Real Gelfand-Mazur algebras

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Contents

List of original publications

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Introduction

The class of complex Gelfand-Mazur algebras was introduced, indepedently of each other, by Mati Abel (see [6]–[8]) and by Anastasios Mallios (see [24]). The structure of complex Gelfand-Mazur algebras has been enough well studied. The class of real Gelfand-Mazur algebras was introduced recently in [18].

Properties of real topological algebras have been studied mainly in case of Banach algebras (see [21, 30]). The main method for the study of real Banach algebras is the following: first to complexify the real Banach algebra A and then to transform the results, that are known for complex Banach algebras, from the complexification $\widetilde{A}$ of A to the initial real Banach algebra A. The same technics is working in case of real Gelfand-Mazur algebras, too.

Using ideas of G. Allan and L. Waelbroeck (see [19, 32]), Mart Abel showed in [2, 4] how to describe closed maximal ideals in complex (not necessarily commutative) Gelfand-Mazur algebras. Using his results on the complexification $\widetilde{A}$ of a real Gelfand-Mazur algebra A, a similar description for a certain kind of closed maximal ideals (in particular, of all closed maximal ideals with codimension 1) is presented in the this Thesis. As an application, the description of closed maximal ideals in subalgebras of $C(X, A; \sigma)$ is given.

This Thesis consists of three Chapters. Properties of the complexification of real topological algebras (in particular, of real locally pseudoconvex and of real galbed algebras) are considered in the first Chapter. Conditions, when a real topological division algebra is a real Gelfand-Mazur division algebra, are given in the last section of this Chapter.

A description of classes of real (commutative and noncommutative) Gelfand-Mazur algebras is given in the second Chapter. Conditions for a real topological algebra A, for which the center of A/P (the quotient algebra of A by a closed primitive ideal P) is homeomorphic to $\mathbb{R}$, are found. Using this result, a description of closed maximal left (right or two-sided) ideals in real unital Gelfand-Mazur algebra is given.

Properties of the topological algebra $C(X, A; \sigma)$ (of A-valued continuous functions on a topological space X in case when σ is a compact cover of X and A a real Gelfand-Mazur algebra) are studied in the third Chapter. A description of closed maximal left (right or two-sided) ideals and of all nontrivial continuous linear multiplicative functionals in subalgebras $\mathfrak{A}(X, A; \sigma)$ of $C(X, A; \sigma)$ are given as an application. Conditions, when $\hom(\mathfrak{A}(X, A; \sigma))$ (the space of all nontrivial continuous homomorphisms from $\mathfrak{A}(X, A; \sigma)$ to $\mathbb{R}$ endowed with the Gelfand topology) and $X \times \hom(A)$ are homeomorphic, are given.

Main results, presented in this Thesis, have been published in [18], [28], and in a coming paper [29]. The author has introduced these results at the following international conferences and workshops: "International Conference on Topological Algebras and its Applications" (Oulu, 2001), "Topological algebras, their applications and related results" (Będlewo, 2003), "International conference dedicated to 125-th anniversary of Hans Hahn" (Chernivtsi, Ukraine, 2004), "International Conference on Topological Algebras and its Applications" (Athens, 2005) and joint workshop "Tartu-Riga" (Riga, 2005).

Chapter 1

Real Gelfand-Mazur division algebras

In this Chapter we consider properties of the complexification of real topological algebras (in particular, of real locally pseudoconvex and of real galbed algebras) and we give the conditions, when a real topological division algebra is a real Gelfand-Mazur division algebra.

Main results in this Chapter are published in [18].

1.1 Preliminary

Here we give definitions of some terms, which are connected with the topological vector space.

Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers, endowed with their usual topologies, X a vector space over $\mathbb{K}$ and τ a topology on X. The pair (X, τ) is called a *topological vector space* over $\mathbb{K}$ if

a) for each neighbourhood O of zero of X in the topology τ there exists another neighbourhood U of zero of X such that $U + U \subset O$;

b) for each neighbourhood O of zero of X in the topology τ there exist a neighbourhood U of zero of $\mathbb{K}$ and a neighbourhood V of zero of X such that $VU \subset O$.

Throughout of this Thesis the zero element in X is denoted by θ_X.

