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# Dynamic Collective Theory of Odd-$A$ Nuclei*

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The unified model and the collective giant-dipole-resonance model are unified. The resulting energy spectrum and the transition probabilities are derived. A new approximate selection rule involving the symmetry of the $\gamma$ vibrations is established. It is verified that the main observable features in the photon-absorption cross section are not influenced by the odd particle, despite the considerably richer spectrum of states as compared to even-even nuclei.

## I. INTRODUCTION

THE previously developed complete collective theory of the nucleus, unifying the low-energy and the high-energy collective degrees of freedom,[1,2] i.e., the rotations, surface vibrations and the dipole oscillations, still lacks one feature, namely, the possibility of having nonvanishing ground-state spin. In the present paper we intend to remedy this situation, and we set ourselves the task of unifying the unified model and the giant-resonance hydrodynamic model in a completely quantum-mechanical treatment.

The reasons for doing it are manyfold. Firstly, experiments are being performed on odd-$A$ nuclei which do have a finite ground-state spin. It has been frequently stated that the last odd particle will have a negligible effect on the giant resonance,[3] but this statement has to be made quantitative. This is particularly important since odd-$A$ nuclei frequently are monoisotopic and therefore the finer details, e.g., the line shape of the lower energy peak, are not washed out as they may be in an isotope mixture. Furthermore, experiments involving nuclear orientation require a finite ground-state spin. For a consistent description of such experiments the incorporation of an odd particle is indispensable. In particular, the tensor polarizability of a nucleus vanishes for a zero-spin ground state. Also, the details of the elastic and the Raman scattering of photons depend essentially on the ground-state spin of the nucleus. Finally, the presence of a further angular momentum, viz., the particle angular momentum, provides for a large number of ways for the system to couple to a given total angular momentum. A considerable enrichment of the structure of the spectrum thus is to be expected, compared to the already quite rich level structure of the even-even nuclei.

Two aspects can be distinguished in the addition of an odd particle. The first is the change of the kinematic features of the system, the change in its symmetries; and the second is the change in the dynamic characteristics, the appearance of new dynamic variables and interaction terms in the Hamiltonian. The first aspect is of quite general validity being based only on angular momentum and parity conservation and on the assumptions of the symmetries of the deformed intrinsic nuclear system. The dynamic aspects depend in detail on the specific assumptions of the model. We are going to use the Nilsson Hamiltonian to describe the odd particle and its interactions with the collective degrees of freedom. Naturally, one cannot expect a quantitative description of the "single-particle" aspects of the nucleus by this simplified treatment.

In Sec. II we write down the Hamiltonian of the system and discuss the magnitude and the importance of the several terms. Omitting the less important terms we establish the Hamiltonain which we then proceed to solve in Sec. III, arriving at the energy spectrum and the wave functions. In Sec. IV we derive the dipole operator in the intrinsic system. We write it as a power-series expansion retaining terms quadratic in the deformation parameter and linear in the vibrational coordinates. In Sec. V we write down the transition matrix elements and indicate the selection rules which arise from the symmetries of the wave function. In Sec. VI we discuss the results obtained and estimate their accuracies. We also give an outline of the possible ways of improving the present treatment.

## II. THE HAMILTONIAN

The total Hamiltonian of an odd nucleus consists of the collective Hamiltonian of the core, the single-particle Hamiltonian of the odd particle and an interaction term between the two. The collective Hamiltonian of the core consists of terms describing rotations $H_{\text{rot}}$, vibrations $H_{\text{vib}}$, dipole oscillations $H_{\text{dip}}$, and various interaction terms between these.

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[1] M. Danos and W. Greiner, Phys. Rev. 134, B285 (1964).

[2] M. Danos and W. Greiner, Phys. Letters 8, 113 (1964).

[3] E. G. Fuller and E. Hayward, in *Nuclear Reactions*, edited by P. M. Endt and P. B. Smith (North-Holland Publishing Company, Amsterdam, 1962), Vol. II.

### 1. Rotations

In the intrinsic coordinate system the shape of a deformed nucleus can be specified by

$$R=R_0[1+(\beta_0+\xi)Y_{20}+\eta(Y_{22}+Y_{2-2})], \quad (1)$$

when $\beta_0$ is the equilibrium deformation parameter and $\xi$ and $\eta$ are the vibrational coordinates.[4] The total angular momentum, $\mathbf{I}$, then is

$$\mathbf{I}=\mathbf{R}+\mathbf{d}+\mathbf{j}, \quad (2)$$

when $\mathbf{R}$, $\mathbf{d}$, and $\mathbf{j}$ are the angular momenta associated with the rotation, the dipole oscillation, and the single-particle, respectively. In the rotational energy $T_{\text{rot}}$,

$$T_{\text{rot}}=\sum_{\nu=1}^{3}\frac{\hbar^2 R_\nu^2}{2J_\nu(\xi,\eta)}, \quad (3)$$

one can express $\mathbf{R}$ in terms of $\mathbf{I}$, $\mathbf{d}$, and $\mathbf{j}$. A straightforward calculation yields

$$\begin{aligned}T_{\text{rot}}=&\frac{\hbar^2}{4}\left(\frac{1}{J_1}+\frac{1}{J_2}\right)[(\mathbf{I}^2-I_3^2)+(\mathbf{j}^2-j_3^2)+(\mathbf{d}^2-d_3^2)]+\frac{\hbar^2}{8}\left(\frac{1}{J_1}-\frac{1}{J_2}\right)[(I_+^2+I_-^2)+(j_+^2+j_-^2)+(d_+^2+d_-^2)]\\&-\frac{\hbar^2}{4}\left(\frac{1}{J_1}+\frac{1}{J_2}\right)[(I_+j_-+I_-j_+)+(I_+d_-+I_-d_+)-(j_+d_-+j_-d_+)]\\&-\frac{\hbar^2}{4}\left(\frac{1}{J_1}-\frac{1}{J_2}\right)[(I_+j_++I_-j_-)+(I_+d_++I_-d_-)-(j_+d_++j_-d_-)]+\frac{\hbar^2}{2J_3}(I_3-j_3-d_3)^2, \quad (4)\end{aligned}$$

where $I_\pm=I_1\pm iI_2$, etc. The moments of inertia $J_\nu$ are given in terms of the shape parameters by[5,6]

$$\begin{aligned}J_{1,2}&=B[2\eta^2+3(\beta_0+\xi)^2\pm(24)^{1/2}(\beta_0+\xi)\eta],\\J_3&=8B\eta^2.\end{aligned} \quad (5)$$

We assume, as usual, that the vibrational amplitudes are small, i.e.,

$$|\xi/\beta_0|\ll 1, \quad |\eta/\beta_0|\ll 1. \quad (6)$$

One then can expand the moments of inertia in (4). The result is

$$\begin{aligned}T_{\text{rot}}=&H_{\text{rot}}+H_{\text{rot vib}}+H_{\text{rot dip}}+H_{\text{rot part}}+H_{\text{int}}\\&+H_{\text{vib dip}}+H_{\text{part dip}}+H_{\text{part vib}}, \quad (7)\end{aligned}$$

where

$$\begin{aligned}H_{\text{rot}}&=(\hbar^2/2J_0)[\mathbf{I}^2-I_3^2-d_3^2-j_3^2]\\&\quad+(\hbar^2/2J_3)[(I_3-j_3-d_3)^2-1],\\H_{\text{rot vib}}&=-\frac{\hbar^2}{J_0}(\mathbf{I}^2-I_3^2)\frac{\xi}{\beta_0}-\frac{1}{(\sqrt{6})}\frac{\hbar^2}{J_0}(I_+^2+I_-^2)\frac{\eta}{\beta_0},\\H_{\text{vib dip}}&=-\frac{\hbar^2}{J_0}(\mathbf{d}^2-d_3^2)\frac{\xi}{\beta_0}-\frac{1}{(\sqrt{6})}\frac{\hbar^2}{J_0}(d_+^2+d_-^2)\frac{\eta}{\beta_0},\\H_{\text{rot dip}}&=-(\hbar^2/2J_0)(I_+d_-+I_-d_+),\\H_{\text{rot part}}&=-(\hbar^2/2J_0)(I_+j_-+I_-j_+),\\H_{\text{part dip}}&=+(\hbar^2/2J_0)(j_+d_-+j_-d_+),\\H_{\text{part vib}}{}^{(1)}&=-\frac{\hbar^2}{J_0}(\mathbf{j}^2-j_3^2)\frac{\xi}{\beta_0}-\frac{1}{(\sqrt{6})}\frac{\hbar^2}{J_0}(j_+^2+j_-^2)\frac{\eta}{\beta_0},\\H_{\text{int}}&=\frac{(\sqrt{6})}{3}\frac{\hbar^2}{J_0}[(I_+j_++I_-j_-)+(I_+d_++I_-d_-)\\&\quad-(j_+d_++j_-d_-)]\frac{\eta}{\beta_0}+\frac{\hbar^2}{J_0}[(I_+j_-+I_-j_+)\\&\quad+(I_+d_-+I_-d_+)-(j_+d_-+j_-d_+)]\frac{\xi}{\beta_0}, \quad (8)\end{aligned}$$

and where

$$J_0=3B\beta_0^2. \quad (8')$$

The meaning of the various terms is clear. $H_{\text{int}}$ describes an interaction of three degrees of freedom; e.g., rotation-particle-vibration, etc. We have done the expansion of the moments of inertia only up to first order in the vibrational coordinates. We further have left out the terms $(\hbar^2/2J_0)(\mathbf{j}^2+\mathbf{d}^2)$ since they are pure single-particle and dipole terms, respectively, and it is understood that they will be contained in the Hamiltonians for the single-particle and for the dipole oscillations.

