DISSERTATIONES MATHEMATICAE UNIVERSITATIS TARTUENSIS
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$M(r, s)$-ideals of compact operators

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Chapter 1

Introduction

1.1 Background

In 1972, E. M. Alfsen and E. G. Effros introduced in their fundamental article "Structure in real Banach spaces" the notion of an M-ideal as follows. Suppose that $\mathcal{L}$ is a real Banach space and $\mathcal{L}^*$ is the dual space of $\mathcal{L}$. It is said that a subspace $\mathcal{K}$ of $\mathcal{L}$ is an *M-summand* of $\mathcal{L}$ if there is a subspace $\mathcal{H} \subset \mathcal{L}$ such that $\mathcal{K} \oplus \mathcal{H} = \mathcal{L}$ and, for all $k \in \mathcal{K},\ h \in \mathcal{H}$,

$$\|k + h\| = \max\{\|k\|, \|h\|\}.$$

Similarly, a subspace $\mathcal{N}$ of $\mathcal{L}^*$ is an *L-summand* of $\mathcal{L}^*$ if there is a subspace $\mathcal{M}$ with $\mathcal{N} \oplus \mathcal{M} = \mathcal{L}^*$ and, for all $p \in \mathcal{N}$, $q \in \mathcal{M}$,

$$\|p + q\| = \|p\| + \|q\|. \tag{1.1}$$

A closed subspace $\mathcal{K}$ of $\mathcal{L}$ is said to be an *M-ideal* if its annihilator $\mathcal{K}^{\perp}$ (see Definition 2.2) is an L-summand in $\mathcal{L}^*$.

The letter "M" in the notion of an M-ideal comes from the word "maximum" and is referring to the norm of $\mathcal{L}$ ([23, p. v]). The word "ideal" seems to be inspired by the connection between M-ideals and algebraic ideals: in C^*-algebras, the M-ideals coincide with the closed two-sided ideals (see, for example, [23, Theorem V.4.4]).

E. M. Alfsen and E. G. Effros introduced M-ideals as an analog and generalization of algebraic ideals. Their approach was designed to encompass structure theories for C^*-algebras, ordered Banach spaces, L^1-preduals, and spaces of affine functions on compact convex sets. However, the M-structure theory was defined solely in terms of norms of Banach spaces, providing a wide range of applicability.

The theory of M-ideals has been widely used for studying the geometry of Banach spaces. It has turned out that $\mathcal{K}$ being an M-ideal in $\mathcal{L}$ has a strong impact on spaces $\mathcal{K}$ and $\mathcal{L}$. Namely, there are a number of important properties shared by M-ideals but not by arbitrary subspaces. For example, for every linear functional defined on an M-ideal, there exists a unique norm-preserving extension to the whole space. In general, this holds for arbitrary subspaces of $\mathcal{L}$ only in the case when $\mathcal{L}^*$ is strictly convex (see [57] and [14]).

In 1993, in the paper [15], G. Godefroy, N. J. Kalton, and P. D. Saphar introduced the notion of an *ideal* (see Definition 2.3) and related it with M-ideals. M-ideals form a subclass of u-ideals (see Definition 5.1) which were introduced by P. G. Casazza and N. J. Kalton earlier in 1990 (see [8]). The M- and u-ideals are ideals satisfying different norm conditions. Nowadays M-ideals are usually defined based on the concept of an ideal (see Definition 2.4).

The approach in [15] inspired J. C. Cabello and E. Nieto (see [4]) to consider a weaker form of the norm condition (1.1), that is, the $M(r,s)$-inequality

$$\|p+q\| \geq r\,\|p\| + s\,\|q\| \quad \forall p \in \mathcal{N},\ \forall q \in \mathcal{M},$$

where $r, s \in (0,1]$. They introduced and studied the notion of an ideal satisfying the $M(r,s)$-inequality, we call it an $M(r,s)$-ideal (see Definition 3.1). Note that if $r = s = 1$, then $M(r,s)$-ideals coincide with M-ideals.

In [4], [5], [7], Cabello, Nieto, and Oja studied whether properties holding for M-ideals carry over to the more general $M(r,s)$-ideals. For example, it turned out that M-ideals, and more generally $M(1,s)$-ideals, (see [4]) have property U (see Definition 3.4) and therefore they also have the unique ideal property (see Definition 3.3). However, e.g., for $r \neq 1$, $M(r,1)$-ideals of compact operators $\mathcal{K}(X)$ need not have property U (see [7, Example 4.5]). Also the 3-ball property (see Definition 2.5) does not hold for $M(r,s)$-ideals in general (see [7, Lemma 2.2]), but according to Alfsen and Effros, being an M-ideal is equivalent to the 3-ball property (see Theorem 2.6). In [7], one can find several examples of $M(r,s)$-ideals which are not M-ideals. $M(r,s)$-ideals have been studied, e.g., in [4], [5], [21], [7], [18], [22], [19], [52], [20].

Over the 40 years, the theory of M-ideals has been well studied. There is also a thorough monograph about M-ideals by P. Harmand, D. Werner, and W. Werner (see [23]) that grew out of the Berlin school of E. Behrends. The studies of u-ideals and ideals in general have been carried out among others by P. G. Casazza, G. Godefroy, K. John, N. J. Kalton, Å. Lima, V. Lima, E. Oja, P. D. Saphar, D. Werner. A recent unified approach regarding the ideals is the concept of an (a, B, c)-ideal (see, for example, [49]) which enables to study the M-, $M(r,s)$-, u-, and other types of ideals simultaneously.

The first example of an *M-ideal of compact operators* origins from 1950. Namely, J. Dixmier showed in [11] that for every Hilbert space H, $\mathcal{K}(H)$ is an M-ideal in $\mathcal{L}(H)$. From the beginning of the M-ideal theory, the problem of identifying for which Banach spaces X and Y the space of compact operators $\mathcal{K}(X,Y)$ is an M-ideal in the space of all bounded linear operators $\mathcal{L}(X,Y)$, has attracted a number of authors. The question is of interest because the existence of M-ideals gives information about the dual space $\mathcal{L}(X,Y)^*$. Not less important is the connection between M-ideals and the theory of approximation properties where even today there are famous unsolved questions which have been open for decades. Regarding Dixmier's result, describing M-ideals of compact operators can also be viewed as a study of the question of how far one can move away from Hilbert spaces without ruining the property of $\mathcal{K}(X,Y)$ being an M-ideal in $\mathcal{L}(X,Y)$ ([23, p. 289]).

The question, for which Banach spaces X and Y the space of compact operators $\mathcal{K}(X,Y)$ is an M-ideal in the space of all bounded linear operators $\mathcal{L}(X,Y)$, in a general formulation, is also a central question in this thesis. We study this question in terms of M-, $M(r,s)$-, and u-ideals.

1.2 Summary of the thesis

The starting point of the investigations in this thesis is the following result by E. Oja. It allows to produce, departing from Banach spaces X such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$, new classes of M-ideals of compact operators.

Corollary 2.29 ([45, Corollary 9]). *If $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are M-ideals in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ respectively, then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.*

The extension of Corollary 2.29 from M-ideals to $M(r,s)$-ideals presents difficulties since the main techniques from the theory of M-ideals involving the 3-ball property do not work in this more general case. For instance, in [23, p. 301], Corollary 2.29 is proven using the 3-ball property.

The objectives of this thesis is to investigate whether a similar result is valid for $M(r,s)$-ideals, how the parameters r and s are affected by forming new

$M(r,s)$-ideals, and to study u-ideals of compact operators using the methodology developed for $M(r,s)$-ideals. The thesis consists of five chapters.

Chapter 1 introduces the background of the problem and the basic notation used throughout this thesis.

The aim of Chapter 2 is to give, based on the schematic original proof in [45], a detailed proof of Corollary 2.29. In Section 2.1, we introduce the notion of an M-ideal through the concept of an ideal, and look at some classical examples of M-ideals of compact operators. Section 2.2 is dedicated to properties (M) and (M^*) which have turned out to be the key structure conditions on X, in order to $\mathcal{K}(X)$ appear as an M-ideal in $\mathcal{L}(X)$. In Section 2.3, the basic background results on approximating the identity operator by compact operators in operator topologies are presented with detailed proofs.

Johnson's lemma, introduced in Section 2.4, guarantees us the existence of an ideal projection whenever there exists a special kind of net of compact operators. As preliminary results, some descriptions of M-ideals of compact operators are regarded in Section 2.5. Corollary 2.29, the main result of Chapter 2, is proved in Section 2.6.

In Chapter 3, we extend and develop the results and methods used in Chapter 2, for M-ideals, to the case of $M(r,s)$-ideals. In Section 3.1, we define $M(r,s)$-ideals. Following [54], the notion of the Johnson projection is introduced in Section 3.3. Properties $M(r,s)$ and $M^*(r,s)$ for $M(r,s)$-ideals are analogs of properties (M) and (M^*) for M-ideals and are discussed in Sections 3.4 and 3.5. The main result of Chapter 3 is Corollary 3.22, in Section 3.6, which extends Corollary 2.29 from M-ideals to $M(r,s)$-ideals.

Corollary 3.22. *Let X and Y be Banach spaces. Assume that $\mathcal{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathcal{L}(X)$ with $r_1+s_1/2>1$ and $\mathcal{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathcal{L}(Y)$ with $r_2+s_2/2>1$. Then $\mathcal{K}(X,Y)$ is an $M(r_1^2r_2,s_1^2s_2)$- and an $M(r_1r_2^2,s_1s_2^2)$-ideal in $\mathcal{L}(X,Y)$.*

Chapter 3 is inspired by [45], [49], [54] and is based on [19].

The parameters $r_1^2r_2$ and $s_1^2s_2$, or $r_1r_2^2$ and $s_1s_2^2$ seem to be not optimal. In Chapter 4, we propose a different approach which will improve the parameters to r_1r_2 and s_1s_2 (see Theorem 4.18 for the case when X or Y is separable and Theorem 4.26 for the general non-separable case).

The key concepts of the new approach are "the ideal projection preserving elementary functionals" (introduced in Section 4.2) and "property $M^*(r,s)$ for operators" (see Section 4.4). An important tool, we are basing on, is the Feder–Saphar description of the dual space of $\mathcal{K}(X,Y)$ (see Section 4.1) which holds whenever X^{**} or Y^* has the Radon–Nikodým property. The reader may notice that this hypothesis is often present also implicitly (as can be seen, e.g., from Proposition 4.21).

Sections 4.2, 4.3, and 4.4 contain necessary auxiliary results regarding the ideal projection preserving elementary functionals and property $M^*(r,s)$ for operators which lead, relying on a vector-valued version of Simons's inequality (see Lemma 4.14), to the main results in the case when one of the spaces X or Y is separable (see Theorems 4.16 and 4.18) in Section 4.5.

In Section 4.6, we prove that $M(r,s)$-ideals of compact operators $\mathcal{K}(X,Y)$ are separably determined for distinct spaces X and Y (see Theorem 4.20; the result seems to be new even for M-ideals). Theorem 4.20 allows us to conclude some results concerning general structure of Banach spaces in Section 4.9.

The fact that $M(r,s)$-ideals of compact operators are separably determined together with Theorem 4.16 lead to the main result of the chapter (Theorem 4.23) asserting that $M^*(r_1,s_1)$-property of X and $M^*(r_2,s_2)$-property of Y imply that $\mathcal{K}(X,Y)$ is an $M(r_1r_2,s_1s_2)$-ideal in $\mathcal{L}(X,Y)$, and to Theorem 4.26 which improves Corollary 3.22.

Theorem 4.26. *Let X and Y be Banach spaces. Let $r_1,s_1,r_2,s_2 \in (0,1]$ satisfy $r_1+s_1/2>1$ and $r_2+s_2/2>1$. If $\mathcal{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathcal{L}(X)$ and $\mathcal{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathcal{L}(Y)$, then $\mathcal{K}(X,Y)$ is an $M(r_1r_2,s_1s_2)$-ideal in $\mathcal{L}(X,Y)$.*

Section 4.8 provides corollaries, which complete and improve some well-known result on M-ideals, from Theorems 4.16, 4.20, and 4.20.

Chapter 4 is inspired by [36], [47], [49] and is based on [20].

In Chapter 5, we apply the methodology developed in Chapter 4 to the case of u-ideals. Relying on [34] and [49] we prove that u-ideals are also separably determined. However, it turns out that the property of creating new u-ideals of compact operators behaves somewhat differently from the case of M- and $M(r,s)$-ideals (see Section 5.4). The chapter is based on [26].

1.3 Notation

Our notation is standard. Throughout the thesis we consider Banach spaces X and Y over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ if not stated otherwise. The identity operator, the closed unit ball, and the unit sphere of a Banach space X are denoted by I_X, B_X, and S_X, respectively. For a set $A \subset X$, its norm closure is denoted by $\overline{A}$, its linear span by $\operatorname{span} A$, and its convex hull by $\operatorname{conv} A$. The closures with respect to other topologies are marked such as $\overline{A}^{w^*}$, for example.

We denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators from a Banach space X to a Banach space Y and by $\mathcal{K}(X, Y)$ its subspace of compact operators. We write $\mathcal{K}(X)$ and $\mathcal{L}(X)$ instead of $\mathcal{K}(X, X)$ and $\mathcal{L}(X, X)$, respectively. The notation $\mathcal{I}(X)$ stands for $\operatorname{span}(\mathcal{K}(X) \cup I_X)$. For an operator $T : X \to Y$, we denote

$$\ker T = \{x \in X : Tx = 0\}$$

the kernel of T, and

$$\operatorname{ran} T = \{Tx \in Y : x \in X\}$$

the range of T. The restriction of T to a subset $K \subset X$ will be denoted by $T|_K$.

Let $\mathcal{L}$ be a subspace of $\mathcal{L}(X, Y)$, and let $x^{**} \in X^{**}$ and $y^* \in Y^*$. Then the functional $x^{**} \otimes y^* \in \mathcal{L}^*$ is defined by $(x^{**} \otimes y^*)(T) = x^{**}(T^* y^*)$ for any $T \in \mathcal{L}$. Note that $\|x^{**} \otimes y^*\| = \|x^{**}\| \|y^*\|$ whenever $\mathcal{L}$ contains the finite-rank operators. By $A \otimes B$, where $A \subset X^{**}$ and $B \subset Y^*$, we mean the set of all $x^{**} \otimes y^*$ such that $x^{**} \in A$ and $y^* \in B$. Thus $A \otimes B \subset \mathcal{L}(X, Y)^*$.

The canonical projection $\pi_X : X^{***} \longrightarrow X^{***}$ is defined by

$$\pi_X = j_{X^*}(j_X)^*,$$

where $j_X : X \longrightarrow X^{**}$ and

$$(j_X x)(x^*) = x^*(x), \quad x^* \in X^*, \ x \in X.$$

Recall that a net $(K_\alpha) \subset \mathcal{K}(X)$ is a *compact approximation of the identity* (*CAI*) provided $K_\alpha \longrightarrow I_X$ strongly (that is, $K_\alpha x \longrightarrow x$ for all $x \in X$). If additionally $K_\alpha^* \longrightarrow I_{X^*}$ strongly, then (K_α) is called a *shrinking CAI*. If X has a CAI such that the convergence is uniform on compact subsets, then X is said to have the *compact approximation property* (*CAP*), and in the case of a shrinking CAI, X^* is said to have the *CAP with conjugate operators*. If (K_α) is a CAI and, moreover, $\|K_\alpha\| \leq \lambda$ for some $\lambda \geq 1$ and for all α, then (K_α) is called a *bounded CAI* (*BCAI*) and a *shrinking BCAI*, respectively. In this case X is said to have the *BCAP* and X^* is said to have the *BCAP with conjugate operators*. In the special case, when $\lambda = 1$, (K_α) is called a *metric CAI* (*MCAI*) and a *shrinking MCAI*, respectively; and X is said to have the *MCAP* and X^* is said to have the *MCAP with conjugate operators*.

We assume that the reader is familiar with well-known basic notions and theorems from the theory of Banach spaces and topological vector spaces (such as a dual space, separability, the Hausdorff theorem, the Hahn–Banach theorem, the bipolar theorem, etc.), and we shall use them without proper references.

Chapter 2

M-ideals

The aim of the chapter is to prove in details that $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$ as soon as $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are, respectively, M-ideals in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$. The result allows to produce, departing from Banach spaces X such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$, new classes of M-ideals of compact operators and is, together with the methodology used here, the starting point for the theory developed for $M(r,s)$-ideals later in Chapter 3. The current chapter relies on the original proof scheme of [45].

2.1 Definition and examples

In this section we introduce the notion of an M-ideal through the concept of an ideal, and look at some classical examples of M-ideals of compact operators.

Definition 2.1. Operator $P \in \mathcal{L}(X)$ is a *projection* on X if $P^2 = P$.

Note that if $P \neq 0$, then $\|P\| \geq 1$.

Definition 2.2. Let $\mathcal{K}$ be a subset of a normed space $\mathcal{L}$, then

$$\{f \in \mathcal{L}^* : \ f|_{\mathcal{K}} = 0\}$$

is called the *annihilator* of $\mathcal{K}$ and is denoted by $\mathcal{K}^{\perp}$.

The annihilator $\mathcal{K}^{\perp}$ is a closed subspace of $\mathcal{L}^*$.

Definition 2.3. A closed subspace $\mathcal{K} \neq \{0\}$ of a Banach space $\mathcal{L}$ is said to be an *ideal* in $\mathcal{L}$ if there exists a norm one projection P on $\mathcal{L}^*$ with $\ker P = \mathcal{K}^\perp$. Such a projection P is called an *ideal projection.*

An ideal projection has the minimal positive projection norm. We emphasize that an ideal projection might not exist even if we allowed $\|P\| > 1$. It follows from the fact that not every closed subspace of a Banach space can be complemented (see, for example, [38, Theorem 3.2.20]).

Definition 2.4. A closed subspace $\mathcal{K} \subset \mathcal{L}$ is said to be an *M-ideal* in $\mathcal{L}$ if $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to some ideal projection P such that

$$\|f\| = \|Pf\| + \|f - Pf\| \quad \forall f \in \mathcal{L}^*. \tag{2.1}$$

Due to the triangular inequality, the inequality "$\leq$" always holds. Thus, the important part in (2.1) is the inequality "$\geq$".

The notion was first introduced and studied by E. M. Alfsen and E. G. Effros in 1972 (see [1]). The definition above is given using the dual space $\mathcal{L}^*$. However, due to Theorem 2.6, discovered in [1], one can avoid the dual space by relying on the 3-ball property.

Definition 2.5. Let $\mathcal{L}$ be a Banach space and $\mathcal{K} \neq \{0\}$ be a closed subspace of $\mathcal{L}$. If for all $k_1, k_2, k_3 \in B_{\mathcal{K}}$, all $l \in B_{\mathcal{L}}$, and all $\varepsilon > 0$ there exists $k \in \mathcal{K}$ satisfying

$$\|l + k_i - k\| \leq 1 + \varepsilon \quad (i = 1, 2, 3),$$

then $\mathcal{K}$ has the *3-ball property* in $\mathcal{L}$.

The following fundamental theorem is due to Alfsen and Effros [1].

Theorem 2.6 ([1] or [23, Theorem I.2.2])**.** *Let $\mathcal{K}$ be a closed subspace of a Banach space $\mathcal{L}$, then $\mathcal{K}$ is an M-ideal in $\mathcal{L}$ if and only if $\mathcal{K}$ has the 3-ball property in $\mathcal{L}$.*

The following facts are verified in [23] relying on Theorem 2.6.

Example 2.7 ([23, Example VI.4.1])**.** Let X be a Banach space, then $\mathcal{K}(X, c_0)$ is an M-ideal in $\mathcal{L}(X, c_0)$.

Example 2.8 ([23, Example VI.4.1])**.** Let $1 < p \leq q < \infty$. Then $\mathcal{K}(\ell_p, \ell_q)$ is an M-ideal in $\mathcal{L}(\ell_p, \ell_q)$.

2.2 Properties (M) and (M^*)

Properties (M) and (M^*) have turned out to be the key factors for describing M-ideals of compact operators (see Theorem 2.27). Next, we will give the definitions of these properties and describe them in connection with relatively compact nets and bounded linear operators.

Definition 2.9. A net $(x_\alpha) \subset X$ is said to *converge weakly* to $x \in X$ if

$$x^*(x_\alpha) \underset{\alpha}{\longrightarrow} x^*(x) \quad \forall x^* \in X^*.$$

The weak convergence of (x_α) to x is denoted by $x_\alpha \xrightarrow{w} x$ or $w\text{-}\lim_\alpha x_\alpha = x$. Note that a weakly convergent net does not have to be bounded, even though every weakly convergent sequence is.

Definition 2.10. A Banach space X has *property* (M) if

$$\limsup_\alpha \|u + x_\alpha\| \le \limsup_\alpha \|v + x_\alpha\|,$$

whenever $u, v \in X$ satisfy $\|u\| \le \|v\|$, and $(x_\alpha) \subset X$ is a bounded net converging weakly to null in X.

In [45], the symbol (sM) is used for the definition above, however, the contemporary notation is (M) (see, for example, [23, Definition VI.4.12]), and the original version of property (M) by N. J. Kalton (see [31]) is the sequential version of property (M). Similar remark holds also for property (M^*) (see Definition 2.12).

Definition 2.11. A net $(x^*_\alpha) \subset X^*$ *converges weak** to $x^* \in X^*$ if

$$x^*_\alpha(x) \underset{\alpha}{\longrightarrow} x^*(x) \quad \forall x \in X.$$

We denote this convergence by $x^*_\alpha \xrightarrow{w^*} x^*$ or $w^*\text{-}\lim_\alpha x^*_\alpha = x^*$. As in the case of the weak convergence, a net which converges weak* does not need to be bounded.

Definition 2.12. A Banach space X has *property* (M^*) if

$$\limsup_\alpha \|u^* + x^*_\alpha\| \le \limsup_\alpha \|v^* + x^*_\alpha\|,$$

whenever $u^*, v^* \in X^*$ satisfy $\|u^*\| \le \|v^*\|$, and $(x^*_\alpha) \subset X^*$ is a bounded net converging weak* to null in X^*.

It is a simple and well-known result that if a Banach space X has property (M^*), then it has property (M) (see, for example, [31, Proposition 2.3] and [45, Proposition 2] or [23, Proposition VI.4.15]). The converse usually does not hold, one of such examples is the sequence space ℓ_1. Also, if X has property (M^*), then X is an M-ideal in X^{**} with respect to the canonical projection (see [31, Proposition 2.3] and [45, Proposition 2], or [23, Proposition VI.4.15]).

Lemma 2.13 ([23, Lemma VI.4.13])**.** *Let X be a Banach space.*
1. *The following conditions are equivalent.*

   (*a*) *X has property (M).*

   (*b*) *If $(u_\alpha), (v_\alpha) \subset X$ are relatively compact nets with $\|u_\alpha\| \leq \|v_\alpha\|$ for every α, then*

$$\limsup_\alpha \|u_\alpha + x_\alpha\| \leq \limsup_\alpha \|v_\alpha + x_\alpha\|,$$

   whenever $(x_\alpha) \subset X$ is a bounded net converging weakly to null in X.

2. *The following conditions are equivalent.*

   (a^*) *X has property (M^*).*

   (b^*) *If $(u^*_\alpha), (v^*_\alpha) \subset X^*$ are relatively compact nets with $\|u^*_\alpha\| \leq \|v^*_\alpha\|$ for every α, then*

$$\limsup_\alpha \|u^*_\alpha + x^*_\alpha\| \leq \limsup_\alpha \|v^*_\alpha + x^*_\alpha\|,$$

   whenever $(x^*_\alpha) \subset X^*$ is a bounded net converging weak to null in X^*.*

Proof. Implications $(b) \Rightarrow (a)$ and $(b^*) \Rightarrow (a^*)$ are trivial. We will show the implication $(a) \Rightarrow (b)$. (The implication $(a^*) \Rightarrow (b^*)$ can be proved analogously.)

If the conclusion were false, then by passing to subnets, we would have

$$\lim_\alpha \|u_\alpha + x_\alpha\| > \lim_\alpha \|v_\alpha + x_\alpha\|.$$

Suitable subnets $(u_\beta) \subset (u_\alpha)$ and $(v_\beta) \subset (v_\alpha)$ converge to $u, v \in X$, respectively. Thus $\|u\| \leq \|v\|$ and we have

$$\lim_\beta \|u + x_\beta\| = \lim_\alpha \|u_\alpha + x_\alpha\| > \lim_\alpha \|v_\alpha + x_\alpha\| = \lim_\beta \|v + x_\beta\|$$

which contradicts property (M). □

Lemma 2.14 ([45, Lemma 4]). *1. Let X and Y be Banach spaces with property (M). If nets $(v_\alpha) \subset X$ and $(u_\alpha) \subset Y$ are relatively compact with $\|u_\alpha\| \le \|v_\alpha\|$ for every α, and $(x_\alpha) \subset X$ is a bounded net converging weakly to null in X, then*

$$\limsup_\alpha \|u_\alpha + Tx_\alpha\| \le \limsup_\alpha \|v_\alpha + x_\alpha\|$$

for any $T \in B_{\mathcal{L}(X,Y)}$.

