CENTER FOR FINANCIAL STUDIES

No. 2006/24

Portfolio Optimization when Risk Factors are Conditionally Varying and Heavy Tailed

Toker Doganoglu, Christoph Hartz,
and Stefan Mittnik

Center for Financial Studies
an der Universität Frankfurt ▪ Mertonstraße 17-21 ▪ D-60325 Frankfurt am Main
Telefon: +49 (0)69 798-30050
Fax: +49 (0)69 798-30077 ▪ http://www.ifk-cfs.de ▪ E-Mail: info@ifk-cfs.de

Center for Financial Studies

The *Center for Financial Studies* is a nonprofit research organization, supported by an association of more than 120 banks, insurance companies, industrial corporations and public institutions. Established in 1968 and closely affiliated with the University of Frankfurt, it provides a strong link between the financial community and academia.

The CFS Working Paper Series presents the result of scientific research on selected topics in the field of money, banking and finance. The authors were either participants in the Center´s Research Fellow Program or members of one of the Center´s Research Projects.

If you would like to know more about the *Center for Financial Studies*, please let us know of your interest.

Prof. Dr. Jan Pieter Krahnen

Prof. Volker Wieland, Ph.D.

Center for Financial Studies
an der Universität Frankfurt ▪ Mertonstraße 17-21 ▪ D-60325 Frankfurt am Main
Telefon: +49 (0)69 798-30050
Fax: +49 (0)69 798-30077 ▪ http://www.ifk-cfs.de ▪ E-Mail: info@ifk-cfs.de

CFS Working Paper No. 2006/24

Portfolio Optimization when Risk Factors are Conditionally Varying and Heavy Tailed*

Toker Doganoglu[1], Christoph Hartz[2],
and Stefan Mittnik[3]

October 2006

Abstract:
Assumptions about the dynamic and distributional behavior of risk factors are crucial for the construction of optimal portfolios and for risk assessment. Although asset returns are generally characterized by conditionally varying volatilities and fat tails, the normal distribution with constant variance continues to be the standard framework in portfolio management. Here we propose a practical approach to portfolio selection. It takes both the conditionally varying volatility and the fat-tailedness of risk factors explicitly into account, while retaining analytical tractability and ease of implementation. An application to a portfolio of nine German DAX stocks illustrates that the model is strongly favored by the data and that it is practically implementable.

JEL Classification: C13, C32, G11, G14, G18

Keywords: Multivariate Stable Distribution, Index Model, Portfolio Optimization, Value-at-Risk, Model Adequacy

* We thank the Deutsche Börse AG for providing data. The work of Hartz and Mittnik was supported by a research grant from the Deutsche Forschungsgemeinschaft via SFB 386.

1 Center for Information and Network Economics, University of Munich, Germany
I would like to thank Volkswagen Stiftung for the generous financial support which made this research possible.

2 Department of Statistics, University of Munich

3 University of Munich, Center for Financial Studies, Frankfurt, and Ifo Institute for Economic Research, Munich
Corresponding author: Institute of Statistics, Ludwig-Maximilians-Universität München, Akademiestr. I/1, D-80799 München, Germany. email: finmetrics@stat.uni-muenchen.de

Non-technical summary

Assumptions about the dynamic and distributional behavior of risk factors are crucial for the construction of optimal portfolios and for risk assessment. It is widely accepted in empirical financial analysis that asset returns are generally characterized by conditionally varying volatilities and a conditional distribution that differs substantially from the normal distribution, exhibiting excess kurtosis (fat–tails) and oftentimes skewness. Despite these phenomena, the normal distribution with constant variance continues to be the basic framework in mean–*variance*–based portfolio management.

This paper presents a practical approach to portfolio selection within the mean–*scale* framework which takes both the conditionally varying volatility and the non–normality of risk factors explicitly into account. The model uses a GARCH–type structure for modeling risk factors' dynamic and utilizes the stable distribution for describing the conditional distribution. The proposed factor model for modeling asset returns generalizes the normal distribution assumption while retaining analytical tractability and ease of implementation. An application to nine stocks from the german DAX illustrates that the model is strongly favored by the data and that it is practically implementable.

Nichttechnische Zusammenfassung

Die Bewertung von Risiken und die optimale Zusammensetzung von Wertpapier–Portfolios hängt insbesondere von den für die Risikofaktoren gemachten Annahmen bezüglich der zugrunde liegenden Dynamik und den Verteilungseigenschaften ab. In der empirischen Finanzmarkt–Analyse ist weitestgehend akzeptiert, daß die Renditen von Finanzmarkt–Zeitreihen zeitvariierende Volatilität (Heteroskedastizität) zeigen und daß die bedingte Verteilung der Renditen von der Normalverteilung abweichende Eigenschaften aufweisen. Insbesondere die Enden der Verteilung weisen eine gegenüber der Normalverteilung höhere Wahrscheinlichkeitsdichte auf ('fat–tails') und häufig ist die beobachtete Verteilung nicht symmetrisch. Trotzdem stellt die Normalverteilungs–Annahme mit konstanter Varianz weiterhin die Basis für den Mittelwert–*Varianz* Ansatz zur Portfolio–Optimierung dar.

In der vorliegenden Studie schlagen wir einen praktikablen Ansatz zur Portfolio–Selektion mit einem Mittelwert–*Skalen* Ansatz vor, der sowohl die bedingte Heteroskedastizität der Renditen, als auch die von der Normalverteilung abweichenden Eigenschaften zu berücksichtigen in der Lage ist. Wir verwenden dazu eine dem GARCH Modell ähnliche Dynamik der Risikofaktoren und verwenden stabile Verteilungen anstelle der Normalverteilung. Dabei gewährleistet das von uns vorgeschlagene Faktor–Modell sowohl gute analytische Eigenschaften und ist darüberhinaus auch einfach zu implementieren. Eine beispielhafte Anwendung des vorgeschlagenen Modells mit neun Aktien aus dem Deutschen Aktienindex veranschaulicht die bessere Anpassung des vorgeschlagenen Modells an die Daten und demonstriert die Anwendbarkeit zum Zwecke der Portfolio–Optimierung.

1 Introduction

The fundamental decision problem faced by investors is how to allocate their wealth over many financial assets. Standard portfolio theory assumes that investors and portfolio managers solve this allocation problem to achieve the highest expected portfolio return for a given expected portfolio risk (or achieve the lowest expected portfolio risk for a given expected portfolio return). The concepts of expected return and risk cannot be defined in isolation from beliefs of the investor on how asset returns will evolve. This issue is often resolved by making assumptions on the evolution of the multivariate distribution governing the asset returns.

The most commonly adopted assumption is that the return vectors are multivariate normal (cf. RiskMetrics Group, 1996). In the conventional mean–variance framework with multivariate normal returns, portfolio risk is measured in terms of the variance or standard deviation of the portfolio return. Clearly, the success of an investment strategy based on the normality assumption is closely related to the ability of the multivariate normal distribution to approximate the data generating process.

Empirical distributions of univariate financial asset returns are shown to exhibit fat tails. Furthermore, large changes in asset returns are often clustered implying state dependent time-varying moments. The issue of time-varying moments is successfully addressed using GARCH models (Engle, 1982; Bollerslev 1986). However, the normal distribution is thin–tailed deeming it an unsuitable candidate to approximate observed asset return distributions. Despite the overwhelming omnipresence of fat tails in empirical return distributions, the popularity of the normal assumption among practitioners persists. Apart from habit, the prevailing use of the normal model has commonly been justified by its analytical tractability. Closure under linear transformation—that is, weighted sums of normally distributed random variables are also normal—, together with the applicability of the central limit theorem make the normal assumption very attractive for theoretical and empirical portfolio analysis.

A natural generalization and extension of the normal framework allowing fat tails is presented by the family of stable Paretian distributions. In numerous empirical studies[1] non–Gaussian stable distributions have been found to be much more appropriate for modeling asset univariate returns, while preserving desirable properties of the normal. First, they are closed under linear transformation, implying that a linear combination of the elements of a stable random vector is again stable. Second, they have domains of attraction and are governed by suitable central limit theorems, implying that stable models possess a degree of robustness against misspecifications (cf. Rachev and Mittnik, 2000, p. 2). Third, an analogous GARCH-stable framework can be adopted to account for time-varying moments.

Despite of these attractive features, the stable model seems to play no role in practical portfolio analysis. Probably the single most important reason for this is the difficulty of estimating multivariate stable distributions from data. Even though the computational complexities—arising due to the lack of a general analytic expression for the stable density and distribution

[1] See, for example, Fama (1965a), Akgiray and Booth (1989), Mittnik and Rachev (1993), McCulloch (1997), and Rachev and Mittnik (2000) and references therein.

functions—are, nowadays, more or less unfounded for univariate stable distributions, given the considerable progress in the computability of stable models during recent years,[2] the estimation of multivariate stable distributions remains as a challenge.