Many of the energies of Eq. (8) occur already for even-even nuclei,[6–9] and it is here only of interest whether a strong coupling exists between the odd particle and the dipole oscillation. We see from the above that $H_{\text{part dip}}$ is of the order of the rotational energies, i.e., 50 keV. If we neglect this coupling, the energies will be uncertain by 50 keV which is about 0.3% for the dipole states, but about 10–15% for the single-particle states on top of the giant resonances relative to each other (the single-particle energies are of the order of 300–500 keV).

Another coupling of the dipole modes with the odd particle takes place via the quadrupole vibrations: the dipole oscillations are strongly coupled to the quadrupole vibrations ($\approx$1 MeV) and the odd particle is coupled to the quadrupole vibrations via $H_{\text{part vib}}{}^{(1)}$ of (8). The latter coupling is, however, very weak ($\sim$5 keV) and therefore the odd-particle structure on top of the giant resonances will be only disturbed by $\sim$1% as a result of this coupling.

[4] The coordinates $\xi$ and $\eta$ introduced here are identical with the coordinates $a_0'$ and $a_2'$ respectively of Refs. 1, 6–9.
[5] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **26**, No. 14 (1952).
[6] A. Faessler and W. Greiner, Z. Physik **168**, 425 (1962).
[7] A. Faessler and W. Greiner, Z. Physik **170**, 105 (1962).
[8] A. Faessler and W. Greiner, Z. Physik **177**, 190 (1964).
[9] A. Faessler, W. Greiner, and R. K. Sheline, University of Maryland Technical Report No. 345, 1963 (unpublished).

### 2. Quadrupole Vibrations

The structure of this energy is clear from earlier papers[6–9] and the quadrupole vibration Hamiltonian is

$$H_{\rm vib}=-\frac{\hbar^2}{2B}\left(\frac{\partial^2}{\partial\xi^2}+\frac{1}{2}\frac{\partial^2}{\partial\eta^2}\right)+\frac{C_0}{2}\xi^2+C_2\eta^2. \quad (9)$$

### 3. Dipole Oscillations

The Hamiltonian of the collective dipole motion in the core has been derived earlier.[1] In the classical three-axial ellipsoid three eigenmodes exist:

$$\psi_\mu=j_1(k_\mu r)\Phi_\mu, \quad \mu=0,\pm1 \quad (10)$$

where

$$\Phi_\mu=(Y_{1\mu}+\mu Y_{1-\mu})/\sqrt{2},$$
$$k_\mu=(2.08/R_\mu)[1+0.08(\Delta R_\mu/R_0)]. \quad (11)$$

If we introduce annihilation and creation operators $b_\mu$ and $b_\mu^\dagger$, respectively, for the states (10), the dipole energy in the adiabatic approximation is

$$H_{\rm dip}=\sum_\mu \hbar\bar\omega_\mu b_\mu^\dagger b_\mu, \quad (12)$$

where the energies $\hbar\bar\omega_\mu$ depend on the deformation parameters, e.g., the vibrational coordinates. Therefore (12) exhibits not only the pure dipole energy but also its interaction energy with core vibrations. If both are separated, one obtains in lowest order in $\xi$, $\eta$

$$H_{\rm dip}+H_{\rm dip\ vib}{}^{(2)}=\sum_\mu \hbar\omega_\mu b_\mu^\dagger b_\mu +\sum_\mu \hbar\omega_\mu G_\mu(\xi-\mu 6^{1/2}\eta)b_\mu^\dagger b_\mu, \quad (13)$$

where, introducing the abbreviation $\beta=(5/4\pi)^{1/2}\beta_0$,

$$\hbar\omega_1=\hbar\omega_{-1}=[(8\kappa/M^*)(NZ/A^2)]^{1/2}(2.08/R_0)\times(1-0.04\beta)(1-0.5\beta)^{-1},$$
$$\hbar\omega_0=[(8\kappa/M^*)(NZ/A^2)]^{1/2}(2.08/R_0)\times(1+0.08\beta)(1+\beta)^{-1}, \quad (14)$$

and

$$G_1=G_{-1}=(5/16\pi)^{1/2}[(1-0.5\beta)^{-1}-0.08],$$
$$G_0=-(5/4\pi)^{1/2}[(1+\beta)^{-1}-0.08]. \quad (15)$$

The last term in (13) describes then the interaction of the dipole oscillations with the quadrupole vibrations. In (14) $\kappa$ is the symmetry energy parameter and $M^*$ is the effective nucleon mass.[1]

### 4. Single-Particle Hamiltonian

The complete single-particle Hamiltonian with its interaction with the quadrupole vibrations has been recently discussed by Faessler.[10] We follow his treatment. Thus we have

$$H_{\rm part}=H_{\rm part}{}^{(0)}+H_{\rm part\ vib}{}^{(2)}, \quad (16)$$

where $H_{\rm part}{}^{(0)}$ is the well-known Nilsson Hamiltonian[11]

$$H_{\rm part}{}^{(0)}=\tfrac{1}{2}F_w(-\nabla^2+\mathcal{K}^2r^2) -F_w\beta_0 r^2Y_{20}+C_w\mathbf{l}\cdot\mathbf{s}+C_w{}^1\mathbf{l}^2, \quad (17)$$

where $F_w$ is the energy parameter of the shell-model potential well, and $H_{\rm part\ vib}{}^{(2)}$ is the interaction of the particle with quadrupole vibrations

$$H_{\rm part\ vib}{}^{(2)}=-F_w\xi r^2Y_{20}-F_w\eta r^2(Y_{22}+Y_{2-2}). \quad (18)$$

The difference between $H_{\rm part\ vib}{}^{(1)}$ in Eq. (8) and $H_{\rm part\ vib}{}^{(2)}$ is that the first arises from the rotational energy while the second arises from the shell-model potential energy. Both terms (8) and (18) together with $H_{\rm dip\ vib}$ lead to a coupling of the dipole oscillations with the odd particle (see the remarks at the end of Sec. II.1). The term $(\hbar^2\mathbf{j}^2)/(2J_0)$ which we mentioned after Eq. (8) is included in (17), since

$$\mathbf{j}^2=(\mathbf{l}+\mathbf{s})^2=(\mathbf{l}^2+\mathbf{s}^2+2\mathbf{l}\cdot\mathbf{s})$$

and the terms $\mathbf{l}^2$ and $2\mathbf{l}\cdot\mathbf{s}$ are understood to be absorbed in the constants $C_w{}^1$ and $C_w$ of (17), respectively. The term $(\mathbf{s}^2\hbar^2)/(2J_0)$ yields only a constant which can be left out, since $\mathbf{s}^2\chi=(3/4)\chi$ for each Nilsson function $\chi$.

## III. WAVE FUNCTIONS AND ENERGIES

### 1. Details on the Solution of the Hamiltonian

In this section we solve the "basic" Hamiltonian

$$\tilde H=H_{\rm rot}+H_{\rm vib}+H_{\rm dip}+H_{\rm part}+H_{\rm dip\ vib}{}^{(2)}. \quad (19)$$

All the other terms discussed so far for completeness, will be neglected, because they result only in small perturbations (at least for low spins). The wave function of (19) consists of terms of the form

$$\mathfrak{D}_{MK}{}^I\psi_\mu\chi_\Omega\tilde\varphi(\xi,\eta). \quad (20)$$

The factors describe rotations, dipole oscillations, the odd particle, and quadrupole vibrations, respectively. $K$ and $\Omega$ describe the projections of the relevant angular momenta along the equilibrium symmetry axis. The subscript $\mu$ denotes, according to (11), certain combinations of projections of the dipole angular momentum on the symmetry axis. Since (19) is invariant under the symmetry operators:

($s1$) rotations through $\pi$ around $z'$ axis;
($s2$) rotations through $\pi$ around $x'$ axis;
($s3$) rotation through $\pi/2$ around $z'$ axis and simultaneous replacement $\eta\to-\eta$;

the wave functions are required to obey the same symmetries.[5] The first symmetry leads to the condition

$$K-\Omega=2\nu\pm\mu, \quad \mu=0,\pm1; \quad \nu=0,\pm1,\pm2\cdots. \quad (21)$$

The second symmetry operation transforms

$$\mathfrak{D}_{MK}{}^I\psi_\mu\chi_\Omega\to(-)^{\frac{1}{2}(1-\mu)\mu+I+K+\mu+j+\Omega+1}\mathfrak{D}_{M-K}{}^I\psi_\mu\chi_{-\Omega}. \quad (22)$$

[10] A. Faessler, Nucl. Phys. **59**, 177 (1964).

[11] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **29**, No. 16 (1955).

