CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Hochschulschriftenserver - Universität Frankfurt am Main

Interacting locally regulated diffusions

Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften

vorgelegt beim Fachbereich Informatik und Mathematik
der Johann Wolfgang Goethe-Universität
in Frankfurt am Main

von
Martin Hutzenthaler
aus Landshut

Frankfurt am Main 2007
(D 30)

vom Fachbereich Informatik und Mathematik
der Johann Wolfgang Goethe – Universität als Dissertation angenommen.

Dekan:	Prof. Dr.-Ing. Detlef Krömker
Gutachter:	Prof. Dr. Anton Wakolbinger Prof. Dr. Götz Kersting Prof. Dr. Alison Etheridge

Datum der Disputation:

A mathematician is a device
for turning coffee into theorems.

Paul Erdős (1913-1996)

Acknowledgments

First and foremost, I wish to express my deepest gratitude to my advisor Prof. Dr. Anton Wakolbinger for his excellent mentoring. In the last three years, I enjoyed to benefit from his intuition and brilliance. In uncounted meetings, he inspired and stimulated me with ideas and pictures for new approaches. He considerably improved my way of writing papers by countless detailed remarks. Furthermore, especially in frustrating moments, Anton Wakolbinger encouraged me with his positive attitude to go on.

Next, I thank my co-authors. The main part of Chapter 2 is joint work with Anton Wakolbinger ([17]). The main part of Chapter 4 is joint work with Roland Alkemper ([1]).

Furthermore, I am grateful to Achim Klenke and three anonymous referees for detailed remarks and useful suggestions which have considerably improved the quality of the above mentioned papers.

This thesis profited much by valuable discussions with Don Dawson, Frank den Hollander and Jan Swart. Their brightness and their skills produced useful ideas. In addition, Chapter 2 is inspired by ideas from the paper [2] of Siva Athreya and Jan Swart.

Last but not least, I thank my next-door colleague Wolfgang Angerer, who moved to Mexico in 2006, for his patience to listen to all my questions and for all the discussions with him.

Contents

Chapter 1

Introduction and main results

1.1 Introduction

In naturally reproducing populations one usually encounters an average number of more than one offspring per individual. However, given non-extinction, classical supercritical branching processes grow beyond all bounds. This is unrealistic because of bounded resources.

An efficient counteraction to unbounded population growth is achieved by a population-size dependent regulation of the reproduction dynamics. An example is the so called *logistic branching process* (Lambert [23]) in which, in addition to the "natural" births and deaths in a supercritical branching mechanism, there are deaths resulting from a competition between any two individuals in the population. In Feller's diffusion limit, this leads to a negative drift term which is proportional to the squared population size. To be more precise, for $N \geq 1$ and $b, d, \gamma, \beta > 0$, let $(Z_t^N)_{t\geq 0}$ be a pure birth-death process with state space $\mathbb{N}_0$ where each particle splits into two particles at rate $\beta + \frac{b}{N}$, each particle dies at rate $\beta + \frac{d}{N}$ and each ordered pair of particles coalesces into one particle at rate $\frac{\gamma}{N^2}$. All these events occur independently of each other. If $\frac{Z_0^N}{N}$ converges weakly to Z_0 as $N \to \infty$ then $\left(\frac{Z_{tN}^N}{N}\right)_{t\geq 0}$ converges weakly to $\left(Z_t\right)_{t\geq 0}$ as $N \to \infty$ where $(Z_t)_{t\geq 0}$ is the solution of

$$dZ_t = (b-d)Z_t\,dt - \gamma Z_t^2\,dt + \sqrt{2\beta Z_t}dB_t. \tag{1.1}$$

Here, $(B_t)_{t\geq 0}$ is a standard Brownian motion. See Section 4.4 for the proof of a similar convergence. The square in (1.1) prevents the population size from escaping to ∞. However, the process $(Z_t)_{t\geq 0}$ converges weakly to zero as $t \to \infty$.

An attempt to combat this extinction is to consider infinite populations modeled by a spatially extended version of the logistic branching process, with subpopulations living in discrete demes arranged in the d-dimensional lattice $\mathbb{Z}^d$, and with a (homogeneous) migration between the demes. This leads to the following system $X = (X_t)_{t\geq 0} = (X_t(i))_{t\geq 0, i\in\mathbb{Z}^d}$ of *interacting Feller diffusions with logistic growth* where $X_t(i) \in [0,\infty)$ denotes the population size of deme $i \in \mathbb{Z}^d$ at time $t \geq 0$:

$$\begin{aligned} dX_t(i) = {} & \alpha \left(\sum_{j\in\mathbb{Z}^d} m(i,j)X_t(j) - X_t(i) \right) dt \\ & + \gamma X_t(i)\big(K - X_t(i)\big)dt + \sqrt{2\beta X_t(i)}\, dB_t(i) \qquad i \in \mathbb{Z}^d. \end{aligned} \tag{1.2}$$

Here, the $B(i)$ are independent standard Brownian motions, m is the transition matrix of a random walk on $\mathbb{Z}^d$, and α, β, γ are nonnegative constants describing the rates of migration, branching and competition, respectively. The constant $K \geq 0$ is called the *capacity*; it is the ratio of the *coefficient of supercriticality*, γK, and the competition rate γ. Interacting Feller diffusions with logistic growth are a prototype example for interacting locally regulated diffusions which we introduce below.

Models with competition have been studied by various authors: Mueller and Tribe [26] and Horridge and Tribe [16] investigated an SPDE analogue of (1.2), with $d = 1$ and $\mathbb{R}^1$ instead of $\mathbb{Z}^1$, and Etheridge [10], motivated by the work of Bolker and Pacala [3], investigated system (1.2) and its measure-valued analogue (with $\mathbb{Z}^d$ replaced by $\mathbb{R}^d$). These models also include long range competition. We emphasize that our methods make use of the fact that the interactions due to competition are solely within the same lattice site.

A central question of this thesis is whether the solution $(X_t)_{t\geq 0}$ of (1.2) suffers extinction as $t \to \infty$. First of all, we clarify what we mean by "extinction". We say that $(X_t)_{t\geq 0}$ suffers *local extinction* if $(X_t)_{t\geq 0}$ converges weakly to the zero configuration as $t \to \infty$. For this, let the topology on $[0,\infty)^{\mathbb{Z}^d}$ be given by the product topology. Furthermore, we speak of *global extinction* if $(|X_t|)_{t\geq 0}$ converges weakly to zero as $t \to \infty$. Throughout the thesis, $|x| := \sum_{i\in\mathbb{Z}^d} x_i$ denotes the total mass of $x \in [0,\infty)^{\mathbb{Z}^d}$. Notice that global extinction implies local extinction. Furthermore, the two notions local extinction and global extinction would coincide if $\mathbb{Z}^d$ was replaced by a finite set. In the context of local extinction, it is typically assumed that the law of X_0 is translation invariant. For global extinction, we assume that $|X_0| < \infty$ almost surely.

Using arguments from oriented percolation, Etheridge [10] shows that system (1.2) and also similar systems with non-local competition, when started from a spatially homogeneous initial state, do not suffer local extinction provided the capacity K is large enough. On the other hand, it was shown in the same paper by a coupling and comparison with subcritical branching (similar as in Mueller and Tribe [26]) that a measure-valued analogue of (1.2) with certain non-local competition mechanisms suffers local extinction. The question whether lattice-based systems like (1.2) suffer local extinction for suitably small K remained open. In Chapter 2, we answer this question in the affirmative for the system (1.2) (Theorem 2). More precisely, we specify a strictly positive constant $\overline{K}$ such that for all capacities $K \leq \overline{K}$ system (1.2) suffers local extinction. The constant $\overline{K}$ is the unique solution of

$$\int_0^\infty \exp\left(\overline{K}\gamma y - \frac{\gamma\beta}{2} y^2\right) \cdot \alpha \exp\left(-\alpha y\right) dy = 1 \tag{1.3}$$

and depends on the rates α, β and γ of migration, branching and competition, respectively, but is uniform in all dimensions d and migration matrices m.

The second main result of Chapter 2 concerns convergence of $(X_t)_{t\geq 0}$ as $t \to \infty$. We construct the maximal process $X^{(\infty)}$, which is the solution of (1.2) entering from infinity at time 0 (Theorem 1). An important property of $X^{(\infty)}$ is that this process dominates every solution of (1.2) in a stochastic order to be introduced below. As time tends to infinity, $(X_t^{(\infty)})_{t\geq 0}$ converges monotonically in distribution to the *upper invariant measure* of (1.2). In Theorem 5, we prove ergodic behaviour of the process $(X_t)_{t\geq 0}$ as $t \to \infty$, that is, the process forgets its initial configuration as $t \to \infty$. More precisely, we show that the solution $(X_t)_{t\geq 0}$ of (1.2), when started in a translation invariant nontrivial initial state, converges weakly to the upper invariant measure as $t \to \infty$. For the proof, we will exploit the following self-duality. Let X be the solution of (1.2) with parameters $\alpha, \beta, \gamma > 0$ and migration matrix m and let $X^\dagger$ be the solution of (1.2) with parameters α, β, γ and migration matrix $m^\dagger$ which is the transpose of m. Theorem 3 states that

$$\mathbf{E}^{\underline{x}} \exp\left(-\frac{\gamma}{\beta}\langle X_t, \underline{y}\rangle\right) = \mathbf{E}^{\underline{y}} \exp\left(-\frac{\gamma}{\beta}\langle \underline{x}, X_t^\dagger\rangle\right) \;\; \forall\, \underline{x} \in \mathbb{E}_\sigma, \underline{y} \in \mathbb{E}_{\sigma^\dagger}, \tag{1.4}$$

where the state spaces $\mathbb{E}_\sigma$ and $\mathbb{E}_{\sigma^\dagger}$ will be defined in Section 1.2. Throughout the thesis, superscripts as in $\mathcal{L}^y$, $\mathbf{P}^y$ or $\mathbf{E}^y$ refer to the initial configuration of a process.

Self-duality was used to prove ergodicity by other authors, e.g. Horridge and Tribe [16] and Athreya and Swart [2]. In the latter paper, self-duality was established for the resampling selection model which is the solution of (1.2) where the Feller term $\sqrt{2\beta X_t(i)}$ is replaced by the Fisher-Wright term $\sqrt{2\beta X_t(i)(1 - X_t(i))}$ and where $K \leq 1$. Furthermore, Athreya and Swart study a branching coalescing particle model which in Feller's diffusion limit leads to the solution of (1.2). For both models, they prove existence of the maximal process and of the upper invariant measure.

We obtain the local extinction result and the result about existence of the maximal process and of the upper invariant measure for a more general class of *interacting locally regulated diffusions.* The system of stochastic differential equations we consider is

$$\begin{aligned} dX_t(i) =& \alpha \left(\sum_{j\in G} m(i,j) X_t(j) - X_t(i) \right) dt \\ &+ h\big(X_t(i)\big)dt + \sqrt{2\cdot g\big(X_t(i)\big)}\, dB_t(i), \qquad i \in G, \end{aligned} \tag{1.5}$$

where G is an at most countable Abelian group. Notice that the two models (1.2) and (1.5) coincide in the case $G = \mathbb{Z}^d$, $h(x) = \gamma x(K - x)$, $g(x) = \beta x$. We will specify an appropriate state space, namely the Liggett-Spitzer space $\mathbb{E}_\sigma \subset [0,\infty)^G$, in Section 1.2 and sufficient conditions on the *regulation function* $h\colon [0,\infty) \to \mathbb{R}$ and on the *diffusion function* $g\colon [0,\infty) \to [0,\infty)$ for existence and uniqueness of the process X in Proposition 1.2.1. Figure 1.1 and 1.2 show generic examples for a regulation function and for a diffusion function, respectively.

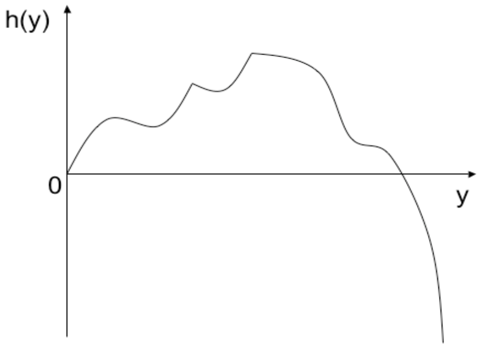


Figure 1.1: A generic example for a regulation function.

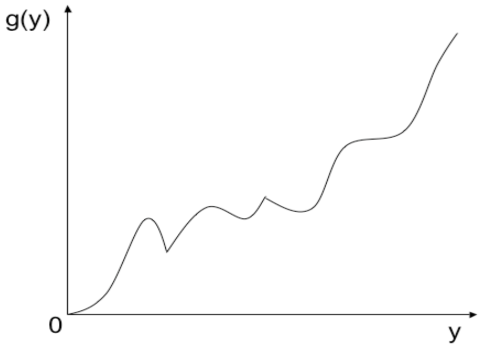


Figure 1.2: A generic example for a diffusion function.

The name "interacting locally regulated diffusions" derives from "interacting diffusions" which denotes the solution of (1.5) in the case $h \equiv 0$. Interacting diffusions have been studied by various authors, among others: Cox and Greven [4], Cox, Fleischmann and Greven [5], Greven, Klenke and Wakolbinger [12]. The process $(X_t)_{t\geq 0}$ is "locally regulated" because the regulation term $h\big(X_t(i)\big)$ depends on X_t only through the local population size $X_t(i)$. If $h(X_t(i))$ in (1.5) was replaced by $h_i(X_t)$ with $h_i\colon [0,\infty)^G \to \mathbb{R}$ then the regulation would be (possibly) long-range.

In Theorem 1, we prove existence of the maximal process $(X_t^{(\infty)})_{t\geq 0}$ and convergence of $(X_t^{(\infty)})_{t\geq 0}$ to the upper invariant measure as $t \to \infty$. For this we need an assumption which ensures that the drift is "sufficiently negative" for large values of $X_t(i)$ so that the process "comes down from infinity". We assume for

Theorem 1 that h is bounded by a function $\hat{h}$ which is negative and concave on some interval $[x_0, \infty)$ and satisfies

$$\int_{x_0}^{\infty} \frac{1}{-\hat{h}(x)}\, dx < \infty. \tag{1.6}$$

Then there exists a solution $(X_t^{(\infty)})_{t\geq 0}$ of (1.5) which starts in $X_0^{(\infty)}(i) = \infty$, $i \in G$, and satisfies $\mathbf{E}X_t^{(\infty)}(i) < \infty$ for all $t > 0$ and $i \in G$, see Theorem 1. Notice that the above condition on h is satisfied in the case of interacting Feller diffusions with logistic growth with $\hat{h}(x) := \gamma x(K - x)$.

Theorem 2 specifies conditions on α, h and g under which the solution $(X_t)_{t\geq 0}$ of (1.5) suffers local extinction. Let the law of X_0 be any distribution on the state space $\mathbb{E}_\sigma$. Assume that h is concave and is bounded by a function $\hat{h}$ which is negative on some interval $[x_0, \infty)$ and satisfies condition (1.6). If

$$\int_0^{\infty} \frac{h(y)}{g(y)} \exp\left(\int_1^y \frac{-\alpha x + h(x)}{g(x)}\, dx\right) dy \leq 0, \tag{1.7}$$

then $(X_t)_{t\geq 0}$ converges weakly to the zero configuration as $t \to \infty$. We mention that, in the case $h(x) = \gamma x(K - x)$ and $g(x) = \beta x$, condition (1.7) is equivalent to $K \leq \overline{K}$ where $\overline{K}$ is the solution of (1.3); see Proposition 2.3.1. The proof of the above local extinction result is achieved by comparing (1.5) with a *mean field model* associated with (1.5), given by the solution $M = (M_t)_{t\geq 0}$ of

$$dM_t = \alpha(\mathbf{E}M_t - M_t)\, dt + h(M_t)\, dt + \sqrt{2{\cdot}g(M_t)}\, dB_t\,, \;\; M_0 \in [0, \infty), \tag{1.8}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. To be more precise, if h is concave and if the law of X_0 is translation invariant and associated (to be defined in (1.36)), then Proposition 1.2.2 shows that

$$\mathbf{E}e^{-\lambda X_t(i)} \geq \mathbf{E}e^{-\lambda M_t}, \quad t, \lambda \geq 0,\, i \in G, \tag{1.9}$$

where $M_0 := X_0(i)$. Consequently, extinction of $(M_t)_{t\geq 0}$ as $t \to \infty$ implies extinction of $(X_t(i))_{t\geq 0}$ as $t \to \infty$ for every $i \in G$. We will see that $(M_t)_{t\geq 0}$ converges weakly to zero as $t \to \infty$ if δ_0 is the only equilibrium distribution of the mean field model. In addition, if h has at most one strictly positive root and is negative in a neighbourhood of infinity, then Proposition 2.3.1 shows that δ_0 is the only equilibrium distribution of the mean field model if and only if inequality (1.7) holds. Furthermore, if inequality (1.7) fails to hold, then we obtain in Proposition 2.3.1 that there exists exactly one nontrivial invariant measure for the mean field model (1.8).

The following approximation illuminates the appearance of the mean field model as a comparison model for interacting locally regulated diffusions. For

$N \geq 1$, let $\Lambda_N := \mathbb{Z}/N\mathbb{Z}$. Denote by $(X_t^N)_{t\geq 0}$ the solution of (1.5) with $G := \Lambda_N$ and with $m(i,j) := \frac{1}{N}$, $i, j \in \Lambda_N$. Furthermore, let $(X_0^N(i))_{i\in\mathbb{Z}/N\mathbb{Z}}$ be independent and identically distributed with common law μ. Then $(X_t^N(i))_{N\geq 1}$ converges weakly to M_t as $N \to \infty$ for every fixed $t \geq 0$ and $i \in \mathbb{Z}$ where M_0 has distribution μ. The proof of this assertion is similar to the proof of Theorem 1.4 in [32]. However, we will not work out the details. Loosely speaking, the mean field model belongs to the closure of the class of interacting locally regulated diffusions and its migration mechanism spreads out mass as uniformly as possible. Motivated by the above approximation, we conjecture that if (1.7) fails to hold then there exists a countable set G and a migration matrix $\big(m(i,j)\big)_{i,j\in G}$ such that the solution of (1.5) does not suffer local extinction.

A consequence of the self-duality (1.4) for the solution of (1.2) is that local extinction is equivalent to global extinction. In Corollary 4, we conclude from the local extinction result that the solution of (1.2) suffers global extinction whenever $K \leq \overline{K}$. For the solution of (1.5), however, there is in general no global extinction result yet. We conjecture that there exists a dominating process for which it is easier to obtain a criterion for global extinction. However, we will not prove this conjecture in this thesis. As a candidate for a comparison model, we now introduce a model which we call *Virgin Island Model*. For this model, we will prove in Chapter 3 a global extinction result. In analogy to the local extinction result, a comparison result of system (1.5) with the Virgin Island Model would lead to a global extinction result for the system of interacting locally regulated diffusions. To motivate the Virgin Island Model, consider, for $N \geq 1$, the solution $(X_t^N)_{t\geq 0}$ of (1.5) with $G := \mathbb{Z}/N\mathbb{Z}$ and with $m(i,j) := \frac{1}{N}$, $i,j \in \mathbb{Z}/N\mathbb{Z}$. Furthermore, let $X_0^N(0) := x_0 \in (0,\infty)$ and $X_0^N(i) := 0$ for $i \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}$. The probability that two emigrants migrate to the same island is equal to $\frac{1}{N}$, which tends to zero as $N \to \infty$. In the Virgin Island Model, every emigrant moves to an unpopulated island.

We characterise the Virgin Island Model by a recursive construction. On the first island evolves a diffusion $Y = (Y_t)_{t\geq 0}$ with state space $[0,\infty)$ given by the strong solution of the stochastic differential equation

$$dY_t = -\alpha Y_t\,dt + h(Y_t)\,dt + \sqrt{2g(Y_t)}dB_t, \qquad Y_0 = y \geq 0, \tag{1.10}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Notice that Y is equal in distribution to $X(0)$ if $m(i,j) = 0$ for all $i,j \in G := \mathbb{Z}^d$ and if $X_0(0) := y$. We assume that Y is regular on $(0,\infty)$ and that zero is an exit boundary for this process, that is, zero is absorbing and is reached in finite time with positive probability. In Assumption A3 below, we give an equivalent condition for this in terms of α, h and g.

Mass emigrates from the first island at rate α, which is modeled by the term $-\alpha Y_t dt$ in (1.10). An emigrant founding the population on an unpopulated island has mass zero in the diffusion limit. The law of excursions of Y from the trap zero is the key ingredient in the construction of the Virgin Island Model. Denote the set of excursions from zero by

$$U := \Big\{\chi \in \mathbf{C}\big([0,\infty),[0,\infty)\big) : T_0 \in (0,\infty],\ \chi_t = 0 \ \ \forall\, t \in \{0\} \cup [T_0,\infty)\Big\} \tag{1.11}$$

where $T_y = T_y(\chi) := \inf\{t > 0 \colon \chi_t = y\}$ is the first hitting time of $y \in [0,\infty)$. The set U is furnished with locally uniform convergence. The *excursion law* $\bar{Q}_Y$ is a σ-finite measure on U. It has been constructed by Pitman and Yor [28] as follows: Under $\bar{Q}_Y$, the trajectories come from zero according to an entrance law and then move according to the law of Y. In Section 3.1, we approximate the excursion measure with a suitably rescaled law of Y. For this, define

$$\bar{s}(z) := \exp\Big(-\int_1^z \frac{-\alpha x + h(x)}{g(x)}\,dx\Big), \quad \bar{S}(y) := \int_0^y \bar{s}(z)\,dz, \quad z, y > 0. \tag{1.12}$$

Note that $\bar{S}$ is a scale function, that is,

$$\mathbf{P}^y\big(T_b(Y) < T_a(Y)\big) = \frac{\bar{S}(y) - \bar{S}(a)}{\bar{S}(b) - \bar{S}(a)} \tag{1.13}$$

holds for all $0 \le a < y < b < \infty$, see Section 6 of [21]. In Theorem 6, we will prove the convergence

$$\lim_{y \to 0} \frac{1}{\bar{S}(y)} \mathbf{E}^y F(Y) = \int F(\chi) \bar{Q}_Y(d\chi) \tag{1.14}$$

for all bounded continuous $F \colon \mathbf{C}\big([0,\infty),[0,\infty)\big) \to \mathbb{R}$ for which there exists an $\varepsilon > 0$ such that $F(\chi) = 0$ whenever $\sup_{t \ge 0} \chi_t < \varepsilon$. Note that the well established Itô excursion theory does not apply here because zero is no regular point.

The existence of $\bar{Q}_Y$ suffices to construct the Virgin Island Model and to formulate results. For the proof of a global extinction result for the Virgin Island Model, however, we need a stronger assertion, namely the convergence stated in (1.17) below. To obtain (1.17), we assume that

$$\mathbf{P}^y\big(T_1(Y) < T_0(Y)\big) \sim cy \qquad \text{as } y \to 0 \tag{1.15}$$

for some constant $c \in (0,\infty)$. Equivalent to (1.15) is that $\bar{S}'(0)$ exists and is positive. Assumption A4 below gives a sufficient condition for (1.15) in terms of α, h and g. Under Assumption A4, we may define

$$Q_Y := \bar{S}'(0) \bar{Q}_Y. \tag{1.16}$$

With this, the convergence (1.14) reads as

$$\lim_{y\to 0}\frac{1}{y}\mathbf{E}^y F(Y) = \int F(\chi) Q_Y(d\chi) \tag{1.17}$$

By an abuse of notation, we denote both Q_Y and $\bar{Q}_Y$ as "the excursion measure of Y".

Employing the excursion measure Q_Y, we now define the Virgin Island Model on subsequent islands. The first island is called the 0-th generation. The $(n+1)$-st generation is the collection of all islands which have been colonised from islands of the n-th generation, $n \geq 0$. We denote the collection of all islands as *Virgin Island Model*. Furthermore, we refer to the total mass of the Virgin Island Model as the Virgin Island process $V = (V_t)_{t\geq 0}$ and to the total mass of the n-th generation as the n-th generation process $V^{(n)} = \big(V_t^{(n)}\big)_{t\geq 0}$. Let $(V_t^{(0)})_{t\geq 0}$ be a random path with distribution $\mathcal{L}^x\big((Y_t)_{t\geq 0}\big)$, $x \geq 0$. For a recursive construction, let the total mass $V^{(n)}$ of the n-th generation, $n \geq 0$, be defined. Conditioned on $V^{(n)}$, let $\Pi^{(n)}$ be a Poisson point process on $[0,\infty)\times U$ with intensity measure $\alpha V_t^{(n)}\, dt \otimes Q_Y(d\chi)$. With this, the $(n+1)$-st generation process is defined as

$$V_t^{(n+1)} := \int \chi_{t-s}\Pi^{(n)}(ds, d\chi) \qquad t \geq 0. \tag{1.18}$$

Emigrants leave islands of the n-th generation at the time dependent rate $\alpha V_t^{(n)}$ and move to unpopulated islands. An island which has been founded at time s contributes mass χ_{t-s} at time t. For definiteness, identify paths $\chi \in U$ with paths $\chi \in \mathbf{C}\big(\mathbb{R}, [0,\infty)\big)$ satisfying $\chi_t = 0$ for all $t \leq 0$. The Virgin Island process V is the total mass of all generation processes:

$$V_t := \sum_{n\geq 0} V_t^{(n)} \qquad t \geq 0. \tag{1.19}$$

The sum in (1.19) has finite expectation and thus is finite almost surely by Lemma 3.3.1.

There are similarities between the Virgin Island Model and the infinitely-many-alleles model (see [11]). In the latter model, every mutant is of a new type, which corresponds to migration to unpopulated islands. The infinitely-many-alleles model can be characterised by a martingale problem. However, we could not construct the Virgin Island Model by a martingale problem with respect to an operator $\mathcal{G}$ with $\mathcal{G} \subset \mathbf{C}_b(E)\times\mathbf{C}_b(E)$ for some complete and separable metric space (E, d). Instead, we give a fairly explicit construction for the total mass process in which the evolution on one single island is incorporated by the excursion law, and in which the different generations may be studied separately.

There is an inherent branching structure in the Virgin Island Model. One offspring island together with all its offspring islands is again a Virgin Island Model but with a typical excursion instead of Y on the first island. This branching structure is similar to Crump-Mode-Jagers branching processes (see [19] under "general branching process") but with continuous mass instead of particles. We recall that a Crump-Mode-Jagers process is a particle process where every particle i gives birth to particles at the time points of a point process ξ_i until its death at time λ_i, and $(\lambda_i, \xi_i)_i$ are independent and identically distributed.

In Theorem 7, we identify conditions under which the Virgin Island Model suffers global extinction. Generally speaking, branching particle processes survive iff the expected number of offspring per particle is strictly greater than one, e.g. a Crump-Mode-Jagers process survives iff $\mathbf{E}\xi_i[0, \lambda_i] > 1$. For the Virgin Island Model, the decisive parameter for survival is α times the expected area under an excursion

$$\int \int_0^\infty \chi_t \, dt \, Q_Y(d\chi). \tag{1.20}$$

We denote the expression in (1.20) also as "expected man-hours" of the excursion law. For the following Theorem 7 and Theorem 8, we assume that the expected man-hours are finite. In Assumption A5 below, we give an equivalent condition for this in terms of α, h and g. In Theorem 7, we will prove that the Virgin Island process suffers global extinction, that is, $(V_t)_{t\geq 0}$ converges weakly to zero as $t \to \infty$, if and only if

$$\int_0^\infty \frac{\alpha y}{g(y)} \exp\Big(\int_0^y \frac{-\alpha u + h(u)}{g(u)} \, du \Big) \, dy \leq 1. \tag{1.21}$$

The method of proof is to study an integro-differential equation (see Lemma 3.3.2) which the Laplace transform of V solves. Furthermore, we will show in Lemma 3.1.5 that α times the expression in (1.20) is equal to the left-hand side of (1.21).

Under Assumption A4, the conditions (1.7) and (1.21) are equivalent, see Proposition 2.3.1. Consequently, under Assumptions A3, A4 and A5, the mean field process suffers extinction if and only if the Virgin Island process dies out globally. We conjecture two more analogies between the mean field model and the Virgin Island Model. Firstly, the mean field model dominates the system of interacting locally regulated diffusions in the sense of (1.9) if the law of X_0 is translation invariant. As mentioned before, we conjecture that the Virgin Island process dominates the total mass of $(X_t)_{t\geq 0}$ in some stochastic order. Secondly, we mentioned above that there is a sequence $(X^N)_{N\in\mathbb{N}}$ of interacting locally regulated diffusions such that $(X_t^N(i))_{N\in\mathbb{N}}$ converges weakly to M_t as $N \to \infty$ for every $t \geq 0, i \in \mathbb{Z}$. For the Virgin Island Model, we conjecture that if $X_0^N(0) = V_0 \geq 0$

and $X_0^N(i) = 0$ for all $i \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}$ then $(|X_t^N|)_{N\in\mathbb{N}}$ converges weakly to V_t as $N \to \infty$ for every $t \geq 0$.

An interesting quantity of the Virgin Island process is the expectation of the total man-hours, i.e., the expected area under the path of V. In Theorem 8, we prove that this quantity is finite exactly in the subcritical situation, that is, (1.21) holds with strict inequality, in which case we give an expression for the expected man-hours in terms of α, h and g. In addition, in the critical case and in the supercritical case, we obtain the asymptotic behaviour of the expected man-hours of V up to time t

$$\int_0^t \mathbf{E}^x V_s \, ds \tag{1.22}$$

as $t \to \infty$ for all $x \geq 0$.

The Virgin Island Model combines the following two properties. On the one hand, it incorporates competition among individuals. On the other hand, there exists a (rather) explicit criterion for the phase transition from extinction to survival. Thus, the Virgin Island Model might be interesting for applications as it is more realistic than models with independent branching but simple enough to bear (rather) explicit formulas.

The self-duality (1.4) is a strong tool for analysing interacting Feller diffusions with logistic growth. We will prove it in Section 2.5 analytically by means of a generator calculation. In Chapter 4, we take a different approach by explaining the dynamics of the processes via *basic mechanisms* on the level of particles. Thereby, we obtain a stochastic picture for the self-duality (1.4) which provides insight into the role of the logistic regulation function $\gamma x(K - x)$ in (1.2) for the self-duality (1.4), and which gives an explanation for the involvement of the function $\exp\left(-\frac{\gamma}{\beta}\langle x, y\rangle\right)$ in (1.4). For simplicity, we only consider the non-spatial case, i.e., $m(i,j) = \mathbb{1}_{i=j}$ for $i, j \in \mathbb{Z}^d$.

In order to state a slightly more general duality than (1.4), let $(X_t)_{t\geq 0}$ denote the strong solution of

$$dX_t = \varsigma X_t \, dt - \gamma X_t^2 \, dt + \sqrt{2\beta X_t} \, dB_t, \tag{1.23}$$

where $\varsigma \in \mathbb{R}$, $\gamma, \beta \geq 0$ and $(B_t)_{t\geq 0}$ is a standard Brownian motion. We call this process the logistic Feller diffusion with parameters $(\varsigma, \gamma, \beta)$. Let $(Y_t)_{t\geq 0}$ be a logistic Feller diffusion with parameters $(\varsigma, r\beta, \gamma/r)$ for some $r > 0$. In Section 4.4, we prove

$$\mathbf{E}^x\left[e^{-rX_t \cdot y}\right] = \mathbf{E}^y\left[e^{-rx \cdot Y_t}\right] \qquad x, y \in [0, \infty), t \geq 0. \tag{1.24}$$

The approach which we introduce below applies not only to (1.24) but also to another duality which has been proven analytically by Athreya and Swart [2]. Let

$b, c, d \geq 0$. Denote by $X_t \in \mathbb{N}_0$ the number of particles at time $t \geq 0$ of the branching-coalescing particle process defined by the initial value $X_0 = n$ and the following dynamics: Each particle splits into two particles at rate b, each particle dies at rate d and each ordered pair of particles coalesces into one particle at rate c. All these events occur independently of each other. In the notation of Athreya and Swart [2], this is the $(1, b, c, d)$-braco-process. Its dual process $(Y_t)_{t\geq 0}$ is the unique strong solution with values in $[0, 1]$ of the one-dimensional stochastic differential equation

$$dY_t = (b - d)Y_t\,dt - bY_t^2\,dt + \sqrt{2cY_t(1 - Y_t)}\,dB_t, \quad Y_0 = y, \tag{1.25}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Athreya and Swart [2] call this process the resampling-selection model with selection rate b, resampling rate c and mutation rate d, or shortly the $(1, b, c, d)$-resem-process. They prove the duality

$$\mathbf{E}^n\big[(1 - y)^{X_t}\big] = \mathbf{E}^y\big[(1 - Y_t)^n\big] \qquad \forall\, n \in \mathbb{N}_0,\ y \in [0, 1],\ t \geq 0. \tag{1.26}$$

The duality relations (1.24) and (1.26) include as special cases (see Remark 4.4.2 and Remark 4.4.4) the duality of Feller's branching diffusion with a deterministic process, the duality of the Fisher-Wright diffusion with Kingman's coalescent, and the duality of the (continuous time) Galton-Watson process with a deterministic process.

Chapter 4 provides a unified stochastic picture for the duality relations (1.24) and (1.26). For every $N \in \mathbb{N}$, we construct approximating Markov processes $\big(X_t^N\big)_{t\geq 0}$ and $\big(Y_t^N\big)_{t\geq 0}$ with càdlàg sample paths and state space $\{0, 1\}^N$ and with the following properties. The processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ are dual in the sense that

$$\mathbf{P}^{x^N}\big[X_t^N \wedge y^N = \underline{0}\big] = \mathbf{P}^{y^N}\big[x^N \wedge Y_t^N = \underline{0}\big], \quad \forall\, x^N, y^N \in \{0, 1\}^N \quad \forall\, t \geq 0. \tag{1.27}$$

The notation $x^N \wedge y^N$ denotes component-wise minimum and $\underline{0}$ denotes the zero configuration. If $|X_0^N| = n$, for some fixed $n \leq N$, then $\big(|X_t^N|\big)_{t\geq 0}$ converges weakly to a branching-coalescing particle process as $N \to \infty$. We use the notation $|x| := \sum_{i=1}^N x_i$ for $x \in \{0, 1\}^N$. Assume that the set of càdlàg-paths is equipped with the Skorohod topology (see e.g. [11]). If $n = n(N)$ depends on N such that $n/N \to x \in [0, 1]$ as $N \to \infty$, then $(|X_t^N|/N)_{t\geq 0}$ converges weakly to a resampling-selection model. If $n = n(N)$ satisfies $n/\sqrt{N} \to x \geq 0$, then $\big(|X_{t\sqrt{N}}^N|/\sqrt{N}\big)_{t\geq 0}$ converges weakly to Feller's branching diffusion with logistic growth. The process $(Y_t^N)_{t\geq 0}$ differs from $(X_t^N)_{t\geq 0}$ only by the set of parameters and by the initial condition.

We will derive the duality (1.26) and the duality (1.24) from (1.27) in the following way. Let the random variable X_0^N be uniformly distributed over all configurations $x^N \in \{0,1\}^N$ with total number of individuals of type 1 equal to $|x^N| = n = n(N)$ for a given $n(N) \leq N$. Similarly, choose Y_0^N uniformly in $\{0,1\}^N$ with $|Y_0^N| = k = k(N)$ for a given $k(N) \leq N$. We will prove in Proposition 4.3.1 that property (1.27) implies a prototype duality relation, namely

$$\lim_{N\to\infty} \mathbf{E}\Big[1 - \frac{k}{N}\Big]^{|X_{tT_N}^N|} = \lim_{N\to\infty} \mathbf{E}\Big[1 - \frac{|Y_{tT_N}^N|}{N}\Big]^n, \quad t \geq 0, \tag{1.28}$$

under some assumptions – including the existence of both limits – on the two processes and on the sequence $(T_N)_{N\geq 1} \subset [0,\infty)$. Choosing n fixed, k such that $\frac{k}{N} \to y \geq 0$ and let $T_N = 1$, we will deduce from (1.28) (and from the convergence properties of $(X_t^N)_{t\geq 0}$ and of $(Y_t^N)_{t\geq 0}$) the duality (1.26) of a branching-coalescing particle process with a resampling-selection model (cf. Theorem 4.4.1). In order to obtain the duality (1.24), choose n, k such that $\frac{n}{\sqrt{N}} \to x \geq 0$, $\frac{k}{\sqrt{N}} \to y \geq 0$ and $T_N = \sqrt{N}$. Notice that $(1 - \frac{y}{\sqrt{N}})^{x\sqrt{N}}$ converges to e^{-xy} uniformly in $0 \leq x, y \leq \tilde{x}$ as $N \to \infty$ for every $\tilde{x} \geq 0$. This together with the weak convergence of the rescaled processes will imply

$$\lim_{N\to\infty} \mathbf{E}\Big[e^{-|X_{t\sqrt{N}}^N|\cdot y/\sqrt{N}}\Big] = \lim_{N\to\infty} \mathbf{E}\Big[e^{-x\cdot|Y_{t\sqrt{N}}^N|/\sqrt{N}}\Big]. \tag{1.29}$$

The approximating processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ are constructed in the following way. We call every function $f\colon \{0,1\}^2 \to \{0,1\}^2$ a *basic mechanism.* A finite tuple $(f_1, ..., f_m)$, $m \in \mathbb{N}$, of basic mechanisms together with rates $\lambda_1, ..., \lambda_m \in [0,\infty)$ defines a process by means of the following graphical representation, which is in the spirit of Harris [14]. With every $k \leq m$ and every ordered pair $(i,j) \in \{1, ..., N\}^2$, $i \neq j$, we associate a Poisson process with rate parameter λ_k. At every time point of this Poisson process, the configuration of (i,j) changes according to f_k. For example, if the configuration was $(1,0)$ before, then it changes to $f_k(1,0) \in \{0,1\}^2$. All Poisson processes are independent. In Section 4.2, we will specify which property (to be called "dual") of a pair of two basic mechanisms leads to the duality relation (1.27). Furthermore, we will identify all dual pairs of basic mechanisms.

1.2 Main results

In this section, we state the main results for the system $(X_t)_{t\geq 0}$ of interacting locally regulated diffusions, which solves (1.5), and for the Virgin Island process $(V_t)_{t\geq 0}$, which has been defined in (1.19). First of all, we introduce an appropriate state space for $(X_t)_{t\geq 0}$, namely the *Liggett-Spitzer space* $\mathbb{E}_\sigma$. Then we provide conditions on the regulation function $h\colon [0,\infty) \to \mathbb{R}$ and on the diffusion function

$g\colon [0,\infty) \to [0,\infty)$ which guarantee existence and uniqueness of a strong $\mathbb{E}_\sigma$-valued solution of (1.5).

Unless stated otherwise, we will assume for the migration matrix m appearing in (1.5) that $\sum_{i\in G} m(0,i) = 1$, that m is translation invariant, i.e., $m(i,j) = m(0,j-i)$, and that m is irreducible, i.e., $\forall\, i,j\ \exists\, n\colon m^{(n)}(i,j) > 0$. Let $\alpha \geq 0$. An appropriate state space for (1.2) and (1.5) is provided by a construction going back to Liggett and Spitzer [25]: For given m, let $\sigma = (\sigma_i)_{i\in G}$ be summable and strictly positive such that

$$\sum_{i\in G} \sigma_i m(i,j) \leq C_{LS}\sigma_j, \qquad j \in G, \tag{1.30}$$

for some $C_{LS} < \infty$. With this, define the *Liggett-Spitzer space*

$$\mathbb{E}_\sigma := \big\{ \underline{x} \in [0,\infty)^G \colon \| \underline{x} \|_\sigma := \sum_{i\in G} \sigma_i |x_i| < \infty \big\}. \tag{1.31}$$

Notice that every translation invariant measure μ on $[0,\infty)^G$ with $\int x_0\, \mu(dx) < \infty$ is supported by $\mathbb{E}_\sigma$.

