ONTOLOGICAL COMMITMENT AND MATHEMATICS

BY

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GRAHAMSTOWN
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By taking ontology and mathematics as the topic for this inaugural address, I hope that a wider audience than only my mathematical colleagues will find something that may be of some interest to them.

Mathematics is generally supposed to be the exact science which allows no difference of opinion that cannot be settled by logical argument. The truth of the matter is that there is more to mathematics than logic, and that there exist differences of opinion on the foundations of mathematics of such a fundamental nature that agreement on these issues seems to be only a remote possibility. These differences centre around the philosophical question as to the mode of existence of mathematical entities such as numbers, points and straight lines, i.e. as to the ontological status of mathematical entities. It may come as a surprise to some of you that a great many mathematical theorems can be regarded as valid or invalid depending on the ontological position the mathematician adopts.

At the end of the previous century and the beginning of this century, this led to violent recriminations among mathematicians. To illustrate this, I wish to mention Kronecker who insisted that numbers like the square root of 2 do not 'exist', while his colleague and adversary, Weierstrass, believed that he had made the square root of 2 as comprehensible and as safe to handle as 2 itself by his work on the foundations of analysis. In a letter to his pupil, Sonja Kowalewski, Weierstrass complained:(1)

"But the worst of it is, that Kronecker uses his authority to proclaim that *all* those who up to now have laboured to establish the theory of functions are sinners before the Lord.

(1) [1] p. 528.

When a whimsical eccentric like Christoffe says that in twenty or thirty years the present theory of functions will be buried and that the whole of analysis will be referred to the theory of forms, we reply with a shrug. But when Kronecker delivers himself of the following verdict, which I repeat word for word: "If time and strength are granted me, I myself will show the mathematical world that not only geometry, but also arithmetic can point the way to analysis, and certainly a more rigorous way. If I cannot do it myself, those who come after me will . . . and they will recognise the incorrectness of *all* those conclusions with which *so-called* analysis works at present"—such a verdict from a man whose eminent talent and distinguished performance in mathematical research I admire as sincerely and with as much pleasure as all his colleagues, is not only humiliating for those whom he adjures to acknowledge as an error and to forswear the substance of what has constituted the object of their thought and unremitting labour, but it is a direct appeal to the younger generation to desert their present leaders, and rally around him as the disciple of a new system which *must* be founded. Truly it is sad, and it fills me with a bitter grief, to see a man, whose glory is without a flaw, let himself be driven by the well justified feeling of his own worth to utterances whose injurious effect upon others he seems not to perceive."

Poincaré commented on Kronecker's views in his own peculiar way when he said that Kronecker had been enabled to do so much fine mathematics because he frequently forgot his own mathematical philosophy.

The causes of these divergent views, hinted at in the above quotations, stem from differences in ontological premises assumed by different mathematicians. It is impossible to cover the vast amount of research done on the foundations of mathematics since the last century in order to clarify these differences in a short lecture like this one. I shall, however, endeavour to explain what turned out to be the basic differences in the various ontological premises, and to point out a few simple consequences of certain ontological positions.

Mathematics, as a science, grew from two roots, viz. geometry and arithmetic. Geometry starts with the basic concepts points, straight lines, planes and space, while arithmetic is based on the concept natural number, i.e. the numbers we use to count objects: 1, 2, 3, 4, 5, ... Now, as we know, points are idealizations of dots, and straight lines are idealizations of strokes drawn with pencil and ruler. Undoubtedly most of us have been taught at our first encounter with geometry that a point has no dimensions and only indicates position, while a straight line indicates direction, has no width, and any finite part of it only has length. The deficiencies of such descriptions of points and straight lines can be made a matter of discussion. It is doubtful, however, whether they can be improved to any great extent, since the concepts straight line and point are the indefinable basic concepts of geometry with supposedly self-evident logical relations between them. It is impossible to give rigorous definitions of the concepts point and straight line, except for the axioms defining the logical relations between these entities, and the teacher has to rely on vague abstractions and the intuition of the pupil to make these concepts and their logical relations clear. Intuition is used here in the sense of an immediate awareness of self-evident truth. Asked what a straight line is, a mathematician, in his more sophisticated moments, will refuse to admit that it is anything more than what the axioms of Euclidean geometry tell us when these are regarded as an extended definition of the relevant geometric entities. He will tell you that it is unnecessary to form an image of a straight line, since a proof in geometry is a logical deduction from the axioms, and these deductions can be made without the aid of any image of the entities defined by the axioms. In his less sophisticated moments he will probably admit that the axioms of Eculidean geometry were obtained from the relations between idealizations of dots and strokes with pencil and ruler, in which such a stroke is deprived of width and is thought of as existing of points which have no dimensions. Furthermore, a straight line is thought of as extending both ways to infinity. It is this image I refer to when speaking of straight lines. The statement that between any two different points A and B on a straight line there exists another point C of the straight line, and that between A and C, and C and B, other points of the straight line can be found, and so on ad infinitum, is taken to be self-evident, to be clearly comprehensible by intuition, and to be without need of any further explanation.