Obstacles regarding the theory of stable portfolios analysis had been overcome much earlier with the development of stable mean–variance analogues. Fama (1965b) investigated the distribution of a portfolio of stably distributed assets governed by a single–index structure and subsequently (Fama, 1971) developed a stable version of the CAPM, which obtains efficient portfolios by minimizing the scale parameter of the portfolio–return distribution for a given mean return. Bawa and Lindenberg (1977) and Harlow and Rao (1989) show that, for stable Paretian portfolios, a capital–market equilibrium exists within a mean–lower partial moment framework, and that it is equivalent to that obtained through the mean–scale framework of Fama (1971). Elton, Gruber and Bawa (1979) provide simple portfolio selection rules under stable assumptions.

All these studies on stable portfolio analysis are of theoretical nature. They do not address the problem of *how* to estimate the joint distribution of the individual asset returns. More recent studies by Belkacem, Vèhel and Walter (1995, 2000) and Gamrowski and Rachev (1999) reformulate the approach of Fama in an estimable framework and constitute a first attempt towards empirical analysis. However, the sole empirical focus of these studies is the estimation of the betas, i.e., the stocks' association with an underlying factor, using covariation–based methods, which represent a generalization of the linear–regression framework. They do not address the question of how to estimate the joint distribution of the stocks and the factor, nor do they consider the construction of optimal portfolios from these stocks. As a consequence, they cannot provide any comparisons of portfolio–selection outcomes under Gaussian and non–Gaussian stable assumptions.

The contributions of this paper are three–fold. First, we adopt the computationally feasible methodology, introduced in Doganoglu and Mittnik (2004), to estimate the parameters of a class of multivariate stable distributions where some moments may vary over time. We extend this methodology here to a factor–GARCH model for multivariate asset returns, thus, taking care of time varying moments. For this purpose, we exploit certain properties of multivariate stable distributions which are governed by multi–factor structures. Specifically, we use the fact that the spectral measure[3] of a factor model has a particular form, which allows us to estimate the distributional parameters along with the factor–association parameters. Second, we compare the stable factor–GARCH model with a normal factor–GARCH to select optimal portfolios using the mean–scale framework introduced by Fama (1965b, 1971).

The remainder of the paper is as follows. In Section 2 we present needful results on multivariate stable distributions. The stable factor–GARCH model is introduced in Section 3. While Section 4 deals with the estimation of the stable factor–GARCH model, Section 5 presents the portfolio optimization problem in the mean–scale framework. An empirical application of the

[2] See, for example, Doganoglu and Mittnik (1998), McCulloch (1998), Mittnik, Doganoglu and Chenyao (1999), Mittnik, Rachev, Doganoglu, and Chenyao (1999), and Nolan (1999).

[3] The spectral measure defines the dependence structure of multivariate stable vectors (see Section 2 below).

model in done in Section 6, while Section 7 concludes.

2 Multivariate Stable Random Vectors

Multivariate stable Paretian distributions—as their univariate counterparts—are commonly defined by their characteristic functions as they lacking general closed–form expressions for both the density and distribution function. The logarithm of the joint characteristic function of a stable random vector $Y = (Y_1, \ldots, Y_q)'$ is given by

$$\ln \Phi_\alpha(\theta) = \begin{cases} i(\theta'\mu) - \int_{\mathcal{S}_q} |\theta' s| \left(1 + i\, \frac{2}{\pi} \text{sign}(\theta' s)\right) \ln |\theta' s| \Gamma(ds), & \text{if } \alpha = 1, \\ i(\theta'\mu) - \int_{\mathcal{S}_q} |\theta' s|^\alpha \left(1 - i\, \text{sign}(\theta' s) \tan \frac{\pi\alpha}{2}\right) \Gamma(ds), & \text{if } \alpha \neq 1, \end{cases} \tag{1}$$

where $\alpha \in (0, 2]$ denotes the characteristic exponent (or shape parameter) of the distribution; Γ is a finite measure on the unit sphere, $\mathcal{S}_q$, in $\mathbb{R}^q$; and μ is the location vector in $\mathbb{R}^q$.[4]

In the case of univariate stable Paretian distributions, i.e., $q = 1$ and Y being a scalar, the sphere, $\mathcal{S}_1$, consists of the two points $\{-1, 1\}$. Denoting the probability masses at these points by $\Gamma(-1)$ and $\Gamma(1)$, expression (1) reduces to

$$\ln \Phi_\alpha(\theta) = i\ \theta\mu - |\theta|^\alpha \left[\Gamma(1) + \Gamma(-1) - i\ \text{sign}(\theta)\left[\Gamma(1) - \Gamma(-1)\right] \tan \frac{\pi\alpha}{2}\right]$$

and coincides with the characteristic function of a univariate stable variable, in which case we write $Y \sim S_\alpha(\sigma, \beta, \mu)$, where

$$\sigma = [\Gamma(1) + \Gamma(-1)]^{1/\alpha} \quad \text{and} \quad \beta = \frac{\Gamma(1) - \Gamma(-1)}{\Gamma(1) + \Gamma(-1)}$$

with σ, β, and μ representing the scale, skewness and location parameters, respectively. It follows from the definition of β that if the spectral measure Γ is symmetric, then the scalar Y is symmetrically distributed, i.e, $\beta = 0$ for $\Gamma(1) = \Gamma(-1)$.

In the following, we use three properties of multivariate stable distributions to develop an estimation method for a stable factor–GARCH model of asset returns, and subsequently, a method to select "optimal" portfolios of these assets. These properties, which we reproduce in the appendix for convenience, are adopted from Samorodnitsky and Taqqu (1994). Property A establishes that a stable random vector with independent elements has a discrete spectral measure over the unit sphere $\mathcal{S}_q$. Given this property, it follows that although they may be dependent, a linear transformation of a stable random vector with independent elements also has a discrete spectral measure. The spectral measure of this linear transformation is easily calculated, and established in Property B. Finally, Property C establishes parameters of the univariate stable distribution by taking a linear combination of the elements of a stable random vector with a given spectral measure. Naturally, this property proves useful in computing the portfolio profit–loss distributions.

[4] The subsequent discussion of the properties of the multivariate stable Paretian distributions closely follows that of Samorodnitsky and Taqqu (1994).

3 The Stable Factor-GARCH Model

A factor model establishes the dependence between the returns of different assets through a set of common market factor.[5] Each return series evolves as a linear combination of the factors and an additive idiosyncratic noise process. Furthermore, we assume that the volatility of the factor is state dependent. In particular, we assume that the scale parameter of the marginal distribution of the factor follows a simple GARCH(1,1)–type process.

Formally, if there are N assets with returns $R_i,\ i = 1, \ldots, N$, and factor returns F, then, in any given period, the return of asset i is given by

$$R_{it} = \mu_{R_i} + b_i(F_t - \mu_F) + \varepsilon_{it}, \quad i = 1, \ldots, N, \tag{2}$$

where ε_{it} denotes the idiosyncratic disturbance for asset i; μ_F is the mean of the factor, F_t; and b_i reflects the systematic influence of the factor on asset i. In matrix notation, the N equations in (2) can be written as

$$R_t = \mu_R + b(F_t - \mu_F) + \varepsilon_t,$$

where $R_t = (R_{1t}, \ldots, R_{Nt})'$, $\mu_R = (\mu_{R_1}, \ldots, \mu_{R_N})'$, $\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Nt})'$, and $b = (b_1, \ldots, b_N)'$.

The factor, unlike individual assets, follows a state dependent process. We assume that the volatility, which is proxied by the scale parameter, of the factor is influenced by its most recent volatility and the magnitude of its most recent return. That is, we adopt the GARCH(1,1)–type structure of Panorkska, Mittnik and Rachev (1995) for modeling the factor dynamics. Let $\sigma_{F,t}$ denote the scale parameter of the factor at time t. Then, the factor return evolves according to

$$F_t = \mu_F + \varphi_t, \qquad \varphi_t = \sigma_{F,t}\phi_t, \tag{3}$$

where

$$\sigma_{F,t} = c_0 + c_1|F_{t-1} - \mu_F| + c_2\sigma_{F,t-1} \tag{4}$$

and ϕ_t is a standardized random disturbance with scale parameter equal to unity.