2. Let X and Y be Banach spaces with property (M^*). If nets $(u_\alpha^*) \subset X^*$ and $(v_\alpha^*) \subset Y^*$ are relatively compact with $\|u_\alpha^*\| \le \|v_\alpha^*\|$ for every α, and $(y_\alpha^*) \subset Y^*$ is a bounded net converging weak to null in X^*, then*

$$\limsup_\alpha \|u_\alpha^* + T^* y_\alpha^*\| \le \limsup_\alpha \|v_\alpha^* + y_\alpha^*\|$$

for any $T \in B_{\mathcal{L}(X,Y)}$.

Proof. 1. Suppose first that $\|T\| = 1$. Fix $\varepsilon > 0$. Then there exists $x \in B_X$ such that

$$\|Tx\| \ge (1-\varepsilon)\|T\| = 1 - \varepsilon.$$

For every index α, let us define $\bar{v}_\alpha = \|v_\alpha\|\, x$. Thus

$$\|\bar{v}_\alpha\| \le \|v_\alpha\|$$

and

$$\|(1-\varepsilon)\,u_\alpha\| = (1-\varepsilon)\ \|u_\alpha\| \le \|Tx\|\ \|v_\alpha\| = \Big\|T(\|v_\alpha\|\ x)\Big\| = \|T\bar{v}_\alpha\|\,.$$

The nets $(\bar{v}_\alpha) \subset X$, $(T\bar{v}_\alpha) \subset Y$, and $((1-\varepsilon)\,u_\alpha) \subset Y$ are relatively compact and (Tx_α) is a bounded net converging weakly to null in Y. (A bounded linear operator transfers a bounded set to a bounded set and a weakly null net to a weakly null net.) We have

$$\begin{aligned}
\limsup_\alpha \|(1-\varepsilon)\,u_\alpha + Tx_\alpha\| &\le \limsup_\alpha \|T\bar{v}_\alpha + Tx_\alpha\| \\
&\le \limsup_\alpha \|T\|\,\|\bar{v}_\alpha + x_\alpha\| \\
&= \limsup_\alpha \|\bar{v}_\alpha + x_\alpha\| \\
&\le \limsup_\alpha \|v_\alpha + x_\alpha\|
\end{aligned}$$

due to Lemma 2.13 and thus,

$$\begin{aligned}\limsup_\alpha \|u_\alpha + Tx_\alpha\| &= \limsup_\alpha \|(1-\varepsilon)\,u_\alpha + Tx_\alpha + \varepsilon\,u_\alpha\| \\ &\leq \limsup_\alpha \|(1-\varepsilon)\,u_\alpha + Tx_\alpha\| + \limsup_\alpha \|\varepsilon\,u_\alpha\| \\ &\leq \limsup_\alpha \|v_\alpha + x_\alpha\| + \varepsilon\,\limsup_\alpha \|u_\alpha\|\,.\end{aligned}$$

Letting $\varepsilon \longrightarrow 0$ we obtain

$$\limsup_\alpha \|u_\alpha + Tx_\alpha\| \leq \limsup_\alpha \|v_\alpha + x_\alpha\|\,. \tag{2.2}$$

Now suppose that $0 \leq \|T\| < 1$. If $T = 0$, let us choose $\overline{T} \in \mathcal{L}(X,Y)$ so that $\left\|\overline{T}\right\| = 1$ (it is always possible to choose such an operator for non-trivial spaces X and Y). If $T \neq 0$, let us define $\overline{T} = \frac{T}{\|T\|}$. We can represent Tx_α by a convex combination of $\overline{T}x_\alpha$ and $-\overline{T}x_\alpha$ as follows

$$Tx_\alpha = \lambda\,\overline{T}\,x_\alpha + (1-\lambda)\,(-\overline{T}\,x_\alpha)$$

where $\lambda = \frac{1+\|T\|}{2} \in (0,1)$. The convexity of the functional $\|u_\alpha + t\,\overline{T}\,x_\alpha\|$, $t \in [-1,1]$, allows us to estimate

$$\begin{aligned}\|u_\alpha + Tx_\alpha\| &= \left\|u_\alpha + \lambda\,\overline{T}\,x_\alpha + (1-\lambda)\,(-\overline{T}\,x_\alpha)\right\| \\ &\leq \max\Big\{\left\|u_\alpha + \overline{T}x_\alpha\right\|, \left\|-u_\alpha + \overline{T}x_\alpha\right\|\Big\}.\end{aligned}$$

Thus,

$$\begin{aligned}\limsup_\alpha \|u_\alpha + Tx_\alpha\| &\leq \limsup_\alpha\ \max\Big\{\left\|u_\alpha + \overline{T}x_\alpha\right\|, \left\|-u_\alpha + \overline{T}x_\alpha\right\|\Big\} \\ &= \max\Big\{\limsup_\alpha \left\|u_\alpha + \overline{T}x_\alpha\right\|, \limsup_\alpha \left\|-u_\alpha + \overline{T}x_\alpha\right\|\Big\},\end{aligned}$$

and since inequality (2.2) holds for $\overline{T}$, we have

$$\limsup_\alpha \|u_\alpha + Tx_\alpha\| \;\leq\; \limsup_\alpha \|v_\alpha + x_\alpha\|\,.$$

2. The following proof is analogous to part 1. Suppose that $\|T\| = 1$, then $\|T^*\| = 1$. Fix $\varepsilon > 0$. There exists $y^* \in B_{Y^*}$ such that

$$\|T^* y^*\| \geq (1-\varepsilon)\,\|T^*\| = 1-\varepsilon.$$

For every index α, let us define $\bar{v}_\alpha^* = \|v_\alpha^*\|\ y^*$. Then

$$\|\bar{v}_\alpha^*\| \le \|v_\alpha^*\|$$

and

$$\|(1-\varepsilon)\, u_\alpha^*\| \le \|T^* \bar{v}_\alpha^*\|\,.$$

The nets $(\bar{v}_\alpha^*) \subset Y^*$, $(T^*\bar{v}_\alpha^*) \subset X^*$, and $((1-\varepsilon)\, u_\alpha^*) \subset X^*$ are relatively compact and $(T^* y_\alpha^*)$ is a bounded net converging weak* to null in X^*. (The dual operator of a bounded linear operator maps a weak* null net into a weak* null net.) Due to Lemma 2.13

$$\begin{aligned}\limsup_\alpha \|(1-\varepsilon)\, u_\alpha^* + T^* y_\alpha^*\| &\le \limsup_\alpha \|T^* \bar{v}_\alpha^* + T^* y_\alpha^*\| \\ &\le \limsup_\alpha \|T^*\|\, \|\bar{v}_\alpha^* + y_\alpha^*\| \\ &\le \limsup_\alpha \|v_\alpha^* + y_\alpha^*\|\end{aligned}$$

and letting $\varepsilon \longrightarrow 0$ we obtain

$$\limsup_\alpha \|u_\alpha^* + T^* y_\alpha^*\| \le \limsup_\alpha \|v_\alpha^* + y_\alpha^*\|\,. \qquad (2.3)$$

In the case when $T = 0$, choose $\overline{T}^* \in \mathcal{L}(Y^*, X^*)$ such that $\|\overline{T}^*\| = 1$, and in the case when $0 < \|T\| < 1$, let $\overline{T}^* = \frac{T^*}{\|T^*\|}$. We represent $T^* y_\alpha^*$ by a convex combination of $\overline{T}^* y_\alpha^*$ and $-\overline{T}^* y_\alpha^*$ as follows

$$T^* y_\alpha^* = \lambda\, \overline{T}^*\, y_\alpha^* + (1-\lambda)\,(-\overline{T}^*\, y_\alpha^*)$$

where $\lambda = \frac{1+\|T^*\|}{2} \in (0,1)$. The functional $\|u_\alpha^* + t\overline{T}^* y_\alpha^*\|,\ t \in [-1,1]$, is convex and thus,

$$\begin{aligned}\|u_\alpha^* + T^* y_\alpha^*\| &= \|u_\alpha^* + \lambda\, \overline{T}^* y_\alpha^* + (1-\lambda)\,(-\overline{T}^* y_\alpha^*)\| \\ &\le \max\Big\{\|u_\alpha^* + \overline{T}^* y_\alpha^*\|, \|-u_\alpha^* + \overline{T}^* y_\alpha^*\|\Big\}.\end{aligned}$$

Inequality (2.3) holds for $\overline{T}^*$ and hence

$$\begin{aligned}\limsup_\alpha \|u_\alpha^* + T^* y_\alpha^*\| &\le \limsup_\alpha\ \max\Big\{\|u_\alpha^* + \overline{T}^* y_\alpha^*\|, \|-u_\alpha^* + \overline{T}^* y_\alpha^*\|\Big\} \\ &= \max\Big\{\limsup_\alpha \|u_\alpha^* + \overline{T}^* y_\alpha^*\|, \limsup_\alpha \|-u_\alpha^* + \overline{T}^* y_\alpha^*\|\Big\} \\ &\le \max\Big\{\limsup_\alpha \|v_\alpha^* + y_\alpha^*\|\,, \limsup_\alpha \|v_\alpha^* + y_\alpha^*\|\Big\} \\ &= \limsup_\alpha \|v_\alpha^* + y_\alpha^*\|\,.\end{aligned}$$

□

2.3 Convergence of operators

Besides properties (M) and (M^*), we also need a special type of net of compact operators which converges in the strong operator topology for describing M-ideals of compact operators (see Theorem 2.27). Let us recall now the basic notions concerning convergence of operators and give some preliminary results regarding it.

Definition 2.15. A net $(K_\alpha) \subset \mathcal{L}(X,Y)$ converges to an operator $K \in \mathcal{L}(X,Y)$ in the *strong operator topology* if

$$K_\alpha x \underset{\alpha}{\longrightarrow} Kx \quad \forall x \in X.$$

Such a convergence is also called the *pointwise convergence.*

Definition 2.16. A net $(K_\alpha) \subset \mathcal{L}(X,Y)$ converges to an operator $K \in \mathcal{L}(X,Y)$ in the *weak operator topology* if

$$y^*(K_\alpha x) \underset{\alpha}{\longrightarrow} y^*(Kx) \quad \forall x \in X,\ \forall y^* \in Y^*.$$

It is a well-known classical fact that in $\mathcal{L}(X,Y)$ the closure of a convex set in the strong operator topology equals its closure in the weak operator topology (see, for example, [12, VI.1.5]). (Sometimes the result is referred to as a generalization of the Mazur theorem (compare, for example, with [38, Theorem 2.5.16])). By passing to convex combinations, this allows to consider a net converging in the weak operator topology as a pointwise converging net.

Lemma 2.17. *Let X be a Banach space and let a net $(K_\alpha)_{\alpha\in A} \subset B_{\mathcal{L}(X)}$ be such that*

$$K_\alpha^* x^* \underset{\alpha}{\longrightarrow} x^* \quad \forall x^* \in X^*.$$

Then there exists a net $(\bar{K}_{\bar{\alpha}})$, $\bar{K}_{\bar{\alpha}} \in \operatorname{conv}(K_\alpha)_{\alpha\succ\gamma}$, $\gamma \in A$, for which

$$\bar{K}_{\bar{\alpha}} x \underset{\bar{\alpha}}{\longrightarrow} x \quad \forall x \in X.$$

Proof. Since

$$\begin{aligned} K_\alpha^* x^* \underset{\alpha}{\longrightarrow} x^* \quad \forall x^* \in X^* \quad &\Rightarrow \quad (K_\alpha^* x^*)x \underset{\alpha}{\longrightarrow} x^*(x) \quad \forall x^* \in X^*,\ \forall x \in X \\ &\Leftrightarrow \quad x^*(K_\alpha x) \underset{\alpha}{\longrightarrow} x^*(x) \quad \forall x^* \in X^*,\ \forall x \in X, \end{aligned}$$

we have $K_\alpha \longrightarrow_\alpha I_X$ in the weak operator topology, i.e., for an arbitrary $\gamma \in A$, we have $I_X \in \overline{\operatorname{conv}}(K_\alpha)_{\alpha\succ\gamma}$ in the weak operator topology. The closures of a convex set in the strong and the weak operator topologies coincide, thus, we conclude that (for an arbitrary $\gamma \in A$) $I_X \in \overline{\text{conv}}(K_\alpha)_{\alpha \succ \gamma}$ in the strong operator topology.

Let $\mathcal{B}$ be some base of neighbourhoods for I_X in the strong operator topology. Then for every $\gamma \in A$ and $U \in \mathcal{B}$, there exists $\bar{K}_{(\gamma,U)} \in \text{conv}(K_\alpha)_{\alpha \succ \gamma}$ such that $\bar{K}_{(\gamma,U)} \in U$. Let us define a partial ordering on the set

$$\bar{A} = \{(\gamma, U) : \gamma \in A,\ U \in \mathcal{B}\}$$

as follows: if $\bar{\alpha}_1 = (\gamma_1, U_1),\ \bar{\alpha}_2 = (\gamma_2, U_2) \in \bar{A}$, then

$$\bar{\alpha}_1 \succ \bar{\alpha}_2 \iff \gamma_1 \succ \gamma_2 \text{ and } U_1 \subset U_2.$$

Thus $\bar{A}$ is a directed set and the net $(\bar{K}_{\bar{\alpha}})_{\bar{\alpha} \in \bar{A}} \subset B_{\mathcal{L}(X)}$ converges to I_X in the strong operator topology. □

Remark 2.18. Since $\bar{K}_{\bar{\alpha}} \in \text{conv}(K_\alpha)_{\alpha \succ \gamma}$, $\gamma \in A$, and $K_\alpha^* x^* \longrightarrow x^*$ for all $x^* \in X^*$, also $\bar{K}^*{}_{\bar{\alpha}} x^* \longrightarrow x^*$ for all $x^* \in X^*$. Thus, by denoting $A := \bar{A}$ and $K_\alpha := \bar{K}_{\bar{\alpha}}$, we may assume without loss of generality that $K_\alpha x \longrightarrow x$ for every $x \in X$ whenever $K_\alpha^* x^* \longrightarrow x^*$ for all $x^* \in X^*$.

Of course, in addition to the convergences in the strong and the weak operator topologies, there is also the convergence in the norm topology of $\mathcal{L}(X)$, i.e.,

$$\lim_\alpha K_\alpha = K.$$

Recall that in general the convergence in the norm topology does not follow from the pointwise convergence. However, Lemma 2.19 below shows how to construct a net which converges in the norm topology from a pointwise converging net.

Lemma 2.19. *Let X and Y be Banach spaces.*
1. *Let $(K_\alpha) \subset B_{\mathcal{L}(X)}$ be a net such that*

$$K_\alpha^* x^* \underset{\alpha}{\longrightarrow} x^* \quad \forall x^* \in X^*.$$

Then

$$\lim_\alpha SK_\alpha = S \quad \forall S \in \mathcal{K}(X, Y).$$

2. *Let $(K_\alpha) \subset B_{\mathcal{L}(X)}$ be a net such that*

$$K_\alpha x \underset{\alpha}{\longrightarrow} x \quad \forall x \in X.$$

Then

$$\lim_\alpha K_\alpha S = S \quad \forall S \in \mathcal{K}(Y, X).$$

Proof. 1. Fix $\varepsilon > 0$. We have to show that

$$\exists \alpha_0 \quad (\|S - SK_\alpha\| \leq \varepsilon \quad \forall \alpha \succ \alpha_0).$$

For an arbitrary α, we have

$$\begin{aligned}\|S - SK_\alpha\| &= \|S^* - K_\alpha^* S^*\| \\ &= \|(I_{X^*} - K_\alpha^*)S^*\| \\ &= \sup_{y^* \in B_{Y^*}} \|(I_{X^*} - K_\alpha^*)S^* y^*\| \\ &= \sup_{x^* \in U} \|x^* - K_\alpha^* x^*\|\end{aligned}$$

where $U = S^*(B_{Y^*})$. By the Schauder theorem (see, e.g., [38, Theorem 3.4.15]), we know that S^* is compact and thus the set U is relatively compact. Hence, due to the Hausdorff theorem, there exist $n \in \mathbb{N}$ and $x_1^*, ..., x_n^* \in U$ such that

$$\forall x^* \in U \; \exists i \in \{1, \dots, n\} \quad \|x^* - x_i^*\| < \frac{\varepsilon}{3}.$$

Let $x^* \in U$, then for every α

$$\begin{aligned}\|x^* - K_\alpha^* x^*\| &= \|x^* - x_i^* + x_i^* - K_\alpha^* x_i^* + K_\alpha^* x_i^* - K_\alpha^* x^*\| \\ &\leq \|x^* - x_i^*\| + \|x_i^* - K_\alpha^* x_i^*\| + \|K_\alpha^* x_i^* - K_\alpha^* x^*\| \\ &\leq \frac{\varepsilon}{3} + \|x_i^* - K_\alpha^* x_i^*\| + \|K_\alpha^*\| \, \|x_i^* - x^*\| \\ &\leq \frac{2\varepsilon}{3} + \|x_i^* - K_\alpha^* x_i^*\| .\end{aligned}$$

Since $K_\alpha^* x_i^* \underset{\alpha}{\longrightarrow} x_i^*$ for all $i = 1, \dots, n$, there exists α_0 such that for $\alpha \succ \alpha_0$ we have

$$\|K_\alpha^* x_i^* - x_i^*\| \leq \frac{\varepsilon}{3} \quad \forall i = 1, \dots, n$$

and thus,

$$\begin{aligned}\|S - SK_\alpha\| &= \sup_{x^* \in U} \|x^* - K_\alpha^* x^*\| \\ &\leq \frac{2\varepsilon}{3} + \max_{1 \leq i \leq n} \|x_i^* - K_\alpha^* x_i^*\| \\ &\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall \alpha \succ \alpha_0.\end{aligned}$$

2. Fix $\varepsilon > 0$. Similarly to the first part, we have to show that

$$\exists \alpha_0 \quad (\|S - K_\alpha S\| \leq \varepsilon \quad \forall \alpha \succ \alpha_0).$$

For an arbitrary index α, we have

$$\begin{aligned}\|S - K_\alpha S\| &= \|(I_X - K_\alpha)S\| \\ &= \sup_{y\in B_Y} \|(I_X - K_\alpha)Sy\| \\ &= \sup_{x\in U} \|x - K_\alpha x\|\end{aligned}$$

where $U = S(B_Y)$. Since U is relatively compact, we can continue as above and find α_0 such that

$$\|S - K_\alpha S\| = \sup_{x\in U} \|x - K_\alpha x\| \leq \varepsilon \quad \forall \alpha \succ \alpha_0.$$

□

Remark 2.20. Note that, by passing to a subnet, we can replace the assumption $\|K_\alpha\| \leq 1$ in Lemma 2.19 with the assumption $\limsup \|K_\alpha\| < \infty$ and the proof still holds.

2.4 Ideal projection and Johnson's lemma

An important well-known property of an ideal projection is that the range space of an ideal projection can be regarded as a dual space of the ideal.

Lemma 2.21. *Let $\mathcal{L}$ be a Banach space and let $\mathcal{K}$ be an ideal in $\mathcal{L}$. Let $P : \mathcal{L}^* \longrightarrow \mathcal{L}^*$ be the corresponding ideal projection. Then $\Phi : \mathcal{K}^* \longrightarrow \operatorname{ran} P$, defined by*

$$\Phi g = Pf, \quad g \in \mathcal{K}^*,$$

where $f \in \mathcal{L}^*$ is any extension of g, is an isometric isomorphism such that $\Phi(f|_\mathcal{K}) = Pf$ for all $f \in \mathcal{L}^*$.

Proof. If $f_1, f_2 \in \mathcal{L}^*$ are some extensions of g, then $Pf_1 = Pf_2$ since $f_1 - f_2 \in \mathcal{K}^\perp = \ker P$ and thus the definition of Φ is correct.

Note that Φ is linear. Indeed, let $\alpha \in \mathbb{K}$, $g_1, g_2 \in \mathcal{K}^*$ and $f_1, f_2 \in \mathcal{L}^*$ be their extensions, respectively. If $f \in \mathcal{L}(X,Y)^*$ is an extension of $g_1 + \alpha g_2$, then $f - (f_1 + \alpha f_2) \in \ker P$ and

$$\Phi(g_1 + \alpha\, g_2) = Pf = P(f_1 + \alpha\, f_2) = Pf_1 + \alpha\, Pf_2 = \Phi g_1 + \alpha\, \Phi g_2.$$

If $f \in \mathcal{L}^*$, then $\Phi(f|_\mathcal{K}) = Pf$ and it is clear that Φ is a surjection.

It remains to prove that Φ is isometric. Let $g \in \mathcal{K}^*$ and let $f \in \mathcal{L}^*$ be a norm-preserving extension of g. Then

$$\|\Phi g\| = \|Pf\| \leq \|P\| \, \|f\| = \|g\| \, ,$$

and, on the other hand,

$$\|\Phi g\| = \|Pf\| \geq \|g\|$$

since Pf is an extension of g. Thus Φ is an isometric isomorphism. □

Remark 2.22. If $\mathcal{K}$ is an M-ideal in $\mathcal{L}$ and P is the corresponding M-ideal projection, then

$$\operatorname{ran} P = \{f \in \mathcal{L}^* : \|f\| = \|f|_{\mathcal{K}}\|\}$$

(see, e.g., [23, Proposition I.1.12]).

As it was stated earlier, an ideal projection need not exist. However, as we can see from *Johnson's lemma*, i.e, Lemma 2.23 below, the existence of a special kind of a net of compact operators always guarantees the existence of an ideal projection.

Lemma 2.23. *Let X and Y be Banach spaces with $\mathcal{K}(X, Y) \neq \{0\}$.*
1. *Let $(K_\alpha) \subset B_{\mathcal{K}(X)}$ be a net such that*

$$K_\alpha^* x^* \underset{\alpha}{\longrightarrow} x^* \quad \forall x^* \in X^*.$$

If the net (K_α) converges weak in $\mathcal{K}(X)^{**}$, then $P : \mathcal{L}(X, Y)^* \longrightarrow \mathcal{L}(X, Y)^*$, where*

$$(Pf)(T) = \lim_\alpha f(TK_\alpha), \quad f \in \mathcal{L}(X, Y)^*, \ T \in \mathcal{L}(X, Y),$$

is a projection with $\|P\| = 1$ and $\ker P = \mathcal{K}(X, Y)^\perp$.

2. *Let $(K_\alpha) \subset B_{\mathcal{K}(X)}$ be a net such that*

$$K_\alpha x \underset{\alpha}{\longrightarrow} x \quad \forall x \in X.$$

If the net (K_α) converges weak in $\mathcal{K}(X)^{**}$, then $P : \mathcal{L}(Y, X)^* \longrightarrow \mathcal{L}(Y, X)^*$, where*

$$(Pf)(T) = \lim_\alpha f(K_\alpha T), \quad f \in \mathcal{L}(Y, X)^*, \ T \in \mathcal{L}(Y, X),$$

is a projection with $\|P\| = 1$ and $\ker P = \mathcal{K}(Y, X)^\perp$.

Remark 2.24. By the Banach–Alaoglu theorem (see, e.g., [38, Theorem 2.6.18]), we can extract from every net $(K_\alpha) \subset B_{\mathcal{K}(X)}$ a subnet so that the subnet converges weak* in $\mathcal{K}(X)^{**}$. Thus, the condition "if the net (K_α) converges weak* in $\mathcal{K}(X)^{**}$" is not really a restriction.

Lemma 2.23 is essentially the same as a result by J. Johnson (see [28, proof of Lemma 1]). We will prove Lemma 2.23 for the sake of completeness.

Proof of Lemma 2.23. 1. Fix $f \in \mathcal{L}(X,Y)^*$ and $T \in \mathcal{L}(X,Y)$. For the correctness of the definition of P, we have to show that the limit $\lim_\alpha f(TK_\alpha)$ exists. Let us define $\mathfrak{T} \in \mathcal{L}(\mathcal{K}(X), \mathcal{L}(X,Y))$ by

$$\mathfrak{T}S = TS, \quad S \in \mathcal{K}(X).$$

The limit

$$\lim_\alpha (\mathfrak{T}^* f)(K_\alpha) = \lim_\alpha f(\mathfrak{T}K_\alpha) = \lim_\alpha f(TK_\alpha)$$

exists because $\mathfrak{T}^* f \in \mathcal{K}(X)^*$ and the net (K_α) converges weak*.

Clearly P is linear, and $\|P\| \leq 1$ since

$$|(Pf)(T)| = |\lim_\alpha f(TK_\alpha)| = \lim_\alpha |f(TK_\alpha)| \leq \limsup_\alpha \|f\| \, \|T\| \, \|K_\alpha\| \leq \|f\| \, \|T\| \, .$$

If $T \in \mathcal{K}(X,Y)$, then Lemma 2.19 implies that $\lim_\alpha TK_\alpha = T$ and thus,

$$f(T) = f(\lim_\alpha TK_\alpha) = \lim_\alpha f(TK_\alpha) = (Pf)(T).$$

Since $f - Pf \in \mathcal{K}(X,Y)^\perp$, we conclude that $\ker P \subset \mathcal{K}(X,Y)^\perp$. We have $\mathcal{K}(X,Y)^\perp \subset \ker P$ because $TK_\alpha \in \mathcal{K}(X,Y)$ and for all $f \in \mathcal{K}(X,Y)^\perp$

$$(Pf)(T) = \lim_\alpha f(TK_\alpha) = \lim_\alpha 0 = 0.$$

Thus, $\ker P = \mathcal{K}(X,Y)^\perp$, and $f - Pf \in \ker P$ for all $f \in \mathcal{L}(X,Y)^*$ proving that $P = P^2$.

