{F}_0} \mathcal{F} \leq \bigwedge_{x\in X} [x](b) = \bigwedge_{x\in X} b(x) = \mathcal{F}_0(b).$$

We define a stratified L-generalized convergence on X as follows.

$$\lim \mathcal{F}(x) = \begin{cases} \bot & \mathcal{F} = \mathcal{F}_0 \\ \top & \mathcal{F} \neq \mathcal{F}_0 \end{cases}$$

Then $(X, \lim)$ does not satisfy (Lpw2):

$$\bigwedge_{\mathcal{F} \neq \mathcal{F}_0} \lim \mathcal{F}(x) = \top \neq \bot = \lim(\bigwedge_{\mathcal{F} \neq \mathcal{F}_0} \mathcal{F})(x) = \lim \mathcal{F}_0(x).$$

Morover we find for $b \in L^X$

$$\mathcal{U}^x(b) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(b)) = \bigwedge_{\mathcal{F} \neq \mathcal{F}_0} \underbrace{(\top \to \mathcal{F}(b))}_{=\mathcal{F}(b)} \wedge \underbrace{(\bot \to \mathcal{F}_0(b))}_{=\top} = \bigwedge_{\mathcal{F} \neq \mathcal{F}_0} \mathcal{F}(b) = \mathcal{F}_0(b)$$

i.e. $\mathcal{U}^x = \mathcal{F}_0$. For the α-level L-neighbourhood filters we find

$$\mathcal{U}^x_\alpha = \begin{cases} \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} \mathcal{F} = \mathcal{F}_0 & \text{for} \quad \alpha = \bot \\ \bigwedge_{\mathcal{F} \neq \mathcal{F}_0} \mathcal{F} = \mathcal{F}_0 & \text{for} \quad \alpha \neq \bot \end{cases}.$$

Therefore for all $\alpha \in L$

$$\bigwedge \{\mathcal{F} \in \mathcal{F}_L^s(X) \mid \mathcal{F}(a) \geq \alpha \wedge \mathcal{F}_0(a)\ \forall a \in L^X\} = \mathcal{F}_0$$

i.e.

$$[\alpha \wedge \mathcal{U}^x] = \mathcal{U}^x_\alpha$$

and (Lpw1) holds.

Example 5.7: This example shows a (Lpw2) space which does not satisfy (Lp) (and consequently also not (Lpw1)). It is interesting in its own right, as it shows a remarkable deviation from the classical case. Let $X = \{x, y\}$ and L be the chain $L = \{\bot, \alpha, \top\}$ such that $\bot < \alpha < \top$. We define the discrete stratified L-generalized convergence on X by [10]

$$\lim \mathcal{F}(x) = \begin{cases} \top & \text{if} \qquad \mathcal{F} \geq [x] \\ \bot & \text{otherwise} \end{cases}$$

The space $(X, \lim)$ satisfies (Lpw2): If $\bigwedge_{\iota \in I} \lim \mathcal{F}_\iota(x) = \top$, then for all $\iota \in I$ we have $\lim \mathcal{F}_\iota(x) = \top$. Hence for all $\iota \in I$ it follows $\mathcal{F}_\iota \geq [x]$. Therefore finally $\bigwedge_{\iota \in I} \mathcal{F}_\iota \geq [x]$ and $\lim \bigwedge_{\iota \in I} \mathcal{F}_\iota(x) = \top$.

We obtain the stratified L-neighborhood filter for this convergence as

$$\begin{aligned} \mathcal{U}^x(a) &= \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \\ &= \bigwedge_{\mathcal{F} \geq [x]} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \wedge \bigwedge_{\mathcal{F} \not\geq [x]} (\lim \mathcal{F}(x) \to \mathcal{F}(a)) \\ &= \bigwedge_{\mathcal{F} \geq [x]} (\top \to \mathcal{F}(a)) \wedge \bigwedge_{\mathcal{F} \not\geq [x]} (\bot \to \mathcal{F}(a)) \\ &= \bigwedge_{\mathcal{F} \geq [x]} \mathcal{F}(a) \wedge \top \ = \ [x](a). \end{aligned}$$