The following assumptions on the regulation function and on the diffusion function guarantee existence and uniqueness of a strong $\mathbb{E}_\sigma$ – valued solution of system (1.5).

Assumption A1. *The functions $h\colon [0,\infty) \to \mathbb{R}$ and $g\colon [0,\infty) \to [0,\infty)$ are locally Lipschitz continuous in $[0,\infty)$ and satisfy $h(0) = g(0) = 0$. In addition, the function h is* upward Lipschitz continuous, *i.e.,*

$$\operatorname{sgn}(x-y)\big(h(x) - h(y)\big) \leq C_h |x-y| \tag{1.32}$$

for all $x, y \geq 0$ and for some constant C_h. Furthermore, g is strictly positive on $(0,\infty)$ and satisfies the growth condition

$$\limsup_{x\to\infty} \frac{\sqrt{g(x)}}{x} < \infty. \tag{1.33}$$

Proposition 1.2.1. *Assume A1. Then, for any $\underline{x} \in \mathbb{E}_\sigma$, the system (1.5) has a unique strong solution $X = (X_t)_{t\geq 0}$ starting in $\underline{x}$ and with paths in $\mathbb{E}_\sigma$ which are a.s. continuous with respect to the norm on $\mathbb{E}_\sigma$.*

This proposition will be proved in Section 2.1. The following theorem, whose proof will be given in Section 2.2, provides for the existence of a maximal process and of a distinguished equilibrium state of (1.5), called the *upper invariant measure*. For the proof of Theorem 1, we will exploit the following assumption. Condition (1.34) ensures that the drift is "sufficiently negative" for large values of $X_t(i)$ so that the process "comes down from ∞".

Assumption A2. *There exists a function* $\hat{h} \geq h$ *such that, for some* $x_0 > 0$, $\hat{h}$ *is negative and concave on* $[x_0, \infty)$ *and satisfies*

$$\int_{x_0}^{+\infty} \frac{1}{-\hat{h}(x)}\, dx < \infty. \tag{1.34}$$

For the interacting Feller diffusions with logistic growth (1.2), the functions h and g are of the form

$$h(x) = \gamma x(K - x), \qquad g(x) = \beta x. \tag{1.35}$$

In this case, Assumptions A1 and A2 are clearly satisfied if $\gamma, \beta > 0$.

To prepare for Theorem 1, we need a bit of notation. If μ_1, μ_2 are probability measures on a partially ordered set S, then we say that μ_1 is *stochastically smaller than or equal to* μ_2, and we write $\mu_1 \leq \mu_2$, if there exists a random pair (Y_1, Y_2) with marginal laws $\mathcal{L}(Y_i) = \mu_i$, $i = 1, 2$ and $Y_1 \leq Y_2$. We say that a sequence of probability measures μ_n *increases stochastically* to a probability measure μ_∞, denoted by $\mu_i \uparrow \mu_\infty$, if there exists a random sequence (Y_i) which a.s. increases to Y_∞ and has marginal distributions $\mathcal{L}(Y_i) = \mu_i$, $i = 1, 2, .., \infty$. Furthermore, a probability measure μ on S is called *associated* if

$$\int f_1 \cdot f_2 \, d\bar{\mu} \geq \int f_1 \, d\bar{\mu} \int f_2 \, d\bar{\mu} \tag{1.36}$$

for all bounded, coordinate-wise nondecreasing $f_1, f_2 \colon \mathbb{E}_\sigma \to \mathbb{R}$.

Theorem 1. *Assume A1 and A2. There exists an* $\mathbb{E}_\sigma$*-valued process* $(X_t^{(\infty)})_{t>0}$ *with the following properties:*

a) *For each* $\varepsilon > 0$, $(X_t^{(\infty)})_{t \geq \varepsilon}$ *is a solution of* (1.5) *starting at time* $t = \varepsilon$.

b) *The first moment of* $\| X_t^{(\infty)} \|_\sigma$ *is finite for every* $t > 0$.

c) *Let* $\underline{x}^{(n)} = (x_i^{(n)})_{i \in G}$, $n = 1, 2, ...$, *be an increasing sequence in* $\mathbb{E}_\sigma$ *such that for all* $i \in G$

$$x_i^{(n)} \uparrow \infty \qquad \text{as } n \to \infty. \tag{1.37}$$

If $(X_t^{(n)})_{t \geq 0}$ *is the solution of* (1.5) *starting in* $\underline{x}^{(n)} \in \mathbb{E}_\sigma$ *at time zero, then*

$$\mathcal{L}\left(X_t^{(n)}\right) \uparrow \mathcal{L}\left(X_t^{(\infty)}\right) \quad \text{as } n \uparrow \infty \qquad (t > 0). \tag{1.38}$$

d) *There exists an equilibrium distribution $\bar{\nu}$ (called the* upper invariant measure*) for the dynamics* (1.5) *such that*

$$\mathcal{L}\left(X_t^{(\infty)}\right) \downarrow \bar{\nu} \quad \textit{as } t \uparrow \infty. \tag{1.39}$$

e) *Any $\mathbb{E}_\sigma$-valued solution $(X_t)_{t\geq 0}$ of* (1.5) *satisfies*

$$\mathcal{L}\big(X_t\big) \leq \mathcal{L}\big(X_t^{(\infty)}\big) \qquad (t > 0). \tag{1.40}$$

In particular, any equilibrium ν is stochastically smaller than or equal to $\bar{\nu}$.

f) *Both the upper invariant measure $\bar{\nu}$ and $\mathcal{L}\big(X_t^{(\infty)}\big)$ are translation invariant and associated.*

Theorem 2 specifies conditions on α, h and g under which the process $(X_t)_{t\geq 0}$ suffers local extinction. A first glance at system (1.2) might tempt one to believe that even for small capacities K (and α fixed), a suitably mobile migration m in the dynamics (1.2) could prevent the system from suffering local extinction. However, Theorem 2 and condition (1.43) below reveal that this is not the case.

Theorem 2. *Assume A1 and A2. Denote by X the solution of equation* (1.5) *for an arbitrarily prescribed initial distribution on $\mathbb{E}_\sigma$. If there exists a concave function $\bar{h} \geq h$ which satisfies*

$$\int_0^\infty \frac{\bar{h}(y)}{g(y)} \exp\left(\int_1^y \frac{-\alpha x + \bar{h}(x)}{g(x)}\, dx\right) dy \leq 0, \tag{1.41}$$

then the process suffers local extinction, i.e.,

$$\mathcal{L}\left(X_t\right) \Longrightarrow \delta_{\underline{0}} \qquad \textit{as } t \to \infty. \tag{1.42}$$

Here, $\underline{0}$ denotes the zero configuration.

In the logistic Feller case (1.35), condition (1.41) simplifies to

$$\int_0^\infty \exp\big(K\gamma y - \frac{\gamma\beta}{2} y^2\big) \cdot \alpha \exp\big(-\alpha y\big)\, dy \leq 1; \tag{1.43}$$

see the proof of Corollary 2.3.2 at the end of Section 2.3.

The proof of Theorem 2 will be given in Section 2.4. Its main idea is a comparison with a *mean field model* corresponding to (1.5), given by the solution M of (1.8). We will show that, for every $t \geq 0$, the marginal distributions of X_t are bounded by the distribution of M_t in the $\leq_{\text{icv}}$ – order (where "icv" stands for "increasing, concave", see [30] for this and related notions). More precisely, in Section 2.4 we will prove the following proposition.

Proposition 1.2.2. *Assume A1 and concavity of h. Let X be a solution of* (1.5) *whose initial distribution* $\bar{\mu}$ *is associated. Assume that the* $X_0(i), i \in G$, *are identically distributed and have finite expectation. Let* $\bar{M}_t = (\bar{M}_t(i))_{i\in G}$ *be a system of processes coupled through the initial state* $\bar{M}_0(i) = X_0(i)$, $i \in G$, *but following independent mean field dynamics, i.e., every* $\bar{M}_t(i)$ *solves equation* (1.8) *with standard Brownian motion* $B(i)$, *where the* $B(i)$, $i \in G$, *are independent. Then*

$$\mathbf{E}^{\bar{\mu}} f\big(X_t\big) \leq \mathbf{E}^{\bar{\mu}} f\big(\bar{M}_t\big), \quad t \geq 0, \tag{1.44}$$

for all bounded, coordinate-wise nondecreasing and concave functions $f\colon \mathbb{E}_\sigma \to \mathbb{R}$ *depending only on finitely many coordinates.*

In the following two theorems, we exploit the specific form of the dynamics (1.2) of the interacting Feller diffusions with logistic growth. As it turns out, the solution of equation (1.2) has a property of *self-duality* which is helpful for the investigation of convergence to equilibria. For the formulation of the self-duality result, write $m^\dagger$ for the transpose of the matrix m, choose a $\sigma^\dagger$ satisfying (1.30) with $m^\dagger$ instead of m, and recall that $\mathbb{E}_{\sigma^\dagger}$ denotes the corresponding Liggett-Spitzer space.

Theorem 3. *Assume* $\beta > 0$. *Let* X *and* $X^\dagger$ *be solutions of* (1.2) *with migration kernels* m *and* $m^\dagger$, *respectively. Then we have the following self-duality:*

$$\mathbf{E}^{\underline{x}} \exp\big(-\frac{\gamma}{\beta}\langle X_t, \underline{y}\rangle\big) = \mathbf{E}^{\underline{y}} \exp\big(-\frac{\gamma}{\beta}\langle \underline{x}, X_t^\dagger\rangle\big) \tag{1.45}$$

for all $\underline{x} \in \mathbb{E}_\sigma, \underline{y} \in \mathbb{E}_{\sigma^\dagger}$, $t \geq 0$.

A similar (though non-self-) duality for *interacting Feller diffusions* (also called *super- random walks*), that is (1.2) with $\gamma = 0$, is given by

$$\mathbf{E}^{\underline{x}} \exp\big(-\langle X_t, \underline{y}\rangle\big) = \exp\left(-\langle \underline{x}, v_t\rangle\right), \tag{1.46}$$

where $v = (v_t(i))$ solves the initial value problem

$$\frac{d}{dt} v_t(i) = \sum_{j\in G} m(i,j)\big(v_t(j) - v_t(i)\big) - v_t(i)^2, \quad i \in G, \qquad v_0 = \underline{y}, \tag{1.47}$$

see e.g. Chapter 4 of [6].

The proof of Theorem 3 is contained in Section 2.5. The main advantage of the self-duality (1.45) is that instead of starting in a configuration with infinite total mass we can analyse the evolution of the process started with finite total mass. For example, choose $\underline{y} = \lambda\delta_0$ and $\underline{x}$ with $x(i) \equiv \text{const}$. Then the self-duality tells us that it makes no difference whether we study the law of $X_t(0)$ started in x, or that of the total mass $|X_t^\dagger| := \sum_i X_t^\dagger(i)$ with $X^\dagger$ started in $\lambda\delta_0$, $\lambda > 0$. This leads to the following corollary (see Lemma 2.5.1 together with Theorem 2):

Corollary 4. *Assume $\beta, \gamma > 0$. Let the parameters α, β, γ, K be such that inequality* (1.43) *holds. Then the solution X of* (1.2) *started from an initial state of finite total mass (i.e., $\sum_i X_0(i) < \infty$) hits $\underline{0}$ in finite time a.s.*

Theorem 3 will be the principal tool for proving convergence to the upper invariant measure specified in Theorem 1. This convergence will be the subject of Theorem 5 below. On an intuitive level, the reason for this convergence is as follows: There are two forces working against each other, supercritical branching and individual competition. The third ingredient is migration which is important for spreading out newly produced mass. Supercritical branching increases mass, whereas competition amongst the individuals decreases it. If a (local) population size is large then competition is stronger, whereas, as long as a local population size is small then competition is negligible in comparison to the mass producing branching. Thus, there should be some attracting equilibrium state in which the two forces balance each other. This is the upper invariant measure.

Theorem 5. *Assume $\beta, \gamma > 0$. Let X be a solution of* (1.2) *and suppose that $\mathcal{L}(X_0) \geq \mu$ where μ is a measure on $\mathbb{E}_\sigma$ which is translation invariant and does not charge the zero configuration $\underline{0}$. Then*

$$\mathcal{L}(X_t) \Longrightarrow \bar{\nu} \qquad \text{as } t \to \infty \tag{1.48}$$

where $\bar{\nu}$ is the upper invariant measure.

From this it is clear that the only extremal translation invariant equilibrium distributions are $\delta_{\underline{0}}$ and $\bar{\nu}$. They coincide in case of local extinction and differ in case of survival. Section 2.6 is devoted to the proof of Theorem 5.

Now we turn to the Virgin Island Model which we introduced in Section 1.1. By Proposition 1.2.1, Assumption A1 guarantees existence and uniqueness of the solution $(Y_t)_{t\geq 0}$ of (1.10). Furthermore, under Assumption A1, zero is an absorbing boundary for (1.10), i.e., $Y_t = 0$ implies $Y_{t+s} = 0$ for all $s \geq 0$. The key ingredient in the construction of the Virgin Island Model is the law of excursions of $(Y_t)_{t\geq 0}$ from the absorbing boundary zero. The excursion measure $\bar{Q}_Y$ is a σ-finite measure on U (defined in (1.11)) and has been constructed by Pitman and Yor [28]. Theorem 6 below proves the approximation result (1.14) which will prove useful in the proofs of our results for the Virgin Island Model. For this approximation, we additionally assume that $(Y_t)_{t\geq 0}$ hits zero in finite time with positive probability. The following assumption formulates a necessary and sufficient condition for this (see Lemma 6.2 of [21]). Recall the scale function $\bar{S}$ from (1.12).

Assumption A3. *The parameter α and the functions g and h satisfy*

$$\int_0^x \bar{S}(y) \frac{1}{g(y)\bar{s}(y)}\, dy < \infty \tag{1.49}$$

for some and then all $x > 0$.

For example, Assumption A1 and Assumption A3 hold whenever $h(y) = \sigma y - \gamma y^2$, $\gamma > 0$ and $g(y) = y^\kappa$ for some $1 \leq \kappa < 2$. Assumption A3 is not met by $h \equiv 0$ and $g(y) = y^2$ because then $\bar{s}(z) = z^\alpha$, $\bar{S}(y) = y^{\alpha+1}/(\alpha+1)$ and condition (1.49) fails to hold.

Pitman and Yor [28] describe the σ-finite measure they construct "in a preliminary way as"

$$\lim_{y \to 0} \frac{1}{\bar{S}(y)} \mathcal{L}^y(Y) \tag{1.50}$$

where the limit indicates weak convergence of finite measures on $\mathbf{C}\big([0,\infty),[0,\infty)\big)$ away from neighbourhoods of the zero-trajectory. Furthermore, they prove that

$$\frac{1}{\bar{S}(y)} \mathbf{E}^y\big[\bar{S}(Y_t) f(Y_t)\big] \to \int f\, d\mu_t \quad \text{as } y \to 0, \qquad \forall\ f \in \mathbf{C}_b([0,\infty)), \tag{1.51}$$

where μ_t is a sub-probability measure on $[0,\infty)$, $t > 0$. We prove the existence of the limit in (1.50) in Theorem 6 below. For this, let the topology on $\mathbf{C}\big([0,\infty),[0,\infty)\big)$ be given by locally uniform convergence. Furthermore, recall the definition of U from (1.11) and the definition of $\bar{S}$ from (1.12).

Theorem 6. *Assume A1 and A3. Then there exists a* σ*-finite measure* $\bar{Q}_Y$ *on* U *such that*

$$\lim_{y \to 0} \frac{1}{\bar{S}(y)} \mathbf{E}^y F(Y) = \int F(\chi) \bar{Q}_Y(d\chi) \tag{1.52}$$

for all bounded continuous $F\colon \mathbf{C}\big([0,\infty),[0,\infty)\big) \to \mathbb{R}$ *for which there exists an* $\varepsilon > 0$ *such that* $F(\chi) = 0$ *whenever* $\sup_{t \geq 0} \chi_t < \varepsilon$.

For our proof of a global extinction result for the Virgin Island Model, we need to have that the scaling function $\bar{S}$ in (1.52) essentially behaves linear in a neighbourhood of zero. More precisely, we need to assume that $\bar{S}'(0)$ exists in $(0,\infty)$. Looking at the definition (1.12) of $\bar{S}$, we see that a sufficient condition for this is given by the following assumption.

Assumption A4. *The integral* $\int_\varepsilon^1 \frac{-\alpha y + h(y)}{g(y)}\, dy$ *has a limit in* $(-\infty,\infty)$ *as* $\varepsilon \to 0$.

It follows from dominated convergence and from the local Lipschitz continuity of h that Assumption A4 holds if $\int_0^1 \frac{y}{g(y)}\, dy$ is finite.

Recall the definition of the Virgin Island process $(V_t)_{t \geq 0}$ and of the n-th generation process $(V_t^{(n)})_{t \geq 0}$ from (1.19) and (1.18), respectively. Lemma 3.3.1 shows that V_t is finite almost surely for every $t \geq 0$. In the next theorem, we give a criterion for extinction of the Virgin Island process. As mentioned in the introduction, the decisive parameter is the expected area under an excursion of Y. The following short calculation gives an idea why this is the right quantity. By equation (1.18), the expected total man-hours of the $(n+1)$-st generation are

$$\begin{aligned} \mathbf{E}^x \int_0^\infty V_s^{(n+1)}\,ds &= \mathbf{E}^x \int \Big(\int_0^\infty \chi_{t-s}\,dt\Big)\Pi^{(n)}(ds,d\chi) \\ &= \mathbf{E}^x \int_0^\infty \int \Big(\int_s^\infty \chi_{t-s}\,dt\Big) Q_Y(d\chi)\alpha V_s^{(n)}\,ds \\ &= \alpha \int_0^\infty \int \chi_t Q_Y(d\chi)\,dt \cdot \mathbf{E}^x \int_0^\infty V_s^{(n)}\,ds. \end{aligned} \tag{1.53}$$

Thus, α times the expected area under an excursion of Y is equal to the ratio of the expected area under the path of the $(n+1)$-st generation process and the expected area under the path of the n-th generation process.

For Theorem 7 and Theorem 8, we assume that the expected man-hours of Y are finite. Lemma 3.1.7 shows that, under Assumptions A1 and A3, an equivalent condition for this is given in Assumption A5 below.

Assumption A5. *The parameter α and the functions g and h satisfy*

$$\int_x^\infty \frac{y}{g(y)\bar{s}(y)}\,dy < \infty \tag{1.54}$$

for some and then for all $x > 0$.

We mention that if Assumptions A1, A3 and A5 hold, then the process Y hits zero in finite time almost surely (see Lemma 3.1.6 and Lemma 3.1.7). A generic example for h and g is $h(y) = c_1 y^{\kappa_1} - c_2 y^{\kappa_2}$, $g(y) = c_3 y^{\kappa_3}$ with $c_1, c_2, c_3 > 0$. The Assumptions A1, A2, A3, A4 and A5 are all satisfied if $\kappa_2 > \kappa_1 \geq 1$ and if $\kappa_3 \in [1, 2)$.

For the formulation of the extinction result, we define

$$s(z) := \exp\Big(-\int_0^z \frac{-\alpha x + h(x)}{g(x)}\,dx\Big), \quad S(y) := \int_0^y s(z)\,dz, \quad z, y > 0, \tag{1.55}$$

which is well-defined under Assumption A4. Notice that $\bar{S}(y) = S(y)\bar{S}'(0)$. Recall the Virgin Island process from (1.19) and the excursion measure Q_Y from (1.16).

Theorem 7. *Assume A1, A3, A4 and A5. Then the Virgin Island process $(V_t)_{t\geq 0}$ started in $x > 0$ dies out (i.e., converges in probability to zero as $t \to \infty$) iff*

$$\int \int_0^\infty \alpha \chi_s\,ds\,Q_Y(d\chi) \leq 1. \tag{1.56}$$

The expression on the left-hand side may be explicitly expressed in terms of α, h *and* g *as*

$$\int_0^\infty \frac{\alpha y}{g(y)s(y)}\,dy = \int\int_0^\infty \alpha\chi_s\,ds\,Q_Y(d\chi). \tag{1.57}$$

In case of survival, V_t *converges weakly as* $t \to \infty$ *to a random variable* V_∞ *satisfying*

$$\mathbf{P}^x(V_\infty = 0) = 1 - \mathbf{P}^x(V_\infty = \infty) = \mathbf{E}^x \exp\left(-q\int_0^\infty \alpha Y_s\,ds\right) \tag{1.58}$$

for all $x \geq 0$ *and some* $q > 0$.

In the critical case, that is, equality in (1.56), V_t converges to zero as $t \to \infty$. However, it turns out that the expected area under the graph of V is infinite. Furthermore, we obtain in Theorem 8 the asymptotic behaviour of the expected man-hours of V up to time t as $t \to \infty$. For this, define

$$w(x) := \int_0^\infty S(x \wedge z)\frac{z}{g(z)s(z)}\,dz,\ \ x \geq 0. \tag{1.59}$$

If Assumptions A1, A3, A4 and A5 hold, then $w(x)$ is finite for fixed $x < \infty$; see Lemma 3.1.7.

Theorem 8. *Assume A1, A3, A4 and A5. If the left-hand side of* (1.56) *is strictly smaller than one, then, for all* $x \geq 0$, *the expected value of the total man-hours of* V *is equal to*

$$\mathbf{E}^x\int_0^\infty V_s\,ds = \frac{\mathbf{E}^x\left(\int_0^\infty Y_s\,ds\right)}{1 - \int\left(\int_0^\infty \alpha\chi_s\,ds\right)Q_Y(d\chi)} = \frac{w(x)}{1 - \int_0^\infty \frac{\alpha z}{g(z)s(z)}\,dz}, \tag{1.60}$$

which is finite. Otherwise, the left-hand side of (1.60) *is infinite. In the critical case, that is, equality in* (1.56),

$$\frac{1}{t}\int_0^t \mathbf{E}^x V_s\,ds \to \frac{\mathbf{E}^x\left(\int_0^\infty Y_u\,du\right)}{\int\left(\int_0^\infty u\alpha\chi_u\,du\right)Q_Y(d\chi)} = \frac{w(x)}{\int_0^\infty \frac{\alpha w(y)}{g(y)s(y)}dy} \in [0,\infty) \tag{1.61}$$

as $t \to \infty$ *where the right-hand side is interpreted as zero if the denominator is equal to infinity. In the supercritical case, i.e., if* (1.56) *fails to be true, let* $\beta > 0$ *be such that*

$$\int_0^\infty e^{-\beta u}\int \alpha\chi_u Q_Y(d\chi)\,du = 1. \tag{1.62}$$

Then the order of convergence of the expected man-hours of V up to time t can be read off from

$$e^{-\beta t}\int_0^t \mathbf{E}^x V_s\,ds \to \frac{\int_0^\infty e^{-\beta u}\mathbf{E}^x \int_0^u Y_s\,ds\,du}{\int_0^\infty u e^{-\beta u}\int \alpha\chi_u Q_Y(d\chi)du} \in (0,\infty) \tag{1.63}$$

as $t\to\infty$ for all $x\geq 0$.

Remark 1.2.3. *The parameter β defined in* (1.62) *is called Malthusian parameter (see [19]).*

1.3 Outline

Fast readers may want to proceed directly to the proof of a specific theorem. Theorem 1, Theorem 2, Theorem 3 and Theorem 5 will be established in Section 2.2, Section 2.4, Section 2.5 and Section 2.6, respectively. The proof of Proposition 1.2.2 is contained in Section 2.4. Furthermore, we prove Theorem 6, Theorem 7 and Theorem 8 in Chapter 3, more precisely in Section 3.1, Section 3.4 and Section 3.2, respectively.

The main results of Chapter 2 are the local extinction result for interacting locally regulated diffusions (Theorem 2) and the convergence result of interacting Feller diffusions with logistic growth (Theorem 5). In Section 2.1, we obtain the existence and uniqueness result of Proposition 1.2.1. Furthermore, Lemma 2.1.3 provides for a comparison of two solutions of equation (1.5) which differ in the regulation function h. This comparison result is an important ingredient in the proof of the existence of the maximal process (Theorem 1) which is included in Section 2.2. Section 2.3 contains an extinction result for the mean field model (1.8). The main step for this is Proposition 2.3.1 which determines the number – depending on the parameters – of equilibrium distributions of the mean field model (1.8). Furthermore, the calculations of Proposition 2.3.1 yield the condition for local extinction, that is, (1.41). Lemma 2.3.3 exploits the properties of the maximal process to conclude that the mean field process dies out if there is no nontrivial equilibrium distribution. Section 2.4 establishes Proposition 1.2.2, that is, the comparison between the mean field model and the system of interacting locally regulated diffusions. Together with the results of Section 2.3, this leads to a proof of Theorem 2. The self-duality stated in Theorem 3 is the key ingredient in our proof of the ergodicity result of Theorem 5. Section 2.5 contains an analytical proof of Theorem 3, and Section 2.6 establishes Theorem 5.

The Virgin Island Model is the subject of Chapter 3. Section 3.1 is devoted to the one-dimensional diffusion (1.10). After proving Theorem 6, we calculate the explicit formulas of both Theorem 7 and of Theorem 8 in Lemma 3.1.3 and in Lemma 3.1.5, respectively. In Section 3.2, we prove Theorem 8 which specifies the asymptotic behaviour of the expected man-hours of V up to time t as $t \to \infty$. Section 3.3 contains the key lemma for the extinction result of Theorem 7. More precisely, we prove in Lemma 3.3.2 that the Laplace transform of the Virgin Island process satisfies a certain integro-differential equation. This equation will then be used in Section 3.4 to prove Theorem 7.

In Chapter 4, we obtain a graphical representation of the two duality relations (1.24) and (1.26). The definition of duality of a pair of basic mechanisms is contained in Section 4.2. In the same section, we construct processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$, which satisfy equation (1.27), by means of a graphical representation. From (1.27), the prototype duality (1.28) is derived in Section 4.3. Finally, we show the convergence of the approximating processes in Section 4.4.

Chapter 2

Local extinction and ergodic behaviour

The system $(X_t)_{t\geq 0}$ of interacting locally regulated diffusions is the solution of equation (1.5). Its state space is the Liggett-Spitzer space $\mathbb{E}_\sigma$ which has been defined in Section 1.2. In Section 2.1, we prove Proposition 1.2.1 which claims existence and uniqueness of a strong solution of (1.5). In the same section, Lemma 2.1.3 provides for a comparison of two solutions of equation (1.5) which differ in the regulation function h. This comparison result is the key ingredient in the proof of the existence of the maximal process (Theorem 1) which we prove in Section 2.2.

In Section 2.4, we prove the local extinction result of Theorem 2. The main steps for this are as follows. An application of Theorem 1 will show that we may assume that $\mathcal{L}(X_0)$ satisfies the assumptions of Proposition 1.2.2, which we prove in Section 2.4. Proposition 1.2.2 asserts that $(X_t)_{t\geq 0}$ is dominated by the mean field model $(M_t)_{t\geq 0}$ which is the solution of (1.8). Hence, it suffices to establish an extinction result for $(M_t)_{t\geq 0}$ which is included in Section 2.3.

The proof of the convergence result of Theorem 5 consists of two steps. First, we prove the duality relation (1.45) of Theorem 3 in Section 2.5. By Theorem 3, it suffices to consider the total mass process defined by $|X_t| := \sum_{i\in\mathbb{Z}^d} X_t(i)$, $t \geq 0$. The second step is to prove that the total mass process with probability one either converges to zero or converges to infinity, see Lemma 2.6.1. Both the proof of Lemma 2.6.1 and the proof of Theorem 5 are contained in Section 2.6.

2.1 Preliminaries

For the proof of existence and uniqueness of the solution of equation (1.5), we need three preliminary lemmas. In the first two of these, we obtain bounds on the first moment and on the second moment of X. For this, we define

$$b_i(\underline{x}) := \alpha\Big(\sum_{j\in G} m(i,j)x_j - x_i\Big) + h(x_i), \quad \underline{x} \in \mathbb{E}_\sigma, \tag{2.1}$$

where $\sigma = (\sigma_i)_{i\in G}$ satisfies (1.30). Denote $z^+ := z \vee 0$. By inequality (1.30) and Assumption A1, there exists a finite constant C_1 such that

$$\sum_{i\in M} \sigma_i \mathbb{1}_{x_i - y_i \geq 0}\big(b_i(\underline{x}) - b_i(\underline{y})\big) \leq C_1 \|\, (x_\cdot - y_\cdot)^+ \,\|_\sigma, \qquad \forall\, \underline{x}, \underline{y} \in \mathbb{E}_\sigma \tag{2.2}$$

for every subset $M \subseteq G$. From inequality (2.2), we will obtain monotonicity in the initial configuration. This monotonicity is a crucial property which we will exploit several times. First, we prove boundedness of second moments.

Lemma 2.1.1. *Suppose that h and g satisfy Assumption A1. Let (X_t) be any weak solution of equation (1.5) with $\mathbf{E}\,\|\, X_0 \,\|_\sigma^2 < \infty$, whose paths are continuous in $\mathbb{E}_\sigma$. Then there exists a constant $C < \infty$ such that for each $T \geq 0$*

$$\sup_{t\leq T} \mathbf{E}\,\|\, X_t \,\|_\sigma^2 \leq \big(1 + \mathbf{E}\,\|\, X_0 \,\|_\sigma^2\big)e^{CT} < \infty. \tag{2.3}$$

Proof. Let G_k be finite subsets of G which monotonically exhaust G as $k \to \infty$. Denote $\|\, x \,\|_{\sigma,k} := \sum_{i\in G_k} \sigma_i |x_i|$. Applying Itô's formula, we obtain

$$\begin{aligned} d\,\|\, X_t \,\|_{\sigma,k}^2 &= 2\,\|\, X_t \,\|_{\sigma,k} \sum_{i\in G_k} \sigma_i \Big(b_i(X_t)\,dt + \sqrt{2g\big(X_t(i)\big)}\,dB_t(i)\Big) \\ &\quad + 2\sum_{i\in G_k} \sigma_i^2 g\big(X_t(i)\big)\,dt. \end{aligned} \tag{2.4}$$

Let $n \in \mathbb{N}$. The continuous function g is bounded on the interval $[0, n/\sigma_i]$ for every $i \in G_k$. Thus, the stochastic integrals on the right hand side of (2.4) are L_2-martingales when stopped at time $\tau_n := \inf_{t\geq 0}\{\|\, X_t \,\|_\sigma \geq n\}$. By path continuity, we have $\tau_n \to \infty$ as $n \to \infty$ almost surely. Taking expectations, inequality (2.2) with $\underline{y} = \underline{0}$ implies

$$\mathbf{E}\|\, X_{t\wedge\tau_n} \,\|_{\sigma,k}^2 \leq \mathbf{E}\|\, X_0 \,\|_{\sigma,k}^2 + 2\mathbf{E}\int_0^{t\wedge\tau_n} \Big(C_1 \|\, X_s \,\|_\sigma^2 + \sum_{i\in G} \sigma_i^2 g\big(X_s(i)\big)\Big)\,ds. \tag{2.5}$$

By the growth condition (1.33), we know that $g(x) \leq C_2(1 + x^2)$ for some constant $C_2 < \infty$. Letting $k \to \infty$ and using monotone convergence, we obtain

$$\mathbf{E}\|\, X_{t\wedge\tau_n} \,\|_\sigma^2 \leq \mathbf{E}\|\, X_0 \,\|_\sigma^2 + C_3 \int_0^t \Big(1 + \mathbf{E}\|\, X_{s\wedge\tau_n} \,\|_\sigma^2 \Big)\,ds \tag{2.6}$$

for some constant $C_3 < \infty$. Applying Gronwall's inequality to the function $t \mapsto 1 + \mathbf{E}\| X_{t\wedge\tau_n} \|_\sigma^2$, we arrive at

$$\mathbf{E}\| X_{t\wedge\tau_n} \|_\sigma^2 \le \big(1 + \mathbf{E}\| X_0 \|_\sigma^2\big) e^{C_3 t} - 1. \tag{2.7}$$

Letting $n \to \infty$, Fatou's lemma completes the proof. □

In the proof of Proposition 1.2.1, we need a stronger uniformity than Lemma 2.1.1 provides.

Lemma 2.1.2. *Assume A1. Let* (X_t) *be any weak solution of equation* (1.5) *satisfying condition* (2.3)*. Then, for each* $T \ge 0$*, there exists a constant* $\tilde{C}_T < \infty$ *such that*

$$\mathbf{E}\sup_{t\le T} \| X_t \|_\sigma \le \tilde{C}_T\big(1 + \mathbf{E} \| X_0 \|_\sigma + \mathbf{E} \| X_0 \|_\sigma^2\big) < \infty. \tag{2.8}$$

Proof. Recall the definition of G_k and $\| \,.\, \|_{\sigma,k}$ from the proof of Lemma 2.1.1. Multiplying by σ_i and summing over $i \in G_k$ in (1.5), we obtain for $t \le T$

$$\| X_t \|_{\sigma,k} - \| X_0 \|_{\sigma,k} = \int_0^t \sum_{i\in G_k} \sigma_i b_i(X_s)\, ds + \int_0^t \sum_{i\in G_k} \sigma_i \sqrt{2g\big(X_s(i)\big)}\, dB_s(i) \tag{2.9}$$

The estimate (2.2) implies that $\sum_{i\in G_k} \sigma_i b_i(X_s) \le C_1 \| X_s \|_\sigma$. Thus, denoting the rightmost term in (2.9) by M_t^k, we obtain

$$\sup_{u\le t} \| X_u \|_{\sigma,k} \le \| X_0 \|_\sigma + \int_0^t C_1 \sup_{r\le s} \| X_r \|_\sigma \, ds + \sup_{u\le T} |M_u^k|. \tag{2.10}$$

The process (M_t^k) is an L_2-martingale since, by the assumption $g(x) \le C(1+x^2)$ and condition (2.3), the integrands $\sqrt{2g(X_s(i))}$ in (2.9) are square integrable, and the second moment $\mathbf{E}|M_T^k|^2 = \int_0^T 2 \sum_{i\in G_k} \sigma_i^2 \mathbf{E} g\big(X_s(i)\big)\, ds$ is bounded by $\bar{C}_T(1 + \mathbf{E}\| X_0 \|_\sigma^2)$ for some constant $\bar{C}_T$. Thus, using the estimate $z \le 1 + z^2$, we conclude from Doob's L_2-inequality that

$$\mathbf{E}\sup_{u\le T} |M_u^k| \le 1 + \mathbf{E}|M_T^k|^2 \le 1 + \bar{C}_T(1 + \mathbf{E}\| X_0 \|_\sigma^2). \tag{2.11}$$

Therefore, taking expectations in (2.10) and applying monotone convergence, we obtain

$$\mathbf{E}\sup_{u\le t} \| X_u \|_\sigma \le \mathbf{E}\| X_0 \|_\sigma + C_1 \int_0^t \mathbf{E}\sup_{r\le s} \| X_r \|_\sigma \, ds + 1 + \bar{C}_T\big(1 + \mathbf{E} \| X_0 \|_\sigma^2\big) \tag{2.12}$$

for all $t \le T$. Now the assertion follows from Gronwall's inequality. □

The following monotone coupling lemma will be an important tool.

