Preferential Fuzzy Sets: A Key to Voting Pattern.

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ABSTRACT

There is a one-to-one correspondence between ordered partitions and kernels of fuzzy subsets under a natural equivalence relation on them called preferential equality, on any n-element set X_n. We discuss some aspects of this correspondence with respect to counting voter's choice or preference through the notions of Flags, Keychains and Pinned-flags.

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Dear Vice-Chancellor, my fellow Senators, Colleagues, Ladies and Gentlemen,
The first question we pose:

In how many ways can a voter exercise her franchise if she is allowed to vote preferentially ranking n candidates contesting an election ? Let that number be J_n. For the first few n

n	$\mathbf{J_n}$
0	**1**
1	**1**
2	**3**
3	**13**
4	**75**
5	**541**
6	**4683**
7	**47293**
8	**545835**
9	**7087261**

Let us crank by hand for $n = 0, 1, 2, 3$:

For $n = 0$ trivially $J_0 = 1$ since there is only one way to exercise preferential voting. That is, no candidates, no choice, and that is one way!!! Some may be worried about this case. Don't, it is not worth it. Take it for granted.

For $n = 1$ Clearly only one choice is possible for one candidate. So $J_1 = 1$.

For $n = 2$ there are two candidates X and Y. We can choose both with equal preference or prefer X over Y or Y over X. Thus we are left with $J_2 = 3$ ways.

For $n = 3$ there are three candidates X, Y, and Z. Things get a bit more complicated but still manageable with some patience. As before we can choose all three with equal preference or prefer X over Y over Z or equally Y over X over Z etc, giving us 6 possibilities one for each permutation of X, Y and Z. Here are all the **SIX**. XYZ, YXZ, YZX, ZYX, XZY and ZXY. Further two candidates may be equally preferred over the third or vice-versa, $XY > Z, Z > XY, XZ > Y, Y > XZ, YZ > X$ or $X > YZ$, giving us six more possibilities. Thats all. Thus $J_3 = 1 + 6 + 6 = 13$.

It is clear that we cannot calculate the numbers J_n cranking by hand for high values of n. In the last 400 hundred years people did a lot of computation by hand before finding the right Mathematics.
It is well known that in 17th century John Napier (1550-1617) had spent almost fourteen years preparing the logarithmic tables. It was periodically updated and were in use even in my life time in the 1960's.
Kummer in the 19th century spent several years patiently calculating numerical calculations on quadratic residues running into thick notebooks. He also worked on FLT (Fermat's Last Theorem) for many years without success, but left a rich legacy.
Ramanujan in the late 19th century did his calculations in 5 note books which contain a wealth of information on number patterns especially partition identities.
Leonardo Euler (1707-1783) in the 18th century purportedly to have written 700 volumes on Mathematics and Science containing many hand calculations.
I have copy of the note book containing the first 10 million primes prepared by D N Lehmer in 1914. I am proud of it and feel like a millionaire.

Surely J_n is dependent on some kind of partitions. We explore more along the "partition" line.

- First there is integer partition.

For instance $10 = 1 + 2 + 3 + 4$ is a distinct four part partition of 10 whereas $10 = 3 + 3 + 4$ is a three part partition with a repetition. There are other partitions of 10, in fact, 42 of them. Generally a partition τ is an integer partition of n if and only if it is a solution of $1.k_1 + 2.k_2 + \cdots + n.k_n = n$ in non-negative integers $k_1, k_2, \cdots, k_n$. In this case we write τ as $\tau \vdash n$ and is

$$\underbrace{1 + \cdots + 1}_{k_1} + \underbrace{2 + \cdots + 2}_{k_2} + \cdots + \underbrace{i + \cdots + i}_{k_i} + \cdots = n.$$

Observe that k_1 is at most n while k_n is at most 1. Let $p(n)$ denote the number of partitions of n. Then

n	**p(n)**
1	**1**
2	**2**
3	**3**
4	**5**
5	**7**
6	**11**
7	**15**
8	**22**
9	**30**
10	**42**

- Secondly there is set partition.