Since $\chi_\Omega$ is not an eigenfunction of $\mathbf{j}^2$, the phase $(-)^j$ has to be understood as an operator which acts on the different $j$'s contained in $\chi_\Omega$. A wave function invariant under operation (22) is therefore of the form

$$\Psi=\tilde{\varphi}_{K\mu\Omega}(\xi,\eta)\psi_\mu \times[\mathfrak{D}_{MK}{}^I\chi_\Omega+(-)^{\frac{1}{2}(1-\mu)\mu+N}\mathfrak{D}_{M-K}{}^I\chi_{-\Omega}],$$
$$N=I+K+\mu+j+\Omega+1. \quad (23)$$

According to (21) we have

$$K=\Omega\pm\mu+2\nu=\Omega-1,\ \Omega+1,\ \Omega+3,\ \cdots$$
$$I=K,\ K+1,\ K+2,\ \cdots. \quad (24)$$

Note, that $K$ can never be zero, since $\Omega$ in (24) is not an integer, while $\mu$ and $2\nu$ are integers. If we insert (23) into

$$\tilde{H}\Psi=E\Psi \quad (25)$$

and multiply from the left by

$$\psi_\mu[\mathfrak{D}_{MK}{}^I\chi_\Omega+(-)^{\frac{1}{2}(1-\mu)\mu+N}\mathfrak{D}_{M-K}{}^I\chi_{-\Omega}]$$

and integrate over all coordinates except $\xi$ and $\eta$ we obtain the following equation for the vibrational wave function:

$$\left\{-\frac{\hbar^2}{2B}\left(\frac{\partial^2}{\partial\xi^2}+\frac{1}{2}\frac{\partial^2}{\partial\eta^2}\right)+\tfrac{1}{2}C_0\xi^2+C_2\eta^2+\hbar\omega_\mu G_\mu(\xi-\mu 6^{1/2}\eta) +\frac{\hbar^2[(K-\Omega)^2+\mu^2-1]}{16B\eta^2}\right\}\tilde{\varphi}_{K\mu n_0 n_2}(\xi,\eta) =\tilde{E}\tilde{\varphi}_{K\mu n_0 n_2}(\xi,\eta). \quad (26)$$

Defining the rotational energy parameter

$$E_R=(\hbar^2/2J_0) \quad (27)$$

the energy, excluding the energy $\tilde{E}$ of (26), is given by

$$E-\tilde{E}=[I(I+1)-K^2-\mu^2-\Omega^2]E_R+\hbar\omega_\mu+\epsilon_{\Omega+}, \quad (28)$$

the single-particle energy $\epsilon_{\Omega+}$ being given by the Nilsson Hamiltonian:

$$H_{\text{part}}\chi_{\Omega+}=\epsilon_{\Omega+}\chi_{\Omega+}. \quad (29)$$

The subscript $\Omega+$ stands for all the relevant quantum numbers which are, in addition to $\Omega$, the principal quantum number $\tilde{P}$, the projection of the spin on the symmetry axis $\Sigma$, and the asymptotic quantum number $n_z$.

Equation (26) separates immediately into equations describing $\xi$ and $\eta$ vibrations. The $\xi$ vibrations ($\beta$ vibrations) are described by a harmonic-oscillator equation where the potential minimum has been shifted because of the term linear in $\xi$ from $\xi=0$ to

$$\xi_\mu=-6\beta_0{}^2G_\mu E_R\hbar\omega_\mu/E_\beta{}^2. \quad (30)$$

The wave functions and energies are thus

$$U_{\mu,n_0}(\xi)=(\alpha/2^{n_0})^{1/2}\pi^{-1/4}He_{n_0}[\alpha(\xi-\xi_\mu)] \times\exp[-\tfrac{1}{2}\alpha^2(\xi-\xi_\mu)^2];$$
$$\alpha^4=BC_0/\hbar^2, \quad (31)$$
$$\epsilon_{\mu n_0}=(n_0+\tfrac{1}{2})E_\beta-A_\mu,$$
$$E_\beta=\hbar(C_0/\beta)^{1/2},$$
$$A_\mu=\tfrac{3}{2}(\hbar\omega_\mu G_\mu\beta_0/E_\beta)^2E_R. \quad (32)$$

The $\eta$ vibrations ($\gamma$ vibrations) are described by the differential equation

$$\left\{-\frac{\hbar^2}{4B}\frac{\partial^2}{\partial\eta^2}+C_2\eta^2+\frac{\hbar^2}{16B}\frac{(K-\Omega)^2+\mu^2-1}{\eta^2} -6^{1/2}\mu G_\mu\hbar\omega_\mu\eta\right\}\varphi(\eta)=\tilde{\epsilon}_{K\mu n_2}\varphi(\eta). \quad (33)$$

Except for the term linear in $\eta$ it has the form of the radial part of a three-dimensional harmonic oscillation with a centrifugal barrier resulting from an angular momentum

$$l=-\tfrac{1}{2}+\tfrac{1}{2}[\mu^2+(K-\Omega)^2]^{1/2}. \quad (34)$$

If we drop the linear term, (33) has the solutions[1,5,6]

$$\varphi(\eta)=\left[\frac{2\bar{\alpha}(l+\frac{3}{2})_n}{n!\Gamma(l+\frac{3}{2})}\right]^{1/2}(\bar{\alpha}\eta)^{l+1} \times\exp[-\tfrac{1}{2}(\bar{\alpha}\eta)^2]{}_1F_1(-n;l+\tfrac{3}{2};\bar{\alpha}^2\eta^2) \quad (35)$$
$$\int_0^\infty\varphi^2d\eta=1,\quad \bar{\alpha}^4=4BC_2/\hbar^2,$$

and the energies

$$\tilde{\epsilon}_{Kn_2\mu}{}^{(0)}=(2n_2+l+\tfrac{3}{2})E_\gamma;\quad E_\gamma=\hbar(C_2/B)^{1/2}. \quad (36)$$

Maximon has treated the Eq. (33) by a perturbation method[12] and has obtained the first two correction terms for the energy

$$\tilde{\epsilon}_{Kn_2\mu}=\tilde{\epsilon}_{Kn_2\mu}{}^{(0)}+[\tfrac{1}{2}\gamma\mu\tilde{\epsilon}_{Kn_2}{}^{(1)}+\tfrac{1}{2}\gamma^2\mu^2\tilde{\epsilon}_{Kn_2}{}^{(2)}]E_\gamma \equiv\tilde{\epsilon}_{Kn_2\mu}{}^{(0)}+A_{Kn_2\mu} \quad (37)$$

as well as the first-order correction to the wave functions. For the first-order correction to the energy, the second term of (37), leads to the splitting of the upper resonance peak reported in an earlier paper[1,2]; it is the first term in the energy which depends on the sign of $\mu$. We give here only the expression $\tilde{\epsilon}^{(1)}$:

$$\tilde{\epsilon}_{Kn_2}{}^{(1)}=\frac{(-\frac{1}{2})_{n_2}}{n_2!}\frac{\Gamma(l+2)}{\Gamma(l+\frac{3}{2})} \times{}_3F_2(-n_2,\tfrac{3}{2},l+2;\tfrac{3}{2}-n_2,l+\tfrac{3}{2};1), \quad (38)$$
$$\gamma=-6\hbar\omega_\mu G_\mu\beta_0(2E_R/E_\gamma{}^3)^{1/2}. \quad (39)$$

[12] L. C. Maximon (private communication, to be published).

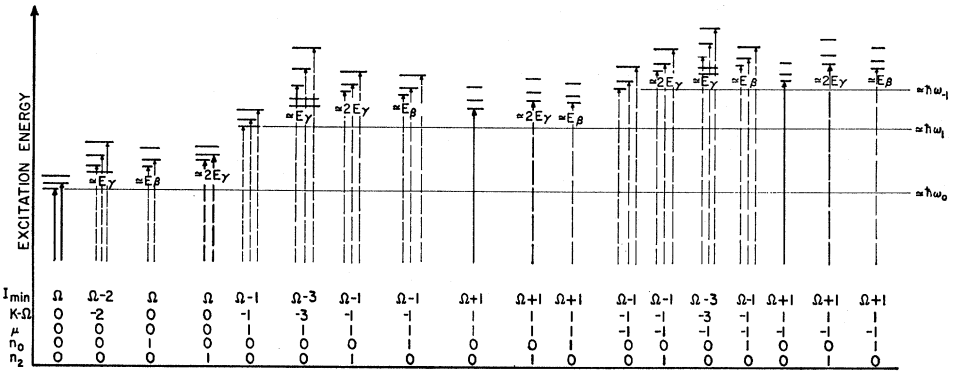


FIG. 1. Schematical level scheme of the giant-resonance region. All states with up to one surface-vibration phonon are shown. The angular momenta of the members of the rotational bands increase in steps of 1 beginning with the value $I_{\min}$; i.e., they are $I_{\min}$, $I_{\min}+1$, $I_{\min}+2$, The intensities of the $E1$ ground-state transitions are also indicated schematically: heavy solid lines represent strong transitions; heavy dashed lines represent weaker transitions; light dashed lines indicate negligible transition strengths.

The expression for $\bar{\epsilon}^{(2)}$ is given in Appendix II. In (38) we have used the abbreviation $(\alpha)_\beta \equiv \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+\beta-1)$.

The potential barrier in Eq. (33), $1/\eta^2$ is impenetrable. The wave function for $\eta>0$ therefore can be chosen independently from that for $\eta<0$. This freedom is very important and must be used to give the total wave function the correct symmetries. The function defined by (35) is normalized for the interval $0\leq\eta\leq\infty$.