Lemma 2.1.3. *Let h_1, h_2 and g satisfy Assumption A1, and let $B = \big(B(i)\big)_{i\in G}$ be a system of independent Brownian motions defined on some filtered probability space. For $\iota = 1,2$, assume that X^ι is defined on the same probability space, satisfies equation (1.5) with Brownian motions $B(i)$, drift function h_ι and initial configuration $\underline{x}^\iota \in \mathbb{E}_\sigma$, and has continuous paths in $\mathbb{E}_\sigma$. Then*

$$h_1 \le h_2 \text{ together with } \underline{x}^1 \le \underline{x}^2 \quad \text{implies} \quad X_t^1 \le X_t^2 \;\; \forall\, t \ge 0 \text{ a.s.} \tag{2.13}$$

Proof. The first part of the proof follows that of Theorem 3.2 in [18]. Let $1 > a_1 > \cdots > a_n > \cdots > 0$ be defined by

$$\int_{a_1}^{1} \frac{1}{u}\,du = 1,\; \int_{a_2}^{a_1} \frac{1}{u}\,du = 2, ..., \int_{a_n}^{a_{n-1}} \frac{1}{u}\,du = n, ... \tag{2.14}$$

Notice that $a_n \to 0$ as $n \to \infty$. For every $n = 1, 2, ..,$ define a continuous function $\psi_n(u)$ with support in (a_n, a_{n-1}) such that

$$0 \le \psi_n(u) \le \frac{2}{nu} \text{ and } \int_{a_n}^{a_{n-1}} \psi_n(u)\,du = 1. \tag{2.15}$$

Furthermore, define

$$\phi_n(x) := \mathbb{1}_{x>0} \int_0^x dy \int_0^y \psi_n(u)\,du, \quad x \in \mathbb{R}. \tag{2.16}$$

These functions satisfy $\phi_n \in C^2(\mathbb{R})$, $|\phi_n'(x)| \le 1$, $\phi_n''(x) = \mathbb{1}_{x>0}\psi_n(x)$, $\phi_n(x) \le x^+$ and $\phi_n(x) \to x^+$ as $n \to \infty$. Fix $i \in G$ and let $\tau_k := \inf\{t \ge 0\colon X_t^1(i) \vee X_t^2(i) \ge k\}$. Write $\Delta_t^i := X_t^1(i) - X_t^2(i)$ and let b_i^ι be as in equation (2.1) with h replaced by h_ι, $\iota = 1, 2$. By Itô's formula,

$$\begin{aligned}
&\phi_n\big(\Delta^i_{t\wedge\tau_k}\big) - \phi_n\big(\Delta^i_0\big) \\
&= \int_0^{t\wedge\tau_k} \phi_n'\big(\Delta_s^i\big)\Big[\sqrt{2g\big(X_s^1(i)\big)} - \sqrt{2g\big(X_s^2(i)\big)}\Big]\,dB_s(i) \\
&+ \int_0^{t\wedge\tau_k} \phi_n'\big(\Delta_s^i\big)\Big[b_i^1\big(X_s^1\big) - b_i^2\big(X_s^2\big)\Big]\,ds \\
&+ \frac{1}{2}\int_0^{t\wedge\tau_k} \phi_n''\big(\Delta_s^i\big)\Big[\sqrt{2g\big(X_s^1(i)\big)} - \sqrt{2g\big(X_s^2(i)\big)}\Big]^2\,ds.
\end{aligned} \tag{2.17}$$

As $n \to \infty$, the left hand side converges to $\big(\Delta^i_{t\wedge\tau_k}\big)^+ - \big(\Delta^i_0\big)^+$ in L_1 by dominated convergence and Lemma 2.1.1. In the rest of the proof, $C_1, C_2, ...$ will be suitably chosen finite constants. By Assumption A1, there exists a constant C_1 such that $g(x) \le C_1(1+x^2)$. Thus, Lemma 2.1.1 implies $\mathbf{E} g(X^\iota_t(i)) < \infty$ and we have by dominated convergence

$$\mathbf{E}\int_0^{t\wedge\tau_k}\left(\mathbb{1}_{\Delta^i_s>0}-\phi^{'}_n(\Delta^i_s)\right)^2\left(\sqrt{2g(X^1_s(i))}-\sqrt{2g(X^2_s(i))}\right)^2 ds \xrightarrow{n\to\infty} 0. \tag{2.18}$$

Hence, the first (stochastic) integral on the right hand side converges in L_2 to the same expression with $\phi^{'}_n(x)$ replaced by $\mathbb{1}_{x>0}$. For the second integral, notice that b^ι_i is globally Lipschitz continuous on $\{\underline{x}\colon x_i \le k\}$. Thus, for $s \le \tau_k$, $|b^\iota_i(X^\iota_s)|$ is bounded by $C_2\| X^\iota_s \|_\sigma$, which has finite expectation by Lemma 2.1.1, and we obtain by dominated convergence

$$\int_0^{t\wedge\tau_k}\left|\mathbb{1}_{\Delta^i_s>0}-\phi^{'}_n(\Delta^i_s)\right| \cdot \left|b^1_i(X^1_s)-b^2_i(X^2_s)\right| ds \xrightarrow{n\to\infty} 0. \tag{2.19}$$

Finally, we consider the third integral on the right hand side of equation (2.17). The local Lipschitz continuity of g implies that $\sqrt{g}$ is globally 1/2-Hölder continuous on the interval $[0,k]$. Therefore, the last integral in (2.17) is bounded by

$$\int_0^{t\wedge\tau_k}\frac{2}{n|\Delta^i_s|}\cdot C_3|\Delta^i_s| \le \frac{2C_3t}{n} \to 0 \quad \text{as } n\to\infty. \tag{2.20}$$

Putting these calculations together, equation (2.17) implies

$$\begin{aligned}(\Delta^i_t)^+ - (\Delta^i_0)^+ &= \int_0^t \mathbb{1}_{\Delta^i_s>0}\left[\sqrt{2g\big(X^1_s(i)\big)}-\sqrt{2g\big(X^2_s(i)\big)}\right] dB_s(i)\\ &\quad+\int_0^t \mathbb{1}_{\Delta^i_s>0}\left[b^1_i\big(X^1_s\big)-b^2_i\big(X^2_s\big)\right] ds\end{aligned} \tag{2.21}$$

for all $t \le \tau_k$ almost surely. By path continuity, we have $\tau_k \to \infty$ almost surely as $k\to\infty$ and thus, equation (2.21) holds for all $t \ge 0$. The stochastic integral on the right hand side is an L_2-martingale because of $g(x) \le C_1(1+x^2)$ and Lemma 2.1.1. Taking expectations, we arrive at

$$\begin{aligned}&\mathbf{E}\left[\sum_{i\in G}\sigma_i\big(\Delta^i_t\big)^+ - \sum_{i\in G}\sigma_i\big(\Delta^i_0\big)^+\right]\\ &= \int_0^t \mathbf{E}\sum_{i\in G}\sigma_i\mathbb{1}_{\Delta^i_s>0}\left[b^1_i\big(X^1_s\big)-b^2_i\big(X^2_s\big)\right] ds\\ &\le C_4\int_0^t \mathbf{E}\sum_{i\in G}\sigma_i\big(\Delta^i_s\big)^+ ds.\end{aligned} \tag{2.22}$$

In the last step, we used $b_i^1 \leq b_i^2$ and inequality (2.2). By Gronwall's inequality, we obtain

$$\mathbf{E}\,\|\,\big(\Delta_t^i\big)^+\,\|_\sigma \leq \|\,\big(\Delta_0^i\big)^+\,\|_\sigma e^{C_4 t}, \quad i \in G. \tag{2.23}$$

For later use, we note that this inequality implies

$$\mathbf{E}\,\|\,\Delta_t^i\,\|_\sigma \leq \|\,\Delta_0^i\,\|_\sigma e^{C_4 t}, \quad i \in G. \tag{2.24}$$

if $b_1 = b_2$. For this, notice that $|x_i^1 - x_i^2| = (x_i^1 - x_i^2)^+ + (x_i^2 - x_i^1)^+$. The right hand side of inequality (2.23) is zero by the assumption $x^1 \leq x^2$, which finishes the proof of the monotonicity result for fixed $t \geq 0$. Finally, $X_t^1 \leq X_t^2$ follows for all $t \in \mathbb{Q}_{\geq 0}$ and then by continuity of paths for all $t \geq 0$ almost surely. □

Proof of Proposition 1.2.1. Let $B = (B_i)_{i \in G}$ be a system of independent Brownian motions, and fix an initial condition $\underline{x} \in \mathbb{E}_\sigma$. We will prove existence of a solution of (1.5) similarly as in [12], where the system (1.5) is studied in the case $h = 0$. To this end, for finite $\Lambda \subseteq G$ and $i, j \in G$, we define $m^\Lambda(i,j) := m(i,j)\mathbb{1}_{i,j\in\Lambda}$ and consider the finite dimensional system

$$\begin{aligned} dX_t^\Lambda(i) =&\alpha \sum_{j\in\Lambda} m^\Lambda(i,j) X_t^\Lambda(j)\,dt - \alpha X_t^\Lambda(i)\,dt \\ &+ h\big(X_t^\Lambda(i)\big)dt + \sqrt{2\cdot g\big(X_t^\Lambda(i)\big)}\,dB_t(i), \qquad i \in \Lambda. \end{aligned} \tag{2.25}$$

Under Assumption A1, equation (2.25) has a unique solution X^Λ starting in $(x_i)_{i\in\Lambda}$. We extend X^Λ to an infinite sequence (still denoted by the same symbol) by putting $X_t^\Lambda(i) := 0$ for $i \in G \setminus \Lambda$. Following the arguments in the proof of Theorem 1 in [12], one can show that there exists a process $X = (X_t(i))$ arising as the monotone limit

$$X_t^\Lambda(i) \uparrow X_t(i) \text{ as } \Lambda \uparrow G. \tag{2.26}$$

To show that X has a.s. continuous paths in $\mathbb{E}_\sigma$, we first note that for each finite $\Lambda \subseteq G$ the process X^Λ, being a finite dimensional diffusion, has a.s. continuous paths and therefore satisfies

$$\lim_{\delta\to 0} \mathbf{P}\Big(\sup_{|t-s|\leq\delta,\, s,t\leq T} \|\,X_t^\Lambda - X_s^\Lambda\,\|_\sigma \geq \varepsilon\Big) = 0 \tag{2.27}$$

for all $\varepsilon > 0$ and $T > 0$.

For all finite $\Lambda \subseteq G$, the process X^Λ satisfies the assumptions of Lemma 2.1.1, with $m(i,j)$ in (1.5) replaced by $m^\Lambda(i,j)$. Consequently, X^Λ also satisfies (2.3), where the constant C can be chosen uniformly in Λ. Therefore, by the monotone convergence (2.26), X satisfies (2.3) and, due to Lemma 2.1.2, also (2.8).

Next, we set out to show that for all $\varepsilon > 0$ and $T \geq 0$

$$\lim_{\Lambda \uparrow G} \mathbf{P}\big(\sup_{t \leq T} \| X_t - X_t^\Lambda \|_\sigma \geq \varepsilon\big) = 0. \tag{2.28}$$

For this purpose, let G_k and $\| x \|_{\sigma,k}$ be as in the proof of Lemma 2.1.1. From (2.26) together with the a.s. component-wise continuity of X and Dini's theorem we conclude that for all $T > 0$ and $k \in \mathbb{N}$:

$$\sup_{t \leq T} \| X_t - X_t^\Lambda \|_{\sigma,k} \to 0 \text{ a.s. as } \Lambda \uparrow G. \tag{2.29}$$

By (2.8) and dominated convergence we therefore have

$$\mathbf{E} \sup_{t \leq T} \| X_t - X_t^\Lambda \|_{\sigma,k} \to 0 \text{ a.s. as } \Lambda \uparrow G. \tag{2.30}$$

For every finite $\Lambda \subseteq G$ and $k \in \mathbb{N}$ we estimate

$$\mathbf{E} \sup_{t \leq T} \| X_t - X_t^\Lambda \|_\sigma \leq \mathbf{E} \sup_{t \leq T} \| X_t - X_t^\Lambda \|_{\sigma,k} + 2\mathbf{E} \sup_{t \leq T} \sum_{i \notin G_k} \sigma_i X_t(i). \tag{2.31}$$

The rightmost term in (2.31) does not depend on Λ and converges to 0, again because of (2.8) and dominated convergence. Together with (2.30) this implies that the left hand side of (2.31) converges to zero, and proves (2.28).

For ε, δ and $T > 0$ we have the estimate

$$\begin{aligned} \mathbf{P}\big(\sup_{|t-s| \leq \delta,\, s,t \leq T} \| X_t - X_s \|_\sigma \geq 3\varepsilon\big) \\ \leq \mathbf{P}\big(\sup_{|t-s| \leq \delta,\, s,t \leq T} \| X_t^\Lambda - X_s^\Lambda \|_\sigma \geq \varepsilon\big) + 2\mathbf{P}\big(\sup_{t \leq T} \| X_t - X_t^\Lambda \|_\sigma \geq \varepsilon\big). \end{aligned} \tag{2.32}$$

Because of (2.27) and (2.28) the left hand side of (2.32) converges to 0 as $\delta \to 0$. This implies almost sure pathwise continuity.

For uniqueness, we proceed as follows. In the situation of Lemma 2.1.3, choose $h_1 = h_2$ and $x^1 = x^2$. Then pathwise uniqueness follows by applying Lemma 2.1.3 twice. Uniqueness in law and strong existence follow then from a Yamada-Watanabe type argument (see [31], Theorem 2.2). For the existence of a strong solution, it remains to show that the dependence of the unique solution on the initial configuration is measurable. This follows from the monotonicity result of Lemma 2.1.3. □

Lemma 2.1.4. *Let h and g satisfy Assumption A1. The strong solution X_t of system* (1.5) *is monotonically continuous in its initial configuration in the following sense: Let $\underline{x}^{(n)}, \underline{x} \in \mathbb{E}_\sigma$ be the starting points of $X_t^{(n)}$ and X_t, such that*

$$\underline{x}^{(n)} \uparrow (\downarrow)\ \underline{x} \quad \textit{as } n \uparrow \infty. \tag{2.33}$$

Then

$$X_t^{(n)} \uparrow (\downarrow)\ X_t \quad \forall\, t \geq 0 \quad as \quad n \uparrow \infty \quad a.s. \tag{2.34}$$

Proof. In equation (2.24), let $h^1 = h^2 := h$, $X_t^1 := X_t$ and $X_t^2 := X_t^{(n)}$. Letting $n \to \infty$, this implies L_1-convergence of $X_t - X_t^{(n)}$ for fixed time $t \geq 0$. The monotonicity result of Lemma 2.1.3 finishes the proof. □

2.2 The upper invariant measure. Proof of Theorem 1

Proof of Theorem 1. To fix notation, let us write $\mathcal{L}^{\underline{x}}(X_t)$ for the distribution of X_t (the solution of (1.5)) starting from an element $\underline{x} \in \mathbb{E}_\sigma$. For $N \in \mathbb{N}$ we define the element $\underline{N} \in \mathbb{E}_\sigma$ by $\underline{N}(i) \equiv N$, $i \in G$. Let $X_t^{\underline{N}}$ be the process started from $\underline{N}$. By Lemma 2.1.3, the sequence $X_t^{\underline{N}}$ is nondecreasing in N for all $t > 0$; let us write $X_t^{(\infty)}$ for its a.s. limit.

Now let $(\underline{x}^{(n)})$ be a sequence as in Theorem 1(c). For all $n \in \mathbb{N}$ we conclude from Lemma 2.1.4 that

$$\mathcal{L}\left(X_t^{(\infty)}\right) \underset{N\to\infty}{\nwarrow} \mathcal{L}^{\underline{N}}(X_t) \geq \mathcal{L}^{\underline{x}^{(n)}\wedge N}(X_t) \underset{N\to\infty}{\nearrow} \mathcal{L}^{\underline{x}^{(n)}}(X_t) \tag{2.35}$$

Again by Lemma 2.1.4 we obtain for all $N \in \mathbb{N}$

$$\mathcal{L}^{\underline{x}^{(n)}\wedge N}(X_t) \underset{n\to\infty}{\nearrow} \mathcal{L}^{\underline{N}}(X_t) \tag{2.36}$$

Thus, by a diagonal argument, there is a subsequence $\underline{x}^{(n_N)}$ of $\underline{x}^{(n)}$ such that

$$\mathcal{L}^{\underline{x}^{(n_N)}\wedge N}(X_t) \underset{N\to\infty}{\nearrow} \mathcal{L}\left(X_t^{(\infty)}\right). \tag{2.37}$$

Together with inequality (2.35) and monotonicity (Lemma 2.1.3), this results in

$$\mathcal{L}^{\underline{x}^{(n_N)}}(X_t) \underset{N\to\infty}{\nearrow} \mathcal{L}\left(X_t^{(\infty)}\right). \tag{2.38}$$

As $(x^{(n)})$ is an increasing sequence, (2.38) is equivalent to (1.38).

The next step shows that the limit is finite almost surely. Let $\hat{h} \geq h$ be the function given by Assumption A2. Notice that $\hat{h}$ may be replaced by $\hat{h} + C$ for every constant $C \geq 0$. Furthermore, h is bounded above. Thus, the function $\hat{h}$

may be modified such that, in addition, $\hat{h}$ is concave. By Itô's formula, Lemma 2.1.1 and translation invariance,

$$\frac{d}{dt}\mathbf{E}X_t^{\underline{N}}(i) = \mathbf{E}h(X_t^{\underline{N}}(i)) \leq \mathbf{E}\hat{h}\big(X_t^{\underline{N}}(i)\big) \leq \hat{h}\big(\mathbf{E}X_t^{\underline{N}}(i)\big). \tag{2.39}$$

For the last step, we applied Jensen's inequality. Therefore, the expectation is bounded above by the deterministic function $y(t,x)$ satisfying

$$\frac{d}{dt}y(t,x) = \hat{h}\big(y(t,x)\big), \quad y(0,x) = x. \tag{2.40}$$

The concave function $\hat{h}(x)$ converges to $-\infty$ as $x \to \infty$. Choose x_0 such that $\hat{h}$ is strictly negative for all $x \geq x_0$. Then for all $x > x_0$ and $t > 0$ we have $x_0 < y(t,x) < x$. From (2.40) we obtain by separation of variables that the solution satisfies

$$t = -\int_{y(t,x)}^{x} \frac{1}{\hat{h}(z)}\,dz \leq \int_{y(t,x)}^{\infty} \frac{1}{-\hat{h}(z)}\,dz \downarrow \int_{\lim_{x\to\infty} y(t,x)}^{\infty} \frac{1}{-\hat{h}(z)}\,dz \quad \text{as } x \to \infty. \tag{2.41}$$

For the monotone convergence, notice that $y(t,x)$ is nondecreasing in x and that all integrals are finite by inequality (1.34). Hence, if $\lim_{x\to\infty} y(t,x)$ was infinite for $t > 0$ then we would face the contradiction $0 < t \leq 0$. Therefore, we arrive at

$$\mathbf{E} \parallel X_t^{(\infty)} \parallel_\sigma = \sum_{i\in G} \sigma_i \uparrow \lim_{N\to\infty} \mathbf{E}X_t^{\underline{N}}(i) \leq \sum_{i\in G} \sigma_i \lim_{x\to\infty} y(t,x) < \infty, \quad t > 0. \tag{2.42}$$

From Lemma 2.1.4 it is then clear that for all $\varepsilon > 0$ the solution of (1.5) which starts at time $t = \varepsilon$ from $X_\varepsilon^{(\infty)}$ is the a.s. monotone limit (as $N \to \infty$) of the solutions of (1.5) starting from $X_\varepsilon^{(\underline{N})}$ at time ε, or equivalently starting from $\underline{N}$ at time 0. At the beginning of the proof we defined $X_t^{(\infty)}$ as this limit; hence we have so far proved parts a), b) and c) of Theorem 1.

A similar argument as in (2.35) proves that the process with initial measure μ is dominated by the maximal process, which is part (e).

To prove part (d), fix $0 < s < t$. By part (e),

$$\mathcal{L}\left(X_r^{(\infty)}\right) \geq \mathcal{L}^{X_{t-s}^{(\infty)}}(X_r). \tag{2.43}$$

Using this with $r = s$, we get the inequality

$$\mathcal{L}\left(X_s^{(\infty)}\right) \geq \mathcal{L}^{X_{t-s}^{(\infty)}}(X_s) = \mathcal{L}\left(X_t^{(\infty)}\right), \tag{2.44}$$

where the last equality follows from the Markov property. We conclude from this monotonicity that $\mathcal{L}\big(X_t^{(\infty)}\big) \downarrow \bar{\nu}$ for some probability measure $\bar{\nu}$ on $\mathbb{E}_\sigma$, which by continuity in the initial configuration (Lemma 2.1.4) is an equilibrium distribution of the dynamics (1.5).

Next, we show that the upper invariant measure is translation invariant and is associated. Both properties are preserved under weak limits. Furthermore, we will argue that these properties are preserved under the dynamics. The constant configuration $X_0^N \equiv N$ is both translation invariant and associated. Hence, both X_t^N and $X_t^{(\infty)}$ have these properties for all $t > 0$. Therefore, the claim follows.

The translation invariance of the migration kernel implies that the dynamics (1.5) preserves translation invariance. To prove the preservation of associated measures, we will argue in a similar way as in [4] where the analogue of (1.5) with $h = 0$ and $[0,1]^G$ instead of $\mathbb{R}_{\geq 0}^G$ was treated. We first consider the approximation scheme (X^Λ, Λ) with finite $\Lambda \subset G$, used to prove the existence part of Proposition 1.2.1. For fixed Λ, Theorem 1.1 in [15] together with a uniform approximation of h and g on compact intervals by smooth and bounded functions h_k and g_k with $\inf_{x\geq 0} g_k(x) > 0$ shows that, for an associated initial distribution $\mathcal{L}(X_0)$, the projections of $\mathcal{L}(X_t^\Lambda)$ to $\mathbb{R}_{\geq 0}^\Lambda$ are associated. Since $\mathcal{L}(X_t^\Lambda)$ approximates $\mathcal{L}(X_t)$ as $\Lambda \uparrow G$, the claim follows. □

2.3 The mean field model

In this section we study the dynamics

$$dM_t = \alpha(\mathbf{E}M_t - M_t)dt + h(M_t)dt + \sqrt{2g(M_t)}dB_t. \tag{2.45}$$

It can be shown (but will not be required for the subsequent proofs) that (2.45) arises as the limit of a sequence of processes following the dynamics (1.5), where G is replaced by a finite set G_n of cardinality n and $m^{(n)}(i,j) = 1/n$ for $i,j \in G_n$. This type of limit is known as *mean field* or *Vlasov-McKean limit*; we will therefore address (2.45) briefly as *mean field model*. Intuitively, a uniform migration which spreads out mass as far as possible should be good for survival, and conversely, extinction of $(M_t)_{t\geq 0}$ governed by (2.45) should imply extinction of $(X_t)_{t\geq 0}$ governed by (1.5). With this motivation in mind, we investigate in this section conditions on h and g under which the dynamics (2.45) admits a nontrivial equilibrium distribution.

To this end, we consider the following

Proposition 2.3.1. *Suppose that Assumption A1 holds and that*

$$\exists\, y_0 > 0\colon h\big|_{[0,y_0]} \geq 0 \quad \textit{and} \quad 0 \not\equiv h\big|_{[y_0,\infty)} \leq 0. \tag{2.46}$$

There is no nontrivial invariant measure for the dynamics (2.45) *if and only if*

$$\int_0^\infty \frac{h(y)}{g(y)} \exp\left(\int_{y_0}^y \frac{-\alpha x + h(x)}{g(x)}\, dx\right) dy \leq 0. \tag{2.47}$$

If condition (2.47) *is not satisfied then there is exactly one nontrivial invariant measure. Under Assumption A1 and Assumption A4, condition* (2.47) *is equivalent to*

$$\int_0^\infty \frac{\alpha y}{g(y)} \exp\left(\int_0^y \frac{-\alpha x + h(x)}{g(x)}\, dx\right) dy \le 1. \tag{2.48}$$

Proof. Let $\theta > 0$ and consider the process given by

$$dM_t^\theta = \alpha(\theta - M_t^\theta)dt + h(M_t^\theta)dt + \sqrt{2g(M_t^\theta)}dB_t. \tag{2.49}$$

By standard theory (e.g. pages 220f and 241 in Karlin and Taylor [21]), the equilibrium distribution of (2.49) is

$$\Gamma_\theta(dy) = \frac{C_\theta}{g(y)} \exp\left(\int_{y_0}^y \frac{\alpha(\theta - x) + h(x)}{g(x)}\, dx\right) dy =: C_\theta\, \Phi(y)\, dy, \tag{2.50}$$

where $C_\theta \in (0, \infty)$ is the normalising constant. Indeed, existence of an equilibrium of (2.49) is clear since the drift in zero is positive in zero and becomes sufficiently negative near ∞; formally, this follows from the finiteness of the integral $\int_0^\infty \Phi(y)\, dy$, which can be checked easily.

Obviously, (2.45) admits a nontrivial equilibrium if and only of $\int y\Gamma_\theta(dy) = \theta$ has a positive solution. Hence, all we need to do is to characterise the situations where

$$\nexists \theta > 0: \quad f(\theta) := \alpha \int \frac{y - \theta}{C_\theta} \Gamma_\theta(dy) = 0. \tag{2.51}$$

We eliminate one occurrence of θ on the left hand side of (2.51) by an integration by parts:

$$\begin{aligned} f(\theta) &= \int_0^\infty \frac{\alpha(y-\theta)}{g(y)} \exp\left(\int_{y_0}^y \frac{\alpha(\theta - x)}{g(x)}\, dx\right) \exp\left(\int_{y_0}^y \frac{h(x)}{g(x)}\, dx\right) dy \\ &= \lim_{\varepsilon \to 0} \left[\exp\left(\int_{y_0}^y \frac{\alpha(\theta - x) + h(x)}{g(x)}\, dx\right)\right]_{1/\varepsilon}^{\varepsilon} \\ &\quad + \int_0^\infty \frac{h(y)}{g(y)} \left(\exp\left(\int_{y_0}^y \frac{\alpha}{g(x)}\, dx\right)\right)^\theta \exp\left(\int_{y_0}^y \frac{-\alpha x + h(x)}{g(x)}\, dx\right) dy. \end{aligned} \tag{2.52}$$

We now analyse the two boundary terms on the right hand side of (2.52). In the following calculations, C_i are finite constants. Recall that h is nonpositive for large arguments. Furthermore, in Assumption A1 we assumed $g(x) \le Cx^2$ for some constant C and all $x \geq y_0 > 0$. With this, the expression coming from the boundary value $1/\varepsilon$ tends to zero as $\varepsilon \to 0$:

$$\exp\left(\int_{y_0}^{1/\varepsilon} \frac{\alpha(\theta - x) + h(x)}{g(x)}\,dx\right) \leq C_1 \exp\left(\int_{y_0\vee\theta}^{1/\varepsilon} \frac{\alpha(\theta - x)}{Cx^2}\,dx\right) \xrightarrow{\varepsilon\to 0} 0. \tag{2.53}$$

For the other boundary term, we recall that h is nonnegative for small arguments and estimate

$$\begin{aligned} 0 \leq \exp&\left(-\int_{\varepsilon}^{y_0} \frac{\alpha(\theta - x) + h(x)}{g(x)}\,dx\right) \leq C_2 \exp\left(-\int_{\varepsilon}^{y_0\wedge(\theta/2)} \frac{\alpha\theta/2}{g(x)}\,dx\right) \\ &\xrightarrow{\varepsilon\to 0} C_3 \exp\left(-\theta\frac{\alpha}{2}\int_{0+} \frac{1}{g(x)}\,dx\right). \end{aligned} \tag{2.54}$$

By assumption, g is locally Lipschitz continuous in zero and thus $g(x) \leq C_4 x$ in a neighbourhood of zero. Together with $\theta > 0$, this implies that all boundary terms vanish. Notice that the expression coming from the boundary value ε does not need to be zero in case $\theta = 0$.

At this point we have seen that f can be rewritten as

$$f(\theta) = \int_0^\infty \frac{h(y)}{g(y)}\left(\exp\left(\int_{y_0}^{y} \frac{\alpha}{g(x)}\,dx\right)\right)^\theta \exp\left(\int_{y_0}^{y} \frac{-\alpha x + h(x)}{g(x)}\,dx\right) dy. \tag{2.55}$$

for $\theta > 0$. We will show that f is strictly decreasing and continuous in $\theta > 0$. For this, consider the function

$$\theta \mapsto h(y)\left(\exp\left(\int_{y_0}^{y} \frac{\alpha}{g(x)}\,dx\right)\right)^\theta \tag{2.56}$$

for fixed $y \geq 0$. If $y < y_0$ then $h(y) \geq 0$ and the integral is negative. If $y > y_0$ then $h(y) \leq 0$ and the integral is positive. In both situations, the function in (2.56) is non-increasing. Furthermore, there is an interval $[y_1, y_2]$ with $y_0 \leq y_1 < y_2$ where $h(y) < 0$ and where the function in (2.56) is strictly decreasing and converging to $-\infty$. The integral over $[0, y_0]$ on the right hand side of (2.55) is continuous and non-increasing in $\theta > 0$, and bounded in $\theta \geq 1$. This follows from dominated convergence and the fact that the integral over $[0, \varepsilon]$ on the right hand side of (2.55) is bounded above by

$$\begin{aligned} &\int_0^\varepsilon \frac{1}{g(y)} \exp\left(\int_\varepsilon^y \frac{\alpha\bar\theta}{2g(x)}\,dx\right) dy \cdot \sup_{x\leq y_0} h(x) \exp\left(\int_{y_0}^\varepsilon \frac{\alpha(\bar\theta - x) + h(x)}{g(x)}\,dx\right) \\ &\leq \exp\left(\int_\varepsilon^y \frac{\alpha\bar\theta}{2g(x)}\,dx\right)\Big|_0^\varepsilon \cdot C < \infty \end{aligned} \tag{2.57}$$

for all $\theta \geq \bar{\theta} > 0$, where $\varepsilon > 0$ is such that $|\alpha x - h(x)| \leq \alpha\bar{\theta}/2$ for all $x \leq \varepsilon$. By monotone convergence, the integral over $[y_0, \infty)$ on the right hand side of (2.55) is continuous and strictly decreasing in $\theta > 0$, and decreases to $-\infty$. Thus, the function f is continuous and strictly decreasing in $\theta > 0$ with $f(\infty) = -\infty$. Hence, condition (2.51) is satisfied if and only if $\lim_{\theta\to 0} f(\theta) \leq 0$. Note that by strict monotonicity of f, there is at most one nontrivial invariant measure.

For the limit $\theta \to 0$ in equation (2.55), we use monotone convergence (for the $\int_0^{y_0}$ part) and dominated convergence (for the $\int_{y_0}^\infty$ part). Thus, we have

$$\lim_{\theta\to 0} f(\theta) = \int_0^\infty \frac{h(y)}{g(y)} \exp\left(\int_{y_0}^y \frac{-\alpha x + h(x)}{g(x)}\, dx\right) dy \tag{2.58}$$

Therefore, $\lim_{\theta\to 0} f(\theta) \leq 0$ is equivalent to condition (2.47).

Now, additionally assume that $\lim_{\varepsilon\to 0} \int_\varepsilon^{y_0} \frac{-\alpha x + h(x)}{g(x)}\, dx$ exists in $(-\infty, \infty]$. Then reversing the calculation in (2.52) with $\theta = 0$, we arrive at

$$\begin{aligned}\lim_{\theta\to 0} f(\theta) = &\int_0^\infty \frac{\alpha y}{g(y)} \exp\left(\int_{y_0}^y \frac{-\alpha x + h(x)}{g(x)}\, dx\right) dy \\ &- \exp\left(-\lim_{\varepsilon\to 0} \int_\varepsilon^{y_0} \frac{-\alpha x + h(x)}{g(x)}\, dx\right).\end{aligned} \tag{2.59}$$

If the limit on the right hand side is ∞ then $\lim_{\theta\to 0} f(\theta) > 0$ and a nontrivial invariant measure exists. The assertion is true in this case because the left hand side of (2.48) is ∞. Otherwise, the limit on the right hand side of (2.59) is finite. Then multiply the equation with $\exp\left(\int_0^{y_0} \frac{-\alpha x + h(x)}{g(x)}\, dx\right)$ and merge the two integrals $\int_0^{y_0}$ and $\int_{y_0}^y$ into one integral. Hence, we see that (2.47) and (2.48) are equivalent. □

We now specialise this result to the logistic Feller case, where condition (2.48) can be simplified.

Corollary 2.3.2. *Consider the mean field model* (2.45) *with* $h(x) = \gamma x(K - x)$ *and* $g(x) = \beta x$. *Assume* $\alpha, \gamma, \beta > 0$ *and let* $\overline{K} > 0$ *be uniquely determined by*

$$\int_0^\infty \exp\left(\overline{K}\gamma y - \frac{\gamma\beta}{2} y^2\right) \cdot \alpha \exp\left(-\alpha y\right) dy = 1. \tag{2.60}$$

There is no nontrivial invariant measure for (2.45) *if and only if* $0 \leq K \leq \overline{K}$.

Proof. First of all, convince yourself that Assumptions A1 and A2 hold. Thus, Proposition 2.3.1 applies if $K > 0$. After an integration and a change of variables ($y \to \beta y$), condition (2.48) takes the form (1.43). The left hand side in (1.43) is strictly increasing in K, tends to ∞, is continuous in K by monotone convergence and is smaller than one for $K = 0$. Hence, $\overline{K}$ exists and is unique. By monotonicity, condition (1.43) holds if and only if $K \leq \overline{K}$. □

For example, in the case $\alpha = \gamma = \beta = 1$ formula (2.60) gives the numerical value $\overline{K} = 0.6973....$

The following extinction result for the mean field dynamics is a fairly direct consequence of Proposition 2.3.1.

Lemma 2.3.3. *Consider the mean field model given by* (2.45)*. Suppose that Assumptions A1, A2 and condition* (2.46) *hold. Then inequality* (2.47) *implies local extinction:*

$$\mathcal{L}(M_t) \Longrightarrow \delta_0 \qquad (as\ t \to \infty) \tag{2.61}$$

for any initial law.

Proof. Paralleling the arguments in Section 2.2, one infers the existence of the maximal process $M^{(\infty)}$ for the dynamics (2.45), which obeys $\mathcal{L}(M_t^{(\infty)}) \geq \mathcal{L}(M_t)$. Again, this maximal process converges to an invariant measure. However, by Proposition 2.3.1 and condition (2.47), the trivial measure δ_0 is the only invariant measure. This implies the assertion. □

2.4 Comparison with the mean field model. Proof of Theorem 2

The main idea for the proof of Theorem 2 is the assertion that the interacting locally regulated diffusions are dominated by the mean field model. The intuition behind this is that a uniform spread of mass reduces competition and therefore is good for survival, and that the mean field model arises as a limit of uniform migration models (see Section 2.3).

We proceed in two steps to prove Theorem 2. Firstly, we establish a comparison between the system of interacting locally regulated diffusions (1.5) and the mean field model (2.45) which implies that it is more likely for the latter to survive. Then we exploit the fact (proved in Section 2.3) that for some parameter configurations not even the mean field model survives.

The proof of the comparison result will first treat the case where the functions h and g satisfy the following assumptions.

Assumption A6. *The set I is a closed finite interval of the form $[0, c]$, $0 < c < \infty$. The functions $h\colon I \to \mathbb{R}$ and $\sqrt{g}\colon I \to \mathbb{R}$ are twice continuously differentiable on I and satisfy $h(0) = g(0) = g(c) = 0 > h(c)$. Furthermore, g is strictly positive on $(0, c)$.*

The proof of Proposition 1.2.2 is based on the following lemma.

Lemma 2.4.1. *Let h and g satisfy Assumption A6. Suppose that h is concave and that the set Λ is finite and nonempty. Then the semigroup of the solution of equation (2.25) preserves the function cone*

$$\mathbf{F} = \{f \in \mathbf{C}^2_{b1}(\mathbb{R}^\Lambda_{\geq 0})\colon \frac{\partial}{\partial x_i} f \geq 0 \ \ \forall\, i, \frac{\partial^2}{\partial x_i \partial x_j} f \leq 0 \ \ \forall\, i,j\}, \tag{2.62}$$

where $\mathbf{C}^2_{b1}(\mathbb{R}^\Lambda_{\geq 0})$ denotes the space of all bounded $\mathbf{C}^2$- functions $f : \mathbb{R}^\Lambda_{\geq 0} \to \mathbb{R}$ with bounded first partial derivatives.

Proof. This lemma is an addendum to Proposition 17 in [5]. There, the preservation of $\mathbf{F}$ was proved for $h \equiv 0$ and matrices m with $\sum_{j\in\Lambda} m(i,j) = 1$ for all $i \in \Lambda$. This proof also works for more general matrices m which only satisfy $\sum_{j\in\Lambda} m(i,j) \leq 1$ for $i \in \Lambda$. To extend the argument to the case $h \neq 0$, let $y(t,x)$ be the solution of

$$\frac{\partial}{\partial t} y(t,x) = h\big(y(t,x)\big) \qquad y(0,x) = x \in I. \tag{2.63}$$

This defines a deterministic Markov process whose semigroup is given by $S_t f(x) := f(y(t,x))$. Similar as in [5] we only need to establish that this semigroup preserves $\mathbf{F}$ if h is twice continuously differentiable. A little calculation shows that it is enough to prove that $y(t,x)$ is increasing and concave in x. To show concavity, notice that differentiating equation (2.63) results in

$$\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2} y(t,x) = h^{''}(y(t,x))\cdot \left(\frac{\partial}{\partial x} y(t,x)\right)^2 + h^{'}(y(t,x))\cdot \frac{\partial^2}{\partial x^2} y(t,x). \tag{2.64}$$

For fixed x, write (2.64) as $z_t^{'} = a_t + b_t \cdot z_t$ with $z_0 = 0$. The solution for this is

$$z_t = \exp\left(\int_0^t b_s\, ds\right) \int_0^t \exp\left(-\int_0^s b_r\, dr\right) a_s\, ds. \tag{2.65}$$

Since $h(x)$ is concave, a_t is negative, implying the claimed concavity. A similar, even simpler argument shows monotonicity. □

Proof of Proposition 1.2.2. We make use of the approximation scheme X^Λ defined in the proof of Proposition 1.2.1; recall that X^Λ is the solution of (2.25). Since $X^\Lambda_t \uparrow X_t$, it suffices to show the inequality (1.44) with X_t replaced by X^Λ_t, and f depending only on the coordinates x_i with $i \in \Lambda$.

Furthermore, we assume for the rest of the proof that h and g satisfy Assumption A6. The general case follows then by approximating h and g pointwise by functions h_k and g_k satisfying Assumption A6. See Lemma 19 of [5] for the details.

In addition, we may assume that $f \in \mathbf{F}$; otherwise approximate f by functions in $\mathbf{F}$ and use dominated convergence. Denote by S_t the strongly continuous semigroup of X^Λ defined on $\mathbf{C}(I^\Lambda)$. When applied to $\varphi \in \mathbf{F}$, the generator of X^Λ takes the form (see e.g. Theorem 7.3.3 of [27])

$$\mathcal{G}\varphi(x) = \sum_{i\in\Lambda}\Big[\alpha\big(\sum_{j\in\Lambda} m(i,j)x_j - x_i\big)\frac{\partial}{\partial x_i} + h(x_i)\frac{\partial}{\partial x_i} + g(x_i)\frac{\partial^2}{\partial x_i^2}\Big]\varphi(x). \tag{2.66}$$

By Proposition 1.1.5 c) of [11], we know that

$$\frac{d}{dt}S_t f = \mathcal{G}S_t f. \tag{2.67}$$

Let $\bar{M}_t = (\bar{M}_t(i))_{i\in\Lambda}$ be a system of processes coupled through the initial state $\bar{M}_0(i) = X_0(i)$, $i \in \Lambda$, but following independent mean field dynamics:

$$\begin{aligned} df(\bar{M}_t) =& \alpha \sum_{i\in\Lambda}\frac{\partial}{\partial x_i}f(\bar{M}_t)\Big(\mathbf{E}^{\bar{\mu}}\bar{M}_t(i) - \bar{M}_t(i) + h\big(\bar{M}_t(i)\big)\Big)\,dt \\ &+ \sum_{i\in\Lambda}\frac{\partial^2}{\partial x_i^2}f(\bar{M}_t)g\big(\bar{M}_t(i)\big)\,dt + \sum_{i\in\Lambda}\frac{\partial}{\partial x_i}f(\bar{M}_t)\sqrt{2g\big(\bar{M}_t(i)\big)}\,dB_t(i), \end{aligned} \tag{2.68}$$

where $B(i)_{i\in\Lambda}$ are independent Brownian motions. Write $\bar{\mu}_t := \mathcal{L}\left(\bar{M}_t\right)$; for brevity we suppress in this notation the dependence on Λ. By equation (2.68) the evolution of $\bar{\mu}_t$ is given by

$$\begin{aligned} \frac{d}{dt}\bar{\mu}_t f = \alpha \sum_{i\in\Lambda}\Big[&\mathbf{E}^{\bar{\mu}}\big[\big(\mathbf{E}^{\bar{\mu}}\bar{M}_t(i) - \bar{M}_t(i)\big)\big(\frac{\partial}{\partial x_i}f\big)(\bar{M}_t)\big] \\ &+ \bar{\mu}_t\big[h(x_i)\frac{\partial}{\partial x_i}f + g(x_i)\frac{\partial^2}{\partial x_i^2}f\big]\Big]. \end{aligned} \tag{2.69}$$

Integration by parts yields

$$\int_0^t \left(\frac{d}{ds}\bar{\mu}_s\right) S_{t-s}f\,ds = \big[\bar{\mu}_s S_{t-s}f\big]_0^t - \int_0^t \bar{\mu}_s \frac{d}{ds}S_{t-s}f\,ds. \tag{2.70}$$

In view of (2.67) this reads as

$$\bar{\mu}_t f - \bar{\mu}S_t f = \int_0^t \left(\frac{d}{ds}\bar{\mu}_s - \bar{\mu}_s\mathcal{G}\right) S_{t-s}f\,ds. \tag{2.71}$$

We will show that the integrand is nonnegative. From Lemma 2.4.1, we know that $\varphi := S_{t-s}f$ (for $0 \le s \le t$ fixed) is an element of $\mathbf{F}$. By equations (2.66) and (2.69),

(2.72)
$$
\begin{aligned}
&\Big(\frac{d}{ds}\bar{\mu}_s - \bar{\mu}_s\mathcal{G}\Big)\varphi \\
&= \alpha\sum_{i\in\Lambda}\mathbf{E}^{\bar{\mu}}\bar{M}_t(i)\mathbf{E}^{\bar{\mu}}\big[\big(\frac{\partial}{\partial x_i}\varphi\big)(\bar{M}_t)\big] - \alpha\sum_{i,j\in\Lambda} m(i,j)\mathbf{E}^{\bar{\mu}}\big[\bar{M}_t(j)\big(\frac{\partial}{\partial x_i}\varphi\big)(\bar{M}_t)\big] \\
&\geq \alpha\sum_{i\in\Lambda}\mathbf{E}^{\bar{\mu}}\bar{M}_t(i)\mathbf{E}^{\bar{\mu}}\big[\big(\frac{\partial}{\partial x_i}\varphi\big)(\bar{M}_t)\big] - \alpha\sum_{i,j\in\Lambda} m(i,j)\mathbf{E}^{\bar{\mu}}\bar{M}_t(j)\mathbf{E}^{\bar{\mu}}\big[\big(\frac{\partial}{\partial x_i}\varphi\big)(\bar{M}_t)\big].
\end{aligned}
$$

Note that under the assumptions on $\bar{\mu}$ we have $\mathcal{L}\left(\bar{M}(i)\right) = \mathcal{L}\left(\bar{M}(0)\right)$ for all $i \in \Lambda$. The right hand side of (2.72) is nonnegative because of $\sum_{j\in\Lambda} m(i,j) \leq 1$. To see the inequality in (2.72), notice that $-\frac{\partial}{\partial x_i}\varphi$ is bounded and component-wise increasing by Lemma 2.4.1. The claimed inequality thus follows from the fact that $\mathcal{L}\left(\bar{M}_t\right)$ is associated, which we now prove. Independent real-valued random variables are associated (see p.78 of [24]), and $\bar{\mu}$ is associated by assumption. Hence

$$
\begin{aligned}
&\mathbf{E}^{\bar{\mu}}\big[f(\bar{M}_t)g(\bar{M}_t)\big] \\
&= \mathbf{E}^{\bar{\mu}}\Big[\mathbf{E}\big[f(\bar{M}_t)g(\bar{M}_t)|\bar{M}_0\big]\Big] \geq \mathbf{E}^{\bar{\mu}}\Big[\mathbf{E}\big[f(\bar{M}_t)|\bar{M}_0\big]\mathbf{E}\big[g(\bar{M}_t)|\bar{M}_0\big]\Big] \\
&\geq \mathbf{E}^{\bar{\mu}}\Big[\mathbf{E}\big[f(\bar{M}_t)|\bar{M}_0\big]\Big]\mathbf{E}^{\bar{\mu}}\Big[\mathbf{E}\big[g(\bar{M}_t)|\bar{M}_0\big]\Big] = \mathbf{E}^{\bar{\mu}}\big[f(\bar{M}_t)\big]\mathbf{E}^{\bar{\mu}}\big[g(\bar{M}_t)\big],
\end{aligned}
\tag{2.73}
$$

showing that $\mathcal{L}\left(\bar{M}_t\right)$ is associated. □

Proof of Theorem 2. As in the proof of Theorem 1, we may w.l.o.g. assume that the function $\hat{h}$ from Assumption A2 is concave. Furthermore, w.l.o.g. we may assume that h itself is concave and satisfies both (1.34), with $\hat{h}$ replaced by h, and (1.41), with $\bar{h}$ replaced by h. Otherwise, by Lemma 2.1.3, $(X_t)_{t\geq 0}$ is dominated by the solution of (1.5) with h replaced by the concave function $\bar{h}\wedge\hat{h}$ which satisfies both (1.34) and (1.41).