Hence, $\mathcal{U}^x = [x]$. We define for $a \in L^X$

$$\mathcal{F}^*(a) = \begin{cases} \top & \text{if} \quad a = 1_X \\ \alpha & \text{if} \quad a(x) = \top, a(y) \neq \top \\ \alpha & \text{if} \quad a(x) = \alpha \\ \bot & \text{if} \quad a(x) = \bot. \end{cases}$$

It is easily verified that $\mathcal{F}^* \in \mathcal{F}_L^s(X)$. Then for $a \in L^X$ defined by $a(x) = \top, a(y) \neq \top$ we have $[x](a) = \top > \alpha = \mathcal{F}(a)$. Hence $[x] \not\leq \mathcal{F}^*$ and therefore $\lim \mathcal{F}^*(x) = \bot$. On the other hand we get

$$\bigwedge_{a \in L^X} (\mathcal{U}^x(a) \to \mathcal{F}^*(a)) = \bigwedge_{a \in L^X} (a(x) \to \mathcal{F}^*(a)) = (\top \to \top) \wedge (\top \to \alpha) \wedge (\bot \to \bot) \wedge (\alpha \to \alpha) = \alpha,$$

which can be checked by considering the nine different L-sets $a \in L^X$. So $(X, \lim)$ satisfies (Lpw2) but not (Lp) (and consequently also not (Lpw1).

6. Kowalski's diagonal condition

A stratified L-principal convergence space is a *stratified L-topological convergence space* [9] ,[10] if the additional axiom

$$\text{(Lt)} \qquad \forall\, x \in X,\ a \in L^X : \qquad \mathcal{U}^x(a) \leq \bigvee \{\mathcal{U}^x(b) \mid b(y) \leq \mathcal{U}^y(a)\ \forall y \in X\}$$

holds. This axiom is equivalent to the topological axiom for Gähler's Φ-topological spaces [5],[6] (in the L-filter case), however our axiom (Lp) seems to be different from the corresponding axiom in this theory. The papers [5],[6] study an interesting theory of convergence spaces from the general viewpoint of monadic topology.

The categories $SL\text{-}TCS$ of stratified L-topological convergence spaces and $SL\text{-}TOP$ of stratified topological spaces [8] are isomorphic (see [9] and [10]). In order to view the axiom (Lt) differently, we introduce a new notation.

Lemma and Definition 6.1: *Let $\mathcal{G} \in \mathcal{F}_L^s(X)$ and for every $y \in X$ let $\mathcal{F}_y \in \mathcal{F}_L^s(X)$. Then $\mathcal{G}(\mathcal{F}_{(\cdot)})$ defined by*

$$\mathcal{G}(\mathcal{F}_{(\cdot)})(a) = \mathcal{G}(\mathcal{F}_{(\cdot)}(a)) \qquad\qquad (a \in L^X)$$

is a stratified L-filter. Here we denote by $\mathcal{F}_{(\cdot)}(a)$ the L-set $y \longmapsto \mathcal{F}_y(a)$.

Proof: As clearly $\mathcal{F}_{(\cdot)}(1_X) = 1_X$ and $\mathcal{F}_{(\cdot)}(1_\emptyset) = 1_\emptyset$, (F1) follows. For $a \leq b$ we have, the $\mathcal{F}_y$ being stratified L-filters, $\mathcal{F}_y(a) \leq \mathcal{F}_y(b)$ for every $y \in X$ and hence $\mathcal{F}_{(\cdot)}(a) \leq \mathcal{F}_{(\cdot)}(b)$. From this we conclude $\mathcal{G}(\mathcal{F}_{(\cdot)})(a) \leq \mathcal{G}(\mathcal{F}_{(\cdot)})(b)$ and (F2) holds. For (F3) we note that for all $y \in X$ $\mathcal{F}_y(a) \wedge \mathcal{F}_y(b) \leq \mathcal{F}_y(a \wedge b)$. Therefore also $\mathcal{F}_{(\cdot)}(a) \wedge \mathcal{F}_{(\cdot)}(b) \leq \mathcal{F}_{(\cdot)}(a \wedge b)$. Hence