How many ways can a set X of n elements be split up into a class of disjoint subsets $S_1, S_2, \cdots, S_k$ with distinct sizes of X? Let the answer be B_n ways. Then B_n is a sum over the set of integer partitions. How? Let $|S_1| = n_1, |S_2| = n_2, \cdots, |S_k| = n_k$ denote the sizes of the sets $S_1, S_2, \cdots, S_k$ respectively. Then clearly $n_1 + n_2 + \cdots + n_k = n$ is an integer partition of n. How many set partitions are possible for this integer partition of n? We argue like this. Firstly factorial notation. It is not difficult to convince oneself that $n! = 1.2. \ldots n$ is the number of ways (each is called a permutation) of arranging n elements of X among themselves in all possible order. Any permutation of X that permutes elements of S_1 within S_1, of S_2 within S_2, $\cdots$, of S_k within S_k will leave the set partition $S_1, S_2, \cdots, S_k$ unchanged and there are $n_1!n_2! \ldots n_k!$ of them. Therefore the number of set partitions of X of sizes $n_1, n_2, \cdots, n_k$ is $\dfrac{n!}{n_1!n_2! \ldots n_k!}$. To account for equal sizes we divide the above expression by factorial of the number of times a size is repeated. Thus if τ is an integer partition of the form $1.k_1 + 2.k_2 + \cdots + m.k_m = n$, the number of set partitions associated with τ is

$$\frac{n!}{k_1!k_2!\dots k_m!\ (1!)^{k_1}(2!)^{k_2}\dots(m!)^{k_m}}$$

This is so because the size 1 is repeated k_1 times, size 2 is repeated k_2 times and so on. Thus the total number of set partitions is an aggregation over all possible integer partitions so that we have

$$B_n = \sum_{\tau \vdash n} \frac{n!}{k_1!k_2!\dots k_m!\ (1!)^{k_1}(2!)^{k_2}\dots(m!)^{k_m}}$$

B_n's are called Bell Numbers

n	$\mathbf{B_n}$
0	**1**
1	**1**
2	**2**
3	**5**
4	**15**
5	**52**
6	**203**
7	**877**
8	**4140**
9	**21147**

- Connection to Calculus.

Bell numbers are obtained from key rules of the Differential Calculus known as the Chain Rule and the Product Rule. Consider the repeated derivatives of a composite of two functions $Y(x) = f(g(x))$ in terms of the derivatives of f and g. Denoting these derivatives with suffixes such as f_k, g_k for the k-th derivatives of f and g respectively, $Y_1 = f_1\, g_1$, $Y_2 = f_1\, g_2 + f_2\, g_2^2$, $Y_3 = f_1 g_3 + f_2(3 g_2 g_1) + f_3 g_1^3$ etc. In general the formula is

$$Y_n = \sum \frac{n! f_k}{k_1! k_2! \cdots k_n!} \left(\frac{g_1}{1!}\right)^{k_1} \left(\frac{g_2}{2!}\right)^{k_2} \cdots \left(\frac{g_n}{n!}\right)^{k_n}$$

where as before the summation to be carried over all τ. The number of parts are identified by the derivatives of the outer function and the sizes of each part are identified by the derivatives of inner function. It is a math. marvel if you know what is going on in the formalism of calculus. It is known as Faa Di Bruno's formula named after a clergyman who did mathematics in his spare time. There is curious history behind this identity spanning the whole of 19th century.

- We are ready to get a handle on J_n.

When somebody ranks the candidates preferentially, the set of candidates is partitioned into a class of mutually disjoint subsets (you cannot give the same candidate two different rankings) arranged in descending (or for that matter ascending) order of preference. Therefore it is clear from the previous discussion, we simply have to order every set partition in all possible ways in order to obtain a value for J_n. Thus to get a formula for J_n we make use of the formula for B_n. With τ given by

$$\tau : \ 1.k_1 + 2.k_2 + \cdots + m.k_m = n$$

there are k_1 one-element subsets, k_2 two-element subsets, etc.. Hence one has $k_1 + k_2 + \cdots + k_m$ mutually disjoint subsets with different rankings associated with τ and these can be permuted in $(k_1+k_2+\cdots+k_m)!$ ways each of which, gives an order of preference. Therefore

$$J_n = \sum_{\tau \vdash n} \frac{(k_1 + k_2 + \cdots + k_m)! \ (n!)}{(k_1!k_2! \ldots k_m!) \ ((1!)^{k_1}(2!)^{k_2} \ldots (m!)^{k_m})}$$