## 2. Complete Wave Functions and Level Scheme

Up to now we have not incorporated the third symmetry ($s3$). Under this operation the wave function (23) transforms

$$\varphi_{K\Omega\mu}(\xi,\eta)\psi_\mu[\mathfrak{D}_{MK}{}^I\chi_\Omega+(-)^{-\frac{1}{2}(1-\mu)\mu+N}\mathfrak{D}_{M-K}{}^I\chi_{-\Omega}]\rightarrow$$
$$\varphi_{K\Omega\mu}(\xi,\eta)\psi_{-\mu}[\mathfrak{D}_{MK}{}^I\chi_\Omega+(-)^{-\frac{1}{2}(1+\mu)\mu+N}\mathfrak{D}_{M-K}{}^I\chi_{-\Omega}](-)^{\frac{1}{2}(K-\Omega-|\mu|)}. \quad (40)$$

Therefore, the complete and properly normalized and symmetrized wave functions are

$$\Psi_{K\Omega n_0 n_2\mu}{}^{IM}=\left(\frac{2I+1}{32\pi^2}\right)^{1/2}\{(\mathfrak{D}_{MK}{}^I\chi_\Omega+(-)^{\frac{1}{2}(1-\mu)\mu+N}\mathfrak{D}_{M-K}{}^I\chi_{-\Omega})\psi_\mu\varphi_{K-\Omega,n_0,n_2}{}^I(\eta)+(-)^{\frac{1}{2}(K-\Omega-|\mu|)}$$
$$\times(\mathfrak{D}_{MK}\chi_\Omega{}^I+(-)^{-\frac{1}{2}(1+\mu)\mu+N}\mathfrak{D}_{M-K}{}^I\chi_{-\Omega})\psi_{-\mu}\varphi_{K-\Omega,-\mu,n_2}{}^I(-\eta)\}, \quad (41)$$
$$K=\Omega,\ \Omega+2,\ \Omega+4\cdots \text{ for } \Omega=\tfrac{1}{2},$$
$$K=\Omega-|\mu|,\ \Omega-|\mu|+2,\ \Omega-|\mu|+4\cdots \text{ for } \Omega\neq\tfrac{1}{2},$$
$$I=K,\ K+1,\ K+2\cdots.$$

The functions $\varphi_{K-\Omega,\mu,n_2}{}^I(\eta)$ and $\varphi_{K-\Omega,-\mu,n_2}{}^I(-\eta)$ are normalized and defined for $\eta\geq0$ and $\eta\leq0$, respectively.

One sees from Eq. (33) that $\varphi_{K-\Omega,\mu,n_2}{}^I(\eta)$ has the same value (up to a phase factor) as $\varphi_{K-\Omega,-\mu,n_2}{}^I(-\eta)$ for the same $|\eta|$. The phase factor has to be chosen, in order that

$$\varphi_{K-\Omega,\mu,n_2}{}^I(+\eta)\equiv\varphi_{K-\Omega,-\mu,n_2}{}^I(-\eta). \quad (42)$$

With this convention (41) fulfills all symmetries. The energies for the wave functions (41) are

$$E_{K,\Omega,n_0,n_2,\mu}{}^I=[I(I+1)-K^2-\mu^2-\Omega^2]E_R+\hbar\omega_\mu$$
$$+\epsilon_{\Omega+}+(n_0+\tfrac{1}{2})E_\beta-A_\mu+(2n_2+l+\tfrac{3}{2}$$
$$+\tfrac{1}{2}\lambda\mu\bar{\epsilon}_{Kn_2}{}^{(1)}+\tfrac{1}{2}\lambda^2\mu^2\bar{\epsilon}_{Kn_2}{}^{(2)})E_\gamma. \quad (43)$$

The level scheme is shown qualitatively in Fig. 1. The wave functions of the different states are given in detail elsewhere.[13] One sees that, compared to the giant-resonance spectrum of a deformed even-even nucleus, many new levels appear in the giant-resonance region of an odd-$A$ nucleus. However, we have to study in detail, which states can be reached with the dipole operator from the ground state. We will see that several selection rules limit the number of such states appreciably.

## IV. DIPOLE OPERATOR

### 1. Introductory Remarks

In Ref. 1 the dipole operator has been derived for an even-even nucleus (the core in the present case) in the lowest order in $\beta_0$ and $\xi$. The $\eta$ dependence was explicitly

[13] M. Danos, W. Greiner, and C. B. Kohr, University of Maryland Technical Report No. 381, 1964 (unpublished).

neglected there, since for transitions from the ground-state linear terms of the dipole operator in $\eta$ do not contribute. In this section we will establish the operator in higher orders in the vibrational coordinates; this is needed for the estimate of the magnitude of diverse correction terms to the transition amplitude, as will become clear later.

The dipole operator in the laboratory system ($\hat{D}_\mu$) is related to the components in the intrinsic system ($D_\nu$) by

$$\hat{D}_\mu=\sum_\nu \mathfrak{D}_{\mu\nu}{}^1 D_\nu, \tag{44}$$

where

$$D_\nu=\int \rho_p r Y_{1\nu} d\tau \tag{45}$$

and is calculated in the classical model. $\rho_p$ is the charge density of the intrinsic nucleus and is given by the sum of the charge density $\rho_p$ (core) of the core and the charge density of the particle $\rho_p$ (particle). The latter is just a $\delta$ function

$$\rho_p(\text{particle})=e_{\text{eff}}\delta(\mathbf{r}_{\text{part}}-\mathbf{r}), \tag{46}$$

where $e_{\text{eff}}=-(Z/A)e$ for an odd neutron and $=(N/A)e$ for an odd proton. Therefore

$$D_\nu=D_\nu(\text{core})+D_\nu(\text{particle}),$$

$$D_\nu(\text{core})=\int \rho_p(\text{core}) r Y_{1\nu} d\tau,$$

$$D_\nu(\text{particle})=e_{\text{eff}}\int \rho_p(\text{part}) r Y_{1\nu} d\tau. \tag{47}$$

We now compute $D_\nu$(core) in the classical model, as done in Ref. 1. The charge density of the core consists of the unperturbed charge density $\rho_p(0)$ and the charge density associated with the dipole oscillation $\rho_{\text{osc}}$:

$$\rho_p(\text{core})=\rho_p(0)+\rho_{\text{osc}}=\rho_p(0)(1+\textstyle\sum_\mu \zeta_\mu f_\mu \psi_\mu). \tag{48}$$

The $f_\mu$ are normalization factors which drop out later and therefore need not to be specified. Here $\zeta_\mu$ are the amplitudes of the dipole oscillations, which are defined by writing the total classical dipole energy

$$\textstyle\sum_\mu [\frac{1}{2}g_\mu|\dot{\zeta}_\mu|^2+\frac{1}{2}h_\mu|\zeta_\mu|^2]=\sum_\mu \hbar\omega_\mu b_\mu{}^\dagger b_\mu. \tag{49}$$

The creation and annihilation operators $b_\mu{}^\dagger$, $b_\mu$ (12) are related to the amplitudes $\zeta_\mu$ by the well-known relation

$$\zeta_\mu=(\hbar\omega_\mu/2h_\mu)^{1/2}(b_\mu{}^\dagger+b_\mu). \tag{50}$$

The procedure for computing $D_\nu$(core) is clear: (48) has to be inserted into (47) and the $\zeta_\mu$ have to be expressed via (50) in terms of the creation and annihilation operators. Therefore the constants $h_\mu$ of the potential energy have to be determined.

### 2. Potential-Energy Constants

They are obtained in the hydrodynamic model from the relation

$$V(\zeta)=\textstyle\sum_\mu \frac{1}{2}h_\mu|\zeta_\mu|^2$$

$$=\kappa\int(|\rho_0-2\rho_p(\text{core})|^2/\rho_0)d\tau+\text{const}, \tag{51}$$

where $\rho_0$ is the unperturbed matter density. Substituting (48) into (51) and expanding yields

$$V(\zeta)=\kappa\int(|\Delta\rho_0|^2/\rho_0)d\tau$$

$$-4\kappa\,\text{Re}\int(\Delta\rho_0)^*(\rho_p(0)/\rho_0)\textstyle\sum_\mu \zeta_\mu f_\mu j_1(k_\mu r)\Phi_\mu d\tau$$

$$+4\kappa\int[|\rho_p(0)\textstyle\sum_\mu \zeta_\mu f_\mu j_1(k_\mu r)\Phi_\mu|^2/\rho_0]d\tau+\text{const}, \tag{52}$$

where $\Delta\rho_0=\rho_0-2\rho_p(0)$. The first term of (52) is a constant and the const can be chosen to cancel it. The second term vanishes because of parity selection rules. Therefore only the last term of (52) needs consideration. We obtain

$$V(\zeta)=(4\kappa|\rho_p(0)|^2/\rho_0)\Big\{\textstyle\sum_\mu|\zeta_\mu f_\mu|^2\int|j_1(k_\mu r)\Phi_\mu|^2 d\tau$$

$$+2\,\text{Re}\int[\zeta_1 f_1 j_1(k_1 r)\Phi_1]^*[\zeta_{-1}f_{-1}j_1(k_{-1}r)\Phi_{-1}]d\tau$$

$$+\text{similar cross terms involving }\Phi_0\Big\}. \tag{53}$$

The cross terms, including the term written down explicitly, all vanish when performing the angular integrations. This can be seen by writing for the relevant integral

$$\zeta_1{}^* f_1{}^* \zeta_{-1} f_{-1}\int\int j_1{}^*(k_1 r) j_1(k_{-1}r) r^2 dr$$

$$\times(Y_{11}{}^*+Y_{1-1}{}^*)(Y_{11}-Y_{1-1})d\Omega.$$

The last two factors in the integrand may be written

$$(Y_{11}{}^*+Y_{1-1}{}^*)(Y_{11}-Y_{1-1})=-(Y_{11}{}^2-Y_{1-1}{}^2).$$

In the expansion of the Bessel functions at the upper limit of integration the symmetry between $Y_{22}$ and $Y_{2-2}$ in the radius $R$ [see Eq. (1)] ensures that the terms $Y_{11}{}^2$ and $Y_{1-1}{}^2$ will cancel after the integration. Now we calculate the diagonal terms of (53). The radial integration yields

$$\tfrac{1}{2}R^3[j_1{}^2(k_\mu R)-j_0(k_\mu R)j_2(k_\mu R)]\equiv R^3 P(k_\mu R). \tag{54}$$

Both arguments, $k_\mu$ and $R$, are functions of the vibrational coordinates, see Eqs. (1) and (11). We expand