Similarly, most mathematicians accept the natural numbers as intuitively given in the course of their ordinary activities. Brouwer made the intuitive evidence of the natural numbers the most fundamental assumption for the whole of mathematics. Brouwer acknowledges only the intuition of time, from which he derives the intuition of number. According to Brouwer, number is derived from "the splitting-up of moments of life into qualitatively different parts which, separated only by time, can be reunited."(2) This is explained to mean that the foremost and primitive act of intellectual construction consists of the following three stages:

1. being conscious of a specific moment of life, which is the totality of impressions that occupy our attention at a given moment;
2. observing that this moment of life is replaced and followed by another one while the memory of the previous moment of life is retained;
3. acknowledging that stages 1 and 2 can be repeated indefinitely.

If the specific moments of life that are observed are now stripped of their properties that distinguish one from the other, then there remains only the fundamental mathematical intuition which consists of the consciousness of a unity that is carried over into a bi-unity and the unending repetition of this process.

(2) [2] p. 208.

Now, according to this point of view, an infinite sequence is an unending or unlimited process of intellectual construction on the lines just described. Hence the sequence of natural numbers is forever in a process of being created by intellectual construction, which presumably means, in words comprehensible to ordinary mortals like myself, that, howver far you count, it is always possible to add another natural number to the numbers you have already used in counting.

The point I wish to emphasis is that Brouwer's intuitionism entails a peculiar idea of the infinite in mathematics. Firstly, only a process can be predicated by the word *infinite,* and secondly, an infinite process can never be taken to be completed, since it is forever in a state of creating new entities. Weyl wrote: "Brouwer made it clear, as I think beyond any doubt, that there is no evidence supporting the belief in the existential character of the totality of all natural numbers . . . The sequence of numbers which grows beyond any stage already reached by passing to the next number, is a manifold of possibilities open towards infinity; it remains forever in the status of creation, but it is not a closed realm of things existing in themselves. That we blindly converted one into the other is the true source of our difficulties, including the antinomies—a source of more fundamental nature than Russell's vicious circle principle indicated. Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the 'absolute' that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence."(3)

In passing I may mention that the antinomies referred to by Weyl include such as the Russell-paradox in which the notion of the set of all sets occur. If the set of all sets is dichotomised into two subsets, namely into the set A of all those sets not containing themselves as members and into the set B of all those containing themselves as members, then it can be proved, with-

(3) [3] p. 48.

out any flaw in the logic used ,that A is a member of itself and also that A is not a member of itself. Contradictions, however, are the death of mathematics. This and other paradoxes therefore helped to spark off the controversy about the concept infinity as it is used by mathematicians.

I hope that I have made it clear that starting with the sequence of natural numbers as being intuitively given, whether derived from the intuition of time as Brouwer does or otherwise, the natural tendency is then to regard an infinite sequence as exactly what these words say, viz. that it is a never-ending row of entities in the sense that, however far you proceed with constructing consecutive entities, it is always possible to add another entity to the already existing row of entities, and that the natural numbers are never present in our consciousness as a complete totality. Starting with the natural numbers as intuitively given, this idea of infinity shows itself to be in accordance with our intuition of natural numbers. Weyl was quoted as saying that the natural numbers do not form "a closed realm of things existing in themselves", and "that there is no evidence supporting the belief in the existential character of the *totality of all natural* numbers." After hearing the intuitionistic approach described above, this may seem quite plausible to you, and it may appear somewhat incomprehensible that the opposite view might conceivably exist.

In order to show that other opinions about the concept infinity exist, we return to geometry. I have mentioned that Brouwer acknowledges only the intuition of time. We need not be afraid to differ from him on this point, since we shall find ourselves in excellent company. Kant, for instance, held that there exists a pure intuition with respect to time and space. I have been told that his use of the word intuition does not concur with mine. However, I shall not use Kant in any way to substantiate my ideas.