Note that if $P = 0$, then $\mathcal{K}(X,Y)^\perp = \ker P = \mathcal{L}(X,Y)^*$ which contradicts $\mathcal{K}(X,Y) \neq \{0\}$. Hence $P \neq 0$.

2. The proof uses the operator $\mathfrak{T} \in \mathcal{L}(\mathcal{K}(X), \mathcal{L}(Y,X))$ defined by

$$\mathfrak{T}S = ST, \quad S \in \mathcal{K}(X),$$

and is symmetric to the proof above. □

Remark 2.25. Due to Remark 2.18, we can replace

$$K_\alpha^* x^* \underset{\alpha}{\longrightarrow} x^* \quad \forall x^* \in X^*$$

in Lemma 2.23, part 1, and the assumption

$$K_\alpha x \underset{\alpha}{\longrightarrow} x \quad \forall x \in X$$

in Lemma 2.23, part 2, with the assumption that (K_α) is a shrinking MCAI of X.

2.5 Descriptions of M-ideals of compact operators

As preliminary results, we will next look at some criteria for compact operators to appear as an M-ideal in the corresponding space of all bounded linear operators. The results are needed for showing, in Section 2.6, that M-ideals of compact operators create new M-ideals of compact operators.

The following result is immediate from [43, Corollary 2.3]. We present a self-contained proof for completeness.

Proposition 2.26 ([43, Corollary 2.3] or [45, Proposition 7]). *Let X and Y be Banach spaces.*
1. *If there exists a shrinking MCAI (K_α) of X such that*

$$\limsup_\alpha \|S + T(I_X - K_\alpha)\| \le 1 \quad \forall S \in B_{\mathcal{K}(X,Y)},\ \forall T \in B_{\mathcal{L}(X,Y)}, \tag{2.4}$$

then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.

2. *If there exists an MCAI (K_α) of X such that*

$$\limsup_\alpha \|S + (I_X - K_\alpha)T\| \le 1 \quad \forall S \in B_{\mathcal{K}(Y,X)},\ \forall T \in B_{\mathcal{L}(Y,X)}, \tag{2.5}$$

then $\mathcal{K}(Y,X)$ is an M-ideal in $\mathcal{L}(Y,X)$.

Proof. Without loss of generality we may assume that the net (K_α) converges weak* in $\mathcal{K}(X)^{**}$ (see Remark 2.24).

1. Let P be the projection on $\mathcal{L}(X,Y)^*$ from Lemma 2.23, part 1. Then $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$ with respect to P and it remains to prove that

$$\|Pf\| + \|f - Pf\| \le \|f\| \quad \forall f \in \mathcal{L}(X,Y)^*.$$

Fix $f \in \mathcal{L}(X,Y)^*$ and $\varepsilon > 0$. Recall that $\|Pf\| = \|Pf|_{\mathcal{K}(X,Y)}\|$ by Lemma 2.21. Now, there exist $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$ such that

$$\begin{aligned}\|Pf|_{\mathcal{K}(X,Y)}\| + \|f - Pf\| - \varepsilon &\le (Pf)(S) + (f - Pf)(T)\\ &= (Pf)(S) + f(T) - (Pf)(T).\end{aligned}$$

Thus, based on the definition of the projection P and Lemma 2.19, we have

$$\begin{aligned}\|Pf\| + \|f - Pf\| - \varepsilon &\le \lim_\alpha f(SK_\alpha) + f(T) - \lim_\alpha f(TK_\alpha)\\ &= f(\lim_\alpha SK_\alpha) + f(T) - \lim_\alpha f(TK_\alpha)\\ &= \lim_\alpha f(S + T - TK_\alpha) \le \limsup_\alpha \|f\| = \|f\|.\end{aligned}$$

Letting $\varepsilon \longrightarrow 0$ we obtain

$$\|Pf\| + \|f - Pf\| \le \|f\|.$$

2. Let P be the projection on $\mathcal{L}(Y,X)^*$ from Lemma 2.23, part 2. Then $\mathcal{K}(Y,X)$ is an ideal in $\mathcal{L}(Y,X)$ with respect to P and the inequality

$$\|Pf\| + \|f - Pf\| \le \|f\| \quad \forall f \in \mathcal{L}(Y,X)^*$$

is verified verbatim to the first part. □

Theorem 2.27 ([45, Theorem 5]). *Let X be a Banach space. The following are equivalent.*

(a) $\mathcal{K}(X)$ *is an M-ideal in* $\mathcal{L}(X)$.

(b) X *has property* (M) *and there exists a shrinking MCAI* (K_α) *of* X *such that*

$$\limsup_{\beta} \ \limsup_{\alpha} \|K_\beta + I_X - K_\alpha\| \le 1.$$

(c) X *has property* (M^*) *and there exists a shrinking MCAI* (K_α) *of* X *such that*

$$\limsup_{\alpha} \|K_\beta + I_X - K_\alpha\| \le 1 \quad \forall \beta. \tag{2.6}$$

(d) $\mathcal{K}(X)$ *is an M-ideal in* $\mathcal{I}(X)$.

The equivalence $(a) \Leftrightarrow (d)$ in Theorem 2.27 was first proved for separable case by N. J. Kalton in [31, Theorem 2.6].

Proof of Theorem 2.27. $(a)\Rightarrow(d)$ is obvious by the 3-ball property (see Theorem 2.6).

$(d)\Rightarrow(c)$. Relying on a strong version of the principle of local reflexivity, due to E. Behrends (see [3, Theorem 3.2]), it is proved in [58, Proposition 2.3] that there exists $(K_\alpha) \subset \mathcal{K}(X)$, $\limsup_\alpha \|K_\alpha\| \le 1$, such that

$$f(K_\alpha) \underset{\alpha}{\longrightarrow} f(I_X) \quad \forall f \in \operatorname{ran} P,$$

where P is the M-ideal projection, and

$$\limsup_{\alpha} \|S + I_X - K_\alpha\| \le 1 \quad \forall S \in B_{\mathcal{K}(X)}. \tag{2.7}$$

From Remark 2.22, it is clear that $x^{**} \otimes x^* \in \operatorname{ran} P$ for all $x^{**} \in X^{**}$ and $x^* \in X^*$. Thus

$$x^{**}(K_\alpha^* x^*) \underset{\alpha}{\longrightarrow} x^{**}(I_{X^*} x^*) \quad \forall x^{**} \in X^{**},\ \forall x^* \in X^*,$$

that is, $K_\alpha^* \longrightarrow I_{X^*}$ in the weak operator topology. By applying a convex combinations argument (see the proof of Lemma 2.17 and Remark 2.18), we may assume that (K_α) is a shrinking CAI, and after norming the operators K_α, we may assume that (K_α) is also an MCAI, in particular, (2.6) holds.

For showing that X has property (M^*), fix $u^*, v^* \in X^*$ satisfying $\|u^*\| < \|v^*\|$ and a bounded net $(x_\nu^*) \subset X^*$ converging weak* to null in X^*. Then

$$\lim_\nu \|K_\alpha^* x_\nu^*\| = 0 \quad \forall \alpha.$$

Choose $v \in X$ such that $v^*(v) = 1$ and $\|u^*\| < \frac{1}{\|v\|}$. Define a finite-rank operator S on X by $Sx = u^*(x)v$, $x \in X$. Then $S \in B_{\mathcal{K}(X)}$, $u^* = S^* v^*$, and $\lim_\nu \|S^* x_\nu^*\| = 0$. We can estimate

$$\begin{aligned}
\limsup_\nu \|u^* + x_\nu^*\| &= \limsup_\nu \|S^* v^* + (I_X - K_\alpha)^* x_\nu^*\| \\
&\le \limsup_\nu \|S^* v^* + S^* x_\nu^* + (I_X - K_\alpha)^* x_\nu^*\| \\
&\le \|S^* + (I_X - K_\alpha)^*\| \limsup_\nu \|v^* + x_\nu^*\| + \|(I_X - K_\alpha)^* v^*\|
\end{aligned}$$

and, thus,

$$\begin{aligned}
&\limsup_\nu \|u^* + x_\nu^*\| \\
&\qquad \le \limsup_\alpha \|S^* + (I_X - K_\alpha)^*\| \limsup_\nu \|v^* + x_\nu^*\| \\
&\qquad\quad + \limsup_\alpha \|(I_X - K_\alpha)^* v^*\| \\
&\qquad \le \limsup_\nu \|v^* + x_\nu^*\|
\end{aligned}$$

because of inequality (2.7) and $K_\alpha^* v^* \longrightarrow v^*$.

Now, if $\|u^*\| \le \|v^*\|$, then $\|u^*\| < \|tv^*\|$ for all $t > 1$ and

$$\begin{aligned}
\limsup_\nu \|u^* + x_\nu^*\| &\le \limsup_\nu \|tv^* + x_\nu^*\| \\
&\le \limsup_\nu \|v^* + x_\nu^*\| + (t-1)\|v^*\|.
\end{aligned}$$

The required inequality follows from letting $t \longrightarrow 1$.

$(c)\Rightarrow(b)$. Recall that property (M) follows from property (M^*) (see Section 2.2).

$(b)\Rightarrow(a)$. By Proposition 2.26, taking also into account that $\lim_\alpha SK_\alpha = S$ (see Lemma 2.19), it is sufficient to prove that for all $S \in B_{\mathcal{K}(X)}$ and $T \in B_{\mathcal{L}(X)}$

$$\limsup_\beta \limsup_\alpha \|SK_\beta + T - TK_\alpha\| \leq 1$$

since

$$\begin{aligned}\limsup_\alpha \|S + T - TK_\alpha\| &\leq \limsup_\beta \|S - SK_\beta\| \\ &\quad + \limsup_\beta \limsup_\alpha \|SK_\beta + T - TK_\alpha\| \\ &= \limsup_\beta \limsup_\alpha \|SK_\beta + T - TK_\alpha\| .\end{aligned}$$

Fix β. By defining $\varepsilon_{(\alpha,n)} = \frac{1}{n}$ and $K_{(\alpha,n)} = K_\alpha$ for every α, $n \in \mathbb{N}$, and replacing (K_α) by the net $(K_{(\alpha,n)})$, which we denote also by (K_α), we may assume without loss of generality that, for the net (K_α), there exists a net (ε_α), $\varepsilon_\alpha > 0$, such that $\varepsilon_\alpha \longrightarrow 0$. Let us choose $(x_\alpha) \subset B_X$ so that

$$\|SK_\beta x_\alpha + T(I_X - K_\alpha)x_\alpha\| \geq \|SK_\beta + T(I_X - K_\alpha)\| - \varepsilon_\alpha.$$

Then

$$\begin{aligned}\limsup_\alpha &\|SK_\beta x_\alpha + T(I_X - K_\alpha)x_\alpha\| \\ &\leq \limsup_\alpha \|SK_\beta + T(I_X - K_\alpha)\| \|x_\alpha\| \\ &\leq \limsup_\alpha (\|SK_\beta x_\alpha + T(I_X - K_\alpha)x_\alpha\| + \varepsilon_\alpha) \\ &= \limsup_\alpha \|SK_\beta x_\alpha + T(I_X - K_\alpha)x_\alpha\| + \limsup_\alpha \varepsilon_\alpha \\ &= \limsup_\alpha \|SK_\beta x_\alpha + T(I_X - K_\alpha)x_\alpha\|,\end{aligned}$$

and hence,

$$\limsup_\alpha \|SK_\beta + T - TK_\alpha\| = \limsup_\alpha \|SK_\beta x_\alpha + (T - TK_\alpha)x_\alpha\| .$$

Note that $(SK_\beta x_\alpha)_\alpha \subset X$ and $(K_\beta x_\alpha)_\alpha \subset X$ are relatively compact nets satisfying $\|SK_\beta x_\alpha\| \leq \|K_\beta x_\alpha\|$ for all α, and $((I_X - K_\alpha)x_\alpha)$ is a bounded net converging weakly to null in X. Since X has property (M), we have by Lemma 2.14

$$\begin{aligned}\limsup_\alpha \|SK_\beta x_\alpha + T(I_X - K_\alpha)x_\alpha\| &\leq \limsup_\alpha \|K_\beta x_\alpha + (I_X - K_\alpha)x_\alpha\| \\ &\leq \limsup_\alpha \|K_\beta + I_X - K_\alpha\|\end{aligned}$$

and thus,

$$\limsup_\beta \limsup_\alpha \|SK_\beta + T - TK_\alpha\| \quad \le \limsup_\beta \limsup_\alpha \|K_\beta + I_X - K_\alpha\| \le 1.$$

□

2.6 M-ideals of compact operators creating new M-ideals

We will prove that $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$ as soon as $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are, respectively, M-ideals in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ (see Corollary 2.29). The result allows to produce, departing from Banach spaces X such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$, new classes of M-ideals of compact operators.

Theorem 2.28 ([45, Theorem 8]). *Let X be a Banach space such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. Then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$ for every Banach space Y having property (M), and $\mathcal{K}(Y,X)$ is an M-ideal in $\mathcal{L}(Y,X)$ for every Banach space Y having property (M^*).*

Proof. From $\mathcal{K}(X)$ being an M-ideal in $\mathcal{L}(X)$, it follows that X has properties (M) and (M^*), and there exists a shrinking MCAI (K_α) of X such that

$$\limsup_\beta \; \limsup_\alpha \|K_\beta + I_X - K_\alpha\| \le 1 \tag{2.8}$$

(see Theorem 2.27).

To prove the first claim, it is sufficient to show that inequality (2.4), from Proposition 2.26, holds. Fix $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$. We have, as in the proof of Theorem 2.27, $(b)\Rightarrow(a)$, that

$$\limsup_\alpha \|S + T(I_X - K_\alpha)\| \quad \le \quad \limsup_\beta \; \limsup_\alpha \|SK_\beta + T(I_X - K_\alpha)\|,$$

and, moreover, we may assume that for every β there exists $(x_\alpha) \subset B_X$ such that

$$\limsup_\alpha \|SK_\beta + T(I_X - K_\alpha)\| = \limsup_\alpha \|SK_\beta x_\alpha + T(I_X - K_\alpha)x_\alpha\|.$$

It remains to prove that

$$\limsup_\beta \; \limsup_\alpha \|SK_\beta + T(I_X - K_\alpha)\| \le 1. \tag{2.9}$$

The nets $(SK_\beta x_\alpha)_\alpha \subset Y$, $(K_\beta x_\alpha)_\alpha \subset X$ are relatively compact, and

$$\|SK_\beta x_\alpha\| \leq \|K_\beta x_\alpha\| \quad \forall \alpha.$$

The net $((I_X - K_\alpha)x_\alpha)$ is bounded and converges weakly to null in X. Since spaces X and Y have property (M), based on Lemma 2.14,

$$\begin{aligned}\limsup_\alpha \|SK_\beta x_\alpha + T(I_X - K_\alpha)x_\alpha\| &\leq \limsup_\alpha \|K_\beta x_\alpha + (I - K_\alpha)x_\alpha\| \\ &\leq \limsup_\alpha \|K_\beta + I_X - K_\alpha\|.\end{aligned}$$

This together with inequality (2.8) implies

$$\limsup_\beta \ \limsup_\alpha \|SK_\beta + T(I_X - K_\alpha)\| \leq \limsup_\beta \ \limsup_\alpha \|K_\beta + I_X - K_\alpha\| \leq 1.$$

Hence, inequality (2.9) holds, and thus, we have proved that $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.

To prove the second claim of the theorem, it is sufficient to prove that inequality (2.5), from Proposition 2.26, holds. Fix $S \in B_{\mathcal{K}(Y,X)}$ and $T \in B_{\mathcal{L}(Y,X)}$. Based on Lemma 2.19, as in the previous part, we see that inequality (2.5) holds if

$$\limsup_\beta \limsup_\alpha \|S^* K_\beta^* + T^*(I_X - K_\alpha)^*\| \leq 1. \tag{2.10}$$

Fix β. Proceeding similarly to the previous proof, we may assume that for some $(x_\alpha^*) \subset B_{X^*}$,

$$\limsup_\alpha \|S^* K_\beta^* + T^*(I_X - K_\alpha)^*\| = \limsup_\alpha \|S^* K_\beta^* x_\alpha^* + T^*(I_X - K_\alpha)^* x_\alpha^*\|.$$

Now, the nets $(S^* K_\beta^* x_\alpha^*)_\alpha \subset Y^*$ and $(K_\beta^* x_\alpha^*)_\alpha \subset X^*$ are relatively compact because the operators $S^* K_\beta^*$ and K_β^* are compact. Moreover,

$$\|S^* K_\beta^* x_\alpha^*\| \leq \|K_\beta^* x_\alpha^*\| \quad \forall \alpha$$

and $((I_X - K_\alpha)^* x_\alpha^*)$ is a bounded net converging weak* to null in X^*.

Both the spaces X and Y have property (M^*). Hence, using Lemma 2.14,

$$\limsup_\alpha \|S^* K_\beta^* x_\alpha^* + T^*(I_X - K_\alpha)^* x_\alpha^*\| \leq \limsup_\alpha \|K_\beta + I_X - K_\alpha\|$$

which together with inequality (2.8) yields inequality (2.10) as needed. □

The main result of this chapter follows directly from Theorems 2.27 and 2.28.

Corollary 2.29 ([45, Corollary 9]). *If $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are M-ideals in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, respectively, then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.*

Proof. Banach space Y has property (M) by Theorem 2.27. Hence, by Theorem 2.28, $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$. □

Chapter 3

$M(r,s)$-ideals

Next, we will extend and develop the results and methods used in Chapter 2, for M-ideals, to the case of $M(r,s)$-ideals. Namely, we will prove that $\mathcal{K}(X,Y)$ is an $M(r_1^2 r_2, s_1^2 s_2)$-ideal and an $M(r_1 r_2^2, s_1 s_2^2)$-ideal in $\mathcal{L}(X,Y)$ whenever $\mathcal{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathcal{L}(X)$ with $r_1 + s_1/2 > 1$ and $\mathcal{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathcal{L}(Y)$ with $r_2 + s_2/2 > 1$. The chapter is motivated by [45], [49], [54] and relies on [19].

3.1 Definition and examples

Let $\mathcal{L}$ be a Banach space.

Definition 3.1. A closed subspace $\mathcal{K} \subset \mathcal{L}$ is said to be an $M(r,s)$-*ideal* in $\mathcal{L}$ if $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to some ideal projection P and there exist $r, s \in (0,1]$ such that

$$\|f\| \geq r\,\|Pf\| + s\,\|f - Pf\| \quad \forall f \in \mathcal{L}^*.$$

In [4] and subsequent works, such a $\mathcal{K}$ was called an ideal satisfying the $M(r,s)$-inequality in $\mathcal{L}$. The shorter name was first used in [19].

In principle, we could also consider the parameters range $(r,s) \in (0,1] \times (0,\infty)$. However, it is clear that $\mathcal{K} = \mathcal{L}$ if and only if $\mathcal{K}$ is an $M(r,s)$-ideal in $\mathcal{L}$ for some $s > 1$ (or for all $(r,s) \in (0,1] \times (0,\infty)$). Nevertheless, occasionally (following [21]) we shall consider $M(r,s)$-ideals also for positive numbers $r \leq 1$ and s, which allows us to obtain some applications to the general structure of Banach spaces in Chapter 4 (see Section 4.9).

Note that M-ideals are precisely $M(1,1)$-ideals.

From Example 2.8, we know that $\mathcal{K}(\ell_p)$ is an M-ideal in $\mathcal{L}(\ell_p)$ for $p > 1$. One of the closest analogs of ℓ_p is the Lorentz sequence space $d(v,p)$.

Let $p \geq 1$, and let $v = (v_n)$ be a non-increasing sequence of positive numbers such that $v_1 = 1$, $\lim_n v_n = 0$, and $\sum_n v_n = \infty$. The Banach space of all sequences of scalars $x = (x_n)$ for which

$$\|x\| = \sup_{\pi} \Big(\sum_{n=1}^{\infty} v_n \left| x_{\pi(n)} \right|^p \Big)^{1/p} < \infty,$$

where π ranges over all the permutations of the natural numbers, is denoted by $d(v,p)$ and is called a *Lorentz sequence space*.

Example 3.2 ([7, Example 4.2])**.** For the Lorentz sequence space $d(v,p)$, $p > 1$, the space of compact operators $\mathcal{K}(d(v,p))$ is an $M(r,s)$-ideal in $\mathcal{L}(d(v,p))$ if $r,s \in (0,1]$ satisfy $r^p + s^p \leq 1$.

It is a well-known result of Hennefeld [25] (see, e.g., [23, p. 305]) that for the Lorentz sequence space $d(v,p)$, $p > 1$, the space of compact operators $\mathcal{K}(d(v,p))$ is not an M-ideal in $\mathcal{L}(d(v,p))$. That $\mathcal{K}(d(v,p))$ is indeed an $M(r,s)$-ideal will be proved in Section 3.3.

For additional examples of $M(r,s)$-ideals which are not M-ideals, see, e.g., [4], [5], or [7].

3.2 $M(r,s)$-ideals of compact operators and the ideal projection

One of the differences between M-ideals and $M(r,s)$-ideals is that in the case of $M(r,s)$-ideals the corresponding ideal projection need not be unique. (For the uniqueness of the ideal projection of M-ideals, see [23, Proposition I.1.2].)

Definition 3.3 (see [36])**.** A closed subspace $\mathcal{K}$ of a Banach space $\mathcal{L}$ has the *unique ideal property* if $\mathcal{K}$ is an ideal in $\mathcal{L}$ and there exists precisely one ideal projection, that is, a unique norm one projection P on $\mathcal{L}^*$ with $\ker P = \mathcal{K}^{\perp}$.

An obvious example of subspaces having the unique ideal property is presented by ideals having Phelps's property U.

Definition 3.4 (see [53])**.** A closed subspace $\mathcal{K}$ of a Banach space $\mathcal{L}$ is said to have *property* U in $\mathcal{L}$ if every $g \in \mathcal{K}^*$ has a unique norm-preserving extension $f \in \mathcal{L}^*$.

Ideals with property U have been studied, e.g., in [25], [41], [42], [43], [46], [51], [54].

Cabello and Nieto showed in the paper [4] that $M(1,s)$-ideals have property U and therefore they also have the unique ideal property. However, e.g., for $r \neq 1$, $M(r,1)$-ideals of compact operators $\mathcal{K}(X)$ need not have property U in $\mathcal{L}(X)$ even if X^* has the Radon–Nikodým property (see [7, Example 4.5]).

3.3 Johnson projection

The result captured in Lemma 2.23 has given a motivation to the following definition (see also Remark 2.25).

Definition 3.5. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K}(X,Y)$. Suppose that (K_α) is a shrinking MCAI of X (respectively, of Y). Then an ideal projection P on $\mathcal{L}^*$ such that

$$(Pf)(T) = \lim_\alpha f(TK_\alpha), \quad f \in \mathcal{L}^*, \quad T \in \mathcal{L}$$

(respectively,

$$(Pf)(T) = \lim_\alpha f(K_\alpha T), \quad f \in \mathcal{L}^*, \quad T \in \mathcal{L})$$

is called the *Johnson projection.*

This is essentially the same concept as in [39] and [54].

The following result extends Proposition 2.26 to $M(r,s)$-ideals.

Proposition 3.6. *Let X and Y be Banach spaces and let $r, s \in (0,1]$. Then $\mathcal{K}(X,Y)$ is an $M(r,s)$-ideal in $\mathcal{L}(X,Y)$ with respect to some Johnson projection whenever there is a shrinking MCAI (K_α) of Y (respectively, of X) with*

$$\limsup_\alpha \|rS + s(T - K_\alpha T)\| \leq 1$$

(respectively,

$$\limsup_\alpha \|rS + s(T - TK_\alpha)\| \leq 1)$$

for any $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$.

Proof. Let (K_α) be a shrinking MCAI of Y (the proof is almost verbatim with obvious changes if we assume that (K_α) is a shrinking MCAI of X). Then, relying on Lemma 2.23 and Remark 2.24, we can assume that $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$ with respect to the Johnson projection P corresponding to (K_α).

Let us fix $f \in \mathcal{L}(X,Y)^*$ and $\varepsilon > 0$. According to Lemma 2.21, $\|Pf\| = \|f|_{\mathcal{K}(X,Y)}\|$ and thus, there exist $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$ such that

$$r\|Pf\| + s\|f - Pf\| - \varepsilon \leq rf(S) + s(f - Pf)(T).$$

Hence, by definition of P, we have

$$\begin{aligned} r\|Pf\| + s\|f - Pf\| - \varepsilon &\leq rf(S) + sf(T) - s\lim_\alpha f(K_\alpha T) \\ &= \lim_\alpha f(rS + s(T - K_\alpha T)) \\ &\leq \|f\| \limsup_\alpha \|rS + s(T - K_\alpha T)\| \\ &\leq \|f\| \end{aligned}$$

whenever $\limsup_\alpha \|rS + s(T - K_\alpha T)\| \leq 1$. □

Note that due to the proof above, we can reformulate Proposition 3.6 for the case when Y has an MCAI without the shrinkingness assumption (see Corollary 3.7 below).