$P(k_\mu R)$ about $k_\mu=k_\mu(0)=2.08/R_0$ and $R=R_0$ to terms of order $\xi$, $\eta$, and $\beta_0^2$. Then

$$R^3\approx R_0^3[1+3(\beta_0+\xi)Y_{20}+3\eta(Y_{22}+Y_{2-2})+3\beta_0^2Y_{20}^2]$$

and

$$P(k_\mu R)\approx P[k_\mu(0)R_0]+\{[k_\mu-k_\mu(0)]R_0+(R-R_0)k_\mu(0)\}\frac{\partial P(\rho)}{\partial\rho}\Big|_{\rho=k_\mu(0)R_0}+[k_\mu-k_\mu(0)](R-R_0)$$
$$+\tfrac{1}{2}\{[k_\mu-k_\mu(0)]^2R_0^2+(R-R_0)^2k_\mu^2(0)+2(R-R_0)[k_\mu-k_\mu(0)]R_0k_\mu(0)\}\frac{\partial^2P(\rho)}{\partial\rho^2}\Big|_{\rho=k_\mu(0)R_0}. \quad (55)$$

After some straightforward computations we obtain

$$I_R{}^{(\mu)}=R^3P(k_\mu R)\approx N_0{}^{(\mu)}+N_1{}^{(\mu)}[(\beta_0+\xi)Y_{20}+\eta(Y_{22}+Y_{2-2})]+N_2{}^{(\mu)}\beta_0^2Y_{20}^2, \quad (56)$$

where

$$N_0{}^{(\mu)}=R_0^3\{P(\rho)+[G_\mu(\xi-\mu(6)^{1/2}\eta)]\times R_0k_\mu(0)P'(\rho)\}_{\rho=k_\mu(0)R_0},$$
$$N_1{}^{(\mu)}=R_0^3[3P(\rho)+R_0k_\mu(0)P'(\rho)]_{\rho=k_\mu(0)R_0}, \quad (57)$$
$$N_2{}^{(\mu)}=R_0^3[3P(\rho)+3R_0k_\mu(0)P'(\rho)+\tfrac{1}{2}R_0{}^2k_\mu{}^2(0)P''(\rho)]_{\rho=k_\mu(0)R_0}.$$

The formula (56) exhibits explicitly the angular dependence of the radial integral in (53). Inserting (56) into (53) and performing the angular integration gives

$$V(\xi)=4\kappa[\rho_p(0)^2/\rho_0]\sum_\mu|\zeta_\mu f_\mu|^2I_\mu,$$

where

$$I_\mu=\sum_{\nu=0}^{3}I_{\mu\nu}, \quad (58)$$
$$I_{\mu0}=N_0+N_0{}^{(\mu)}(\xi-\mu(6)^{1/2}\eta),$$
$$I_{\mu1}=N_1{}^{(\mu)}(\beta_0+\xi)(5/4\pi)^{1/2}\langle201\mu|1\mu\rangle\langle2010|10\rangle\equiv N_1{}^{(\mu)}(\beta_0+\xi)f_{1\mu},$$
$$I_{\mu2}=\mu N_1{}^{(\mu)}\eta(5/4\pi)^{1/2}\langle221-1|11\rangle\langle2010|10\rangle\equiv\mu N_1{}^{(\mu)}\eta f_{2\mu},$$
$$I_{\mu3}=N_2{}^{(\mu)}\beta_0^2(5/4\pi)[\langle2020|00\rangle^2+\langle2020|20\rangle\langle201\mu|1\mu\rangle\langle2010|10\rangle]\equiv N_2{}^{(\mu)}\beta_0^2f_{3\mu}. \quad (59)$$

Comparing (57) with (49) yields for the desired potential energy coefficients $h_\mu$

$$h_\mu=8\kappa[\rho_p(0)^2/\rho_0]f_\mu{}^2\sum_\nu I_{\mu\nu}\equiv Q_\mu f_\mu{}^2. \quad (60)$$

### 3. Intrinsic Components of the Dipole Operator

We calculate now the quantities $D_\nu$(core) of (47). Inserting (48) into (47) yields

$$D_\nu(\text{core})=\rho_p(0)\int[1+\sum_\mu\zeta_\mu f_\mu j_1(k_\mu r)\Phi_\mu]rY_{1\nu}d\tau. \quad (61)$$

The radial integration gives

$$D_\nu(\text{core})=\rho_p(0)\sum_\mu\zeta_\mu f_\mu\int_\Omega(R^3/k_\mu)j_2(k_\mu R)\Phi_\mu Y_{1\nu}d\Omega. \quad (62)$$

We again expand the integrand about $k_\mu(0)R_0$ and obtain after straightforward calculations

$$[R^3j_2(k_\mu R)/k_\mu]\approx M_0+M_1[(\beta_0+\xi)Y_{20}+\eta(Y_{22}+Y_{2-2})]+M_2\beta_0^2Y_{20}^2, \quad (63)$$

where

$$M_0=[R_0^3/k_\mu(0)]\{(1-G_\mu[\xi-\mu(6)^{1/2}\eta])j_2(\rho)+G_\mu[\xi-\mu(6)^{1/2}\eta]R_0k_\mu(0)j_2'(\rho)\}_{\rho=k_\mu(0)R_0},$$
$$M_1=[R_0^3/k_\mu(0)][3j_2(\rho)+R_0k_\mu(0)j_2'(\rho)]_{\rho=k_\mu(0)R_0}, \quad (64)$$
$$M_2=[R_0^3/k_\mu(0)][3j_2(\rho)+3R_0k_\mu(0)j_2'(\rho)+R_0{}^2k_\mu{}^2(0)j_2''(\rho)]_{\rho=k_\mu(0)R_0}.$$

We now write (63) as follows:

$$[R^3j_2(k_\mu R)/k_\mu]=\sum_{\sigma=0}^{3}J_{\mu\sigma}, \quad (65)$$

where

$$J_{\mu0}=M_0+M_0{}^{(\mu)}(\xi-\mu(6)^{1/2}\eta)\equiv\tilde{J}_{\mu0}+\mu\tilde{J}_{\mu0}',$$
$$J_{\mu1}=M_1(\beta_0+\xi)Y_{20},$$
$$J_{\mu2}=M_1\eta(Y_{22}+Y_{2-2}),$$
$$J_{\mu3}=M_2\beta_0^2Y_{20}^2. \quad (66)$$

Equation (62) then becomes

$$D_\nu(\text{core})=\rho_p(0)\sum_{\mu\sigma}\zeta_\mu f_\mu I_{\mu\sigma\nu}, \quad (67)$$

where

$$I_{\mu\sigma\nu}=\int J_{\mu\sigma}\Phi_\mu Y_{1\nu}d\Omega. \quad (68)$$

These quantities are given for all relevant indices in Appendix I. Using these results and inserting them into (69) we obtain

$$D_1(\text{core})=\rho_p(0)\{(\zeta_1f_1+\zeta_{1-}f_{-1})(I_{101}+I_{111}+I_{131})+(\zeta_1f_1-\zeta_{-1}f_{-1})(I_{121}+I_{101}')\},$$
$$D_{-1}(\text{core})=\rho_p(0)\{(\zeta_1f_1-\zeta_{-1}f_{-1})(I_{101}+I_{111}+I_{131})+(\zeta_1f_1+\zeta_{-1}f_{-1})(I_{121}+I_{101}')\}, \quad (69)$$
$$D_0(\text{core})=\rho_p(0)\zeta_0f_0(I_{000}+I_{010}+I_{020}+I_{030}).$$

With Eq. (47) we now can introduce the creation and annihilation operators of the dipole quanta. Using (60) for the constants $h_\mu$ we find

$$\begin{aligned}
D_1(\text{core}) &= \rho_p(0)\{[(\hbar\omega_1/2Q_1)^{1/2}(b_1^\dagger+b_1)+(\hbar\omega_{-1}/2Q_{-1})^{1/2}(b_{-}{}^\dagger+b_{-1})](I_{101}+I_{111}+I_{131}) \\
&\quad +[(\hbar\omega_1/2Q_1)^{1/2}(b_1^\dagger+b_1)-(\hbar\omega_{-1}/2Q_{-1})^{1/2}(b_{-1}^\dagger+b_{-1})](I_{121}+I_{101}')\}. \\
D_{-1}(\text{core}) &= \rho_p(0)\{[(\hbar\omega_1/2Q_1)^{1/2}(b_1^\dagger+b_1)-(\hbar\omega_{-1}/2Q_{-1})^{1/2}(b_{-1}^\dagger+b_{-1})](I_{101}+I_{111}+I_{131}) \\
&\quad +[(\hbar\omega_1/2Q_1)^{1/2}(b^\dagger+b_1)+(\hbar\omega_{-1}/2Q_{-1})^{1/2}(b_{-1}^\dagger+b_{-1})](I_{121}+I_{101}')\}, \qquad (70) \\
D_0(\text{core}) &= \rho_p(0)(\hbar\omega_0/2Q_0)^{1/2}(b_0^\dagger+b_0)(I_{000}+I_{010}+I_{020}+I_{030}).
\end{aligned}$$

These formulas can be written in a short way:

$$\begin{aligned}
D_\nu(\text{core}) &= \rho_p(0)\{[(\hbar\omega_{|\nu|}/2Q_{|\nu|})^{1/2}(b_{|\nu|}^\dagger+b_{|\nu|})+\nu(\hbar\omega_{-|\nu|}/2Q_{-|\nu|})^{1/2}(b_{-|\nu|}^\dagger+b_{-|\nu|})](I_{|\nu|0|\nu|}+I_{|\nu|1|\nu|}+I_{|\nu|3|\nu|}) \\
&\quad +[(\hbar\omega_{|\nu|}/2Q_{|\nu|})^{1/2}(b_{|\nu|}^\dagger+b_{|\nu|})-\nu(\hbar\omega_{-|\nu|}/2Q_{-|\nu|})^{1/2}(b_{-|\nu|}^\dagger+b_{-|\nu|})](I_{|\nu|2|\nu|}+I_{|\nu|0|\nu|}')\}, \quad \nu=\pm1, 0. \qquad (71)
\end{aligned}$$