We call the above a factorial formula since it involves, and entirely made up of, factorials. Also we can write each of the fractions as multinomial coefficients, viz.,

$$\frac{(k_1 + k_2 + \cdots + k_m)!}{(k_1!k_2! \ldots k_m!)} = \binom{(k_1 + k_2 + \cdots + k_m)}{k_1, k_2, \cdots, k_m}$$

- What is next?

Case One : Consider the case when a voter prefers not to rank at all certain candidate or candidates for various reasons.

If this choice is available to the voter in addition to the normal choice of ranking all candidates, the number of voter's preferences would double that of the normal preferences. This is so because for every choice of ranking all the candidates, there will be another alternative to leave the candidates with least ranking out of the ranking system. In fact one could leave out any one block of preferred candidates out of the ranking system without affecting the preferences on other candidates. The preferential ranking is only relative and not absolute. Thus we have $2J_n$ number of preferential rankings that one can exercise when voting in this case.

Case Two: In a complementary sense the above arguments can equally be applied to the case when a voter has a choice to vote certain candidates or a candidate with absolute confidence, that is to a degree one. In this complementary case also the number of preferences available to a voter is $2J_n$.

- The two cases together.

Here we suppose the voter has a chance to say a definite **“NO”** to all, some or none of the candidates and a definite **“YES”** to none, some others or all in addition to the usual ranking of the rest all or none with a **“SO–SO ”** preference. If we argue as before we should have a double of double the number of preferences as in J_n. So it looks like it is $4J_n$. But there is only one exception that occurs. Is it obvious to you? Yes if you shut your eyes and pause a few moments.
A voter cannot say **“YES”** and **“NO”** to all the candidates at once; however can say YES or NO or SO–SO with the same degree of preference to all.
We have a formula for $F(n)$, the number of preferences ranging from a definite ”yes” indicated by a 1 to definite ”no” by a 0 and a ”So-So” a number anywhere in between 1 and 0, as in J_n, over all integer partitions $\tau \vdash n$ of n:

$$F(n) = \sum_{\tau \vdash n} \frac{4\,(k_1 + k_2 + \cdots + k_n)!\; n!}{(k_1!k_2! \cdots k_n!)(1!^{k_1}2!^{k_2} \cdots n!^{k_n})} - 1$$

where τ is the same as before. There are other cases such as when a voter has exactly k preferences for a specific k. Sounds ”Fuzzy Stuff”, I mean ”Fuzzy Sets”.

- Prelude to Fuzzy Sets.

The first paper on Fuzzy Sets was written by Lotfi Zadeh in 1965 which he reinforced for many years. He proposed "the Principle of incompatibility of complex systems"in 1973 as, I quote, "The essence of this principle is that as the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance or relavence become almost mutually exclusive characteristics. It is in this sense that precise quantitative analyses of the behavior of humanistic systems are not likely to have much relevance to the real-world societal, political, economic, and other types of problems which involve humans either as individuals or in groups."
His student Goguen (1974) is more categorical:
The inexactness of the description is not a liability; on the contrary it is a blessing in that sufficient information can be conveyed with less effort. The vague description is also easier to remember. That is, inexactness makes for greater efficiency.
Bellman calls for "We must balance the needs for exactness and simplicity, and reduce complexity without oversimplification. This is essential to communication and decision making."

- Classical Logic vs Fuzzy Logic.

