Second Cluster Integral and Excluded Volume Effects for the Pion Gas

A. Kostyuk [1,2], M. Gorenstein [1,2], H. Stöcker[1] and W. Greiner [1]
[1] *Institut für Theoretische Physik, Universität Frankfurt, Germany*
[2] *Bogolyubov Institute for Theoretical Physics, Kyiv, Ukraine*

Abstract

The quantum mechanical formula for Mayer's second cluster integral for the gas of relativistic particles with hard-core interaction is derived. The proper pion volume calculated with quantum mechanical formula is found to be an order of magnitude larger than its classical evaluation.

The second cluster integral for the pion gas is calculated in quantum mechanical approach with account for both attractive and hard-core repulsive interactions. It is shown that, in the second cluster approximation, the repulsive $\pi\pi$-interactions as well as the finite width of resonances give important but almost canceling contributions. In contrast, an appreciable deviation from the ideal gas of pions and pion resonances is observed beyond the second cluster approximation in the framework of the Van der Waals excluded-volume model.

I. INTRODUCTION

Thermal models (”fireball”) have been popular for decades (see, e.g. [1]) to fit the data on multiparticle production in high energy nucleus–nucleus collisions (see, e.g. [2] and references therein). The ideal gas (IG) model of noninteracting hadrons and resonances which has mostly been empoyed to extract temperature T, baryonic chemical potential μ_B, etc., from fits to the data is however not adequate for this purpose. This is among other things because of the following two reasons:

- The ideal gas model ignores the finite width of the resonances, while most of them have a width comparable to or even larger than the typical temperatures of the hadron gas $120 \div 180$ MeV. This leads to underestimation of the attraction between hadrons.
- The IG model does not take into account nonresonance interaction between hadrons, in particular the repulsion. As a result, in the description of the hadron yields data for the AGS and SPS energies the IG model leads to artificially large particle number densities, e.g. $\rho \cong 1.25$ fm^{-3} at $T = 185$ MeV and $\mu_B = 270$ MeV [3], which hardly can be consistent with a picture of a gas of point-like, noninteracting hadrons.

The procedure of Ref. [4] introduces the Breit–Wigner mass spectrum of resonances. However, this widely used procedure works for narrow resonances only. It was found that it is insufficient in the realistic case [5] as it does not take into account correlated particle pairs appearing along with resonances in the hadron gas. Therefore, the standard procedure underestimates tha attractive part of hadron interactions.

To solve the second problem the procedure, which allows to take into acount finite particcle volume, was proposed by Hagedorn and Rafelski [6]. The excluded-volume Van der Waals equation of state was derived in Ref. [7] and used by several authors (see, e.g. [2,3,8] and references therein). Recently, this procedure was generalized to multicomponent [9] and relativistic particle systems [10]. Still, the proper particle volume was so far calculated by *classical* statistical mechanics formulae.

Our aim is to calculate Mayer's cluster integrals (CIs) for the hadron gas from the available data on the hadron scatterings using correct *quantum* formulae and use them for fixing the parameters of the Van der Waals excluded volume model. In the present paper a first step in this direction is made, namely we calculate the second cluster integral (CI) in the case of a pure pion gas for a wide temperature range and consider the contribution of the repulsive part of the $\pi\pi$ interactions into the CI as an excluded volume of the Van der Waals model.

The article is organized as follows: in section II we derive the formula for the second cluster integral taking into account relativistic effects as well as the isospin of the pion. Section III is devoted to the hard-core repulsion at the quantum level. The domain of applicability of the classical formulae is found. The resonance attraction will be considered in section IV. The conditions which allow to use the narrow resonance approximation (NRA) and the Bright–Wigner formula of Ref. [4] will be studied. In section V the CI for the pion gas is calculated from the experimental data on the $\pi\pi$-scattering. The results are compared with various approximations. In section VI the interacting pion gas is studied in the framework of the excluded-volume Van der Waals approach. The conclusions and a discussion of the results are given in section VII.

II. GENERAL FORMULAE

Quantum mechanical formula for a calculation of Mayer's second cluster integral b_2 in the case of nonrelativistic zero isospin $I_0 = 0$ particles was considered in Ref. [11] (see also [12]). The pions, however, have nonzero isospin $I_0 = 1$ and the temperature of interest can be comparable to or larger than the pion mass. Therefore, for an adequate description of the pion gas a generalization of the formulae given in Refs. [11,12] for relativistic particles carrying nonzero isospin is needed.

We start from the canonical partition function for N identical particles in the volume V at the temperature T

$$Z(V,T,N) = \int d^{3N}r \sum_{\alpha} \Psi_{\alpha}^{*}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) \exp\left(-\frac{H}{T}\right) \Psi_{\alpha}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N), \tag{1}$$

where H is the Hamiltonian operator and $\{\Psi_\alpha\}$ is a complete set of orthonormal wave functions in co-ordinate representation. With the notations

$$W_N(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) \equiv N! \sum_{\alpha} \Psi_{\alpha}^{*}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) \exp\left(-\frac{H}{T}\right) \Psi_{\alpha}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) \tag{2}$$

one gets

$$Z(V,T,N) \; = \; \frac{1}{N!}\int d^{3N}r W_N(\mathbf{r}_1,\mathbf{r}_2,\ldots \mathbf{r}_N) \; . \tag{3}$$

The function $W_1(\mathbf{r}_1)$ can be calculated in the thermodynamical limit $V \to \infty$

$$\begin{aligned} W_N(\mathbf{r}_1) &= \sum_{\mathbf{p},t_I} \frac{e^{-i(\mathbf{p},\mathbf{r}_1)}}{\sqrt{V}} \exp\left(-\frac{H}{T}\right) \frac{e^{i(\mathbf{p},\mathbf{r}_1)}}{\sqrt{V}} \\ &= (2I_0+1)\int \frac{d^3p}{(2\pi)^3} \exp\left(-\frac{\sqrt{\mathbf{p}^2+m^2}}{T}\right) = g\phi(T;m), \end{aligned} \tag{4}$$

where I_0 and m are, respectively, the particle isospin and mass ($t_I = -I_0, ..., +I_0$ is the isospin projection), $g \equiv (2I_0+1)$ is the isospin degeneration factor[1] and $\phi(T;m)$ can be expressed via K_2 modified Bessel function

$$\phi(T;m) \; = \; \frac{1}{2\pi^2}\int_0^\infty p^2 dp \; \exp\left(-\frac{\sqrt{p^2+m^2}}{T}\right) \; = \; \frac{m^2 T}{2\pi^2}\; K_2\left(\frac{m}{T}\right) \; . \tag{5}$$

The asymptotics of $\phi(T;m)$ in the nonrelativistic, $m >> T$, and ultra-relativistic, $m << T$, limits are

$$\phi(T;m) \; \simeq \; \begin{cases} \left(\frac{mT}{2\pi}\right)^{3/2} \; \exp(-m/T) \; , \; m >> T \\ \\ \frac{T}{\pi^2}\left(T^2 - \frac{m^2}{4}\right) \; , \qquad\qquad m << T \end{cases} \tag{6}$$

Following Refs. [12,13] we introduce the functions $U_l(\mathbf{r}_1,\mathbf{r}_2,\ldots \mathbf{r}_l)$:

$$\begin{aligned} W_1(\mathbf{r}_1) &= U_1(\mathbf{r}_1) \\ W_2(\mathbf{r}_1,\mathbf{r}_2) &= U_1(\mathbf{r}_1)U_1(\mathbf{r}_2) + U_2(\mathbf{r}_1,\mathbf{r}_2) \\ W_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) &= U_1(\mathbf{r}_1)U_1(\mathbf{r}_2)U_1(\mathbf{r}_3) + U_1(\mathbf{r}_1)U_2(\mathbf{r}_2,\mathbf{r}_3) \\ &\quad + U_1(\mathbf{r}_2)U_2(\mathbf{r}_3,\mathbf{r}_1) + U_1(\mathbf{r}_3)U_2(\mathbf{r}_1,\mathbf{r}_2) + U_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) \\ &\text{etc.} \end{aligned} \tag{7}$$

and define Mayer's CIs[2]

$$b_l(V,T) \; = \; \frac{1}{l!V[g\phi(T;m)]^l}\int d^{3l}r \; U_l(\mathbf{r}_1,\mathbf{r}_2,\ldots \mathbf{r}_l). \tag{8}$$

---

[1] If not only the isospin but also the spin has a nonzero value, J_0, the degeneration factor has the form $g \equiv (2I_0+1)(2J_0+1)$.

[2] The normalizations of the CIs in Eq. (8) are different from that of Ref. [12] and correspond to the definition used in [14]. The CIs (8) have dimensionality [volume]$^{l-1}$, while in Ref. [12] CIs are dimensionless.

It is easy to see that

$$b_1 \;\equiv\; 1 \;. \tag{9}$$

Substituting the expression (7) for the function W_2 into Eq. (3) one gets the two-particle partition function expressed via the CIs

$$Z(V,T,2) \;=\; g^2 \left[\frac{1}{2}(b_1(V,T)\phi(T;m)V)^2 + b_2(V,T)\phi^2(T;m)V\;\right] . \tag{10}$$

For arbitrary N the expression of the partition function via b_l reads [12]

$$Z(V,T,N) \;=\; \sum_{\{m_l\}} \prod_{l=1}^{N} \frac{1}{m_l!}\left(b_l(V,T)\left[g\phi(T;m)\right]^l V\right)^{m_l} \;, \tag{11}$$

where the sum runs over all sets of nonnegative integer numbers $\{m_l\}$ satisfying the condition

$$\sum_{l=1}^{N} l m_l \;=\; N \;. \tag{12}$$

Introducing the absolute activity [14], which in the relativistic case takes the form

$$z \;\equiv\; g\phi(T;m)\exp\left(\frac{\mu}{T}\right) \;, \tag{13}$$

where μ is the chemical potential, the grand canonical partition function has the form

$$\mathcal{Z}(V,T,\mu) \;=\; \sum_{N=1}^{\infty} \exp\left(\frac{\mu N}{T}\right) Z(V,T,N) \;=\; \exp\left(V\sum_{l=1}^{\infty} b_l(V,T) z^l\right) \;. \tag{14}$$

From Eq. (14) one can find the *cluster* expansion of the pressure and the particle density:

$$p(T,\mu) \;=\; T\lim_{V\to\infty}\frac{\log\mathcal{Z}(V,T,\mu)}{V} \;=\; T\sum_{l=1}^{\infty} b_l(T) z^l \;, \tag{15}$$

$$n(T,\mu) \;=\; \lim_{V\to\infty}\frac{T}{V}\frac{\partial\log\mathcal{Z}(V,T,\mu)}{\partial\mu} \;=\; \sum_{l=1}^{\infty} l b_l(T) z^l \;, \tag{16}$$

where

$$b_l(T) \equiv \lim_{V\to\infty} b_l(V,T). \tag{17}$$

Substituting the particle density (16) into the *virial* expansion[3] for the pressure

---

[3] Sometimes the term 'virial expansion' is used in place of 'cluster expansion' [21,22]. We prefer to use the standard terminology [12,14,15]: 'cluster expansion' for the expansion in powers of the activity and 'virial expansion' for that in powers of the particle density.

$$p(T,n) = T\sum_{i=1}^{\infty} a_i n^i \tag{18}$$

and equating the coefficient of each power of z with Eq.(15) one obtains the following expressions for the virial coefficients [12] in terms of the CIs:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= -b_2 \\ a_3 &= 4b_2^2 - 2b_3 \\ a_4 &= -20b_2^3 + 18b_2b_3 - 3b_4 \\ &\ldots \end{aligned} \tag{19}$$