Our first extension to the standard factor model is the adoption of GARCH(1,1)–type dynamics for the factor returns. Under the usual normality assumption, the asset and factor disturbances would follow independent univariate normal distributions, that is, with the present notation, $\varepsilon_{it}, \phi_t \sim S_2(\cdot,\cdot,\cdot)$. In our analysis, we relax the normality assumption by allowing the asset and factor disturbances follow independent heavy-tailed stable distributions, i.e., $\varepsilon_{it}, \phi_t \sim S_\alpha(\cdot,\cdot,\cdot)$, with $0 < \alpha \leq 2$. Thus, in a symmetric setting, we generalize the normal factor-GARCH model by simply relaxing a single parameter. That is, instead of imposing the Gaussian restriction $\alpha = 2$, we allow $\alpha \in (0, 2]$. A second parameter is added, namely $\beta \in [-1, 1]$, if we allow for asymmetry.

Let $\epsilon_{it} = \varepsilon_{it}/\sigma_{\varepsilon_i}$, $\epsilon_t = (\epsilon_{1t}, \ldots, \epsilon_{Nt})'$ and $\phi_t = \varphi_t/\sigma_{F,t}$. We make the following assumption on the process generating $(\epsilon_t', \phi_t)'$:

[5] The model and the methodology we develop below are easily extended to accommodate multiple factors. Since we use a single factor in our empirical application, and exposition is simpler, we present a single factor model.

Assumption 1 *At each instant t, the random variables $\{\epsilon_{1t}, \ldots, \epsilon_{Nt}, \phi_t\}$ are independent and follow (univariate) standard-stable distributions*[6] *with a common shape parameter $0 < \alpha \leq 2$. That is, $\epsilon_{it} \sim S_\alpha(1, \beta_{\varepsilon_i}, 0)$ $(i = 1, \ldots, N)$, and $\phi_t \sim S_\alpha(1, \beta_F, 0)$,*

Notice that combining returns from equation (2) and the factor from equation (3) and rearranging yields

$$\begin{bmatrix} R_t - \mu_R \\ F_t - \mu_F \end{bmatrix} = \begin{bmatrix} \Sigma_\varepsilon & \sigma_{F,t} b \\ \mathbf{0}' & \sigma_{F,t} \end{bmatrix} \begin{bmatrix} \epsilon_t \\ \phi_t \end{bmatrix}. \tag{5}$$

where Σ_ε is a diagonal matrix whose elements are equal to the scale parameters of the idiosyncratic shocks to each asset return, i.e.

$$\Sigma_\varepsilon = \begin{bmatrix} \sigma_{\varepsilon_1} & 0 & \ldots & 0 \\ 0 & \sigma_{\varepsilon_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \sigma_{\varepsilon_N} \end{bmatrix}.$$

Given Assumption (1), the model in (5) implies that the excess return on an individual asset, $R_{it} - \mu_{R_i}$, is a linear combination of 2 independent standard stable random variables, namely the the factor disturbance, ϕ_t, and the idiosyncratic disturbance, ϵ_{it}. However, note that the linear combination changes at each time instant t following changes to $\sigma_{F,t}$.

4 Estimation

We will make use of Properties A and B to estimate the stable factor–GARCH model. In order to facilitate discussion, let

$$A_t = \begin{bmatrix} \Sigma_\varepsilon & \sigma_{F,t} b \\ \mathbf{0}' & \sigma_{F,t} \end{bmatrix},$$

$$Y_t = \begin{bmatrix} R_t - \mu_R \\ F_t - \mu_F \end{bmatrix}$$

and

$$X_t = \begin{bmatrix} \epsilon_t \\ \phi_t \end{bmatrix}.$$

Then (5) can be re–rewritten as $Y_t = A_t X_t$ with X_t having independent elements. We can now state a result characterizing the spectral measure and the location vector for $(R_t', F_t)'$.

Proposition 1 *Let $(R_t', F_t)'$ be a vector of asset and factor returns generated by the stable factor–GARCH model (5). Moreover, let $a_{\cdot k}^t$ denote the kth column of the matrix A_t. Define $\|a_{\cdot k}^t\| = (\sum_{i=1}^{N+1} a_{ik}^t)^{1/2}$, $\iota_k^t = \frac{a_{\cdot k}^t}{\|a_{\cdot k}^t\|}$ and let $\delta(\iota_k^t)$ denote Dirac–function of unit size at a point*

[6] A standard stable random variable has a location parameter that is equal to zero and a scale parameter that is unity.

with coordinates given by ι_k. If Assumption 1 holds, then $(R_t', F_t)'$ follows a multivariate stable law whose spectral measure and location vector at time t is given by

$$\Gamma_t = \frac{1}{2}\sum_{k=1}^{N}\sigma_{\varepsilon_k}^{\alpha}\left[(1+\beta_{\varepsilon_k})\delta(\iota_k^t)+(1-\beta_{\varepsilon_k})\delta(-\iota_k^t)\right] \\ +\frac{1}{2}\sigma_{F,t}^{\alpha}(1+\sum_{j=1}^{N}b_j^2)^{\alpha/2}\left[(1+\beta_F)\,\delta\big(\iota_{N+1}^t\big)\;+(1-\beta_F)\;\delta\left(-\iota_{N+1}^t\right)\right], \tag{6}$$

and

$$\mu = \begin{bmatrix} \mu_R \\ \mu_F \end{bmatrix}, \tag{7}$$

respectively.

Proof. Since $(\epsilon_t', \phi_t)'$ is an $(N+1)\times 1$ vector of independent standard stable random variables, Property A implied that its spectral measure has masses distributed on $2(N+1)$ points on the $N+1$ dimensional unit sphere. It is easy to verify that $(R_t' - \mu_R', F_t - \mu_F)'$ has a spectral density given by (6) and a zero location vector by applying Property B to (5). Thus, the spectral measure and the location vector of $(R_t', F_t)'$ are given by (6) and (7), respectively.

■

Since A_t is square matrix, it is possible to characterize the joint probability density function of Y_t in terms of the density of X_t. Due to the independence of its elements, vector X_t has the joint density $f_X(x_t) = \prod_{i=1}^{d} f_{X_i}(x_{it})$; and, if A_t is nonsingular—which is always the case for non–degenerate distributions—then $X_t = A_t^{-1}Y_t$, and

$$f_Y(y_t) = f_X(A_t^{-1}y_t)|\det(A_t^{-1})|, \tag{8}$$

where $\det(\cdot)$ denotes the determinant of a matrix. Hence, the evaluation of the joint multivariate density of Y_t only involves the computation of univariate stable densities, $f_{X_i}(\cdot)$. The inverse of A_t is given by

$$A_t^{-1} = \begin{bmatrix} \Sigma_\varepsilon^{-1} & -\Sigma_\varepsilon^{-1}b \\ \mathbf{0}' & \sigma_{F,t}^{-1} \end{bmatrix} \tag{9}$$

and

$$\det(A^{-1}) = \left(\left(\prod_{i=1}^{N}\sigma_{\varepsilon_i}\right)\sigma_{F,t}\right)^{-1}.$$

Assumption 1 implies that the vector $(\epsilon_t', \phi_t)'$ has independent components with standard–stable densities $f_{\epsilon_{it}}(\cdot;\alpha,\beta_{\varepsilon_i})$ and $f_{\phi_t}(\cdot;\alpha,\beta_F)$. Given the return and factor realizations $r_t = (r_{1t},\ldots,r_{Nt})'$ and f_t, for period t, the value of the joint density for the t-th observation, $(r_t', f_t)'$, is given by

$$f_{R_t,F_t}(r_t, f_t;\theta) = f_\phi\left(\frac{f_t-\mu_F}{\sigma_{F,t}};\alpha,\beta_F\right)\prod_{i=1}^{N}\frac{1}{\sigma_{\varepsilon_i}}f_{\epsilon_{it}}\left(\frac{r_{it}-b_i(f_t-\mu_F)-\mu_R}{\sigma_{\varepsilon_i}};\alpha,\beta_{\varepsilon_i}\right)$$

where $\theta = (\alpha, \beta'_\varepsilon, \beta_F, \mu'_R, \mu_F, b', \sigma'_\varepsilon, c_0, c_1, c_2)'$, with $\beta_\varepsilon = (\beta_{\varepsilon 1}, \ldots, \beta_{\varepsilon N})'$, collects all $3N + 6$ parameters of the stable factor–GARCH model.