This form of dipole operator is not suitable for actual calculations, since the quantities $\omega_{|\nu|}$, $B_{|\nu|}$, and $I_{\mu\nu\lambda}$ depend on the vibrational coordinates $\xi$, $\eta$. In order to obtain a form showing explicitly this dependence we expand these quantities to first order in $\xi$, $\eta$ and to second order in $\beta_0$. Using (59), (60), and (13) we find for $\hbar\omega_\nu/2Q_\nu$

$$\begin{aligned}
\left[\frac{\hbar\omega_\nu}{2Q_\nu}\right]^{1/2} &= \left[\frac{\hbar\bar{\omega}_\nu}{16\kappa}\frac{\rho_0}{\rho_p(0)^2}\frac{1+G[\xi-\mu(6)^{1/2}\eta]}{N_0+(N_1f_{1\nu}+N_0{}^{(\nu)})(\beta_0+\xi)+[N_1f_{2\nu}-(6)^{1/2}N_{0\nu}]\nu\eta+N_2\beta_0{}^2f_{3\nu}}\right]^{1/2} \\
&\approx \frac{1}{4\rho_p(0)}\left(\frac{\rho_0}{\kappa}\right)^{1/2}\left[\frac{\hbar\bar{\omega}_\nu}{N_0+N_1\beta_0f_{1\nu}+N_2\beta_0{}^2f_{3\nu}}\right]^{1/2} \\
&\quad \times\left\{1+\tfrac{1}{2}\left[G_\nu-\frac{N_1f_{1\nu}+N_0{}^{(\nu)}}{N_0+N_1f_{1\nu}\beta_0+N_2f_{3\nu}\beta_0{}^2}\right]\xi-\tfrac{1}{2}\nu\left[(6)^{1/2}G_\nu+\frac{N_1f_{2\nu}-(6)^{1/2}N_0{}^{(\nu)}}{N_0+N_1f_{1\nu}\beta_0+N_2f_{3\nu}\beta_0{}^2}\right]\eta\right\}. \qquad (72)
\end{aligned}$$

The dependence of $I_{\nu\mu\lambda}$ on the vibrational coordinates can be found in the Appendix. Using (72) and the Appendix and inserting this into (71) yields after some calculations

$$\begin{aligned}
D_\nu(\text{core}) &= d_\nu\{[(b_{|\nu|}^\dagger+b_{|\nu|})+\nu(b_{-|\nu|}^\dagger+b_{-|\nu|}][S_0(\nu)+S_1(\nu)\xi] \\
&\quad +[(b_{|\nu|}^\dagger+b_{|\nu|}-\nu(b_{-|\nu|}^\dagger+b_{-|\nu|})][S_2(\nu)+S_3(\nu)]\eta\}, \quad \nu=\pm1, 0, \qquad (73)
\end{aligned}$$

where

$$\begin{aligned}
d_\nu &\equiv \frac{1}{4}\left(\frac{\rho_0}{\kappa}\right)^{1/2}\left[\frac{\hbar\bar{\omega}_\nu}{N_0+N_1\beta_0f_{1\nu}+N_2\beta_0{}^2f_{3\nu}}\right]^{1/2}, \\
S_0(\pm1) &= -2^{-1/2}\{M_0+M_1f_{11}\beta_0+M_2f_{31}\beta_0{}^2\}, \\
S_0(0) &= M_0+M_1f_{10}\beta_0+M_2f_{30}\beta_0{}^2, \\
S_1(\nu) &= \tfrac{1}{2}\left[G_\nu-\frac{N_1f_{1\nu}+N_0{}^{(\nu)}}{N_0+N_1f_{1\nu}\beta_0+N_2f_{3\nu}\beta_0{}^2}\right]S_0(\nu)+(-)^{|\nu|}(2)^{-1/2|\nu|}(M_1+M_0{}^{(\nu)})f_{1|\nu|}, \qquad (74) \\
S_2(\nu) &= -\tfrac{1}{2}\nu\left[6^{1/2}G_\nu+\frac{N_1f_{2\nu}-(6)^{1/2}N_0{}^{(\nu)}}{N_0+N_1f_{1\nu}\beta_0+N_2f_{3\nu}\beta_0{}^2}\right]S_0(\nu), \\
S_3(\nu) &= -|\nu|2^{-1/2}[M_1-(6)^{1/2}M_0{}^{(\nu)}]f_{2\nu}.
\end{aligned}$$

Equation (73) is the dipole operator to be used in actual calculations. It is easily checked that in the case of $\xi=\eta=0$ it is identical with the result derived in (1).

## V. DIPOLE TRANSITIONS FROM THE GROUND STATE

To obtain the dipole absorption cross section it is necessary to compute the transition matrix element from the ground state $\psi_{K=\Omega,n_0=0,n_2=0,\Omega}{}^{IM}$ to the dipole state $\psi_{K'n'n_2'\mu'\Omega'}{}^{I'M'}$ of Eq. (38),

$$\langle\psi_{K=\Omega,n_0=0,n_2=0,\Omega}{}^{IM}|\hat{D}_\nu|\psi_{K'n'n_2'\mu'\Omega'}{}^{I'M'}\rangle\equiv\mathfrak{M}. \qquad (75)$$

We are here interested only in the collective transitions and therefore the single-particle part of the dipole operator

(44) is omitted. We further restrict ourselves to the main term of the intrinsic operator (73), i.e., we neglect its dependence on $\xi$, $\eta$. Using (38) and (73) thus the matrix element (75) is

$$\mathfrak{M}=\frac{[(2I+1)(2I'+1)]^{1/2}}{32\pi^2}\langle[\mathfrak{D}_{M\Omega}{}^{I}\chi_{\Omega+}-(-)^{I+j}\mathfrak{D}_{M-\Omega}{}^{I}\chi_{-\Omega+}][\varphi_{000}(\eta)+\varphi_{000}(-\eta)]u_0(\xi)\psi^{(0)}$$
$$\times|\textstyle\sum_{\nu'}\mathfrak{D}_{\nu\nu'}{}^{1}d_{\nu'}[(b_{|\nu'|}{}^{\dagger}+b_{|\nu|})+\nu'(b_{-|\nu'|}{}^{\dagger}+b_{-|\nu'|})]|[\mathfrak{D}_{M'K'}{}^{I'}\chi_{\Omega'+}+(-)^{\frac{1}{2}\mu'(1-\mu')+N'}\mathfrak{D}_{M'-K'}{}^{I'}\chi_{-\Omega'+}]$$
$$\times\psi_{\mu'}\varphi_{K'-\Omega',\mu',n_2'}{}^{I'}(\eta)+(-)^{\frac{1}{2}(K'-\Omega'-|\mu'|)}[\mathfrak{D}_{M'K'}{}^{I'}\chi_{\Omega'+}-(-)^{-\frac{1}{2}(1+\mu')\mu'+N'}\mathfrak{D}_{M'-K'}{}^{I'}\chi_{-\Omega'+}]$$
$$\times\psi_{-\mu'}\varphi_{K'-\Omega',-\mu',n_2'}{}^{I'}(-\eta)\}u_{\mu'n_0'}(\xi)\rangle. \quad (76)$$

On collecting the different terms (76) becomes

$$\mathfrak{M}=\frac{[(2I+1)(2I'+1)]^{1/2}}{32\pi^2}$$
$$\times\langle u_0(\xi)|u_{\mu',n_0}(\xi)\rangle\delta_{\Omega+,\Omega'+}\textstyle\sum_{\nu'}\{\langle\mathfrak{D}_{M\Omega}{}^{I}|\mathfrak{D}_{\nu\nu'}{}^{1}|\mathfrak{D}_{M'K'}{}^{I'}\rangle(\delta_{|\nu'|,\mu'}+\nu'\delta_{-|\nu'|,\mu'})$$
$$\times d_{\nu'}\langle\varphi_{000}(\eta)|\varphi_{K'-\Omega',\mu',n_2'}{}^{I'}(\eta)\rangle+(-)^{\frac{1}{2}(K'-\Omega'-|\mu'|)}\langle\mathfrak{D}_{M\Omega}{}^{I}|\mathfrak{D}_{\nu\nu'}{}^{1}|\mathfrak{D}_{M'K'}{}^{I'}\rangle(\delta_{-\mu',|\nu'|}+\nu'\delta_{\mu',|\nu'|})$$
$$\times d_{\nu'}\langle\varphi_{000}(-\eta)|\varphi_{K'-\Omega',-\mu',n_2'}{}^{I'}(-\eta)\rangle-(-)^{\frac{1}{2}(1-\mu')\mu'+I'+'j+N'}\langle\mathfrak{D}_{M-\Omega}{}^{I'}|\mathfrak{D}_{\nu\nu'}{}^{1}|\mathfrak{D}_{M'-K'}{}^{I}\rangle(\delta_{\mu',|\nu'|}+\nu'\delta_{\mu',-|\nu'|})$$
$$\times d_{\nu'}\langle\varphi_{000}(\eta)|\varphi_{K'-\Omega',\mu',n_2'}{}^{I'}(\eta)\rangle-(-)^{\frac{1}{2}[K'-\Omega'-|\mu'|-(1+\mu')\mu']+I'+j'+N'}\langle\mathfrak{D}_{M-\Omega}{}^{I}|\mathfrak{D}_{\nu\nu'}{}^{1}|\mathfrak{D}_{M'-K'}{}^{I'}\rangle(\delta_{-\mu',|\nu'|}+\nu'\delta_{\mu',|\nu'|})$$
$$\times d_{\nu'}\langle\varphi_{000}(-\eta)|\varphi_{K'-\Omega',-\mu',n_2'}{}^{I'}(-\eta)\rangle\}. \quad (77)$$