Let us represent the CIs as a sum of two terms:

$$b_l = b_l^{(0)} + b_l^{(i)} \ , \qquad l > 1, \tag{20}$$

where $b_l^{(0)}$ are the CIs for the IG and $b_l^{(i)}$ appear due to the particle interaction. In the classical (Boltzmann) gas one obtains $b_l^{(0)} = 0$ for all $l > 1$. In the quantum case, $b_l^{(0)}$ are nonzero due to Bose (Fermi) effects and can be easily found for arbitrary l. For noninteracting particles the logarithm of the expression (14) should coincide with the well-known expression for the logarithm of the ideal gas grand canonical partition function

$$\log \mathcal{Z}^{(0)}(V,T,\mu) \ = \ \pm\, gV \int \frac{d^3p}{(2\pi)^3} \log\left[1 \pm \exp\left(\frac{\mu - \sqrt{\mathbf{p}^2 + m^2}}{T}\right)\right] \tag{21}$$

(the upper (lower) sign corresponds to Fermi-Dirac (Bose-Einstein) statistics). One can expand the logarithm in the integrand and perform the integration

$$\begin{aligned} \log \mathcal{Z}^{(0)}(V,T,\mu) \ &= \ gV \sum_{l=1}^{\infty} \frac{(\mp 1)^{l+1}}{l} \int \frac{d^3p}{(2\pi)^3} \exp\left(\frac{l\left(\mu - \sqrt{\mathbf{p}^2 + m^2}\right)}{T}\right) \\ &= \ gV \sum_{l=1}^{\infty} \frac{(\mp 1)^{l+1}}{l} \exp\left(\frac{l\mu}{T}\right) \phi(T/l;m) \ . \end{aligned} \tag{22}$$

Comparing the last expression with Eq. (14) gives

$$b_l^{(0)} \ = \ \frac{(\mp 1)^{l+1}}{l g^{l-1}} \, \frac{\phi(T/l;m)}{[\phi(T;m)]^l}. \tag{23}$$

In the nonrelativistic limit Eq.(23) is reduced to

$$b_l^{(0)} \ = \ (\mp 1)^{l+1} l^{-5/2} \left[\frac{\lambda^3}{g}\right]^{l-1} \ , \tag{24}$$

where λ is the thermal wave length

$$\lambda \ = \ \sqrt{\frac{2\pi}{mT}} \ . \tag{25}$$

The expression (24) coincides for $I_0 = 0$ with the corresponding formulae of Ref. [12] (up to the dimensional factor $\lambda^{3(l-1)}$, because of different normalization in Eq. (8)).

Using Eqs. (10) and (20) one can express $b_2^{(i)}$ via differences of the two-particle partition functions for real and ideal gases:

$$b_2^{(i)} \;=\; \frac{Z(V,T,2) - Z^{(0)}(V,T,2)}{V[g\phi(T;m)]^2} \tag{26}$$

Let us calculate $Z(V,T,2)$. A complete set of the orthonormal state vectors in the two particle system can be constructed from the following wave functions

$$|\alpha\rangle \;\equiv\; |\mathbf{P},\tilde{\alpha}\rangle \;=\; \frac{e^{i(\mathbf{P},\mathbf{R})}}{\sqrt{V}}|\tilde{\alpha}\rangle \;, \tag{27}$$

where $\mathbf{P}$ is the total momentum of the system, $\mathbf{R}$ is the radius-vector of its center of mass, and $|\tilde{\alpha}\rangle$ form a complete set of orthonormal state vectors of the system in the center of mass frame (c.m.f.) satisfying the Schrödinger equation

$$H\;|\tilde{\alpha}\rangle \;=\; \varepsilon_{\tilde{\alpha}}\;|\tilde{\alpha}\rangle \tag{28}$$

with the normalization condition

$$\langle\tilde{\alpha}'|\tilde{\alpha}\rangle \;=\; \delta_{\tilde{\alpha}'\tilde{\alpha}} \;. \tag{29}$$

The wave function (27) thus satisfies the following equations

$$H\;|\mathbf{P},\tilde{\alpha}\rangle = \sqrt{\mathbf{P}^2+\varepsilon_{\tilde{\alpha}}^2}\;|\mathbf{P},\tilde{\alpha}\rangle \;, \tag{30}$$

$$\langle\mathbf{P}',\tilde{\alpha}'|\mathbf{P},\tilde{\alpha}\rangle \;=\; \delta_{\tilde{\alpha}'\tilde{\alpha}}\delta_{\mathbf{P}'\mathbf{P}} \;. \tag{31}$$

The expression for $Z(V,T,2)$ in terms of the introduced wave functions has the form

$$Z(V,T,2) \;=\; \sum_{\mathbf{P},\tilde{\alpha}}\langle\mathbf{P},\tilde{\alpha}|\exp\left(-\frac{H}{T}\right)|\mathbf{P},\tilde{\alpha}\rangle \;=\; \sum_{\mathbf{P},\tilde{\alpha}}\exp\left(-\frac{\sqrt{\mathbf{P}^2+\varepsilon_{\tilde{\alpha}}^2}}{T}\right) \;. \tag{32}$$

In the thermodynamical limit $V \to \infty$ the summation over $\mathbf{P}$ can be replaced by the integration and one finds

$$Z(V,T,2) \;=\; \sum_{\tilde{\alpha}} V\int\frac{d^3P}{(2\pi)^3}\exp\left(-\frac{\sqrt{\mathbf{P}^2+\varepsilon_{\tilde{\alpha}}^2}}{T}\right) \;=\; V\sum_{\tilde{\alpha}}\phi(T;\varepsilon_{\tilde{\alpha}}) \;. \tag{33}$$

The states of two spinless particles in their c.m.f. can be enumerated by the following quantum numbers: the radial momentum q (or, alternatively, the energy in c.m.f. $\varepsilon(q) = \sqrt{q^2+m^2}$), the orbital angular momentum L, its projection m_L, the total isospin I and its projection t_I, e.g. $\tilde{\alpha} = (q, L, m_L, I, t_I)$. Eq.(33) can be rewritten explicitly [4]

[4]We assume that the particles do not form bound states with energy $\varepsilon < 2m$.

$$Z(V,T,2) \;=\; V\sum_{I=0}^{2I_0}\sum_{t_I=-I}^{I}\sum_{L}{}'\sum_{m_L=-L}^{L}\int_0^\infty dq g_{Lm_LIt_I}(q)\phi(T;\varepsilon(q))\;, \tag{34}$$

where $g_{Lm_LIt_I}(q)$ is the density of states with the given set of quantum numbers. The sum $\sum'$ extends over only those values of L that satisfy the symmetry properties of the wave function. For spinless bosons it takes even value if the isospin part of the wave function is symmetric and odd values if it is antisymmetric. In the case of integer isospin particles like pions this means [5]

$$L \;=\; \begin{cases} 0,2,4,6,\ldots & \text{for even } I \\ 1,3,5,7,\ldots & \text{for odd } I \end{cases}\;, \tag{35}$$

Substituting Eq.(34) into (26) one gets the following expression for the second CI

$$b_2^{(i)} \;=\; \frac{1}{[g\phi(T;m)]^2}\sum_{I=0}^{2I_0}\sum_{t_I=-I}^{I}\sum_{L}{}'\sum_{m_L=-L}^{L}\int_0^\infty dq(g_{Lm_LIt_I}(q)-g^{(0)}_{Lm_LIt_I}(q))\phi(T;\varepsilon(q))\;, \tag{36}$$

where $g^{(0)}_{Lm_LIt_I}$ is the state density for the IG. The difference $g_{Lm_LIt_I}(q)-g^{(0)}_{Lm_LIt_I}(q)$ in the thermodynamical limit can be expressed via phase shifts of two-particle scattering $\delta_{Lm_LIt_I}(q)$ [11,12]:

$$g_{Lm_LIt_I}(q)-g^{(0)}_{Lm_LIt_I}(q) \;=\; \frac{1}{\pi}\frac{d\delta_{Lm_LIt_I}(q)}{dq}\;. \tag{37}$$

Using the expression for $\phi(T;m)$ one gets

$$b_2^{(i)} \;=\; \frac{2\pi}{m^4T[gK_2(m/T)]^2}\sum_{I=0}^{2I_0}\sum_{t_I=-I}^{I}\sum_{L}{}'\sum_{m_L=-L}^{L}\int_0^\infty dq\varepsilon^2(q)\frac{d\delta_{Lm_LIt_I}(q)}{dq}K_2(\varepsilon(q)/T)\;. \tag{38}$$

In the case of hadron gas the phase shift does not depend on the angular momentum projection m_L (no external fields) and the isospin projection t_I (if only strong interactions are taken into account). This simplifies the last formula:

$$b_2^{(i)} \;=\; \frac{2\pi}{m^4T[gK_2(m/T)]^2}\sum_{I=0}^{2I_0}\sum_{L}{}'(2I+1)(2L+1)\int_0^\infty dq\varepsilon^2(q)\frac{d\delta_{LI}(q)}{dq}K_2\left(\frac{\varepsilon(q)}{T}\right)\;. \tag{39}$$

Performing a partial integration and taking into account the properties of the Bessel functions, one gets

$$b_2^{(i)} \;=\; \frac{2\pi}{m^4T^2[gK_2(m/T)]^2}\sum_{I=0}^{2I_0}\sum_{L}{}'(2I+1)(2L+1)\int_{2m_\pi}^\infty d\varepsilon\varepsilon^2\delta_{L,I}(\varepsilon)K_1\left(\frac{\varepsilon}{T}\right)\;. \tag{40}$$

In the nonrelativistic limit the formula (39) is reduced to

$$b_2^{(i)} = \frac{2\sqrt{2}}{\pi g^2}\lambda^3\sum_{I=0}^{2I_0}\sum_{L}{}'(2I+1)(2L+1)\int_0^\infty dq\frac{d\delta_{LI}(q)}{dq}\exp\left(-\frac{q^2}{mT}\right), \tag{41}$$

which again at $I_0=0$ coincides with the corresponding formulae of Refs. [11,12], up to the factor λ^3.

---

[5] In the case of half-odd isospin scalar bosons we would have the opposite rule: even L for odd I and odd L for even I.

III. HARD CORE REPULSION

The hard core repulsion plays an important role in the phenomenological description of the $\pi\pi$-scattering: the phase shift data for the isospin state $I = 2$ can be successfully described assuming hard core repulsion between two particles [16]. The best fit of the phase shift in S_0-state ($I = L = 0$) can be obtained by assuming hard core repulsion in addition to resonance attraction [17,18]. Hence we start our analysis from applying Eqs.(39–41) to hard sphere model.[6]

The radial part of the wave function in the c.m.f. of two particles interacting by hard core potential in the state with orbital momentum L and radial momentum q can be represented in the following way

$$\phi(r) = \begin{cases} 0 & \text{for } r \le r_0 \\ C(\cos\delta_L \; j_L(qr) + \sin\delta_L \; y_L(qr)) & \text{for } r > r_0 \end{cases} , \tag{42}$$

where r is the distance between particle centers, r_0 is the minimal admitted value for r (that is the doubled radius of the particle considered as a hard sphere), $j_n(z)$ and $y_n(z)$ are spherical Bessel functions, C is the normalization constant and δ_L is fixed by the condition

$$\cos\delta_L \; j_L(qr_0) + \sin\delta_L \; y_L(qr_0) = 0 \; . \tag{43}$$

Using asymptotic properties of the spherical Bessel functions, it is easy to see that δ_L has a meaning of the phase shift describing scattering of two hard spheres:

$$\phi(r) \simeq \frac{\sin\left(qr - \frac{l\pi}{2} + \delta_L\right)}{qr}, \quad r \to \infty \; . \tag{44}$$

The derivative of the phase shift is found to be

$$\frac{d\delta_L}{dq} = \frac{d}{dq} \arctan\left(\frac{j_L(qr_0)}{y_L(qr_0)}\right) = -\frac{r_0}{(qr_0)^2[j_L^2(qr_0) + y_L^2(qr_0)]} , \tag{45}$$

where the formula for the Wronskian [19]

$$W(j_L(z), y_L(z)) = z^{-2} \tag{46}$$

has been used.