Corollary 3.7. *Let X and Y be Banach spaces and let $r, s \in (0,1]$. Then $\mathcal{K}(X,Y)$ is an $M(r,s)$-ideal in $\mathcal{L}(X,Y)$ whenever there is an MCAI (K_α) of Y with*

$$\limsup_\alpha \|rS + s(T - K_\alpha T)\| \leq 1$$

for any $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$.

Remark 3.8. Based on [7, Theorem 3.1], whenever we assume that $X = Y$ and $r + s/2 > 1$, also the converse of Proposition 3.6 holds: if $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}(X)$, then X admits a shrinking MCAI (K_α) such that

$$\limsup_\alpha \|rS + s(T - TK_\alpha)\| \leq 1$$

for any $S \in B_{\mathcal{K}(X)}$ and $T \in B_{\mathcal{L}(X)}$.

Next, relying on Corollary 3.7, we will prove Example 3.2.

Proof of Example 3.2. Let us denote the canonical coordinate projection by $P_n : d(v,p) \longrightarrow d(v,p)$ where

$$P_n(x_1, x_2, \dots) = (x_1, \dots, x_n, 0, 0, \dots), \quad (x_k) \in d(v,p).$$

For every $n \in \mathbb{N}$, we have $P_n \in \mathcal{K}(d(v,p))$, $\|P_n\| = 1$, and $P_n x \longrightarrow x$ for all $x \in d(v,p)$. Hence,

$$\begin{aligned} \left\|rP_n x + s(I_{d(v,p)} - P_n)y\right\|^p &\le \|P_n(rx)\|^p + \left\|(I_{d(v,p)} - P_n)(sy)\right\|^p \\ &\le r^p \|P_n\|^p \|x\|^p + s^p \left\|(I_{d(v,p)} - P_n)\right\|^p \|y\|^p \\ &\le r^p + s^p \end{aligned}$$

whenever $\|x\|, \|y\| \le 1$. Relying on Lemma 2.19, we have

$$\begin{aligned} &\limsup_n \left\|rS + s(I_{d(v,p)} - P_n)T\right\| \\ &\qquad \le \limsup_n \left[\, \|rS - rP_nS\| + \left\|rP_nS + s(I_{d(v,p)} - P_n)T\right\| \,\right] \\ &\qquad \le \limsup_n \sup_{\|z\|\le 1} \left\|rP_nSz + s(I_{d(v,p)} - P_n)Tz\right\| \\ &\qquad \le \limsup_n \sup_{\|x\|,\|y\|\le 1} \left\|rP_nx + s(I_{d(v,p)} - P_n)y\right\| \end{aligned}$$

for every $S \in B_{\mathcal{K}(d(v,p))}$ and $T \in B_{\mathcal{L}(d(v,p))}$. Thus,

$$\limsup_n \left\|rS + sT(I_{d(v,p)} - P_n)\right\| \le 1$$

for every $S \in B_{\mathcal{K}(d(v,p))}$ and $T \in B_{\mathcal{L}(d(v,p))}$ whenever $r^p + s^p \le 1$, which proves, due to Corollary 3.7, that $\mathcal{K}(d(v,p))$ is an $M(r,s)$-ideal in $\mathcal{L}(d(v,p))$ if $r^p + s^p \le 1$. □

3.4 Properties $M(r,s)$ and $M^*(r,s)$

Analogously to properties (M) and (M^*) for M-ideals, Cabello and Nieto introduced properties $M(r,s)$ and $M^*(r,s)$ for $M(r,s)$-ideals (see [6]).

Definition 3.9. Let $r, s \in (0,1]$. A Banach space X has *property* $M(r,s)$ if

$$\limsup_\nu \|ru + sx_\nu\| \le \limsup_\nu \|v + x_\nu\|,$$

whenever $u, v \in X$ satisfy $\|u\| \le \|v\|$, and $(x_\nu) \subset X$ is a bounded net converging weakly to null in X.

Definition 3.10. Let $r, s \in (0,1]$. A Banach space X has *property* $M^*(r,s)$ if

$$\limsup_\nu \|ru^* + sx^*_\nu\| \leq \limsup_\nu \|v^* + x^*_\nu\|,$$

whenever $u^*, v^* \in X^*$ satisfy $\|u^*\| \leq \|v^*\|$, and $(x^*_\nu) \subset X^*$ is a bounded net converging weak* to null in X^*.

Properties $M(1,1)$ and $M^*(1,1)$ clearly coincide with their prototypical properties (M) and (M^*). A much more general version of property (M^*), namely property $M^*(a, B, c)$, was introduced and studied in [49] (see also [48]). It can be easily seen that property $M^*(s, \{-s\}, r)$ is precisely property $M^*(r,s)$.

Analogously to properties (M) and (M^*) (see Section 2.2), one can prove that property $M^*(r,s)$ implies property $M(r,s)$.

Lemma 3.11. *1. Let X and Y be Banach spaces with properties $M(r_1, s_1)$ and $M(r_2, s_2)$, respectively. If $(u_\alpha) \subset X$ and $(v_\alpha) \subset Y$ are relatively compact nets with $\|v_\alpha\| \leq \|u_\alpha\|$ for every α, and (x_α) is a bounded net converging weakly to null in X, then*

$$\limsup_\alpha \|r_1 r_2 v_\alpha + s_1 s_2 T x_\alpha\| \leq \limsup_\alpha \|u_\alpha + x_\alpha\|$$

for any $T \in B_{\mathcal{L}(X,Y)}$.

2. Let X and Y be Banach spaces with properties $M^*(r_1, s_1)$ and $M^*(r_2, s_2)$, respectively. If $(v^*_\alpha) \subset X^*$ and $(u^*_\alpha) \subset Y^*$ are relatively compact nets with $\|v^*_\alpha\| \leq \|u^*_\alpha\|$ for every α, and (y^*_α) is a bounded net converging weak to null in Y^*, then*

$$\limsup_\alpha \|r_1 r_2 v^*_\alpha + s_1 s_2 T^* y^*_\alpha\| \leq \limsup_\alpha \|u^*_\alpha + y^*_\alpha\|$$

for any $T \in B_{\mathcal{L}(X,Y)}$.

Proof. We only give a proof of the first half of the lemma; the other half is a matter of similarity. We first do the case $\|T\| = 1$. Suppose that, contrary to our claim,

$$\lim_\alpha \|r_1 r_2 v_\alpha + s_1 s_2 T x_\alpha\| > \lim_\alpha \|u_\alpha + x_\alpha\|$$

for some relatively compact nets $(u_\alpha) \subset X$ and $(v_\alpha) \subset Y$ with $\|v_\alpha\| \leq \|u_\alpha\|$ for every α, and for some bounded weakly null net $(x_\alpha) \subset X$. By passing to subnets, we may assume that $u_\alpha \longrightarrow u$ in X and $v_\alpha \longrightarrow v$ in Y. Consequently,

$$\lim_\alpha \|r_1 r_2 v + s_1 s_2 T x_\alpha\| > \lim_\alpha \|u + x_\alpha\|.$$

For any ε, choose $x \in B_X$ so that $(1+\varepsilon)\|Tx\| > 1$. Note that (Tx_α) is a bounded weakly null net in Y. Applying property $M(r_2, s_2)$ we have

$$\begin{aligned}\lim_\alpha \|r_1 r_2 v + s_1 s_2 T x_\alpha\| &\le \limsup_\alpha \Big\| r_1(1+\varepsilon)\|v\| Tx + s_1 T x_\alpha \Big\| \\ &\le \limsup_\alpha \Big\| r_1 \|v\| x + s_1 x_\alpha \Big\| + \varepsilon \|v\|,\end{aligned}$$

and applying property $M(r_1, s_1)$ we have

$$\limsup_\alpha \Big\| r_1 \|v\| x + s_1 x_\alpha \Big\| \le \lim_\alpha \|u + x_\alpha\|.$$

This leads to

$$\lim_\alpha \|r_1 r_2 v + s_1 s_2 T x_\alpha\| \le \lim_\alpha \|u + x_\alpha\|$$

which is a contradiction.

The general case follows now by writing $T \in B_{\mathcal{L}(X,Y)}$ in the form $T = \lambda T' + (1-\lambda)T''$ for some $\lambda \in [0,1]$ and T', T'' with $\|T'\| = \|T''\| = 1$ (see the proof of Lemma 2.14). □

Remark 3.12. In the special case of $r_1 = s_1 = r_2 = s_2 = 1$, Lemma 3.11 reduces to Lemma 2.14.

3.5 Descriptions of $M(r,s)$-ideals of compact operators

The following lemma (inspired by [45, Theorem 5, (d)⇒(e)]) shows how to fulfill the lim sup assumptions of Proposition 3.6.

Lemma 3.13. *1. Let X and Y be Banach spaces with properties $M(r_1, s_1)$ and $M(r_2, s_2)$, respectively. If there exists a shrinking MCAI (K_α) of X such that*

$$\limsup_\beta \limsup_\alpha \|\tilde{r} K_\beta + \tilde{s}(I_X - K_\alpha)\| \le 1$$

for some $\tilde{r}, \tilde{s} \ge 0$, then

$$\limsup_\alpha \|rS + s(T - TK_\alpha)\| \le 1$$

for any $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$, where $r = r_1 r_2 \tilde{r}$ and $s = s_1 s_2 \tilde{s}$.

2. Let X and Y be Banach spaces with properties $M^*(r_1, s_1)$ and $M^*(r_2, s_2)$, respectively. If there exists an MCAI (K_α) of Y such that

$$\limsup_\beta \limsup_\alpha \|\tilde{r}K_\beta + \tilde{s}(I_Y - K_\alpha)\| \leq 1$$

for some $\tilde{r}, \tilde{s} \geq 0$, then

$$\limsup_\alpha \|rS + s(T - K_\alpha T)\| \leq 1$$

for any $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$, where $r = r_1 r_2 \tilde{r}$ and $s = s_1 s_2 \tilde{s}$.

Proof. 1. Assume that (K_α) is a shrinking MCAI of X. Fix $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$. Since $SK_\alpha \longrightarrow S$ (see Lemma 2.19),

$$\limsup_\alpha \|rS + s(T - TK_\alpha)\| \leq \limsup_\beta \limsup_\alpha \|rSK_\beta + s(T - TK_\alpha)\|.$$

Fix β. Proceeding similarly to the proof of Theorem 2.27, $(b)\Rightarrow(a)$, we may assume that there exists $(x_\alpha) \subset B_X$ such that

$$\limsup_\alpha \|rSK_\beta + s(T - TK_\alpha)\| = \limsup_\alpha \|rSK_\beta x_\alpha + s(T - TK_\alpha)x_\alpha\|.$$

Note that $(SK_\beta x_\alpha)_\alpha \subset Y$ and $(K_\beta x_\alpha)_\alpha \subset X$ are relatively compact nets with $\|SK_\beta x_\alpha\| \leq \|K_\beta x_\alpha\|$ for any α, and $((I_X - K_\alpha)x_\alpha)$ is a bounded weakly null net in X. Hence, by Lemma 3.11,

$$\begin{aligned}\limsup_\alpha \|rSK_\beta x_\alpha + s(T - TK_\alpha)x_\alpha\| &\leq \limsup_\alpha \|\tilde{r}K_\beta x_\alpha + \tilde{s}(I_X - K_\alpha)x_\alpha\| \\ &\leq \limsup_\alpha \|\tilde{r}K_\beta + \tilde{s}(I_X - K_\alpha)\| \leq 1\end{aligned}$$

and the claim follows.

2. Assume now that (K_α) is an MCAI of Y. Fix $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$. Since $K_\alpha S \longrightarrow S$ (see Lemma 2.19),

$$\begin{aligned}\limsup_\alpha \|rS + s(T - K_\alpha T)\| &\leq \limsup_\beta \limsup_\alpha \|rK_\beta S + s(T - K_\alpha T)\| \\ &= \limsup_\beta \limsup_\alpha \|rS^* K_\beta^* + s(T^* - T^* K_\alpha^*)\|.\end{aligned}$$

Fix β. Analogously to the proof of the first part, we may assume that there is a net $(y_\alpha^*) \subset B_{Y^*}$ such that

$$\limsup_\alpha \|rS^* K_\beta^* + s(T^* - T^* K_\alpha^*)\| = \limsup_\alpha \|rS^* K_\beta^* y_\alpha^* + s(T^* - T^* K_\alpha^*)y_\alpha^*\|.$$

Note that $(S^*K_\beta^*y_\alpha^*)_\alpha \subset X^*$ and $(K_\beta^*y_\alpha^*)_\alpha \subset Y^*$ are relatively compact nets with $\|S^*K_\beta^*y_\alpha^*\| \le \|K_\beta^*y_\alpha^*\|$ for any α, and $((I_Y - K_\alpha)^*y_\alpha^*)$ is a bounded net converging weak* to null in Y^*. Hence, by Lemma 3.11,

$$\begin{aligned}\limsup_\alpha \|rS^*K_\beta^*y_\alpha^* + s(T^* - T^*K_\alpha^*)y_\alpha^*\| &\le \limsup_\alpha \|\tilde{r}K_\beta^*y_\alpha^* + \tilde{s}(I_Y - K_\alpha)^*y_\alpha^*\| \\ &\le \limsup_\alpha \|\tilde{r}K_\beta^* + \tilde{s}(I_Y - K_\alpha)^*\| \le 1\end{aligned}$$

and the claim follows. □

For proving that $M(r,s)$-ideals create new $M(r,s)$-ideals, we shall need, as auxiliary results, the following two lemmas together with their corollaries.

Lemma 3.14 ([49, Corollary 4.4])**.** *Let X be a Banach space. If $r,s \in (0,1]$ satisfy $r + s/2 > 1$, then the following assertions are equivalent.*

1° *$\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$.*

2° *X has an MCAI and property $M^*(r,s)$.*

Lemma 3.15 ([7, Theorem 3.1])**.** *Let X be a Banach space and let $\mathcal{L} \subset \mathcal{L}(X)$ be a closed subspace containing $\mathcal{I}(X)$. If $r,s \in (0,1]$ satisfy $r + s/2 > 1$, then the following assertions are equivalent.*

1° *$\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}$.*

2° *There exists a shrinking MCAI (K_α) of X such that*

$$\limsup_\alpha \|rSK_\alpha + s(T - TK_\alpha)\| \le 1 \quad \forall S,T \in B_{\mathcal{L}}. \tag{3.1}$$

Corollary 3.16. *Let X be a Banach space and let $r,s \in (0,1]$ satisfy $r + s/2 > 1$. If $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$, then X has property $M^*(r,s)$ and there is a shrinking MCAI (K_α) of X with*

$$\limsup_\alpha \|rS + s(I_X - K_\alpha)\| \le 1 \quad \forall S \in B_{\mathcal{K}(X)}. \tag{3.2}$$

Proof. By Lemma 3.14, X has property $M^*(r,s)$. By Lemma 3.15, there exists a shrinking MCAI (K_α) satisfying (3.1) with $T = I_X$. □

Corollary 3.17. *Let X be a Banach space and let $r,s \in (0,1]$ satisfy $r + s/2 > 1$. If $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}(X)$, then $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$.*

Proof. Based on Lemma 3.15, X has a shrinking MCAI (K_α) such that inequality (3.1) holds with $\mathcal{L} = \mathcal{L}(X)$. Hence, it also holds with $\mathcal{L} = \mathcal{I}(X)$. Therefore, by Lemma 3.15, $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$. □

3.6 $M(r,s)$-ideals of compact operators creating new $M(r,s)$-ideals

The M-ideal prototype (that is the case when $r_1 = s_1 = r_2 = s_2 = 1$) of the following Theorems 3.18 and 3.19 is Theorem 2.28.

Theorem 3.18. *Let X and Y be Banach spaces and let $r_1, s_1, r_2, s_2 \in (0,1]$. Assume that $\mathcal{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathcal{I}(X)$ with $r_1 + s_1/2 > 1$, and Y has property $M(r_2,s_2)$. Then $\mathcal{K}(X,Y)$ is an $M(r_1^2 r_2, s_1^2 s_2)$-ideal in $\mathcal{L}(X,Y)$.*

Proof. By Corollary 3.16, X has property $M^*(r_1,s_1)$, recall that this implies property $M(r,s)$, and there is a shrinking MCAI (K_α) of X with

$$\limsup_\alpha \|r_1 K_\beta + s_1(I_X - K_\alpha)\| \leq 1 \quad \forall \beta.$$

By the first part of Lemma 3.13,

$$\limsup_\alpha \|r_1^2 r_2 S + s_1^2 s_2 (T - TK_\alpha)\| \leq 1$$

for any $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$. The claim now follows from Proposition 3.6. □

Theorem 3.19. *Let X and Y be Banach spaces and let $r_1, s_1, r_2, s_2 \in (0,1]$. Assume that X has property $M^*(r_1,s_1)$ and $\mathcal{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathcal{I}(Y)$ with $r_2+s_2/2 > 1$. Then $\mathcal{K}(X,Y)$ is an $M(r_1 r_2^2, s_1 s_2^2)$-ideal in $\mathcal{L}(X,Y)$.*

Proof. The proof is analogous to the previous one. By Corollary 3.16, Y has property $M^*(r_2,s_2)$ and there is a shrinking MCAI (K_α) of Y with

$$\limsup_\alpha \|r_1 K_\beta + s_1(I_Y - K_\alpha)\| \leq 1 \quad \forall \beta,$$

when we take $S = K_\beta$. By the second part of Lemma 3.13,

$$\limsup_\alpha \|r_1^2 r_2 S + s_1^2 s_2 (T - K_\alpha T)\| \leq 1$$

for any $S \in B_{\mathcal{K}(X,Y)}$ and $T \in B_{\mathcal{L}(X,Y)}$. The claim now follows from Proposition 3.6. □

Recall that $M(1,1)$-ideals are just M-ideals. Hence, the following Corollary 3.20 is immediate from Theorems 3.18 and 3.19 by Corollary 3.17.

Corollary 3.20. *Let X be a Banach space such that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ and let $r, s \in (0,1]$. Then $\mathcal{K}(X,Y)$ is an $M(r,s)$-ideal in $\mathcal{L}(X,Y)$ for all Banach spaces Y with property $M(r,s)$, and $\mathcal{K}(Y,X)$ is an $M(r,s)$-ideal in $\mathcal{L}(Y,X)$ for all Banach spaces Y with property $M^*(r,s)$.*

Gathering the assumptions of Theorems 3.18 and 3.19 together and using Lemma 3.14 yield our main result in Chapter 3.

Theorem 3.21. *Let X and Y be Banach spaces and let $r_1, s_1, r_2, s_2 \in (0,1]$. Assume that $\mathcal{K}(X)$ is an $M(r_1, s_1)$-ideal in $\mathcal{I}(X)$ with $r_1 + s_1/2 > 1$ and $\mathcal{K}(Y)$ is an $M(r_2, s_2)$-ideal in $\mathcal{I}(Y)$ with $r_2 + s_2/2 > 1$. Then $\mathcal{K}(X,Y)$ is an $M(r_1^2 r_2, s_1^2 s_2)$- and an $M(r_1 r_2^2, s_1 s_2^2)$-ideal in $\mathcal{L}(X,Y)$.*

Using Corollary 3.17, this immediately implies:

Corollary 3.22. *Let X and Y be Banach spaces and let $r_1, s_1, r_2, s_2 \in (0,1]$. Assume that $\mathcal{K}(X)$ is an $M(r_1, s_1)$-ideal in $\mathcal{L}(X)$ with $r_1 + s_1/2 > 1$ and $\mathcal{K}(Y)$ is an $M(r_2, s_2)$-ideal in $\mathcal{L}(Y)$ with $r_2 + s_2/2 > 1$. Then $\mathcal{K}(X,Y)$ is an $M(r_1^2 r_2, s_1^2 s_2)$- and an $M(r_1 r_2^2, s_1 s_2^2)$-ideal in $\mathcal{L}(X,Y)$.*

Corollary 3.22 extends Corollary 2.29 from M-ideals to $M(r,s)$-ideals.

The following is immediate from Corollary 3.17 and Theorem 3.21.

Corollary 3.23. *Let $r, s \in (0,1]$ satisfy $r + s/2 > 1$. If $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}(X)$, then $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$. If $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$, then $\mathcal{K}(X)$ is an $M(r^3, s^3)$-ideal in $\mathcal{L}(X)$*

In the special case of $r = s = 1$, Corollary 3.23 reduces to Kalton's theorem (the equality (a)$\Leftrightarrow$(d) in Theorem 2.27): $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ if and only if $\mathcal{K}(X)$ is an M-ideal in $\mathcal{I}(X)$. The following problem remains unsolved in this thesis.

Problem 3.24. Can Corollary 3.23 be improved to yield the desirable result: $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}(X)$ if and only if $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$?

Chapter 4

Improved parameters of $M(r, s)$-ideals

In Chapter 3, we proved that $\mathcal{K}(X, Y)$ is an $M(r_1^2 r_2, s_1^2 s_2)$- and an $M(r_1 r_2^2, s_1 s_2^2)$-ideal in $\mathcal{L}(X, Y)$ whenever $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are $M(r_1, s_1)$- and $M(r_2, s_2)$-ideals in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, respectively, with $r_1 + s_1/2 > 1$ and $r_2 + s_2/2 > 1$. The parameters $r_1^2 r_2$ and $s_1^2 s_2$, or $r_1 r_2^2$ and $s_1 s_2^2$ seem to be not optimal. In this chapter, we propose a different approach that improves the parameters to $r_1 r_2$ and $s_1 s_2$. The key concepts of the new approach are "the ideal projection preserving elementary functionals" and "property $M^*(r, s)$ for operators". An important tool, we are basing on, is the Feder–Saphar description of the dual space of $\mathcal{K}(X, Y)$ which holds whenever X^{**} or Y^* has the Radon–Nikodým property. The chapter relies on [20].

4.1 The Feder–Saphar description of $\mathcal{K}(X, Y)^*$

The Feder–Saphar theorem, in [13], is formulated in the terminology of tensor products. Here we introduce some basic notation of it, for definitions and the theory in general, see, for example, [17] or [55].

Let X and Y be Banach spaces. Denote by $X \hat{\otimes} Y$ the *projective tensor product* of X and Y, and by $\|.\|_\pi$ the *projective norm* on it. If $u \in X \hat{\otimes} Y$, then there is a representation (see, e.g., [17], [56], or [55])

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n, \quad x_n \in X,\ y_n \in Y,$$

with $\sum_{n=1}^{\infty} \|x_n\|\, \|y_n\| < \infty$.

Let Z be another Banach space and let $T \in \mathcal{L}(Y, Z)$. One defines

$$Tu = \sum_{n=1}^{\infty} x_n \otimes Ty_n, \quad u = \sum_{n=1}^{\infty} x_n \otimes y_n \in X \hat{\otimes} Y.$$

Then $Tu \in X \hat{\otimes} Z$ and $\|Tu\|_\pi \leq \|T\|\, \|u\|_\pi$. For every $u \in X \hat{\otimes} X^*$, $u = \sum_{n=1}^{\infty} x_n \otimes x_n^*$, we denote

$$\operatorname{trace} u = \sum_{n=1}^{\infty} x_n^*(x_n).$$

The functional $u \longmapsto \operatorname{trace} u$ is well defined, linear, and of norm less than or equal to 1.

Definition 4.1. The dual space X^* has the *Radon–Nikodým property* if from the separability of a subspace Z of X it follows that also Z^* is separable.

Theorem 4.2 ([13, Theorem 1])**.** *Let X and Y be Banach spaces. If X^{**} or Y^* has the Radon–Nikodým property, then $V : Y^* \hat{\otimes} X^{**} \longrightarrow \mathcal{K}(X, Y)^*$ defined by*

$$(Vu)(S) = \operatorname{trace}(S^* u), \quad u \in Y^* \hat{\otimes} X^{**},\ S \in \mathcal{K}(X, Y),$$

is a quotient map such that

$$\forall f \in \mathcal{K}(X, Y)^*\ \exists u \in Y^* \hat{\otimes} X^{**}, \quad f = Vu\ \wedge\ \|f\| = \|u\|_\pi.$$

A useful conclusion of the Feder–Saphar theorem is the following result, also named as the Feder–Saphar theorem, which was observed, e.g., in [16] and [43].

Theorem 4.3. *Let X and Y be Banach spaces, and let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X, Y)$ containing $\mathcal{K}(X, Y)$. If X^{**} or Y^* has the Radon–Nikodým property, then*

$$\mathcal{K}(X, Y)^* = \overline{\operatorname{span}}\{x^{**} \otimes y^*|_{\mathcal{K}(X,Y)} : x^{**} \in X^{**},\ y^* \in Y^*,\ x^{**} \otimes y^* \in \mathcal{L}^*\}.$$

Proof. Let $V : Y^*\hat{\otimes}X^{**} \longrightarrow \mathcal{K}(X,Y)^*$ be defined as in Theorem 4.2. Note that $V(y^*\otimes x^{**}) = x^{**}\otimes y^*|_{\mathcal{K}(X,Y)}$ for every $x^{**}\in X^{**}$ and $y^*\in Y^*$, $x^{**}\otimes y^*\in \mathcal{L}^*$. Indeed,

$$(V(y^*\otimes x^{**}))(S) = \text{trace}(S^*(y^*\otimes x^{**})) = x^{**}(S^*y^*) = (x^{**}\otimes y^*)(S)$$

for every $S\in\mathcal{K}(X,Y)$.