$$\mathcal{G}(\mathcal{F}_{(\cdot)})(a) \wedge \mathcal{G}(\mathcal{F}_{(\cdot)})(b) = \mathcal{G}(\mathcal{F}_{(\cdot)})(a)) \wedge \mathcal{G}(\mathcal{F}_{(\cdot)})(b)) \leq \mathcal{G}(\mathcal{F}_{(\cdot)})(a) \wedge (\mathcal{F}_{(\cdot)})(b))$$

$$\leq \mathcal{G}(\mathcal{F}_{(\cdot)}(a \wedge b)) = \mathcal{G}(\mathcal{F}_{(\cdot)})(a \wedge b).$$

Finally, the stratification condition (Fs) can be verified as follows.

$$\alpha \wedge \mathcal{G}(\mathcal{F}_{(\cdot)})(a) \leq \mathcal{G}(\alpha \wedge \mathcal{F}_{(\cdot)}(a)) \leq \mathcal{G}(\mathcal{F}_{(\cdot)}(\alpha \wedge a)) = \mathcal{G}(\mathcal{F}_{(\cdot)})(\alpha \wedge a).$$

□

With this notation and using (F2), we can state the axiom (Lt) in concise form.

$$\text{(Lt)} \qquad \forall\, x \in X: \qquad \mathcal{U}^x \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)}).$$

We are now going to characterize this axiom by a diagonal condition. We need two Lemmas and leave their straightforward proofs for the reader.

Lemma 6.2: *If $\mathcal{F}_y \leq \mathcal{H}_y$ for all $y \in X$, then $\mathcal{G}(\mathcal{F}_{(\cdot)}) \leq \mathcal{G}(\mathcal{H}_{(\cdot)})$.*

Lemma 6.3: *If $\mathcal{G} \leq \mathcal{H}$ then $\mathcal{G}(\mathcal{F}_{(\cdot)}) \leq \mathcal{H}(\mathcal{F}_{(\cdot)})$.*

We introduce a new axiom.

$$\text{(LK)} \quad \forall \mathcal{G} \in \mathcal{F}_L^s(X), \forall \mathcal{F}_y \in \mathcal{F}_L^s(X)\ (y \in X): \quad \lim \mathcal{G}(x) \wedge \bigwedge_{y \in X} \lim \mathcal{F}_y(y) \leq \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x).$$

Lemma 6.4 (Diagonal condition): *For an (Lp) space $(X, \lim)$ the following are equivalent.*

(1) *(Lt)*

(2) *(LK)*

Proof:

(1)⇒(2): Let $\beta = \bigwedge_{y \in X} \lim \mathcal{F}_y(y)$. Then for every $y \in Y$ we have

$$\beta \leq \lim \mathcal{F}_y(y) = \bigwedge_{a \in L^X} (\mathcal{U}^y(a) \to \mathcal{F}_y(a))$$

We now fix an L-set $a \in L^X$. Then $\beta \wedge \mathcal{U}^y(a) \leq \mathcal{F}_y(a)$ and therefore $\beta \wedge \mathcal{U}^{(\cdot)}(a) \leq \mathcal{F}_{(\cdot)}(a)$. Since $\mathcal{G}$ is a stratified L-filter, this leads to

$$\mathcal{G}(\mathcal{F}_{(\cdot)}) \geq \mathcal{G}(\beta \wedge \mathcal{U}^{(\cdot)}(a)) \geq \beta \wedge \mathcal{G}(\mathcal{U}^{(\cdot)}(a)).$$