Let us introduce the functions

$$\kappa^{\pm}(z) = {\sum_L}' \frac{1}{z^2[j_L^2(z) + y_L^2(z)]} , \tag{47}$$

where the sum ${\sum_L}'$ runs over either even (superscript '+') or odd (superscript '−') nonnegative numbers.

---

[6] Relativistic consideration of a hard sphere model by no means can be consistent. Still, as far as this model describes experimental data on low energy $\pi\pi$-scattering, we find it phenomenologically satisfactory.

Expanding $\kappa^{\pm}$ around zero, one gets

$$\kappa^{+}(z) \simeq 1 + \frac{5}{9}z^4 + O(z^6) \ , \tag{48}$$

$$\kappa^{-}(z) \simeq 3z^2 - 3z^4 + \frac{682}{225}z^6 + O(z^8) \ . \tag{49}$$

It has been checked numerically that the asymptotic behavior of $\kappa^{\pm}(z)$ at large z with a high accuracy can be presented by the formula

$$\kappa^{\pm}(z) \simeq \frac{1}{3}z^2 + \frac{\pi}{4}z + \frac{1}{3} + O(z^{-1}) \ . \tag{50}$$

The CI (39) for the case of hard sphere model can be represented in the form[7]

$$b_2^{(i)} = \frac{1}{g^2}\sum_{I=0}^{2I_0}(2I+1)b_2^{\pm}(r_0,T,m) \ , \tag{51}$$

where superscript '+'('−') corresponds to even (odd) values[8] of I and $b_2^{\pm}(r_0)$ is expressed via the function $\kappa^{\pm}(z)$

$$b_2^{\pm} = -\frac{2\pi r_0}{m^4 T[K_2(m/T)]^2}\int_0^{\infty} dq[\varepsilon(q)]^2\kappa^{\pm}(qr_0)K_2\left(\frac{\varepsilon(q)}{T}\right) . \tag{52}$$

In the nonrelativistic approximation the expression for $b_2^{\pm}(r_0)$ is reduced to

$$b_2^{\pm}(r_0,T,m) = -\frac{2\sqrt{2}}{\pi}\lambda^3 r_0\int_0^{\infty} dq\kappa^{\pm}(qr_0)\exp\left(-\frac{q^2}{mT}\right) . \tag{53}$$

Substituting Eq. (48) into the last formula one gets the nonrelativistic expression for $b_2^{\pm}(r_0,T,m)$ at small r_0 ($r_0 << \lambda$)

$$b_2^{+}(r_0,T,m) \simeq -2\lambda^2 r_0\left(1 + \frac{5}{3}\pi^2(r_0/\lambda)^4 + O\left[(r_0/\lambda)^6\right]\right) , \tag{54}$$

$$b_2^{-}(r_0,T,m) \simeq -6\pi r_0^3\left(1 - 3\pi(r_0/\lambda)^2 + \frac{682}{45}\pi^2(r_0/\lambda)^4 + O\left[(r_0/\lambda)^6\right]\right) , \tag{55}$$

in the opposite case $r_0 >> \lambda$ the nonrelativistic expression for $b_2^{\pm}(r_0,T,m)$ can be obtained using Eq.(50):

$$b_2^{\pm}(r_0,T,m) = -\frac{2}{3}\pi r_0^3\left(1 + \frac{3\sqrt{2}}{4}\frac{\lambda}{r_0} + \frac{3}{2\pi}\frac{\lambda^2}{r_0^2} + O\left[\left(\frac{\lambda}{r_0}\right)^3\right]\right) \tag{56}$$

---

[7] In the simplest version of the hard sphere model when the core radius r_0 does not depend on I, the formula (51) could be reduced to $b_2^{(i)} = b_2^{\pm}(r_0,T,m)$. However, in the case of realistic $\pi\pi$ interaction every isospin state has its own $r_0(I)$ [17,16,18].

[8] Again, for the case of half-odd isospin particles the opposite rule would be valid.

The pion thermal wave length at the temperature $T = 50 \div 200$ MeV ranges between $3 \div 6$ fm. From Eq. (56) one sees that the classical formula [15]

$$b_2^{\pm}(r_0, T, m) \approx -\frac{2}{3}\pi r_0^3 \tag{57}$$

(the particle volume multiplied by 4) would give a reasonable approximation only at unrealistically large hard core radius $r_0 \geq 50$ fm. The hard core radii found from the $\pi\pi$-scattering are much smaller: $r_0 = 0.60$ fm in the S_0 state [18] and $r_0 = 0.17$ fm at $I = 2$ [16]. (No evidence of hard core repulsion was found in P_1-state (I=L=1)). In this case the value of $b_2^{+}(r_0, T, m)$ can be estimated from formula (54). The results are presented in Fig.1. It is seen that in contrast to the classical case the quantum treatment leads to a rather strong dependence of the second CI on the temperature (approximately proportional to $1/T$) even in the nonrelativistic approximation. Numerical calculations show that relativistic effects make this dependence even stronger and essentially reduce at high temperature the CI with respect to its nonrelativistic value (see Fig. 1). Both relativistic and nonrelativistic quantum formulae give a much ($1 \div 2$ orders of magnitude) larger value than those given by the classical formula (57): 0.45 fm^3 and 0.01 fm^3 for $r_0 = 0.60$ fm $r_0 = 0.17$ fm, respectively.

As can be seen from Fig.1, the relativistic effects cannot be ignored even at relatively low temperatures. Therefore, only the relativistic formula (52) is used in the following calculations.

IV. RESONANCE ATTRACTION

The phase shifts of $\pi\pi$ elastic scattering can be approximately described by assuming that the attractive parts of the interaction appear due to the propagation of resonances in the s-channel of the reaction (see [16–18,20] and references therein).

The distinctive feature of the resonance interaction is the rapid growth of the phase shift by π radian in the vicinity of the pion momentum q_r, which is related to the resonance mass M_r:

$$M_r = 2\sqrt{q_r^2 + m^2} \ . \tag{58}$$

In the limit $\Gamma_r \to 0$ (Γ_r is the resonance width) the derivative of the phase shift can be approximated by the Dirac delta-function:

$$\frac{d\delta_{L,I}(q)}{dq} \approx \pi\delta(q - q_r) \ . \tag{59}$$

In this approximation, which we will call the 'narrow resonance approximation' (NRA), the expression (39) for $b_2^{(i)}$ yields:

$$b_2^{(i)} \approx \frac{1}{g^2}\sum_r (2I_r + 1)(2L_r + 1)\frac{\phi(T; M_r)}{[\phi(T; m)]^2} \ , \tag{60}$$

where I_r, L_r and M_r are, respectively, the resonance's isospin, spin and mass, with the index r running over all resonances in the two-pion system.

Eq.(60) allows to rewrite the expression for the grand canonical partition function (14) in the following form

$$\mathcal{Z}(V,T,\mu) = \exp\left[V\left(\sum_{l=1}^{\infty} b_l^{(0)} z^l + \sum_r z_r\right)\right] , \tag{61}$$

where

$$z_r = \phi(T; M_r) \exp\left(\frac{2\mu}{T}\right) \tag{62}$$

is the absolute activity of the resonance r with degeneration factor $g_r = (2I_r + 1)(2L_r + 1)$. The expression (61) is nothing else than the partition function for a mixture of ideal gases of pions and two-pion resonances[9]. This recovers the well known result of Ref. [21] that narrow resonances contribute to the partition function as an ideal gas of stable particles. The quantitative criterion for an applicability of the NRA was found to be [21]

$$\Gamma_r << T \ . \tag{63}$$

The resonances appearing in $\pi\pi$-scattering do not satisfy this criterion: most of them ($\rho(770)$, $f_0(980)$, $f_2(1270)$, $\rho_3(1690)$) have widths comparable with a typical temperature of the hadron gas and the width of $f_0(400 - 1200)$ (known also as the σ) is a few times larger then the temperature. Therefore, it is necessary to take into account the finite width of the resonances.

The scalar-isoscalar resonances contribution to the $\pi\pi$-phase shift in the S_0 state can be parametrized in the following way [17]:

$$\tan\delta_r(q) = \frac{q}{q_r}\frac{M_r^2}{\varepsilon(q)}\frac{\Gamma_r}{M_r^2 - \varepsilon^2(q)} \ , \qquad r = \sigma,\ f_0(980) \ . \tag{64}$$

For parametrization of nonzero (iso-)spin resonances we shall use the following formula [20]

$$\tan\delta_r(q) = \left(\frac{q}{q_r}\right)^{2L_r+1} \frac{M_r x_r \Gamma_r}{M_r^2 - \varepsilon^2(q)}\frac{D_{L_r}(q_r R_r)}{D_{L_r}(q R_r)} \ , \qquad r = \rho(770),\ f_2(1270),\ \rho_3(1690), \tag{65}$$

where x_r is the inelasticity, i.e. the decay fraction of the resonance into two pions, R_r is the so-called interaction radius and the functions $D_L(z)$ are given by the formulae

$$\begin{aligned} D_1(z) &= 1 + z^2 \\ D_2(z) &= 9 + 3z^2 + z^4 \\ D_3(z) &= 225 + 45z^2 + 6z^4 + z^6 \ . \end{aligned} \tag{66}$$

The resonance parameters are given in the Table I.

[9] The fact that the resonance gases are classical is an artifact of the second cluster approximation. The quantum correction would appear from the 4-th and higher cluster integrals for the interacting pions.