Let $y_0 := \max\{y \geq 0 : h(y) = 0\}$. Assume for the moment that $y_0 > 0$. The measure $\bar{\mu} := \mathcal{L}\big(X_1^{(\infty)}\big)$ is associated, shift invariant and its first moment is finite by Theorem 1. Let (M_t) be the solution of (2.45) with initial distribution $\mu := \mathcal{L}\left(X_1^{(\infty)}(0)\right)$. Theorem 1(e), (a) and Proposition 1.2.2 imply

$$
\mathbf{E}e^{-\lambda X_{t+1}(i)} \geq \mathbf{E}e^{-\lambda X_{t+1}^{(\infty)}(i)} = \mathbf{E}^{\bar{\mu}}e^{-\lambda X_t^{(\infty)}(i)} \geq \mathbf{E}^{\mu}e^{-\lambda M_t}. \tag{2.74}
$$

It follows from Lemma 2.3.3 that, under the stated assumptions, $\mathbf{E}e^{-\lambda M_t} \to 1$ for all $\lambda > 0$ as $t \to \infty$. This proves the assertion for the case $y_0 > 0$.

If $y_0 = 0$ then $\tilde{h}(x) := \big(h(x) - h(1)\big) \wedge 0$ satisfies $h \leq \tilde{h} \leq 0$ because h is concave. Let $\tilde{X}$ be the solution of (1.5) with h replaced by $\tilde{h}$ and with the same family of Brownian motions. By the previous step, $\tilde{X}$ suffers local extinction. Lemma 2.1.3 implies $X \leq \tilde{X}$ which completes the proof. □

2.5 Self-duality. Proof of Theorem 3

In the rest of the paper, we exploit the specific form of the dynamics (1.2) for the interacting Feller diffusions with logistic growth. Theorem 3 states that the process is "dual to itself" via

$$\mathbf{E}^x \exp\big(-\frac{\gamma}{\beta}\langle X_t, y\rangle\big) = \mathbf{E}^y \exp\big(-\frac{\gamma}{\beta}\langle x, X_t^\dagger\rangle\big). \tag{2.75}$$

We will prove this for the solution X^Λ of (2.25). By (2.27), we know that the process X^Λ monotonically approximates X. Hence, the assertion follows by dominated convergence.

For the rest of the proof, we consider X^Λ. We write X instead of X^Λ and x, y instead of x^Λ, y^Λ. The duality function is $H(x,y) = \exp\big(-\frac{\gamma}{\beta}\langle x, y\rangle\big)$. Recall the definition of $\mathbf{C}_{b1}^2(\mathbb{R}_{\geq 0}^\Lambda)$ from Section 2.4. Define the linear operator $\mathcal{G}^X \colon \mathbf{C}_{b1}^2(\mathbb{R}_{\geq 0}^\Lambda) \to \mathbf{C}(\mathbb{R}_{\geq 0}^\Lambda)$ by

$$\mathcal{G}^X f(x) = \sum_{i\in\Lambda}\left[\alpha\left(\sum_{j\in\Lambda} m^\Lambda(i,j)x_j - x_i\right)\frac{\partial}{\partial x_i}f + \gamma x_i(K - x_i)\frac{\partial}{\partial x_i}f + \beta x_i \frac{\partial^2}{\partial x_i^2}f\right]. \tag{2.76}$$

By Itô's formula, the process $(X_t)_t$ is a solution of the martingale problem for $\big(\mathcal{G}^X, \mathbf{C}_{b1}^2(\mathbb{R}_{\geq 0}^\Lambda)\big)$. In order to apply Theorem 4.4.11 of [11] (with the choice $\alpha, \beta = 0$), we will show that

$$\mathcal{G}^X H(\cdot, y)(x) = \mathcal{G}^{X^\dagger} H(x, \cdot)(y) \qquad \forall\, x, y \in \mathbb{R}_{\geq 0}^\Lambda. \tag{2.77}$$

We prove equation (2.77) by considering the different parts of (2.76) separately. Since H is a function $\rho(\langle x, y\rangle)$ of the scalar product, it is easy to see that the migration terms of both sides are equal. To establish equation (2.77), it remains to show that

$$\begin{aligned} &\gamma x_i(K - x_i)\frac{\partial}{\partial x_i}H(x,y) + \beta x_i \frac{\partial^2}{\partial x_i^2}H(x,y) \\ &= \gamma y_i(K - y_i)\frac{\partial}{\partial y_i}H(x,y) + \beta y_i \frac{\partial^2}{\partial y_i^2}H(x,y) \end{aligned} \tag{2.78}$$

for all $i \in \Lambda$. Observe that this equation is symmetric in x and y. Consider the left hand side of equation (2.78) divided by $H(x,y)$:

$$\gamma x_i(K - x_i)\cdot(-\frac{\gamma}{\beta}y_i) + \beta x_i\cdot y_i^2\big(\frac{\gamma}{\beta}\big)^2 = -\frac{\gamma^2 K}{\beta}x_i y_i + \frac{\gamma^2}{\beta}x_i^2 y_i + \frac{\gamma^2}{\beta}x_i y_i^2. \tag{2.79}$$

The right hand side of (2.79) is symmetric in x and y and therefore, by interchanging the roles of x and y, is also equal to the right hand side of equation (2.78) divided by $H(x,y)$.

Theorem 4.4.11 of [11] is applicable if we prove that

$$\sup_{s,t\le T} |\mathcal{G}^X H(X_s, X_t^\dagger)| \tag{2.80}$$

is integrable for all $T < \infty$ where X and $X^\dagger$ are independent. It is not hard to see that

$$|\mathcal{G}^X H(x,y)| \le C\big(|x||y| + |x| + |y|\big), \qquad \forall\, x,y \in \mathbb{R}_{\ge 0}^\Lambda \tag{2.81}$$

for a finite constant C. For this, use that $z\exp(-z)$ is bounded in $z \ge 0$. Integrability of (2.80) therefore follows from the independence of X and $X^\dagger$ and from Lemma 2.1.2. □

Let us write $\mathcal{M}_c(\mathbb{Z}^d)$ for the set of configurations in $\mathbb{R}_{\ge 0}^{\mathbb{Z}^d}$ with finite support. As a consequence of the self-duality, we prove the following characterisation of the upper invariant measure in terms of the finite mass process.

Lemma 2.5.1. *Assume $\beta, \gamma > 0$. The upper invariant measure $\bar\nu$ of* (1.2) *is uniquely determined by*

$$\int \exp\big(-\frac{\gamma}{\beta}\langle x, \lambda\rangle\big)\,\bar\nu(dx) = \mathbf{P}^\lambda(\exists t \ge 0 \text{ such that } X_t^\dagger = \underline{0}), \qquad |\lambda| < \infty, \tag{2.82}$$

where $X^\dagger$ is the solution of (1.2) *with the transpose migration matrix $m^\dagger$.*

Proof. Fix a configuration $\lambda \in \mathcal{M}_c(\mathbb{Z}^d)$ and consider the process $\big(X_t^{(n)}\big)$ started in the constant configuration $\underline{n}(i) \equiv n$. This process converges to the maximal process as $n \to \infty$. Therefore, the self-duality implies for $t > 0$

$$\begin{aligned} &\mathbf{E}\exp\big(-\frac{\gamma}{\beta}\langle X_t^{(\infty)}, \lambda\rangle\big) = \lim_{n\to\infty}\mathbf{E}\exp\big(-\frac{\gamma}{\beta}\langle X_t^{(n)}, \lambda\rangle\big) \\ &= \mathbf{E}^\lambda \lim_{n\to\infty}\exp\big(-\frac{\gamma}{\beta}\langle X_t^\dagger, \underline{n}\rangle\big) = \mathbf{P}^\lambda\big(X_t^\dagger = \underline{0}\big) = \mathbf{P}^\lambda\big(\exists s \le t\colon X_s^\dagger = \underline{0}\big). \end{aligned} \tag{2.83}$$

For the second equality, we used monotone convergence. Letting $t \to \infty$, the assertion follows from Theorem 1(d). For general λ with $|\lambda| < \infty$, use monotone convergence. □

2.6 Convergence to the upper invariant measure. Proof of Theorem 5

Proof of Theorem 5. Let μ be a translation invariant distribution on $\mathbb{Z}^d$ which satisfies $\mu(\underline{0}) = 0$. For analysing the long-term behaviour of the interacting Feller diffusion with logistic growth started in μ we can assume without loss of generality that μ has finite first moment and satisfies $\mu(x_0 = 0) = 0$. Otherwise we let the system run for a little time $\varepsilon > 0$, obtaining

$$\lim_{t\to\infty} \mathcal{L}^{\mu}(X_t) = \lim_{t\to\infty} \mathcal{L}^{\mathcal{L}^{\mu}(X_\varepsilon)}(X_t). \tag{2.84}$$

A comparison with the maximal process (see Theorem 1 (e), (b)) yields $\mathbf{E}^{\mu} X_\varepsilon(0) < \infty$. Furthermore, after a fixed positive time $\varepsilon > 0$ every component is strictly positive almost surely (see Lemma 2.6.2).

Let X and $X^\dagger$ be solutions of (1.2) with migration matrix m and its transpose $m^\dagger$, respectively. In Lemma 2.6.1 we will show that the total mass hits zero in finite time or tends to infinity. Hence, we get by self-duality (Theorem 3)

$$\begin{aligned}\mathbf{E}^{\mu}\exp\big(-\frac{\gamma}{\beta}\langle X_t,\lambda\rangle\big) = \int \mu(dx)\Big(&\mathbf{E}^{\lambda}\big[1_{|X_s^\dagger|\to\infty}\exp\big(-\frac{\gamma}{\beta}\langle x, X_t^\dagger\rangle\big)\big]\\ &+\mathbf{E}^{\lambda}\big[1_{\exists s:\, X_s^\dagger=\underline{0}}\exp\big(-\frac{\gamma}{\beta}\langle x, X_t^\dagger\rangle\big)\big]\Big).\end{aligned} \tag{2.85}$$

We treat the two terms on the right hand side separately and begin with the first term. Apply Hölder's inequality to the integral with respect to μ. For this, let $1/p_i = X_t^\dagger(i)/|X_t^\dagger|$ if this is positive. Thus, we obtain

$$\begin{aligned}&\mathbf{E}^{\lambda}\Big[1_{|X_s^\dagger|\to\infty}\int \mu(dx)\exp\big(-\frac{\gamma}{\beta}\langle x, X_t^\dagger\rangle\big)\Big]\\ &\le \mathbf{E}^{\lambda}\Big[1_{|X_s^\dagger|\to\infty}\prod_{i\in\mathbb{Z}^d}\Big(\int \mu(dx)\exp\big(-\frac{\gamma}{\beta}x_i|X_t^\dagger|\big)\Big)^{\frac{X_t^\dagger(i)}{|X_t^\dagger|}}\Big]\\ &= \mathbf{E}^{\lambda}\Big[1_{|X_s^\dagger|\to\infty}\int \mu(dx)\exp\big(-\frac{\gamma}{\beta}x_0|X_t^\dagger|\big)\Big] \to 0 \quad \text{as } t\to\infty.\end{aligned} \tag{2.86}$$

The equality is a consequence of the translation invariance of μ. The last expression tends to zero because of dominated convergence and the assumption $\mu(x_0 = 0) = 0$. As to the second term on the right hand side of (2.85), dominated convergence gives

$$\int \mu(dx)\mathbf{E}^{\lambda}\big[1_{\exists s:\, X_s^\dagger=\underline{0}}\exp\big(-\frac{\gamma}{\beta}\langle x, X_t^\dagger\rangle\big)\big] \to \int \mu(dx)\mathbf{E}^{\lambda}\big[1_{\exists s:\, X_s^\dagger=\underline{0}}\big] \tag{2.87}$$

as $t \to \infty$. Using Lemma 2.5.1 we arrive at

$$\lim_{t\to\infty} \mathbf{E}^{\mu} \exp\left(-\frac{\gamma}{\beta}\langle X_t, \lambda\rangle\right) = \mathbf{P}^{\lambda}(\exists t \geq 0 \text{ s.t. } X_t^{\dagger} = \underline{0}) = \int \exp\left(-\frac{\gamma}{\beta}\langle x, \lambda\rangle\right) \bar{\nu}(dx).$$

Starting in $\mathcal{L}(X_0) \geq \mu$, the process $\mathcal{L}(X_t)$ is bounded below by $\mathcal{L}^{\mu}(X_t)$ (Lemma 2.1.3) and is bounded above by $\mathcal{L}\big(X_t^{(\infty)}\big)$ (Theorem 1(e)) which both converge to $\bar{\nu}$. This concludes the proof of Theorem 5. □

We have to append

Lemma 2.6.1. *Assume $\beta > 0$. Let X be a solution of* (1.2) *starting in $x \in \mathbb{E}_{\sigma}$ with finite total mass $|x| < \infty$. Then with probability* 1 *either*

- *there is a $t \geq 0$ such that $X_s = \underline{0}$ for all $s \geq t$ or*
- *$|X_t| \to \infty$ as $t \to \infty$.*

Proof. The intuition behind this is the following. The process always has a positive probability of hitting the lower trap. Whenever the total mass stays bounded, the process will seize its chance.

This is made precise in Theorem 2 of [20]. In order to apply this result, we only need to verify that there always is the risk of extinction in the following sense:

$$\forall\, y\colon \inf_{|x|\leq y} \mathbf{P}^{x}\big(\exists t\colon X_t = \underline{0}\big) > 0. \tag{2.88}$$

Let Y_t be a solution of (1.5) with $G = \mathbb{Z}^d$, $h(x) = \gamma K x$ and $g(x) = \beta x$. By Lemma 2.1.3, X and Y may be coupled such that X_t is bounded above by Y_t almost surely. Furthermore, $|Y_t|$ is equal in distribution to Feller's branching diffusion F_t with super-criticality γK started in $|x|$. The extinction probability of F_t is strictly positive, see e.g. Appendix 6.2 of [7]. Therefore condition (2.88) follows from

$$\mathbf{P}^{x}\big(\exists t \geq 0\colon X_t = \underline{0}\big) \geq \mathbf{P}^{|x|}\big(\exists t \geq 0\colon F_t = 0\big) \geq \mathbf{P}^{y}\big(\exists t \geq 0\colon F_t = 0\big) > 0 \tag{2.89}$$

for every x with $|x| \leq y$. □

Lemma 2.6.2. *Suppose that h and g satisfy Assumption A1. Let X be a solution of* (1.5)*. If its initial law μ is translation invariant and does not charge the zero configuration $\underline{0}$, then, for every fixed time $t_0 > 0$,*

$$X_{t_0}(i) > 0 \quad \forall\, i \in G \qquad \mathbf{P}^{\mu} - a.s. \tag{2.90}$$

Proof. Assume, that $h \le 0$. Otherwise, compare X_t with the process defined with $h \wedge 0$ instead of h.

Let $\tilde{h}(\theta) = -\alpha\theta + h(\theta)$. For $\varepsilon > 0$, define the solution of

$$dY_t^{\varepsilon,i} = \alpha\varepsilon dt + \tilde{h}\big(Y_t^{\varepsilon,i}\big)dt + \sqrt{2g(Y_t^{\varepsilon,i})}dB_t^i, \qquad Y_0^{\varepsilon,i} \ge 0, \tag{2.91}$$

on the same probability space as X by using the same system of Brownian motions. This system satisfies $\mathbf{P}^0(Y_t^{\varepsilon,i} > 0) = 1$ for all $t > 0$. Otherwise, continuity in the initial value would imply that there is a $t > 0$ and a θ_0 such that $\mathbf{P}^\theta(Y_t^{\varepsilon,i} = 0) > 0$ for all $\theta \le \theta_0$. Integrating this with the equilibrium distribution Γ_ε (see equation (2.50), it exists because of $h \le 0$) yields $\Gamma_\varepsilon(0) > 0$ which is false. Thus, we have

$$Y_\delta^{\varepsilon m(i,j),i} > 0 \quad \forall\, \varepsilon \in (0,1)\cap\mathbb{Q} \ \ \forall\, \delta \in (0,1)\cap\mathbb{Q} \ \ \forall\, i,j \text{ s.t. } m(i,j) > 0 \text{ a.s.} \tag{2.92}$$

Denote the event $\{X_t(j) \ge \varepsilon \ \ \forall\, t \in [t_0-\delta, t_0]\}$ by $A_{\varepsilon,\delta}$. On $A_{\varepsilon,\delta}$ we compare X with the solution of (2.91):

$$\begin{aligned} X_t(i) &= X_{t_0-\delta}(i) + \int_{t_0-\delta}^t \alpha \sum_k m(i,k)X_s(k)\,ds \\ &\qquad + \int_{t_0-\delta}^t \tilde{h}\big(X_s(i)\big)\,ds + \int_{t_0-\delta}^t \sqrt{2g\big(X_s(i)\big)}\,dB_s(i) \\ &\ge \int_{t_0-\delta}^t \alpha m(i,j)\varepsilon\,ds + \int_{t_0-\delta}^t \tilde{h}\big(X_s(i)\big)\,ds + \int_{t_0-\delta}^t \sqrt{2g\big(X_s(i)\big)}\,dB_s(i). \end{aligned} \tag{2.93}$$

for all $t \in [t_0-\delta, t_0]$. By standard comparison results (e.g. Theorem (V.43.1) in [29] and a stopping argument), this implies $X_{t_0}(i) \ge Y_\delta^{m(i,j)\varepsilon}$ on $A_{\varepsilon,\delta}$ a.s. By path continuity, $A_{\varepsilon,\delta}$ approximates $\{X_{t_0}(j) > 0\}$ as $\delta, \varepsilon \to 0$. It follows that on $X_{t_0}(j) > 0$ we have $X_{t_0}(i) > 0$ for all i such that $m(i,j) > 0$ a.s. With the migration kernel being irreducible every site can be reached from j. By induction we conclude that every component of X_{t_0} is positive a.s. given $X_{t_0}(j) > 0$.

Starting in a nontrivial translation invariant measure the system a.s. never hits $\underline{0}$. Therefore, there is a location j with $X_{t_0}(j) > 0$ a.s. This proves the lemma. □

Chapter 3

The Virgin Island Model

Chapter 3 is devoted to the Virgin Island process $(V_t)_{t\geq 0}$ which has been defined in (1.19) as total sum over all n-th generation processes $(V_t^{(n)})_{t\geq 0}$, $n \geq 0$. The 0-th generation process is the one-dimensional diffusion $(Y_t)_{t\geq 0}$ which is the solution of (1.10). The key ingredient in the construction of the Virgin Island process is the law $\bar{Q}_Y$ of excursions of $(Y_t)_{t\geq 0}$ from the absorbing boundary zero. The excursion measure $\bar{Q}_Y$ is defined through Theorem 6 which we prove in Section 3.1. In addition, Section 3.1 contains a number of preliminary lemmas. Fast readers may want to proceed directly to Section 3.2.

Section 3.4 includes our proof of the extinction result (Theorem 7). The key step for this proof is Lemma 3.3.2 which asserts that the Laplace transform of the Virgin Island process satisfies a certain integro-differential equation. This key equation is related to a concave function which is studied in Lemma 3.4.1. The concavity of this function is the second important observation in the proof of Theorem 7.

In Section 3.2, we prove Theorem 8 which specifies the asymptotic behaviour of the expected man-hours of V up to time t as $t \to \infty$. We will show that the expression in (1.22) satisfies a renewal equation, see equation (3.72). Thus, the main part of the proof of Theorem 8 consists of known results from renewal theory. The explicit formulas in (1.60) and in (1.61) are derived in Lemma 3.1.3 and in Lemma 3.1.5.

3.1 Excursions from a trap of one-dimensional diffusions. Proof of Theorem 6

Recall the Assumptions A1, A3, A4 and A5 from Section 1.2. The process $(Y_t)_{t\geq 0}$, the scale function $\bar{S}$ and the excursion set U have been defined in (1.10), in (1.12) and in (1.11), respectively.

In this section, we define the excursion measure $\bar{Q}_Y$ and prove the convergence result of Theorem 6. We follow Pitman and Yor [28] in the construction of the excursion measure. Under Assumptions A1 and A3, zero is an absorbing point for Y. Thus, we cannot simply start in zero and wait until the process returns to zero. Informally speaking, we instead condition the process to converge to infinity. One way to achieve this is by Doob's h-transformation. Note that $\big(\bar{S}(Y_{t\wedge T_\varepsilon})\big)_{t\geq 0}$ is a bounded martingale for every $\varepsilon > 0$, see Section V.28 in [29]. In particular,

$$\mathbf{E}^y\big[\bar{S}(Y_{t\wedge T_\varepsilon})\big] = \bar{S}(y) \tag{3.1}$$

for every $y < \varepsilon$ by the optional stopping theorem. For $\varepsilon > 0$, consider the diffusion $(Y_t^{\uparrow,\varepsilon})_{t\geq 0}$ on $[0,\infty)$ – to be called the $\uparrow$-diffusion stopped at time T_ε – defined by the semigroup $(T_t^\varepsilon)_{t\geq 0}$

$$T_t^\varepsilon f(y) := \frac{1}{\bar{S}(y)}\mathbf{E}^y\big[\bar{S}(Y_{t\wedge T_\varepsilon})f(Y_{t\wedge T_\varepsilon})\big], \quad y > 0, t \geq 0. \tag{3.2}$$

The sequence of processes $\big((Y_t^{\uparrow,\varepsilon})_{t\geq 0}, \varepsilon > 0\big)$ is consistent in the sense that

$$\mathcal{L}^y\left(Y^{\uparrow,\varepsilon+\delta}_{\boldsymbol{\cdot}\wedge T_\varepsilon}\right) = \mathcal{L}^y\left(Y^{\uparrow,\varepsilon}_{\boldsymbol{\cdot}}\right) \tag{3.3}$$

for all $y, \varepsilon, \delta > 0$. Therefore, we may define a process $Y^\uparrow = (Y_t^\uparrow)_{0\leq t\leq T_\infty}$ which coincides with $(Y_t^{\uparrow,\varepsilon})_{t\geq 0}$ until time T_ε for every $\varepsilon > 0$. Note that the $\uparrow$-diffusion possibly explodes in finite time.

The following important observation of Williams has been quoted by Pitman and Yor [28]. Because we assume that zero is an exit boundary for the 0-diffusion, zero is an entrance boundary but not an exit boundary for the $\uparrow$-diffusion. Indeed, the $\uparrow$-diffusion started at its entrance boundary zero and run up to the last time it hits a level $y > 0$ is described by Theorem 2.5 of Williams [33] as the time reversal back from T_0 of the $\downarrow$-diffusion started at y, where the $\downarrow$-diffusion is the 0-diffusion conditioned on $T_0 < \infty$. Hence, the process $\big(Y_t^\uparrow\big)_{t\geq 0}$ may be started in zero but takes strictly positive values at positive times.

Pitman and Yor [28] define the excursion measure $\bar{Q}_Y$ as follows. Under

$$\bar{Q}_Y(\boldsymbol{\cdot}|T_\varepsilon < T_0), \tag{3.4}$$

that is, conditional on "excursions reach level ε", an excursion follows the $\uparrow$-diffusion until time T_ε and then follows the 0-diffusion. With this in mind, define a process $\hat{Y}^\varepsilon := \big(\hat{Y}_t^\varepsilon\big)_{t\geq 0}$ which satisfies

$$\mathcal{L}^y\big((\hat{Y}^\varepsilon_{t\wedge T_\varepsilon})_{t\geq 0}\big) = \mathcal{L}^y\big((Y_t^{\uparrow,\varepsilon})_{t\geq 0}\big) \tag{3.5}$$

$$\mathcal{L}^y\big((\hat{Y}^\varepsilon_{T_\varepsilon+t})_{t\geq 0}\big) = \mathcal{L}^\varepsilon\big((Y_t)_{t\geq 0}\big) \tag{3.6}$$

for $y \geq 0$. In addition, $(\hat{Y}_t^\varepsilon, t \leq T_\varepsilon)$ and $(\hat{Y}_t^\varepsilon, t \geq T_\varepsilon)$ are independent. Define the excursion measure $\bar{Q}_Y$ on U by

$$\mathbb{1}_{T_\varepsilon < T_0} \bar{Q}_Y(d\chi) := \frac{1}{\bar{S}(\varepsilon)} \mathbf{P}^0\big(\hat{Y}^\varepsilon \in d\chi\big), \quad \varepsilon > 0. \tag{3.7}$$

This is well-defined if

$$\mathbb{1}_{T_{\varepsilon+\delta} < T_0} \frac{1}{\bar{S}(\varepsilon)} \mathbf{P}^0\big(\hat{Y}^\varepsilon \in d\chi\big) = \frac{1}{\bar{S}(\varepsilon+\delta)} \mathbf{P}^0\big(\hat{Y}^{\varepsilon+\delta} \in d\chi\big) \tag{3.8}$$

holds for all $\varepsilon, \delta > 0$. The critical part here is the path between T_ε and $T_{\varepsilon+\delta}$. Therefore, (3.8) follows from

$$\begin{aligned} &\frac{1}{\bar{S}(\varepsilon)} \mathbf{E}^\varepsilon\big[F(Y) \mathbb{1}_{T_{\varepsilon+\delta} < T_0}\big] = \frac{1}{\bar{S}(\varepsilon+\delta)} \mathbf{E}^\varepsilon\big[F(Y) | T_{\varepsilon+\delta} < T_0\big] \\ &= \frac{1}{\bar{S}(\varepsilon+\delta)} \mathbf{E}^\varepsilon\big[F(\hat{Y}^{\varepsilon+\delta})\big] = \frac{1}{\bar{S}(\varepsilon+\delta)} \mathbf{E}^0\big[F(\hat{Y}^{\varepsilon+\delta}_{T_\varepsilon + \cdot})\big]. \end{aligned} \tag{3.9}$$

The first equality follows from equation (1.13) with $a = 0$, $y = \varepsilon$ and $b = \varepsilon+\delta$. The last equality is the strong Markov property of $Y^{\uparrow, \varepsilon+\delta}$. The last but one equality is the following lemma.

Lemma 3.1.1. *Assume A1 and A3. Let* $0 < y < \varepsilon$. *Then*

$$\mathcal{L}^y\big(\, Y \,|\, T_\varepsilon < T_0\big) = \mathcal{L}^y\big(\hat{Y}^\varepsilon\big). \tag{3.10}$$

Proof. We begin with the proof of independence of $(\hat{Y}_t^\varepsilon, t \leq T_\varepsilon)$ and of $(\hat{Y}_t^\varepsilon, t \geq T_\varepsilon)$. Let F and G be two bounded continuous functions on the path space. Denote by $\mathcal{F}_{T_\varepsilon}$ the σ-algebra generated by $(Y_t)_{t \leq T_\varepsilon}$. Then

$$\begin{aligned} &\mathbf{E}^y\big[F\big(Y_{T_\varepsilon \wedge \cdot}\big) G\big(Y_{T_\varepsilon + \cdot}\big) | T_\varepsilon < T_0\big] \\ &= \mathbf{E}^y\Big[F\big(Y_{T_\varepsilon \wedge \cdot}\big) \mathbf{E}^y\big[G\big(Y_{T_\varepsilon + \cdot}\big) | \mathcal{F}_{T_\varepsilon}\big] | T_\varepsilon < T_0\Big] \\ &= \mathbf{E}^y\big[F\big(Y_{T_\varepsilon \wedge \cdot}\big) | T_\varepsilon < T_0\big] \mathbf{E}^\varepsilon\big[G\big(Y_\cdot\big)\big]. \end{aligned} \tag{3.11}$$

The last equality is the strong Markov property of Y. Choosing $F \equiv 1$ in (3.11) proves that the left-hand side of (3.10) satisfies (3.6). In addition, equation (3.11) proves the desired independence. For the proof of

$$\mathbf{P}^y\big((Y_t^{\uparrow,\varepsilon})_{t \geq 0}\big) = \mathbf{P}^y\big((Y_{t \wedge T_\varepsilon})_{t \geq 0} | T_\varepsilon < T_0\big), \tag{3.12}$$

we exploit the fact that

$$\mathbf{E}^y\left[\prod_{i=1}^n f_i\big(Y_{t_i}^{\uparrow,\varepsilon}\big)\right] = \frac{1}{\bar{S}(y)} \mathbf{E}^y\left[\bar{S}(Y_{t_n \wedge T_\varepsilon}) \prod_{i=1}^n f_i\big(Y_{t_i \wedge T_\varepsilon}\big)\right] \tag{3.13}$$

for bounded, continuous functions $f_1, ..., f_n$ and time points $0 \leq t_1 < ... < t_n$. By equation (1.13) with $a := 0$,

$$\bar{S}(Y_{t_n \wedge T_\varepsilon}) = \bar{S}(\varepsilon)\mathbf{P}^{Y_{t_n \wedge T_\varepsilon}}\big[T_\varepsilon < T_0\big] = \bar{S}(\varepsilon)\mathbf{E}^y\big[\mathbb{1}_{T_\varepsilon < T_0} | \mathcal{F}_{t_n \wedge T_\varepsilon}\big] \tag{3.14}$$

$\mathbf{P}^y$–almost surely where $\mathcal{F}_{t_n \wedge T_\varepsilon}$ is the σ-algebra generated by $(Y_s)_{s \leq t_n \wedge T_\varepsilon}$. Insert this identity in the right-hand side of (3.13) to obtain

$$\mathbf{E}^y\left[\prod_{i=1}^n f_i\big(Y_{t_i}^{\uparrow,\varepsilon}\big)\right] = \frac{1}{\mathbf{P}^y\big(T_\varepsilon < T_0\big)}\mathbf{E}^y\left[\mathbb{1}_{T_\varepsilon < T_0}\prod_{i=1}^n f_i\big(Y_{t_i \wedge T_\varepsilon}\big)\right]. \tag{3.15}$$

This proves (3.12) because finite dimensional distributions determine the law of a process. □

Now we prove convergence to the excursion measure $\bar{Q}_Y$.

Proof of Theorem 6. Let $F\colon \mathbf{C}\big([0,\infty),[0,\infty)\big) \to \mathbb{R}$ be a bounded continuous function for which there exists an $\varepsilon > 0$ such that $F(\chi)\mathbb{1}_{T_0 < T_\varepsilon} = 0$ for every path χ. Let $0 < y < \varepsilon$. With Lemma 3.1.1, we obtain

$$\begin{aligned} \frac{1}{\bar{S}(y)}\mathbf{E}^y F(Y) &= \frac{1}{\bar{S}(\varepsilon)\mathbf{P}^y(T_\varepsilon < T_0)}\mathbf{E}^y\big[F(Y)\mathbb{1}_{T_\varepsilon < T_0}\big] \\ &= \frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^y F(\hat{Y}^\varepsilon) = \frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^0 F(\hat{Y}^\varepsilon_{T_y + \bullet}). \end{aligned} \tag{3.16}$$

The last equality is the strong Markov property of the $\uparrow$-diffusion. The random time T_y converges to zero almost surely as $y \to 0$. Another observation we need is that every continuous path $(\chi_t)_{t \geq 0}$ is uniformly continuous on any compact set $[0,T]$. Hence, the sequence of paths $\big((\chi_{T_y+t})_{t\geq 0}, y > 0\big)$ converges locally uniformly to the path $\big(\chi_t\big)_{t \geq 0}$ almost surely as $y \to 0$. Therefore, the dominated convergence theorem implies

$$\lim_{y \to 0}\mathbf{E}^0 F(\hat{Y}^\varepsilon_{T_y+\bullet}) = \mathbf{E}^0 \lim_{y \to 0} F(\hat{Y}^\varepsilon_{T_y+\bullet}) = \mathbf{E}^0 F(\hat{Y}^\varepsilon_\bullet). \tag{3.17}$$

Putting (3.16) and (3.17) together, we arrive at

$$\begin{aligned} \lim_{y \to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y F(Y) &= \frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^0 F(\hat{Y}^\varepsilon) = \int F(\chi)\mathbb{1}_{T_\varepsilon < T_0}\bar{Q}_Y(d\chi) \\ &= \int F(\chi)\bar{Q}_Y(d\chi), \end{aligned} \tag{3.18}$$

which proves the theorem. □

We require the convergence (1.52) of Theorem 6 to hold for functionals F which are not included in the assertion of Theorem 6. For example, we will prove in Lemma 3.1.5 together with Lemma 3.1.7 that

$$\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y\Big(\int_0^\infty Y_s\,ds\Big) = \int\Big(\int_0^\infty \chi_s\,ds\Big)\bar{Q}_Y(d\chi) = \int_0^\infty \frac{z}{g(z)\bar{s}(z)}\,dz \tag{3.19}$$

provided that Assumptions A1, A3 and A5 hold. The first equality in equation (3.19) cannot be concluded directly from Theorem 6 because the functional $(\chi_s)_{s\geq 0}\mapsto \int_0^\infty \chi_s\,ds$ is neither bounded nor is it equal to zero whenever $\sup_{t\geq 0}\chi_t\leq\varepsilon$ for some $\varepsilon>0$. The following lemmas prepare for the proof of (3.19).

Lemma 3.1.2. *Assume A1 and A3. Let the continuous function f have compact support in $(0,\infty)$. Furthermore, let the continuous function $\phi\colon[0,\infty)\to[0,\infty)$ be either nonnegative and nondecreasing, or Lebesgue-integrable. Then*

$$\frac{1}{\bar{S}(y)}\mathbf{E}^y\Big(\int_0^{T_b}\phi(s)f(Y_s)\,ds\Big)\longrightarrow\int\Big(\int_0^{T_b}\phi(s)f(\chi_s)\,ds\Big)\bar{Q}_Y(d\chi)\quad(y\to 0) \tag{3.20}$$

for every $b\leq\infty$.

Proof. Let $\varepsilon>0$ be such that $\varepsilon<\inf\operatorname{supp} f$ and let $y<\varepsilon$. W.l.o.g. we assume $f\geq 0$. Using Lemma 3.1.1, we see that the left-hand side of (3.20) is equal to

$$\begin{aligned}&\frac{1}{\bar{S}(y)}\mathbf{E}^y\Big[\int_0^{T_b}\phi(s)f(Y_s)\,ds\mathbb{1}_{T_\varepsilon<T_0}\Big]=\frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^y\Big[\int_0^{T_b}\phi(s)f(\hat{Y}_s^\varepsilon)\,ds\Big]\\&=\frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^0\Big[\int_{T_y}^{T_b}\phi(s-T_y)f(\hat{Y}_s^\varepsilon)\,ds\Big]\xrightarrow{y\to 0}\int\int_0^{T_b}\phi(s)f(\chi_s)\,ds\mathbb{1}_{T_\varepsilon<T_0}\bar{Q}_Y(d\chi).\end{aligned}$$

The second equality is the strong Markov property of $Y^{\uparrow,\varepsilon}$ and the change of variable $s\mapsto s-T_y$. For the convergence, we applied the dominated convergence theorem or the monotone convergence theorem, respectively, depending on whether ϕ is Lebesgue-integrable or not. □

The explicit formula on the right-hand side of (3.19) originates in the explicit formula (3.21) below, which we recall from the literature. The proof of the second equality in (3.19) is essentially contained in Lemma 3.1.5 below.

Lemma 3.1.3. *Assume A1 and A3. If $f\in\mathbf{C}_b[0,\infty)$ or if $f\in\mathbf{C}\big([0,\infty),[0,\infty)\big)$, then*

$$\mathbf{E}^y\Big(\int_0^{T_0\wedge T_b}f(Y_s)\,ds\Big)=\int_0^b\Big(f(z)\frac{\bar{S}(b)-\bar{S}(y\vee z)}{\bar{S}(b)}\frac{\bar{S}(y\wedge z)}{g(z)\bar{s}(z)}\Big)\,dz \tag{3.21}$$

for every $0\leq y\leq b$.

Proof. See e.g. Section 15.3 of Karlin and Taylor [21]. □

Let $(Y_t)_{t\geq 0}$ be a Markov process with càdlàg sample paths and state space E which is equipped with a Polish topology. For an open set $O \subset E$, denote by τ the first exit time of $(Y_t)_{t\geq 0}$ from the set O. Notice that τ is a stopping time. For $m \in \mathbb{N}_0$, define

$$w_m(y) := \mathbf{E}^y\Big(\int_0^\tau s^m f(Y_s)\,ds\Big),\ y \in E, \tag{3.22}$$

for a given function $f \in \mathbf{C}\big(O,[0,\infty)\big)$. In the following lemma, we derive expressions for w_1 and w_2 for which Lemma 3.1.3 is applicable.