Since V is a bounded linear surjection,

$$\begin{aligned}\mathcal{K}(X,Y)^* &= V(Y^*\hat{\otimes}X^{**})\\ &= V\Big(\overline{\text{span}}\{y^*\otimes x^{**} : y^*\in Y^*,\ x^{**}\in X^{**}\}\Big)\\ &\subset \overline{\text{span}}\Big\{V(y^*\otimes x^{**}) : y^*\in Y^*,\ x^{**}\in X^{**}\Big\}\\ &= \overline{\text{span}}\{x^{**}\otimes y^*|_{\mathcal{K}(X,Y)} : x^{**}\in X^{**},\ y^*\in Y^*,\ x^{**}\otimes y^*\in\mathcal{L}^*\}.\end{aligned}$$

Thus,

$$\mathcal{K}(X,Y)^* = \overline{\text{span}}\{x^{**}\otimes y^*|_{\mathcal{K}(X,Y)} : x^{**}\in X^{**},\ y^*\in Y^*,\ x^{**}\otimes y^*\in\mathcal{L}^*\}.$$

□

4.2 Ideal projection preserving elementary functionals

Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$. Assume that $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to an ideal projection P.

Definition 4.4. If $P(x^{**}\otimes y^*) = x^{**}\otimes y^*$ for all $x^{**}\in X^{**}$ and $y^*\in Y^*$, then we say that P *preserves elementary functionals.*

Example 4.5. The Johnson projection is an ideal projection preserving elementary functionals.

Proof. By Lemma 2.23, the Johnson projection is an ideal projection. Let us denote it by P. Consider any $x^{**}\otimes y^*\in\mathcal{L}^*$, and let $T\in\mathcal{L}$. If (K_α) is a shrinking MCAI of X, then

$$\begin{aligned}(P(x^{**}\otimes y^*))(T) &= \lim_\alpha (x^{**}\otimes y^*)(TK_\alpha)\\ &= \lim_\alpha x^{**}(K_\alpha^* T^* y^*) = x^{**}(T^*y^*)\\ &= (x^{**}\otimes y^*)(T).\end{aligned}$$

If, respectively, (K_α) is a shrinking MCAI of Y, then

$$\begin{aligned}(P(x^{**}\otimes y^*))(T) &= \lim_\alpha (x^{**}\otimes y^*)(K_\alpha T)\\ &= \lim_\alpha x^{**}(T^*K_\alpha^* y^*) = x^{**}(T^*y^*)\\ &= (x^{**}\otimes y^*)(T).\end{aligned}$$

□

In contrast with the Johnson projection, an ideal projection preserving elementary functionals may also be defined departing from a (generally) unbounded net of compact operators, as the following example shows.

Example 4.6 (see [35, Theorem 5.1] and [36, proof of Theorem 4.6])**.** Let X and Y be Banach spaces such that X^{**} or Y^* has the Radon–Nikodým property. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$. If X^* or Y^* has the CAP with conjugate operators, then $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to an ideal projection preserving elementary functionals.

Proof. Suppose first that X^* has the CAP with conjugate operators. Let a net $(K_\alpha)\subset\mathcal{K}$ be such that $K_\alpha^* \longrightarrow I_X^*$ uniformly on compact subsets of X^*. According to Theorem 4.2, for any $g\in\mathcal{K}^*$, there exists $u=\sum_{n=1}^\infty y_n^*\otimes x_n^{**}\in Y^*\hat{\otimes}X^{**}$ such that

$$g(S)=\sum_{n=1}^\infty x_n^{**}(S^*y_n^*)\quad \forall S\in\mathcal{K}$$

and $\|g\| = \|u\|_\pi$. Without loss of generality we may assume that $\sum_{n=1}^\infty \|x_n^{**}\|<\infty$ and $\|y_n^*\|\longrightarrow 0$. Let $T\in\mathcal{L}$. Then $TK_\alpha\in\mathcal{K}$ and

$$\begin{aligned}|\mathrm{trace}(T^*u)-g(TK_\alpha)| &= \left|\sum_{n=1}^\infty \Big[x_n^{**}(T^*y_n^*)-x_n^{**}(K_\alpha^*T^*y_n^*)\Big]\right|\\ &\le \sup_n \|(I_{X^*}-K_\alpha^*)(T^*y_n^*)\|\sum_{n=1}^\infty \|x_n^{**}\|\,.\end{aligned}$$

We have

$$\sup_n \|(I_{X^*}-K_\alpha^*)(T^*y_n^*)\| \underset{\alpha}{\longrightarrow} 0$$

because $\{0, T^*y_1^*, T^*y_2^*,\dots\}$ is a compact subset of X^*. As a result, we can conclude that

$$\lim_\alpha |\mathrm{trace}(T^*u)-g(TK_\alpha)|=0\quad \forall T\in\mathcal{L}.$$

Let us define $P : \mathcal{L}^* \longrightarrow \mathcal{L}^*$ by

$$(Pf)(T) = \lim_{\alpha} f(TK_\alpha) = \operatorname{trace}(T^* u), \quad f \in \mathcal{L}^*,\ T \in \mathcal{L},$$

where $u \in Y^* \hat{\otimes} X^{**}$ corresponds to $f|_{\mathcal{K}} \in \mathcal{K}^*$ due to the Feder–Saphar theorem.

Clearly P is well-defined and linear, and since

$$\|\operatorname{trace}(T^* u)\| \le \|T^* u\|_\pi \le \|T^*\| \, \|u\|_\pi = \|T\| \, \|f|_{\mathcal{K}}\| \le \|T\| \, \|f\|$$

for every $T \in \mathcal{L}$, we have $\|P\| \le 1$.

Recall that $\lim SK_\alpha = S$ for every $S \in \mathcal{K}$ (see Lemma 2.19 and Remark 2.20) providing

$$f(S) = f(\lim_{\alpha} SK_\alpha) = \lim_{\alpha} f(SK_\alpha) = (Pf)(S) \quad \forall f \in \mathcal{L}^*.$$

Thus, $f - Pf \in \mathcal{K}^\perp$ and we conclude that $\ker P \subset \mathcal{K}^\perp$. The operator P is a projection if $\mathcal{K}^\perp \subset \ker P$. The latter holds since $TK_\alpha \in \mathcal{K}$ and for $f \in \mathcal{K}^\perp$ we have

$$(Pf)(T) = \lim_{\alpha} f(TK_\alpha) = \lim_{\alpha} 0 = 0.$$

Projection P preserves elementary functionals since

$$(P(x^{**} \otimes y^*))(T) = \lim_{\alpha} x^{**}(K_\alpha^* T^* y^*) = x^{**}(T^* u^*) = (x^{**} \otimes y^*)(T)$$

for all $T \in \mathcal{L}$, $x^{**} \in X^{**}$, and $y^* \in Y^*$.

The case when Y^* has the CAP with conjugate operators is analogous to the previous case.

Let a net $(K_\alpha) \subset \mathcal{K}$ be such that $K_\alpha^* \longrightarrow I_Y^*$ uniformly on compact subsets of Y^*. As in the previous part, for any $g \in \mathcal{K}^*$, there exists $u = \sum_{n=1}^\infty y_n^* \otimes x_n^{**}$ with $\sum_{n=1}^\infty \|x_n^{**}\| < \infty$ and $\|y_n^*\| \longrightarrow 0$ such that

$$g(S) = \sum_{n=1}^\infty x_n^{**}(S^* y_n^*) \quad \forall\, S \in \mathcal{K}$$

and $\|g\| = \|u\|_\pi$. Let $T \in \mathcal{L}$. Then $K_\alpha T \in \mathcal{K}$ and

$$\begin{aligned}|\operatorname{trace}(T^* u) - g(K_\alpha T)| &= \left| \sum_{n=1}^\infty \Big[x_n^{**}(T^* y_n^*) - x_n^{**}(T^* K_\alpha^* y_n^*) \Big] \right| \\ &\le \sup_n \|(I_{Y^*} - K_\alpha^*)(y_n^*)\| \sum_{n=1}^\infty \|x_n^{**}\| \, \|T^*\| \,.\end{aligned}$$

Since $\{0, y_1^*, y_2^*, \dots\}$ is compact, we have

$$\sup_n \|(I_{Y^*} - K_\alpha^*)(y_n^*)\| \underset{\alpha}{\longrightarrow} 0$$

providing

$$\lim_\alpha |\mathrm{trace}(T^*u) - g(K_\alpha T)| = 0 \quad \forall\, T \in \mathcal{L}.$$

Defining $P : \mathcal{L}^* \longrightarrow \mathcal{L}^*$ by

$$(Pf)(T) = \lim_\alpha f(K_\alpha T) = \mathrm{trace}(T^*u), \quad f \in \mathcal{L}^*,\ T \in \mathcal{L},$$

where $u \in Y^* \hat{\otimes} X^{**}$ corresponds to $f|_{\mathcal{K}} \in \mathcal{K}^*$ due to the Feder–Saphar theorem, we can conclude the proof as in the previous part. □

In the sequel, we shall need the fact that in many important cases, the ideal of compact operators enjoys the unique ideal property with respect to ideal projections preserving elementary functionals.

Proposition 4.7. *Let X and Y be Banach spaces. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$. If X^{**} or Y^* has the Radon–Nikodým property, then for $\mathcal{K}$ in $\mathcal{L}$ there is at most one ideal projection preserving elementary functionals.*

Proof. Let Q and P be ideal projections on $\mathcal{L}^*$ preserving elementary functionals with $\ker Q = \ker P = \mathcal{K}^\perp$. Let $\Phi : \mathcal{K}^* \longrightarrow \operatorname{ran} Q$ and $\Psi : \mathcal{K}^* \longrightarrow \operatorname{ran} P$ be the corresponding isometric isomorphisms such that $Pf = \Phi(f|_{\mathcal{K}})$ and $Qf = \Psi(f|_{\mathcal{K}})$, where $f \in \mathcal{L}^*$ (see Lemma 2.21). Therefore, we need to prove that

$$\Phi g = \Psi g \quad \forall g \in \mathcal{K}^*.$$

The desired equality is immediate from the fact that

$$\mathcal{K}^* = \overline{\mathrm{span}}\{(x^{**} \otimes y^*)|_{\mathcal{K}} \colon x^{**} \in X^{**},\ y^* \in Y^*,\ x^{**} \otimes y^* \in \mathcal{L}^*\}$$

(see Theorem 4.3) and the equality

$$\Phi((x^{**} \otimes y^*)|_{\mathcal{K}}) = P(x^{**} \otimes y^*) = x^{**} \otimes y^* = Q(x^{**} \otimes y^*) = \Psi((x^{**} \otimes y^*)|_{\mathcal{K}})$$

which holds for all $x^{**} \in X^{**}$ and $y^* \in Y^*$. □

Definition 4.8. The *dual weak operator topology* on $\mathcal{L}(X,Y)$ is defined by the functionals $A \longmapsto x^{**}(A^*y^*),\ y^* \in Y^*,\ x^{**} \in X^{**}$.

Clearly the dual weak operator topology is not weaker than the weak operator topology which is defined by the functionals $A \longmapsto y^*(Ax),\ y^* \in Y^*,\ x \in X$. However, under the assumption of the Radon–Nikodým property, we obtain from convergence in the dual weak operator topology, the convergence in the weak topology induced by the projection preserving elementary functionals.

Proposition 4.9. *Let X and Y be Banach spaces, and suppose that X^{**} or Y^* has the Radon–Nikodým property. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$, and suppose that $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to an ideal projection P preserving elementary functionals. If for an operator $T \in \mathcal{L}$ there exists a bounded net $(T_\alpha) \subset \mathcal{L}$ such that $T_\alpha \longrightarrow T$ in the dual weak operator topology, then*

$$(Pf)(T_\alpha) \underset{\alpha}{\longrightarrow} (Pf)(T) \quad \forall f \in \mathcal{L}^*.$$

Proof. As in the previous proof, we shall apply the fact that $\operatorname{span}\{(x^{**} \otimes y^*)|_{\mathcal{K}} : x^{**} \in X^{**},\ y^* \in Y^*,\ x^{**} \otimes y^* \in \mathcal{L}^*\}$ is dense in $\mathcal{K}^*$ (see Theorem 4.3). Using the associated isomorphism $\Phi : \mathcal{K}^* \longrightarrow \operatorname{ran} P$ satisfying $\Phi(f|_{\mathcal{K}}) = Pf,\ f \in \mathcal{L}^*$ (see Lemma 2.21), and that P preserves the elementary functionals, we get that $\operatorname{span}\{x^{**} \otimes y^* : x^{**} \in X^{**},\ y^* \in Y^*\} \subset \operatorname{ran} P$ is dense in $\operatorname{ran} P \subset \mathcal{L}^*$.

Every $A \in \mathcal{L}$ can be viewed as an element of $(\operatorname{ran} P)^*$ with the same norm, defining

$$\langle A, h \rangle = h(A), \quad h \in \operatorname{ran} P.$$

Since the net (T_α) is bounded and for all $x^{**} \in X^{**}$, $y^* \in Y^*$,

$$\begin{aligned} \langle T_\alpha, x^{**} \otimes y^* \rangle &= (x^{**} \otimes y^*)(T_\alpha) \\ &= x^{**}(T_\alpha^* y^*) \underset{\alpha}{\longrightarrow} x^{**}(T^* y^*) \\ &= \langle T, x^{**} \otimes y^* \rangle, \end{aligned}$$

we have $\langle T_\alpha, h \rangle \longrightarrow_\alpha \langle T, h \rangle$ for all $h \in \operatorname{ran} P$. This means that $(Pf)(T_\alpha) \longrightarrow_\alpha (Pf)(T)$ for all $f \in \mathcal{L}^*$. $\square$

Proposition 4.9 extends Lemma 1.2 of [54] from the case of the Johnson projection (involving the shrinking MCAI assumptions for X or Y) to an arbitrary ideal projection preserving elementary functionals.

4.3 $M(r,s)$-ideals of compact operators and the ideal projection preserving elementary functionals

We shall apply Proposition 4.9 to deduce the following criteria for $M(r,s)$-ideals of compact operators with respect to the ideal projection preserving elementary functionals. The result will be needed in Section 4.6.

Theorem 4.10. *Let X and Y be Banach spaces, and suppose that X^{**} or Y^* has the Radon–Nikodým property. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$, and suppose that $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to an ideal projection P preserving elementary functionals. Let $r \le 1$ and s be positive numbers. If for every operator $T \in S_{\mathcal{L}}$ there exists a bounded net $(T_\alpha) \subset \mathcal{K}$ such that $T_\alpha \longrightarrow T$ in the dual weak operator topology, then the following assertions are equivalent.*

(a) *$\mathcal{K}$ is an $M(r,s)$-ideal in $\mathcal{L}$ with respect to P.*

(b) *For every $\varepsilon > 0$, $S \in B_{\mathcal{K}}$, $T \in B_{\mathcal{L}}$, and every index α (in the corresponding net (T_α)), there exists*

$$K \in \operatorname{conv}\{T_\beta \colon \beta \ge \alpha\}$$

such that

$$\|rS + s(T - K)\| \le 1 + \varepsilon.$$

(c) *For every $S \in S_{\mathcal{K}}$ and $T \in S_{\mathcal{L}}$, there exists a net $(K_\nu) \subset \mathcal{K}$ such that $K_\nu \longrightarrow T$ in the dual weak operator topology and*

$$\limsup_{\nu} \|rS + s(T - K_\nu)\| \le 1.$$

Proof. (a) $\Rightarrow$ (b). This implication follows from a general $M(r,s)$-inequality criterion (see [21, Proposition, (a) $\Rightarrow$ (b)]. If the conclusion is false, then there are $\varepsilon > 0$, $S \in B_{\mathcal{K}}$, $T \in B_{\mathcal{L}}$, and α such that for $C := \operatorname{conv}\{T_\beta \colon \beta \ge \alpha\}$, we have

$$sC \cap B(rS + sT, 1+\varepsilon) = \varnothing,$$

where $B(rS+sT, 1+\varepsilon)$ is the open ball with center $rS+sT$ and radius $1+\varepsilon$. By the Hahn–Banach theorem, there exists $f \in S_{\mathcal{L}^*}$ such that

$$\begin{aligned} \operatorname{Re} f(rS+sT) - (1+\varepsilon) &= \inf\{\operatorname{Re} f(U) \colon U \in B(rS+sT, 1+\varepsilon)\} \\ &\ge s \operatorname{Re} f(K) = s \operatorname{Re} Pf(K) \quad \forall K \in C, \end{aligned}$$

because $C \subset \mathcal{K}$ and $f - Pf \in \ker P = \mathcal{K}^\perp$. Hence,

$$\begin{aligned}1 + \varepsilon &\le \operatorname{Re} f(rS + sT) - s \operatorname{Re} Pf(K) \\ &= r \operatorname{Re} Pf(S) + s \operatorname{Re}(f - Pf)(T) + s \operatorname{Re} Pf(T - K) \\ &\le 1 + s \operatorname{Re} Pf(T - K) \quad \forall K \in C.\end{aligned}$$

Since $Pf(T) = \lim_\alpha Pf(T_\alpha)$ (see Proposition 4.9), this implies that $\varepsilon \le 0$, a contradiction.

(b) $\Rightarrow$ (c). Consider the set of all pairs $\nu = (\varepsilon, \alpha)$, where $\varepsilon > 0$ and where (T_α) corresponds to T, directed in the natural way, and choose $K_\nu \in \operatorname{conv}\{T_\beta \colon \beta \ge \alpha\}$ from condition (b).

(c) $\Rightarrow$ (a). Let us fix $f \in \mathcal{L}^*$ and $\varepsilon > 0$. Recalling that $\|Pf\| = \|f|_\mathcal{K}\|$ (see Lemma 2.21), we choose $S \in S_\mathcal{K}$ and $T \in S_\mathcal{L}$ so that

$$r\|Pf\| + s\|f - Pf\| - \varepsilon \le rf(S) + s(f - Pf)(T).$$

Let (K_ν) be given by (c). By passing to a subnet, we may assume that (K_ν) is bounded. By Proposition 4.9, $(Pf)(T) = \lim_\nu (Pf)(K_\nu) = \lim_\nu f(K_\nu)$, because $K_\nu \in \mathcal{K}$ and $Pf - f \in \ker P = \mathcal{K}^\perp$. It follows that

$$\begin{aligned}r\|Pf\| + s\|f - Pf\| - \varepsilon &\le rf(S) + sf(T) - s \lim_\nu f(K_\nu) \\ &= \lim_\nu f(rS + s(T - K_\nu)) \\ &\le \|f\| \limsup_\nu \|rS + s(T - K_\nu)\| \\ &\le \|f\|.\end{aligned}$$

□

Historically, for M-ideals, conditions similar to (b) and (c) of Theorem 4.10 were first considered in [37, Proposition 2.8], [58, Theorem 3.1 and Remark], and [47, proof of Theorem 2].

4.4 Property $M^*(r, s)$ for operators

In [32, Section 6], an operator version of property (M) was introduced and studied (see also [27] and [29] for applications of this property). We need to extend its (M^*) prototype as follows.

Definition 4.11. Let X and Y be Banach spaces, and let $r, s \in (0, 1]$. We say that an operator $T \in B_{\mathcal{L}(X,Y)}$ has *property $M^*(r, s)$* if

$$\limsup_\nu \|rx^* + sT^*y_\nu^*\| \leq \limsup_\nu \|y^* + y_\nu^*\|,$$

whenever $x^* \in X^*$, $y^* \in Y^*$ satisfy $\|x^*\| \leq \|y^*\|$, and $(y_\nu^*) \subset Y^*$ is a bounded net converging weak* to null in Y^*.

If Y is separable, then $T \in B_{\mathcal{L}(X,Y)}$ has property $M^*(r, s)$ if and only if T has the sequential version of property $M^*(r, s)$ (i.e., the nets (y_ν^*) being replaced with the weak* null sequences (y_n^*)). This can be easily checked using the fact that the bounded subsets of Y^* are weak* metrizable.

Clearly, an operator T has property (M^*) if and only if T has property $M^*(1, 1)$, and a Banach space X has property $M^*(r, s)$ if and only if its identity operator I_X has property $M^*(r, s)$. A much more general notion, namely an operator having property $M^*(a, B, c)$, was introduced and studied in [49] (see also [48]). As in the case of spaces, property $M^*(r, s)$ for operators is precisely property $M^*(s, \{-s\}, r)$.

Properties $M^*(r, s)$ for spaces and operators are related similarly to the (M^*) case (see Lemma 2.14 part 2).

Proposition 4.12. *Let X and Y be Banach spaces, and let $r_1, s_1, r_2, s_2 \in (0, 1]$. If X has property $M^*(r_1, s_1)$ and Y has property $M^*(r_2, s_2)$, then every $T \in B_{\mathcal{L}(X,Y)}$ has property $M^*(r_1r_2, s_1s_2)$.*

Proof. Take $x^* \in X^*$, $y^* \in Y^*$ satisfying $\|x^*\| \leq \|y^*\|$, and let $u_\alpha^* = x^*$ and $v_\alpha^* = y^*$ for every α in Lemma 3.11 part 2. □

Proposition 4.13. *Let X and Y be Banach spaces, and let $r, s \in (0, 1]$. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X, Y)$ containing $\mathcal{K} := \mathcal{K}(X, Y)$. If an operator $T \in B_\mathcal{L}$ has property $M^*(r, s)$ and there is a net $(T_\alpha) \subset \mathcal{K}$ such that $T_\alpha^* \longrightarrow T^*$ strongly, then*

$$\limsup_\alpha |f(rS + s(T - T_\alpha))| \leq 1$$

for all $S \in B_\mathcal{K}$ and $f \in \overline{B_{X^{**}} \otimes B_{Y^*}}^{weak^*} \subset \mathcal{L}^*$.

Proof. Let $f = w^*\text{-}\lim x_\nu^{**} \otimes y_\nu^*$, i.e., $x_\nu^{**}(A^*y_\nu^*) \longrightarrow f(A)$, $A \in \mathcal{L}$, with $x_\nu^{**} \in B_{X^{**}}$, $y_\nu^* \in B_{Y^*}$. By passing to a subnet, we may assume that (y_ν^*) converges weak* to some $y^* \in B_{Y^*}$. Property $M^*(r, s)$ implies that

$$\limsup_\nu \|rS^*y^* + sT^*y_\nu^* - sT^*y^*\| \leq \limsup_\nu \|y_\nu^*\| \leq 1.$$

Hence, for any fixed α,

$$\begin{aligned}|f(rS+s(T-T_\alpha))| &= \lim_\nu |x_\nu^{**}((rS+s(T-T_\alpha))^*y_\nu^*)| \\ &\le \limsup_\nu \|(rS+s(T-T_\alpha))^*y_\nu^*\| \\ &\le \limsup_\nu \Big(\|rS^*y_\nu^* - rS^*y^*\| \\ &\quad + \|rS^*y^* + sT^*y_\nu^* - sT^*y^*\| \\ &\quad + \|sT^*y^* - sT_\alpha^*y^*\| + \|sT_\alpha^*y_\nu^* - sT_\alpha^*y^*\|\Big) \\ &\le 1 + \|sT^*y^* - sT_\alpha^*y^*\|,\end{aligned}$$

which implies

$$\limsup_\alpha |f(rS+s(T-T_\alpha))| \le 1.$$

□

In the sequential case in Proposition 4.13, one may go further, by applying the following vector-valued version of Simons's inequality due to [33], to obtain a similar norm condition: see Lemma 4.15 below.

Lemma 4.14 (see [33, Corollary 4] and its proof)**.** *Let X and Y be Banach spaces. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ and let (A_n) be a bounded sequence in $\mathcal{L}$. If*

$$\limsup_n \operatorname{Re} f(A_n) \le \lambda$$

for some $\lambda \ge 0$ and for all $f \in \overline{S_X \otimes S_{Y^*}}^{weak^*} \subset \mathcal{L}^*$, then there exists $B_n \in \operatorname{conv}\{A_n, A_{n+1}, \dots\}$ such that

$$\limsup_n \|B_n\| \le \lambda.$$

Lemma 4.15. *Let X and Y be Banach spaces, and let $r,s \in (0,1]$. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$. If $T \in B_{\mathcal{L}}$ has property $M^*(r,s)$ and there is a sequence $(T_n) \subset \mathcal{K}$ such that $T_n^* \longrightarrow T^*$ strongly, then for all $S \in B_{\mathcal{K}}$ there exists $S_n \in \operatorname{conv}\{T_n, T_{n+1}, \dots\}$ such that*

$$\limsup_n \|rS + s(T-S_n)\| \le 1.$$

4.5 Improved parameters of $M(r,s)$-ideals for the separable case

The next theorem is one of our main results. As we shall see in Section 4.8, in the M-ideal case, its Corollary 4.28 complements [32, Theorem 6.3], and its Corollaries 4.29 and 4.31 improve the dual version of [32, Theorem 6.3; see p. 171] and [27, Theorem 2.4].