It is easy to see that if a resonance lies far from the threshold:

$$M_r - 2m >> \Gamma_r \tag{67}$$

both formulae are reduced to

$$\tan \delta_r(q) \approx \frac{\Gamma_r/2}{M_r - \varepsilon} \tag{68}$$

In this case the activity of the resonance can be represented in the form

$$\sum_r z_r = \sum_{I=0}^{2I_0} \sum_L{}' \int_{2m}^{\infty} d\varepsilon \zeta(\varepsilon)(2I+1)(2L+1)\phi(T;\varepsilon)\ \exp\left(\frac{2\mu}{T}\right), \tag{69}$$

where the resonance profile function is given by the Breit-Wigner formula:

$$\zeta(\varepsilon) = \frac{1}{2\pi} \frac{\Gamma_r}{(\varepsilon - M_r)^2 + (\Gamma_r/2)^2} \tag{70}$$

From this we conclude that the procedure of Ref. [4] where the profile function was postulated to be

$$\zeta(\varepsilon) = \frac{\xi \Gamma_r}{(\varepsilon - M_r)^2 + (\Gamma_r/2)^2} \tag{71}$$

with normalization constant ξ fixed by the condition

$$\int_{2m}^{\infty} d\varepsilon \frac{\xi \Gamma_r}{(\varepsilon - M_r)^2 + (\Gamma_r/2)^2} = 1 \tag{72}$$

becomes valid in the limit (67). The σ-resonance obviously does not satisfy this condition, even for the $\rho(770)$ the difference $M_r - 2m$ is only about 3 times larger then the width. We have calculated the contributions of these two resonances into the second cluster integral $b_2^{(i)}$ using the parametrizations (64) and (65) and compare them with the corresponding approximate values found in the framework of the procedure of Ref. [4]:

$$b_2^{(i)} = \frac{1}{m^4 T [g K_2(m/T)]^2} \sum_r \int_0^{\infty} dq \varepsilon^2(q) \frac{\xi \Gamma_r}{(\varepsilon - M_r)^2 + (\Gamma_r/2)^2} \frac{d\delta_{LI}(q)}{dq} K_2\left(\frac{\varepsilon(q)}{T}\right) . \tag{73}$$

and in the NRA (60). The results are shown in Figs. 2 and 3. As can be seen from Fig.2, for σ-resonance the both approximations essentially underestimate the CI. It is interesting to mention that at $T > 150$ MeV the formula of Ref. [4] gives slightly worse result than even that of the NRA. In the case of the ρ-resonance this formula systematically overestimates the CI in contrast to the σ case. The role of the resonance width becomes small at high temperatures and all three formulae give comparable results in both (σ and ρ) cases.

V. INTERACTING PION GAS

Both type of interaction: hard core repulsion and resonance attraction are present in the pion gas. The phase shift for $\pi\pi$-scattering in the S_0 state at the center of mass energy below 1 GeV can be represented as a sum of three terms [16–18]:

$$\delta_{00}(q) = \delta_\sigma(q) + \delta_{f_0}(q) + \delta_{BG}(q) \ . \tag{74}$$

The background term δ_{BG} is related to the hard core repulsion

$$\delta_{BG}(q) = -r_0 q \ , \qquad r_0 = 3.03 \ \mathrm{GeV}^{-1} \tag{75}$$

and two first terms describe attraction due to the resonances σ and f_0 and are parametrized by the formula (64). The contributions of the S_0 state to the CI are shown in Fig.4. The attractive part is larger in absolute value than the repulsive part, so that total contribution of the S_0 state is positive.

The interaction in the S_2 state has a purely repulsive nature (no exotic resonances with isospin $I = 2$ have been found). The phase shift can be successfully fitted by the hard core formula:

$$\delta_{02}(q) = -r_0 q \ , \qquad r_0 = 0.87 \ \mathrm{GeV}^{-1} \ . \tag{76}$$

As it is seen from Fig.4 the absolute value of the negative contribution of S_2 state into the CI is slightly larger than that of the positive contribution of the S_0 state, so that these two quantities almost cancel each other. The resulting contribution of the S-state into CI is negative and a few times smaller than those of the S_0- and S_2-states separately. Due to the small value of the total S-state contribution into CI the P_1 state becomes important already at relatively low temperature.

The phase shifts of $\pi\pi$-scattering in P_1, D_2 and F_1 states can be parametrized by the formula (65) [20][10]. The results are presented in Fig.5. At small temperatures $T < 80$ MeV both the S- and P-wave give comparable contributions to the CI. At higher temperatures, the P-wave dominates. The D- and F-waves add small corrections to the CI at $T > 140$ MeV. The contribution of higher waves is assumed to be negligible.

It should be mentioned that at very low temperatures, $T < 30$ MeV, the total CI becomes negative, in agreement with the results of Ref. [23] for the isotopically symmetric pion gas, while at high temperatures attraction dominates over repulsion.

The exact CI is also compared in Fig.6 to various approximations widely used for the hadron gas analysis. It is seen that ignoring repulsion between pions overestimates the CI by more than 35%. On the other hand, the NRA underestimates the CI by at least 20%. At low temperatures $T < 120$ MeV both approximations become completely unreasonable. If one ignores both the finite resonance width and the repulsion (it corresponds to the ideal gas of pions and resonances) these two errors partially cancel each other. The simplest

[10]The background and higher pole terms which are present in the formulae of Ref. [20] were found to give a negligible contribution to the CI and are dropped in the present consideration.

approximation, surprisingly enough, appears to give better results than the both more complicated ones. (There remains, however, discrepancy up to about 15% at high temperatures). This means that both effects, the repulsion and the finite resonance width, should be taken into account simultaneously. Including either of these effects without another one increases rather than decreases the numerical errors with respect to the simplest IG model of pion and two-pion resonances.

Comparing the CI $b_2^{(i)}$ with the ideal gas CI $b_2^{(0)}$ (see Fig. 8) one observes that the interactions give essential contribution to the CI already at $T = 70$ MeV. At $T > 150$ MeV the interaction part $b_2^{(i)}$ clearly dominates over Bose effects related to $b_2^{(0)}$.

To estimate the influence of the second CI on the thermodynamical properties of the pion gas we have calculated the particle density in the second cluster approximation

$$n = n_0 + 2b_2^{(i)} z^2, \tag{77}$$

where

$$n_0 = \sum_{l=1}^{\infty} l b_l^{(0)} z^l = \frac{g}{2\pi^2} \int_0^{\infty} dp\, p^2 \, \frac{1}{\exp\left(\frac{\sqrt{p^2+m^2}-\mu}{T}\right) - 1} \tag{78}$$

is the density of the ideal pion gas (without resonances). The calculations were done assuming that the chemical equilibrium, $\mu = 0$, is reached.

The temperature dependence of the ratio n/n_0 is shown in Fig.7. It can be seen that all approximations give consistent results up to $5 \div 15\%$. A rather small errors is explained by the fact that at low temperatures ($T < 120$ MeV), where ignoring either the finite resonance width or the repulsion between pions leads to huge errors in the value of cluster integral, the activity of the equilibrium pion gas is small and the contribution of the second term of the cluster expansion into the value of particle density is not important. On the other hand, at large temperatures, where the second term becomes comparable with the first one, the both approximation provide more exact value of the cluster integral. Again, the ideal gas model provides the best approximation at all temperatures, except very large ones $T > 180$ MeV.

This conclussion is close to that of Ref. [8], where it was pointed out that the interacting pion gas *in the second cluster approximation* only slightly differs from the ideal gas of pions and ρ-mesons due to the nearly exact cancellation of the contributions from S-wave attractive and repulsive channels. That is the repulsive interactions and the contribution of the broad σ-resonans can be dropped simultaneously. The aim of our further consideration is to take properly into account the hard-core repulsive interactions. In this case the σ-meson contribution must be retained.

It is seen from Fig.7 that the contribution of the second term of cluster expansion to the particle density is comparable to that of the ideal gas. There is no reason to expect that the higher terms are negligible. The purpose of the next section is to go beyond the second cluster approximation.

VI. VAN DER WAALS EQUATION

Taking into account the attractive parts of higher cluster terms is straightforward: assuming that the attractive interaction of three and more pions is dominated by resonance interaction (as it was in the two pion case) we just add the activities of all the lightest pion resonances to the grand canonical partition function logarithm:

$$\log \mathcal{Z}(V,T,\mu) = \log \mathcal{Z}^{(0)}(V,T,\mu) + V \sum_r z_r, \tag{79}$$

where the first term in the right hand side is the partition function of the ideal pion gas and the sum in the second term runs over not only the two pion resonances (Table I), but also includes the resonances decaying into three and more pions (Table II). For the activity of each resonance species we use the following expression

$$z_r = \int_{N_r m}^{\infty} d\varepsilon \zeta(\varepsilon) g_r \phi(T;\varepsilon)\ \exp\left(\frac{\mu_r}{T}\right). \tag{80}$$

The chemical potential μ_r of a resonance decaying into N_r pions is proportional to the pion chemical potential:

$$\mu_r = N_r \mu \tag{81}$$

In our calculations we assume chemical equilibrium: $\mu_r = \mu = 0$. For the two-pion resonances from Table I we put

$$\zeta(\varepsilon) = \frac{1}{\pi}\frac{d\delta}{d\varepsilon} \tag{82}$$

and use the parametrization (64) and (65), so that the contribution of two pion resonances is reduced to the $b_2^{(a)} z^2$, where the $b_2^{(a)}$ is an attractive part of the CI $b_2^{(i)}$ shown in Fig. 8.

For the resonances from Table II we use the Breit-Wigner profile function (71) with the normalization (72) [11]. In the NRA the expression (80) is reduced to Eq. (62).

It follows from Eq. (79) that the pressure and the pion density are calculated from the ideal gas model for pion and pion resonances with finite width (that is the repulsive interactions are ignored):

$$p(T,\mu) = p_0(T,\mu) + T\sum_r z_r = p_0(T,\mu) + \sum_r p_r(T,\mu_r) \tag{83}$$

$$n(T,\mu) = n_0(T,\mu) + \sum_r N_r z_r = n_0(T,\mu) + \sum_r N_r n_r(T,\mu_r), \tag{84}$$

where the ideal pion gas pressure p_0 is given by the formula

$$p_0(T,\mu) = -gT \int \frac{d^3p}{(2\pi)^3} \log\left[1 - \exp\left(\frac{\mu - \sqrt{\mathbf{p}^2 + m^2}}{T}\right)\right], \tag{85}$$

[11] The formula of Ref. [4] does not lead to large error because of the dominating contribution comes from the narrow resonance $\omega(782)$, for which both creteria (63) and (67) are fulfilled. On the other hand, lack of detailed experimental information on the phase shifts in the vicinity on the broad $\pi\rho$- and $\pi\sigma$-resonances as well as large uncertainties in their masses, width and decay fractions make impossible and useless the application of the more exact formula.

with the particle density of the ideal pion gas given by Eq. (78).

To take into account the hard core repulsion between pions and resonances we use the excluded volume Van der Waals model [24]. In the framework of this model the pressure $p^{VdW}(T,\mu)$ of one-component gas of particles with the excluded-volume parameter v_0 can be found from the transcendental equation

$$p^{VdW}(T,\mu) = p^{id}\left(T, \mu - v_0 p^{VdW}(T,\mu)\right), \tag{86}$$

where $p^{id}(T,\mu)$ is the pressure of corresponding ideal gas. The particle density $n^{VdW}(T,\mu)$ is related to that of ideal gas by the expression

$$n^{VdW}(T,\mu) = \frac{n^{id}\left(T, \mu - v_0 p^{VdW}(T,\mu)\right)}{1 + v_0 n^{id}\left(T, \mu - v_0 p^{VdW}(T,\mu)\right)}, \tag{87}$$

The above model can be straightforwardly generalized to a multi-component gas, if one assumes that all particle species have the same excluded-volume parameter. In our calculations we put it to be the same for the pions and pion resonances. The standard procedure of derivation of the Van der Waals equation in the statistical physics shows that the excluded-volume parameter is equal to the absolute value of the repulsive part of the second virial coefficient [15]. Therefore, the excluded-volume parameter for the pions can be identified with the repulsive part of the CI $b_2^{(i)}$ (See Fig. 8):

$$v_0 = \left|b_2^{(r)}\right| \ . \tag{88}$$

Hence, to find the pressure of the interacting pion gas in the framework of the Van der Waals excluded volume model we solve the transcendental equation

$$\begin{aligned} p^{VdW}(T,\mu) &= p_0(T,\tilde{\mu}) + \sum_r p_r(T,\tilde{\mu}_r) \\ \tilde{\mu} &= \mu - v_0 p^{VdW}(T,\mu) \ , \\ \tilde{\mu}_r &= \mu_r - v_0 p^{VdW}(T,\mu) \ . \end{aligned} \tag{89}$$

The particle density of the pions is found from

$$n^{VdW}(T,\mu) \ = \ \frac{n_0(T,\tilde{\mu}) + \sum_r N_r n_r(T,\tilde{\mu}_r)}{1 + v_0\left[n_0(T,\tilde{\mu}) + \sum_r n_r(T,\tilde{\mu}_r)\right]} \tag{90}$$

The result of the calculation is shown in Fig. 9. It is seen that essential deviation from the second order cluster expansion take place at the temperatures $T \gtrsim 140$ MeV. A comparison with the ideal gas model of pions and pion resonances shows that the effects of the hard-core repulsion are not cancelled by the effects of the finite resonance width. This leads to an essential (up to 30%) suppression of the pion density with respect to the ideal gas of pions and pion resonances.