Let us consider the existence of a straight line as intuitively evident. We have been taught at school to identify the points of a straight line with numbers. When we were taught graphs, we were shown an x–axis and a y–axis. The point where the two axes intersect was identified with the number 0. After choosing a suitable length as a unit for measurement, other points were identified with the natural numbers. Let us consider now the line segment on the x–axis with end points corresponding to the numbers 0 and 1. The point halfway between the endpoints is identified with the number $\frac{1}{2}$. The midpoints of the two resulting line segments are identified with the numbers $\frac{1}{4}$ and $\frac{3}{4}$. This process can be continued ad infinitum. It is easy to give a method by which, to every rational number between 0 and 1 we wish to name, a point of the line segment can be allotted. We have now reached the stage where we have an infinity of points, every one identified with a rational number, in front of us. This infinity of points lies before us, open for inspection. They exist in their totality, and any one we may choose to pick is there for the picking. By identifying the rational numbers between 0 and 1 with points on a line segment, it has become plausible to think of these numbers as a given totality existing independently of the human mind. It is now possible to think of the existence of these numbers as not dependent on the reflecting subject. Since this tendency asserted itself especially in the philosophy of Plato, it is called "platonism".

To show that the natural numbers can be viewed from a platonist position, consider the points on the line segment between 0 and 1 associated with the rational numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, . . . etc. They are merely the inverses of the natural numbers 1, 2, 3, . . . etc. Now, instead of labelling the points under consideration with 1, $\frac{1}{2}$, $\frac{1}{3}$, . . . etc., associate the numbers 1, 2, 3, . . . etc., with these points in the same order. In this way we have found a set of points on the line segment between 0 and 1 representing the natural numbers. Again, these points might be thought of as existing in their totality independently of the mind of the reflecting mathematician. And since we have associated with each of these points a natural number, it is possible to do what Weyl would not like us to do, namely, to think of the natural numbers as "a closed realm of things existing in themselves".

Hence we have the two opposing views about the concept infinity. The first, which we shall call the *intuitionistic view,* is that the concept is related to a never-ending *process* in which an act can be repeated without any limit. According to this view, the natural numbers do not form a complete, finished totality of entities, but are forever in the process of being constructed by intellectual acts based on our intuition of time. Hence we can never speak of *the set of all natural numbers* in the context of the intuitionistic mathematics of Brouwer. The idea of an infinite set as a complete totality is not allowed.

Kronecker's denial of the 'existence' of the number 'square root of 2' is an extreme consequence of this point of view. Since $\sqrt{2}$ is ordinarily defined as an infinite decimal fraction, one can reason that the digits following the decimal point are never given in their totality but are always in the process of being calculated by given rules. This process never stops, since, if it stops, i.e. if every digit following after a certain one is zero, it will mean that $\sqrt{2}$ is a rational number, which is not the case, as can be proved easily. Consequently the calculation of the digits following the decimal point is an infinite process. At any stage of the calculation we have only a finite number of decimal figures, representing a number whose square is not quite equal to two. Since to speak of 'the infinite set of all digits following the decimal point' has no meaning, the assertion of the existence of $\sqrt{2}$ is nonsensical.

On similar grounds Kronecker denied the existence of all irrational numbers. Discussing Lindemann's proof that π is transcendental, Kronecker asked, 'Of what use is your beautiful investigation regarding π? Why study such problems, since irrational numbers do not exist?'(4)

Brouwer and his disciples do not subscribe to this extreme finitistic view. They have introduced notions by which it is possible to avoid such extreme conclusions. However, we shall not pursue this matter any further.

Taking platonism as our point of departure, we may regard the phrase 'the set of *all* natural numbers' as meaningful, since

(4) [1] p. 627.

an infinite set as a given complete toatlity is allowed as a meaningful concept.

The natural numbers are there already, existing independently of the reflecting subject. They are existing, say, in some heaven for numbers, all of them sitting in a row, and the mathematician can go exploring among them: the totality of natural numbers forming a complete and finished set.

The decimal expansion of $\sqrt{2}$ is regarded from this point of view as complete, every digit already existing and waiting to be discovered by the mathematician—hence Weierstrass's confident assertion that $\sqrt{2}$ 'exists'.

It is my opinion that these two contrasting ontologies of the infinite, viz. platonism and Brouwer's intuitionism, are rooted in the different intuitive evidence forming the bases of the concepts space and number, the latter essentially reflecting our intuition of discreteness, the former reflecting, among other things, our intuition of continuity.