Given T observations and defining the $N \times T$ matrix $\mathbf{r} = (r_1, \ldots, r_T)$ and the $1 \times T$ vector $\mathbf{f} = (f_1, \ldots, f_T)$, the joint density of $\mathbf{r}$ and $\mathbf{f}$ is given by

$$f(\mathbf{r}, \mathbf{f}; \theta) = \prod_{t=1}^{T} f_{R_t, F_t}(r_t, f_t; \theta).$$

The maximum likelihood (ML) estimator of parameter vector θ is obtained by maximizing the log–likelihood function

$$\mathcal{L}(\theta; \mathbf{r}, \mathbf{f}) = \sum_{t=1}^{T} \log f(\mathbf{r}, \mathbf{f}; \theta) \tag{10}$$

with respect to θ.

Even though feasible, maximization of the likelihood is not a trivial task. The univariate stable Paretian densities lack a closed form expression. The most straightforward way to compute the likelihood of an observation given parameters is to invert the Fourier integral relating characteristic function and the probability density function. Mittnik, Doganoglu and Chenyao (1999) have demonstrated that this can be accurately and efficiently accomplished by using fast Fourier transforms. In this paper, we use the polynomial approximation developed in Doganoglu and Mittnik (1998) based on the accurate fast Fourier transform computations of Mittnik, Doganoglu and Chenyao (1999). For a sample size of 1000 and 10 assets, the maximization of the log–likelihood function (10) is completed in about five minutes on an Intel Pentium 4 computer with a 2.4GHz cpu using MATLAB as the computing platform. Thus, ML estimation of multivariate stable distributions can be performed in practical situations without hesitation. Clearly, with dedicated and optimized routines and faster computers, the computational cost can be reduced even further.

In our empirical application we use nine assets and one factor. After restricting all skewness parameters to zero, we have a total of 23 parameters which we estimate via maximization of (10). However, in many practical applications, the large number of parameters to be estimated, namely, $3N + 6$, may render the ML estimator infeasible, due to the computational complexity that arises. An alternative and practically more feasible estimation strategy consists of a combination of ordinary least squares (OLS) estimation of the b_i coefficients and ML estimation of the distributional parameters.[7] The consistency of the OLS estimates of b_i in this setting where regressors are also α-stable distributed is established in Kurz-Kim, Rachev and Samorodnitsky (2004). Also, for $1 < \alpha \leq 2$, which appears to hold for financial applications, the mean vectors μ_R and μ_F are consistently estimated by the sample means $\bar{r}_t$ and $\bar{f}_t$, respectively. Then, the ML estimator can be used to estimate the remaining $2N + 3$ distributional parameters, that is, $\sigma_{\varepsilon_i}, \sigma_F, \beta_{\varepsilon_i}, \beta_F$, and α.

[7] Blattberg and Sargent (1973) show that the coefficients of a regression model with stable disturbances can be consistently estimated via OLS. However, our setup differs from their's in that we have a stochastic regressors (see Doganoglu and Mittnik, 2002).

5 Portfolio Selection and Risk Assesment

Because the standard mean–*variance* approach is not applicable to non–Gaussian stable portfolios, Fama (1971) and Bawa and Lindenberg (1977) develop a mean–*scale* analogue. For the expected portfolio return to be finite, they assume that $\alpha \in (1, 2]$. This assumption is justified on empirical grounds and will be adopted in the following.[8]

In solving the portfolio selection problem we make use of Property C, which allows us to derive the parameters of the distribution of a linear combination of stable random variables. Letting $\mu_{p,t}$ and $\sigma_{p,t}$, respectively, denote the expected mean and the scale parameter of the portfolio return at time t, the set of efficient portfolios is derived by finding the weight vector $w_t = (w_{1,t}, \dots, w_{N,t})'$ which solves the optimization problem:

$$\max_{w_t} \mu_{p,t} = w_t' \mu \tag{11}$$

subject to

$$\sigma_{p,t} \leq \sigma_p^*,$$

$$w_t' \mathbf{1}_N = 1,$$

and, if short–selling is not allowed,

$$w_i \geq 0, \qquad i = 1, \dots, N,$$

where σ_p^* is the risk limit; and $\mathbf{1}_N$ denotes an $N \times 1$ vector of ones. For $\alpha < 2$, relationship (14) in Appendix A is used to compute the portfolio scale, $\sigma_{p,t}$. Also in the normal case, when $\alpha = 2$, we use the stable Paretian representation of the normal distribution and use Property B to compute the portfolio scale. Alternatively, one can use the portfolio standard deviation $\sigma_{p,t} = (w_t' \Sigma_t w_t)^{1/2}$, where Σ_t denotes the conditional covariance matrix implied by the normal factor–GARCH model at time t instead of (14).

An important point to highlight here is that the multivariate distribution of the asset and factor returns change in time due to state dependence in the evolution of the factor. Thus, an investor would incorporate information up to time t to construct an estimate of the underlying data generation process. Given parameter estimates, one can construct first a forecast of $\sigma_{F,t+1}$, and then using the estimated model (5) and the result presented in Proposition 1 to form a forecast of the distribution of asset and factor returns at time $t + 1$. One can then solve the portfolio optimization problem in (11) using this forecast of the distribution to obtain the optimal portfolio weights.

Given that investors base their decisions solely on expected risk–return considerations, the decision problem can be placed in any suitable expected return–risk space. In the mean–variance framework, the minimum–variance set reflects the minimum–risk portfolios—with risk being measures in terms of the portfolio variance or standard deviation—that are associated with feasible expected portfolio returns. To handle the stable case, Fama (1971) and Bawa and

[8] If $\alpha \leq 1$, the analysis would have to be placed in a *location*–scale framework.

Lindenberg (1977) adopt the mean–scale framework. Unfortunately, scale parameters of stable distributions with different characteristic exponents cannot be meaningfully compared. Therefore, the implications of different stable assumptions (here, $\alpha = 2$ versus $\alpha < 2$) are not easily accessible in this framework.

In order to evaluate the impact of the distributional assumptions on the portfolio selection problem, we first find optimal portfolio weights by solving (11), but then use the portfolio's Value–at–Risk (VaR) as a risk measure in comparing the normal model with the stable. In this way, we can examine the consequences of alternative distributional assumptions in a common risk–return setting. For a given target probability, λ, there is a strictly affine one–to–one correspondence between the scale parameter, σ_p, and VaR_λ–level (i.e., the negative $100\times\lambda\%$–quantile) of the stable profit–and–loss distribution.[9]

6 An Application to DAX Stocks

6.1 The Data

We consider portfolios constructed from a set of nine stocks belonging to the German DAX index. The sample consists of $T = 2273$ daily observations of (dividend–corrected) returns on the 9 stocks and the DAX, the market index, covering the period from January 2, 1996 to December 30, 2004.

The returns of our sample's assets are shown in Figure 1. The names of the stocks and various summary statistics, are listed in Table 1. For all stocks and the DAX index the sample kurtosis exceeds 3—the value compatible with the normal assumption—significantly at the 99% level. For four out of the nine stocks, as well as for the DAX index, we find significant asymmetries of the returns at the 95% level. The need of time varying moments is shown by the Ljung–Box test statistics for the absolute and squared returns who are significant on the 99% level for all assets.

6.2 Estimation results

Assuming that the dependence of the nine stock–return series can be captured by a single–factor model, with the DAX being the underlying factor, we estimate the model parameters under both the normal and (non–Gaussian) stable assumption. For the latter we estimate a symmetric ($\beta = 0$ for all the asset return and factor return distributions) version for simplicity. Using the entire sample both models, the normal model and the stable one, are estimated by maximizing the log-likelihood given by (10).

The estimated tail index for the symmetric stable model is $\widehat{\alpha} = 1.7312$. The estimated time varying scale for the factor is plotted in Figure 2. The upper part refers to the normal model, while the lower part refers to the stable case. Note that we present the normal results

[9] By equating the VaR_λ–level with the (negative) $100 \times \lambda\%$–quantile of the return distribution, the analysis assumes an initial investment of unity and, thus, is independent of the size of the initial investment.