VII. CONCLUSION

A quantum mechanical formula for the second cluster integral for the gas of relativistic particles with hard-core interaction was derived and analyzed. In the nonrelativistic classical limit, this formula is reduced to the expression used in Refs. [2,3]. In the quantum case, however, the value of the cluster integral appears to be much larger in magnitude than the corresponding classical value and, in contrast to the classical case, depends on the temperature even in the nonrelativistic limit. It has been demonstrated that the second cluster integral for the pion gas all reasonable temperatures is far away from the classical limit. Its repulsive part, which can be interpreted as proper particle volume, is an order of magnitude larger than it could be expected from the classical evaluation. It should be mentioned that not only quantum effects but also relativistic ones are important in the case of pion gas. Surprisingly, they essentially modify the proper pion volume even at relatively low temperatures $T \sim 30$ MeV.

The role of finite resonance width in the second cluster integral was studied. It was established that the widely used *add hoc* formula [4] with the normalized Breit-Wigner resonance profile is unsuitable for broad resonances lying close to the threshold, the parametrization of the experimental phase shifts should be used instead. The most striking example of this kind is the σ-resonance. Our analysis shows that in the case of σ-resonance the calculations with the normalized Breit-Wigner profile can give even worse result than simple zero-width approximation.

At the second order of cluster expansion, due to the presence of broad resonances in the $\pi\pi$-system, the negative contribution of the hard core repulsion into the cluster integral almost canceled by positive contributions of finite resonance widths in a rather broad temperature range. Because of this fact, the thermodynamical properties of the interacting pion gas in the second cluster approximation appear to be quite similar to those of the ideal gas of pions and two-pion resonances: the error in the value of the particle density does not exceed a few percents. Surprisingly, the account for finite resonance widths without account for the hard core repulsion as well as consideration of the hard core repulsion when the resonance widths are neglected worsen rather than improve a simple ideal gas model of pions and zero-width pion resonances. Both effects should be either neglected or taken into account simultaneously.

This does not mean, however, that we can restrict ourselves to the simple ideal gas picture of pions and zero-width pion resonances at all temperatures. As it has been demonstrated, when the temperature is sufficiently high ($T \gtrsim 140$ MeV), the pion density becomes so large that the cluster expansion cannot be truncated at the second order. In contrast to the second cluster approximation, an appreciable deviation from the ideal gas model is observed, when the higher order terms are taken into account. In the framework of Van der Waals excluded-volume model, the pion density appears to be up to 30% lower than that of the ideal pion-resonance gas. Hence, at high particle densities the correct model of the pion gas must include all pion resonances and the resonance width as well as the repulsive interactions between the particles must be taken into account.

It should be emphasized that, if the model takes properly into account the hard core repulsion, there is no reason to drop the σ-resonance. It must be included into the model along with other resonances.

The developed in the present paper approach will be used for calculation of excluded volumes of other hadrons. This will allow us to study the influence of hard core repulsion on the properties of realistic hadron gas including (anti-)nucleons and strange particles by means of multicomponent Van der Waals equation [9]. We expect essentially larger excluded volume effects for nucleons: preliminary calculations show that the proper volume of the nucleon is essentially (by the factor $2 \div 2.5$) larger than that of the pion. Thus the hard core repulsion may essentially modify the particle number ratios in comparison to widely used ideal resonance gas model.

ACKNOWLEDGMENTS

We thank K.Bugaev for helpful discussions and comments. We acknowledge the financial support of DAAD and DFG, Germany. The research described in this publication was made possible in part by Award No. UP1-2119 of the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).

REFERENCES

[1] H. Stöcker, W. Greiner and W. Scheid, Z. Physik **A286**, 121 (1978).
H. Stöcker, A.A. Ogloblin and W. Greiner, Z. Phys. **A303**, 259 (1981).
D. Hahn and H. Stöcker, Nucl. Phys. **A452** 723 (1986).

[2] G. D. Yen and M. I. Gorenstein, Phys. Rev. **C59**, 2788 (1999).
P. Braun-Munzinger, I. Heppe and J. Stachel, Phys. Lett. **B465**, 15 (1999) nucl-th/9903010.

[3] G. D. Yen, M. I. Gorenstein, W. Greiner, S.N. Yang, Phys. Rev. **C56**, 2210 (1997).
G. D. Yen, M. I. Gorenstein, H. Stöcker, W. Greiner, S.N. Yang, J. Phys. **G24** 1777 (1998).

[4] K.G. Denisenko and S. Mrowczynski, Phys. Rev. **C35**, 1932 (1987).

[5] W. Weinhold, B. Friman and W. Norenberg, Phys. Lett. **B433**, 236 (1998) nucl-th/9710014.

[6] R. Hagedorn and J. Rafelski, Phys. Lett. **B97**, 136 (1980).

[7] M. I. Gorenshtein, V. K. Petrov and G. M. Zinovev, Phys. Lett. **B106**, 327 (1981).
D. H. Rischke, M. I. Gorenstein, H. Stocker and W. Greiner, Z. Phys. **C51**, 485 (1991).

[8] R. Venugopalan and M. Prakash, Nucl. Phys. **A546**, 718 (1992).

[9] M. I. Gorenstein, A. P. Kostyuk and Y. D. Krivenko, J. Phys. G **G25**, L75 (1999) [nucl-th/9906068].

[10] K. A. Bugaev, M. I. Gorenstein, H. Stöcker and W. Greiner, Phys. Lett. **B485**, 121 (2000) [nucl-th/0004061].

[11] E. Beth and G.E. Uhlenbeck, Physica **4**, 915 (1937).

[12] K. Huang, "Statistical mechanics" *John Wiley & Sons* (1963).

[13] B. Kahn and G.E. Uhlenbeck, Physica **5**, 399 (1938).

[14] J.E. Mayer and M. Goeppert Mayer, "Statistical mechanics" *John Wiley & Sons (1977).*

[15] W. Greiner, L. Neise and H. Stöcker "Thermodynamics and Statistical Mechanics" *Springer, NY* (1995).

[16] S. Ishida, T. Ishida, M. Ishida, K. Takamatsu and T. Tsuru, Prog. Theor. Phys. **98**, 1005 (1997).

[17] S. Ishida, M. Ishida, H. Takahashi, T. Ishida, K. Takamatsu and T. Tsuru, particle," Prog. Theor. Phys. **95**, 745 (1996) hep-ph/9610325.

[18] S. Ishida, hep-ph/9905260.

[19] M. Abramowitz and I. Stegun, "Handbook of Mathematical Functions" *Pover Publications (1968).*

[20] B. Hyams *et al.*, Nucl. Phys. **B64**, 134 (1973).

[21] R. Dashen, S.K. Ma, H.J. Bernstein Phys. Rev. **187**, 345 (1969).

[22] A. Dobado and J. R. Pelaez, Phys. Rev. **D59**, 034004 (1999) [hep-ph/9806416].

[23] I.N. Mishustin and W. Greiner, J. Phys. G **G19**, L101 (1993).

[24] D. H. Rischke, M. I. Gorenstein, H. Stöcker and W. Greiner, Z. Phys. **C51**, 485 (1991).

TABLES

TABLE I. Parameters for the lightest resonances in $\pi\pi$-system.

Resonance	Isospin I_r	Spin L_r	Mass M_r (MeV)	Width Γ_r (MeV)	Elasticity x_r	Interaction radius R_r (GeV^{-1})
σ	0	0	585	385	—	—
$f_0(980)$	0	0	993.2	54.32	—	—
$\rho(770)$	1	1	777	155	1	3.09
$f_2(1270)$	0	2	1281	205	0.84	4.94
$\rho_3(1690)$	1	3	1713	228	0.26	6.38

TABLE II. Parameters for the lightest resonances decaying into 3 and more pions.

Resonance	Degeneration factor g_r	Mass M_r (MeV)	Width Γ_r (MeV)	Elasticity x_r	Number of pions in the final state N_r
$\omega(782)$	3	782	8.41	0.888	3
$\phi(1020)$	3	1019	4.43	0.155	3
$h_1(1170)$	3	1170	360	~ 0.5	3
$b_1(1235)$	9	1230	142	~ 1	4
$a_1(1260)$	9	1230	425	~ 1	3
$f_1(1285)$	3	1282	24	0.35	4
$\pi(1300)$	3	1300	400	~ 1	3
$a_2(1320)$	15	1318	107	0.70	3