Lemma 3.1.4. *Let $(Y_t)_{t\geq 0}$ be a time homogeneous Markov process with càdlàg sample paths and state space E which is equipped with a Polish topology. Let w_m be as in (3.22) with an open set $O \subset E$ and with a function $f \in \mathbf{C}\big(O,[0,\infty)\big)$. Then*

$$\mathbf{E}^y\Big(\int_0^\tau s f(Y_s)\,ds\Big) = \mathbf{E}^y\Big(\int_0^\tau w_0(Y_s)\,ds\Big) \tag{3.23}$$

$$\mathbf{E}^y\Big(\int_0^\tau s^2 f(Y_s)\,ds\Big) = \mathbf{E}^y\Big(\int_0^\tau 2w_1(Y_s)\,ds\Big) \tag{3.24}$$

for all $y \in E$.

Proof. Let $y \in E$ be fixed. For the proof of (3.23), we apply Fubini to obtain

$$\begin{aligned} w_1(y) &= \mathbf{E}^y\Big(\int_0^\tau \int_0^s dr f(Y_s)\,ds\Big) = \mathbf{E}^y\Big(\int_0^\tau \int_r^\tau f(Y_s)\,ds\,dr\Big) \\ &= \int_0^\infty \mathbf{E}^y\Big(\mathbb{1}_{r<\tau}\int_0^\infty \mathbb{1}_{s+r<\tau} f(Y_{s+r})ds\Big)\,dr \end{aligned} \tag{3.25}$$

The last equality follows from Fubini and a change of variables. The stopping time τ can be expressed as $\tau = F\big((Y_u)_{u\geq 0}\big)$ with a suitable path functional F. Furthermore, τ satisfies

$$\{r<\tau\}\cap\{s+r<\tau\} = \{r<\tau\}\cap\{s<F\big((Y_{u+r})_{u\geq 0}\big)\} \tag{3.26}$$

for $r, s \geq 0$. Therefore, the right-hand side of (3.25) is equal to

$$\begin{aligned} &\int_0^\infty \mathbf{E}^y\Big(\mathbb{1}_{r<\tau}\int_0^\infty \mathbb{1}_{s<F\left((Y_{u+r})_{u\geq 0}\right)} f(Y_{s+r})\,ds\Big)\,dr \\ &= \int_0^\infty \mathbf{E}^y\Big(\mathbb{1}_{r<\tau}\mathbf{E}^{Y_r}\Big[\int_0^\infty \mathbb{1}_{s<\tau} f(Y_s)\,ds\Big]\Big)\,dr = \mathbf{E}^y\Big(\int_0^\tau w_0(Y_r)\,dr\Big). \end{aligned} \tag{3.27}$$

The last but one equality is the Markov property of $(Y_t)_{t\geq 0}$. This proves (3.23). For the proof of (3.24), break the symmetry in the square of $w_2(y)$ to see that $w_2(y)$ is equal to

$$\begin{aligned}(3.28)\qquad &\mathbf{E}^y\Big(\int_0^\tau f(Y_s)\int_0^s\int_0^s 2\mathbb{1}_{r\leq v}dv\,dr\,ds\Big) = 2\mathbf{E}^y\Big(\int_0^\tau\int_r^\tau (s-r)f(Y_s)\,ds\,dr\Big)\\ &= 2\int_0^\infty \mathbf{E}^y\Big(\mathbb{1}_{r<\tau}\int_0^{\tau-r} sf(Y_{s+r})ds\Big)\,dr = \mathbf{E}^y\Big(\int_0^\tau 2w_1(Y_r)dr\Big).\end{aligned}$$

This finishes the proof. □

The following lemma proves the second equality in (3.19). For this, denote the monotone limit $\lim_{y\to\infty}\bar S(y)$ by $\bar S(\infty)$ and define

$$(3.29)\qquad w(z) := \int_0^\infty f(u)\frac{\bar S(z\wedge u)}{g(u)\bar s(u)}\,du,\quad z\geq 0$$

for $f\in\mathbf{C}\big([0,\infty),[0,\infty)\big)$. If $\bar S(\infty)=\infty$, then $w(z)$ is the monotone limit of the right-hand side of (3.21) as $b\to\infty$.

Lemma 3.1.5. *Assume A1, A3 and* $\bar S(\infty)=\infty$. *Let* $f\in\mathbf{C}\big([0,\infty),[0,\infty)\big)$. *Then*

$$(3.30)\qquad \int\Big(\int_0^\infty f(\chi_s)\,ds\Big)\bar Q_Y(d\chi) = \int_0^\infty f(z)\frac{1}{g(z)\bar s(z)}\,dz$$

$$(3.31)\qquad \int\Big(\int_0^\infty sf(\chi_s)\,ds\Big)\bar Q_Y(d\chi) = \int_0^\infty w(z)\frac{1}{g(z)\bar s(z)}\,dz.$$

If (3.30) *is finite, then* (3.30) *is equal to*

$$(3.32)\qquad \int_0^\infty f(z)\frac{1}{g(z)\bar s(z)}\,dz = \lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y\Big(\int_0^\infty f(Y_s)\,ds\Big).$$

If (3.31) *is finite, then* (3.31) *is equal to*

$$(3.33)\qquad \int_0^\infty w(z)\frac{1}{g(z)\bar s(z)}\,dz = \lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y\Big(\int_0^\infty sf(Y_s)\,ds\Big).$$

Proof. Choose $f_\varepsilon\in\mathbf{C}\big([0,\infty),[0,\infty)\big)$ with compact support in $(0,\infty)$ for every $\varepsilon>0$ such that $f_\varepsilon\uparrow f$ as $\varepsilon\to 0$. Fix $\varepsilon>0$ and $b\in(0,\infty)$. Lemma 3.1.2 proves that

$$(3.34)\qquad \lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y\Big(\int_0^{T_b} f_\varepsilon(Y_s)\,ds\Big) = \int\Big(\int_0^{T_b} f_\varepsilon(\chi_s)\,ds\Big)\bar Q_Y(d\chi).$$

Lemma 3.1.3 provides us with an expression for the left-hand side of equation (3.34). Hence,

$$\begin{aligned}\int \Big(\int_0^{T_b} f_\varepsilon(\chi_s)\,ds\Big)\bar{Q}_Y(d\chi) &= \lim_{y\to 0}\frac{1}{\bar{S}(y)}\int_0^b f_\varepsilon(z)\frac{\bar{S}(b)-\bar{S}(y\vee z)}{\bar{S}(b)}\frac{\bar{S}(y\wedge z)}{g(z)\bar{s}(z)}\,dz\\ &= \int_0^b f_\varepsilon(z)\Big(1-\frac{\bar{S}(z)}{\bar{S}(b)}\Big)\frac{1}{g(z)\bar{s}(z)}\,dz.\end{aligned}$$

The last equation follows from dominated convergence and Assumption A3. Notice that $T_b\big((\chi_t)_{t\geq 0}\big)\to\infty$ as $b\to\infty$ for every continuous path $(\chi_t)_{t\geq 0}$. Letting $b\to\infty$ and $\varepsilon\to 0$, apply monotone convergence to arrive at equation (3.30).

Now we prove (3.32). By the monotone convergence theorem, the right-hand side of (3.32) is equal to

$$(3.35)\qquad\begin{aligned}\lim_{y\to 0}\frac{1}{\bar{S}(y)}\lim_{b\to\infty}\mathbf{E}^y\Big(\int_0^{T_b} f(Y_s)\,ds\Big) &= \lim_{y\to 0}\frac{1}{\bar{S}(y)}\int_0^\infty f(z)\frac{\bar{S}(y\wedge z)}{g(z)\bar{s}(z)}\,dz\\ = \int_0^\infty f(z)\frac{\lim_{y\to 0}\frac{1}{\bar{S}(y)}\bar{S}(y\wedge z)}{g(z)\bar{s}(z)}\,dz &= \int_0^\infty f(z)\frac{1}{g(z)\bar{s}(z)}\,dz.\end{aligned}$$

The first equality is Lemma 3.1.3 and monotone convergence. The second equality follows from dominated convergence and the assumption that (3.30) is finite.

Similar arguments prove (3.31) and (3.33). Instead of (3.34), consider

$$(3.36)\qquad \lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y\Big(\int_0^{T_b} sf_\varepsilon(Y_s)\,ds\Big) = \int\Big(\int_0^{T_b} sf_\varepsilon(\chi_s)\,ds\Big)\bar{Q}_Y(d\chi)$$

which is implied by Lemma 3.1.2. Furthermore, instead of applying Lemma 3.1.3 to equation (3.34), apply equation (3.23) together with equation (3.21). □

We will need that $(Y_t)_{t\geq 0}$ dies out in finite time. The following lemma gives a condition for this. Recall $\bar{S}(\infty):=\lim_{y\to\infty}\bar{S}(y)$.

Lemma 3.1.6. *Assume A1 and A3. Let $y>0$. Then the solution $(Y_t)_{t\geq 0}$ of equation* (1.10) *hits zero in finite time almost surely if and only if $\bar{S}(\infty)=\infty$. If $\bar{S}(\infty)<\infty$, then $(Y_t)_{t\geq 0}$ converges to infinity as $t\to\infty$ on the event $\{T_0=\infty\}$ almost surely.*

Proof. On the event $\{Y_t\leq K\}$, we have that

$$(3.37)\qquad \mathbf{P}^{Y_t}\big(\exists s\colon Y_s=0\big)\geq\mathbf{P}^K\big(T_0<\infty\big)>0$$

almost surely. The last inequality follows from Lemma 15.6.2 of [21] and Assumption A3. Therefore, Theorem 2 of Jagers [20] implies that with probability one either $(Y_t)_{t\geq 0}$ hits zero in finite time or converges to infinity as $t \to \infty$. With equation (1.13), we obtain

$$\mathbf{P}^y\big(\lim_{t\to\infty} Y_t = \infty\big) = \lim_{b\to\infty} \mathbf{P}^y\big(Y \text{ hits } b \text{ before } 0\big) = \lim_{b\to\infty} \frac{\bar{S}(y)}{\bar{S}(b)} = \frac{\bar{S}(y)}{\bar{S}(\infty)}. \tag{3.38}$$

This proves the assertion. □

The following lemma provides sufficient conditions under which the expected area under $(Y_t)_{t\geq 0}$ and the expected area under an typical excursion of $(Y_t)_{t\geq 0}$ are finite. Recall $(Y_t)_{t\geq 0}$ and $\bar{Q}_Y$ from (1.10) and Theorem 6, respectively.

Lemma 3.1.7. *Assume A1 and A3. Assumption A5 holds if and only if*

$$\mathbf{E}^y\Big(\int_0^\infty Y_s\,ds\Big) < \infty \qquad \forall\, y > 0. \tag{3.39}$$

If Assumption A5 holds, then $\bar{S}(\infty) = \infty$,

$$\int \Big(\int_0^\infty \chi_s\,ds\Big)\bar{Q}_Y(d\chi) = \int_0^\infty \frac{z}{g(z)\bar{s}(z)}\,dz < \infty \tag{3.40}$$

and

$$\mathbf{E}^y\Big(\int_0^\infty Y_s\,ds\Big) = \int_0^\infty \bar{S}\big(y\wedge z\big)\frac{z}{g(z)\bar{s}(z)}\,dz < \infty \tag{3.41}$$

for all $y \geq 0$.

Proof. In equation (3.21) with $f(z) := z$, let $b \to \infty$ and apply monotone convergence to obtain

$$\begin{aligned} \mathbf{E}^y\Big(\int_0^\infty Y_s\,ds\Big) &= \int_0^\infty \Big(z\Big[1 - \frac{\bar{S}(y\vee z)}{\bar{S}(\infty)}\Big]\frac{\bar{S}(y\wedge z)}{g(z)\bar{s}(z)}\Big)\,dz \\ &\leq y\int_0^y \frac{\bar{S}(z)}{g(z)\bar{s}(z)}\,dz + \bar{S}(y)\int_y^\infty \frac{z}{g(z)\bar{s}(z)}\,dz. \end{aligned} \tag{3.42}$$

Hence, if Assumption A5 holds, then Assumption A3 implies that the right-hand side of (3.42) is finite and thus the left-hand side of (3.42) is finite. Furthermore, $(Y_t)_{t\geq 0}$ does not converge to infinity with positive probability as $t \to \infty$. Lemma 3.1.6 implies $\bar{S}(\infty) = \infty$. Thus, the equality in (3.42) implies (3.41). The equation (3.40) follows from Lemma 3.1.5 with $f(y) := y$.

Now we prove that Assumption A5 holds if the left-hand side of (3.42) is finite. Again, Lemma 3.1.6 implies $\bar{S}(\infty) = \infty$. Using monotonicity of S, we obtain for $x > 0$

$$\int_x^\infty \frac{z}{g(z)\bar{s}(z)}\,dz \leq \frac{1}{\bar{S}(x)}\int_0^\infty z\frac{\bar{S}(x\wedge z)}{g(z)\bar{s}(z)}\,dz. \tag{3.43}$$

The right-hand side is finite because the left-hand side of (3.42) is finite. Therefore, Assumption A5 holds. □

The convergence (1.52) of Theorem 6 also holds for $(\chi_s)_{s\geq 0} \mapsto f(\chi_t)$, t fixed, if $f(y)/y$ is a bounded function. For this, we first estimate the first two moments of $(Y_t)_{t\geq 0}$.

Lemma 3.1.8. *Assume A1. Let $(Y_t)_{t\geq 0}$ be a solution of equation* (1.10) *and let T be finite. Then there exists a constant C_T such that*

$$\sup_{t\leq T} \mathbf{E}^y\big[Y_{\tau\wedge t}\big] \leq C_T y, \quad \mathbf{E}^y\big[\sup_{t\leq T} {Y_t}^2\big] \leq C_T(y+y^2) \tag{3.44}$$

for all $y \geq 0$ and every stopping time τ.

Proof. We begin with the proof of the second inequality in (3.44). Let τ be an arbitrary stopping time and choose C_h such that $h(y) \leq C_h y$ for all $y \geq 0$. The process $(Y_t)_{t\geq 0}$ is almost surely bounded by the solution $(Z_t)_{t\geq 0}$ of

$$dZ_t = C_h Z_t\,dt + \sqrt{2g(Z_t)}\,dB_t, \quad Z_0 = y, \tag{3.45}$$

where $(B_t)_{t\geq 0}$ is the same Brownian motion as in (1.10). See Lemma 2.1.3 for this comparison. By Itô's formula,

$$d{Z_t}^2 = 2Z_t C_h Z_t\,dt + 2g(Z_t)\,dt + 2Z_t\sqrt{2g(Z_t)}\,dB_t. \tag{3.46}$$

The stochastic integral on the right-hand side is a martingale when stopped at the stopping time $\tau_K := \inf\{t \geq 0\colon Z_t \geq K\}$, $K \geq 0$. By Assumption A1, $g(y) \leq C_g(y+y^2)$ for all $y \geq 0$ and for some constant $C_g < \infty$. Taking expectations, we obtain for every $t \leq T$

$$\begin{aligned} &\mathbf{E}Z^2_{t\wedge\tau\wedge\tau_K} \\ &\leq y^2 + \int_0^t \mathbf{E}\Big[2C_h Z^2_{s\wedge\tau\wedge\tau_K}\Big]\,ds + \int_0^t \mathbf{E}2g(Z_{s\wedge\tau\wedge\tau_K})\,ds \\ &\leq y^2 + 2C_g T\sup_{s\leq T}\mathbf{E}Z_{s\wedge\tau\wedge\tau_K} + 2(C_h+C_g)\int_0^t \mathbf{E}Z^2_{s\wedge\tau\wedge\tau_K}\,ds. \end{aligned} \tag{3.47}$$

By Gronwall's inequality and the first inequality in (3.44), we conclude

$$\mathbf{E}Z^2_{t\wedge\tau\wedge\tau_K} \leq (y^2 + \tilde{C}_T y)e^{2(C_h+C_g)T} \tag{3.48}$$

for some finite constant $\tilde{C}_T$. Notice that $\tau_K \to \infty$ as $K \to \infty$ almost surely. Apply Fatou's lemma and Doob's L_2-inequality to the submartingale $(Z_{t\wedge\tau_K})_{t\geq 0}$ to obtain

$$\begin{aligned} \mathbf{E}^y\big[\sup_{t\leq T}{Y_t}^2\big] &\leq \mathbf{E}^y\big[\liminf_{K\to\infty}\sup_{t\leq T} Z^2_{t\wedge\tau_K}\big] \leq \liminf_{K\to\infty}\mathbf{E}^y\big[\sup_{t\leq T} Z^2_{t\wedge\tau_K}\big] \\ &\leq \liminf_{K\to\infty} 4\mathbf{E}^y\big[Z^2_{T\wedge\tau_K}\big] \leq C_T(y+y^2) \end{aligned} \tag{3.49}$$

for some finite constant C_T. The last inequality is (3.48).

The proof of the first inequality in (3.44) is similar to the proof of inequality (3.48). Instead of considering (3.46), stop equation (3.45) at τ_K and take expectations. □

Lemma 3.1.9. *Assume A1, A3 and A4. Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous function such that $C_f := sup_{y>0} \frac{|f(y)|}{y} < \infty$. Then*

$$\int f(\chi_t)\bar{Q}_Y(d\chi) = \lim_{y\to 0} \frac{1}{\bar{S}(y)} \mathbf{E}^y f(Y_t) = \mathbf{E}^0 \big[\frac{1}{\bar{S}(Y_t^{\uparrow})} f(Y_t^{\uparrow}) \mathbb{1}_{t<T_\infty} \big] < \infty \tag{3.50}$$

for all $t > 0$.

Proof. W.l.o.g. we may assume $f \geq 0$. Choose $f_\varepsilon \in \mathbf{C}\big([0,\infty),[0,\infty)\big)$ with compact support in $(0,\infty)$ for every $\varepsilon > 0$ such that $f_\varepsilon \uparrow f$ pointwise as $\varepsilon \to 0$. By Theorem 6,

$$\int f_\varepsilon(\chi_t)\bar{Q}_Y(d\chi) = \lim_{y\to 0} \frac{1}{\bar{S}(y)} \mathbf{E}^y f_\varepsilon(Y_t). \tag{3.51}$$

The left-hand side of (3.51) converges to the left-hand side of (3.50) as $\varepsilon \to 0$ by the monotone convergence theorem. Hence, the first equality in (3.50) follows from (3.51) if the limits $\lim\limits_{\varepsilon\to 0}$ and $\lim\limits_{y\to 0}$ interchange. For this, we prove the second equality in (3.50).

Let $b \in (0,\infty)$. The $\uparrow$-diffusion is a strong Markov process. Thus, by equation (3.2),

$$\begin{aligned} &\lim_{y\to 0} \frac{1}{\bar{S}(y)} \mathbf{E}^y \big[f(Y_t) \mathbb{1}_{t<T_b} \big] = \lim_{y\to 0} \mathbf{E}^y \big[\frac{f(Y_t^{\uparrow})}{\bar{S}(Y_t^{\uparrow})} \mathbb{1}_{t<T_b} \big] \\ &= \mathbf{E}^0 \big[\lim_{y\to 0} \frac{f(Y_{t+T_y}^{\uparrow})}{\bar{S}(Y_{t+T_y}^{\uparrow})} \mathbb{1}_{t+T_y<T_b} \big] = \mathbf{E}^0 \big[\frac{f(Y_t^{\uparrow})}{\bar{S}(Y_t^{\uparrow})} \mathbb{1}_{t<T_b} \big]. \end{aligned} \tag{3.52}$$

The second equality follows from the dominated convergence theorem because

$$\sup_{0<y\leq b} \frac{f(y)}{\bar{S}(y)} \leq C_f \sup_{0<y\leq b} \frac{y}{\bar{S}(y)} < \infty. \tag{3.53}$$

For the last equality in (3.52), we used right-continuity of the function $t \mapsto \frac{f(Y_t^{\uparrow})}{\bar{S}(Y_t^{\uparrow})} \mathbb{1}_{t<T_b}$. Now we let $b \to \infty$ in (3.52) and apply monotone convergence to obtain

$$\lim_{b\to\infty} \lim_{y\to 0} \frac{1}{\bar{S}(y)} \mathbf{E}^y \big[f(Y_t) \mathbb{1}_{t<T_b} \big] = \mathbf{E}^0 \big[\frac{f(Y_t^{\uparrow})}{\bar{S}(Y_t^{\uparrow})} \mathbb{1}_{t<T_\infty} \big]. \tag{3.54}$$

The following estimate justifies the interchange of the limits $\lim_{b\to\infty}$ and $\lim_{y\to 0}$

$$\begin{aligned}
&\big|\lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y f(Y_t) - \lim_{b\to\infty}\lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y\big[f(Y_t)\mathbb{1}_{t<T_b}\big]\big| \\
&\le C_f \lim_{b\to\infty}\sup_{y\le 1}\frac{1}{\bar S(y)}\mathbf{E}^y\big[Y_t\mathbb{1}_{\sup_{s\le t}Y_s\ge b}\big] \\
&\le C_f \lim_{b\to\infty}\frac{1}{b}\sup_{y\le 1}\frac{y}{\bar S(y)}\sup_{y\le 1}\frac{1}{y}\mathbf{E}^y\sup_{s\le t}Y_s^2 = 0.
\end{aligned} \tag{3.55}$$

The last equality follows from $\bar S'(0)\in(0,\infty)$ and from Lemma 3.1.8. Putting (3.55) and (3.54) together, we obtain

$$\lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y\big[f(Y_t)\big] = \lim_{b\to\infty}\lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y\big[f(Y_t)\mathbb{1}_{t<T_b}\big] = \mathbf{E}^0\Big[\frac{f(Y_t^\uparrow)}{\bar S(Y_t^\uparrow)}\mathbb{1}_{t<T_\infty}\Big]. \tag{3.56}$$

Note that (3.56) is finite because of $f(y)\le C_f y$, Lemma 3.1.8 and because of $\bar S'(0)\in(0,\infty)$.

We finish the proof of the first equality in (3.50) by proving that the limits $\lim_{\varepsilon\to 0}$ and $\lim_{y\to 0}$ on the right-hand side of (3.51) interchange.

$$\begin{aligned}
&\big|\lim_{\varepsilon\to 0}\lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y f_\varepsilon(Y_t) - \lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y f(Y_t)\big| \\
&\le \lim_{\varepsilon\to 0}\lim_{y\to 0}\frac{1}{\bar S(y)}\mathbf{E}^y\big[f(Y_t)-f_\varepsilon(Y_t)\big] = \lim_{\varepsilon\to 0}\mathbf{E}^0\Big[\frac{f(Y_t^\uparrow)-f_\varepsilon(Y_t^\uparrow)}{\bar S(Y_t^\uparrow)}\mathbb{1}_{t<T_\infty}\Big] = 0.
\end{aligned} \tag{3.57}$$

The first equality is (3.56) with f replaced by $f-f_\varepsilon$. The last equality follows from the dominated convergence theorem. The function $f_\varepsilon/\bar S$ converges to $f/\bar S$ for every $y>0$ as $\varepsilon\to 0$. Note that $Y_t^\uparrow>0$ almost surely for $t>0$. Integrability of $\frac{f(Y_t^\uparrow)}{\bar S(Y_t^\uparrow)}\mathbb{1}_{t<T_\infty}$ follows from the finiteness of (3.56). □

We have settled equation (3.19) in Lemma 3.1.5 and in Lemma 3.1.7. A consequence of the finiteness of this equation is that $\liminf_{t\to\infty}\int\chi_t\,d\bar Q_Y = 0$. In the proof of the extinction result for the Virgin Island Model, we will need that $\int\chi_t\,d\bar Q_Y$ converges to zero as $t\to\infty$. This convergence will follow from equation (3.19) if $[0,\infty)\ni t\mapsto\int\chi_t\,d\bar Q_Y$ is globally upward Lipschitz continuous. We first prove that this function is bounded in t. Lemma 3.1.9 implies this boundedness if the right-hand side of (3.50) with $f(y)=y$ is bounded. Thus, we need to prove boundedness of the function $y\mapsto y/\bar S(y)$.

Lemma 3.1.10. *Assume A1, A3, A4 and A5. Then*

$$\sup_{y\in(0,\infty)}\frac{y}{\bar S(y)}<\infty. \tag{3.58}$$

Proof. It suffices to prove $\liminf_{y\to\infty}\frac{\bar S(y)}{y}>0$ because $\frac{y}{\bar S(y)}$ is locally bounded in $(0,\infty)$ and $\bar S'(0)\in(0,\infty)$ by Assumption A4. By Assumption A1, $g(y)\le C_g y^2$ for all $y\ge 1$ and a constant $C_g<\infty$. Together with A5, this implies

$$\infty>\int_1^\infty \frac{y}{g(y)\bar s(y)}\,dy\ge\frac{1}{C_g}\int_1^\infty\frac{1}{y\bar s(y)}\,dy. \tag{3.59}$$

The function $\mathbb{R}\ni x\mapsto\psi(x):=1-(1-x)^+\wedge 1$ is continuous. From the estimate (3.59), we see that

$$\infty>\int_1^\infty\frac{1}{y\bar s(y)}\,dy\ge\int_1^\infty\frac{1}{y}\big(1-\psi(\bar s(y))\big)\,dy. \tag{3.60}$$

The last inequality follows from $\frac{1}{y}\ge\mathbb{1}_{y\le 1}\ge 1-\psi(y)$. Consequently,

$$1=\lim_{z\to\infty}\frac{\int_1^z\frac{1}{y}\psi(y)\,dy}{\log(z)}=\lim_{z\to\infty}\frac{\frac{1}{z}\psi(\bar s(z))}{\frac{1}{z}}=\lim_{z\to\infty}\psi(\bar s(z)). \tag{3.61}$$

The proof of the second equation in (3.61) is similar to the proof of the lemma of L'Hospital. From (3.61), we conclude $\liminf_{y\to\infty}\bar s(y)\ge 1$ which implies

$$\liminf_{z\to\infty}\frac{\int_0^z\bar s(y)\,dy}{z}\ge 1. \tag{3.62}$$

This finishes the proof. □

Lemma 3.1.11. *Assume A1, A3, A4 and A5. Then*

$$\lim_{t\to\infty}\int\chi_t\,\bar Q_Y(d\chi)=0. \tag{3.63}$$

Proof. We will prove that the function $[0,\infty)\ni t\mapsto\int\chi_t d\bar Q_Y$ is globally upward Lipschitz continuous; see Assumption A1 for a definition of this notion. The assertion then follows from the finiteness of the integrals in equation (3.40). Let $\tau_K:=\inf\{t\ge 0\colon Y_t\ge K\}$, $K\ge 0$, let C_S be the upper bound from Lemma 3.1.10 and choose a constant C_h such that $h(y)\le C_h y$ for all $y\ge 0$. From (1.10), we obtain for $y\ge 0$ and $0\le s\le t$

$$\frac{1}{\bar S(y)}\mathbf{E}^y\big(Y_{t\wedge\tau_K}\big)-\frac{1}{\bar S(y)}\mathbf{E}^y\big(Y_{s\wedge\tau_K}\big)\le C_h\int_s^t\frac{1}{\bar S(y)}\mathbf{E}^y\big(Y_{r\wedge\tau_K}\big)\,dr. \tag{3.64}$$

Letting $K\to\infty$ and then $y\to 0$, we conclude from the dominated convergence theorem, Lemma 3.1.8 and Lemma 3.1.9 that

$$\int\chi_t\,\bar Q_Y(d\chi)-\int\chi_s\,\bar Q_Y(d\chi)\le C_h\int_s^t\mathbf{E}^0\big[\frac{Y_r^\uparrow}{\bar S(Y_r^\uparrow)}\mathbb{1}_{r<T_\infty}\big]\,dr\le C_hC_S|t-s|. \tag{3.65}$$

The last inequality follows from Lemma 3.1.10. Inequality (3.65) implies upward Lipschitz continuity which finishes the proof. □

Fix $\lambda, \kappa, t > 0$ and $f \in \mathbf{C}_b\big([0,\infty),[0,\infty)\big)$. In the following lemma, we obtain the convergence (1.52) of Theorem 6 for the functional

$$\mathbf{C}\big([0,\infty),[0,\infty)\big) \ni (\chi_s)_{s\geq 0} \mapsto 1 - \exp\Big(-\lambda\chi_t - \kappa\int_0^t \chi_s f(s)\,ds\Big) \tag{3.66}$$

which is bounded and continuous but for which there is no $\varepsilon > 0$ such that the functional vanishes whenever $\sup_{t\geq 0}\chi_t \leq \varepsilon$. Furthermore, Lemma 3.1.12 is an essential step in establishing equation (3.93), which is the key equation for the proof of the extinction result of Theorem 7.

Lemma 3.1.12. *Assume A1, A3 and A4. Let* $\lambda, \kappa \geq 0$, *let* $Y = (Y_t)_{t\geq 0}$ *be as in* (1.10) *and let* Q_Y *be as in* (1.16). *Then*

$$\begin{aligned} &-\frac{d}{dy}\Big|_{y=0}\mathbf{E}^y \exp\Big(-\lambda Y_t - \kappa\int_0^t Y_s f(s)\,ds\Big)\\ &= \int \Big[1 - \exp\Big(-\lambda\chi_t - \kappa\int_0^t \chi_s f(s)\,ds\Big)\Big] Q_Y(d\chi) \end{aligned} \tag{3.67}$$

for every $f \in \mathbf{C}_b\big([0,\infty),[0,\infty)\big)$ *and for all* $t \in [0,\infty)$.

Proof. Let $\phi_\varepsilon \in \mathbf{C}^\infty(\mathbb{R}_{\geq 0})$ be such that $\phi_\varepsilon(x) = 0$ for all $x \leq \varepsilon$, $\phi_\varepsilon(x) = x$ for all $x \geq 2\varepsilon$ and $\phi_\varepsilon(x) \uparrow x$ as $\varepsilon \to 0$. By Theorem 6 and equation (1.16), we know that

$$\begin{aligned} &-\frac{d}{dy}\Big|_{y=0}\mathbf{E}^y \exp\Big(-\lambda\phi_\varepsilon(Y_t) - \kappa\int_0^t \phi_\varepsilon(Y_s) f(s)\,ds\Big)\\ &= \int \Big[1 - \exp\Big(-\lambda\phi_\varepsilon(\chi_t) - \kappa\int_0^t \phi_\varepsilon(\chi_s) f(s)\,ds\Big)\Big] Q_Y(d\chi). \end{aligned} \tag{3.68}$$

The right-hand side of (3.68) converges to the right-hand side of (3.67) as $\varepsilon \to 0$ by the monotone convergence theorem. We will prove that the left-hand side of equation (3.68) converges to the left-hand side of equation (3.67) as $\varepsilon \to 0$. Define $\bar\phi_\varepsilon := x - \phi_\varepsilon(x) \geq 0$. The absolute difference of the left-hand sides of equations (3.68) and (3.67) is bounded by

$$\begin{aligned} &\limsup_{y\to 0}\frac{1}{y}\mathbf{E}^y\Big[\exp\Big(-\lambda\phi_\varepsilon(Y_t) - \kappa\int_0^t \phi_\varepsilon(Y_s) f(s)\,ds\Big)\\ &\qquad\qquad - \exp\Big(-\lambda Y_t - \kappa\int_0^t Y_s f(s)\,ds\Big)\Big]\\ &\leq \lim_{y\to 0}\frac{1}{y}\mathbf{E}^y\Big[\lambda\bar\phi_\varepsilon(Y_t) + \kappa\int_0^t \bar\phi_\varepsilon(Y_s) f(s)\,ds\Big]\\ &= \lambda\int \bar\phi_\varepsilon(\chi_t) Q_Y(d\chi) + \kappa\int_0^t\int \bar\phi_\varepsilon(\chi_s) Q_Y(d\chi) f(s)\,ds. \end{aligned} \tag{3.69}$$

The last step follows from the dominated convergence theorem together with Lemma 3.1.8 and from Lemma 3.1.9 because the function $\bar{\phi}_\varepsilon(x)/\bar{S}(x)$ is bounded by Assumption A4. The integrand of the second summand on the right-hand side of (3.69) is bounded by $\kappa C_f \chi_s$ uniformly in $\varepsilon > 0$, for some upper bound C_f of f, which is integrable with respect to $ds \otimes Q_Y(d\chi)$ by Lemma 3.1.8. Thus, we are allowed to apply dominated convergence. Letting $\varepsilon \to 0$ in inequality (3.69) finishes the proof. □

3.2 Proof of Theorem 8

Recall $(V_t)_{t\geq 0}$, $(V_t^{(n)})_{t\geq 0}$, $(Y_t)_{t\geq 0}$ and Q_Y from (1.19), (1.18), (1.10) and (1.16), respectively. Fix $x \geq 0$. A calculation similar to (1.53) shows that

$$\int_0^t \mathbf{E}^x V_s^{(n+1)}\, ds = \int_0^t \int \alpha \chi_u Q_Y(d\chi) \int_0^{t-u} \mathbf{E}^x V_s^{(n)}\, ds\, du \tag{3.70}$$

for $n \geq 0$ and $t \geq 0$. Summing over $n \geq 0$, this results in

$$\int_0^t \mathbf{E}^x V_s\, ds = \int_0^t \mathbf{E}^x Y_s\, ds + \int_0^t \int \alpha \chi_u Q_Y(d\chi) \int_0^{t-u} \mathbf{E}^x V_s\, ds\, du \tag{3.71}$$

for $t \geq 0$. Define

$$x(t) := \int_0^t \mathbf{E}^x V_s\, ds, \quad f(t) := \int_0^t \mathbf{E}^x Y_s\, ds, \quad \mu(du) := \int \alpha \chi_u Q_Y(d\chi)\, du \tag{3.72}$$

for $t \geq 0$. In this notation, equation (3.71) reads as renewal equation

$$x(t) = f(t) + \int_0^t x(t-u)\mu(du), \quad t \geq 0. \tag{3.73}$$

From this, (1.63) and the first equation in (1.60) follow from Theorem 5.2.8 and Theorem 5.2.9 of Jagers [19], respectively. Lemma 3.1.7 implies the second equation in (1.60). The denominator on the right-hand side of (1.63) is finite because of $ue^{-\beta u} \leq \frac{1}{\beta} e^{-1}$, $u \geq 0$, and Lemma 3.1.7.

For the proof of (1.61), define $\theta := \int_0^\infty u\mu(du)$. Corollary 5.2.14 of [19] with $c := f(\infty) < \infty$ and $n := 0$ implies that

$$\frac{1}{t} x(t) \longrightarrow \frac{c}{\theta} \qquad (\text{as } t \to \infty). \tag{3.74}$$

Note that the assumption $\theta < \infty$ of this corollary is not necessary for this conclusion. By Lemma 3.1.6, we know that $\lim_{y\to\infty} \bar{S}(y) = \infty$. Lemma 3.1.7 and equation (3.31) with $f(y) := y$ show that $\frac{c}{\theta}$ is equal to the right-hand side of (1.61). This finishes the proof. □

3.3 Recursion for the Virgin Island process

Recall the definition of $(Y_t)_{t\geq 0}$ and of $(V_t^{(k)})_{t\geq 0}$, $k \geq 0$, from (1.10) and from (1.18), respectively. We mentioned in the introduction that there is an inherent branching structure in the Virgin Island Model. One offspring island together with all its offspring islands is again a Virgin Island Model but with a typical excursion instead of $(Y_t)_{t\geq 0}$ on the first island. In this section, we exploit this branching structure to obtain a recursive equation for the Laplace transform of the Virgin Island process in Lemma 3.3.2 below. This recursive equation is the key equation for the proof the extinction result of Theorem 7.

For the proof of Lemma 3.3.2, we will need a bound on the first moment of the Virgin Island process $(V_t)_{t\geq 0}$.

Lemma 3.3.1. *Assume A1 and A4. For every $T < \infty$, there exists a constant $C_T < \infty$ such that*

$$\sup_{t\leq T}\sup_{x>0}\frac{1}{x}\mathbf{E}^x V_t \leq \sup_{t\leq T}\sum_{k=0}^{\infty}\sup_{x>0}\frac{1}{x}\mathbf{E}^x V_t^{(k)} \leq C_T. \tag{3.75}$$

Consequently, the Virgin Island process $(V_t)_{t\geq 0}$ is finite for finite time points almost surely.

Proof. Let $\tilde{C}_T$ be the constant of Lemma 3.1.8. Recall from Section 1.1 that – conditioned on $(V_t^{(n)})_{t\geq 0}$ – $\Pi^{(n)}$ is a Poisson point process with intensity measure $\alpha V_r^{(n)}\,dr \otimes Q_Y(d\chi)$. Using the definition (1.18) and Lemma 3.1.8, we obtain for $t \leq T$

$$\begin{aligned}
\sum_{k=0}^{n}\sup_{x>0}\frac{1}{x}\mathbf{E}^x V_t^{(k)} &\leq \tilde{C}_T + \sum_{k=1}^{n}\sup_{x>0}\frac{1}{x}\mathbf{E}^x\Big[\int \chi_{t-r}\,\Pi^{(k-1)}(dr,d\chi)\Big]\\
&\leq \tilde{C}_T + \sum_{k=0}^{n}\sup_{x>0}\frac{1}{x}\mathbf{E}^x\Big[\int_0^t\Big(\alpha V_r^{(k)}\int \chi_{t-r}\,Q_Y(d\chi)\Big)\,dr\Big]\\
&\leq \tilde{C}_T + \alpha\tilde{C}_T\int_0^t\sum_{k=0}^{n}\sup_{x>0}\frac{1}{x}\mathbf{E}^x V_r^{(k)}\,dr.
\end{aligned}\tag{3.76}$$

By Gronwall's lemma, this implies

$$\sup_{t\leq T}\sup_{x>0}\frac{1}{x}\mathbf{E}^x V_t \leq \sup_{t\leq T}\sup_{n>0}\sum_{k=0}^{n}\sup_{x>0}\frac{1}{x}\mathbf{E}^x V_t^{(k)} \leq \tilde{C}_T e^{\alpha\tilde{C}_T T}, \tag{3.77}$$

which proves the lemma. □

In Lemma 3.3.2, we establish an equation for the Laplace transform of the Virgin Island process. This equation will then be used in Section 3.4 to prove the extinction result of Theorem 7.

Lemma 3.3.2. *The Laplace transform* $v(t,x) := \mathbf{E}^x \exp\big(-\lambda V_t\big)$, $\lambda \geq 0$, *of the Virgin Island process is differentiable in* $x = 0$ *for every* $t > 0$. *Furthermore, it solves the equation*

$$v(t,x) = \mathbf{E}^x \exp\Big(-\lambda Y_t + \alpha \int_0^t Y_s \frac{d}{dx} v(t-s,0)\, ds\Big) \tag{3.78}$$

for all $\lambda, t, x \geq 0$.