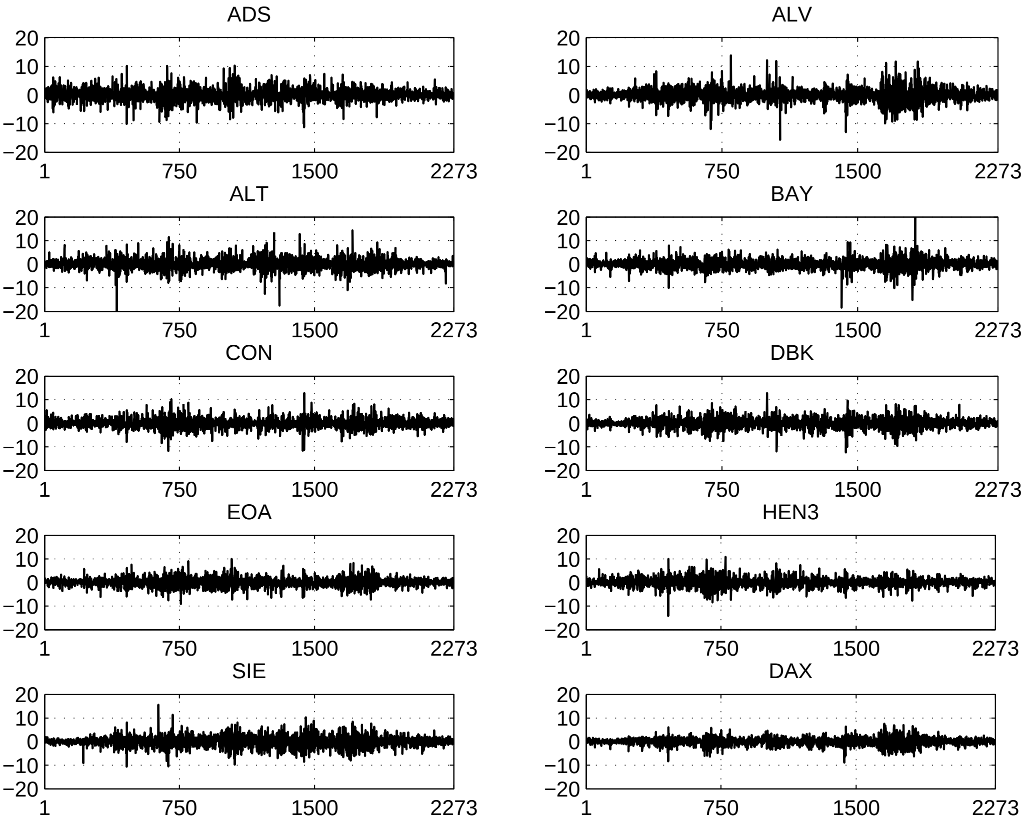

Figure 1: Asset returns

as a stable distribution with $\alpha = 2$. The relationship to the standard normal representation is $S_2(\sigma, \cdot, \cdot) \equiv N(\cdot, 2\sigma^2)$.

Calculating the likelihood ratio test statistic

$$LR_{N,SS} = -2(\mathcal{L}_N - \mathcal{L}_{SS}),$$

where $\mathcal{L}_N$ ($\mathcal{L}_{SS}$) is the log-likelihood value of the normal (symmetric stable), we see that the normal model is outperformed by the symmetric stable model at any reasonable level by a likelihood ratio test statistic of $LR_{N,SS} = 2,535.1$.

6.3 Portfolio Selection Results

We use the estimated parameter values to solve the optimal portfolio problem for the the minimum risk (or minimum scale) portfolio (MRP) stated in (11), to calculate optimal portfolio weights, w_t, and obtain the corresponding portfolio mean and scale, $\mu_{p,t}$ and $\sigma_{p,t}$. Observe that these value are in–sample value, i.e., they are based on parameter estimates for the sample $t = 1, \ldots, T$.

To overcome the problem of initial values we dismiss the first 273 observations and calculate the optimal portfolio weights for $t = 274, \ldots, 2273$, i.e., for the last 2000 observations of the sample. The resulting portfolio weights for the different assets are given in Figure 3 for both distributional assumptions. We see that there are small differences between the resulting portfolio

Table 1: Summary Statistics

Name	Mean	Variance	Skewness	Kurtosis	LB1(20)	LB2(20)
Adidas	0.0494	4.9124	-0.0166	5.4481***	587.14***	449.14***
Allianz	-0.0111	5.8080	-0.0113	7.1145***	2025.96***	932.49***
Altana	0.0601	5.8955	-0.4215***	12.9306***	446.61***	57.24***
Bayer	0.0111	5.0559	0.8694***	25.3716***	670.18***	133.30***
Continental	0.0668	4.3562	0.0768	6.2435***	592.52***	413.38***
Deutsche Bank	0.0290	4.9642	-0.1034**	5.7079***	1641.64***	965.06***
EON	0.0338	3.6195	0.1090**	5.0883***	1310.61***	669.66***
Henkel	0.0370	3.7904	0.0447	6.4847***	1228.06***	774.42***
Siemens	0.0372	5.7672	0.0815	5.1445***	1956.08***	669.43***
DAX	0.0280	2.8057	-0.2214***	5.3899***	2823.56***	1926.52***

LB1 (LB2) refers to the Ljung–Box test statistic for the absolute (squared) returns

* (**,***) indicates significance at the 90% (95%,99%) level

weights between the normal and the stable model. Even though the differences of the resulting portfolios are small, the different distributional assumptions can effect risk assessment for these portfolios considerably, as we will see below.

Value–at–Risk Coverage

Using $\mu_{p,t}$ and $\sigma_{p,t}$, we are able to calculate the Value–at–Risk of the MRP for different risk levels λ. In our case, the VaR is defined as the λ-quantile VaR_λ of the distribution of $r_{p,t}$, i.e., either $S_{\widehat{\alpha}}(r_{p,t}; \widehat{\mu}_{p,t}, \widehat{\sigma}_{p,t})$ for the stable or $S_2(r_{p,t}; \widehat{\mu}_{p,t}, \widehat{\sigma}_{p,t})$ for the normal model.

Using the actual portfolio return, r_t, we define the hit (or violation) sequence of VaR violations by

$$I_t = \begin{cases} 1, & \text{if } r_t < -\widehat{\text{VaR}}_{\lambda,t} \\ 0, & \text{if } r_t \geq -\widehat{\text{VaR}}_{\lambda,t} \end{cases}$$

and the empirical shortfall probability as

$$\hat{\lambda} = \frac{1}{T-273} \sum_{t=274}^{T} I_t$$

with $\widehat{\text{VaR}}_{\lambda,t}$ the VaR estimate for time t. Ideally, we have

$$I_t \overset{i.i.d.}{\sim} \text{Bernoulli}(\lambda).$$

There are three tests (see, Christoffersen (2003)) with which one can test this hypothesis:

1. Unconditional coverage test: Under the null we have $f(I_t; \lambda) = (1-\lambda)^{1-I_t}\lambda^{I_t}$. The likelihood under the null of i.i.d. Bernoulli is

$$L(\lambda) = \prod_{t=274}^{T} (1-\lambda)^{1-I_t}\lambda^{I_t} = (1-\lambda)^{T_0}\lambda^{T_1}$$

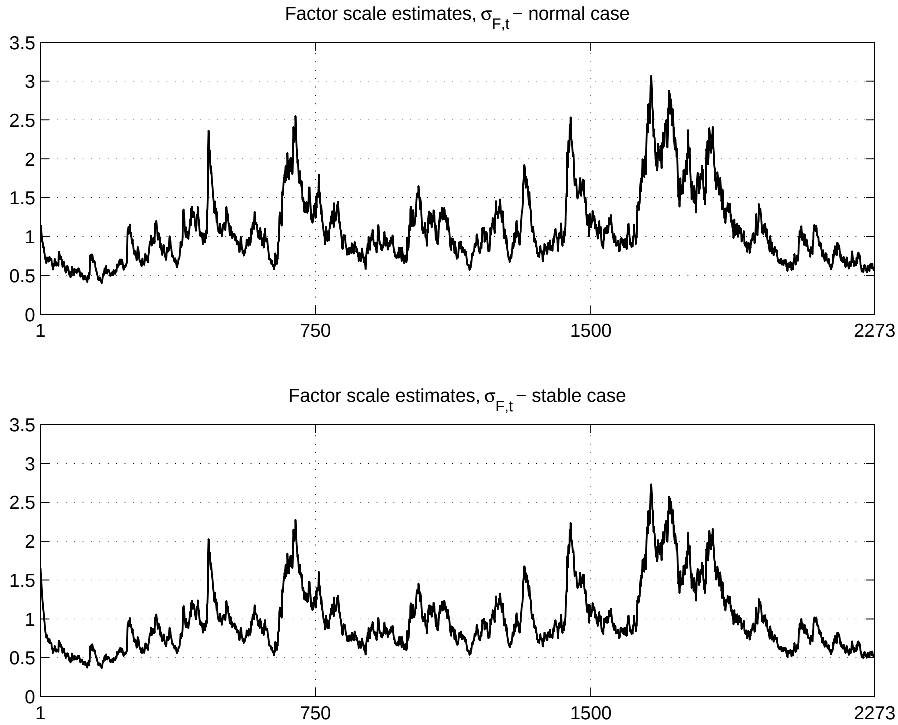

Figure 2: Estimated factor scales $\widehat{\sigma}_{F,t}$

while the observed likelihood value is given by

$$L(\hat{\lambda}) = \prod_{t=274}^{T} (1-\hat{\lambda})^{1-I_t}\widehat{\lambda}^{I_t} = (1-\hat{\lambda})^{T_0}\hat{\lambda}^{T_1},$$

where T_0 and T_1 are the number of zeros and ones observed in the hit sequence. The likelihood ratio test statistic and the corresponding p-value are

$$LR_{uc} = -2\ln[L(\lambda)/L(\hat{\lambda})] \sim \chi_1^2 \qquad \text{and} \qquad \mathrm{P}_{uc} = 1 - F_{\chi_1^2}(LR_{uc}).$$

P_{uc} is the the probability of getting a sample that conforms less to the null hypothesis than the sample observed. If P_{uc} is below the specified significance level then we reject the null.