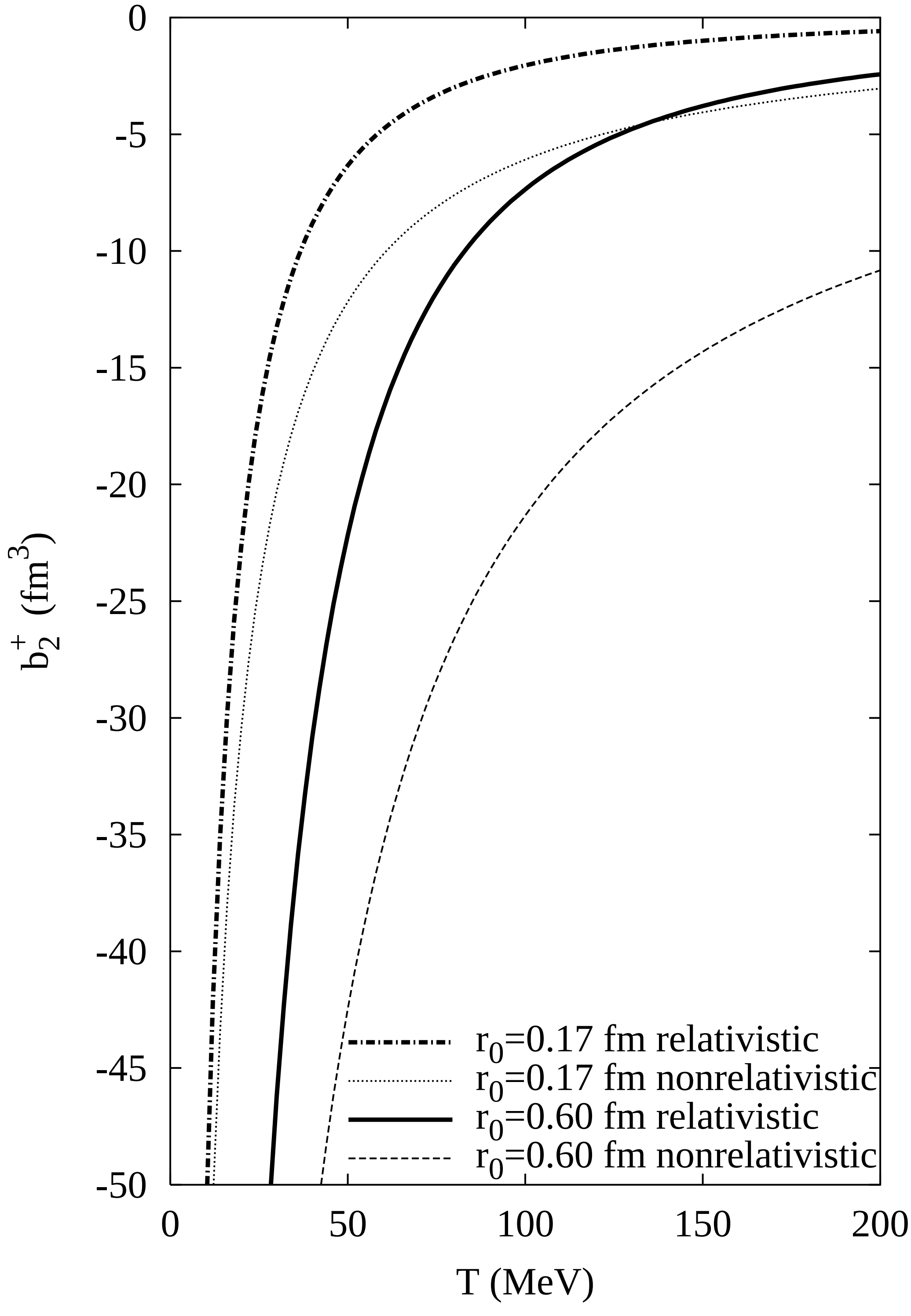

FIG. 1. The dependence of the CI $b_2^+(r_0, T, m)$ on the temperature in the hard sphere model. The formula (52) is used for the relativistic calculations. The results are compared to the nonrelativistic approximation (53).

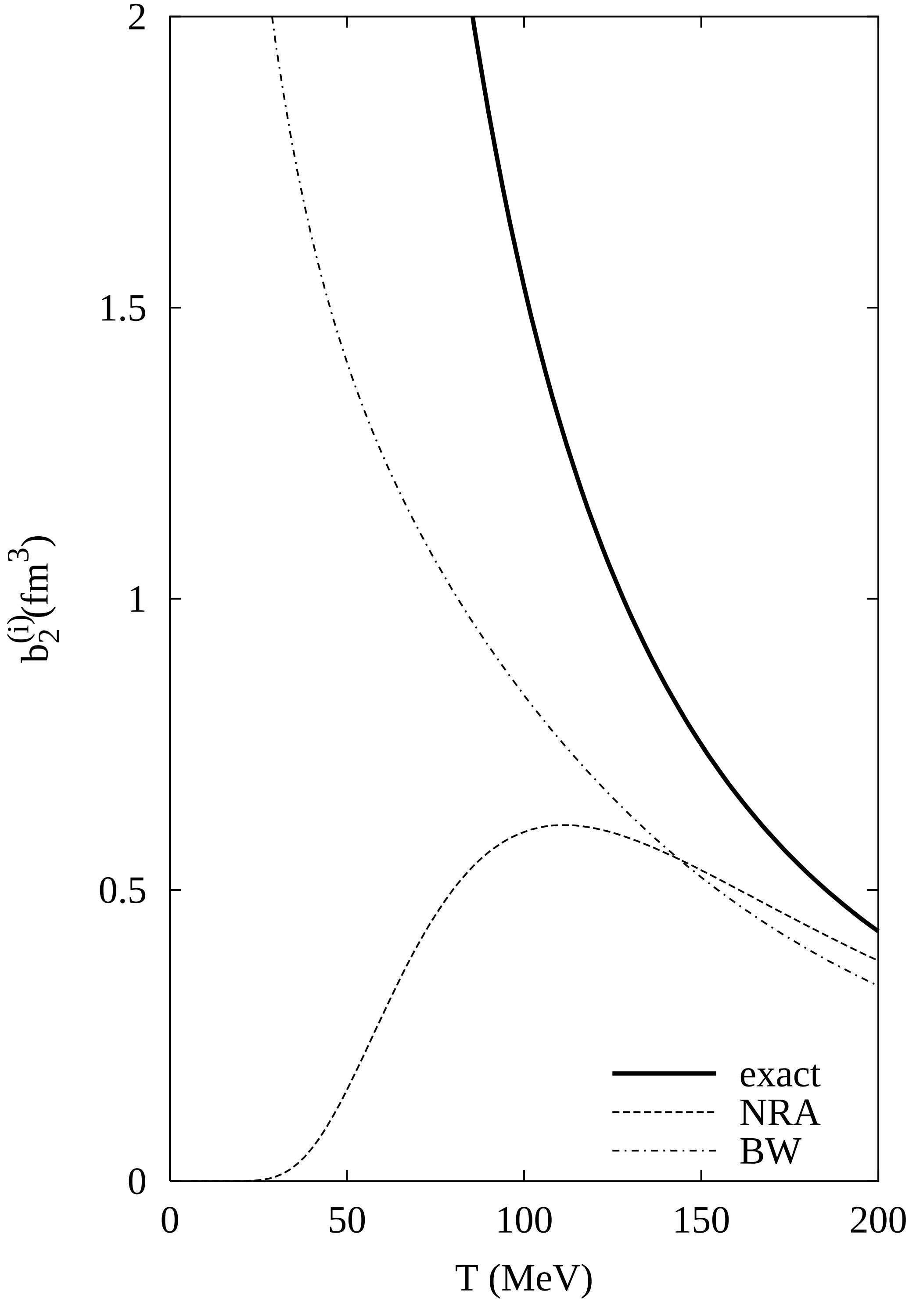

FIG. 2. The contribution of σ-resonance to $b_2^{(i)}(T)$. The exact value is compared to those calculated in narrow resonance approximation (NRA) and using normalized Breit-Wigner (BW) profile of the resonance (73).

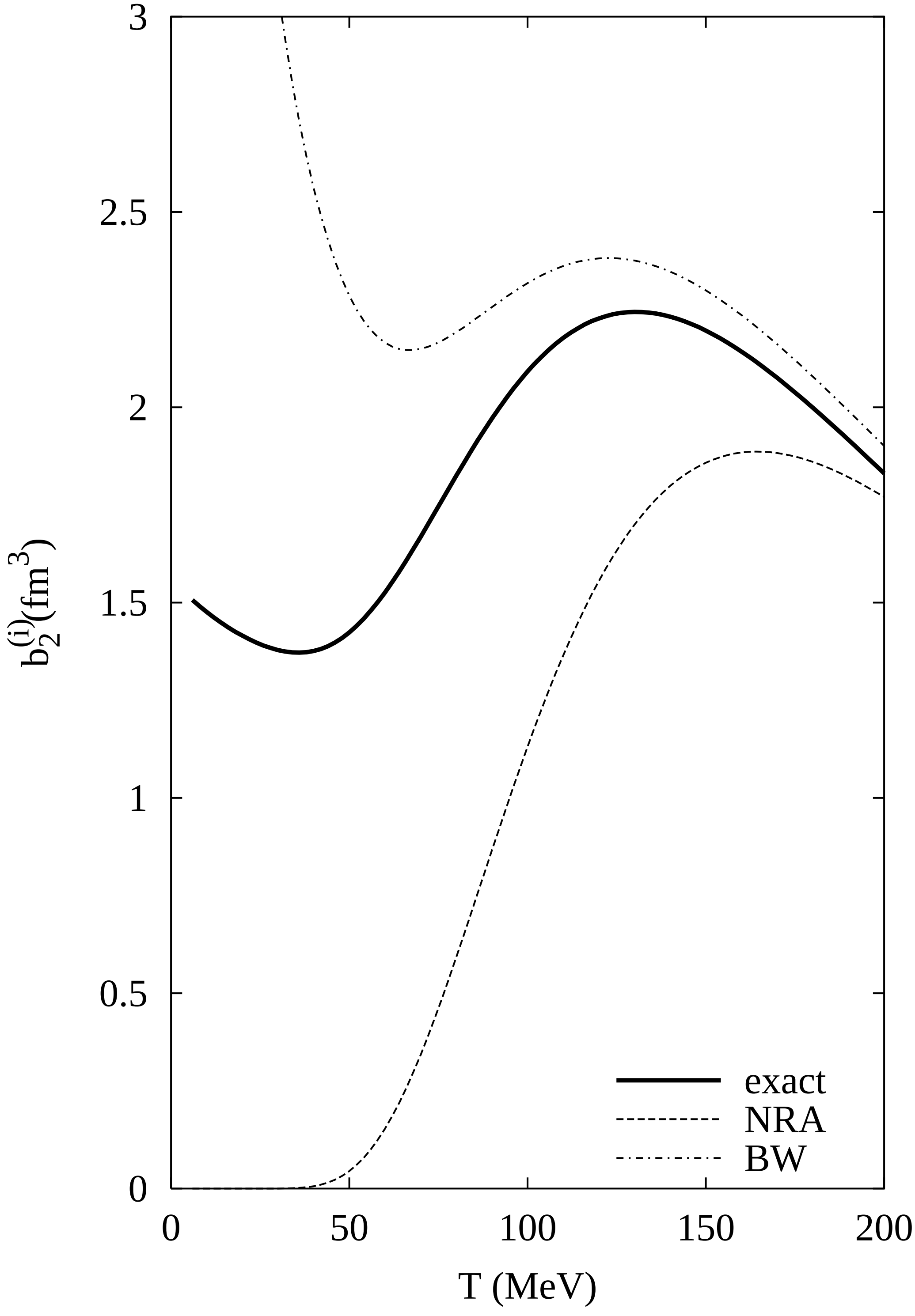

FIG. 3. The contribution of $\rho(770)$ to $b_2^{(i)}(T)$. The exact value is compared to those calculated in narrow resonance approximation (NRA) and using normalized Breit-Wigner (BW) profile of the resonance (73).