Proof. Fix $\lambda \geq 0$ and define

$$v_n(t,x) := \mathbf{E}^x \exp\Big(-\lambda \sum_{l=0}^{n} V_t^{(l)}\Big), \quad t,x \geq 0, \;\; n \in \mathbb{N}_0. \tag{3.79}$$

We will prove by induction on n that

$$-\frac{d}{dx} v_n(t,0) = \int Q_Y(d\chi)\Big[1 - \exp\big(-\lambda\chi_t + \alpha \int_0^t dr\, \chi_r \frac{d}{dx} v_{n-1}(t-r,0)\big)\Big] \tag{3.80}$$

for all $t > 0$ and that for every $0 \leq m \leq n$ and all $t,x \geq 0$

$$v_n(t,x) = \mathbf{E}^x \exp\Big(-\lambda \sum_{k=0}^{n-m} V_t^{(k)} + \alpha \int_0^t ds\, V_s^{(n-m)} \frac{d}{dx} v_{m-1}(t-s,0)\Big) \tag{3.81}$$

where $v_{-1} \equiv 0$. If $n = 0$, then (3.80) follows from Lemma 3.1.12 with $\kappa := 0$ and (3.81) is trivial. For the induction step, suppose that (3.80) and (3.81) hold for all $0 \leq \tilde{n} \leq n-1$, $n \geq 1$. We prove (3.81) by induction on m, $0 \leq m \leq n$. The case $m = 0$ is trivial. Let $m \geq 1$. Assume that (3.81) is true for all $0 \leq \tilde{m} \leq m-1$. By the induction hypothesis and (1.18), we have for $t,x \geq 0$

$$\begin{aligned}
&v_n(t,x) \\
&= \mathbf{E}^x \exp\Big(-\lambda \sum_{k=0}^{n-(m-1)} V_t^{(k)} + \alpha \int_0^t ds \Big(V_s^{(n-(m-1))} \frac{d}{dx} v_{m-2}(t-s,0)\Big)\Big) \\
&= \mathbf{E}^x \exp\Big(-\lambda \sum_{k=0}^{n-m} V_t^{(k)} - \lambda \int \chi_{t-r} \Pi^{(n-m)}(dr,d\chi) \\
&\qquad + \alpha \int_0^t ds \Big(\int \chi_{s-r} \Pi^{(n-m)}(dr,d\chi) \frac{d}{dx} v_{m-2}(t-s,0)\Big)\Big).
\end{aligned} \tag{3.82}$$

Condition on $(V^{(i)})_{i=0,\dots,n-m}$ and rewrite the Laplace transform of the Poisson point process $\Pi^{(n-m)}$ to conclude that $v_n(t,x)$ is equal to

$$\mathbf{E}^x\Bigg(\exp\Big(-\lambda\sum_{k=0}^{n-m}V_t^{(k)}\Big)\mathbf{E}\Big[\exp\Big(\int\Big[-\lambda\chi_{t-r} + \alpha\int_0^t ds\chi_{s-r}\frac{d}{dx}v_{m-2}(t-s,0)\Big]\,\Pi^{(n-m)}(dr,d\chi)\Big)\Big|V^{(n-m)}\Big]\Bigg)$$

$$(3.83)\qquad = \mathbf{E}^x\exp\Bigg(-\lambda\sum_{k=0}^{n-m}V_t^{(k)} - \alpha\int_0^t dr\,V_r^{(n-m)} \int Q_Y(d\chi)\Big[1-\exp\big(-\lambda\chi_{t-r}+\alpha\int_0^t ds\,\chi_{s-r}\frac{d}{dx}v_{m-2}(t-s,0)\big)\Big]\Bigg)$$

$$= \mathbf{E}^x\exp\Big(-\lambda\sum_{k=0}^{n-m}V_t^{(k)} + \alpha\int_0^t dr\,\Big(V_r^{(n-m)}\frac{d}{dx}v_{m-1}(t-r,0)\Big)\Big).$$

In the last step, we substituted $s-r\to s$ and applied the induction hypothesis (3.80) with n and t replaced by $m-1$ and $t-r$, respectively. Equation (3.83) proves (3.81) which finishes the induction on m. For the proof of (3.80), notice that we have just shown $(m=n)$

$$(3.84)\qquad v_n(t,x) = \mathbf{E}^x\exp\Big(-\lambda Y_t + \alpha\int_0^t ds\,Y_s\frac{d}{dx}v_{n-1}(t-s,0)\Big).$$

Lemma 3.1.12 with $\kappa := \alpha$ and $f(s) := -\frac{d}{dx}v_{n-1}(t-s,0)$ implies (3.80). This concludes the induction on n.

Finally, let $n\to\infty$ in equation (3.84) and use monotone convergence to obtain

$$\mathbf{E}^x e^{-\lambda V_t} = \mathbf{E}^x\exp\Big(-\lambda Y_t - \alpha\int_0^t ds\,Y_s\lim_{n\to\infty}\lim_{x\to 0}\frac{1}{x}\mathbf{E}^x\Big(1-\exp\big(-\lambda\sum_{k=0}^{n-1}V_{t-s}^{(k)}\big)\Big)\Big)$$

for $t,x\geq 0$. Lemma 3.3.1 implies that the limits on the right-hand side may be interchanged. This proves the assertion. □

3.4 Extinction and survival in the Virgin Island Model. Proof of Theorem 7

Recall the definition of $(Y_t)_{t\geq 0}$ from (1.10). As we pointed out in Section 1.2, the expected area under an excursion of $(Y_t)_{t\geq 0}$ play an important role. The following lemma provides us with some properties of the modified Laplace transform $k(z)$ of the total man-hours. We will see later that these properties are crucial for our proof of Theorem 7. Recall from Section 1.1 the excursion measure Q_Y of the solution $(Y_t)_{t\geq 0}$ of equation (1.10).

Lemma 3.4.1. *Assume A1, A3, A4 and A5. The function*

$$k(z) := \int 1 - \exp\Big(-z\alpha \int_0^\infty \chi_s \, ds\Big) Q_Y(d\chi), \quad z \geq 0, \tag{3.85}$$

is concave with at most two fixed point. Zero is the only fixed point iff

$$k^{'}(0) = \alpha \int \int_0^\infty \chi_s \, ds Q_Y(d\chi) \leq 1. \tag{3.86}$$

Denote by q the maximal fixed point. Then we have for all $z \geq 0$:

$$z \leq k(z) \implies z \leq q \tag{3.87}$$

$$z \geq k(z) \wedge z > 0 \implies z \geq q. \tag{3.88}$$

Proof. The function k has finite values because of $1 - e^{-c} \leq c$, $c \geq 0$, and Lemma 3.1.7. Concavity of k is inherited from the concavity of $x \mapsto 1 - e^{-xc}$, $c \geq 0$. Using dominated convergence (with Lemma 3.1.7), we see that

$$\frac{k(z)}{z} = \int \frac{1 - \exp\big(-z\alpha \int_0^\infty \chi_s \, ds\big)}{z} Q_Y(d\chi) \xrightarrow{z\to\infty} 0. \tag{3.89}$$

In addition, dominated convergence (with Lemma 3.1.7) implies

$$k^{'}(z) = \int \Big[\int_0^\infty \alpha\chi_s \, ds \, \exp\Big(-z\alpha \int_0^\infty \chi_s \, ds\Big)\Big] Q_Y(d\chi) \quad z \geq 0. \tag{3.90}$$

If $\alpha > 0$, then k is strictly concave. Thus, k has a fixed point which is not zero if and only if $k^{'}(0) > 1$. The implications (3.87) and (3.88) follow from the concavity of k. □

The method of proof (cf. Section 6.5 in [19]) of the extinction result for a Crump-Mode-Jagers process $(J_t)_{t\geq 0}$ is to study an equation for $(\mathbf{E}e^{-\lambda J_t})_{t\geq 0,\lambda>0}$. The Laplace transform $(\mathbf{E}e^{-\lambda J_t})_{\lambda>0}$ converges monotonically to $\mathbf{P}(J_t = 0)$ as $\lambda \to \infty$, $t \geq 0$. Furthermore, $\mathbf{P}(J_t = 0) = \mathbf{P}(\exists s \leq t \colon J_s = 0)$ converges monotonically to the extinction probability $\mathbf{P}(\exists s \geq 0 \colon J_s = 0)$ as $t \to \infty$. Taking monotone limits in the equation for $(\mathbf{E}e^{-\lambda J_t})_{t\geq 0,\lambda>0}$ results in an equation for the extinction probability. In our situation, there is an equation for the modified Laplace transform $(L_t^\lambda)_{t>0,\lambda>0}$ as defined in (3.91) below. However, the monotone limit of L_t^λ as $\lambda \to \infty$ might be infinite. Thus, it is not clear how to transfer the above method of proof. The following proof of Theorem 7 directly establishes the convergence of the modified Laplace transform.

Proof of Theorem 7. Recall the definition of q from Lemma 3.4.1. In the first step, we will prove

$$L_t := L_t^\lambda := \lim_{x\to 0}\frac{1}{x}\mathbf{E}^x\Big(1-e^{-\lambda V_t}\Big) \to q \qquad (\text{as } t\to\infty) \tag{3.91}$$

for all $\lambda > 0$. It follows from Lemma 3.3.1 that $(L_t)_{t\le T}$ is bounded for every finite T. By Lemma 3.3.2, the Laplace transform $v(t,x) := \mathbf{E}^x \exp\big(-\lambda V_t\big)$ of the Virgin Island process satisfies

$$v(t,x) = \mathbf{E}^x \exp\Big(-\lambda Y_t + \alpha \int_0^t Y_s \frac{d}{dx} v(t-s,0)\,ds\Big) \tag{3.92}$$

for all $\lambda, t, x \ge 0$. Notice that $L_t = -\frac{d}{dx}v(t,0)$. Take derivatives in (3.92) with respect to x in $x=0$ and apply Lemma 3.1.12 to arrive at

$$L_t = \int \Big[1-\exp\Big(-\lambda\chi_t - \alpha\int_0^t \chi_s L_{t-s} ds\Big)\Big] Q_Y(d\chi). \tag{3.93}$$

Based on (3.93), the idea of the proof of (3.91) is as follows. The term $\lambda\chi_t$ vanishes as $t\to\infty$. If L_t converges to some limit, then the limit has to be a fixed point of the function

$$k(z) = \int \Big[1-\exp\Big(-z\alpha\int_0^\infty \chi_s\,ds\Big)\Big] Q_Y(d\chi). \tag{3.94}$$

By Lemma 3.4.1, this function is concave. Therefore, it has exactly one attracting fixed point. Furthermore, this fact forces L_t to converge as $t\to\infty$.

We will need the finiteness of $L_\infty := \limsup_{t\to\infty} L_t$. Seeking for a contradiction, we assume $L_\infty = \infty$. Then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n\to\infty$ such that $L_{t_n} = \max_{t\le t_n} L_t$. We estimate

$$\begin{aligned} L_{t_n} &\le \int \Big[1-\exp\Big(-\lambda\chi_{t_n} - \alpha\int_0^\infty \chi_s \sup_{r\le t_n} L_r\,ds\Big)\Big] Q_Y(d\chi) \\ &\le k(L_{t_n}) + \int \exp\Big(-\alpha\int_0^\infty \chi_s L_{t_n}\,ds\Big)\Big(1-e^{-\lambda\chi_{t_n}}\Big) Q_Y(d\chi) \\ &\le k(L_{t_n}) + \int \lambda\chi_{t_n} Q_Y(d\chi). \end{aligned} \tag{3.95}$$

The last summand is bounded in n by Lemma 3.1.11. Inequality (3.95) leads to the contradiction

$$1 \le \lim_{n\to\infty}\frac{k(L_{t_n})}{L_{t_n}} + \lim_{n\to\infty}\frac{C}{L_{t_n}} = 0. \tag{3.96}$$

The last equation is a consequence of (3.89) and the assumption $L_\infty = \infty$. Using boundedness of L_t, we prove $L_\infty \le q$. Let $(t_n)_{n\in\mathbb{N}}$ be such that $\lim_{n\to\infty} L_{t_n} = L_\infty < \infty$. Then a calculation as in (3.95) results in

$$\begin{aligned}\lim_{n\to\infty} L_{t_n} \le \limsup_{n\to\infty} \int \Big[1-\exp\Big(-\alpha\int_0^\infty \big(\chi_s \sup_{t\ge t_n} L_{t-s}\big)\,ds\Big)\Big] Q_Y(d\chi)\\ + \limsup_{n\to\infty}\int \lambda\chi_{t_n} Q_Y(d\chi).\end{aligned} \tag{3.97}$$

The last summand is equal to zero by Lemma 3.1.11. The first summand on the right-hand side of (3.97) is dominated by

$$\Big(\sup_{t>0} L_t\Big)\int\Big(\int_0^\infty \alpha\chi_s\,ds\Big)Q_Y(d\chi) < \infty. \tag{3.98}$$

Applying dominated convergence, we conclude that L_∞ is bounded by

$$\int\Big[1-\exp\Big(-\alpha\int_0^\infty \big(\chi_s \varlimsup_{t\to\infty} L_{t-s}\big)\,ds\Big)\Big]Q_Y(d\chi) = k\big(L_\infty\big). \tag{3.99}$$

Thus, Lemma 3.4.1 implies $\limsup_{t\to\infty} L_t \le q$. This proves Theorem 7 in the case of $q = 0$.

Assume $q > 0$ and suppose that $m := \liminf_{t\to\infty} L_t = 0$. Let $(t_n)_{n\in\mathbb{N}}$ be such that $L_{t_n} = \min_{t\le t_n} L_t \to 0$ as $n\to\infty$ and $t_n \le t_{n+1}\to\infty$. We estimate

$$\begin{aligned}L_{t_n} &\ge \int\Big[1-\exp\Big(-\alpha\int_0^{t_n}\big(\chi_s \inf_{t\le t_n} L_t\big)\,ds\Big)\Big]Q_Y(d\chi)\\ &\ge \int\Big[1-\exp\Big(-\alpha\int_0^{t_{n_0}}\big(\chi_s L_{t_n}\big)\,ds\Big)\Big]Q_Y(d\chi)\end{aligned} \tag{3.100}$$

for all $n \ge n_0$. By Lemma 3.4.1, there is a n_0 such that $\int\int_0^{t_{n_0}} \alpha\chi_s\,ds Q_Y(d\chi) > 1$. Using dominated convergence, the assumption $m = 0$ results in the contradiction

$$\begin{aligned}1 &\ge \lim_{n\to\infty}\frac{1}{L_{t_n}}\int\Big[1-\exp\Big(-L_{t_n}\int_0^{t_{n_0}}\alpha\chi_s\,ds\Big)\Big]Q_Y(d\chi)\\ &= \int\Big(\int_0^{t_{n_0}}\alpha\chi_s\,ds\Big)Q_Y(d\chi) > 1.\end{aligned} \tag{3.101}$$

In order to prove $m \ge q$, let $(t_n)_{n\in\mathbb{N}}$ be such that $\lim_{n\to\infty} L_{t_n} = m > 0$. An estimate as above together with dominated convergence yields

$$\begin{aligned}m = \lim_{n\to\infty} L_{t_n} &\ge \lim_{n\to\infty}\int\Big[1-\exp\Big(-\alpha\int_0^{t_n}\big(\chi_s \inf_{t\ge t_n} L_{t-s}\big)\,ds\Big)\Big]Q_Y(d\chi)\\ &= \int\Big[1-\exp\Big(-\alpha\int_0^\infty\big(\chi_s \liminf_{t\to\infty} L_t\big)\,ds\Big)\Big]Q_Y(d\chi) = k(m).\end{aligned} \tag{3.102}$$

Therefore, Lemma 3.4.1 implies $\liminf_{t\to\infty} L_t = m \geq q$, which yields (3.91).

Now we finish the proof of Theorem 7. Using Lemma 3.3.2 and the first step, we see that

$$\mathbf{E}^x e^{-\lambda V_t} = \mathbf{E}^x \exp\left(-\lambda Y_t - \alpha \int_0^t Y_s L_{t-s}\, ds\right) \to \mathbf{E}^x \exp\left(-q\alpha \int_0^\infty Y_s\, ds\right) \tag{3.103}$$

as $t \to \infty$. For this, we used dominated convergence and the fact that $Y_t \to 0$ almost surely as $t \to \infty$ (see Lemma 3.1.6). Hence, the Virgin Island process $(V_t)_{t\geq 0}$ started in $x \geq 0$ converges weakly as $t \to \infty$ to a random variable V_∞^x which only takes values in $\{0, \infty\}$ and satisfies

$$\mathbf{P}(V_\infty^x = 0) = \mathbf{E}^x \exp\left(-q\alpha \int_0^\infty Y_s\, ds\right). \tag{3.104}$$

Thus, the Virgin Island Model dies out iff $q = 0$ which by Lemma 3.4.1 is the case iff $k'(0) \leq 1$. This is condition (1.56). Equation (1.57) follows from Lemma 3.1.7 and from $Q_Y = \bar{s}(0)\bar{Q}_Y$. This proves Theorem 7. □

Chapter 4

Graphical representation of two duality relations

4.1 Introduction

The self-duality (1.4) of interacting Feller diffusions with logistic growth is the key ingredient in the proof of the ergodicity result of Theorem 5. Because of this, we wish to gain more insight into (1.4). In this chapter, we complement the analytical proof of Section 2.5 with a stochastic picture for the self-duality (1.4). As mentioned in the introduction, only the non-spatial case is considered, that is, $m(i,i) = 1$ for all $i \in \mathbb{Z}^d$. In the rest of the chapter, we refer to the slightly more general duality (1.24).

Two processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ with state spaces E_1 and E_2, respectively, are called dual with respect to the duality function H if $H\colon E_1 \times E_2 \to \mathbb{R}$ is a measurable and bounded function and if $\mathbf{E}^x[H(X_t, y)] = \mathbf{E}^y[H(x, Y_t)]$ holds for all $x \in E_1$, $y \in E_2$ and all $t \geq 0$ (see e.g. [24]). Superscripts as in $\mathbf{P}^x$ or in $\mathbf{E}^x$ indicate the initial value of a process. In this chapter, E_1 and E_2 will be subsets of $[0,\infty)$ or will be equal to $\{0,1\}^N$. We speak of a *moment duality* if $H(x,y) = y^x$ or $H(x,y) = (1-y)^x$, $x \in E_1 \subset \mathbb{N}_0$, $y \in [0,1]$, and of a *Laplace duality* if $H(x,y) = \exp(-\lambda x{\cdot}y)$, $x, y \in E_1 = E_2 \subset [0,\infty)$, for some $\lambda > 0$.

We provide a unified stochastic picture for the moment duality (1.26) and for the Laplace duality (1.24). In Section 4.2, we construct Markov processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ with càdlàg sample paths and state space $\{0,1\}^N$ by means of a graphical representation such that

$$(4.1) \quad \mathbf{P}^{x^N}\left[X_t^N \wedge y^N = \underline{0}\right] = \mathbf{P}^{y^N}\left[x^N \wedge Y_t^N = \underline{0}\right] \quad \forall\, x^N, y^N \in \{0,1\}^N \;\; \forall\, t \geq 0$$

for every $N \geq 1$. The notation $x^N \wedge y^N$ denotes component-wise minimum and $\underline{0}$ denotes the zero configuration. In Proposition 4.3.1, we prove that property (4.1) implies a prototype duality relation namely

$$\lim_{N\to\infty} \mathbf{E}\Big[1-\frac{k}{N}\Big]^{\left|X^N_{tT_N}\right|} = \lim_{N\to\infty} \mathbf{E}\Big[1-\frac{\left|Y^N_{tT_N}\right|}{N}\Big]^n, \quad t\geq 0, \tag{4.2}$$

under some assumptions – including the convergence of both sides – on the two processes and on the sequence $(T_N)_{N\geq 1}\subset[0,\infty)$. This prototype duality – together with certain convergence properties of the processes $(X^N_t)_{t\geq 0}$ and $(Y^N_t)_{t\geq 0}$ – will lead to the duality relations (1.24) and (1.26).

For the construction of the approximating processes, we interpret the elements of $\{1,\ldots,N\}$ as "individuals" and the elements of $\{0,1\}$ as the "type" of an individual. In the terminology of population genetics, individuals are denoted as "genes", whereas in population dynamics, the statement "individual i is of type 1 (resp. 0)" would be phrased as "site i is occupied (resp. not occupied) by a particle". Throughout the paper, we assume that in any change of the configuration at most two individuals are involved. We call every function $f\colon\{0,1\}^2\to\{0,1\}^2$ a *basic mechanism*. A finite tuple $(f_1,...,f_m)$, $m\in\mathbb{N}$, of basic mechanisms together with rates $\lambda_1,...,\lambda_m\in[0,\infty)$ defines a process with state space $\{0,1\}^N$ by means of the following graphical representation, which is in the spirit of Harris [14]. With every $k\leq m$ and every ordered pair $(i,j)\in\{1,...,N\}^2$, $i\neq j$, of individuals, we associate a Poisson process with rate parameter λ_k. At every time point of this Poisson process, the configuration of (i,j) changes according to f_k. For example, if the pair of types was $(1,0)$ before, then it changes to $f_k(1,0)\in\{0,1\}^2$. All Poisson processes are independent. This construction can be visualised by drawing arrows from i to j at the time points of the Poisson processes associated with the pair (i,j) (cf. Figure 4.1).

As an example, consider the following continuous time *Moran model* $(M^N_t)_{t\geq 0}$ with state space $\{0,1\}^N$. This is a population genetic model where ordered pairs of individuals resample at rate β/N, $\beta>0$. When a resampling event occurs at (i,j), individual i bequeaths its type to individual j. Thus, the basic mechanism is f^R defined by

$$f^R(1,\cdot):=(1,1),\ f^R(0,\cdot):=(0,0). \tag{4.3}$$

Figure 4.1 shows a realisation with three resampling events. At time t_1, the pair $(2,1)$ resamples. The arrow in Figure 4.1 at time t_1 indicates that individual 2 bequeaths its type to individual 1. Furthermore, individual 5 inherits the type of individual 3 at time t_3. The dual process of the Moran model is a coalescent process. This process is defined by the coalescent mechanism f^C given by

$$f^C(1,\cdot):=(0,1),\ f^C(x):=x,\quad x\in\{(0,0),(0,1)\}, \tag{4.4}$$

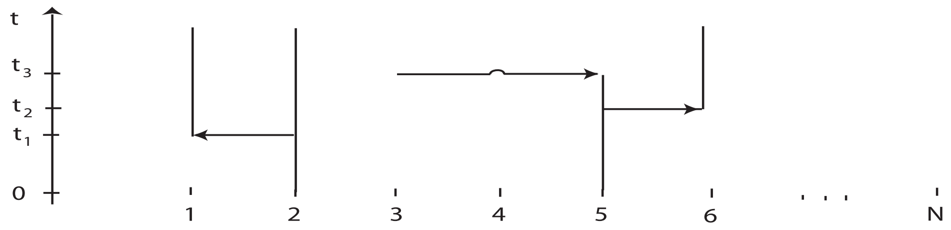


Figure 4.1: Three resampling events. Type 1 is indicated by black lines, absent lines correspond to type 0.

and by the rate β/N. To put it differently, the coalescent process is a coalescing random walk on the complete oriented graph of $\{1,\dots,N\}$. In Section 4.2, we will specify in which sense f^R and f^C are dual, and why this implies (4.1) (see Proposition 4.2.3). More generally, we will identify all dual pairs of basic mechanisms.

Our method elucidates the role of the square in (1.23) for the duality of the logistic Feller diffusion with another logistic Feller diffusion. We illustrate this by the Laplace duality of Feller's branching diffusion $(F_t)_{t\geq 0}$, which is the logistic Feller diffusion with parameters $(0,0,\beta)$, $\beta>0$. Its dual process $(y_t)_{t\geq 0}$ is the logistic Feller diffusion with parameters $(0,\beta,0)$, i.e., the solution of the ordinary differential equation

$$\frac{d}{dt}y_t = -\beta\, y_t^2, \qquad y_0 = y \in [0,\infty). \tag{4.5}$$

The duality relation between these two processes is $\mathbf{E}^x[e^{-F_t y}] = e^{-xy_t}$, $t\geq 0$. In Theorem 4.4.3, we prove that the rescaled Moran model $\big(|M^N_{t\sqrt{N}}|/\sqrt{N}\big)_{t\geq 0}$ converges weakly to $(F_t)_{t\geq 0}$ as $N\to\infty$. To get an intuition for this convergence, notice that $(|M_t^N|)_{t\geq 0}$ is a pure birth-death process with size-dependent transition rates ("birth" corresponds to creation of an individual with type 1, whereas "death" corresponds to creation of an individual with type 0). It remains to prove that the birth and death events become asymptotically independent as $N\to\infty$. It is known, e.g. Section 2 in [9], that the dual process of the Moran model $(M_t^N)_{t\geq 0}$, $N\geq 1$, is a coalescing random walk. Furthermore, the total number of particles of this coalescing random walk is a pure death process on $\{1,...,N\}$ which jumps from k to $k-1$ at exponential rate $\frac{\beta}{N}k(k-1)$, $2\leq k\leq N$. This rate is essentially quadratic in k for large k. We will see that a suitably rescaled pure death process converges to a solution of (4.5); see Remark 4.4.5. The square in (4.5) originates in the quadratic rate of the involved pure death process; see the equations (4.35) and (4.23) for details.

In the literature, e.g. [24], the duality function $H(x^N, y^N) = \mathbb{1}_{x^N \leq y^N}$, $x^N, y^N \in \{0,1\}^N$, can be found frequently, where $x^N \leq y^N$ denotes component-wise comparison. Processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ with state space $\{0,1\}^N$ are dual with respect to this duality function if they satisfy

$$\mathbf{P}^{x^N}\left[X_t^N \leq y^N\right] = \mathbf{P}^{y^N}\left[x^N \leq Y_t^N\right] \qquad \forall\, x^N, y^N \in \{0,1\}^N, t \geq 0. \tag{4.6}$$

The biased voter model is dual to a coalescing branching random walk in this sense (see [22]). Property (4.6) could also be used to derive the Laplace duality (1.24) and the moment duality (1.26).

dualities mentioned in this introduction. In fact, the two properties (4.1) and (4.6) are equivalent in the following sense: If $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ satisfy (4.1) then $(X_t^N)_{t\geq 0}$ and $(\underline{1} - Y_t^N)_{t\geq 0}$ satisfy (4.6) and vice versa. In the configuration $\underline{1}$ every individual has type 1 and $\underline{1} - y$ denotes component-wise subtraction. The dynamics of the process $(\underline{1} - Y_t^N)_{t\geq 0}$ is easily obtained from the dynamics of $(Y_t^N)_{t\geq 0}$ by interchanging the roles of the types 0 and 1.

4.2 Dual basic mechanisms

Fix $m \in \mathbb{N}$ and let $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ be two processes defined by basic mechanisms $(f_1, ..., f_m)$ and $(g_1, ..., g_m)$, respectively. Suppose that the Poisson processes associated with $k \leq m$ have the same rate parameter $\lambda_k \geq 0$, $k = 1, \ldots, m$. We introduce a property of basic mechanisms which will imply (4.1).

Definition 4.2.1. *Let $f, g : \{0,1\}^2 \to \{0,1\}^2$ and for $x = (x_1, x_2) \in \{0,1\}^2$ let $x^\dagger := (x_2, x_1)$. The basic mechanisms f and g are said to be **dual** iff the following two conditions hold:*

$$\forall\, x, y \in \{0,1\}^2: \quad y \wedge \big(f(x)\big)^\dagger = (0,0) \quad \Longrightarrow \quad g(y) \wedge x^\dagger = (0,0), \tag{4.7}$$

$$\forall\, x, y \in \{0,1\}^2: \quad x \wedge \big(g(y)\big)^\dagger = (0,0) \quad \Longrightarrow \quad f(x) \wedge y^\dagger = (0,0). \tag{4.8}$$

To see how this connects to the duality relation in (4.1), we illustrate this definition by an example.

Example 4.2.2. The resampling mechanism f^R and the coalescent mechanism f^C defined in (4.3) and in (4.4), respectively, are dual. We check condition (4.7) with $f = f^R$ and $g = f^C$ by looking at Figure 4.2. The resampling mechanism acts in upward time (solid lines), the coalescent mechanism in downward time (dashed lines). There are three nontrivial configurations for x, i.e., $(1,1)$, $(1,0)$ and $(0,1)$. In the first two cases, we have $f^R(x) = (1,1)$. Then only $y = (0,0)$ satisfies $y \wedge (f^R(x))^\dagger = (0,0)$. In the third case, every y satisfies $y \wedge (f^R(0,1))^\dagger = (0,0)$

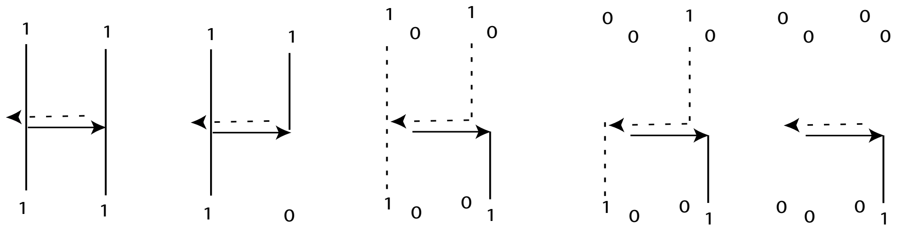


Figure 4.2: The resampling mechanism and the coalescent mechanism satisfy (4.7)

and has to be checked separately. We see that whenever the configuration y is disjoint from $(f(x))^\dagger$, i.e., $y \wedge (f(x))^\dagger = (0,0)$, then $g(y)$ is disjoint from $x^\dagger$. The coalescent mechanism is the natural dual mechanism of the resampling mechanism. Type 1 of the coalescent mechanism "traces back" the lines of descent of type 0 of the resampling mechanism. The "birth event" $(0,1) \mapsto (0,0)$ of an individual of type 0 results in a coalescent event of ancestral lines.

Figure 4.3 is useful to verify condition (4.8). Again, the coalescent mechanism is drawn with dashed lines. Here, the coalescent process is started in the nontrivial configurations $(1,1)$, $(1,0)$ and $(0,1)$. In any case we obtain $(f^C(y))^\dagger = (1,0)$. Hence, all admissible x are of the form $(0,\cdot)$. Condition (4.8) then follows from $f^R(0,\cdot) = (0,0)$.

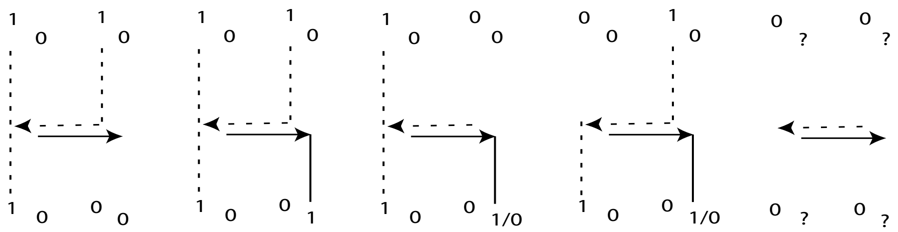


Figure 4.3: The resampling mechanism and the coalescent mechanism satisfy (4.8)

The following proposition shows that two processes are dual in the sense of (4.1) if their defining basic mechanisms are dual (cf. Definition 4.2.1). The proofs of both Proposition 4.2.3 and Proposition 4.3.1 follow similar ideas as in [13].

Proposition 4.2.3. *Let $m \in \mathbb{N}$ and let the processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ be defined by basic mechanisms $(f_1, ..., f_m)$ and $(g_1, ..., g_m)$, respectively. Suppose that the Poisson processes associated with $k \in \{1, \dots, m\}$ in $(X_t^N)_{t\geq 0}$ and in $(Y_t^N)_{t\geq 0}$ have the same rate parameter $\lambda_k \geq 0$. If f_k and g_k are dual for every $k = 1, \dots, m$, then $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ satisfy the duality relation (4.1).*

Proof. Fix $T > 0$ and initial values $X_0^N, Y_0^N \in \{0,1\}^N$. Assume for simplicity that $m = 1$ and let $f := f_1$, $g := g_1$. Define the process $\big(\hat{Y}_t^N\big)_{0 \le t \le T}$ in backward time in the following way. Reverse all arrows in the graphical representation of $(X_t^N)_{t \ge 0}$. At (forward) time T, start with a type configuration given by $\hat{Y}_0^N := Y_0^N$. Now proceed until (forward) time 0: Whenever you encounter an arrow, change the configuration according to g. Recall that the direction of the arrow indicates the order the involved individuals. We show that the processes $(X_t^N)_{t \ge 0}$ and $(\hat{Y}_t^N)_{0 \le t \le T}$ satisfy

$$X_0^N \wedge \hat{Y}_T^N = \underline{0} \iff X_T^N \wedge \hat{Y}_0^N = \underline{0} \qquad \forall\, X_0^N, \hat{Y}_0^N \in \{0,1\}^N, \tag{4.9}$$

for every realisation. We prove the implication "$\Longrightarrow$" by contradiction. Hence, assume that for some initial configuration there is a (random) time $t \in [0,T]$ such that

$$X_0^N \wedge \hat{Y}_T^N = \underline{0} \text{ and } X_t^N \wedge \hat{Y}_{T-t}^N \neq \underline{0}. \tag{4.10}$$

There are only finitely many arrows until time T and no two arrows occur at the same time almost surely. Hence, there is a first time τ such that the processes are disjoint before this time but not after this time. The arrow at time τ points from i to j, say. Denote by $(x_i^-, x_j^-) \in \{0,1\}^2$ and (x_i^+, x_j^+) the types of the pair $(i,j) \in \{1,...,N\}^2$ according to the process $(X_t^N)_{t \ge 0}$ immediately before and after forward time τ, respectively. By the definition of the process, we then have $f(x_i^-, x_j^-) = (x_i^+, x_j^+)$. Furthermore, denote by (y_j^-, y_i^-) the types of the pair (j,i) according to $(Y_t^N)_{t \ge 0}$ immediately before backward time $T - \tau$. We have chosen τ, i, j such that

$$(x_i^-, x_j^-) \wedge \big(g(y_j^-, y_i^-)\big)^\dagger = (0,0) \quad \text{and} \quad (x_i^+, x_j^+) \wedge (y_i^-, y_j^-) \neq (0,0). \tag{4.11}$$

However, this contradicts the duality of f and g. The proof of the other implication is analogous.

It remains to prove that Y_T^N and $\hat{Y}_T^N$ are equal in distribution. The assertion then follows from

$$\begin{aligned} \mathbf{P}\big[X_0^N \wedge Y_T^N = \underline{0}\big] &= \mathbf{P}\big[X_0^N \wedge \hat{Y}_T^N = \underline{0}\big] \\ &\overset{(4.9)}{=} \mathbf{P}\big[X_T^N \wedge \hat{Y}_0^N = \underline{0}\big] = \mathbf{P}\big[X_T^N \wedge Y_0^N = \underline{0}\big]. \end{aligned} \tag{4.12}$$

If a Poisson process is conditioned on its value at some fixed time $T > 0$, then the time points are uniformly distributed over the interval $[0,T]$. The uniform distribution is invariant under time reversal. In addition, the Poisson processes of $(Y_t^N)_{t \ge 0}$ nd $(X_t^N)_{t \ge 0}$ have the same rate parameter. Thus, $(Y_t^N)_{0 \le t \le T}$ and $(\hat{Y}_t^N)_{0 \le t \le T}$ have the same one-dimensional distributions. □

We will now give a list of those maps $f : \{0,1\}^2 \to \{0,1\}^2$ for which there exists a dual basic mechanism (see Definition 4.2.1). The maps f and g in every row of the following table are dual to each other. As in Example 4.2.2, it is elementary to check this.

N^o	$f(0,0)$	$f(0,1)$	$f(1,0)$	$f(1,1)$	$g(0,0)$	$g(0,1)$	$g(1,0)$	$g(1,1)$
i)	(0,0)	(0,0)	(1,1)	(1,1)	(0,0)	(0,1)	(0,1)	(0,1)
ii)	(0,0)	(0,1)	(1,1)	(1,1)	(0,0)	(0,1)	(1,1)	(1,1)
iii)	(0,0)	(0,0)	(0,1)	(0,1)	(0,0)	(0,0)	(0,1)	(0,1)
iv)	(0,0)	(0,1)	(1,0)	(1,1)	(0,0)	(0,1)	(1,0)	(1,1)
v)	(0,0)	(1,1)	(1,1)	(1,1)	(0,0)	(1,1)	(1,1)	(1,1)
vi)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Check that the pair (f, g) is dual if and only if the pair $(f^\dagger, g^\dagger)$ is dual where $f^\dagger(x) := (f(x^\dagger))^\dagger$. Furthermore, the pair (f, g) is dual if and only if $(\hat{f}, \hat{g}^\dagger)$ is dual where $\hat{f}(x) := f(x^\dagger)$ and $\hat{g}^\dagger(x) = (g(x))^\dagger$ for $x \in \{0,1\}^2$. Thus, for each of the listed dual pairs (f, g), the pairs $(f^\dagger, g^\dagger)$, $(\hat{f}, \hat{g}^\dagger)$ and $(\hat{f}^\dagger, \hat{g})$ are also dual. Modulo this relation, the listing of dual basic mechanisms is complete. The proof of this assertion is elementary but somewhat tedious and is thus omitted.

Of particular interest are the dualities in i)-iii). The first of these is the duality between the resampling mechanism and the coalescent mechanism, which we already encountered in Example 4.2.2. The duality in ii) is the self-duality of the **pure birth mechanism**

(4.13)
$$f^B : \{0,1\}^2 \to \{0,1\}^2,\ (1,0) \mapsto (1,1) \text{ and } x \mapsto x\ \forall\, x \in \{(0,0),(0,1),(1,1)\}$$

and iii) is the self-duality of the **death/coalescent mechanism**

$$f^{DC} : \{0,1\}^2 \to \{0,1\}^2,\ (1,\cdot) \mapsto (0,1) \text{ and } (0,\cdot) \mapsto (0,0). \tag{4.14}$$

We are only interested in the effect of an basic mechanism on the total number of individuals of type 1. The identity map in iv) does not change the number of individuals of type 1 in the configuration. The effect of v) and vi) on the number of individuals of type 1 is similar to the effect of ii) and iii), respectively. Furthermore, both $f^\dagger$ and $\hat{f}$ have the same effect on the number of individuals of type 1 as f.

Closing this section, we define processes which satisfy the duality relation (4.1). These processes will play a major role in deriving the dualities (1.26) and (1.24) in Section 4. For $u, e, \gamma, \beta \geq 0$, let $(X_t^N)_{t\geq 0} = (X_t^{N,(u,e,\gamma,\beta)})_{t\geq 0}$ be the process on $\{0,1\}^N$ with the following transition rates (of independent Poisson processes):

- With rate $\frac{u}{N}$, the pure birth mechanism f^B occurs (cf.(4.13)).

- With rate $\frac{e}{N}$, the death/coalescent mechanism f^{DC} occurs (cf. (4.14)).
- With rate $\frac{\gamma}{N}$, the coalescent mechanism f^{C} occurs (cf. (4.4)).
- With rate $\frac{\beta}{N}$, the resampling mechanism f^{R} occurs (cf. (4.4)).

Together with an initial configuration, this defines the process. The processes $(X_t^{N,(u,e,\gamma,\beta)})_{t\geq 0}$ and $(X_t^{N,(u,e,\beta,\gamma)})_{t\geq 0}$ are defined by the basic mechanisms (f^B, f^{DC}, f^C, f^R) and (f^B, f^{DC}, f^R, f^C), respectively. Proposition 4.2.3 then yields the following corollary.

Corollary 4.2.4. *Let $u, e, \gamma, \beta \geq 0$. Then the two processes $(X_t^{N,(u,e,\gamma,\beta)})_{t\geq 0}$ and $(X_t^{N,(u,e,\beta,\gamma)})_{t\geq 0}$ satisfy the duality relation* (4.1)*.*

4.3 Prototype duality

In this section, we derive a prototype duality from (4.1). The main idea for this is to integrate equation (4.1) in the variables x^N and y^N with respect to a suitable measure. Furthermore, we will exploit the fact that drawing from an urn with replacement and without replacement, respectively, is almost surely the same if the urn contains infinitely many balls.