Using well-known relations of angular-momentum theory, Eq. (77) can be rewritten as

$$\mathfrak{M}=\left(\frac{2I'+1}{2I+1}\right)^{1/2}\delta_{\Omega+,\Omega'+}\langle u_0(\xi)|u_{\mu'n_0'}(\xi)\rangle\textstyle\sum_{\nu'}(I'M'1\nu|IM)(I'K'1\nu'|I\Omega)(\delta_{\mu',|\nu'|}+\nu'\delta_{\mu',-|\nu'|})$$
$$\times d_{\nu'}\langle\varphi_{000}(\eta)|\varphi_{K'-\Omega',\mu',n_2'}{}^{I'}(\eta)\rangle\tfrac{1}{2}[1+(-)^{\frac{1}{2}(K'-\Omega'-|\mu'|)}]. \quad (78)$$

The notation for the overlap integrals in (77) and (78) is the following:

$$\langle\varphi_A(\eta)|\varphi_B(\eta)\rangle=\langle\varphi_A(-\eta)|\varphi_B(-\eta)\rangle=\int_0^{\infty}\varphi_A(\eta)\varphi_B(\eta)d\eta. \quad (79)$$

Because of angular momentum conservation the sum reduces to the one term $\nu=\Omega-K'$. Explicit expressions for (79) are given in Appendix II.

The most interesting aspect of (78) is the selection rule implied by the bracket containing only phase factors. The origin of this rule is the symmetry of the vibrational wave function under substitution $\eta\rightarrow-\eta$, which thus can be called $\eta$ parity. Transitions in which the $\eta$-parity changes are forbidden owing to the vanishing of the overlap integral. The possibility of having vibrational states degenerate in energy but of opposite parity is a consequence of the impenetrability of the potential barrier at $\eta=0$ in (33). Without this barrier the symmetry requirements of certain states could not be fulfilled.

The $\eta$-parity selection rule is, however, not exact. It is broken by the terms in the dipole operator having $\eta$ as factor, as well as by the term $H_{\text{rot vib}}$ neglected in the present treatment. Both these terms are small and the strength of $\eta$-parity forbidden transitions is of the order of 10% of that of allowed transitions. The parameter which determines the strength of the forbidden transition is $E_R/E_\gamma$. The selection rule thus loses validity as one approaches the region of vibrational nuclei.

## VI. RESULTS AND DISCUSSION

We now summarize the main results of this paper. To begin with, the assertion that the odd particle does not have an important influence on the giant dipole resonance has been borne out. As a matter of fact, this assertion is even better fulfilled than expected in that the $\eta$-parity selection rule discussed in the last section limits the number of the important upper dipole transitions to two, the same number as in the even-even nuclei, while the lower peak splits into two roughly equally strong components separated by about 100 keV, a splitting masked completely by the width which for the lower peak is about 2 MeV. This is true despite the fact that here the number of states which can be reached by $E1$ transitions when considering only angular momentum and parity conservation is considerably greater than in even-even nuclei. In fact, the $\eta$-parity selection rule is not exact and the photon absorption spectrum in

odd-$A$ nuclei will be somewhat different than in even-even nuclei. Because of the smallness of the symmetry-breaking terms this difference will, however, be so small that it is not clear whether it can be experimentally detected at all.

The structure of the Hamiltonian and of the transition matrix elements further shows that Brink's hypothesis[14] is fulfilled to a large extent for our model. This hypothesis asserts that besides fulfilling the sum rule, dipole transitions originating in excited states of the nucleus resemble the transitions from the ground state even more closely in that in such transitions also a giant resonance appears at the same photon energy. This is supposed to be true for transitions starting at every excited state. This hypothesis turns out to be fulfilled for the giant resonance "based on" the excited single-particle states while only very small changes occur in the giant resonances "based on" the lower rotational and vibrational states.

We would now like to discuss the consistency of the model, the accuracy of the solutions, and the possibility of refinements of the theory.

The model consists of two kinds of collective degrees of freedom and of a single particle moving in a potential well. The first two, the surface and the dipole modes, can be considered as resulting from the quantization of a continuous system, i.e., the treatment of their degrees of freedom may be called quantum hydrodynamics. The parameters of the theory are to be considered as arbitrary parameters to be determined separately, either from more fundamental theory or from experiment. This system will of necessity fulfill the classical dipole sum rule, and by introduction of an effective mass one may even include the effects of exchange forces. Depending on whether the odd particle is a proton or a neutron the sum rule must be taken either as $(N-1)Z/A$ or $N(Z-1)/A$; the odd particle does not participate in the collective motions. However, in the denominator one has to retain $A$ rather than changing to $A-1$ since the odd particle participates in the recoil motion. The addition of the odd particle to the model is, however, not completely consistent: *de facto*, all particles participate in the collective excitations. For example, an $f_{7/2}$ valence nucleon can make a dipole transition to a $g_{9/2}$ state. When treating the dipole state in the shell model, this transition has to be admixed to the states making up the dipole state. It would therefore be wrong to expect to see a $\frac{9}{2}^+$ state with a single-particle $E1$ strength at the energy corresponding to the independent particle transition energy which here is the energy separation of the major shells, i.e., at about 7 MeV. However, such a state would be predicted by the model. The reason for this inconsistency is ovbious: the model Hamiltonian is not symmetric in all particles; the $A-1$ particles which make up the core are described by the collective variables while the valence particle is treated as an independent particle in a potential well. It is also not clear how to formulate the antisymmetrization of the wave function in the model, the coordinates of the core particles being hidden away in a nontransparent manner.

The above rather obvious remarks were made to indicate the limited validity of the model: it can be used to describe only a very limited number of "single-particle excitations." On the other hand, it is complete for the purpose of defining all the possible kinds of symmetries of the wave function of a deformed odd-$A$ nucleus. If one allows also integer values of $\Omega$ one can similarly investigate the symmetries of the wave function for odd-odd nuclei. In other words, this model can describe all the kinematics of heavy nuclei. It also can be expected to describe very well the dynamics of the collective aspects; however, the model is too primitive to describe the dynamics of the "particle-excitation spectrum." The kinematic aspects of the odd particle are, however, very important. They are, for example, indispensable in the description of the elastic and the Raman scattering of photons on nuclei and of experiments involving nuclear orientation. Fortunately, the kinematic aspects are sufficient for the description of these phenomena, and the limitations of the model here are of no consequence.

We now turn to the discussion of the accuracy of our solutions and of the model Hamiltonian. Concerning the energies, the rotations and the $\beta$ vibrations could be treated exactly. The treatment of the main splitting, i.e., the determination of $\hbar\bar{\omega}_0$ and $\hbar\bar{\omega}_{\pm 1}$, is very accurate. The only relatively large uncertainty is associated with the subsidiary splitting, which is associated with the interaction of the dipole mode with the $\gamma$ vibrations, i.e., with the solution of Eq. (33). However, even that is not very important. According to Maximon,[12] an educated guess gives for the uncertainty of the energies of the states with $\mu=\pm 1$ a value of about $\pm 50$ keV, or about 0.3% of the dipole energy. Again, considering the width of the states, this uncertainty is trivially small.

The accuracy of the dipole intensities is somewhat smaller. The inaccuracies are here associated with the dipole operator, with the matrix elements of the dipole operator $(\psi, D\psi)$ and with the evaluation of the overlap $(\varphi|\varphi)$ of Eq. (80) which, again, involves the solution of (33). The dipole operator has in the present treatment been evaluated by expanding it in powers of $\beta_0$, $\xi$, and $\eta$. The matrix elements were evaluated using spherical functions rather than spheroidal functions. Together with the uncertainties resulting from the evaluation of the above overlap integrals and the different small terms dropped from the Hamiltonian the accuracy of the line intensities is therefore not better than about $\pm 20\%$. The accuracy for the main transition is somewhat better. Some of these inaccuracies are eliminated if one

[14] D. Brink, thesis, Oxford University, Oxford, England, 1955 (unpublished).

computes only relative line strengths where the accuracy is about 10%.

As to the possible refinements of the theory, very little can be done in the improvement of the model Hamiltonian as long as one keeps the two kinds of dynamic variables, the collective and the single particle, which are not independent of each other. In order to improve the description of the "single-particle" aspects one could think, for example, of including more than the minimum number of particles in the "odd"-particle part of the Hamiltonian, viz., one for odd-$A$ nuclei and two for odd-odd nuclei. One could then do configuration mixing to improve the dynamics of the "single-particle spectrum." However, this is not advisable since one would then rob the collective degrees of freedom even further of their completeness. The possible improvements of the collective Hamiltonian, as, e.g., the inclusion of the diverse coupling terms dropped in the present treatment, are of minor importance in the giant-resonance region. Also, improvements in the wave functions will have practically no effect concerning the energies of the states. However, the transition probabilities are much more sensitive to the accuracy of the wave functions. A more accurate solution therefore seems desirable. This would have to include a numerical treatment of the $\gamma$ vibrations. Also, certain selection rules forbidding, e.g., the transitions involving a change in $\eta$ parity are broken by some of the neglected coupling terms. This last effect results again in a rather small change in the cross section. However, all these small inaccuracies will have to be cleaned up if one aspires to compute the intensities to better than, say, 10%.