Let now X be a topological vector space and U an arbitrary set in X. Then U is

- *balanced* if $\lambda U \subset U$, whenever $|\lambda| \leqslant 1$;
- *absorbing* if for each $a \in X$ there exists a number $\mu > 0$ such that $a \in \lambda U$, whenever $|\lambda| \geqslant \mu$;
- *convex* if $\lambda a + (1-\lambda)b \in U$ for each $a, b \in U$ and $0 \leqslant \lambda \leqslant 1$;
- *absolutely k-convex* if $\lambda a + \mu b \in U$ for each $a, b \in U$ and $\lambda, \mu \in \mathbb{K}$ such that $|\lambda|^k + |\mu|^k \leqslant 1$ and $k \in (0, 1]$;
- *pseudoconvex* if $U + U \subset 2^{\frac{1}{k}} U$ for some $k \in (0, 1]$;
- *bounded* in X if for each neighbourhood O of zero of X there exists a number $\lambda_O > 0$ such that $U \subset \lambda_O O$.

Let k be a positive real number. The map $p : X \to \mathbb{R}^+$ is a *k-homogeneous seminorm*[1] on X if

a) $p(x) \geqslant 0$ for each $x \in X$;

b) $p(\lambda x) = |\lambda|^k p(x)$ for each $x \in X$ and $\lambda \in \mathbb{K}$;

c) $p(x + y) \leqslant p(x) + p(y)$ for each $x, y \in X$.

Let M be an absorbing subset in X. The map $p_M : X \to \mathbb{R}^+$ (the set of nonnegative real numbers), defined by

$$p_M(x) = \inf\{|\lambda|^k > 0 : x \in \lambda M\}$$

for each $x \in X$, is called a *k-homogeneous Minkowski functional* of M on X.

It is easy to see that, if M is an absolutely k-convex, absorbing and balanced set in X, then the k-homogeneous Minkowski functional p_M of M is a k-homogenous seminorm on X (see, for example, [20], Propostion 4.1.10).

[1] A k-homogeneuos seminorm p on X is a *k-homogeneous norm* if from $p(x) = 0$ it follows that $x = \theta_X$.

1.2 Real topological algebras and their complexifications

1. A topological vector space A over $\mathbb{K}$ is called a *topological algebra*[2] over $\mathbb{K}$ (shortly, *topological algebra*) if there has been defined in A an associative multiplication such that

a) A is an algebra over $\mathbb{K}$ with respect to this multiplication;

b) this multiplication is separately continuous.

The condition b) means that for any $a \in A$ and every neighbourhood U of zero in A there is another neighbourhood V of zero in A such that $Va, aV \subset U$.

In particular, when $\mathbb{K} = \mathbb{R}$, we will speak about real topological algebras, and when $\mathbb{K} = \mathbb{C}$, about complex topological algebras.

In case when the multiplication of A is *jointly continuous* (that is, for any neighbourhood U of zero in A there is another neighbourhood V of zero in A such that $VV \subset U$), then A is called a *topological algebra with jointly continuous multiplication.*

Throughout of this Thesis the unit element in A is denoted by e_A.

For any unital topological algebra A we will use the following notations:

- $m(A)$ is the set of all closed two-sided ideals of A, which are maximal as left or right ideals;
- $m_l(A)$ ($m_r(A)$ or $m_t(A)$) is the set of all closed maximal left (respectively, right or two-sided) ideals of A.

2. Let A be a real (not necessarily topological) algebra and $\widetilde{A} = A + iA$ the complexification of A. Then every element $\widetilde{a}$ of $\widetilde{A}$ is representable in the form $\widetilde{a} = a + ib$, where $a, b \in A$ and $i^2 = -1$. If the addition, the scalar multiplication and the multiplication in $\widetilde{A}$

[2]In case, when we have already specified the topology τ on X, then we talk about a topological algebra (A, τ).

to define by

$$\begin{aligned}(a+ib)+(c+id) &= (a+c)+i(b+d),\\ (\alpha+i\beta)(a+ib) &= (\alpha a-\beta b)+i(\alpha b+\beta a),\\ (a+ib)(c+id) &= (ac-bd)+i(ad+bc)\end{aligned}$$

for all elements $a, b, c, d \in A$ and $\alpha, \beta \in \mathbb{R}$, then $\widetilde{A}$ is a complex algebra with the zero element $\theta_{\widetilde{A}} = \theta_A + i\theta_A$. In case, when A has the unit element e_A, then $e_{\widetilde{A}} = e_A + i\theta_A$ is the unit element of $\widetilde{A}$. Herewith, $\widetilde{A}$ is an associative (commutative) algebra if A is an associative (respectively, commutative) algebra. Moreover, we can considered A as a real subalgebra of $\widetilde{A}$ if we embed A into $\widetilde{A}$ by the map ν defined by $\nu(a) = a + i\theta_A$ for each $a \in A$.