Proposition 4.3.1. *Let $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ be processes with state space $\{0,1\}^N$ for every $N \geq 1$. Assume that $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ satisfy the duality relation* (4.1)*. Choose $n, k \in \{0, ..., N\}$ which may depend on N. Define $\mu_n^N(x^N) := \binom{N}{n}^{-1} \mathbb{1}_{|x^N|=n}$ for every $x^N \in \{0,1\}^N$ where $|x^N| = \sum_{i=1}^N x_i^N$ is the total number of individuals of type* 1*. Assume $\mathcal{L}\left(X_0^N\right) = \mu_n^N$ and $\mathcal{L}\left(Y_0^N\right) = \mu_k^N$. Suppose that the process $(X_t^N)_{t\geq 0}$ satisfies*

$$\frac{n}{N} \to 0 \quad \text{and} \quad \frac{\mathbf{E}\left[|X_{t_N}^N|\right]}{N} \longrightarrow 0 \qquad \text{as } N \to \infty, \tag{4.15}$$

where $t_N \geq 0$. Then

$$\lim_{N\to\infty} \mathbf{E}\left[\left(1 - \tfrac{k}{N}\right)^{\left|X_{t_N}^N\right|}\right] = \lim_{N\to\infty} \mathbf{E}\left[\left(1 - \tfrac{\left|Y_{t_N}^N\right|}{N}\right)^{n}\right] \tag{4.16}$$

under the assumption that the limits exist.

Proof. A central idea of the proof is to make use of the well known fact that the hypergeometric distribution $\mathrm{Hyp}(N, R, l)$, $R, l \in \{0, ..., N\}$, can be approximated by the binomial distribution $\mathrm{B}(l, \frac{R}{N})$ as $N \to \infty$ provided that l is sufficiently small compared to N. In fact, by Theorem 4 of [8],

$$\begin{aligned}(4.17)\quad &\Big|\mathrm{B}(l, \tfrac{R}{N})\big[\{0\}\big] - \mathrm{Hyp}(N, R, l)\big[\{0\}\big]\Big| \\ &\qquad \le d_{TV}\Big(\mathrm{B}\big(l, \tfrac{R}{N}\big), \mathrm{Hyp}(N, R, l)\Big) \le \frac{4 \cdot l}{N} \qquad \forall\, R, l \le N,\end{aligned}$$

where d_{TV} is the total variation distance. By assumption (4.15), we have (with $R := k, l := \big|X^N_{t_N}\big|$)

$$\begin{aligned}(4.18)\quad \mathbf{E}\Big[\Big(1 - \tfrac{k}{N}\Big)^{\left|X^N_{t_N}\right|}\Big] &= \mathbf{E}\Big[\mathrm{B}\Big(\big|X^N_{t_N}\big|, \tfrac{k}{N}\Big)\big[\{0\}\big]\Big] \\ &= \mathbf{E}\Big[\mathrm{Hyp}\big(N, k, \big|X^N_{t_N}\big|\big)\big[\{0\}\big]\Big] + o(1)\end{aligned}$$

as $N \to \infty$. Similarly, we have (with $R := \big|Y^N_{t_N}\big|$, $l := n$)

$$\begin{aligned}(4.19)\quad \mathbf{E}\Big[\Big(1 - \tfrac{\left|Y^N_{t_N}\right|}{N}\Big)^n\Big] &= \mathbf{E}\Big[\mathrm{B}\Big(n, \tfrac{\left|Y^N_{t_N}\right|}{N}\Big)\big[\{0\}\big]\Big] \\ &= \mathbf{E}\Big[\mathrm{Hyp}\big(N, \big|Y^N_{t_N}\big|, n\big)\big[\{0\}\big]\Big] + o(1)\end{aligned}$$

as $N \to \infty$. For fixed $t \ge 0$, $\mathrm{Hyp}(N, |Y^N_t|, n)[\{0\}]$ is the probability of drawing no individual i with $Y^N_t(i) = 1$ when picking n individuals at random without replacement. Thus, it follows that

$$\begin{aligned}(4.20)\quad \mathrm{Hyp}\big(N, \big|Y^N_t\big|, n\big)\big[\{0\}\big] &= \binom{N}{n}^{-1} \sum_{x^N :\, |x^N| = n} \mathbb{1}_{\{x^N \wedge Y^N_t = \underline{0}\}} \\ &= \mu^N_n\big[x^N : x^N \wedge Y^N_t = \underline{0}\big].\end{aligned}$$

By the same argument as before, we also obtain

$$\begin{aligned}(4.21)\quad \mathrm{Hyp}\big(N, k, \big|X^N_t\big|\big)\big[\{0\}\big] &= \mathrm{Hyp}\big(N, \big|X^N_t\big|, k\big)\big[\{0\}\big] \\ &= \mu^N_k\big[y^N : X^N_t \wedge y^N = \underline{0}\big].\end{aligned}$$

We denote by $\mathbf{P}^{x^N}$ the law of the process $(X^N_t)_{t \ge 0}$ started in the fixed initial configuration $x^N \in \{0, 1\}^N$. Starting from the left-hand side of (4.16), the above considerations yield

$$
\begin{aligned}
&\mathbf{E}\Big[\big(1-\frac{k}{N}\big)^{|X^N_{t_N}|}\Big]+o(1) \overset{(4.18)}{=} \mathbf{E}\Big[\mathrm{Hyp}(N,k,|X^N_{t_N}|)[\{0\}]\Big]\\
&\overset{(4.21)}{=} \int \mathbf{E}^{x^N}\Big[\mu^N_k\big[X^N_{t_N}\wedge y^N=\underline{0}\big]\Big]\mu^N_n(dx^N)\\
&\overset{(4.1)}{=} \int\int \mathbf{P}^{y^N}\Big[x^N\wedge Y^N_{t_N}=\underline{0}\Big]\mu^N_k(dy^N)\,\mu^N_n(dx^N)=\mathbf{E}\Big[\mu^N_n\big[x^N\wedge Y^N_{t_N}=\underline{0}\big]\Big]\\
&\overset{(4.20)}{=} \mathbf{E}\Big[\mathrm{Hyp}(N,|Y^N_{t_N}|,n)[\{0\}]\Big] \overset{(4.19)}{=} \mathbf{E}\bigg[\Big(1-\frac{|Y^N_{t_N}|}{N}\Big)^n\bigg]+o(1),
\end{aligned}
$$

which proves the assertion. □

4.4 Various scalings

Recall the definition of the process $(X_t^{N,(u,e,\gamma,\beta)})_{t\geq 0}$ from the end of Section 4.2. Define $X_t^N := X_t^{N,(u,e,\gamma,\beta)}$ and $Y_t^N := X_t^{N,(u,e,\beta,\gamma)}$ for $t \geq 0$ and $N \in \mathbb{N}$. Notice that the Poisson process attached to the resampling mechanism in the process $(Y_t^N)_{t\geq 0}$ has rate γ. By Corollary 4.2.4, the two processes $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ satisfy the duality relation (4.1). Let $\mathcal{L}\left(X_0^N\right) = \mu_n^N$ and $\mathcal{L}\left(Y_0^N\right) = \mu_k^N$ for some $n, k \in \mathbb{N}$ to be chosen later, where μ_n^N is defined in Proposition 4.3.1. In order to apply Proposition 4.3.1, we essentially have to prove existence of the limits in (4.16). Depending on the scaling, this will result in the moment duality (1.26) of a resampling-selection model with a branching-coalescing particle process and in the Laplace duality (1.24) of the logistic Feller diffusion with another logistic Feller diffusion, respectively. Both dualities could be derived simultaneously. However, in order to keep things simple, we consider the two cases separately.

Theorem 4.4.1. *Assume $b, c, d \geq 0$. Let $X_0 = n \in \mathbb{N}_0$ and $Y_0 = y \in [0,1]$. Furthermore, denote by $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ the $(1,b,c,d)$-braco-process and the $(1,b,c,d)$-resem-process, respectively. Then*

$$
\mathbf{E}^n\left[(1-y)^{X_t}\right] = \mathbf{E}^y\left[(1-Y_t)^n\right], \qquad t \geq 0. \tag{4.22}
$$

Remark 4.4.2. *In the special case $b = 0 = d$ and $c > 0$, this is the moment duality of the Fisher-Wright diffusion with Kingman's coalescent. Furthermore, choosing $c = 0$ and $b, d > 0$ results in the moment duality of the Galton-Watson process with a deterministic process.*

Proof. Choose $u, e, \beta \geq 0$ and $\gamma = \gamma(N)$ such that $b = u + \beta$, $d = e + \beta$ and $\gamma/N \to c$ as $N \to \infty$. In the first step, we prove that the process $(|X_t^N|)_{t\geq 0}$ of the total number of individuals of type 1 converges weakly to $(X_t)_{t\geq 0}$. The total number of individuals of type 1 increases by one if a "birth event" occurs (f^B or f^R) and if the type configuration of the respective ordered pair of individuals is $(1,0)$. If the total number of individuals of type 1 is equal to k, then the probability of the type configuration of a randomly chosen ordered pair to be $(1,0)$ is $\frac{k}{N}\frac{N-k}{N-1}$. The number of Poisson processes associated with a fixed basic mechanism is $N(N-1)$. Thus, the process of the total number of individuals of type 1 has the following transition rates:

$$\begin{aligned} k &\to k+1: \quad \tfrac{u+\beta}{N}\cdot N(N-1)\cdot \tfrac{k}{N}\tfrac{N-k}{N-1} \\ k &\to k-1: \quad \tfrac{e+\beta}{N}\cdot N(N-1)\cdot \tfrac{N-k}{N}\tfrac{k}{N-1} + \tfrac{e+\gamma}{N}\cdot N(N-1)\cdot \tfrac{k}{N}\tfrac{k-1}{N-1}, \end{aligned} \tag{4.23}$$

where $k\in\mathbb{N}_0$. Notice that the coalescent mechanism produces the quadratic term $k(k-1)$ because the probability of the type configuration of a randomly chosen ordered pair to be $(1,1)$ is $\frac{k}{N}\frac{k-1}{N-1}$ if there are k individuals of type 1. The transition rates determine the generator $\mathcal{G}^N=\mathcal{G}^{N,(u,e,\gamma,\beta)}$ of $(|X_t^N|)_{t\geq 0}$, namely

$$\begin{aligned} \mathcal{G}^N f(k) =& \frac{u+\beta}{N}\cdot k(N-k)\cdot\big(f(k+1)-f(k)\big) \\ &+\frac{e+\beta}{N}\cdot k(N-k)\cdot\big(f(k-1)-f(k)\big) \\ &+\frac{e+\gamma}{N}\cdot k(k-1)\cdot\big(f(k-1)-f(k)\big), \quad k\in\{0,\dots,N\}, \end{aligned} \tag{4.24}$$

for $f\colon\{0,\dots,N\}\to\mathbb{R}$. The $(1,u+\beta,c,e+\beta)$-braco-process $(X_t)_{t\geq 0}$ is the unique solution of the martingale problem for $\mathcal{G}$ (see [2]) where

$$\mathcal{G}f(k) := (u+\beta)k\,\big(f(k+1)-f(k)\big)+\big((e+\beta)+c(k-1)\big)k\,\big(f(k-1)-f(k)\big),$$

for $k\in\mathbb{N}_0$ and for $f\colon\mathbb{N}_0\to\mathbb{R}$ with finite support. Letting $N\to\infty$, we see that

$$\mathcal{G}^N f(k) \longrightarrow \mathcal{G}f(k) \qquad \text{as } N\to\infty, \quad k\in\mathbb{N}_0, \tag{4.25}$$

for $f\colon\mathbb{N}_0\to\mathbb{R}$ with finite support. We aim at using Lemma 4.5.1 which is given below (with $E_N=\{0,\dots,N\}$ and $E=\mathbb{N}_0$), to infer from (4.24) the weak convergence of the corresponding Markov processes. A coupling argument shows that $(|X_t^N|)_{t\geq 0}$ is dominated by $(Z_t^N)_{t\geq 0} := (|X_t^{N,(u,0,0,\beta)}|)_{t\geq 0}$. The process $(Z_t^N)_{t\geq 0}$ solves the martingale problem for $\mathcal{G}^{N,(u,0,0,\beta)}$. Thus, we obtain

$$Z_t^N - Z_0^N = \int_0^t \mathcal{G}^{N,(u,0,0,\beta)} Z_s^N\,ds + C_t^N = \int_0^t uZ_s^N\tfrac{N-Z_s^N}{N}\,ds + C_t^N \tag{4.26}$$

where $(C_t^N)_{t\geq 0}$ is a martingale. Hence, $(Z_t^N)_{t\geq 0}$ is a submartingale. Taking expectations, Gronwall's inequality implies

$$\mathbf{E}[Z_t^N] \leq \mathbf{E}[Z_0^N]e^{ut}, \qquad \forall\, t\geq 0. \tag{4.27}$$

Let $S_N = T_N = 1$, $s_N = u$ and recall $|X_0^N| = n$. With this, the assumptions of Lemma 4.5.1 are satisfied. Thus, Lemma 4.5.1 implies that $(|X_t^N|)_{t\geq 0}$ converges weakly to $(X_t)_{t\geq 0}$ as $N \to \infty$. Let $k = k_N \in \{0, ..., N\}$ be such that $k/N \to y$ as $N \to \infty$. For every $\bar{n} \in \mathbb{N}$, $(1 - \frac{k}{N})^n$ converges uniformly in $n \leq \bar{n}$ to $(1-y)^n$ as $N \to \infty$. In general, if the sequence $(\tilde{X}_n)_{n\in\mathbb{N}}$ of random variables with complete and separable state space converges weakly to $\tilde{X}$ and if the sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in C_b$, converges uniformly on compact sets to $f \in C_b$, then $\mathbf{E}[f_n(\tilde{X}_n)] \to \mathbf{E}[f(\tilde{X})]$ as $n \to \infty$. Hence,

$$\mathbf{E}^n\Big[(1-y)^{X_t}\Big] = \lim_{N\to\infty} \mathbf{E}\bigg[\Big(1 - \frac{k}{N}\Big)^{|X_t^N|}\bigg]. \tag{4.28}$$

The next step is to prove that the rescaled processes $(|Y_t^N|/N)_{t\geq 0}$ converge weakly to $(Y_t)_{t\geq 0}$ as $N \to \infty$. The generator of $(|Y_t^N|/N)_{t\geq 0}$ is given by

$$\begin{aligned}
&\mathcal{G}^{N,(u,e,\beta,\gamma)} f\big(\tfrac{k}{N}\big)\\
&= \gamma k\, \frac{N-k}{N}\Big(f\big(\tfrac{k+1}{N}\big) + f\big(\tfrac{k-1}{N}\big) - 2f\big(\tfrac{k}{N}\big)\Big)\\
&\quad + uk\, \frac{N-k}{N}\bigg(f\big(\tfrac{k+1}{N}\big) - f\big(\tfrac{k}{N}\big)\bigg) + ek\, \frac{N-k}{N}\Big(f\big(\tfrac{k-1}{N}\big) - f\big(\tfrac{k}{N}\big)\Big)\\
&\quad + \tfrac{e+\beta}{N}\, k(k-1)\Big(f\big(\tfrac{k-1}{N}\big) - f\big(\tfrac{k}{N}\big)\Big), \qquad k \in \{0, ..., N\},
\end{aligned} \tag{4.29}$$

for $f \in C_c^2([0,1])$. Choose $k = k_N \leq N$ such that $\frac{k}{N} \to y \in [0,1]$ as $N \to \infty$. Notice that

$$N^2 \cdot \Big(f\big(\tfrac{k+1}{N}\big) + f\big(\tfrac{k-1}{N}\big) - 2f\big(\tfrac{k}{N}\big)\Big) \to f^{''}(y) \quad \text{as } N \to \infty. \tag{4.30}$$

As $N \to \infty$, the right-hand side of (4.29) converges to

$$\begin{aligned}
&cy(1-y)\cdot f^{''}(y) + (u-e)y(1-y)\cdot f^{'}(y) - (e+\beta)y^2 \cdot f^{'}(y)\\
&= (u-e)y \cdot f^{'}(y) - (u+\beta)y^2 \cdot f^{'}(y) + cy(1-y)\cdot f^{''}(y) =: \mathcal{G}f(y)
\end{aligned} \tag{4.31}$$

for every $f \in C_c^2([0,1])$. Athreya and Swart [2] show that the $(1,b,c,d)$-resem-process $(Y_t)_{t\geq 0}$ solves the martingale problem for $\mathcal{G}$ and that this solution is unique. Let $E_N = \{0,1,\ldots,N\}$, $E = [0,1]$, $Z_t^N := |X_t^{N,(u,0,0,\gamma)}|$, $S_N = N$ and $T_N = 1$. With this, the assumptions of Lemma 4.5.1 are satisfied and we conclude that $(|Y_t^N|/N)_{t\geq 0}$ converges weakly to $(Y_t)_{t\geq 0}$. It follows that, for $k = k_N \in \{0, ..., N\}$ with $k/N \to y$,

$$\lim_{N\to\infty} \mathbf{E}\left[\left(1 - \frac{|Y_t^N|}{N}\right)^n\right] = \mathbf{E}^y\Big[(1-Y_t)^n\Big]. \tag{4.32}$$

This proves existence of the limits in (4.16) with $t_N := t$. Inequality (4.27) and $|X_0^N| = n << N$ imply condition (4.15). Thus, Proposition 4.3.1 establishes equation (4.16). The assertion follows from equations (4.28), (4.16) and (4.32). □

Next, we derive the Laplace duality of a logistic Feller diffusion with another logistic Feller diffusion. Recall that the logistic Feller diffusion with parameters $(\varsigma, \gamma, \beta)$ solves equation (1.23).

Theorem 4.4.3. *Suppose that $\varsigma, \gamma, \beta \geq 0$, $r > 0$ and $X_0 = x \geq 0, Y_0 = y \geq 0$. Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be logistic Feller diffusions with parameters $(\varsigma, \gamma, \beta)$ and $(\varsigma, r\beta, \gamma/r)$, respectively. Then*

$$\mathbf{E}^x\left[e^{-rX_t\cdot y}\right] = \mathbf{E}^y\left[e^{-rx\cdot Y_t}\right] \tag{4.33}$$

for all $t \geq 0$.

Remark 4.4.4. *(a) For $\beta, \gamma > 0$ and $r = \gamma/\beta$, Theorem 4.4.3 yields the self-duality of the logistic Feller diffusion.*

(b) For $\varsigma = 0$, $\gamma = 0$, $r = 1$ and $\beta > 0$, Theorem 4.4.3 specialises to the Laplace duality of Feller's branching diffusion.

Proof. Choose $u = u_N \geq 0$ and $e = e_N \geq 0$ such that $(u - e)\sqrt{N} \to \varsigma$ as $N \to \infty$. We prove that the rescaled process $(|Y^N_{t\sqrt{N}}|/(r\sqrt{N}))_{t\geq 0}$ converges weakly to $(Y_t)_{t\geq 0}$ as $N \to \infty$. The generator of the rescaled process is given by (cf. (4.29))

$$\begin{aligned}
&\sqrt{N}\mathcal{G}^N f\big(\tfrac{k}{r\sqrt{N}}\big)\\
&= \sqrt{N}\cdot\gamma\cdot k\frac{(N-k)}{N}\cdot\Big(f\big(\tfrac{k+1}{r\sqrt{N}}\big) + f\big(\tfrac{k-1}{r\sqrt{N}}\big) - 2f\big(\tfrac{k}{r\sqrt{N}}\big)\Big)\\
&\quad + \sqrt{N}u_N\cdot k\frac{(N-k)}{N}\cdot\Big(f\big(\tfrac{k+1}{r\sqrt{N}}\big) - f\big(\tfrac{k}{r\sqrt{N}}\big)\Big)\\
&\quad + \sqrt{N}e_N\cdot k\frac{(N-k)}{N}\cdot\Big(f\big(\tfrac{k-1}{r\sqrt{N}}\big) - f\big(\tfrac{k}{r\sqrt{N}}\big)\Big)\\
&\quad + \sqrt{N}\cdot(e_N+\beta)\cdot\frac{k(k-1)}{r^2N}r^2\cdot\frac{r\sqrt{N}}{r\sqrt{N}}\Big(f\big(\tfrac{k-1}{r\sqrt{N}}\big) - f\big(\tfrac{k}{r\sqrt{N}}\big)\Big),
\end{aligned} \tag{4.34}$$

for $k \in \{0, \ldots, N\}$ and for $f \in C_c^2([0,\infty))$. Let $k = k(N) \in \{0, \ldots, N\}$ be such that $k/(r\sqrt{N}) \to y$. Letting $N \to \infty$, the right-hand side converges to

$$\tfrac{\gamma}{r}\, y\cdot f''(y) + \varsigma\, y\cdot f'(y) - \beta r\, y^2\cdot f'(y) =: \mathcal{G}f(y) \tag{4.35}$$

for every $f \in C_c^2([0,\infty))$. Notice that the quadratic term y^2 originates in the quadratic term $k(k-1)$. Hutzenthaler and Wakolbinger [17] prove that $(Y_t)_{t\geq 0}$ In

Section 2.5, we proved – in the case $r = 1$ – that $(Y_t)_{t\geq 0}$ is the unique solution of the martingale problem for $\mathcal{G}$. The proof for general $r > 0$ is analogous. Let $|Y_0^N| = k = k(N)$ be such that $k/(r\sqrt{N}) \to y \in [0,1]$ as $N \to \infty$ and define $Z_0^N := k$. As before, $(Z_t^N)_{t\geq 0} := (|X_t^{N,(u,0,0,\gamma)}|)_{t\geq 0}$ is a submartingale which dominates $(Y_t^N)_{t\geq 0}$ and which satisfies

$$\sup_N \frac{1}{r\sqrt{N}} \mathbf{E}[Z^N_{t\sqrt{N}}] \leq \sup_N \frac{1}{r\sqrt{N}} \mathbf{E}[Z_0^N] e^{u_N t\sqrt{N}} < \infty, \qquad \forall\, t \geq 0. \tag{4.36}$$

Let $E_N := \{0, \ldots, N\}$, $E := [0, \infty)$, $s_N := u_N$, $S_N := r\sqrt{N}$ and $T_N := \sqrt{N}$. The assumptions of Lemma 4.5.1 are satisfied and we conclude that $(|Y^N_{t\sqrt{N}}|/(r\sqrt{N}))_{t\geq 0}$ converges weakly to $(Y_t)_{t\geq 0}$. This also proves that $(|X^N_{t\sqrt{N}}|/\sqrt{N})_{t\geq 0}$ converges weakly to $(X_t)_{t\geq 0}$ if $|X_0^N| = n = n(N)$ is such that $n/\sqrt{N} \to x$ as $N \to \infty$. It is not hard to see that, for every $\tilde{z} \geq 0$,

$$\left(1 - r\frac{k/(r\sqrt{N})}{\sqrt{N}}\right)^{\sqrt{N}z} \longrightarrow e^{-rzy} \quad \text{and} \quad \left(1 - r\frac{z}{\sqrt{N}}\right)^{\sqrt{N}\frac{n}{\sqrt{N}}} \longrightarrow e^{-rxz} \tag{4.37}$$

uniformly in $0 \leq z \leq \tilde{z}$ as $N \to \infty$. Together with the weak convergence of the rescaled processes, this implies

$$\mathbf{E}^x\left[e^{-rX_t\cdot y}\right] = \lim_{N\to\infty} \mathbf{E}^n\left[\left(1 - r\frac{k/(r\sqrt{N})}{\sqrt{N}}\right)^{\sqrt{N}\cdot X^N_{t\sqrt{N}}/\sqrt{N}}\right] \tag{4.38}$$

and

$$\lim_{N\to\infty} \mathbf{E}^k\left[\left(1 - r\frac{Y^N_{t\sqrt{N}}/(r\sqrt{N})}{\sqrt{N}}\right)^n\right] = \mathbf{E}^y\left[e^{-rx\cdot Y_t}\right] \tag{4.39}$$

for $t \geq 0$. This proves existence of the limits in (4.16) with $t_N := t\sqrt{N}$. Inequality (4.36) and $|X_0^N| = n << N$ imply condition (4.15). Thus, Proposition 4.3.1 establishes equation (4.16). The assertion follows from equations (4.38), (4.16) and (4.39). □

Remark 4.4.5. *Assume $u = e = \gamma = \varsigma = 0$ and $r = 1$ in the proof of Theorem 4.4.3. Then $(|Y_t^N|)_{t\geq 0}$ is a pure death process on $\{1, ..., N\}$ which jumps from k to $k-1$ at exponential rate $\frac{\beta}{N}k(k-1)$, $2 \leq k \leq N$. Furthermore, $(Y_t)_{t\geq 0}$ is a solution of (4.5). We have just shown that the rescaled pure death process $\left(|Y^N_{t\sqrt{N}}|/\sqrt{N}\right)_{t\geq 0}$ converges weakly to $(Y_t)_{t\geq 0}$ as $N \to \infty$.*

4.5 Weak convergence of processes

In the proofs of Theorem 4.4.1 and Theorem 4.4.3, we have established convergence of generators plus a domination principle. In this section, we prove that this implies weak convergence of the corresponding processes. For the weak convergence of processes with càdlàg paths, let the topology on the set of càdlàg paths be given by the Skorohod topology (see [11], Section 3.5).

Lemma 4.5.1. *Let $E \subset \mathbb{R}_{\geq 0}$ be closed. Assume, that the martingale problem for $(\mathcal{G}, \nu)$ has at most one solution where $\mathcal{G}\colon C_c^2(E) \to C_b(E)$ is a linear operator and ν is a probability measure on E. Furthermore, for $N \in \mathbb{N}$, let $E_N \subset \mathbb{R}_{\geq 0}$ and let $(Y_t^N)_{t\geq 0}$ be an E_N-valued Markov process with càdlàg paths and generator $\mathcal{G}^N$. Let $(S_N)_{N\in\mathbb{N}}$ and $(T_N)_{N\in\mathbb{N}}$ be sequences in $\mathbb{R}_{>0}$ with $y^N/S_N \in E$ for all $y^N \in E_N$ and $N \in \mathbb{N}$. Suppose that*

$$y^N \in E_N,\ \lim_{N\to\infty} \tfrac{y^N}{S_N} = y \in E \quad \textit{implies} \quad T_N \mathcal{G}^N f\big(\tfrac{y^N}{S_N}\big) \to \mathcal{G}f(y) \ \textit{as}\ N \to \infty, \tag{4.40}$$

for every $f \in C_c^2(E)$. Assume that, for $N \in \mathbb{N}$, $(Y_t^N)_{t\geq 0}$ is dominated by a process $(Z_t^N)_{t\geq 0}$, i.e., $Y_t^N \leq Z_t^N$ for all $t \geq 0$ almost surely, which is a submartingale satisfying $\mathbf{E}[Z_t^N] \leq \mathbf{E}[Z_0^N]e^{ts_N}$ for all $t \geq 0$ and some constant s_N. In addition, suppose that $\limsup_{N\to\infty} s_N T_N < \infty$ and $\limsup_{N\to\infty} \frac{\mathbf{E}[Z_0^N]}{S_N} < \infty$. If Y_0^N/S_N converges weakly to ν as $N \to \infty$, then

$$\mathcal{L}\Big(\big(Y_{tT_N}^N/S_N\big)_{t\geq 0}\Big) \Longrightarrow \mathcal{L}^\nu\Big(\big(Y_t\big)_{t\geq 0}\Big) \qquad \textit{as } N \to \infty \tag{4.41}$$

where $(Y_t)_{t\geq 0}$ is a solution of the martingale problem $(\mathcal{G}, \nu)$ with initial distribution ν.

Proof. We aim at applying Corollary 4.8.16 of Ethier and Kurtz [11]. For this, define

$$\tilde{E}_N := \{\tfrac{y^N}{S_N} \colon y^N \in E_N\}, \quad \tilde{\mathcal{G}}^N f(\tilde{y}^N) := T_N \mathcal{G}^N f\big(\tfrac{y^N}{S_N}\big)\Big|_{y^N = \tilde{y}^N S_N}, \tilde{y}^N \in \tilde{E}^N, \tag{4.42}$$

for $f \in C_c^2(E)$ and let $\eta_N \colon \tilde{E}_N \to E$ be the embedding function. The process $\big(Y_{tT_N}^N/S_N\big)_{t\geq 0}$ has state space $\tilde{E}_N$ and generator $\tilde{\mathcal{G}}^N$. Now we prove the compact containment condition, i.e., for fixed $\varepsilon, t > 0$ we show

$$\big(\exists K > 0\big)\,\big(\forall N \in \mathbb{N}\big)\ \mathbf{P}\Big[\sup_{s\leq t} \frac{Y_{sT_N}^N}{S_N} \leq K\Big] \geq 1 - \varepsilon. \tag{4.43}$$

Using $Y_t^N \leq Z_t^N$, $t \geq 0$, and Doob's Submartingale Inequality, we conclude for all $N \in \mathbb{N}$

$$\begin{aligned} \mathbf{P}\Big[\sup_{s\leq t} Y_{sT_N}^N \geq KS_N\Big] &\leq \mathbf{P}\Big[\sup_{s\leq t} Z_{sT_N}^N \geq KS_N\Big] \leq \frac{1}{KS_N}\mathbf{E}\big[Z_{tT_N}^N\big] \\ &\leq \frac{1}{K} \sup_{N\in\mathbb{N}} \frac{\mathbf{E}\big[Z_0^N\big]}{S_N} \cdot \exp\big(t \cdot \sup_{N\in\mathbb{N}}(s_N T_N)\big) =: \frac{C}{K}. \end{aligned} \tag{4.44}$$

Thus, choosing $K := \frac{C}{\varepsilon}$ completes the proof of the compact containment condition.

It remains to verify condition (f) of Corollary 4.8.7 of [11]. Condition (4.40) implies that for every $f \in C_c^2$ and every compact set $K \subset E$

$$\sup_{y \in K \cap \tilde{E}_N} |\tilde{\mathcal{G}}^N f(y) - \mathcal{G} f(y)| \to 0 \qquad \text{as } N \to \infty. \tag{4.45}$$

Choose a sequence K_N such that (4.45) still holds with K replaced by K_N. This together with the compact containment condition implies condition (f) of Corollary 4.8.7 of [11] with $G_N := K_N \cap \tilde{E}_N$ and $f_N := f|_{\tilde{E}_N}$. Furthermore, notice that $C_c^2(E)$ is an algebra that separates points and E is complete and separable. Now Corollary 4.8.16 of Ethier and Kurtz [11] implies the assertion. □

Open Question: Athreya and Swart [2] prove a self-duality of the resem-process given by (1.25). We were not able to establish a graphical representation for this duality. Thus, the question whether our technique also works in this case yet waits to be answered.

Bibliography

[1] Roland Alkemper and Martin Hutzenthaler. Graphical representation of some duality relations in stochastic population models. *Electron. Comm. Probab.*, 12:206–220, 2007.

[2] Siva R. Athreya and Jan M. Swart. Branching-coalescing particle systems. *Probab. Theory Related Fields*, 131(3):376–414, 2005.

[3] Benjamin M. Bolker and Stephen W. Pacala. Using moment equations to understand stochastically driven spatial pattern formation in ecological systems. *Theor. Popul. Biol.*, 52(3):179–197, 1997.

[4] J. T. Cox and Andreas Greven. Ergodic theorems for infinite systems of locally interacting diffusions. *Ann. Probab.*, 22(2):833–853, 1994.

[5] J. Theodore Cox, Klaus Fleischmann, and Andreas Greven. Comparison of interacting diffusions and an application to their ergodic theory. *Probab. Theory Related Fields*, 105(4):513–528, 1996.

[6] Donald A. Dawson. Measure-valued Markov processes. In *École d'Été de Probabilités de Saint-Flour XXI—1991*, volume 1541 of *Lecture Notes in Math.*, pages 1–260. Springer, Berlin, 1993.

[7] Donald A. Dawson, Luis G. Gorostiza, and Anton Wakolbinger. Hierarchical equilibria of branching populations. *Electron. J. Probab.*, 9:no. 12, 316–381 (electronic), 2004.

[8] Persi Diaconis and David A. Freedman. Finite exchangeable sequences. *Ann. Probab.*, 8(4):745–764, 1980.

[9] Peter Donnelly. The transient behavior of the Moran model in population genetics. *Math. Proc. Cambridge Philos. Soc.*, 95(2):349–358, 1984.

[10] Alison M. Etheridge. Survival and extinction in a locally regulated population. *Ann. Appl. Probab.*, 14(1):188–214, 2004.

[11] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes: Characterization and convergence.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986.

[12] Andreas Greven, Achim Klenke, and Anton Wakolbinger. Interacting diffusions in a random medium: comparison and longtime behavior. *Stochastic Process. Appl.*, 98(1):23–41, 2002.

[13] David Griffeath. *Additive and cancellative interacting particle systems*, volume 724 of *Lecture Notes in Mathematics.* Springer, Berlin, 1979.

[14] Theodore E. Harris. Additive set-valued Markov processes and graphical methods. *Ann. Probability*, 6(3):355–378, 1978.

[15] Ira Herbst and Loren Pitt. Diffusion equation techniques in stochastic monotonicity and positive correlations. *Probab. Theory Related Fields*, 87(3):275–312, 1991.

[16] Paul Horridge and Roger Tribe. On stationary distributions for the KPP equation with branching noise. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(6):759–770, 2004.

[17] Martin Hutzenthaler and Anton Wakolbinger. Ergodic behavior of locally regulated branching populations. *Ann. Appl. Probab.*, 17(2):474–501, 2007.

[18] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1981.

[19] Peter Jagers. *Branching processes with biological applications.* Wiley-Interscience [John Wiley & Sons], London, 1975. Wiley Series in Probability and Mathematical Statistics—Applied Probability and Statistics.

[20] Peter Jagers. Stabilities and instabilities in population dynamics. *J. Appl. Probab.*, 29(4):770–780, 1992.

[21] Samuel Karlin and Howard M. Taylor. *A second course in stochastic processes.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.

[22] Stephen M. Krone and Claudia Neuhauser. Ancestral processes with selection. *Theor. Popul. Biol.*, 51(3):210–237, 1997.

[23] Amaury Lambert. The branching process with logistic growth. *Ann. Appl. Probab.*, 15(2):1506–1535, 2005.

[24] Thomas M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[25] Thomas M. Liggett and Frank Spitzer. Ergodic theorems for coupled random walks and other systems with locally interacting components. *Z. Wahrsch. Verw. Gebiete*, 56(4):443–468, 1981.

[26] Carl Mueller and Roger Tribe. A phase transition for a stochastic PDE related to the contact process. *Probab. Theory Related Fields*, 100(2):131–156, 1994.

[27] Bernt Øksendal. *Stochastic differential equations – An introduction with applications*. Universitext. Springer-Verlag, Berlin, fifth edition, 1998.

[28] Jim Pitman and Marc Yor. A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete*, 59(4):425–457, 1982.

[29] L. C. G. Rogers and David Williams. *Diffusions, Markov processes and martingales. Vol. 2.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[30] Moshe Shaked and J. George Shanthikumar. *Stochastic orders and their applications*. Probability and Mathematical Statistics. Academic Press Inc., Boston, MA, 1994.

[31] Tokuzo Shiga and Akinobu Shimizu. Infinite-dimensional stochastic differential equations and their applications. *J. Math. Kyoto Univ.*, 20(3):395–416, 1980.

[32] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[33] David Williams. Path decomposition and continuity of local time for one-dimensional diffusions. I. *Proc. London Math. Soc. (3)*, 28:738–768, 1974.

Deutsche Zusammenfassung

In Populationen mit natürlicher Fortpflanzung ist die durchschnittliche Zahl an Nachkommen pro Individuum üblicherweise strikt größer als eins. Bedingt auf Überleben wachsen klassische superkritische Verzweigungsmodelle jedoch über alle Grenzen. Dies ist unrealistisch, da Ressourcen wie beispielsweise Nahrung beschränkt sind.

Eine effektive Gegenmaßnahme gegen unbeschränktes Populationswachstum ist eine Regulierung der Dynamik in Abhängigkeit von der Populationsgröße. Ein Beispiel hierfür ist der sog. *logistische Verzweigungsprozess*, bei dem, zusätzlich zu den „natürlichen“ Geburten und Todesfällen eines superkritischen Verzweigungsprozesses, Todesfälle aus dem Konkurrenzkampf zwischen je zwei Individuen einer Population resultieren. Dies führt in Fellers Diffusionslimes zu einem negativen Driftterm, welcher proportional zur quadrierten Populationsgröße ist. Um dies zu präzisieren, betrachte, für $N \geq 1$ und $b, d, \gamma, \beta > 0$, einen reinen Geburts-Todes-Prozess $(Z_t^N)_{t \geq 0}$ mit Zustandsraum $\mathbb{N}_0$, bei dem sich jedes Teilchen mit Rate $\beta + \frac{b}{N}$ in zwei neue Teilchen spaltet, jedes Teilchen mit Rate $\beta + \frac{d}{N}$ stirbt und jedes geordnete Paar von Teilchen mit Rate $\frac{\gamma}{N^2}$ zu einem Teilchen verschmilzt. Alle diese Ereignisse geschehen unabhängig voneinander. Falls $\frac{Z_0^N}{N}$ in Verteilung für $N \to \infty$ gegen Z_0 konvergiert, so konvergiert $\left(\frac{Z_{tN}^N}{N}\right)_{t \geq 0}$ in Verteilung für $N \to \infty$ gegen $\big(Z_t\big)_{t \geq 0}$, wobei $(Z_t)_{t \geq 0}$ die stochastische Differentialgleichung

$$dZ_t = (b - d) Z_t \, dt - \gamma Z_t^2 \, dt + \sqrt{2\beta Z_t} dB_t \tag{Z.1}$$

löst. Dabei bezeichnet $(B_t)_{t \geq 0}$ eine Standard-Brownsche Bewegung. Das Quadrat in (Z.1) verhindert ein unbeschränktes Anwachsen der Populationsgröße. Leider konvergiert $(Z_t)_{t \geq 0}$ in Verteilung für $t \to \infty$ gegen Null und ist somit als Populationsmodell nur bedingt geeignet.

Um dem Aussterben entgegenzuwirken, betrachtet man eine räumlich erweiterte Version des logistischen Verzweigungsprozesses. Dabei leben Teilpopulationen auf räumlich isolierten „Inseln“, welche im d-dimensionalen Gitter $\mathbb{Z}^d$ angeordnet sind und durch einen (homogenen) Migrationsmechanismus verbunden sind. Dies führt zu folgendem System $X = (X_t)_{t \geq 0} = (X_t(i))_{t \geq 0, i \in \mathbb{Z}^d}$ von *wechselwir-*

kenden Feller-Diffusionen mit logistischem Wachstum, wobei $X_t(i) \in [0,\infty)$ die Populationsgröße auf der Insel $i \in \mathbb{Z}^d$ zur Zeit $t \geq 0$ bezeichnet:

$$\begin{aligned} dX_t(i) &= \alpha \left(\sum_{j\in\mathbb{Z}^d} m(i,j)X_t(j) - X_t(i) \right) dt \\ &\quad + \gamma X_t(i)\big(K - X_t(i)\big)dt + \sqrt{2\beta X_t(i)}\, dB_t(i) \qquad i \in \mathbb{Z}^d. \end{aligned} \tag{Z.2}$$

Dabei sind $B(i)$ unabhängige Standard-Brownsche Bewegungen, m ist die Übergangsmatrix einer Irrfahrt auf $\mathbb{Z}^d$ und α, β, γ sind nichtnegative Konstanten, die die Raten von Migration, Verzweigung beziehungsweise Kompetition beschreiben. Die Konstante $K \geq 0$ nennt man *Kapazität.* Die *Migrationsmatrix* m sei eine translationsinvariante, irreduzible stochastische Matrix. Wechselwirkende Feller-Diffusionen mit logistischem Wachstum sind ein generisches Beispiel für ein System *wechselwirkender lokal regulierter Diffusionen* (engl. interacting locally regulated diffusions). Im Mittelpunkt dieser Arbeit steht das folgende System stochastischer Differentialgleichungen:

$$\begin{aligned} dX_t(i) =&\alpha \left(\sum_{j\in G} m(i,j)X_t(j) - X_t(i) \right) dt \\ &+ h\big(X_t(i)\big)dt + \sqrt{2\cdot g\big(X_t(i)\big)}\, dB_t(i), \qquad i \in G, \end{aligned} \tag{Z.3}$$

wobei G eine höchstens abzählbare Abelsche Gruppe ist. Es sei bemerkt, dass die zwei Modelle (Z.2) und (Z.3) übereinstimmen, falls $G = \mathbb{Z}^d$, $h(x) = \gamma x(K-x)$, $g(x) = \beta x$ gilt. Die folgenden Figuren 4.4 und 4.5 zeigen generische Beispiele für eine *Regulierungsfunktion* h beziehungsweise für eine *Diffusionsfunktion* g.