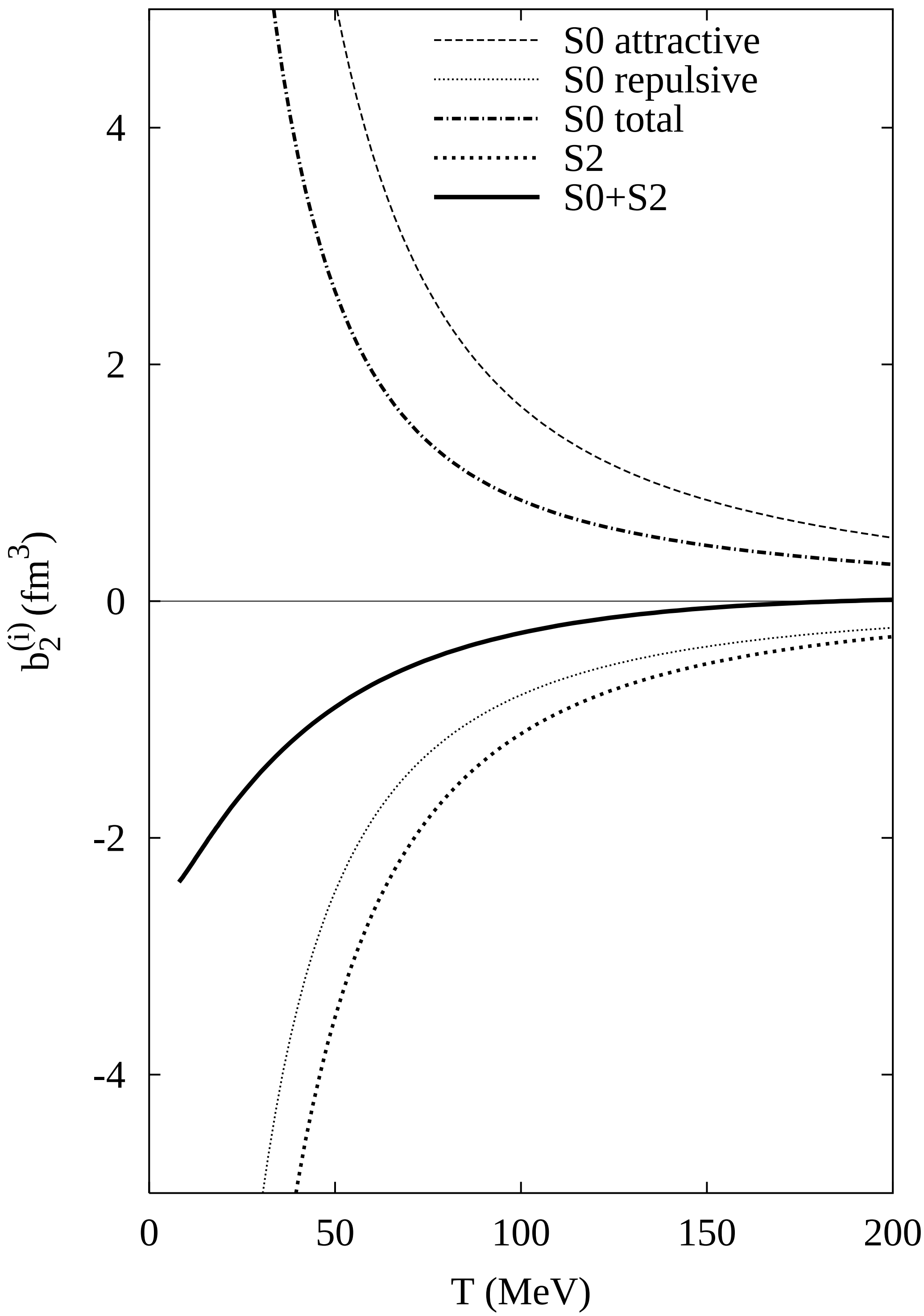

FIG. 4. The contribution of S_0 and S_2 states of $\pi\pi$-scattering into the second CI $b_2^{(i)}$.

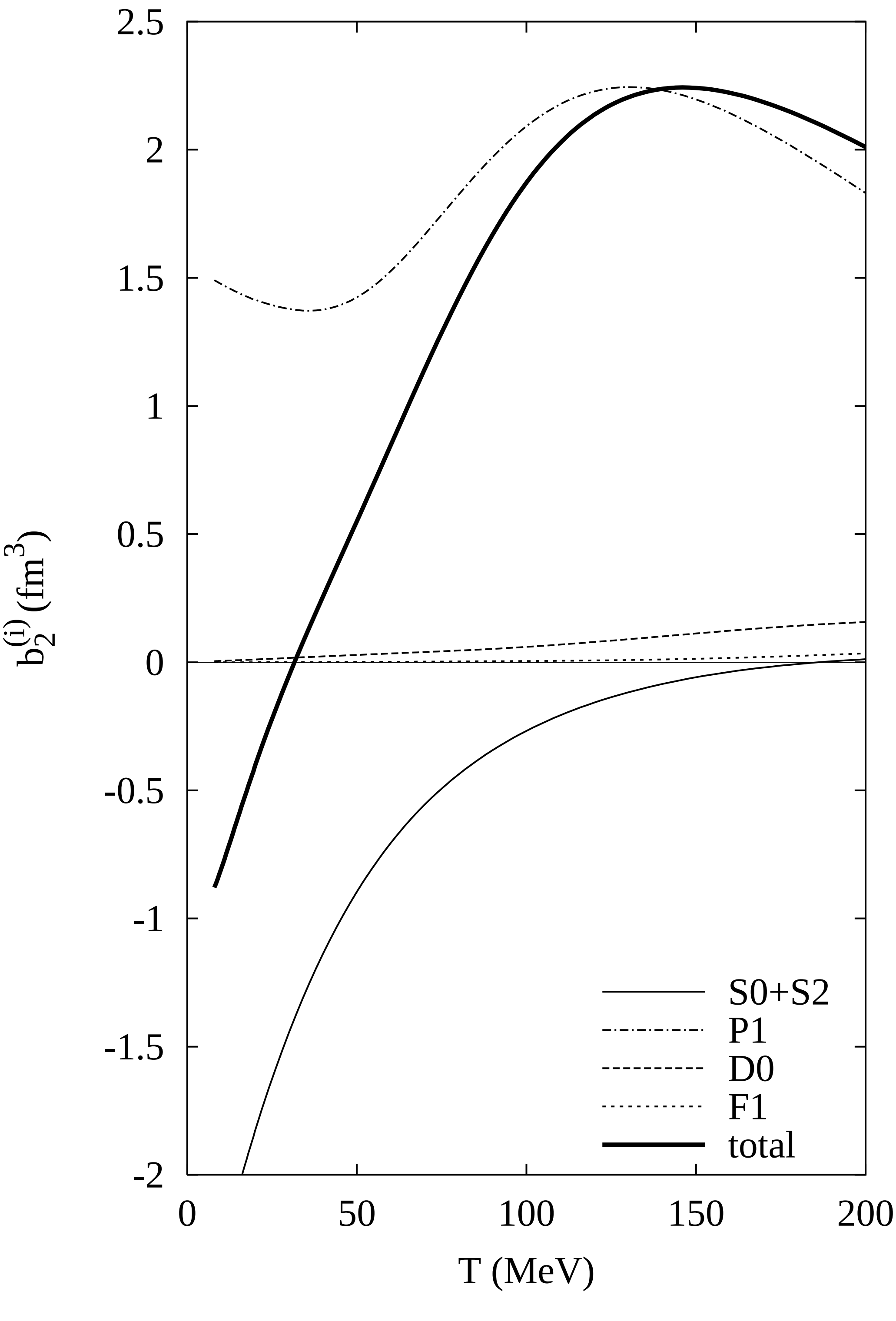

FIG. 5. The partial $\pi\pi$-wave contribution into the second CI $b_2^{(i)}$ and its total value.

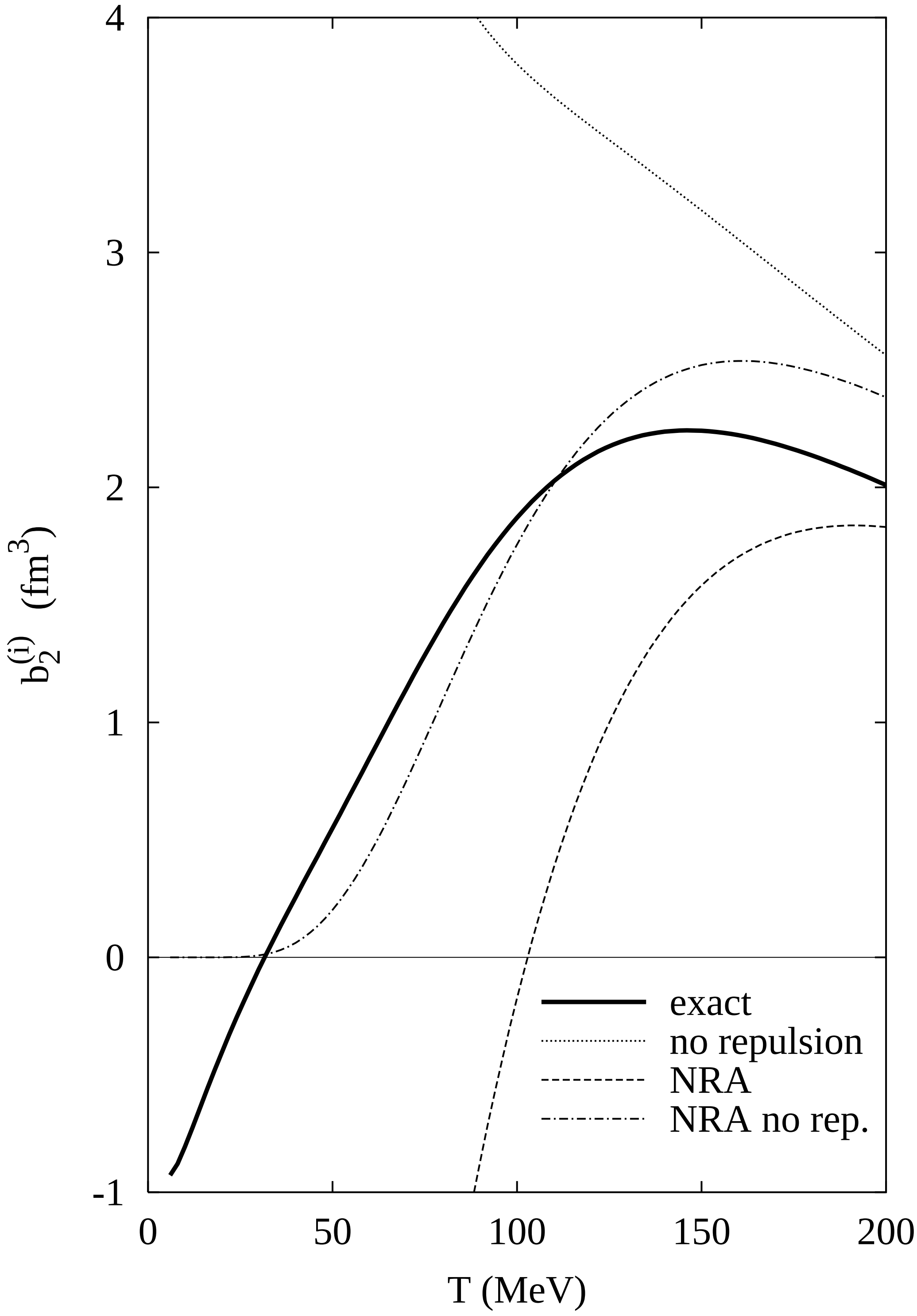

FIG. 6. The exact value of the CI $b_2^{(i)}$ compared to various approximations: "No repulsion" — the repulsive part of the $\pi\pi$-interaction is dropped out; "NRA" — narrow resonance approximation; "NRA no repulsion" — both repulsion and final resonance width are ignored.