2. Independence test: Let $T_{ij},\ (i,j=0,1)$ the number of observed pairs in the hit sequence where j follows i, and define the probabilities $\pi_{ij} = \mathrm{Prop}(I_t = i \text{ and } I_{t+1} = j),\ i,j = 0,1$. Their estimates are given by

$$\hat{\lambda}_{01} = \frac{T_{01}}{T_{00}+T_{01}} \qquad \hat{\lambda}_{11} = \frac{T_{11}}{T_{10}+T_{11}}$$
$$\hat{\lambda}_{00} = 1-\hat{\lambda}_{01}, \qquad \hat{\lambda}_{10} = 1-\hat{\lambda}_{11}$$

Under the null the likelihood is given by

$$L(\hat{\lambda}) = \prod_{t=274}^{T} (1-\hat{\lambda})^{1-I_t}\hat{\lambda}^{I_t} = (1-\hat{\lambda})^{T_0}\hat{\lambda}^{T_1};$$

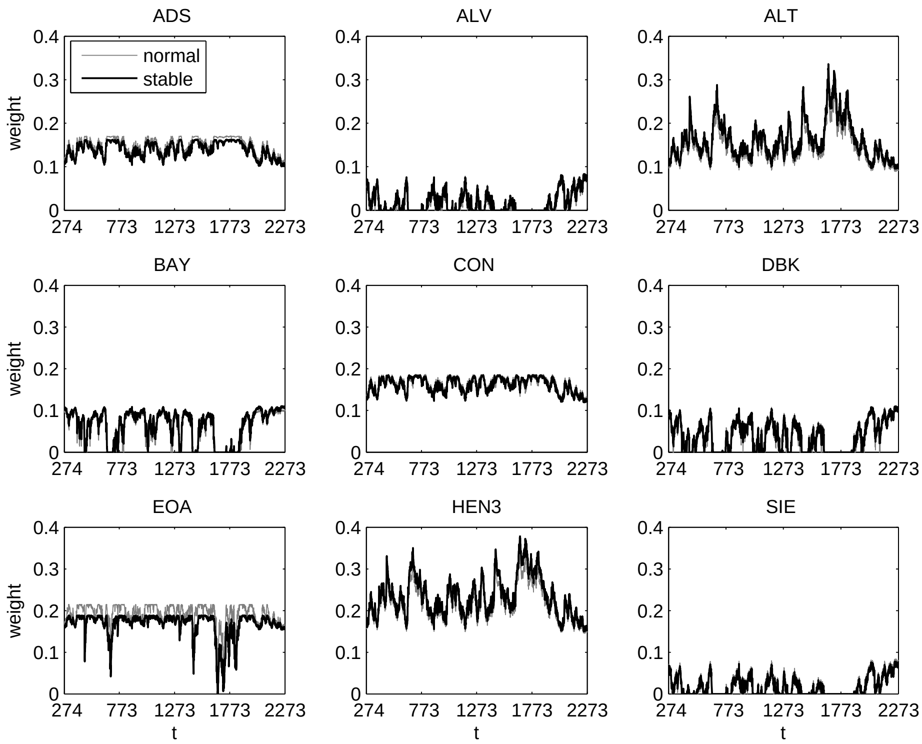

Figure 3: Portfolio weights for the MRP

the observed likelihood is

$$L(\hat{\Lambda}) = (1-\hat{\lambda}_{01})^{T_{00}}\hat{\lambda}_{01}^{T_{01}}(1-\hat{\lambda}_{11})^{T_{10}}\hat{\lambda}_{11}^{T_{11}}.$$

The likelihood ratio test statistic and the corresponding p-value are given by

$$LR_{ind} = -2\ln[L(\hat{\lambda})/L(\hat{\Lambda})] \sim \chi^2_1 \qquad \text{and} \qquad \mathrm{P}_{ind} = 1 - F_{\chi^2_1}(LR_{ind}).$$

3. Conditional coverage test: By combining the unconditional and the independence test statistics we can test for conditional coverage. The resulting likelihood ratio test statistic and the p-value are given by

$$LR_{cc} = -2\ln[L(\lambda)/L(\hat{\Lambda})] = LR_{uc} + LR_{ind} \sim \chi^2_2 \qquad \text{and} \qquad \mathrm{P}_{cc} = 1 - F_{\chi^2_2}(LR_{cc}).$$

We examine the VaR predictions for three shortfall probabilities, namely $\lambda = 0.01, 0.05, 0.10$, which are most commonly found in practice. The shortfall probabilities, $\widehat{\lambda}$ and the p-values for the unconditional coverage, the independence and the conditional coverage test statistics are reported in Table 2.

For the smaller risk levels, $\lambda = 0.01$, and $\lambda = 0.05$, the normal model underestimates the unconditional coverage significantly. Only for the 10% risk level the unconditional coverage of the VaR estimates are accurate for the normal model.

For the stable model the unconditional coverage on the 1% level is significantly overestimated, i.e., we have a conservative VaR estimate, while for the 5% and 10% risk level the VaR estimates are insignificantly different from the nominal level.

Table 2: VaR coverage for the MRP: Unconditional Coverage, Independence, Conditional Coverage

		$\lambda = 0.01$	$\lambda = 0.05$	$\lambda = 0.10$
normal	$\hat{\lambda}$	0.0205	0.0620	0.1015
	P_{uc}	0.0000	0.0174	0.8234
	P_{ind}	0.0580	0.0556	0.0438
	P_{cc}	0.0000	0.0095	0.1278
stable	$\hat{\lambda}$	0.0055	0.0555	0.1015
	P_{uc}	0.0270	0.2671	0.8234
	P_{ind}	0.7156	0.1204	0.0039
	P_{cc}	0.0812	0.1617	0.0153

$\hat{\lambda}$: empirical downfall probability. If the model underestimates (overestimates) the risk, the empirical downfall is higher (smaller) than the VaR level, λ.
P_{uc}: p-value for the unconditional coverage test statistic.
P_{ind}: p-value for the independent test statistic.
P_{cc}: p-value for the conditional coverage test statistic.
p-values should be higher than the suggested significance level.

For both distributional assumptions we find significant dependency in the violations for the 10% VaR level. For the normal model we also find significant dependency on the 10% significance level for the 1% and 5% VaR level, while for the stable model there is no dependency indicated.

Turning to the conditional coverage, we find the normal model unable to predict the VaR correctly for the lower VaR levels. Only for the 10% VaR level the conditional coverage test statistic is below the 10% significance level. For the stable model the conditional coverage on the 1% VaR level is slightly significant on the 10% significance level, due to its conservative unconditional coverage. The conditional coverage on the 5% VaR level is insignificantly different from the nominal, while on the 10% VaR level the accurate unconditional coverage is vitiate by the dependency of the violations.

"Cost" of Portfolio Optimization

To assess the impact of lowering the Gaussian assumption on the portfolio weights, we calculate the overall variability of portfolio weight changes by

$$V = \sum_{t=2}^{T} \Delta w_t' \Delta w_t \,, \tag{12}$$

where $\Delta w_t = w_t - w_{t-1}$. The values are $V_N = 0.9660$ and $V_{SS} = 0.7197$ for the normal and the stable model, respectively, i.e., there is less fluctuation in the weights of optimal portfolio of the stable model.