3. Let (A, τ) be a real topological algebra and $\mathcal{B}_A = \{U_\alpha : \alpha \in \mathcal{A}\}$ a base of neigbourhoods of zero in (A, τ). As usual (see [20] or [36]), we endow $\widetilde{A}$ with the topology $\widetilde{\tau}$, in which a base of neighbourhoods of zero is $\mathcal{B}_{\widetilde{A}} = \{U_\alpha + i\,U_\alpha : \alpha \in \mathcal{A}\}$. It is easy to see that $(\widetilde{A}, \widetilde{\tau})$ is a topological algebra and the multiplication in $(\widetilde{A}, \widetilde{\tau})$ is jointly continuous if the multiplication in (A, τ) is jointly continuous (see Proposition 2.2.10 from [20]). Moreover, the underlying topological space of $\widetilde{A}$ is a Hausdorff space if the underlying topological space of A is a Hausdorff space.

4. A usual method for the study of properties of a real topological algebra A is the following: complexify and then apply results known for complex topological algebras to the complexification $\widetilde{A}$ of A and deduce similar results for the original topological algebra A.

1.3 Real locally pseudoconvex algebras and their complexifications

1. A topological algebra (A, τ) is a *locally pseudoconvex algebra* if its underlying topological vector space is locally pseudoconvex. It means that (A, τ) has a base

$$\mathcal{B}_A = \{U_\alpha : \quad \alpha \in \mathcal{A}\}$$

of neighbourhoods of zero, consisting of balanced and pseudoconvex sets. The topology τ of a locally pseudoconvex algebra (A, τ) is usually given by a family

$$\mathcal{P}_A = \{p_\alpha : \quad \alpha \in \mathcal{A}\}$$

of k_α-homogeneous seminorms, where $k_\alpha \in (0, 1]$ for each $\alpha \in \mathcal{A}$ (see [33], p. 4).

Now we define two paticular cases of locally pseudoconvex algebras. A locally pseudoconvex algebra (A, τ) is

- *locally absorbingly pseudoconvex* (shortly, *locally A-pseudoconvex*) if for each $U_\alpha \in \mathcal{B}_A$ and $a \in A$ there is a positive number μ_a such that $aU_\alpha,\ U_\alpha a \subset \mu_a U_\alpha$ or, in terms of seminorms, if every seminorm $p \in \mathcal{P}_A$ is *A-multiplicative*, it means that for each $a \in A$ there are positive numbers $M(a, p)$ and $N(a, p)$ such that

  $$p(ab) \leqslant M(a, p)p(b) \quad \text{and} \quad p(ba) \leqslant N(a, p)p(a)$$

  for all $b \in A$;

- *locally multiplicatively pseudoconvex* (shortly, *locally m-pseudoconvex*) if $U_\alpha^2 \subset U_\alpha$ for each $U_\alpha \in \mathcal{B}_A$ or, in terms of seminorms, if every seminorm $p \in \mathcal{P}_A$ is *submultiplicative*, it means that

  $$p(ab) \leqslant p(a)p(b)$$

  for all $a, b \in A$.

It is easy to see that every locally m-pseudoconvex algebra is locally A-pseudoconvex. Indeed, let (A, τ) be a locally m-pseudoconvex algebra, U an arbitrary set in $\mathcal{B}_A$ and $a \in A$. Then there is a number $\lambda > 0$ such that $a \in \lambda U$. Therefore $aU \subset \lambda UU \subset \lambda U$ (similarily, $Ua \subset \lambda U$). Hence, (A, τ) is a locally A-pseudoconvex algebra.

2. Let now A be a real locally pseudoconvex algebra, $\widetilde{A}$ the complexification of A,

$$\Gamma_{k_\alpha}(U_\alpha + i\theta_A) = \Big\{\sum_{k=1}^{n} \lambda_k(u_k + i\theta_A) : \quad n \in \mathbb{N},$$

$$u_1, ..., u_n \in U_\alpha, \quad \lambda_1, ..., \lambda_n \in \mathbb{C} \quad \text{with} \quad \sum_{k=1}^{n} |\lambda_k|^{k_\alpha} \leqslant 1\Big\}$$

and

$$q_\alpha(a+ib) = \inf\{|\,\lambda\,|^{k_\alpha}: \quad (a+ib) \in \lambda\Gamma_{k_\alpha}(U_\alpha + i\theta_A)\}$$

for each $a+ib \in \widetilde{A}$. Then $\Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ is the absolutelly k_α-convex hull of $U_\alpha + i\theta_A$ for each $\alpha \in \mathcal{A}$ and q_α is a k_α-homogeneous Minkowski functional of $\Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ on $\widetilde{A}$. For real normed algebras the following result has been proved in [21], p. 68−69 (see also [30], p. 8) and for k-seminormed algebras with $k \in (0,1]$ in [20], p. 183−184.