Theorem 4.16. *Let X and Y be Banach spaces. Suppose that X^{**} or Y^* has the Radon–Nikodým property and that X or Y has a shrinking compact approximating sequence. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$ and let $r,s \in (0,1]$. If every $T \in S_{\mathcal{L}}$ has property $M^*(r,s)$, then $\mathcal{K}$ is an $M(r,s)$-ideal in $\mathcal{L}$ with respect to an ideal projection preserving elementary functionals.*

Remark 4.17. The assumptions enforce X^* (and X) or Y^* (and Y) to be separable. In the latter case, Y^* automatically has the Radon–Nikodým property and, as was mentioned before, property $M^*(r,s)$ for operators is equivalent to its sequential version (see Section 4.4).

Proof of Theorem 4.16. By Example 4.6, $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to an ideal projection P preserving elementary functionals.

For every operator $T \in S_{\mathcal{L}}$, let us define $T_n = TK_n$ (respectively, $T_n = K_nT$) if (K_n) is the shrinking compact approximating sequence of X (respectively, of Y). Then clearly $T_n^* \longrightarrow T^*$ strongly. Let $S \in S_{\mathcal{K}}$. By Lemma 4.15, there exists $S_n \in \operatorname{conv}\{T_n, T_{n+1}, \dots\}$ such that

$$\limsup_n \|rS + s(T - S_n)\| \leq 1.$$

Since also $S_n^* \longrightarrow T^*$ strongly, by Theorem 4.10, (c) $\Rightarrow$ (a), $\mathcal{K}$ is an $M(r,s)$-ideal in $\mathcal{L}$ with respect to P. □

Now, using Theorem 4.16 and Proposition 4.12, we can prove the desired improvement of Corollary 3.22 for the case if one of the Banach spaces X or Y is *separable.*

Theorem 4.18. *Let X and Y be Banach spaces such that X or Y is separable. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1 + s_1/2 > 1$ and $r_2 + s_2/2 > 1$. If $\mathcal{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathcal{L}(X)$ and $\mathcal{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathcal{L}(Y)$, then $\mathcal{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathcal{L}(X,Y)$.*

Proof. If $r + s/2 > 1$ and $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{L}(X)$, then, by [7, Lemma 2.3 and Proposition 2.1], $X^* = \overline{\operatorname{span}}(w^*\text{-}\operatorname{sexp} B_{X^*})$ (i.e., the weak* strongly exposed points of B_{X^*} span a norm dense subspace of X^*) and X^* has the Radon–Nikodým property. Therefore, by [7, Proposition 3.2] and [49, Theorem 4.1, $1° \Rightarrow 2°$], X has the MCAP and property $M^*(r,s)$. Hence, in our case, both X and Y have the MCAP, X has property $M^*(r_1, s_1)$, and Y has property $M^*(r_2, s_2)$. From Proposition 4.12 we get that every $T \in B_{\mathcal{L}(X,Y)}$ has property $M^*(r_1 r_2, s_1 s_2)$.

We can now apply Theorem 4.16 to show that $\mathcal{K}(X,Y)$ is an $M(r_1 r_2, s_1 s_2)$-ideal in $\mathcal{L}(X,Y)$. Indeed, as we saw above, Y^* has the Radon–Nikodým property. If, e.g., X is separable, since X has the MCAP, X clearly has a metric compact approximating sequence $(K_n)_{n=1}^{\infty}$. Then $(K_n)_{n=1}^{\infty}$ is shrinking because $X^* = \overline{\operatorname{span}}(w^*\text{-}\operatorname{sexp} B_{X^*})$ (this fact is well-known and can be easily checked). □

The proof of Theorem 4.18 clearly shows that *if* Theorem 4.16 held true also in the non-separable case (i.e., with the assumption "X or Y has a shrinking compact approximating sequence" being replaced by "X^* or Y^* has the BCAP with conjugate operators"), *then* in Theorem 4.18 the separability assumption ("X or Y is separable") could be dropped. However, we do not know whether the non-separable case of Theorem 4.16 is true. Nevertheless, in Section 4.7, we shall establish the general non-separable case of Theorem 4.18 (see Theorem 4.23) using different methods.

4.6 $M(r,s)$-ideals of compact operators are separably determined

It is well-known that M-ideals of compact operators are separably determined [47]: *if a Banach space X has the MCAP and $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$ for all separable closed subspaces E of X having the MCAP, then $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$.* This theorem and its proof have served as a prototype to obtain similar results on certain general approximations of the identity [49] (see also [48]) and ideals of compact operators having Phelps's uniqueness property U [54]. The next result shows that $M(r,s)$-ideals of compact operators are also separably determined. For its proof, we shall develop ideas from [47] and [49, proofs of Lemmas 3.2 and 4.2] but (following an idea in [54, proofs of Theorems 2.2 and 2.3]) there are no precise ε-nets of certain compact subsets. One inconvenience to be overcome is that in the $M(r,s)$-ideal case, unlike the M-ideal and property U cases, the ideal projection need not be unique.

Definition 4.19. If a Banach space X is an $M(r,s)$-ideal in X^{**} with respect to the canonical ideal projection on X^{***}, then we say that X *satisfies the $M(r,s)$-inequality*.

Theorem 4.20. *Let X and Y be Banach spaces. Let positive numbers $r \leq 1$ and s satisfy $r+s>1$, and let $\varrho, \sigma \in (0,1]$ satisfy $\varrho+\sigma>1$. Suppose that Y satisfies the $M(\varrho,\sigma)$-inequality and has the MCAP. If $\mathcal{K}(E,F)$ is an $M(r,s)$-ideal in $\mathcal{L}(E,F)$ with respect to an ideal projection preserving elementary functionals for all separable closed subspaces E of X and F of Y such that F has the MCAP, then $\mathcal{K}(X,Y)$ is an $M(r,s)$-ideal in $\mathcal{L}(X,Y)$.*

For proving Theorem 4.20, we shall need the following auxiliary result.

Proposition 4.21 (see [7, Proposition 2.1] and [49, proof of Corollary 1.7])**.** *Let $r,s \in (0,1]$. If a Banach space X satisfies the $M(r,s)$-inequality (in particular, if X has property $M^*(r,s)$) for $r+s>1$, then X^* has the Radon–Nikodým property and every MCAI of X is shrinking.*

Proof of Theorem 4.20. We are going to apply Theorem 4.10. Let (K_α) be an MCAI of Y. By Proposition 4.21, (K_α) is shrinking and Y^* has the Radon–Nikodým property. Further, $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$ with respect to an ideal projection preserving elementary functionals (see Example 4.6) and $K_\alpha T \longrightarrow T$ in the dual weak operator topology for every $T \in B_{\mathcal{L}(X,Y)}$.

Assume for contradiction that $\mathcal{K}(X,Y)$ is not an $M(r,s)$-ideal in $\mathcal{L}(X,Y)$. Then condition (b) of Theorem 4.10 is not satisfied: there are $\varepsilon>0$, $S \in B_{\mathcal{K}(X,Y)}$, $T \in B_{\mathcal{L}(X,Y)}$, and α_0 such that

$$\|rS + s(T-KT)\| > 1+3\varepsilon \quad \forall K \in \operatorname{conv}\{K_\alpha \colon \alpha \geq \alpha_0\}.$$

We shall define separable closed subspaces E of X and F of Y such that F has the MCAP, but $\mathcal{K}(E,F)$ cannot be an $M(r,s)$-ideal in $\mathcal{L}(E,F)$ with respect to any ideal projection preserving elementary functionals. This will contradict the assumption and complete the proof.

To begin, let $E_0=\{0\}\subset X$ and $F_0=\{0\}\subset Y$. Pick $x_0 \in B_X$ such that

$$\|(rS+s(T-K_{\alpha_0}T))x_0\| > \|rS+s(T-K_{\alpha_0}T)\| - \varepsilon > 1+2\varepsilon.$$

Denote $E_1 = E_0 \cup \{x_0\}$ and $F_1 = F_0 \cup K_{\alpha_0}(F_0) \cup S(E_1) \cup T(E_1)$. Then choose $\alpha_1 \geq \alpha_0$ such that

$$\|K_{\alpha_1}y - y\| < 1 \quad \forall y \in F_1.$$

Also choose a finite ε/s-net Λ_1 in $\operatorname{conv}\{K_{\alpha_0}, K_{\alpha_1}\}$, and for every $L \in \Lambda_1$ pick $x_L \in B_X$ such that

$$\|(rS + s(T - LT))x_L\| > \|rS + s(T - LT)\| - \varepsilon > 1 + 2\varepsilon.$$

Denote

$$E_2 = E_1 \cup \{x_L \colon L \in \Lambda_1\}$$

and

$$F_2 = F_1 \cup K_{\alpha_0}(F_1) \cup K_{\alpha_1}(F_1) \cup S(E_2) \cup T(E_2).$$

Continuing similarly, we obtain, for all $n \in \mathbb{N}$, an index α_n, a finite ε/s-net Λ_n in $\operatorname{conv}\{K_{\alpha_0}, \ldots, K_{\alpha_n}\}$, a finite subset $\{x_L \colon L \in \Lambda_n\} \subset B_X$ such that

$$\|(rS + s(T - LT))x_L\| > 1 + 2\varepsilon, \quad L \in \Lambda_n,$$

and finite subsets $E_n \subset X$ and $F_n \subset Y$ such that

$$E_{n+1} = E_n \cup \{x_L \colon L \in \Lambda_n\},$$

$$F_{n+1} = F_n \cup K_{\alpha_0}(F_n) \cup \ldots \cup K_{\alpha_n}(F_n) \cup S(E_{n+1}) \cup T(E_{n+1}),$$

and

$$\|K_{\alpha_n} y - y\| < \frac{1}{n} \quad \forall y \in F_n.$$

Denote $E = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} E_n$ and $F = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} F_n$. It can be easily seen that $S(E) \subset F$, $T(E) \subset F$, $K_{\alpha_n}(F) \subset F$ for all $n \in \mathbb{N}$, and $K_{\alpha_n} y \longrightarrow y$ for all $y \in F$. Consider $S|_E \in B_{\mathcal{K}(E,F)}$, $T|_E \in B_{\mathcal{L}(E,F)}$, and $K_{\alpha_n}|_F \in B_{\mathcal{K}(F)}$.

Since Y satisfies the $M(\varrho, \sigma)$-inequality, also F does (this fact, which is similar to that of the M-embedded spaces (see, e.g., [23, p. 111]), was observed in [4, Proposition 2.1]). Consequently, as in the beginning of the proof, we are in position to apply Theorem 4.10 to $\mathcal{K}(E,F)$ in $\mathcal{L}(E,F)$. According to Theorem 4.10, if $\mathcal{K}(E,F)$ were an $M(r,s)$-ideal in $\mathcal{L}(E,F)$ with respect to an ideal projection preserving elementary functionals, then there would exist $K \in \operatorname{conv}\{K_{\alpha_1}, \ldots, K_{\alpha_n}\}$, for some $n \in \mathbb{N}$, such that

$$\|(rS + s(T - KT))|_E\| \leq 1 + \varepsilon.$$

Let $L \in \Lambda_n$ satisfy $\|K - L\| < \varepsilon/s$. Then

$$\begin{aligned} 1 + 2\varepsilon &< \|(rS + s(T - LT))|_E\| \\ &\leq \|(rS + s(T - KT))|_E\| + \varepsilon \\ &\leq 1 + 2\varepsilon, \end{aligned}$$

a contradiction. □

Remark 4.22. From the proof of Theorem 4.20 it is clear that the assumption "Y satisfies the $M(\varrho,\sigma)$-inequality with $\varrho+\sigma>1$" can be replaced by any assumption guaranteeing that Y^* has the Radon–Nikodým property and every MCAI of any closed subspace F of Y is shrinking.

4.7 Improved parameters of $M(r,s)$-ideals for the general case

Let us now turn to the promised main results of the present chapter.

Theorem 4.23. *Let X and Y be Banach spaces. Assume that Y has the MCAP. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1+s_1>1$ and $r_2+s_2>1$. If X has property $M^*(r_1,s_1)$ and Y has property $M^*(r_2,s_2)$, then $\mathcal{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathcal{L}(X,Y)$.*

Proof. Property $M^*(r_2,s_2)$ of Y implies that Y satisfies the $M(r_2,s_2)$-inequality (see Proposition 4.21). Let $E\subset X$ and $F\subset Y$ be separable closed subspaces, and assume that F has the MCAP. Property $M^*(r,s)$ is inherited by closed subspaces (see [49, p. 2804]). Hence, E has property $M^*(r_1,s_1)$ and F has property $M^*(r_2,s_2)$. From Proposition 4.12, we know that then every $T\in B_{\mathcal{L}(E,F)}$ has property $M^*(r_1r_2,s_1s_2)$. Since F is separable and has the MCAP, it has a metric compact approximating sequence which is shrinking, because F satisfies the $M(r_2,s_2)$-inequality (see Proposition 4.21). It follows that F^* is separable. Applying Theorem 4.16, we get that $\mathcal{K}(E,F)$ is an $M(r_1r_2,s_1s_2)$-ideal in $\mathcal{L}(E,F)$ with respect to an ideal projection preserving elementary functionals. Hence, according to Theorem 4.20, $\mathcal{K}(X,Y)$ is an $M(r_1r_2,s_1s_2)$-ideal in $\mathcal{L}(X,Y)$. □

A basic theorem of the theory of M-ideals of compact operators asserts that *$\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ if and only if X has property (M^*) and the MCAP.* It was established in [32] for separable X, in [34] for reflexive X, and extended to arbitrary (non-separable) X in [47]. A self-contained and "the shortest known proof" (we quote [39] here) is given in [49], another self-contained proof based on a new structure theorem for Borel probability measures can be found in a very recent paper [39]. The above theorem together with Theorem 2.28 immediately yields a more general result: *$\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$ whenever X and Y have property (M^*), and Y has the MCAP.* A self-contained measure-theoretic proof of this result is given in [39]. Keeping in mind that property (M^*) is precisely property

$M^*(1,1)$, Theorem 4.23 contains the latter result as a special case, yielding another self-contained proof of it. It would be interesting to study whether the measure-theoretic approach by Nygaard and Põldvere [39] could be used to give an alternative proof of Theorem 4.23.

For the M-ideal prototype of the next result, see Theorem 2.28.

Corollary 4.24. *Let X and Y be Banach spaces. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1 + s_1 > 1$ and $r_2 + s_2/2 > 1$. If X has property $M^*(r_1, s_1)$ and $\mathcal{K}(Y)$ is an $M(r_2, s_2)$-ideal in $\mathcal{I}(Y)$, then $\mathcal{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathcal{L}(X,Y)$.*

Proof. This is immediate from Theorem 4.23 and Lemma 3.14. □

Corollary 4.24 improves Theorem 3.19 from parameters $r_1r_2^2$ and $s_1s_2^2$ to parameters r_1r_2 and s_1s_2.

The next theorem, which is one of the main results of the current chapter, is also immediate from Theorem 4.23 and Lemma 3.14. It improves parameters of Theorem 3.21.

Theorem 4.25. *Let X and Y be Banach spaces. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1+s_1/2 > 1$ and $r_2+s_2/2 > 1$. If $\mathcal{K}(X)$ is an $M(r_1, s_1)$-ideal in $\mathcal{I}(X)$ and $\mathcal{K}(Y)$ is an $M(r_2, s_2)$-ideal in $\mathcal{I}(Y)$, then $\mathcal{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathcal{L}(X,Y)$.*

From Theorem 4.25 and Corollary 3.17 we immediately get the desired extension of Theorem 4.18 to arbitrary (non-separable) spaces. Let us spell it out.

Theorem 4.26. *Let X and Y be Banach spaces. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1+s_1/2 > 1$ and $r_2+s_2/2 > 1$. If $\mathcal{K}(X)$ is an $M(r_1, s_1)$-ideal in $\mathcal{L}(X)$ and $\mathcal{K}(Y)$ is an $M(r_2, s_2)$-ideal in $\mathcal{L}(Y)$, then $\mathcal{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathcal{L}(X,Y)$.*

Remark that Theorem 4.26 extends Corollary 2.29 from M-ideals to $M(r,s)$-ideals and improves Corollary 3.22 in the sense of parameters.

Corollary 4.27. *Let X be a Banach space and let $r, s \in (0,1]$ satisfy $r + s/2 > 1$. If $\mathcal{K}(X)$ is an $M(r,s)$-ideal in $\mathcal{I}(X)$, then $\mathcal{K}(X)$ is an $M(r^2, s^2)$-ideal in $\mathcal{L}(X)$.*

Corollary 4.27 improves Corollary 3.23 where the claim is that $\mathcal{K}(X)$ is an $M(r^3, s^3)$-ideal in $\mathcal{L}(X)$.

4.8 Applications to the theory of M-ideals

Next, we will make several corollaries from Theorems 4.16, 4.20, and 4.23 which complete and improve some well-known results on M-ideals.

According to a theorem due to Kalton and Werner [32, Theorem 6.3], *if X is Banach space having an unconditional shrinking compact approximating sequence and Y is a Banach space such that every $T \in S_{\mathcal{L}(X,Y)}$ has property (M), then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.* The following immediate special case of our Theorem 4.16 completes the Kalton–Werner theorem showing that the unconditionality assumption is superfluous if one assumes that Y^* has the Radon–Nikodým property and strengthens property (M) up to (M^*).

Corollary 4.28. *Let X and Y be Banach spaces. Suppose that X^{**} or Y^* has the Radon–Nikodým property and that X has a shrinking compact approximating sequence. If every $T \in S_{\mathcal{L}(X,Y)}$ has property (M^*), then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.*

The dual version of the Kalton–Werner theorem states (see [32, p. 171] and [27, pp. 54–55]): *if Y is a Banach space having an unconditional shrinking compact approximating sequence and X is a Banach space such that every $T \in S_{\mathcal{L}(X,Y)}$ has property (M^*), then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.* The following immediate special case of Theorem 4.16 improves this theorem showing that the unconditionality assumption is superfluous.

Corollary 4.29. *Let X and Y be Banach spaces. Suppose that Y has a shrinking compact approximating sequence. If every $T \in S_{\mathcal{L}(X,Y)}$ has property (M^*), then $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.*

Definition 4.30. A Banach space Y has *property* (wM^*) (introduced by Lima [34]) if

$$\limsup_{\nu} \|y^*_\nu\| = \limsup_{\nu} \|2y^* - y^*_\nu\|,$$

whenever $y^* \in Y^*$ and $(y^*_\nu) \subset Y^*$ is a bounded net converging weak* to y^* in Y^*.

Corollary 4.31 below is an improvement of a theorem due to John and Werner [27, Theorem 2.4]: its assumption that Y has an unconditional shrinking compact approximating sequence (which easily implies property (wM^*) of Y) will be weakened up to the assumption that Y has property (wM^*), showing, e.g., that there is no need for a separability requirement of Y^*.

If Y is separable, then again (due to the weak* metrizability of bounded subsets of Y^*) the sequential version of (wM^*) is equivalent to property

(wM^*), and the same concerns the property of Y (introduced by John and Werner [27]) described in the following.

Corollary 4.31. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let Y be a Banach space having property (wM^*) and let*

$$\limsup_\nu (\|y^*\|^q + \|y^*_\nu\|^q)^{1/q} \le \limsup_\nu \left(\frac{\|y^* + y^*_\nu\|^q + \|y^* - y^*_\nu\|^q}{2} \right)^{1/q},$$

whenever $y^* \in Y^*$ and $(y^*_\nu) \subset Y^*$ is a bounded net converging weak to null in Y^*. Then $\mathcal{K}(\ell_p, Y)$ is an M-ideal in $\mathcal{L}(\ell_p, Y)$.*

Proof. Based on Corollary 4.28, it is sufficient to show that every $T \in S_{\mathcal{L}(\ell_p,Y)}$ has property (M^*).

Let $x^* \in \ell_q$ and $y^* \in Y^*$ be such that $\|x^*\| \le \|y^*\|$, and let $(y^*_\nu) \subset Y^*$ be a bounded net such that $y^*_\nu \longrightarrow 0$ weak*. Then for every $T \in S_{\mathcal{L}(\ell_p,Y)}$,

$$\begin{aligned} \limsup_\nu \|x^* + T^* y^*_\nu\| &= \limsup_\nu (\|x^*\|^q + \|T^* y^*_\nu\|^q)^{1/q} \\ &\le \limsup_\nu (\|y^*\|^q + \|y^*_\nu\|^q)^{1/q} \\ &\le \limsup_\nu \left(\frac{\|y^* + y^*_\nu\|^q + \|y^* - y^*_\nu\|^q}{2} \right)^{1/q}. \end{aligned}$$

Since Y has property (wM^*),

$$\limsup_\nu \|y^* + y^*_\nu\| = \limsup_\nu \|y^* - y^*_\nu\|.$$

Hence,

$$\limsup_\nu \|x^* + T^* y^*_\nu\| \le \limsup_\nu \|y^* + y^*_\nu\|.$$

□

It is well-known to be true that if Y has property U in its bidual Y^{**}, then Y^* has the Radon–Nikodým property and every MCAI of any closed subspace F of Y is shrinking (see [50, Corollary 5] and, e.g., [54, Lemma 2.1]).

From Theorem 4.20, by applying Remark 4.22, we obtain the following result which shows that M-ideals of compact operators $\mathcal{K}(X, Y)$ are separably determined not only for $X = Y$ but also for *distinct* spaces X and Y.

Corollary 4.32. *Let X and Y be Banach spaces. Suppose that Y has property U in its bidual and has the MCAP. If $\mathcal{K}(E, F)$ is an M-ideal in $\mathcal{L}(E, F)$ for all separable closed subspaces E of X and F of Y such that F has the MCAP, then $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$.*

Proof. This is immediate from Theorem 4.20 and Remark 4.22 because M-ideals enjoy the unique ideal property, and under the assumptions on E and F, $\mathcal{K}(E,F)$ is an ideal in $\mathcal{L}(E,F)$ with respect to an ideal projection preserving elementary functionals (see Example 4.5 or 4.6). □

Remark 4.33. The prototype of Corollary 4.32 is [54, Theorem 2.3] asserting that property U of $\mathcal{K}(X,Y)$ in $\mathcal{L}(X,Y)$ is separably determined.

Definition 4.34. A Banach space X is said to have the λ-*commuting BCAP* (with $\lambda \geq 1$) if X has a CAI (K_α) such that $K_\alpha K_\beta = K_\beta K_\alpha$ for all indexes α and β, and $\limsup \|K_\alpha\| \leq \lambda$.

It follows from [52, Theorem 4.4] that X has the MCAP whenever X satisfies the $M(r,s)$-inequality and has the λ-commuting BCAP with $\lambda < r+s$. Therefore we can make the following essential remark.

Remark 4.35. The assumption of the MCAP of Y in Theorem 4.23 (and also in Theorem 4.20) can be replaced by the assumption that Y has the λ-commuting BCAP with $\lambda < \varrho + \sigma$ (and $\lambda < r_2 + s_2$, respectively).

Both results described in Remark 4.35 are new for M-ideals. Since a corollary of Theorem 4.23 represents a version of the basic theorem of the theory of M-ideals of compact operators, let us spell it out as follows.

Corollary 4.36. *Let X and Y be Banach spaces having property (M^*). If Y has the λ-commuting BCAP with $\lambda < 2$, then $\mathcal{K}(X,Y)$ is an M-ideal $\mathcal{L}(X,Y)$.*

Remark that the special case of Corollary 4.36 when $X = Y$ is proven in [52, Corollary 4.10].

4.9 Applications to the general structure of Banach spaces

Theorem 4.20 allows us to conclude some results concerning the general structure of Banach spaces.

There exist infinite-dimensional Banach spaces X and Y for which $\mathcal{K}(X,Y) = \mathcal{L}(X,Y)$. This is the case, for instance, when $X = \ell_p$, $Y = \ell_q$ with $p > q$ (Pitt's theorem); $X = \ell_p$, $Y = d(w,q)$ with $p > q$ and $w \notin \ell_{p/(p-q)}$ [44] (other Pitt's type theorems for Lorentz and Orlicz sequence spaces can be found in

[2]). A consequence of Theorem 4.20 is that the property $\mathcal{K}(X, Y) = \mathcal{L}(X, Y)$ is also separably determined.

Corollary 4.37. *Let X and Y be Banach spaces. Suppose that Y has property U in its bidual and has the MCAP. If $\mathcal{K}(E, F) = \mathcal{L}(E, F)$ for all separable closed subspaces E of X and F of Y such that F has the MCAP, then $\mathcal{K}(X, Y) = \mathcal{L}(X, Y)$.*

Proof. Apply Remark 4.22 and Theorem 4.20 to any $s > 1$. □

It is a well-known consequence of the Eberlein–Šmulian theorem that a Banach space is reflexive whenever all its separable closed subspaces are (for an alternative easy proof see [21, Corollary 2]). The next corollary shows that for $\mathcal{L}(X, Y)$ to be reflexive, it suffices that the separable subspaces of the form $\mathcal{K}(E, F)$ are reflexive.