For translating the differences in portfolio weights into transaction cost equivalents, we cal-

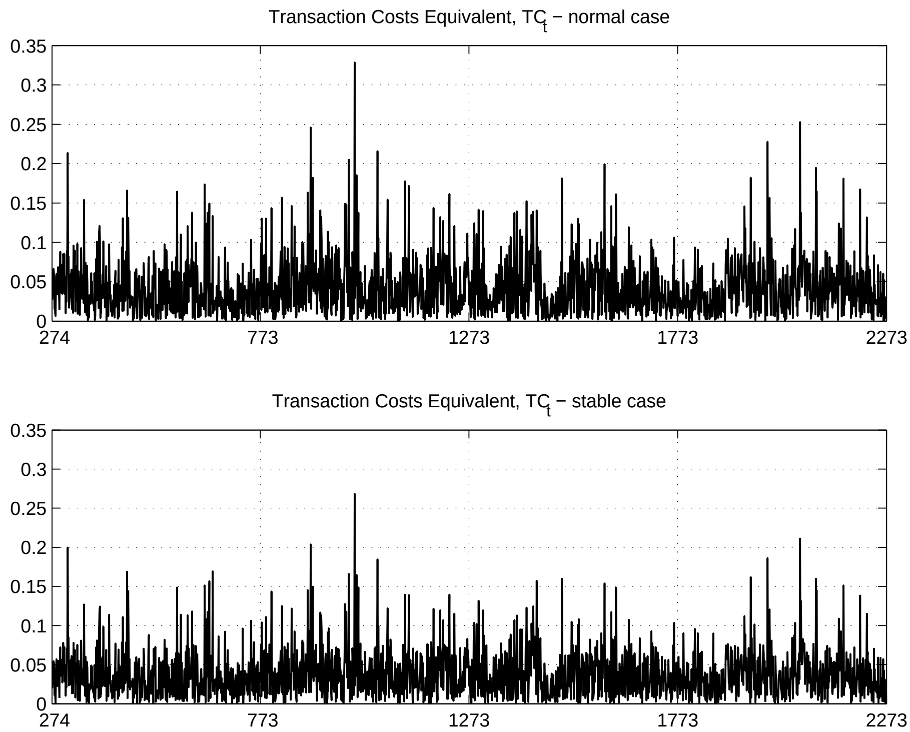

Figure 4: Transaction Cost Equivalents

culate

$$TC_t = \sum_{i=1}^{N} |\Delta w_t|$$

and the cumulative transaction cost equivalent as

$$CTC_t = \sum_{\tau=1}^{t} TC_\tau \, .$$

Figure 4 plots the resulting transaction cost equivalents. The upper figure refers to the normal case, while the lower figure is for the stable model. Looking at the cumulative transaction cost equivalents in Figure 5 it is obvious that the transaction costs for the stable model are lower than for the normal model.

7 Conclusion

It is well established that returns on financial assets are generally heavy tailed. In practice, however, portfolio–selection strategies commonly assume joint normality for the underlying assets in order to avoid difficulties in estimation and optimization. In this paper, we have proposed a practical approach to estimating the parameters of portfolios governed by a conditional multivariate non–Gaussian stable distribution characterized by a GARCH–factor structure. It encompasses the standard Gaussian case, gives rise to straightforward estimation procedures, and enables us to derive optimal portfolio weights under stable GARCH–factor assumptions in a practically feasible manner.

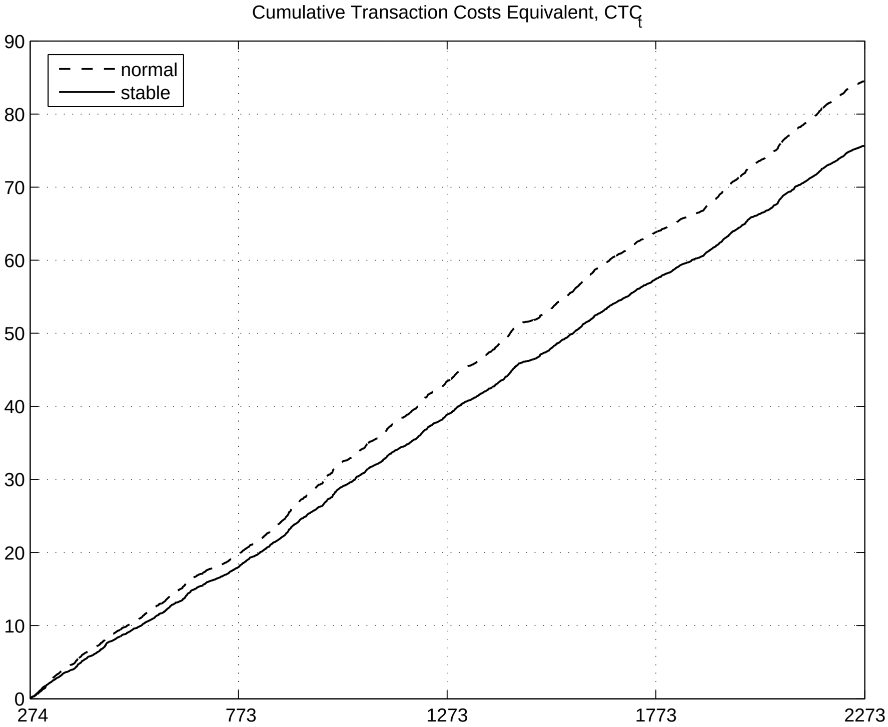

Figure 5: Cumulative Transaction Cost Equivalents

Using a set of nine stocks belonging to the German DAX index, we find overwhelming statistical evidence against a Gaussian GARCH–factor model in favor of its stable (non–Gaussian) generalization. The extremely significant improvement is achieved by relaxing only a single model parameter, namely the characteristic exponent α, which determines the heavy–tailedness.

To assess the consequences for portfolio optimization we optimally rebalance the portfolio over a period of 2,000 trading days. Examining the accuracy in VaR–based risk assessment we find that the Gaussian GARCH–factor model leads to inadequate results for the practically relevant 1% and 5% shortfall probabilities—both in terms of coverage and serial dependence of VaR violations. The results are greatly improved when allowing returns to follow a stable non–Gaussian distribution. Only for the 10% shortfall probability—in practice of little relevance—the Gaussian model has an adequate coverage and, in fact, less serial dependence in the VaR violations than the stable model.

Finally, we examined the cost of optimally rebalancing the portfolio under the two distributional assumption. Using the variability of the weight vectors over time and the cumulative transaction volume as proxies, the Gaussian model—with 34% more variability and about 13% increase in transaction volume—is less attractive for portfolio managers.

Clearly, transaction volume can only be a proxy for transaction cost. The latter depends on the specific trading strategy adopted. To what extent one can use our results to develop such strategies remains to be investigated.

Although we do not expect asset returns to exactly follow any stable distribution, our empirical results indicate that the heavy–tailed stable GARCH–factor model provides a more realistic framework for dynamic portfolio optimization without giving up analytical tractability and practical implementability.

References

Akgiray, V. and Booth, G.G. (1988). The Stable Law Model of Stock Returns. *Journal of Business and Economics Statistics*, **6**, 51–57.

Anderson, T. and Darling, D. (1952). Asymptotic theory of certain 'goodness of fit' criteria based on stochastic processes. *Annals of Mathematical Statatistics*, **23**, 193–212.

Basle Committee. (1996a). Overview of the of the Capital Accord to Incorporate Market Risk, Basle Committee on Bank Supervision.

Basle Committee. (1996b). Supervisory Framework for the use of "Backtesting" in Conjunction with the Internal Models Approach to Market Risk Capital Requirements, Basle Committee on Bank Supervision.

Bawa, V.S., and Lindenberg, E.B. (1977). Capital Market Equilibrium in a Mean-Lower Partial Moment Framework. *Journal of Financial Economics*, **5**, 189–200.

Lotfi, B., Lèvy-Vèhel, J. and Walter, C. (1995). Generalized Market Equilibrium: Stable CAPM. unpublished manuscript.

Lotfi, B., Lèvy-Vèhel, J. and Walter, C. (2000). CAPM, Risk and Portfolio Selection in α-Stable Markets. *Fractals*, **8**, 99–116.

Blattberg, R. and Sargent, T.J. (1971). Regression with Non-Gaussian Stable Disturbances: Some Sampling Results. *Econometrica*, **39**, 501–510.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, **31**, 307–327.

Brown, S.J. (1989). The Number of Factors in Security Returns. *Journal of Finance*, **44**, 1247–1262.

Chamberlain, G. (1983). A Characterization of the Distributions That Imply Mean-variance Utility Functions. *Journal of Economic Theory*, **29**, 185–201.

Christoffersen, Peter F. (2003). *Elements of Financial Risk Management.* Academic Press, London.

D'Agostino, R. and Stephens, M. (1986). *Goodness of Fit Techniques.* Marcel Dekker, New York.

DeGroot, M.H. (1986). *Probability and Statistics*, 2nd edn. Addison-Wesley, Reading, Massachusetts.

Dhrymes, P.J., Friend, I. and Gültekin, N.B. (1988). A Critical Examination of the Empirical Evidence of the Arbitrage Pricing Theory. *Journal of Finance*, **39**, 323–346.