Classical logic is based on "Principle of Dichotomy". Every proposition is either true or false with the law of excluded middle (LEM). Number of Logicians have examined LEM carefully from time to time. But the status quo prevailed for nearly two thousand years from the Greek times. But the modern day complex systems require multiple truth values for knowledge representation. Jan Łukasiewicz proposed 3-valued logic, true or false or undecided to denote somewhere in between true and false. Zadeh proposed "Fuzzy Logic" to model the truth values of such propositions as the ones below, involving "linguistic wedges" as he called them.
x should be $substantially\ larger$ than 10.
$Very\ few$ can afford a house at the present bond rate.
The price of petrol is $marginally$ increased.
The set of $tall$ people.
So Zadeh used the unit interval $[0, 1]$, all numbers between 0 and 1, as possible degree of truth values of propositions including the classical complementary truth values zero and one. A study of these propositions together with multitudes of concepts of logic with extended truth values constitute "Fuzzy Logic".

- Crisp Sets vs Fuzzy Sets.

A crisp set is simply a subset of a set of elements, entities or objects. Let us consider an example from Economics. Suppose we have a basket of currencies

$$\mathbb{X} = \{D, E, R, Re, Ri, Ye, Yu, Z, Pu, Kw\}$$

which constitutes a set and $\mathbb{Y} = \{D, E, Ye, Pu\}$ consists of those currencies that are stronger than R the Rand, then $\mathbb{Y}$ is a subset of $\mathbb{X}$. We describe $\mathbb{Y}$ as a function $\chi_{\mathbb{Y}}$ called the characteristic function from $\chi : \mathbb{X} \to \{0, 1\}$

$$\chi_{\mathbb{Y}}(x) = \begin{cases} 1 \text{ if } x = D, E, Ye \text{ or } Pu \\ 0 \text{ if } x = R, Re, Ri, Yu, Z, Kw \end{cases}$$

If we want to describe those currencies that are very much stronger than Rand, then we have a fuzzy subset μ of $\mathbb{X}$ which attaches a degree of strength between 1 and 0, with 1 for absolutely strong to 0 absolutely weak and other numbers between 1 and 0 appropriately. Thus μ is a function from $\mathbb{X} \to [0, 1]$. We call the number $\mu(x)$ the degree of membership of x to the fuzzy subset μ of $\mathbb{X}$. Thus μ is given by

$$\mu(x) = \begin{cases} 1 \text{ if } x = D \text{ or } E \\ 3/4 \text{ if } x = Ye \\ 1/2 \text{ if } x = R, Yu, \text{ or } Pu \\ 1/4 \text{ if } x = Re, \text{ or } Ri \\ 1/100 \text{ if } x = Kw \\ 0 \text{ if } x = Z \end{cases}$$

We refer to the subset of $\mathbb{X}$ consisting of those x for which $\mu(x) = 1$ as *core* and to the subset of those x for which $\mu(x) > 0$ as *support*. Thus in this example Z is not in the support while D and E are the core currencies. An early notation used by Yager for fuzzy sets is

$$\mu = \{1/D, 1/E, \frac{3}{4}/Ye, \cdots, \frac{1}{100}/Kw, 0/Z\}$$

But we use the more accepted modern functional notation $\mu : \mathbb{X} \to [0,1]$ for a fuzzy set μ without specifying the degrees of membership values.
Crisp subsets are thought of as fuzzy subsets taking only the two extreme truth values, namely the two element set $\{0,1\}$ for every element x of $\mathbb{X}$. So in some sense the theory of fuzzy sets subsumes that of crisp set theory, but there are also some important differences.

- Boolean Algebra vs Fuzzy Algebra.

The propositional calculus is the study of conjunction (**and**), disjunction (**or**), negation (**not**) of propositions and their truth values. Thus we are lead to the Boolean Algebra of propositions, named after the nineteenth century British Mathematician George Boole who wrote on "Laws of Thought"in 1854. His ideas further developed by A. De Morgan (incidentally was born in 1806, Madurai, Tamil Nadu, India close to where I come from) in Britain and C. S. Peirce in the United States in the late nineteenth century are profoundly influential in modern day Computer Science.
The truth values of conjunction and disjunction are given by maximum and minimum of $\{0, 1\}$ respectively while negation of 1 is 0 and of 0 is 1. These are now the subject matter of first year courses throughout the world.