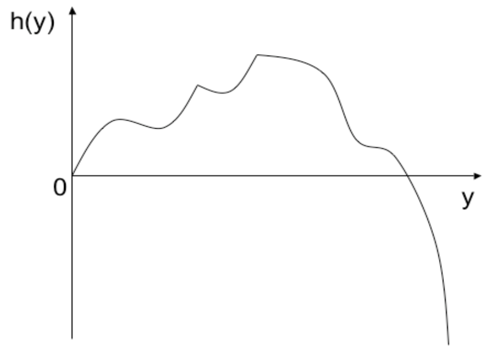


Abbildung 4.4: Ein generisches Beispiel für eine Regulierungsfunktion.

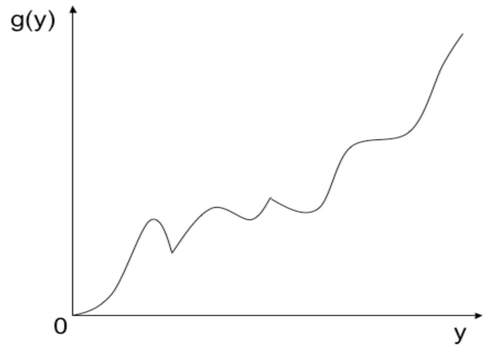


Abbildung 4.5: Ein generisches Beispiel für eine Diffusionsfunktion.

Einen geeigneten Zustandsraum für das System (Z.3) erhält man durch folgende Konstruktion, welche auf Liggett und Spitzer [25] zurückgeht. Wähle zu gegebener Migrationsmatrix m eine summierbare und strikt positive Folge $\sigma = (\sigma_i)_{i\in G}$, für die

$$\text{(Z.4)} \qquad \sum_{i\in G} \sigma_i m(i,j) \le C_{LS}\sigma_j, \qquad j \in G,$$

für eine Konstante $C_{LS} < \infty$ gilt. Definiere hiermit als Zustandsraum den *Liggett-Spitzer-Raum*

$$\text{(Z.5)} \qquad \mathbb{E}_\sigma := \Big\{ \underline{x} \in [0,\infty)^G \colon \| \underline{x} \|_\sigma := \sum_{i\in G} \sigma_i |x_i| < \infty \Big\}.$$

Der Liggett-Spitzer-Raum hat die wichtige Eigenschaft, dass jedes translationsinvariante Maß auf $[0,\infty)^G$ mit $\int x_0 \, \mu(dx) < \infty$ Träger in $\mathbb{E}_\sigma$ hat. Die folgenden Annahmen an die Regulierungsfunktion h und an die Diffusionsfunktion g garantieren die Existenz und die Eindeutigkeit einer Lösung von Gleichung (Z.3).

Annahme A1. *Die Funktionen* $h\colon [0,\infty) \to \mathbb{R}$ *und* $g\colon [0,\infty) \to [0,\infty)$ *sind lokal Lipschitz stetig und erfüllen* $h(0) = 0 = g(0)$. *Desweiteren ist* h *nach oben global Lipschitz stetig, d. h., es gilt* $\operatorname{sgn}(x-y)\big(h(x) - h(y)\big) \le C_h|x-y|$ *für alle* $x, y \ge 0$ *und eine Konstante* $C_h < \infty$. *Die Funktion* g *ist strikt positiv auf* $(0,\infty)$ *und erfüllt* $g(x) \le C_g(1+x^2)$ *für alle* $x \ge 0$ *und ein* $C_g < \infty$.

Unter diesen Annahmen zeigt Proposition 1.2.1, dass das System (Z.3) eine eindeutige starke Lösung mit Werten in $\mathbb{E}_\sigma$ hat.

In Theorem 1 wird der Maximalprozess $X^{(\infty)}$ konstruiert, welcher der Gleichung (Z.3) gehorcht und welcher zur Zeit Null von unendlich „herunter kommt". Hierfür wird eine Bedingung an h benötigt, die sicher stellt, dass die Drift „hinreichend negativ" für große Werte von $X_t^{(\infty)}(i)$ ist. Diese Bedingung wird in folgender Annahme formuliert.

Annahme A2. *Es existiert eine Funktion* $\hat{h} \ge h$ *derart, dass* $\hat{h}$ *für ein* $x_0 > 0$ *auf* $[x_0,\infty)$ *negativ und konkav ist und*

$$\text{(Z.6)} \qquad \int_{x_0}^{+\infty} \frac{1}{-\hat{h}(x)} \, dx < \infty$$

erfüllt.

Die entscheidende Eigenschaft von $X^{(\infty)}$ ist, dass dieser Prozess jede Lösung von (Z.3) in einer stochastischen Ordnung dominiert, welche nun vorgestellt wird. Sind μ_1, μ_2 Wahrscheinlichkeitsmaße auf der partiell geordneten Menge $\mathbb{E}_\sigma$, dann heisst μ_1 *stochastisch kleiner oder gleich* μ_2, geschrieben als $\mu_1 \le \mu_2$, falls ein zufälliges Paar (Y_1, Y_2) mit Randverteilungen $\mathcal{L}(Y_i) = \mu_i$, $i = 1, 2$, existiert, für welches $Y_1 \leq Y_2$ fast sicher gilt. Desweiteren sagt man, dass eine Folge $(\mu_i)_{i\in\mathbb{N}}$ von Wahrscheinlichkeitsmaßen *stochastisch* gegen das Wahrscheinlichkeitsmaß μ_∞ *anwächst*, falls eine zufällige Folge $(Y_i)_{i\in\mathbb{N}}$ existiert, welche fast sicher monoton steigend gegen Y_∞ konvergiert und welche $\mathcal{L}(Y_i) = \mu_i$, $i = 1, 2, \ldots, \infty$, erfüllt. Man schreibt hierfür $\mu_i \uparrow \mu_\infty$. In dieser Notation lässt sich die Existenz des Maximalprozesses wie folgt formulieren.

Theorem 1. *Die Annahmen A1 und A2 seien erfüllt. Dann existiert ein $\mathbb{E}_\sigma$-wertiger Prozess $(X_t^{(\infty)})_{t>0}$ mit den folgenden Eigenschaften:*

a) *Für jedes $\varepsilon > 0$ ist $(X_t^{(\infty)})_{t\geq\varepsilon}$ eine Lösung von* (Z.3)*, welche zur Zeit $t = \varepsilon$ startet.*

b) *Das erste Moment von $\| X_t^{(\infty)} \|_\sigma$ ist für jedes $t > 0$ endlich.*

c) *Sei $\underline{x}^{(n)} = (x_i^{(n)})_{i\in G}$, $n = 1, 2, ...$, eine monoton steigende Folge in $\mathbb{E}_\sigma$ mit der Eigenschaft, dass für alle $i \in G$*

$$x_i^{(n)} \uparrow \infty \qquad \text{für } n \to \infty. \tag{Z.7}$$

Falls $(X_t^{(n)})_{t\geq 0}$ die Lösung von (Z.3) *mit Startpunkt $\underline{x}^{(n)} \in \mathbb{E}_\sigma$ zur Zeit Null ist, dann*

$$\mathcal{L}\left(X_t^{(n)}\right) \uparrow \mathcal{L}\left(X_t^{(\infty)}\right) \quad \text{für } n \uparrow \infty \qquad (t > 0). \tag{Z.8}$$

d) *Es existiert eine Gleichgewichtsverteilung $\bar{\nu}$ (bezeichnet als* oberes invariantes Maß*) für die Dynamik* (Z.3)*, sodass*

$$\mathcal{L}\left(X_t^{(\infty)}\right) \downarrow \bar{\nu} \quad \text{für } t \uparrow \infty. \tag{Z.9}$$

e) *Jede $\mathbb{E}_\sigma$-wertige Lösung $(X_t)_{t\geq 0}$ von* (Z.3) *erfüllt*

$$\mathcal{L}\big(X_t\big) \leq \mathcal{L}\big(X_t^{(\infty)}\big) \qquad (t > 0). \tag{Z.10}$$

Insbesondere ist jede Gleichgewichtsverteilung ν stochastisch kleiner oder gleich $\bar{\nu}$.

Sowohl der Maximalprozess als auch das obere invariante Maß spielen für die folgenden Resultate eine wichtige Rolle.

Eine zentrale Frage dieser Arbeit ist, ob $(X_t)_{t\geq 0}$ für $t \to \infty$ ausstirbt oder überlebt. Zuerst klären wir, was wir unter „Aussterben“ verstehen. Wir sprechen von *lokalem Aussterben*, falls $(X_t)_{t\geq 0}$ in Verteilung für $t \to \infty$ gegen die Nullkonfiguration konvergiert. Hierfür sei die Topologie auf $[0, \infty)^G$ gleich der Produkttopologie.

Desweiteren sprechen wir von *globalem Aussterben*, falls $(|X_t|)_{t\geq 0}$ für $t \to \infty$ gegen Null konvergiert. Die gesamte Arbeit hindurch wird mit $|x| := \sum_{i\in G} x_i$ die Gesamtmasse von $x \in [0,\infty)^G$ bezeichnet. Es sei bemerkt, dass globales Aussterben lokales Aussterben impliziert. Darüberhinaus stimmen diese beiden Eigenschaften überein, falls G eine endliche Menge ist. Im Zusammenhang mit lokalem Aussterben wird typischerweise Translationsinvarianz der Verteilung von X_0 angenommen. Für globales Aussterben wird im Allgemeinen angenommen, dass fast sicher $|X_0| < \infty$ gilt. Von *lokalem* beziehungsweise *globalem Überleben* sprechen wir, falls das System nicht lokal beziehungsweise nicht global ausstirbt.

Mit Hilfe von Argumenten aus der Perkolationstheorie zeigt Etheridge [10], dass das System (Z.2) und ebenso ähnliche Modelle mit nichtlokaler Kompetition nicht lokal ausstirbt, falls die Kapazität groß genug ist und falls die Anfangsverteilung translationsinvariant ist. Darüberhinaus wurde in derselben Arbeit mit Hilfe einer Kopplung und eines Vergleiches mit subkritischer Verzweigung (ähnlich wie in Mueller und Tribe [26]) bewiesen, dass ein maßwertiges Analogon zu (Z.2) mit gewissen nichtlokalen Kompetitionsmechanismen lokal ausstirbt. Die Frage, ob Systeme wie (Z.2), die auf Gitter beruhen, für sehr kleine Werte von K lokal aussterben, blieb unbeantwortet. In Kapitel 2 wird diese Frage für das System (Z.2) mit Ja beantwortet. Genauer gesagt wird eine strikt positive Konstante $\bar{K}$ spezifiziert, so dass das System (Z.2) für jedes $K \leq \bar{K}$ lokal ausstirbt. Die Konstante $\bar{K}$ ist die eindeutige Lösung der Gleichung

$$\int_0^\infty \exp\left(\overline{K}\gamma y - \frac{\gamma\beta}{2}y^2\right) \cdot \alpha \exp\left(-\alpha y\right) dy = 1 \tag{Z.11}$$

und hängt von den Parametern α, β und γ der Migration, der Verzweigung beziehungsweise der Kompetition ab, ist jedoch uniform in der Dimension d und in der Migrationsmatrix m. Für das allgemeinere Modell (Z.3) wird ein Kriterium für lokales Aussterben in Theorem 2 formuliert.

Theorem 2. *Die Annahmen A1 und A2 seien erfüllt. Sei X eine Lösung der Gleichung* (Z.3) *mit einer beliebigen Anfangsverteilung auf $\mathbb{E}_\sigma$. Falls eine konkave Funktion $\bar{h} \geq h$ existiert, welche*

$$\int_0^\infty \frac{\bar{h}(y)}{g(y)} \exp\left(\int_1^y \frac{-\alpha x + \bar{h}(x)}{g(x)}\,dx\right) dy \leq 0 \tag{Z.12}$$

erfüllt, dann stirbt der Prozess X lokal aus, d. h.,

$$\mathcal{L}\left(X_t\right) \Longrightarrow \delta_{\underline{0}} \quad \text{für } t \to \infty. \tag{Z.13}$$

Dabei bezeichnet $\underline{0}$ die Nullkonfiguration.

Im Fall $h(x) = \gamma x(K - x)$ und $g(x) = \beta x$ ist die Bedingung (Z.12) mit $\bar{h} := h$ äquivalent zu $K \le \bar{K}$, wobei $\bar{K}$ die Gleichung (Z.11) löst.

In den folgenden beiden Theoremen wird die spezielle Form der Dynamik (Z.2) von wechselwirkenden Feller-Diffusionen mit logistischem Wachstum ausgenutzt. Das zweite Hauptresultat von Kapitel 2 beweist Ergodizität der Lösung $(X_t)_{t\ge 0}$ von Gleichung (Z.2) für $t \to \infty$, das heisst, der Prozess vergisst seine Anfangsverteilung im Grenzübergang $t \to \infty$. Genauer gesagt konvergiert $(X_t)_{t\ge 0}$ in Verteilung gegen das obere invariante Maß für $t \to \infty$ wann immer der Prozess in einer translationsinvarianten und nichttrivialen Anfangsverteilung startet.

Theorem 5. *Sei $\beta, \gamma > 0$. Desweiteren sei $X = (X_t)_{t\ge 0}$ die Lösung der Gleichung (Z.2). Es gelte $\mathcal{L}(X_0) \ge \mu$, wobei μ ein translationsinvariantes Wahrscheinlichkeitsmaß auf $\mathbb{E}_\sigma$ ist, welches keine Masse auf die Nullkonfiguration legt. Dann*

$$\mathcal{L}(X_t) \Longrightarrow \bar{\nu} \qquad \text{für } t \to \infty, \tag{Z.14}$$

wobei $\bar{\nu}$ das obere invariante Maß ist.

Im Beweis von Theorem 5 spielt folgende Selbstdualität eine zentrale Rolle. Sei X die Lösung von (Z.2) mit Parametern $\alpha, \beta, \gamma \ge 0$ und Migrationsmatrix m, und sei $X^\dagger$ die Lösung von (Z.2) mit Parametern $\alpha, \beta, \gamma \ge 0$ und Migrationsmatrix $m^\dagger$, welches die transponierte Matrix von m ist. Desweiteren sei $\mathbb{E}_{\sigma^\dagger}$ ein zu $m^\dagger$ passender Liggett-Spitzer-Raum.

Theorem 3. *Es gelte $\beta > 0$. Seien X und $X^\dagger$ Lösungen von (Z.2) mit Migrationsmatrizen m beziehungsweise $m^\dagger$. Dann gilt die folgende Selbstdualität:*

$$\mathbf{E}^{\underline{x}} \exp\big(-\frac{\gamma}{\beta}\langle X_t, \underline{y}\rangle\big) = \mathbf{E}^{\underline{y}} \exp\big(-\frac{\gamma}{\beta}\langle \underline{x}, X_t^\dagger\rangle\big) \tag{Z.15}$$

für alle $\underline{x} \in \mathbb{E}_\sigma, \underline{y} \in \mathbb{E}_{\sigma^\dagger}$, $t \ge 0$.

Beispielsweise führt die Wahl $\underline{y} = \lambda\delta_0$ und $\underline{x} \equiv \kappa$ zu der Gleichung

$$\mathbf{E}^{\underline{x}} \exp\big(-\frac{\gamma}{\beta}\lambda X_t(0)\big) = \mathbf{E}^{\lambda\delta_0} \exp\big(-\frac{\gamma}{\beta}\kappa|X_t^\dagger|\big) \tag{Z.16}$$

für alle $\lambda, \kappa \ge 0$, wobei δ_0 die Punktmasse in $0 \in \mathbb{Z}^d$ bezeichnet. In Abschnitt 2.6 wird gezeigt, dass $|X_t^\dagger|$ mit Wahrscheinlichkeit eins für $t \to \infty$ entweder gegen Null oder gegen unendlich konvergiert. Dies impliziert die Konvergenz der rechten Seite von Gleichung (Z.16). Lemma 2.5.1 zeigt den Zusammenhang des Grenzwertes der rechten Seite von (Z.16) mit dem oberen invarianten Maß.

Eine direkte Konsequenz von Theorem 3 ist, dass ein System wechselwirkender Feller-Diffusionen mit logistischem Wachstum genau dann lokal ausstirbt wenn es global ausstirbt. Dies verhält sich bei wechselwirkenden lokal regulierten Diffusionen anders. Hierfür gilt Theorem 3 im Allgemeinen nicht. Bisher ist kein allgemeines Kriterium für globales Aussterben des Systems (Z.3) bekannt. Wir stellen nun das *Virgin Island Modell* vor. Für dieses Modell wird in Theorem 7 ein Kriterium für globales Aussterben hergeleitet. Desweiteren vermuten wir, dass das Virgin Island Modell die Lösung von (Z.3) in einer geeigneten stochastischen Ordnung dominiert. Zusammen mit Theorem 7 würde diese Vermutung zu einer Bedingung für globales Aussterben für das System (Z.3) führen.

Wir charakterisieren das Virgin Island Modell durch eine rekursive Konstruktion. Auf der ersten Insel entwickelt sich eine Diffusion $Y = (Y_t)_{t\geq 0}$ mit Zustandsraum $[0,\infty)$, welche gegeben wird durch die stochastische Differentialgleichung

$$dY_t = -\alpha Y_t\,dt + h(Y_t)\,dt + \sqrt{2g(Y_t)}dB_t, \qquad Y_0 = y \geq 0. \tag{Z.17}$$

Dabei ist $(B_t)_{t\geq 0}$ eine Standard-Brownsche Bewegung. Der Prozess Y sei regulär auf $(0,\infty)$ und Null sei ein Austrittsrand, das heisst, Null ist ein absorbierender Rand und wird mit positiver Wahrscheinlichkeit in endlicher Zeit erreicht. Äquivalent hierzu ist die folgende Bedingung an α, h, g.

Annahme A3. *Der Parameter α und die Funktionen g und h erfüllen*

$$\int_0^x \bar{S}(y)\frac{1}{g(y)\bar{s}(y)}\,dy < \infty \tag{Z.18}$$

für ein und damit für alle $x > 0$, wobei

$$\bar{S}(y) := \int_0^y \bar{s}(z)\,dz, \quad \bar{s}(z) := \exp\Big(-\int_1^z \frac{-\alpha x + h(x)}{g(x)}\,dx\Big),\ y,z \geq 0. \tag{Z.19}$$

Beispielsweise gelten A1 und A3, falls $h(y) = \sigma y - \gamma y^2$, $\gamma > 0$, und $g(y) = y^\kappa$ für ein $1 \leq \kappa < 2$. Annahme A3 ist jedoch nicht erfüllt im Fall $h \equiv 0$ und $g(y) = y^2$, denn dann ist $\bar{s}(z) = z^\alpha$, $\bar{S}(y) = y^{\alpha+1}/(\alpha+1)$ und Bedingung (Z.18) ist verletzt.

Masse emigriert von der ersten Insel mit Rate α. Dies wird durch den Term $-\alpha Y_t\,dt$ in (Z.17) modelliert. Jeder Emigrant landet auf einer unbesiedelten Insel. Im Diffusionslimes hat ein Emigrant Masse Null. Allerdings kann die von einem Emigranten gegründete Population im Diffusionslimes nach positiver Zeit positiv sein. Das Gesetz der Exkursionen von Y vom Rand Null ist deshalb ein wichtiger Bestandteil der Konstruktion des Virgin Island Modells. Die Menge der Exkursionspfade von Null sei bezeichnet mit

$$U := \Big\{\chi \in \mathbf{C}\big([0,\infty),[0,\infty)\big) : T_0 \in (0,\infty],\ \chi_t = 0\ \ \forall\, t \in \{0\} \cup [T_0,\infty)\Big\} \tag{Z.20}$$

wobei $T_y = T_y(\chi) := \inf\{t > 0\colon \chi_t = y\}$ die erste Treffzeit von $y \in [0,\infty)$ sei. Die Menge U sei versehen mit uniformer Konvergenz. Das *Exkursionsmaß* Q_Y ist ein σ-endliches Maß auf U. Wir definieren es durch Theorem 6. Hierfür benötigen wir eine weitere Voraussetzung. Wir nehmen an, dass

$$\text{(Z.21)} \qquad \mathbf{P}^y\big(T_1(Y) < T_0(Y)\big) \sim cy \qquad \text{für } y \to 0$$

für eine Konstante $c \in (0,\infty)$ gilt. Genauer gesagt setzen wir die Gültigkeit der folgenden etwas stärkeren Annahme voraus.

Annahme A4. *Das Integral* $\int_\varepsilon^1 \frac{-\alpha y + h(y)}{g(y)}\,dy$ *hat einen Grenzwert in* $(-\infty,\infty)$ *für* $\varepsilon \to 0$.

Theorem 6. *Die Annahmen A1, A3 und A4 seien erfüllt. Dann existiert ein* σ*-endliches Maß* Q_Y *auf* U*, sodass*

$$\text{(Z.22)} \qquad \lim_{y\to 0} \frac{1}{y}\mathbf{E}^y F(Y) = \int F(\chi) Q_Y(d\chi)$$

für alle beschränkten, stetigen Funktionen $F\colon \mathbf{C}\big([0,\infty),[0,\infty)\big) \to \mathbb{R}$ *für die ein* $\varepsilon > 0$ *existiert so dass* $F(\chi) =$ *gilt wann immer* $\sup_{t\geq 0}\chi_t < \varepsilon$.

Mit Hilfe des Exkursionsmaßes definieren wir nun das Virgin Island Modell auf den nachfolgenden Inseln. Die erste Insel bezeichnen wir als 0-te Generation. Die $(n+1)$-ste Generation ist die Menge aller Inseln, welche von Inseln der n-ten Generation besiedelt worden sind. Die Menge aller Inseln schließlich bezeichnen wir als *Virgin Island Modell.* Desweiteren verstehen wir unter dem *Virgin Island Prozess* den Prozess der Gesamtmasse aller Inseln des Virgin Island Modells. Sei $(V_t^{(0)})_{t\geq 0}$ ein zufälliger Pfad mit Verteilung $\mathcal{L}^x\big((Y_t)_{t\geq 0}\big)$, $x \geq 0$. Für jedes $n \geq 1$ definieren wir nun rekursiv einen Prozess $V^{(n)} = (V_t^{(n)})_{t\geq 0}$. Dies ist die Gesamtmasse aller Inseln der n-ten Generation. Gegeben $V^{(n)}$ sei $\Pi^{(n)}$ ein Poisson Punktprozess auf $[0,\infty) \times U$ mit Intensitätsmaß $\alpha V_t^{(n)}\,dt \otimes Q_Y(d\chi)$. Hiermit wird $(V_t^{(n+1)})_{t\geq 0}$ definiert durch

$$\text{(Z.23)} \qquad V_t^{(n+1)} := \int \chi_{t-s}\Pi^{(n)}(ds,d\chi) \qquad t \geq 0.$$

Emigranten verlassen Inseln der n-ten Generation mit der zeitabhängigen Rate $\alpha V_t^{(n)}$ und landen auf unbesiedelten Inseln. Eine Insel, welche zur Zeit $s \geq 0$ besiedelt wurde, trägt zur Zeit $t \geq 0$ Masse χ_{t-s} zur Gesamtmasse bei. Der Virgin Island Prozess ist die Gesamtmasse aller Generationen:

$$\text{(Z.24)} \qquad V_t := \sum_{n\geq 0} V_t^{(n)} \qquad t \geq 0.$$

Nach Lemma 3.3.1 ist der Erwartungswert dieser Summe endlich.

In Theorem 7 identifizieren wir Bedingungen an α, h und g, unter welchen das Virgin Island Modell global ausstirbt. Der entscheidende Parameter ist hierbei die erwartete Fläche unter einer Exkursion

$$\int \int_0^\infty \chi_t \, dt \, Q_Y(d\chi). \tag{Z.25}$$

Für die folgenden Theoreme 7 und Theorem 8 nehmen wir an, dass der Ausdruck in (Z.25) endlich ist. Falls die Bedingungen A1, A3 und A4 erfüllt sind, ist hierzu die folgende Bedingung an α, h und g äquivalent.

Annahme A5. *Der Parameter α und die Funktionen g und h erfüllen*

$$\int_x^\infty \frac{y}{g(y)s(y)} \, dy < \infty \tag{Z.26}$$

für ein und damit für alle $x > 0$, wobei

$$s(y) := \exp\Big(-\int_0^y \frac{-\alpha x + h(x)}{g(x)} \, dx\Big), \; y \geq 0. \tag{Z.27}$$

Ein generisches Beispiel für h und g ist $h(y) = c_1 y^{\kappa_1} - c_2 y^{\kappa_2}$, $g(y) = c_3 y^{\kappa_3}$ mit $c_1, c_2, c_3 > 0$. Die Annahmen A1, A2, A3, A4 und A5 sind alle erfüllt, falls $\kappa_2 > \kappa_1 \geq 1$ und $\kappa_3 \in [1, 2)$ gilt.

Theorem 7. *Die Annahmen A1, A3, A4 und A5 seien erfüllt. Dann stirbt der Virgin Island Prozess $(V_t)_{t\geq 0}$ für jeden Startpunkt $x > 0$ global aus genau dann wenn*

$$\int \int_0^\infty \alpha \chi_s \, ds \, Q_Y(d\chi) \leq 1 \tag{Z.28}$$

gilt. Der Ausdruck auf der linken Seite ist gleich

$$\int_0^\infty \frac{\alpha y}{g(y)s(y)} \, dy = \int \int_0^\infty \alpha \chi_s \, ds \, Q_Y(d\chi). \tag{Z.29}$$

Im Falle des Überlebens konvergiert $(V_t)_{t\geq 0}$ in Verteilung für $t \to \infty$ gegen eine Zufallsvariable V_∞, deren Verteilung charakterisiert wird durch

$$\mathbf{P}^x(V_\infty = 0) = 1 - \mathbf{P}^x(V_\infty = \infty) = \mathbf{E}^x \exp\big(-q \int_0^\infty \alpha Y_s \, ds\big) \tag{Z.30}$$

für alle $x \geq 0$ und ein $q > 0$.

Eine interessante Größe des Virgin Island Prozesses ist die erwartete Fläche unter dem Graphen von $(V_t)_{t\geq 0}$. In Theorem 8 wird die Asymptotik der erwarteten Fläche unter dem Graphen von $(V_s)_{s\leq t}$ für $t \to \infty$ ermittelt. Definiere

$$w(x) := \int_0^\infty S(x \wedge z) \frac{z}{g(z)s(z)}\, dz \tag{Z.31}$$

für $x \geq 0$.

Theorem 8. *Die Annahmen A1, A3, A4 und A5 seien erfüllt. Falls die linke Seite von Ungleichung* (Z.28) *strikt kleiner als eins ist, dann ist, für jedes* $x \geq 0$, *die Fläche unter dem Graphen von* $(V_t)_{t\geq 0}$ *gleich*

$$\mathbf{E}^x \int_0^\infty V_s\, ds = \frac{\mathbf{E}^x\Big(\int_0^\infty Y_s\, ds\Big)}{1 - \int \Big(\int_0^\infty \alpha\chi_s\, ds\Big) Q_Y(d\chi)} = \frac{w(x)}{1 - \int_0^\infty \frac{\alpha z}{g(z)s(z)}\, dz} \in (0, \infty). \tag{Z.32}$$

Andernfalls ist die linke Seite von (Z.32) *gleich unendlich. Im kritischen Fall, das heisst Gleichheit in* (Z.28), *gilt*

$$\frac{1}{t}\int_0^t \mathbf{E}^x V_s\, ds \to \frac{\mathbf{E}^x\Big(\int_0^\infty Y_u\, du\Big)}{\int \Big(\int_0^\infty u\alpha\chi_u\, du\Big) Q_Y(d\chi)} = \frac{w(x)}{\int_0^\infty \frac{\alpha w(y)}{g(y)s(y)} dy} \in [0, \infty) \tag{Z.33}$$

für $t \to \infty$, *wobei die rechte Seite als Null interpretiert wird, falls der Zähler gleich unendlich ist. Im superkritischen Fall, das heisst, falls Ungleichung* (Z.28) *nicht erfüllt ist, sei* $\beta > 0$ *die eindeutige Lösung von*

$$\int_0^\infty e^{-\beta u} \int \alpha\chi_u Q_Y(d\chi)\, du = 1. \tag{Z.34}$$

Damit kann die Konvergenzordnung der erwarteten Fläche unter dem Pfad von $(V_s)_{s\leq t}$ *abgelesen werden von*

$$e^{-\beta t}\int_0^t \mathbf{E}^x V_s\, ds \to \frac{\int_0^\infty e^{-\beta u}\mathbf{E}^x \int_0^u Y_s\, ds\, du}{\int_0^\infty u e^{-\beta u} \int \alpha\chi_u Q_Y(d\chi) du} \in (0, \infty) \tag{Z.35}$$

für $t \to \infty$ *für alle* $x \geq 0$.

Das Virgin Island Modell vereinigt auf sich die folgenden zwei Eigenschaften. Einerseits beinhaltet es Kompetition unter Individuen. Andererseits existiert ein explizites Kriterium für den Phasenübergang von Aussterben zu Überleben. Somit ist das Virgin Island Modell möglicherweise interessant für Anwendungen, denn es ist realistischer als klassische Verzweigungsmodelle, ist aber noch so einfach, dass es explizite Formeln hat.

Die Selbstdualität (Z.15) ist ein starkes Werkzeug, um wechselwirkende Feller-Diffusionen mit logistischem Wachstum zu untersuchen. Abschnitt 2.5 beinhaltet einen analytischen Beweis dieser Selbstdualität, welcher auf einer Generatorrechnung beruht. In Kapitel 4 verfolgen wir einen anderen Ansatz, wobei wir die Dynamik der Prozesse durch sogenannte *Grundmechanismen* auf der Ebene von Teilchen darstellen. Dadurch erhalten wir ein stochastisches Bild für die Selbstdualität (Z.15), welches das Verständnis der Rolle der logistischen Regulierungsfunktion $\gamma x(K-x)$ in (Z.2) für die Selbstdualität (Z.15) vertieft und welches eine Erklärung für das Auftreten der Dualitätsfunktion $\exp\left(-\frac{\gamma}{\beta}\langle x, y\rangle\right)$ in (Z.15) liefert. Der Einfachheit halber betrachten wir nur den nichträumlichen Fall, das heisst, $m(i,j) = \mathbb{1}_{i=j}$ für $i, j \in \mathbb{Z}^d$.

Für eine etwas allgemeinere Dualität als (Z.15) betrachten wir die starke Lösung $(X_t)_{t\geq 0}$ von

$$dX_t = \varsigma X_t\, dt - \gamma X_t^2\, dt + \sqrt{2\beta X_t}\, dB_t, \tag{Z.36}$$

wobei $(B_t)_{t\geq 0}$ eine Standard-Brownsche Bewegung ist. Wir bezeichnen den Prozess $(X_t)_{t\geq 0}$ als logistische Feller-Diffusion mit Parametern $(\varsigma, \gamma, \beta)$. Sei $(Y_t)_{t\geq 0}$ eine logistische Feller-Diffusion mit Parametern $(\varsigma, r\beta, \gamma/r)$ für ein $r > 0$. In Abschnitt 4.4 beweisen wir

$$\mathbf{E}^x\left[e^{-rX_t\cdot y}\right] = \mathbf{E}^y\left[e^{-rx\cdot Y_t}\right] \qquad x, y \in [0,\infty), t \geq 0. \tag{Z.37}$$

Der Ansatz, den wir im Folgenden vorstellen, ist nicht nur auf (Z.37) anwendbar, sondern auch auf eine andere Dualität, welche analytisch von Athreya und Swart [2] bewiesen wurde. Seien $b, c, d \geq 0$. Es bezeichne $X_t \in \mathbb{N}_0$ die Anzahl der Teilchen zur Zeit $t \geq 0$ eines Verzweigungs-Verschmelzungs-Teilchenprozesses, welcher durch die anfängliche Anzahl $X_0 = n$ und durch folgende Dynamik definiert wird: Jedes Teilchen spaltet sich mit Rate b in zwei neue Teilchen, jedes Teilchen stirbt mit Rate d und jedes geordnete Paar von Teilchen verschmilzt mit Rate c zu einem Teilchen. Alle diese Ereignisse geschehen voneinander unabhängig. Athreya und Swart [2] bezeichnen diesen Prozess als $(1, b, c, d)$-braco-Prozess. Dessen dualer Prozess $(Y_t)_{t\geq 0}$ mit Zustandsraum $[0,1]$ ist die eindeutige starke Lösung der stochastischen Differentialgleichung

$$dY_t = (b-d)Y_t\, dt - bY_t^2\, dt + \sqrt{2cY_t(1-Y_t)}\, dB_t, \quad Y_0 = y. \tag{Z.38}$$

Athreya und Swart [2] beweisen die Dualität

$$\mathbf{E}^n\left[(1-y)^{X_t}\right] = \mathbf{E}^y\left[(1-Y_t)^n\right] \qquad \forall\, n \in \mathbb{N}_0,\, y \in [0,1],\, t \geq 0. \tag{Z.39}$$

Spezialfälle der Dualitäten (Z.37) und (Z.39) sind (siehe Remark 4.4.2 und Remark 4.4.4) die Dualität von Fellers Verzweigungsdiffusion mit einem deterministischen Prozess, die Dualität der Fisher-Wright Diffusion mit Kingmans Coalescent und die Dualität des Galton-Watson Prozesses (in kontinuierlicher Zeit) mit einem deterministischen Prozess.

Kapitel 4 zeichnet ein einheitliches Bild für die Dualitäten (Z.37) und (Z.39). Für jedes $N \in \mathbb{N}$ konstruieren wir Prozesse $(X_t^N)_{t\geq 0}$ und $(Y_t^N)_{t\geq 0}$ mit Zustandsraum $\{0,1\}^N$. Je nach Reskalierung approximieren die Prozesse $(X_t^N)_{t\geq 0}$ und $(Y_t^N)_{t\geq 0}$ für $N \to \infty$ einen $(1,b,c,d)$-braco-Prozess, eine logistische Feller-Diffusion oder einen $(1,b,c,d)$-resem-Prozess. Desweiteren sind $(X_t^N)_{t\geq 0}$ und $(Y_t^N)_{t\geq 0}$ für jedes $N \in \mathbb{N}$ dual zueinander. Im Abschnitt 4.4 werden wir aus dieser Dualität – je nach Reskalierung – im Grenzübergang $N \to \infty$ die Dualität (Z.39) beziehungsweise die Dualität (Z.37) folgern.

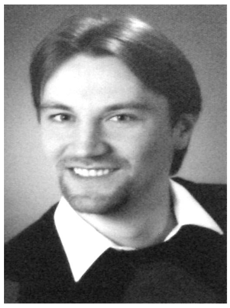
Der Prozess $(X_t^N)_{t\geq 0}$ wird durch folgende graphische Repräsentation konstruiert, welche im Geiste von Harris [14] ist. Als Grundmechanismus bezeichnen wir jede Funktion $f\colon \{0,1\}^2 \to \{0,1\}^2$. Ein endliches Tupel $(f_1,\ldots,f_m)$, $m \in \mathbb{N}$, von Grundmechanismen und ein Tupel $(\lambda_1,\ldots,\lambda_m) \in [0,\infty)^m$ von Raten definieren wie folgt einen Prozess. Mit jedem $k \leq m$ und jedem geordneten Paar $(i,j) \in \{1,\ldots,N\}^2$, $i \neq j$, wird ein Poisson-Prozess mit Rate λ_k assoziiert. Zu jedem Zeitpunkt dieses Poisson-Prozesses ändert sich die Konfiguration von (i,j) gemäß f_k. War die Konfiguration zuvor beispielsweise gleich $(1,0)$, so ändert sie sich in $f_k(1,0) \in \{0,1\}^2$. Alle Poisson-Prozesse sind voneinander unabhängig. Der Prozess $(Y_t^N)_{t\geq 0}$ wird mit Hilfe derselben Poisson-Prozesse definiert, jedoch in umgekehrter Zeit. Ob $(X_t^N)_{t\geq 0}$ und $(Y_t^N)_{t\geq 0}$ dual zueinander sind, erkennt man somit durch Verfolgen von Vorwärts- und Rückwärtspfaden. Dies führt zu einer Dualitätsbedingung an korrespondierende Paare von Grundmechanismen. Diese Dualitätsbedingung stellen wir in Abschnitt 4.2 vor. Desweiteren identifizieren wir in Abschnitt 4.2 alle dualen Paare von Grundmechanismen.

Wie in der Literatur bekannt ist, ist das Moran Modell dual zu Kingmans Coalescent. In der Sprache der Grundmechanismen besagt diese Dualität, dass der Resampling-Mechanismus dual zum Coalescent-Mechanismus ist. Es stellt sich heraus, dass es im Wesentlichen diese Dualität ist, aus der man die Dualität (Z.37) folgern kann. Der Resampling-Mechanismus führt dabei im Diffusionslimes zum Term $\sqrt{2\beta X_t}\, dB_t$ in Gleichung (Z.36). Die Gesamtanzahl der Teilchen eines Coalescent-Prozesses ist ein reiner Todesprozess, welcher von k nach $k-1$ mit exponentieller Rate $\frac{\gamma}{N^2}k(k-1)$, $k \geq 2$, springt. Diese Rate ist im Wesentlichen quadratisch in k für große k und führt im Diffusionslimes zum quadratischen Term $\gamma X_t^2\, dt$ in Gleichung (Z.36).

Lebenslauf

Name

Martin Hutzenthaler



Angaben zur Person

Geburtsdatum	20. Oktober 1978
Geburtsort	Landshut
Staatsangehörigkeit	deutsch

Bildungsweg

09/1985 - 07/1989	Grundschule Niederaichbach
09/1989 - 06/1998	Hans-Leinberger-Gymnasium Landshut Abschluss: Abitur (Note 1,8)
10/1998 - 04/2004	Studium der Mathematik mit Nebenfach Informatik an der Friedrich-Alexander-Universität Erlangen-Nürnberg
10/2000	Vordiplom (Note 1,0)
02/2002 - 07/2002	Auslandssemester an der University of Bath in England
04/2004	Abschluss: Diplom („mit Auszeichnung"), Betreuer der Diplomarbeit: Prof. Dr. A. Greven
05/2004 - 06/2007	Wissenschaftlicher Mitarbeiter an der J.W. Goethe-Universität Frankfurt, Institut für Stochastik und mathematische Informatik, Betreuer der Dissertation: Prof. Dr. A. Wakolbinger

Diplomarbeit

„Finite System Scheme in der kritischen Dimension: Clusterwachstum verzweigender Prozesse unter der Palmverteilung"

Akademische Lehrer

Professoren der Universität Erlangen-Nürnberg: Zowe, Barth, Greven, Sauer, Berens, Keller, Duzaar, Leutwiler, Plaumann, Strambach, Weidner