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## APPENDIX I

The $I_{\mu\sigma\lambda}$ are given by

$$I_{101}=-M_0/\sqrt{2}\,,$$

$$I_{111}=-\frac{M_1}{\sqrt{2}}(\beta_0+\xi)\left(\frac{5}{4\pi}\right)^{1/2}(2011|11)(2010|00)=\frac{M_1}{2(10\pi)^{1/2}}(\beta_0+\xi)\,,$$

$$I_{121}=-\frac{M_1}{\sqrt{2}}\eta\left(\frac{5}{4\pi}\right)^{1/2}(2-211|1-1)(2010|10)=\left(\frac{3}{20\pi}\right)^{1/2}M_1\eta\,,$$

$$I_{131}=-\frac{M_2}{\sqrt{2}}\beta_0{}^2\frac{5}{4\pi}(2020|00)^2(0011|11)(0010|10)+(2020|20)^2(2011|11)(2010|10)=\frac{-5M_2\beta_0{}^2}{28\pi\sqrt{2}}\,,$$

$$I_{000}=M_0\,,$$

$$I_{010}=M_1(\beta_0+\xi)\left(\frac{5}{4\pi}\right)^{1/2}(2010|10)^2=\frac{M_1}{\sqrt{5\pi}}(\beta_0+\xi)\,,$$

$$I_{020}=0\,,$$

$$I_{030}=M_2\beta_0{}^2\frac{5}{4\pi}(2020|00)(0010|10)^2+(2020|20)(2010|10)^2=\frac{11}{28\pi}M_2\beta_0{}^2\,.$$

## APPENDIX II

We quote here the results obtained by Maximon[12] for the energies and the overlap integrals by means of a perturbation treatment for the coefficients of the power-series expansion of the solutions of Eq. (33). The notation is the same as in (37) and (39) except that we omit the subscript 2 in $n_2$, and we write

$$\nu=l+1\,. \tag{II.1}$$

The energies of Eq. (37) are

$$\tilde{\epsilon}_{Kn}{}^{(2)}=-\frac{1}{4}\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})}\frac{(\nu+\frac{1}{2})_n}{n!}\sum_{j=0}^{n}\sum_{j'=0}^{n}\frac{(-n)_j(-n)_{j'}(\nu+1)_j}{j!j'!(\nu+\frac{1}{2})_j(\nu+\frac{1}{2})_{j'}}$$
$$\times\left\{\frac{\Gamma(\nu+j+j'+\frac{5}{2})}{n!}\sum_{m=0}^{n-1}(-1)^m(n-m-1)!\frac{(\frac{3}{2}+j-n)_m(-\frac{1}{2}-j')_m}{\Gamma(\nu+j+m+2)}-\frac{(-\frac{1}{2}-j)_n(-\frac{1}{2}-j')_n}{n!}\right.$$
$$\left.\times\sum_{m=0}^{n}\frac{1}{m+\frac{1}{2}}\frac{\Gamma(\nu+j'+m+\frac{3}{2})}{\Gamma(\nu+n+m+1)}\right\}+\frac{1}{4}\frac{\Gamma^2(\nu+1)}{\Gamma^2(\nu+\frac{1}{2})}\frac{[(-\frac{1}{2})_n]^2}{(n!)^2}\,{}_3F_2(-n,\nu+1,\tfrac{3}{2};\nu+\tfrac{1}{2},\tfrac{3}{2}-n;1)$$
$$\times\sum_{j=0}^{n}\frac{(-n)_j(\nu+1)_j(\frac{3}{2})_j}{j!(\nu+\frac{1}{2})_j(\frac{3}{2}-n)_j}[\psi(n-j-\tfrac{1}{2})-\psi(\tfrac{1}{2})+\psi(1)-\psi(n+1)]+\frac{1}{2}\frac{\Gamma(\nu+1)(\nu+\frac{1}{2})_n}{\Gamma(\nu+\frac{1}{2})(n!)^2}$$
$$\times[(-\tfrac{1}{2})_n]^2\,{}_3F_2(-n,\nu+1,\tfrac{3}{2};\nu+\tfrac{1}{2},\tfrac{3}{2}-n;1)\sum_{j'=0}^{n}\frac{(-n)_{j'}(\frac{3}{2})_{j'}\Gamma(2\nu+2+2j')2^{n-\frac{1}{2}-j'}}{j'!(\nu+\frac{1}{2})_{j'}(\frac{3}{2}-n)_{j'}\Gamma(2\nu+\frac{3}{2}+j'+n)}$$
$$\times\left[\frac{\partial}{\partial\zeta}\,{}_2F_1(\tfrac{3}{2}+j'-n-\zeta,n-\tfrac{1}{2}-j';2\nu+\tfrac{3}{2}+j'+n;\tfrac{1}{2})\right]_{\zeta=0}. \quad \text{(II.2)}$$

The overlap integrals, Eq. (79), are here given for transitions from the ground state, i.e., one of the functions is always the ground-state function $\varphi_{000}(x)$, which is known exactly since the linear term is absent in the potential of the equation for the ground state; see Eq. (33). The wave functions for the excited states are normalized up to an accuracy linear in the perturbation parameter $\gamma$, i.e.,

$$\int_0^\infty[\varphi_{\alpha,\mu,n}(x)]^2dx=1+0(\gamma^2). \quad \text{(II.3)}$$

Then, up to terms linear in $\gamma$, there holds

$$\int_0^\infty\varphi_{000}(x)\varphi_{k-\Omega,1,n}(x)dx$$
$$=\frac{\Gamma(\frac{1}{2}\nu+\frac{3}{4})(\frac{1}{2}\nu-\frac{1}{4})_n}{[n!\Gamma(\nu+\frac{1}{2})(\nu+\frac{1}{2})_n]^{1/2}}\left\{1+\frac{\gamma}{4}\frac{(-\frac{1}{2})_n\Gamma(\nu+1)}{n!\Gamma(\nu+\frac{1}{2})}\left[\sum_{r=0}^{n-1}\frac{(-n)_r(\frac{3}{2})_r(\nu+1)_r}{r!(\frac{3}{2}-n)_r(\nu+\frac{1}{2})_r}[\psi(n-r-\tfrac{1}{2})-\psi(-\tfrac{1}{2})]\right.\right.$$
$$\left.+{}_3F_2\frac{(\nu+\frac{1}{2})_n}{(\frac{1}{2}\nu-\frac{1}{4})_n}\sum_{j=0}^{n-1}\frac{(-n)_j(\frac{1}{2}\nu+\frac{3}{4})_j}{j!(\nu+\frac{1}{2})_j}[\psi(1)-\psi(n+1-j)]\right]+\frac{\gamma}{4}\frac{(\nu+\frac{1}{2})_n}{(\frac{1}{2}\nu-\frac{1}{4})_n}\frac{\Gamma(\frac{1}{2}\nu+\frac{5}{4})}{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}$$
$$\times\sum_{j=1}^{n}\frac{(\frac{3}{2}-n)_{j-1}(\frac{1}{2}\nu+\frac{5}{4})_j}{(\frac{3}{2})_j(\nu+1)_j}\sum_{r=0}^{j-1}\frac{(-n)_r(\frac{3}{2})_r(\nu+1)_r}{r!(\frac{3}{2}-n)_r(\nu+\frac{1}{2})_r}+\frac{\gamma}{4}\frac{(-\frac{1}{2})_n(\nu+\frac{1}{2})_n}{n!(\frac{1}{2}\nu-\frac{1}{4})_n}\,{}_3F_2\sum_{j=1}^{n}(-1)^j(j-1)!(\tfrac{1}{2}\nu-\tfrac{1}{4})_{n-j}$$
$$\times\left[\frac{(\frac{1}{2}\nu+\frac{5}{4})_j\Gamma(\frac{1}{2}\nu+\frac{5}{4})}{(\frac{3}{2})_j(\nu+1)_n\Gamma(\frac{1}{2}\nu+\frac{3}{4})}-\frac{(\frac{1}{2}\nu+\frac{3}{4})_j\Gamma(\nu+1)}{(\nu+\frac{1}{2})_nj!\Gamma(\nu+\frac{1}{2})}\right]+\frac{\gamma}{2}\frac{(-\frac{1}{2})_n}{n!}\left[\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})}-\frac{(\nu+\frac{1}{2})_n}{(\nu+1)_n}\frac{\Gamma(\frac{1}{2}\nu+\frac{5}{4})}{\Gamma(\frac{1}{2}\nu+\frac{3}{4})}\right]{}_3F_2$$
$$\left.+\frac{\gamma}{4}\frac{(-\frac{1}{2})_n}{n!}\frac{(\nu+\frac{1}{2})_n\Gamma(\nu+1)\Gamma(\nu+\frac{3}{2})2^{\frac{1}{2}\nu+n+\frac{3}{4}}}{\Gamma(\frac{1}{2}\nu+\frac{3}{4})\Gamma(\frac{3}{2}\nu+n+\frac{5}{4})}\,{}_3F_2\left[\frac{\partial}{\partial\zeta}\,{}_2F_1(\tfrac{5}{4}-\tfrac{1}{2}\nu-n-\zeta,\tfrac{1}{2}\nu-\tfrac{1}{4}+n;\tfrac{3}{2}\nu+n+\tfrac{5}{4};\tfrac{1}{2})\right]_{\zeta=0}\right\}, \quad \text{(II.4)}$$

where

$${}_3F_2\equiv{}_3F_2(-n,\tfrac{3}{2},\nu+1;\tfrac{3}{2}-n,\nu+\tfrac{1}{2};1). \quad \text{(II.5)}$$

Note that for $n=0$ the first four summations are to be dropped.

Very few terms in the series expansion of $[\partial_2F_1/\partial\zeta]_{\zeta=0}$ are needed to obtain a good numerical accuracy for (II.2) and (II.4) since the argument of the function is $\frac{1}{2}$.