conjunction	Intersection	*and*	$\bigcap$	$\wedge$	*min*	*wedge*	*meet*
disjunction	Union	*or*	$\bigcup$	$\vee$	*max*	*vee*	*join*
negation	Complement	*not*	$()^c$	$\neg$	*dual*	*not*	*neg*

In the above table we have collected the various equivalent symbolisms. The first and third columns are used in the context of Propositional Calculus, the second and fourth

in Set Theory, fifth and sixth in Boolean Algebra and the last two in Lattice Theory.
Fuzzy Algebra is concerned with the study of Fuzzy Logic of Propositions with extended truth values from $\mathbf{I} = [0, 1]$ and of Fuzzy Set Theory with similar truth values from the unit interval $\mathbf{I}$ for the degree of membership of elements belonging to a fuzzy set of an universal set X. The same table of terms and symbolisms is applicable to the Calculus of Fuzzy Algebra. The only difference is the extended truth values of numbers from the unit interval $\mathbf{I}$. If μ and ν are two fuzzy sets of X and $x \in X$, then

$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x) = max(\mu(x), \nu(x))$$

$$(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x) = min(\mu(x), \nu(x))$$

But for the uninitiated we need to highlight the *negation rule* or the *complement* of a fuzzy set in Fuzzy Algebra. The truth value of $\neg p$ is $1-$ the truth value of p. Similarly the complement μ^c of a fuzzy set μ is defined by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. This is a great and vast departure in direction from the Classical Logic and Boolean Algebra, sometimes fruitful and at other times very irksome. It is reflective of doing away with the LEM. Instead of Boolean Algebra of sets we get a Complete Lattice of Fuzzy Sets.

- Alpha-Cuts.

For a real number $\alpha \in \mathbf{I}$, the alpha-cut of a fuzzy set μ is the crisp set containing all elements of X that belong to μ at least to a degree α, that is, $\{x \in X : \mu(x) \geq \alpha\}$. We denote it by μ^α or A^α. It is clear that "higher the degree, fewer the elements". Hence if $\alpha_0 > \alpha_1 > \cdots > \alpha_n$ are the degrees of membership then the various alpha-cuts form an increasing chain of subsets $A^{\alpha_0} \subset A^{\alpha_1} \subset \cdots \subset A^{\alpha_n}$ where A^{α_n} may or may not be equal to X depending on α_n is zero or not respectively. The use of alpha cuts enables one to decompose a fuzzy set into its distinct significant constituent parts. Conversely given such an increasing chain of subsets of X with decreasing real numbers one can construct a fuzzy set by $\mu = \displaystyle\bigvee_{i=0}^{n} \alpha \chi_{A^{\alpha_i}}$. It looks horrible and my intention is not to scare you away but to show that a germ of an idea behind it, which we exploited in our research.

The kernel $Ker(\mu)$ is a partition Π of X whose blocks are just the collection of elements of X with the same degree of membership to μ. The blocks are ordered in a way reverse to the ordering of membership values. The decompositions are related to the kernels in an interesting way.

- Preferential Equality.

Fuzzy sets inherently have preferences built into it. For example if an element x has a higher degree of membership than another y, we can interpret it as x is preferred more than y. So the relative degrees of membership of elements automatically reflect a preference relation among the elements of X with respect a fuzzy set. We capture this notion by stipulating or requiring that two fuzzy sets are $preferentially\, equal$, ($\sim$) if they preserve relative membership degrees between any two elements. Technically, $\mu \sim \nu$ if and only if, they have the same core and support, and in addition, $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$ for any two x and y in X.
It is easily checked that this relation is indeed an equivalence relation on the set of fuzzy sets on X and when restricted to crisp sets of X coincides with equality of sets. That is, '$\sim$' preserves the Boolean Algebra of sets. The relation $\mu \sim \nu$ is true if and only if for each $\alpha > 0$ there exists an $\beta > 0$ such that $\mu^\alpha = \nu^\beta$ and conversely.
A class of fuzzy sets on X that are preferentially equal to each other is called a preferential fuzzy set. Different preferential fuzzy sets express different preferences that one can exercise on the elements of X.

- Pinned-flags.