Hence we conclude with (Lt) that

$$\begin{aligned}
\lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x) &= \bigwedge_{a\in L^X} \left(\mathcal{U}^x(a) \to \mathcal{G}(\mathcal{F}_{(\cdot)})a))\right) \\
&\overset{(Lt)}{\geq} \bigwedge_{a\in L^X} \Big(\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \to (\beta \wedge \mathcal{G}(\mathcal{U}^{(\cdot)}(a))\Big) \\
&= \bigwedge_{a\in L^X} \left(\underbrace{(\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \to \beta)}_{\geq\beta} \wedge (\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \to \mathcal{G}(\mathcal{U}^{(\cdot)}(a)) \right) \\
&\geq \beta \wedge \bigwedge_{a\in L^X} \Big(\mathcal{U}^x(\mathcal{U}^{(\cdot)}(a)) \to \mathcal{G}(\mathcal{U}^{(\cdot)}(a))\Big) \\
&\geq \beta \wedge \bigwedge_{b\in L^X} (\mathcal{U}^x(b) \to \mathcal{G}(b)) \\
&= \beta \wedge \lim \mathcal{G}(x) \\
&= \bigwedge_{y\in X} \lim \mathcal{F}_y(y) \wedge \lim \mathcal{G}(x).
\end{aligned}$$

(2)$\Rightarrow$(1): Put $\mathcal{G} = \mathcal{U}^x$ and $\mathcal{F}_y = \mathcal{U}^y$ for all $y \in X$. Then $\lim\mathcal{U}^x(x) = \top = \lim\mathcal{U}^y(y)$ for all $y \in X$ and hence by (LK)

$$\top = \lim\mathcal{U}^x(\mathcal{U}^{(\cdot)})(x) = \bigwedge_{a\in L^X} \Big(\mathcal{U}^x(a) \to \mathcal{U}^x(\mathcal{U}^{(\cdot)})(a)\Big).$$

Therefore for every $a \in L^X$ we have $\mathcal{U}^x(a) \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)})(a)$. This means $\mathcal{U}^x \leq \mathcal{U}^x(\mathcal{U}^{(\cdot)})$, i.e. (Lt) holds. $\square$

Remark: The case $L = \{0,1\}$

We identify an $\{0,1\}$-set $\varphi \in \{0,1\}^Z$ with the set $\{y \in Z \mid \varphi(y) = 1\}$. Hence a stratified $\{0,1\}$-filter $\mathcal{F}$ is identified with the filter (again denoted by $\mathcal{F}$) defined by

$$F \in \mathcal{F} \iff \mathcal{F}(F) = 1.$$

We then have in the sense of this identification $\mathcal{F}_{(\cdot)}(A) = \{z \in X \mid A \in \mathcal{F}_z\}$ and the filter $\mathcal{G}(\mathcal{F}_{(\cdot)})$ is defined by the relation

$$A \in \mathcal{G}(\mathcal{F}_{(\cdot)}) \iff \{z \in X \mid A \in \mathcal{F}_z\} \in \mathcal{G}.$$

Kowalski [12] defined a "compression operator" for a family of filters $(\mathcal{F}_y)_{y\in X}$ and a filter $\mathcal{G}$ on X by

$$\kappa(\mathcal{G}, (\mathcal{F}_y)_{y\in X}) = \bigvee_{F\in\mathcal{G}} \bigwedge_{z\in F} \mathcal{F}_z.$$

(Kowalski's notation is different) and stated his *diagonal condition*

$$(K) \qquad \mathcal{G} \longrightarrow x,\ \mathcal{F}_y \longrightarrow y\ \forall y \in X \quad \Rightarrow \quad \kappa(\mathcal{G}, (\mathcal{F}_y)_{y\in X}) \longrightarrow x$$

(where we have written $\mathcal{G} \longrightarrow x$ for $x \in \lim \mathcal{G}$).