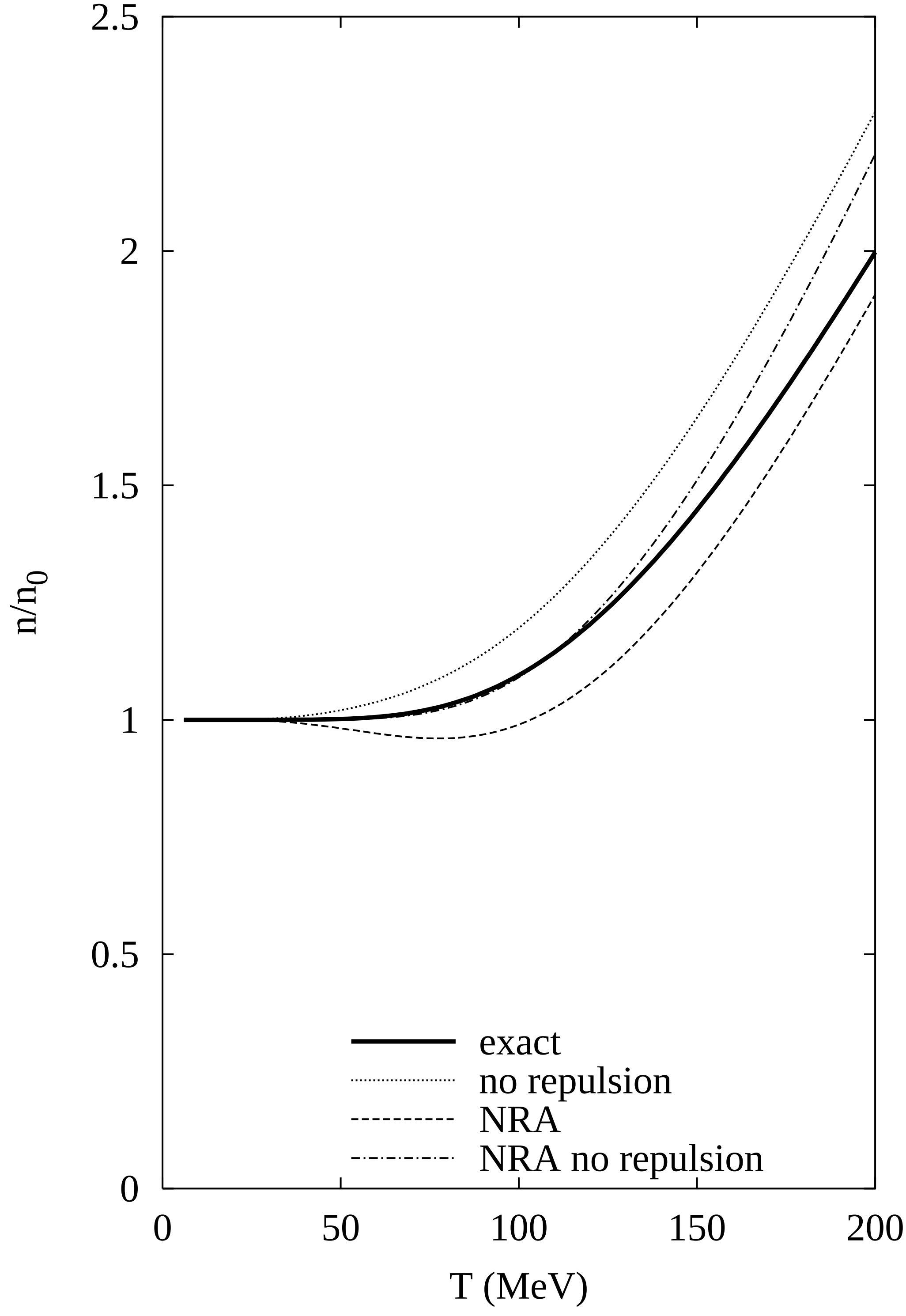

FIG. 7. The ratio of the particle density of the pion gas calculated in the second order cluster expansion to the particle density of the ideal pion gas. Various approximations are shown: "No repulsion" — the repulsive part of the $\pi\pi$-interaction is dropped out; "NRA" — narrow resonance approximation; "NRA no repulsion" — both repulsion and final resonance width are ignored.

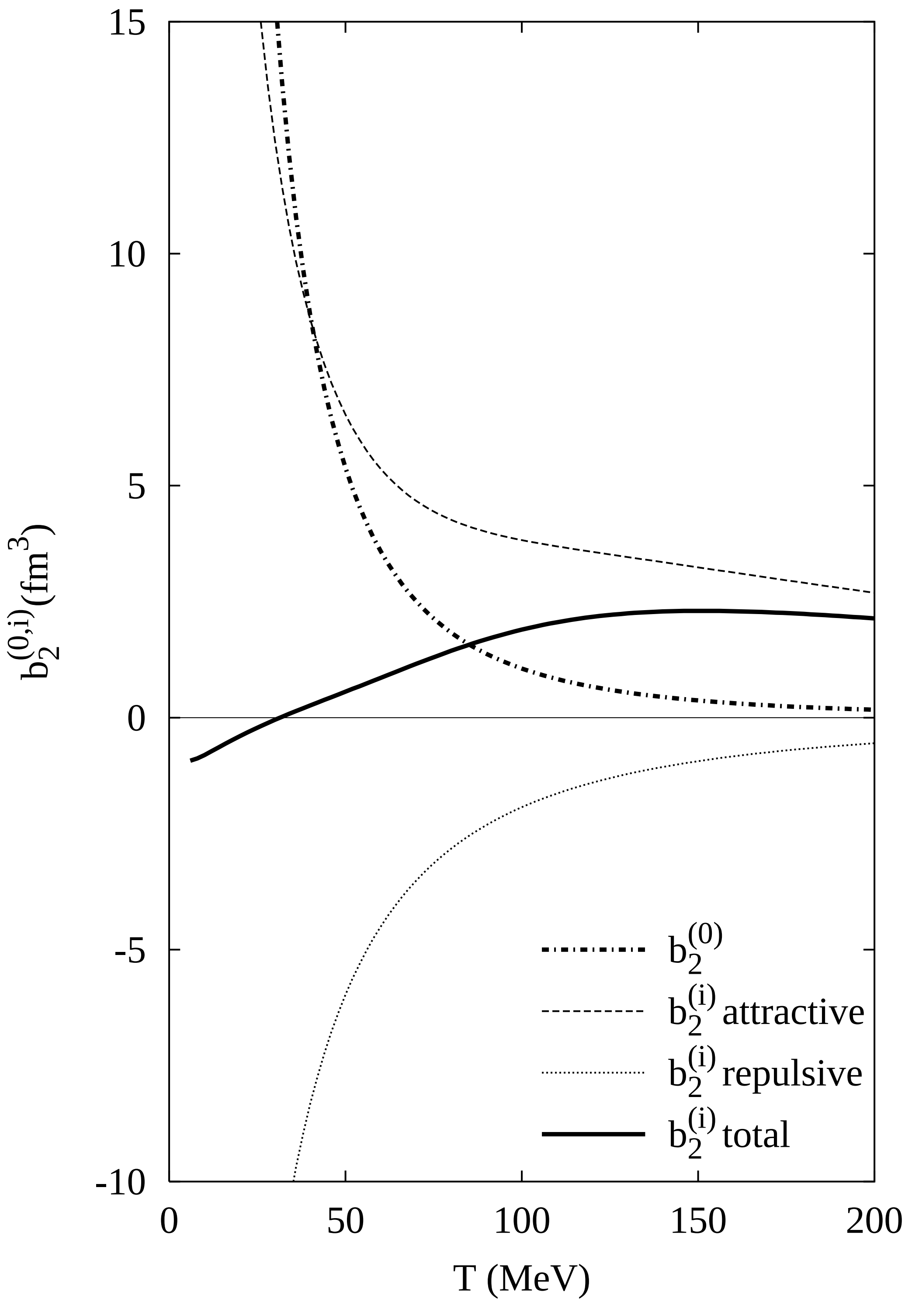

FIG. 8. The attractive and repulsive part of CI $b_2^{(i)}$ and its total value. The ideal gas CI $b_2^{(0)}$ is also shown.

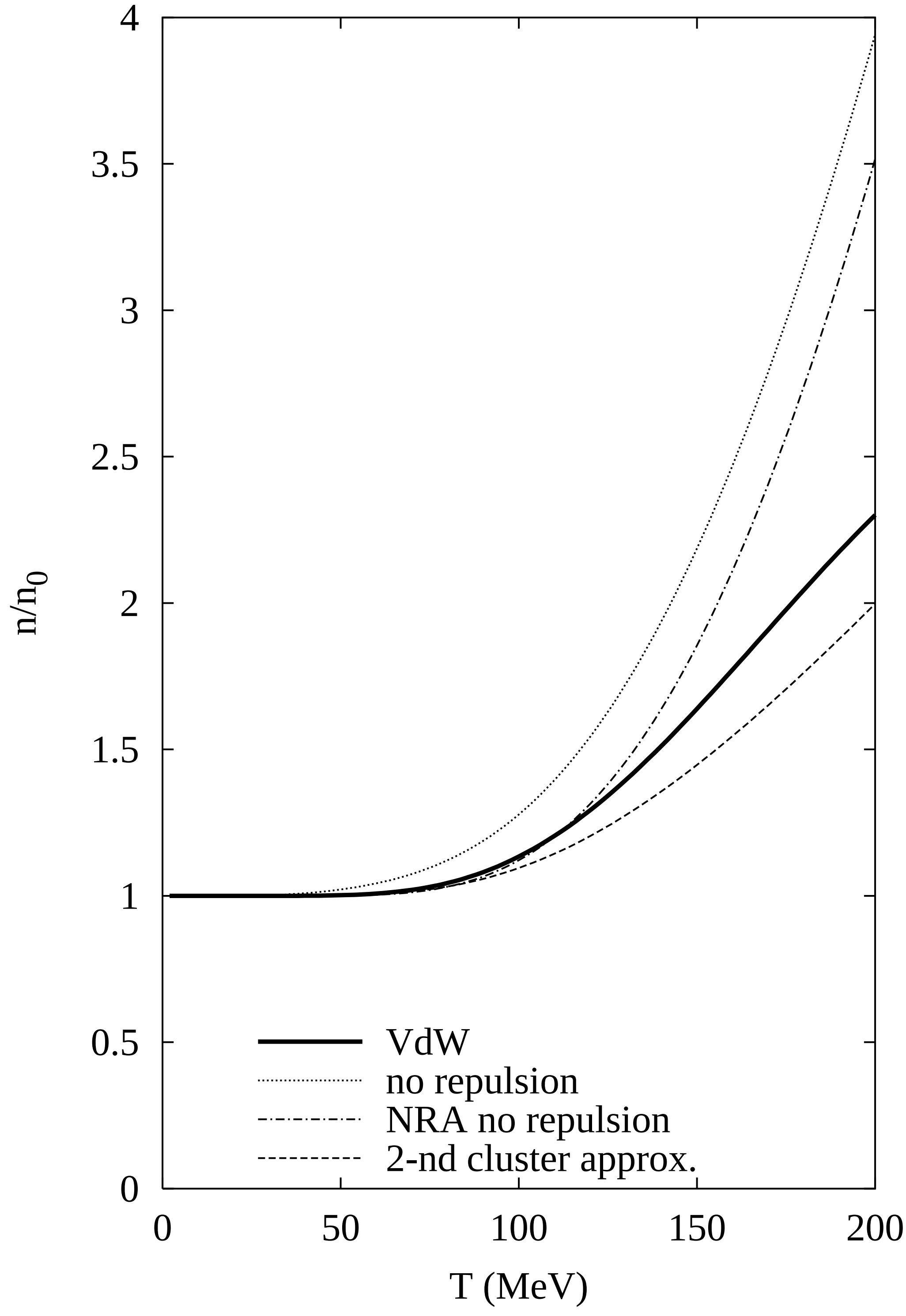

FIG. 9. The pion density in the Van der Waals excluded volume model compared to the ideal gas of pions and broad pion resonances (no repulsion), to the ideal gas of pions and zero width pion resonances (NRA no repulsion) and to the pion gas in the second order cluster approximation.