The weakest platonic assumption is the existence of the totality of integers. However, platonic conceptions in mathematics extend far beyond the assumption of the existence of the totality of members of a sequence of integers. The totality of real numbers is taken to exist in the platonic sense, and in Cantor's theories of sets the platonic point of view assumes truly majestic proportions.

Cantor not only assumed the existence of infinite sets in a platonic way, but went further and assigned magnitude to these sets. To understand how this is done, we shall first consider two finite sets, each with an equal number of elements. To take concrete examples, let the set A consist of ten men and let the set B consist of ten chairs. Since each set has the same number of elements, it is possible to assign to each man a chair, and to each chair a man. We say that the set A has been brought into one-one correspondence with the set B. Evidently the elements of any two finite sets with an equal number of elements can always be brought into one-one correspondence. Conversely, if they can be brought into one-one correspondence, the two sets obviously have an equal number of elements.

The notion of one-one correspondence is used to define equality of the 'number of elements' for infinite sets. Thus, if the elements of two infinite sets can be brought into one-one correspondence, the sets are said to be of equal power, which, in the case of finite sets, means they have the same number of elements. This seems a natural enough extension of the idea of equal numbers of elements of finite sets to infinite sets. There is one unexpected consequence, namely that an infinite set can have the same power as one of its proper subsets. Consider, for example, the set A of all natural numbers and the Set E of all even natural numbers, viz. 2, 4, 6, . . . A can be brought into one-one correspondence with E by assigning to 1 the even number 2, to 2 the even number 4, to 3 the number 6, to 4 the number 8, etc. Clearly, to every natural number is assigned one even number; conversely, to every even number is assigned one natural number. Hence A and E have the same power or, stated more colloquially: A has the same number of elements as E, although E is a proper subset of A.

If the elements of some set A cannot be brought into one-one correspondence with the elements of another Set B, but A can be brought into one-one correspondence with a proper subset of B, then B is said to be of a higher power than A. Stated less rigorously, we may say B is of a greater infinity than A.

Let A now be the set of all natural numbers. Consider all possible subsets of A, and make these the elements of a new set B, i.e. B is the set of all subsets of A. It can be shown that B is of a higher power than A. Similarly, all the subsets of B can be taken to be the elements of a third set C which, again is of a higher power than B, and so on ad infinitum. In this way a hierarchy of infinite sets is created, each set of greater infinity than the previous one. Without going into further detail, it is worth mentioning that Cantor also introduced ordinal numbers for infinite sets. It is possible to prove that there is no greatest ordinal. Unfortunately, Burali-Forti also proved that a greatest ordinal does exist. Now this is serious in any mathematical theory, especially since there is nothing wrong with the logic of both proofs. This made mathematicians somewhat uneasy about the platonic ontology, and even more so since in the Russell-paradox the platonic ontology is also an implicit assumption. Mathematicians, like Brouwer, found this so disturbing that they devoted their lives to putting the whole of mathematics on foundations that would make contradictions such as the Russell- and Burali-Forti-paradoxes impossible.

Nevertheless, the application of platonism in mathematics is so widespread that it is not an exaggeration to say that platonism reigns today in mathematics. A great many definitions and proofs of theorms in mathematics depend for their validity on platonism. A case in point is Poincaré's impredicative definitions. An impredicative definition of a specific real number appeals to the hypothesis that *all* real numbers have a certain property P, or the hypothesis that there exists a real number with the property T. In both cases we have the underlying assumption of the existence of an infinite set in the platonic sense. It is used in particular to prove the fundamental theorem that a bounded set of real numbers always has a least upper bound.

The consistent development of Brouwer's intuitionism leads to a very restricted mathematics. For example, in the context of Brouwer's intuitionistic mathematics, one may not generally make use of disjunctions such as: a series of positive terms is either convergent or divergent. Also, a number of usual theorems must be abandoned; for example, the fundamental theorem that every continuous function has a maximum in a closed interval. Very few results in set theory remain valid in intuitionistic mathematics. Furthermore, intuitionistic mathematics gives rise to theorems that are blatant untruths when the concepts occurring in the theorem are viewed from a platonic standpoint. Such a theorem is: "A real function that is everywhere defined on a closed interval of the real numbers is uniformly continuous on that interval."