Corollary 4.38. *Let X and Y be reflexive Banach spaces. Suppose that Y has the CAP. If $\mathcal{K}(E, F)$ is reflexive for all separable closed subspaces E of X and F of Y such that F has the CAP, then $\mathcal{L}(X, Y)$ is reflexive.*

Proof. It is known (see [9] or [16]) that a reflexive Banach space with the CAP actually has the MCAP. Since F has the CAP, by [16, Corollary 1.3], $\mathcal{K}(E, F)^{**} = \mathcal{L}(E, F)$, and by this identification, $j_{\mathcal{K}(E,F)}(T) = T$, for all $T \in \mathcal{K}(E, F)$. Since $\mathcal{K}(E, F)$ is reflexive, we have $\mathcal{K}(E, F) = \mathcal{L}(E, F)$. By Corollary 4.37, $\mathcal{K}(X, Y) = \mathcal{L}(X, Y)$. Hence, according to a classical theorem proved independently by Heinrich [24] and Kalton [30], $\mathcal{L}(X, Y)$ is reflexive. Alternatively, we have as above, $\mathcal{K}(X, Y)^{**} = \mathcal{L}(X, Y) = \mathcal{K}(X, Y)$, meaning that $\mathcal{K}(X, Y)$ is reflexive, and also so is $\mathcal{L}(X, Y)$. □

Chapter 5

u-Ideals

Let us apply now the methods developed for $M(r,s)$-ideals in Chapter 4 to the case of u-ideals. Similarly to M- and $M(r,s)$-ideals, we will prove that u-ideals are separably determined. However, the property of creating new u-ideals of compact operators behaves somewhat differently compared with M- and $M(r,s)$-ideals. This chapter is based on [26]

5.1 The definition

Let us start with introducing the notion.

Definition 5.1. Let $\mathcal{L}$ be a Banach space. A closed subspace $\mathcal{K} \subset \mathcal{L}$ is said to be a *u-ideal* in $\mathcal{L}$ if $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to some ideal projection P satisfying

$$\|I_{\mathcal{L}^*} - 2P\| = 1.$$

Equivalently, it can be said that for $\mathcal{K}^\perp$ there exists a closed subspace $\mathcal{M} \subset \mathcal{L}^*$ such that $\mathcal{K}^\perp \oplus \mathcal{M} = \mathcal{L}^*$ and

$$\|p+q\| = \|p-q\|$$

for every $p \in \mathcal{K}^\perp$, $q \in \mathcal{M}$.

Note that if $\mathcal{K}$ is an M-ideal in $\mathcal{L}$, then $\mathcal{K}$ is a u-ideal in $\mathcal{L}$. Indeed, in the case of M-ideal we have for every $p \in \mathcal{K}^\perp$ and $q \in \mathcal{M}$ the norm condition

$$\|p+q\| = \|p\| + \|q\|,$$

but also

$$\|p - q\| = \|p\| + \|q\| .$$

Thus, u-ideals are a generalization of M-ideals.

The notion of a u-ideal was introduced by P. G. Casazza and N. J. Kalton in [8]. The letter "u" comes from the word "unconditional" and refers to the existence of a special kind of unconditionally converging series (see [8, Theorem 3.8] and [8, Theorem 3.9]).

5.2 u-Ideals and the ideal projection

Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$. If (K_α) is a shrinking MCAI of X (respectively, a shrinking MCAI of Y), then due to Lemma 2.23, $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to the Johnson projection P.

Note that P satisfies the condition $\|I_{\mathcal{L}^*} - 2P\| = 1$ whenever we assume that $\limsup_\alpha \|I_X - 2K_\alpha\| \le 1$ (respectively, $\limsup_\alpha \|I_Y - 2K_\alpha\| \le 1$). Indeed, let (K_α) be a shrinking MCAI of X (respectively, a shrinking MCAI of Y), then for every $\varepsilon > 0$ there exist $f \in S_{\mathcal{L}^*}$ and $T \in S_{\mathcal{L}}$ such that

$$\begin{aligned}\|I_{\mathcal{L}^*} - 2P\| - \varepsilon &\le |f(T) - 2\lim_\alpha f(TK_\alpha)| \\ &= |\lim_\alpha f(T - 2TK_\alpha)| \\ &\le \|f\|\,\|T\| \limsup_\alpha \|I_X - 2K_\alpha\| \le 1\end{aligned}$$

(respectively,

$$\|I_{\mathcal{L}^*} - 2P\| - \varepsilon \le |f(T) - 2\lim_\alpha f(K_\alpha T)| \le 1).$$

Thus, we can state that $\mathcal{K}$ is a u-ideal in $\mathcal{L}$ if there exists a shrinking MCAI (K_α) either in X or in Y such that $\limsup_\alpha \|I - 2K_\alpha\| \le 1$, where I denotes, respectively, the identity operator of X or of Y. Following [40, proof of Theorem 1] or [43, Theorem 1.3], we can formulate the result even in more general terms.

Lemma 5.2 (cf. [40, proof of Theorem 1] or [43, Theorem 1.3]). *Let $\mathcal{L}$ be a Banach space and let $\mathcal{K}$ be a closed subspace of $\mathcal{L}$. Suppose that $(U_\alpha) \subset \mathcal{L}(\mathcal{L})$ is a net with*

$$\limsup_\alpha \|U_\alpha\| \le 1$$

and

$$\operatorname{ran} U_\alpha \subset \mathcal{K} \quad \forall \alpha, \quad \lim_\alpha g(U_\alpha y) = g(y) \quad \forall y \in \mathcal{K},\ \forall g \in \mathcal{K}^*.$$

Then there exists a projection $P \in \mathcal{L}(\mathcal{L}^*)$ *with* $\|P\| = 1$ *and* $\ker P = \mathcal{K}^\perp$ *such that*

$$(Pf)(x) = \lim_\beta f(U_{\alpha(\beta)}x), \quad f \in \mathcal{L}^*,\ x \in \mathcal{L},$$

for some subnet $(U_{\alpha(\beta)})$ *of* (U_α). *Moreover, if* $\limsup_\alpha \|I_\mathcal{L} - 2U_\alpha\| \le 1$, *then* $\|I_{\mathcal{L}^*} - 2P\| \le 1$.

Remark 5.3. If we assume that $\limsup_\alpha \|I_\mathcal{L} - 2U_\alpha\| \le 1$, then due to the estimate

$$2\|U_\alpha\| \le \|2U_\alpha - I_\mathcal{L}\| + 1$$

we can conclude that $\limsup_\alpha \|U_\alpha\| \le 1$.

Proof of Lemma 5.2. Since $\limsup_\alpha \|U_\alpha\| \le 1$, we may assume without loss of generality that $\sup_\alpha \|U_\alpha\| < \infty$. By the Banach–Alaoglu theorem (see Remark 2.25), we can extract from (U_α) a subnet $(U_{\alpha(\beta)})$ which converges weak* in $\mathcal{L}(\mathcal{L})^{**}$.

Fix $x \in \mathcal{L}$ and $f \in \mathcal{L}^*$. Let us define

$$\varphi(A) = f(Ax), \quad A \in \mathcal{L}(\mathcal{L}).$$

Clearly $\varphi \in \mathcal{L}(\mathcal{L})^*$. Since $\operatorname{ran} U_{\alpha(\beta)} \subset \mathcal{K}$ for all β, we have

$$\varphi(U_{\alpha(\beta)}) = f(U_{\alpha(\beta)}x) = g(U_{\alpha(\beta)}x),$$

where $g = f|_\mathcal{K}$. The limit $\lim_\beta \varphi(U_{\alpha(\beta)})$ exists because $(U_{\alpha(\beta)})$ converges weak* in $\mathcal{L}(\mathcal{L})^{**}$ and thus, also the limit

$$\lim_\beta f(U_{\alpha(\beta)}x) = \lim_\beta \varphi(U_{\alpha(\beta)})$$

exists. If we define

$$(Pf)(x) = \lim_\beta f(U_{\alpha(\beta)}x), \quad f \in \mathcal{L}^*,\ x \in \mathcal{L},$$

then $P \in \mathcal{L}(\mathcal{L}^*)$, and $\|P\| \le 1$ since for every $f \in \mathcal{L}^*$ and $x \in \mathcal{L}$

$$|(Pf)(x)| = |\lim_\beta f(U_{\alpha(\beta)}x)| \le \limsup_\beta \|f\| \left\|U_{\alpha(\beta)}\right\| \|x\| \le \|f\| \|x\|.$$

We have for every $f \in \mathcal{L}^*$ and $y \in \mathcal{K}$

$$(f - Pf)(y) = f(y) - \lim_\beta g(U_{\alpha(\beta)}y) = g(y) - g(y) = 0,$$

where $g = f|_{\mathcal{K}}$ and thus, $f - Pf \in \mathcal{K}^{\perp}$. The operator P is a projection with $\ker P = \mathcal{K}^{\perp}$ if $\mathcal{K}^{\perp} \subset \ker P$. The latter holds since $\operatorname{ran} U_{\alpha(\beta)} \subset \mathcal{K}$ for all β and

$$(Pf)(x) = \lim_{\beta} f(U_{\alpha(\beta)}x) = \lim_{\beta} 0 = 0 \quad \forall x \in \mathcal{L}$$

for all $f \in \mathcal{K}^{\perp}$.

Let us assume that $\limsup_{\alpha} \|I_{\mathcal{L}} - 2U_{\alpha}\| \leq 1$. Then, in the same way as in the beginning of this section, for every $\varepsilon > 0$ there exist $f \in S_{\mathcal{L}^*}$ and $x \in S_{\mathcal{L}}$ such that

$$\|I_{\mathcal{L}^*} - 2P\| - \varepsilon \leq |f(x) - 2\lim_{\beta} f(U_{\alpha(\beta)}x)| \leq 1.$$

□

Remark 5.4. Johnson's lemma (see Lemma 2.23) is a special case of Lemma 5.2. Indeed, in the first part of Lemma 2.23 we assume the existence of a net $(K_{\alpha}) \subset B_{\mathcal{K}(X)}$ such that

$$K_{\alpha}^* x^* \longrightarrow x^* \quad \forall x^* \in X^*.$$

Let us define

$$U_{\alpha}(A) = AK_{\alpha}, \quad A \in \mathcal{L}(X, Y).$$

Then clearly $U_{\alpha} \in \mathcal{L}(\mathcal{L}(X, Y))$ and $\operatorname{ran} U_{\alpha} \subset \mathcal{K}(X, Y)$ for all α. Since $\|K_{\alpha}\| \leq 1$ for all α, we have

$$\|U_{\alpha}\| = \sup_{\|A\|=1} \|U_{\alpha}(A)\| = \sup_{\|A\|=1} \|AK_{\alpha}\| \leq \sup_{\|A\|=1} \|A\| \, \|K_{\alpha}\| \leq 1 \quad \forall \alpha.$$

Moreover, due to Lemma 2.19, for all $S \in \mathcal{K}(X, Y)$ and $g \in \mathcal{K}(X, Y)^*$, we have

$$\lim_{\alpha} g(U_{\alpha}(S)) = \lim_{\alpha} g(SK_{\alpha}) = g(\lim_{\alpha} SK_{\alpha}) = g(S).$$

For the second part of Lemma 2.23, where $(K_{\alpha}) \subset B_{\mathcal{K}(Y)}$ and

$$K_{\alpha} y \longrightarrow y \quad \forall y \in Y,$$

we define

$$U_{\alpha}(A) = K_{\alpha}A, \quad A \in \mathcal{L}(X, Y).$$

Then similarly $U_{\alpha} \in \mathcal{L}(\mathcal{L}(X, Y))$, $\operatorname{ran} U_{\alpha} \subset \mathcal{K}(X, Y)$ and $\|U_{\alpha}\| \leq 1$ for all α, and for all $S \in \mathcal{K}(X, Y)$, $g \in \mathcal{K}(X, Y)^*$ we have $\lim_{\alpha} g(U_{\alpha}(S)) = \lim_{\alpha} g(S)$.

Corollary 5.5. *Let X and Y be Banach spaces, and let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X, Y)$ containing $\mathcal{K} := \mathcal{K}(X, Y)$. If X^* has the CAP with conjugate operators and*

$$\limsup_{\alpha} \|I_X - 2K_\alpha\| \leq 1,$$

where (K_α) is a shrinking CAI of X, then $\mathcal{K}$ is a u-ideal in $\mathcal{L}$.

Proof. Let us define

$$U_\alpha(A) = AK_\alpha, \quad A \in \mathcal{L},$$

where (K_α) is a shrinking CAI of X. Then $U_\alpha \in \mathcal{L}(\mathcal{L})$, $\operatorname{ran} U_\alpha \subset \mathcal{K}$ for all α, and $\lim_\alpha g(U_\alpha S) = g(S)$ for all $g \in \mathcal{K}^*$, $S \in \mathcal{K}$ (see Remark 5.4). We have

$$\begin{aligned} \|I_{\mathcal{L}} - 2U_\alpha\| &= \sup_{\|A\|=1} \|(I_{\mathcal{L}} - 2U_\alpha)(A)\| \\ &= \sup_{\|A\|=1} \|A - 2AK_\alpha\| \\ &\leq \sup_{\|A\|=1} \|A\| \, \|I_X - 2K_\alpha\| \\ &= \|I_X - 2K_\alpha\| \end{aligned}$$

and thus, $\limsup_\alpha \|I_{\mathcal{L}} - 2U_\alpha\| \leq 1$. Due to Lemma 5.2, $\mathcal{K}$ is a u-ideal in $\mathcal{L}$. □

Corollary 5.6. *Let X and Y be Banach spaces, and let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X, Y)$ containing $\mathcal{K} := \mathcal{K}(X, Y)$. If Y has the CAP and*

$$\limsup_{\alpha} \|I_Y - 2K_\alpha\| \leq 1,$$

where (K_α) is a CAI of Y, then $\mathcal{K}$ is a u-ideal in $\mathcal{L}$.

Proof. Similarly to the proof of Corollary 5.5, we define

$$U_\alpha(A) = K_\alpha A, \quad A \in \mathcal{L},$$

where (K_α) is a CAI of Y. Then also $U_\alpha \in \mathcal{L}(\mathcal{L})$, $\operatorname{ran} U_\alpha \subset \mathcal{K}$ for all α, $\lim_\alpha g(U_\alpha S) = g(S)$ for all $g \in \mathcal{K}^*$, $S \in \mathcal{K}$, and

$$\limsup_{\alpha} \|I_{\mathcal{L}} - 2U_\alpha\| \leq \limsup_{\alpha} \|I_Y - 2K_\alpha\| \leq 1.$$

Due to Lemma 5.2, $\mathcal{K}$ is a u-ideal in $\mathcal{L}$. □

Corollaries 5.5 and 5.6 are essentially known from [8] (see [8, Theorem 3.9]).

5.3 u-Ideals and the ideal projection preserving elementary functionals

We can describe u-ideals of compact operators in terms of ideal projections preserving elementary functionals and bounded nets converging in the dual weak operator topology as follows.

Theorem 5.7. *Let X and Y be Banach spaces, and suppose that X^{**} or Y^* has the Radon–Nikodým property. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$, and suppose that $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to an ideal projection P preserving elementary functionals. If for every operator $T \in S_{\mathcal{L}}$ there exists a bounded net $(T_\alpha) \subset \mathcal{K}$ such that $T_\alpha \longrightarrow T$ in the dual weak operator topology, then the following assertions are equivalent.*

(a) *$\mathcal{K}$ is a u-ideal in $\mathcal{L}$ with respect to P.*

(b) *For every $\varepsilon > 0$, $T \in B_{\mathcal{L}}$, and every index α (in the corresponding net (T_α)), there exists*

$$K \in \operatorname{conv}\{T_\beta \colon \beta \geq \alpha\}$$

such that

$$\|T - 2K\| \leq 1 + \varepsilon.$$

(c) *For every $T \in S_{\mathcal{L}}$, there exists a net $(K_\nu) \subset \mathcal{K}$ such that $K_\nu \longrightarrow T$ in the dual weak operator topology and*

$$\limsup_{\nu} \|T - 2K_\nu\| \leq 1.$$

Theorem 5.7 for u-ideals is analogous to Theorem 4.10 for $M(r,s)$-ideals.

Proof of Theorem 5.7. (a) $\Rightarrow$ (b). If the conclusion is false, then there are $\varepsilon > 0$, $T \in B_{\mathcal{L}}$, and α such that for $C := \operatorname{conv}\{T_\beta \colon \beta \geq \alpha\}$, we have

$$2C \cap B(T, 1+\varepsilon) = \varnothing,$$

where $B(T, 1+\varepsilon)$ is the open ball with center T and radius $1+\varepsilon$. By the Hahn–Banach theorem, there exists $f \in S_{\mathcal{L}^*}$ such that

$$\begin{aligned} \operatorname{Re} f(T) - (1+\varepsilon) &= \inf\{\operatorname{Re} f(U) \colon U \in B(T, 1+\varepsilon)\} \\ &\geq \operatorname{Re} 2f(K) \quad \forall K \in C. \end{aligned}$$

Since $f - Pf \in \ker P = \mathcal{K}^\perp$ for every $f \in \mathcal{L}^*$, we have

$$\operatorname{Re} 2f(K) = \operatorname{Re} 2Pf(K) \quad \forall K \in C$$

because $C \subset \mathcal{K}$. Hence, by (a),

$$\begin{aligned} 1 + \varepsilon &\le \operatorname{Re} f(T) - \operatorname{Re} 2Pf(K) \\ &= \operatorname{Re}(f - 2Pf)(T) + \operatorname{Re} 2Pf(T - K) \\ &\le 1 + \operatorname{Re} 2Pf(T - K) \quad \forall K \in C. \end{aligned}$$

Since $Pf(T) = \lim_\alpha Pf(T_\alpha)$ (see Proposition 4.9), this implies that $\varepsilon \le 0$, a contradiction.

(b) $\Rightarrow$ (c). Consider the set of all pairs $\nu = (\varepsilon, \alpha)$, where $\varepsilon > 0$ and where (T_α) corresponds to T, directed in the natural way, and choose $K_\nu \in \operatorname{conv}\{T_\beta \colon \beta \ge \alpha\}$ from condition (b).

(c) $\Rightarrow$ (a). Fix $\varepsilon > 0$. Recall that $\|Pf\| = \|f|_{\mathcal{K}}\|$ where $f \in \mathcal{L}^*$ (see Lemma 2.21). We choose $f \in S_{\mathcal{L}^*}$ and $T \in S_{\mathcal{L}}$ so that

$$\|I_{\mathcal{L}^*} - 2P\| - \varepsilon \le (f - 2Pf)(T).$$

Let (K_ν) be given by (c). By passing to a subnet, we may assume that (K_ν) is bounded. By Proposition 4.9, $(Pf)(T) = \lim_\nu (Pf)(K_\nu) = \lim_\nu f(K_\nu)$ because $K_\nu \in \mathcal{K}$. It follows that

$$\begin{aligned} \|I_{\mathcal{L}^*} - 2P\| - \varepsilon &\le f(T) - 2 \lim_\nu f(K_\nu) \\ &= \lim_\nu f(T - 2K_\nu) \\ &\le \|f\| \limsup_\nu \|T - 2K_\nu\| \\ &\le 1. \end{aligned}$$

Thus, $\mathcal{K}$ is a u-ideal in $\mathcal{L}$. □

Remark 5.8. Historically condition similar to (c) of Theorem 5.7 were first considered in [49, Theorem 4.1, 5°].

Corollary 5.9. *Let X be a Banach space, and suppose that X^* or X^{**} has the Radon–Nikodým property. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X)$ containing $\mathcal{K} := \mathcal{K}(X)$ and I_X, and suppose that $\mathcal{K}$ is a u-ideal in $\mathcal{L}$ with respect to an ideal projection P preserving elementary functionals. If X^* has the BCAP with conjugate operators, then there exists a shrinking MCAI (K_ν) of X such that* $\limsup_\nu \|I_X - 2K_\nu\| \le 1$.

Proof. If X^* has the BCAP with conjugate operators, then, by passing to convex combinations, we may assume that X has a shrinking BCAI (S_α). Thus, for every $T \in S_{\mathcal{L}}$, we can define a net $(TS_\alpha) \subset \mathcal{K}$ which is bounded and $TS_\alpha \longrightarrow T$ in the dual weak operator topology. By Theorem 5.7, (a) $\Rightarrow$ (c), there exists $(K_\nu) \subset \mathcal{K}$ such that $K_\nu \longrightarrow I_X$ in the dual weak operator topology and

$$\limsup_\nu \|I_X - 2K_\nu\| \leq 1.$$

Since

$$x^{**}(K_\nu^* x^*) \longrightarrow x^{**}(I_X^* x^*) = x^{**}(I_{X^*} x^*) \quad \forall x^{**} \in X^{**},\ \forall x^* \in X^*,$$

we have $K_\nu^* \longrightarrow I_{X^*}$ in the weak operator topology. By the argument of convex combinations (see the proof of Lemma 2.17), we may assume that

$$K_\nu^* x^* \longrightarrow x^* \quad \forall x^* \in X^*, \quad K_\nu x \longrightarrow x \quad \forall x \in X.$$

It remains to observe that $\limsup \|K_\nu\| \leq 1$ (see Remark 5.3). $\square$

Remark 5.10. Note that in general from X^{**} having the Radon–Nikodým property does not follow that X^* has the Radon–Nikodým property (see, e.g., [16, Remark 1.8]) or consider, for example, c_0 and $c_0^{**} = \ell_\infty$, which do not have the Radon–Nikodým property, but $c_0^* = \ell_1$ has (see, e.g., [10, pp. 218–219]). Thus, the assumptions X^* having the Radon–Nikodým property and X^{**} having the Radon–Nikodým property are independent.

Now, we will see that the assumption "X^* has the BCAP with conjugate operators" of Corollary 5.9 is in fact redundant. The next proposition relies on a method developed in [7] (see [7, Proposition 3.2]), and allows us to give a necessary and sufficient condition for $\mathcal{K}(X)$ being a u-ideal in $\mathcal{L}(X)$ (see Corollary 5.12).

Proposition 5.11. *Let X be a Banach space, and suppose that X^* or X^{**} has the Radon–Nikodým property. Suppose that $\mathcal{K} = \mathcal{K}(X)$ is an ideal in $\mathcal{L} = \mathcal{L}(X)$ with respect to an ideal projection preserving elementary functionals. Then there exists a shrinking MCAI (K_α) of X.*

Proof. Let $P : \mathcal{L}^* \longrightarrow \mathcal{L}^*$ be an ideal projection preserving elementary functionals. Then $Pf = \Phi(f|_{\mathcal{K}})$ for every $f \in \mathcal{L}^*$ where $\Phi : \mathcal{K}^* \longrightarrow \operatorname{ran} P$ is the isometric isomorphism (see Lemma 2.21). Let us consider on $\mathcal{L}$ the weak topology induced by the ideal projection, i.e., the topology $\sigma(\mathcal{L}, \operatorname{ran} P)$. Then, due to the bipolar theorem, $B_{\mathcal{K}}$ is $\sigma(\mathcal{L}, \operatorname{ran} P)$-dense in $B_{\mathcal{L}}$ and thus, there exists a net $(K_\alpha) \subset B_{\mathcal{K}}$ such that $K_\alpha \longrightarrow I_X$ in the $\sigma(\mathcal{L}, \operatorname{ran} P)$-topology. By the Feder–Saphar theorem (see Theorem 4.3),

$$\mathcal{K}^* = \overline{\operatorname{span}}\{x^{**} \otimes x^*|_{\mathcal{K}} : x^{**} \in X^{**},\ x^* \in X^*,\ x^{**} \otimes x^* \in \mathcal{L}^*\},$$

and since P preserves elementary functionals, we have $\Phi(x^{**} \otimes x^*|_{\mathcal{K}}) = x^{**} \otimes x^*$ for every $x^{**} \in X^{**}$, $x^* \in X^*$. Hence,

$$\Phi(x^{**} \otimes x^*|_{\mathcal{K}})(K_\alpha - I_X) = x^{**}(K_\alpha^* - I_X^*)x^* \longrightarrow 0$$

for every $x^{**} \in X^{**}$, $x^* \in X^*$, i.e., $K_\alpha^* \longrightarrow I_X^*$ in the weak operator topology. By applying the argument of convex combinations (see Lemma 2.17 and Remark 2.18), we may assume that (K_α) is a shrinking MCAI of X. □

We can conclude from Proposition 5.11, Corollary 5.9, and Corollary 5.5 the following result.

Corollary 5.12. *Let X be a Banach space, and suppose that X^* or X^{**} has the Radon–Nikodým property. Then $\mathcal{K}(X)$ is a u-ideal in $\mathcal{L}(X)$ with respect to an ideal projection preserving elementary functionals if and only if X has a shrinking MCAI (K_α) with* $\limsup_\alpha \|I_X - 2K_\alpha\| \leq 1$.

Corollary 5.12 is essentially contained in [49, Corollary 4.5], however, we do not assume that $X^* = \overline{\operatorname{span}}(w^*\text{-}\operatorname{sexp} B_{X^*})$ as in [49], where $w^*\text{-}\operatorname{sexp} B_{X^*}$ denotes the set of all weak* strongly exposed points of B_{X^*}.

5.4 u-Ideals of compact operators creating new u-ideals

Recall that any Banach space X is an ideal in X^{**} with respect to the canonical projection.