Doganoglu, T. and Mittnik, S. (1998). An Approximation Procedure for Asymmetric Stable Densities. *Computational Statistics*, **13**, 463–475.

Doganoglu, T., and Mittnik, S. (2002). The Estimation of Multivariate Stable Paretian Index Models. unpublished manuscript.

Elton, E.J., Gruber, M.J. and Bawa, V.S. (1979). Simple Rules for Optimal Portfolio Selection in Stable Paretian Markets. *Journal of Finance*, **34**, 1041–1047.

Fama, Eugene F. (1965a). The Behavior of Stock Market Prices. Journal of Business, **38**, 34–105.

Fama, Eugene F. (1965b). Portfolio Analysis in a Stable Paretian Market. *Management Science*, **11**, 404–419.

Fama, Eugene F. (1971) Risk, Return and Equilibrium. *Journal of Political Economy*, **77**, 31–55.

Gamrowski, B. and Rachev, S.T. (1999) A Testable Version of the Pareto-Stable CAPM. *Mathematical and Computer Modelling*, **29**, 61–82.

Harlow, W. V. and Rao, R.K.S. (1989). Asset Pricing in a Generalized Mean–Lower Partial Moment Framework: Theory and Evidence. *Journal of Financial and Quantitative Analysis*, **24**, 394–419.

Kurz-Kim, J.-R., Rachev, S.T. and Samorodnitsky, G. (2004) Asymptotic distribution of unbiased linear estimators in the presence of heavy-tailed stochastic regressors and residuals. mimeo.

McCulloch, J. Huston (1998). Numerical Approximation of the Symmetric Stable Distribution and Densitiy. In *A Practical Guide to Heavy Tails*, R.J. Adler, R. Feldman and M. S. Taqqu (eds.), Boston,MA: Birkhauser.

Mittnik, S., Doganoglu, T. and Chenyao, D. (1999). Computing the Probability Density Function of the Stable Paretian Distribution. *Mathematical and Computer Modelling*, **29** , 235–240.

Mittnik, S. and Rachev, S.T. (1993). Modeling Stock Returns with Alternative Stable Distribution. *Econometric Reviews*, **12**, 261–330.

Mittnik, S., Rachev, S.T., Doganoglu, T. and Chenyao, D. (1999). Maximum Likelihood Estimation of the Stable Paretian Models. *Mathematical and Computer Modelling*, **29**, 275–293.

Nolan, J.P. (1999). An Algorithm for Evaluating Stable Densities in Zolotarev's (M) Parameterization. *Mathematical and Computer Modelling*, **29**, 229–233.

Panorska, A., Mittnik, S. and Rachev, S. (1995). Stable ARCH modles for financial time series. *Applied Mathematic Letters*, **8**(4), 33–37.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P. (1991). *Numerical Recipes in Fortran: The Art of Scientific Computing*, 2nd edn. Cambridge University Press, New York.

Rachev, S.T., and Mittnik, S. (2000). *Stable Paretian Models in Finance*. Wiley, Chichester.

RiskMetrics Group (1996). RiskMetrics—Technical Document. 4th edition. http://www.riskmetrics.com/research/techdoc/index.cgi

Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes*. Chapman & Hall, New York.

Tanaka, K. (1996). *Time Series Analysis, Nonstationarity and Noninvertible Distribution Theory*. John Wiley & Sons, New York.

Telser, L.G. (1955). Safety First and Hedging. *Review of Economic Studies*, **25**, 65–86.

A Some Useful Properties of Multivariate Stable Paretian Random Vectors

The following property gives rise to a straightforward procedure for modeling asset–return vectors governed by multivariate stable distributions.

Property A (Samorodnitsky and Taqqu, 1994, p. 70) *The spectral measure associated with the stable vector Y is composed of a finite number of atoms on the unit sphere, if and only if Y can be represented by a linear transformation of independent stable random variables.*

There is a natural relationship between Property A and the class of index models put forth in portfolio theory; and it is the discreteness of the spectral measure that gives rise to the estimation strategy we adopt.

The next property enables us to derive the spectral measure of a multivariate stable Paretian vector in terms of the spectral measure of a stable random vector with independent elements.

Property B (Samorodnitsky and Taqqu, 1994, p. 69) *Let $X = \left(X_1, \ldots, X_p\right)'$ with $X_k \sim S_\alpha(\sigma_k, \beta_k, \mu_k)$, $k = 1, \ldots p$, be a vector of independent random variables with common characteristic exponent α (but possibly different scale, skewness and location parameters); and let $A = \{a_{jk}\}$, $j = 1, \ldots, q$, $k = 1, \ldots, p$, be a real matrix. Then, the vector $Y = (Y_1, \ldots, Y_q)'$ of linear combination of the independent stable variables X_k, $k = 1, \ldots, p$, given by*

$$Y = AX,$$

is also stable and has the spectral measure

$$\Gamma = \frac{1}{2}\sum_{k=1}^{p} (\sigma\|a_{\cdot k}\|)^{\alpha} \left[(1+\beta_k)\,\delta\!\left(\frac{a_{\cdot k}}{\|a_{\cdot k}\|}\right) + (1-\beta_k)\,\delta\!\left(\frac{-a_{\cdot k}}{\|a_{\cdot k}\|}\right)\right], \tag{13}$$

where $a_{\cdot k}$ denotes the k–th column of matrix A; $\|a_{\cdot k}\| = \left(\sum_{j=1}^{q} a_{jk}^2\right)^{1/2}$ is the length of the vector $a_{\cdot k}$, such that $a_{\cdot k}/\|a_{\cdot k}\|$ represents the coordinates of a point on the unit sphere $\mathcal{S}_q$; and $\delta(\cdot)$ denotes the Dirac–delta function.

If the returns of the assets in a portfolio are characterized by a joint multivariate stable distribution, the aggregate return of the portfolio is given by a linear combination of jointly stable Paretian random variables.

Property C (Samorodnitsky and Taqqu, 1994, p. 67) *Let $w = (w_1, \ldots, w_q)' \in \mathbb{R}^q$ denote a vector of weights. Then, any linear combination $w'Y$ of the components of a stable vector $Y = (Y_1, \ldots, Y_q)'$ with spectral measure $\Gamma(ds)$ and location vector μ, follows the (univariate) stable distribution $w'Y \sim S_\alpha(\sigma(w'Y), \beta(w'Y), \mu(w'Y))$ with*

$$\sigma(w'Y) = \left(\int_{\mathcal{S}_q} |w's|^{\alpha}\Gamma(ds)\right)^{1/\alpha}, \tag{14}$$

$$\beta(w'Y) = \frac{\int_{\mathcal{S}_q} |w's|^{\alpha} sign(w's)\Gamma(ds)}{\int_{\mathcal{S}_q} |w's|^{\alpha}\Gamma(ds)}, \tag{15}$$

$$\mu(w'Y) = w'\mu. \tag{16}$$

This result allows us to express the portfolio–return distribution as a function of the multivariate stable distribution of the underlying vector of asset returns.

CFS Working Paper Series:

No.	Author(s)	Title
2006/24	**Toker Doganoglu** **Christoph Hartz** **Stefan Mittnik**	Portfolio Optimization when Risk Factors are Conditionally Varying and Heavy Tailed
2006/23	**Christoph Hartz** **Stefan Mittnik** **Marc S. Paolella**	Accurate Value-at-Risk Forecasting Based on the (good old) Normal-GARCH Model
2006/22	**Dirk Krueger** **Hanno Lustig** **Fabrizio Perri**	Evaluating Asset Pricing Models with Limited Commitment using Household Consumption Data
2006/21	**Juan Carlos Conesa** **Sagiri Kitao** **Dirk Krueger**	Taxing Capital? Not a Bad Idea After All!
2006/20	**Annamaria Lusardi** **Olivia S. Mitchell**	Baby Boomer Retirement Security: The Roles of Planning, Financial Literacy, and Housing Wealth
2006/19	**Carol C. Bertaut** **Michael Haliassos**	Credit Cards: Facts and Theories
2006/18	**Dirk Krueger** **Alexander Ludwig**	On the Consequences of Demographic Change for Rates of Returns to Capital, and the Distribution of Wealth and Welfare
2006/17	**Franklin Allen** **Elena Carletti**	Mark-to-Market Accounting and Liquidity Pricing
2006/16	**Erik Hurst** **Arthur Kennickell** **Annamaria Lusardi** **Francisco Torralba**	Precautionary Savings and the Importance of Business Owners
2006/15	**Arthur Kennickell** **Annamaria Lusardi**	Disentangling the Importance of the Precautionary Saving Motive

Copies of working papers can be downloaded at http://www.ifk-cfs.de