I have tried to give an indication of the far-reaching consequences certain ontological presuppositions have for mathematics. To this end I have chosen the platonic view of the existence of infinite sets as opposed to Brouwer's intuitionistic conception of the infinite. These are not the only ontological standpoints with important bearings on mathematics. Other problems such as the relation between logic and mathematics, and to what extent mathematics is characterized as a formal game with symbols, manipulated according to certain rules, are dependent on definite ontological and epistemological commitments for answers. The answers differ according to the ontological position the mathematician assumes.

Sometimes certain positions are weakened by unacceptable consequences of such positions. For example, the platonic position is under suspicion because of paradoxes such as the Russell-paradox. It is, however, seldom necessary to discard a position entirely, since the situation is usually saved by introducing suitable limitations. As to the question of which position is the 'correct' one, it seems to be an impossibility to decide, since the various positions can be developed consistently, and, as is agreed by almost all mathematicians, the only grounds on which such a question can be settled is whether a certain position entails contradictions or not. It is unlikely that any new mathematical or metamathematical results will ever definitely refute any ontological standpoint, though they might conceivably have some influence on the readiness to adopt such a standpoint, for reasons which are purely subjective.

The time when mathematicians seriously quarrelled over which standpoint was the 'true' one has passed long ago. Most contemporary mathematicians satisfy themselves with a platonic standpoint of sorts, when they are engaged in research which is not specifically directed towards the foundations of mathematics. Many mathematicians will cheerfully adopt some standpoint just to see how far they get with the implications of this particular standpoint. Though the foundations of set theory are still somewhat shaky, most mathematicians continue to apply its concepts successfully in most branches of mathematics, adopting an attitude of confidence that future research will smooth out most difficulties in a way that will leave their work essentially unaffected.

How do these different ontological standpoints affect the applicability of mathematics to physics, engineering and other applied sciences?

The applicability of mathematics to the physical world is a problem that has received much attention from mathematicians and philosophers alike, and, in keeping with the tradition of philosophy, they differ widely in their answers to this problem. Again the difference arise from various ontological and epistemological commitments.

Without trying to analyse my own position in this respect, I shall try to give in as few words as possible an indication of a trend of thought that may provide an answer.

Propositions of pure mathematics are not dependent on perception for their confirmation. Empirical propositions are, unlike mathematical propositions, capable of being confirmed or refuted by experiment and observation.

Application of mathematics to physical phenomena consists, in my opinion, of replacing empirical concepts and propositions by mathematical concepts and propositions, then deducing consequences from the mathematical premises so obtained, and finaly replacing some of the deduced mathematical propositions by corresponding empirical propositions which can be verified experimentally.

Mathematical theories may differ because of different ontologies or postulates. All of them, however, may be applicable to physical phenomena, provided that the mathematical concepts and propositions concerned are suitable to replace empirical concepts and propositions.

Running the risk of over-simplifying the issue, the following analogy may be helpful in understanding the application of mathematics.

Essentially, the application of mathematics to empirical data is like fitting a conceptual map to these data. Just as a geographer reflects in his map certain relations between cities, rivers, mountains, etc., by corresponding relations between dots, curves and shaded areas, mathematics can be construed to reflect similarly, but certainly in a much more complex way, certain relations between empirical data by means of corresponding relations between sets of numbers. Just as it is immaterial whether the geographer draws his map on a square sheet of paper or on a sphere, provided the relations he wants be portray are faithfully represented on his map by corresponding relations, so it is immaterial what conceptual map the mathematician constructs, provided the relevant relations are faithfully portrayed.

In this analogy I have not tried to be exact, and I leave it to the philosophically inclined to give content to such phrases as 'conceptual maps'.

Mr. Vice-Chancellor and members of the Council
and Senate of Rhodes University,

I came to occupy a chair in mathematics at Rhodes University through somewhat unusual circumstances. I sincerely hope that the trust you have so readily shown me will be jusitfied. I shall serve Rhodes University to the best of my ability, and I hope to contribute in this way to the welfare of our country.

Professor van der Walt,

I wish to thank you for your comradeship and the way you have accepted me as a colleague. It is with deep regret that I see you leaving Rhodes University.

Colleagues in the Department of Mathematics,

Each of us has a part in building a mathematics department worthy of Rhodes University. My wish is that we shall work together to achieve this, as we have done in the past.

Students,

I am happy to be at your service. I shall assist you by word and deed to the best of my ability.

Ladies and Gentlemen,

I thank you for your presence and patient attention.

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