Lemma 6.5: *Let $\mathcal{G}, \mathcal{F}_y \in \mathcal{F}(X)$ $(y \in X)$. Then $\kappa(\mathcal{G}, (\mathcal{F}_y)_{y\in X}) = \mathcal{G}(\mathcal{F}_{(\cdot)})$.*

Proof: For $F \subset X$ we denote $\mathcal{H}_F = \bigwedge_{z\in F} \mathcal{F}_z$. First let $A \in \mathcal{G}(\mathcal{F}_{(\cdot)})$. Then $F = \{z \in X \mid A \in \mathcal{F}_z\} \in \mathcal{G}$. So, for every $z \in F$ we have $A \in \mathcal{F}_z$ and hence

$$A \in \bigwedge_{z\in F} \mathcal{F}_z \leq \bigvee_{F\in\mathcal{G}} \bigwedge_{z\in F} \mathcal{F}_z = \kappa(\mathcal{G}, (\mathcal{F}_z)_{z\in X}).$$

Conversely, let $A \in \kappa(\mathcal{G}, (\mathcal{F}_y)_{y\in X})$. Then $A \supset H_1 \cap H_2 \cap ... \cap H_n$ with $H_k \in \mathcal{H}_{F_k}$ where $F_k \in \mathcal{G}$ for all k. Then $H_1 \in \mathcal{F}_z$ for all $z \in F_1$ and $H_2 \in \mathcal{F}_z$ for all $z \in F_2$ and... and $H_n \in \mathcal{F}_z$ for all $z \in F_n$. Hence $\{z|H_1 \in \mathcal{F}_z\} \supset F_1$ and $\{z|H_2 \in \mathcal{F}_z\} \supset F_2$ and ... and $\{z|H_n \in \mathcal{F}_z\} \supset F_n$. Therefore $H_1 \in \mathcal{G}(\mathcal{F}_{(\cdot)}), H_2 \in \mathcal{G}(\mathcal{F}_{(\cdot)}), ..., H_n \in \mathcal{G}(\mathcal{F}_{(\cdot)})$ and, $\mathcal{G}(\mathcal{F}_{(\cdot)})$ being a filter, finally $H_1 \cap H_2 \cap ... \cap H_n \in \mathcal{G}(\mathcal{F}_{(\cdot)})$. So also the superset $A \in \mathcal{G}(\mathcal{F}_{(\cdot)})$. $\square$

The Lemma shows, that for $L = \{0, 1\}$ the axiom (LK) is the same as Kowalski's diagonal condition (K).

7. Conclusions

Flores et al. [4] define the category *SL-P-CS* of stratified L-pretopological spaces as the reflective subcategroy of *SL-CS* with objects all spaces which satisfy the additional axiom:

$$\mathcal{H} \stackrel{q_\alpha}{\to} x \quad \iff \quad \mathcal{H} \geq \bigwedge\{\mathcal{F} \in \mathcal{F}_X^s \mid \mathcal{F} \stackrel{q_\alpha}{\to} x\}$$

Restricting to left-continuous spaces, in view of the isomorphism $\Phi : SL\text{-}GCS \longrightarrow SL\text{-}L\text{-}CS$ with $\mathcal{F} \stackrel{q_\alpha}{\to} x \iff \lim \mathcal{F}(x) \geq \alpha$, this axiom translates to

$$\lim \mathcal{H}(x) \geq \alpha \iff \mathcal{H} \geq \mathcal{U}_\alpha^x.$$

This is our axiom (Lpw2). We showed in this paper that this category is different from the category *SL-PCS* of stratified L-principal convergence spaces (as defined earlier [9],[10]). We also showed that the latter category, however, is isomorphic to the category of stratified L-interior spaces. Only by requiring the additional axiom (Lpw1), which is a slight weakening of the axiom (Lp), Flores et al.'s spaces will be the same as stratified L-principal convergence spaces. This poses a natural question: Are Flores et al.'s spaces related to some kind of lattice-valued interior spaces?

We generalized further Kowalski's diagonal condition for principal convergence spaces to the lattice valued case. The resulting axiom is, also in terms of the classical situation with $L = \{0, 1\}$, a nice formulation of the diagonal condition. In the classical case of convergence spaces, there is a generalization of Kowalski's condition, usually called the condition (F), as it is attributed to H.R. Fischer (first published in [3]). This axiom (F) is equivalent to the fact that a convergence space is topological [11]. We will address the generalization of this axiom to the lattice-valued case in our future work.

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