The kernel of a fuzzy subset of X generates an ordered partition on X, the blocks of which are formed by elements with same membership value and the ordering is opposite to that of membership values. Conversely an ordered partition gives rise to a fuzzy set on X when n real numbers from the unit interval **I**, one for each block, are assigned in such a way that the $alpha$-cuts of it are in the reverse order to the numbers in the unit interval **I**. Using the kernels we further develop preferential fuzzy sets in yet another useful way called $pinned - flags$. It is a pair ($\mathcal{C}, \ell$)

$$X_0^1 \subset X_1^{\lambda_1} \subset X_2^{\lambda_2} \cdots \subset X_n^{\lambda_n} : \bigvee \{\lambda_i \chi_{X_i} : 0 \leq i \leq n\} = \mu$$

where $\mathcal{C}$ is a flag on X, that is a maximal chain of subsets

$$\mathcal{C} : X_0 \subset X_1 \subset X_2 \cdots \subset X_n = X$$

and ℓ is a keychain from **I**, that is,

$$\ell : 1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

Preferential fuzzy sets and pinned-flags are in one-to-one correspondence. Hence we could count the number of pinned-flags to get the number of preferential fuzzy sets which is the same as the number of preferences a voter can exercise in choosing a candidate.

- Combinatorics of $F(n)$.

By carefully analysing the pinned-flags which in fact are preferential fuzzy sets of X in another guise, we get the number $F(n)$ in different forms. We write down some of these expressions without going through the mathematical calculations. This derivation is based on the principle of inclusion and exclusion.

$$F(n) = 4\left(\sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^{n}\right) - 1$$

Using Umbral Calculus, in particular, the shift and difference operators, we have another formula,

$$F(n) = \sum_{i=2}^{\infty} i^{n}\, 2^{(1-i)}$$

A recurrence relation for $F(n)$ is given by

$$F(n+1) = \sum_{j=0}^{n}\binom{n+1}{j}F(j) + 2^{(n+1)}$$

Finally an exponential generating function is of the form

$$\frac{e^{2x}}{2-e^{x}} = \sum_{n=0}^{\infty}\frac{F(n)}{n!}x^{n}$$

This last identity has many interesting properties mostly mathematical. It is one of $F_t(x) = e^{tx}/2 - e^x$ for $t = 2$. There are conjunctures on other integral values of t. For $t = 0, 1, 2$ and $t = 3$ we have satisfactory answers. So we have come a full circle starting with a combinatorial problem, identifying it as question in ordered partitions and then dealing with it from preferential fuzzy sets point view.

Some of the other questions in this area. Firstly we could restrict the number of preferences, say, $1 \leq k \leq n$. How many preferences are available? The answer is $k!S(n, k)$ where $S(n, k)$ are the Stirling numbers of the second kind. These numbers come from the product of integers

$$n(n-1)\ldots(n-k+1).$$

They are named after the inventor, James Stirling, 18th century Scottish Mathematician.

Preferential fuzzy sets can be considered on sets with some algebraic operations such as groups, vector spaces. These pose some very interesting questions for further development of preferential fuzzy algebra along the combinatorial and algebraic lines.

Then there are geometric questions regarding representations of preferential fuzzy sets as simplexes, in particular the keychains as simplexes. Also lattice diagrams and tree diagrams have a geometric role to play in preferential fuzzy sets.

To expand further in the direction of voting patterns, one should investigate a set of candidates belonging to different parties to be preferentially ranked, not necessarily along party line but mixing up the ranking of candidates across the parties and come up with preferential voting. This will be a truer reflection of a voter's mind in choosing candidates. The Mathematics involved is even more complex but the tools of preferential fuzzy sets are adequate. So far we managed to come up with an answer for a number of candidates belong to only two parties. More work is needed for three or more parties containing at least three candidates each.

Then there are the competition outcomes as preferential rankings. That will be very challenging given that there are several criteria that are vague and are expressed in terms of linguistic wedges.

The dynamics of decision-making in the context of preferential fuzzy sets goes deep into the Mathematics of combinatorics, Umbrel Calculus and their associated ideas. But I stop here. Thank you.

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