Definition 5.13. If X is a u-ideal in X^{**} with respect to the canonical projection, then we say that X *is a u-ideal.*

Theorem 5.14. *Let X and Y be Banach spaces. If $\mathcal{K}(Y)$ is a u-ideal in $\mathcal{L}(Y)$ and Y is a u-ideal, or if $\mathcal{K}(X)$ is a u-ideal in $\mathcal{L}(X)$ and X is a u-ideal, then $\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$.*

Proof. Let us assume that $\mathcal{K}(Y)$ is a u-ideal in $\mathcal{L}(Y)$ and Y is a u-ideal, the proof in the other case is analogous.

By [49, Corollary 4.5, $1^{\circ\circ} \Rightarrow 3^{\circ}$], there exists a shrinking MCAI (K_α) of Y such that

$$\limsup_\alpha \|I_Y - 2K_\alpha\| \leq 1.$$

Thus, $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$ with respect to the Johnson projection P and (see the beginning of Section 5.2)

$$\left\|I_{\mathcal{L}(X,Y)^*} - 2P\right\| = 1.$$

□

A separable prototype for Theorem 5.14 is [15, Proposition 8.2].

5.5 Property (wM^*)

Analogously to property (M^*) concerning M-ideals or property $M^*(r,s)$ concerning $M(r,s)$-ideals, there is property (wM^*) for spaces concerning u-ideals. Let us recall the definition of property (wM^*).

Definition 5.15. A Banach space X has *property* (wM^*) if

$$\limsup_\nu \|x^*_\nu - 2x^*\| = \limsup_\nu \|x^*_\nu\|,$$

whenever $x^* \in X^*$ and $(x^*_\nu) \subset X^*$ is a bounded net converging weak* to x^* in X^*.

Property (wM^*) was introduced by Lima in [34]. The letter "w" in the notation (wM^*) comes from the word "weak". It denotes the fact that property (wM^*) follows from property (M^*). Indeed, if $(x^*_\nu) \subset X^*$ is a bounded net converging weak* to $x^* \in X^*$ in X^*, then $x^*_\nu - x^* \xrightarrow{w^*} 0$ and for X with property (M^*) we have

$$\begin{aligned}\limsup_\nu \|-x^* + (x^*_\nu - x^*)\| &= \limsup_\nu \|x^* + (x^*_\nu - x^*)\| \\ &= \limsup_\nu \|x^*_\nu\|\end{aligned}$$

and

$$\limsup_\nu \|-x^* + (x^*_\nu - x^*)\| = \limsup_\nu \|x^*_\nu - 2x^*\|.$$

Let us introduce also the operator version of property (wM^*).

Definition 5.16. An operator $T \in B_{\mathcal{L}(X,Y)}$ has *property* (wM^*) if

$$\limsup_{\nu} \|T^*(y^*_\nu - 2y^*)\| \leq \limsup_{\nu} \|y^*_\nu\|,$$

whenever $y^* \in Y^*$ and $(y^*_\nu) \subset Y^*$ is a bounded net converging weak* to y^* in Y^*.

We say that T has the *sequential version of property* (wM^*) if the nets $(y^*_\nu) \subset Y^*$, in the definition, are replaced with the sequences $(y^*_n) \subset Y^*$ which converge weak* to some $y^* \in Y^*$. Analogously to property $M^*(r,s)$ for operators, in the case when Y is separable, $T \in B_{\mathcal{L}(X,Y)}$ has property (wM^*) if and only if it has the sequential version of the property.

Clearly, a Banach space X has property (wM^*) if and only if its identity operator I_X has (wM^*). It is immediate that if either X or Y has property (wM^*), then every $T \in B_{\mathcal{L}(X,Y)}$ has property (wM^*). Indeed, let $T \in B_{\mathcal{L}(X,Y)}$, $y^* \in Y^*$, and let $(y^*_\nu) \subset Y^*$ be a bounded net converging weak* to y^* in Y^*. Then also $(T^*y^*_\nu) \subset X^*$ is a bounded net converging weak* to $T^*y^* \in X^*$. Now, if X has property (wM^*), then

$$\limsup_{\nu} \|T^*(y^*_\nu - 2y^*)\| = \limsup_{\nu} \|T^*(y^*_\nu)\| \leq \limsup_{\nu} \|y^*_\nu\|,$$

and if Y has property (wM^*), then

$$\limsup_{\nu} \|T^*(y^*_\nu - 2y^*)\| \leq \limsup_{\nu} \|y^*_\nu - 2y^*\| = \limsup_{\nu} \|y^*_\nu\|.$$

Properties (wM^*) for spaces and for operators are precisely properties $M^*(a,B,c)$ for spaces and for operators, when $a = 1$, $B = \{-2\}$, and $c = 0$ (see [49] and/or [48]).

In [34, Corollary 4.4], Lima showed that for reflexive Banach spaces X the MCAP assumption together with property (wM^*) (for spaces) is equivalent to $\mathcal{K}(X)$ being a u-ideal in $\mathcal{L}(X)$. The following theorem proves that for arbitrary Banach spaces X and Y, the MCAP assumption together with property (wM^*) for one of the spaces is sufficient for $\mathcal{K}(X,Y)$ being a u-ideal in $\mathcal{L}(X,Y)$.

Theorem 5.17. *Let X and Y be Banach spaces. If X or Y has the MCAP and property (wM^*), then $\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$.*

Proof. If Y has the MCAP and property (wM^*), then, due to [49, Corollary 4.5, $2^\circ \Rightarrow 3^\circ$], there exists a shrinking MCAI (K_α) of Y such that

$$\limsup_{\alpha} \|I_Y - 2K_\alpha\| \leq 1.$$

From Corollary 5.6 follows that $\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$.

In the case when X has the MCAP and property (wM^*), apply Corollary 5.5. □

The following Proposition 5.18 and Theorem 5.20 are inspired by similar results for $M(r,s)$-ideals (see Section 4.4).

Proposition 5.18. *Let X and Y be Banach spaces. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$. If an operator $T \in B_{\mathcal{L}}$ has property (wM^*) and there is a net $(T_\alpha) \subset \mathcal{K}$ such that $T_\alpha^* \longrightarrow T^*$ strongly, then*

$$\limsup_\alpha |f(T - 2T_\alpha)| \le 1$$

for all $f \in \overline{B_{X^{**}} \otimes B_{Y^*}}^{weak^*} \subset \mathcal{L}^*$.

Proof. Let $f = w^*\text{-}\lim x_\nu^{**} \otimes y_\nu^*$, i.e., $x_\nu^{**}(A^* y_\nu^*) \longrightarrow f(A)$, $A \in \mathcal{L}$, with $x_\nu^{**} \in B_{X^{**}}$, $y_\nu^* \in B_{Y^*}$. By passing to a subnet, we may assume that (y_ν^*) converges weak* to some $y^* \in B_{Y^*}$. From property (wM^*) we get that

$$\limsup_\nu \|2T^* y^* - T^* y_\nu^*\| \le \limsup_\nu \|y_\nu^*\| \le 1.$$

Hence, for any fixed α,

$$\begin{aligned} |f(T - 2T_\alpha)| &= \lim_\nu |x_\nu^{**}((T - 2T_\alpha)^* y_\nu^*)| \\ &\le \limsup_\nu \|(T - 2T_\alpha)^* y_\nu^*\| \\ &\le \limsup_\nu \Big(\|2T^* y^* - T^* y_\nu^*\| \\ &\quad + 2(\|T_\alpha^* y^* - T_\alpha^* y_\nu^*\| \\ &\quad + \|T^* y^* - T_\alpha^* y^*\|) \Big) \\ &\le 1 + 2\|T^* y^* - T_\alpha^* y^*\|, \end{aligned}$$

which implies

$$\limsup_\alpha |f(T - 2T_\alpha)| \le 1.$$

□

By applying the vector-valued version of Simons's inequality (see Lemma 4.14) to a sequential version of Proposition 5.18, we obtain the following result.

Lemma 5.19. *Let X and Y be Banach spaces. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$. If $T \in B_{\mathcal{L}}$ has property (wM^*) and there is a sequence $(T_n) \subset \mathcal{K}$ such that $T_n^* \longrightarrow T^*$ strongly, then for all $n \in \mathbb{N}$ there exists $S_n \in \operatorname{conv}\{T_n, T_{n+1}, \dots\}$ such that*

$$\limsup_n \|T - 2S_n\| \leq 1.$$

The next theorem is analogous to Theorem 4.16 for $M(r,s)$-ideals. Note that its assumptions enforce X^* (and X) or Y^* (and Y) to be separable.

Theorem 5.20. *Let X and Y be Banach spaces. Suppose that X^{**} or Y^* has the Radon–Nikodým property and that X or Y has a shrinking compact approximating sequence. Let $\mathcal{L}$ be a closed subspace of $\mathcal{L}(X,Y)$ containing $\mathcal{K} := \mathcal{K}(X,Y)$. If every $T \in S_{\mathcal{L}}$ has property (wM^*), then $\mathcal{K}$ is a u-ideal in $\mathcal{L}$ with respect to an ideal projection preserving elementary functionals.*

Proof. Based on Example 4.6 we get that $\mathcal{K}$ is an ideal in $\mathcal{L}$ with respect to an ideal projection P preserving elementary functionals.

For every operator $T \in S_{\mathcal{L}}$ and $n \in \mathbb{N}$, let us define $T_n = TK_n$ (respectively, $T_n = K_nT$) where (K_n) is the shrinking compact approximating sequence of X (respectively, of Y). Then clearly $T_n^* \longrightarrow T^*$ strongly. By Lemma 5.19, there exists $S_n \in \operatorname{conv}\{T_n, T_{n+1}, \dots\}$ such that

$$\limsup_n \|T - 2S_n\| \leq 1.$$

Since also $S_n^* \longrightarrow T^*$ strongly, by Theorem 5.7, (c) $\Rightarrow$ (a), $\mathcal{K}$ is a u-ideal in $\mathcal{L}$ with respect to P. ☐

Corollary 5.21. *Let X and Y be Banach spaces. Suppose that X^{**} or Y^* has the Radon–Nikodým property. If X or Y has a shrinking compact approximating sequence and property (wM^*), then $\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$.*

Proof. Since X or Y has property (wM^*), every $T \in B_{\mathcal{L}(X,Y)}$ has property (wM^*) and thus, by Theorem 5.20, $\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$. ☐

Note that Corollary 5.21 is a separable version of Theorem 5.17. However, Corollary 5.21 assumes the existence of a shrinking compact approximating sequence, but in Theorem 5.17 we have the MCAP assumption.

5.6 u-Ideals of compact operators are separably determined

Next, we will prove that u-ideals of compact operators are separably determined. The result is analogous to Theorem 4.20 for $M(r,s)$-ideals.

Theorem 5.22. *Let X and Y be Banach spaces. Let Y have property (wM^*) and the MCAP. If $\mathcal{K}(E,F)$ is a u-ideal in $\mathcal{L}(E,F)$ with respect to an ideal projection preserving elementary functionals for all separable closed subspaces E of X and F of Y such that F has the MCAP, then $\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$.*

Proof. Let (K_α) be an MCAI of Y. Since Y has property (wM^*), by [34, Proposition 4.1], (K_α) is shrinking and Y^* has the Radon–Nikodým property. Also, we know that $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$ with respect to the Johnson projection and $K_\alpha T \longrightarrow T$ for all $T \in S_{\mathcal{L}(X,Y)}$ in the dual weak operator topology.

Assume for contradiction that $\mathcal{K}(X,Y)$ is not a u-ideal in $\mathcal{L}(X,Y)$. Then condition (b) of Theorem 5.7 is not satisfied: there are $\varepsilon > 0$, $T \in B_{\mathcal{L}(X,Y)}$, and α_0 such that

$$\|T - 2KT\| > 1 + 3\varepsilon \quad \forall K \in \operatorname{conv}\{K_\alpha \colon \alpha \geq \alpha_0\}.$$

We shall define separable closed subspaces E of X and F of Y such that F has the MCAP, but $\mathcal{K}(E,F)$ cannot be a u-ideal in $\mathcal{L}(E,F)$ with respect to an ideal projection preserving elementary functionals. This will contradict the assumption and complete the proof.

To begin, let $E_0 = \{0\} \subset X$ and $F_0 = \{0\} \subset Y$. Pick $x_0 \in B_X$ such that

$$\|(T - 2K_{\alpha_0}T)x_0\| > \|T - 2K_{\alpha_0}T)\| - \varepsilon > 1 + 2\varepsilon.$$

Continuing similarly to the proof of Theorem 4.20 (where $S = 0$), we obtain for all $n \in \mathbb{N}$ an index α_n, a finite $\varepsilon/2$-net Λ_n in $\operatorname{conv}\{K_{\alpha_0}, \ldots, K_{\alpha_n}\}$, a finite subset $\{x_L \colon L \in \Lambda_n\} \subset B_X$ such that

$$\|(T - 2LT)x_L\| > 1 + 2\varepsilon, \quad L \in \Lambda_n,$$

and finite subsets $E_n \subset X$ and $F_n \subset Y$ such that

$$E_{n+1} = E_n \cup \{x_L \colon L \in \Lambda_n\},$$

$$F_{n+1} = F_n \cup K_{\alpha_0}(F_n) \cup \ldots \cup K_{\alpha_n}(F_n) \cup T(E_{n+1}),$$

and

$$\|K_{\alpha_n} y - y\| < \frac{1}{n} \quad \forall y \in F_n.$$

Denote $E = \overline{\text{span}} \bigcup_{n=1}^{\infty} E_n$ and $F = \overline{\text{span}} \bigcup_{n=1}^{\infty} F_n$. It can be easily seen that $T(E) \subset F$, $K_{\alpha_n}(F) \subset F$ for all $n \in \mathbb{N}$, and $K_{\alpha_n} y \longrightarrow y$ for all $y \in F$. Consider $T|_E \in B_{\mathcal{L}(E,F)}$, and $K_{\alpha_n}|_F \in B_{\mathcal{K}(F)}$.

It follows from a general fact about property $M^*(a, B, c)$ (see [49, p. 2804]) that if Y has property (wM^*), then also F has it. As in the beginning of the proof, note that F^* has the Radon–Nikodým property. Consequently we are in position to apply Theorem 5.7 to $\mathcal{K}(E,F)$ in $\mathcal{L}(E,F)$. According to Theorem 5.7, if $\mathcal{K}(E,F)$ were a u-ideal in $\mathcal{L}(E,F)$ with respect to an ideal projection preserving elementary functionals, then there would exist $K \in \text{conv}\{K_{\alpha_1}, \ldots, K_{\alpha_n}\}$, for some $n \in \mathbb{N}$, such that

$$\|(T - 2KT)|_E\| \leq 1 + \varepsilon.$$

Let $L \in \Lambda_n$ satisfy $\|K - L\| < \varepsilon/2$. Then

$$\begin{aligned} 1 + 2\varepsilon &< \|(T - 2LT))|_E\| \\ &\leq \|(T - 2KT))|_E\| + \varepsilon \\ &\leq 1 + 2\varepsilon, \end{aligned}$$

a contradiction. □

Lima showed in [34, Proposition 4.1] that if X is a u-ideal, then X^* has the Radon–Nikodým property and $X^* = \overline{\text{span}}(w^*\text{- sexp}\, B_{X^*})$. The latter yield that every MCAI of X is shrinking (cf. the proof of [49, Corollary 1.7]).

From the proof of Theorem 5.22, it is clear that we can replace the assumption "Y has property (wM^*)" by any assumption guaranteeing that Y^* has the Radon–Nikodým property and every MCAI of any closed subspace of Y is shrinking. Both are fulfilled if we assume that Y is a u-ideal.

Corollary 5.23. *Let X and Y be Banach spaces. Let Y be a u-ideal and have the MCAP. If $\mathcal{K}(E,F)$ is a u-ideal in $\mathcal{L}(E,F)$ with respect to an ideal projection preserving elementary functionals for all separable closed subspaces E of X and F of Y such that F has the MCAP, then $\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$.*

The following Lemma 5.24 is needed for proving Theorem 5.25 below.

Lemma 5.24 ([49, Theorem 3.5]). *A Banach space X has a shrinking MCAI (K_α) such that*

$$\limsup_{\alpha} \|I_X - 2K_\alpha\| \leq 1$$

if and only if X has the MCAP, and for every separable closed subspace E having the MCAP, there exists a shrinking MCAI (K_n) of E such that

$$\limsup_{n} \|I_E - 2K_n\| \leq 1.$$

Theorem 5.25. *Let X be a Banach space such that X is a u-ideal. If $\mathcal{K}(X)$ is a u-ideal in $\mathcal{L}(X)$ with respect to an ideal projection preserving elementary functionals, then $\mathcal{K}(E)$ is a u-ideal in $\mathcal{L}(E)$ for all separable closed subspaces E of X having the MCAP.*

Proof. If X is a u-ideal, then X^* has the Radon–Nikodým property. By Corollary 5.12, X^* has the MCAP with conjugate operators and

$$\limsup_{\alpha} \|I_X - 2K_\alpha\| \leq 1,$$

where (K_α) is a shrinking MCAI of X. By Lemma 5.24, every separable closed subspace E has a shrinking MCAI (K_n) such that $\limsup_n \|I_E - 2K_n\| \leq 1$. Thus, based on Corollary 5.5, $\mathcal{K}(E)$ is a u-ideal in $\mathcal{L}(E)$ with respect to an ideal projection preserving elementary functionals. □

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Kompaktsete operaatorite $M(r,s)$-ideaalid

Kokkuvõte

Käesoleva väitekirja keskne küsimus on järgmine. Milliste Banachi ruumide X ja Y korral kõigi ruumist X ruumi Y tegutsevate kompaktsete operaatorite alamruum $\mathcal{K}(X,Y)$ osutub M-ideaaliks või, veelgi üldisemalt, $M(r,s)$-ideaaliks kõigi pidevate lineaarsete operaatorite ruumis $\mathcal{L}(X,Y)$? See probleem on huvipakkuv näiteks seetõttu, et M-ideaalil või üldisemalt $M(1,s)$-ideaalil määratud igal pideval lineaarsel funktsionaalil leidub ühene normi säilitav jätk kogu ruumile. Teiseks annab $M(r,s)$-ideaalide struktuuri olemasolu teavet kaasruumi $\mathcal{L}(X,Y)^*$ ehituse kohta. Sugugi vähetähtis pole ka kompaktsete operaatorite $M(r,s)$-ideaalide teooria seos aproksimatsiooniomaduste teooriaga, kus veel tänapäevalgi on aastakümnetevanuseid kuulsaid lahendamist ootavaid probleeme.

Väitekirja lähtekohaks on järgmine E. Oja tulemus 1993. aastal ilmunud artiklis, mis on publitseeritud ka P. Harmand, D. Werner ja W. Werner monograafias "M-ideals in Banach Spaces and Banach Algebras" [23, lk. 301]. See tulemus näitab, kuidas Banachi ruumid X, mille korral $\mathcal{K}(X) := \mathcal{K}(X,X)$ on M-ideaal ruumis $\mathcal{L}(X) := \mathcal{L}(X,X)$, tekitavad uusi kompaktsete operaatorite M-ideaale.

Teoreem ([45]). *Olgu X ja Y Banachi ruumid. Kui $\mathcal{K}(X)$ ja $\mathcal{K}(Y)$ on M-ideaalid vastavalt ruumides $\mathcal{L}(X)$ ja $\mathcal{L}(Y)$, siis $\mathcal{K}(X,Y)$ on M-ideaal ruumis $\mathcal{L}(X,Y)$.*

Käesolevas väitekirjas uuritakse, kas analoogiline tulemus kehtib ka kompaktsete operaatorite $M(r,s)$-ideaalide korral, milliseks kujunevad sellisel juhul uute tekkinud $M(r,s)$-ideaalide parameetrid, ja rakendatakse kompaktsete operaatorite u-ideaalidele $M(r,s)$-ideaalide jaoks väitekirjas loodud metoodikat.

Käesolev väitekiri koosneb viiest peatükist. Väitekirja esimene peatükk sisaldab probleemi tausta tutvustust, väitekirja kokkuvõtet ning üldiste kasutatud tähistuste kirjeldust.

Teises peatükis tõestatakse üksikasjalikult Teoreem tuginedes originaaltõestuse skeemile artiklist [45]. Selleks tuuakse sisse M-ideaali mõiste tuginedes ideaaliprojektori mõistele. Tutvustatakse omadusi (M) ja (M^*), mis on osu-

tunud võtmeteguriteks kompaktsete operaatorite M-ideaalide kirjeldamisel. Käsitletakse operaatorite ruumi erinevaid topoloogiaid ja koondumisi nendes ning tõestatakse Johnsoni lemma, mis garanteerib teatavat tüüpi kompaktsete operaatorite pere olemasolul ideaaliprojektori eksisteerimise. Teoreem järeldatakse kompaktsete operaatorite M-ideaalide kirjeldusest, kus üks ruumidest on omadusega (M) või (M^*) ja teine ruum rahuldab tingimust, et $\mathcal{K}(X)$ on M-ideaal ruumis $\mathcal{L}(X)$.

Kolmandas peatükis üldistatakse M-ideaalide tõestusmetoodikat $M(r,s)$-ideaalide jaoks. Tutvustatakse $M(r,s)$-ideaali mõistet ning tuuakse näiteid mõningatest M-ideaalide ja $M(r,s)$-ideaalide erinevustest. Analoogiliselt artikli [54] käsitlusele defineeritakse Johnsoni projektor. Käsitletakse $M(r,s)$-ideaalidega seotud omadusi $M(r,s)$ ja $M^*(r,s)$, mis on omaduste (M) ja (M^*) üldistused. Tuginedes artiklile [49] tõestatakse kolmanda peatüki põhitulemus.

Järeldus 3.22. *Olgu X ja Y sellised Banachi ruumid, et $\mathcal{K}(X)$ on $M(r_1,s_1)$-ideaal ruumis $\mathcal{L}(X)$ ja $\mathcal{K}(Y)$ on $M(r_2,s_2)$-ideaal ruumis $\mathcal{L}(Y)$, kus $r_1 + s_1/2 > 1$ ja $r_2 + s_2/2 > 1$. Siis $\mathcal{K}(X,Y)$ on nii $M(r_1^2 r_2, s_1^2 s_2)$- kui ka $M(r_1 r_2^2, s_1 s_2^2)$-ideaal ruumis $\mathcal{L}(X,Y)$.*

Kolmas peatükk on inspireeritud artiklitest [45], [49], [54] ja tugineb artiklile [19].

Parameetrid $r_1^2 r_2$, $s_1^2 s_2$ ja $r_1 r_2^2$, $s_1 s_2^2$ järelduses 3.22 tunduvad olevat mitteoptimaalsed. Seetõttu pakutakse neljandas peatükis välja teistsugune lähenemisviis, mis võimaldab saada paremad parameetrid, nimelt $r_1 r_2$ ja $s_1 s_2$. Seejuures on võtmemõisteteks "elementaarfunktsionaale säilitav ideaaliprojektor" ja "operaatorite omadus $M^*(r,s)$". Olulisteks töövahenditeks on Simonsi võrratuse operaatorvariant [33] ja Feder–Saphari kirjeldus [13] kompaktsete operaatorite ruumi $\mathcal{K}(X,Y)$ kaasruumile, mis kehtib eeldusel, et X^{**} või Y^* on Radon–Nikodými omadusega.

Tuginedes elementaarfunktsionaale säilitavaid ideaaliprojektoreid ja operaatorite omadust $M^*(r,s)$ puudutavatele tulemustele, tõestatakse, kasutades Simonsi võrratust, neljanda peatüki põhitulemuse separaabel versioon. Edasi tõestatakse, et kompaktsete operaatorite $M(r,s)$-ideaalid on separaablilt määratud. See võimaldab põhitulemuse separaablilt versioonilt minna edasi üldisele juhule ning näidata, et kompaktsete operaatorite $M(r,s)$-ideaalid tekitavad uusi kompaktsete operaatorite $M(r,s)$-ideaale järgmisel viisil.

Teoreem 4.26. *Olgu X ja Y sellised Banachi ruumid, et $\mathcal{K}(X)$ on $M(r_1,s_1)$-ideaal ruumis $\mathcal{L}(X)$ ja $\mathcal{K}(Y)$ on $M(r_2,s_2)$-ideaal ruumis $\mathcal{L}(Y)$, kus $r_1 +$*

$s_1/2 > 1$ *ja* $r_2 + s_2/2 > 1$. *Siis* $\mathcal{K}(X,Y)$ *on* $M(r_1r_2, s_1s_2)$*-ideaal ruumis* $\mathcal{L}(X,Y)$.

Neljanda peatüki tulemuste rakendusena on muuhulgas parendatud mõningaid klassikalisi M-ideaalide teooria tulemusi.

Neljas peatükk on inspireeritud artiklitest [36], [47], [49] ja tugineb artiklile [20].

Viiendas peatükis rakendatakse $M(r,s)$-ideaalide jaoks arendatud tõestusmetoodikat u-ideaalidele, mis on üks $M(r,s)$-ideaalidest erinev M-ideaalide üldistus. Osutub, et nii nagu M- ja $M(r,s)$-ideaalid, on ka u-ideaalid separaablilt määratud, kuid omadus tekitada uusi kompaktsete operaatorite u-ideaale erineb mõnevõrra M- ja $M(r,s)$-ideaalide juhust.

Viienda peatüki põhitulemusi sisaldava artikli eelvariant on [26].

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List of original publications

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