Theorem 1.3.1. *Let (A,τ) be a real locally pseudoconvex algebra, $\mathcal{P}_A = \{p_\alpha : \alpha \in \mathcal{A}\}$ a family of k_α-homogeneuos seminorms on A (with $k_\alpha \in (0,1]$ for each $\alpha \in \mathcal{A}$), which defines the topology τ of A, and let $U_\alpha = \{a \in A : p_\alpha(a) < 1\}$. Then the following statements are true for each $\alpha \in \mathcal{A}$:*

a) *$(\widetilde{A}, \widetilde{\tau})$ is a complex locally pseudoconvex algebra, which topology $\widetilde{\tau}$ is defined by the set $\{q_\alpha : \alpha \in \mathcal{A}\}$ of k_α-homogeneuos seminorm on $\widetilde{A}$;*

b) *$\max\{p_\alpha(a), p_\alpha(b)\} \leqslant q_\alpha(a+ib) \leqslant 2\max\{p_\alpha(a), p_\alpha(b)\}$ for each $a, b \in A$;*

c) *$q_\alpha(a + i\theta_A) = p_\alpha(a)$ for each $a \in A$;*

d) *$\Gamma_{k_\alpha}(U_\alpha + i\theta_A) = \{a+ib \in \widetilde{A} : q_\alpha(a+ib) < 1\}$.*

Proof. a) We will show that q_α is a k_α-homogeneuos seminorm on $\widetilde{A}$ for each $\alpha \in \mathcal{A}$. For it, it is enough to show, that $\Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ is an absorbing set. Let $\alpha \in \mathcal{A}$, $(a+ib) \in \widetilde{A} \setminus \{\theta_{\widetilde{A}}\}$ and

$$\mu_\alpha^{k_\alpha} > \max\{p_\alpha(a), p_\alpha(b)\}.$$

Then $\dfrac{a}{\mu_\alpha}, \dfrac{b}{\mu_\alpha} \in U_\alpha$. Since

$$2^{-\frac{1}{k_\alpha}}\Big(\frac{a}{\mu_\alpha} + i\frac{b}{\mu_\alpha}\Big) = 2^{-\frac{1}{k_\alpha}}\left(\frac{a}{\mu_\alpha} + i\theta_A\right) + i2^{-\frac{1}{k_\alpha}}\left(\frac{b}{\mu_\alpha} + i\theta_A\right)$$

and

$$|2^{-\frac{1}{k_\alpha}}|^{k_\alpha} + |i2^{-\frac{1}{k_\alpha}}|^{k_\alpha} = 1,$$

then

$$(a+ib) \in 2^{\frac{1}{k_\alpha}} \mu_\alpha \Gamma_{k_\alpha}(U_\alpha + i\theta_A). \tag{1.3.1}$$

Hence, $(a+ib) \in \lambda_\alpha \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ for each $\alpha \in \mathcal{A}$ if $|\lambda_\alpha| \geqslant 2^{\frac{1}{k_\alpha}} \mu_\alpha$. It means that the set $\Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ is an absorbing set. Consequently (see [20], Proposition 4.1.10), q_α is a k_α-homogeneous seminorm on $\widetilde{A}$.

b) Let now $(a+ib) \in \widetilde{A} \setminus \{\theta_{\widetilde{A}}\}$. Then from (1.3.1) it follows that

$$q_\alpha(a+ib) \leqslant 2\mu_\alpha^{k_\alpha}.$$

Since this inequality is valid for each $\mu_\alpha^{k_\alpha} > \max\{p_\alpha(a), p_\alpha(b)\}$, then

$$q_\alpha(a+ib) \leqslant 2\max\{p_\alpha(a), p_\alpha(b)\}. \tag{1.3.2}$$

Let now $a+ib \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$. Then

$$a+ib = \sum_{k=1}^{n} (\lambda_k + i\mu_k)(a_k + i\theta_A) = \sum_{k=1}^{n} \lambda_k a_k + i \sum_{k=1}^{n} \mu_k a_k$$

for some $a_1, \ldots, a_n \in U_\alpha$ and real numbers $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ such that

$$\sum_{k=1}^{n} |\lambda_k + i\mu_k|^{k_\alpha} \leqslant 1.$$

Since $|\lambda_k| \leqslant |\lambda_k + i\mu_k|$ and $|\mu_k| \leqslant |\lambda_k + i\mu_k|$ for each $k = 1, \ldots, n$, then

$$a = \sum_{k=1}^{n} \lambda_k a_k \ \text{ and } \ b = \sum_{k=1}^{n} \mu_k a_k$$

belong to $\Gamma_{k_\alpha}(U_\alpha) = U_\alpha$.

Let now $\varepsilon > 0$ and

$$\mu_\alpha > \Big(\frac{1}{q_\alpha(a+ib) + \varepsilon}\Big)^{\frac{1}{k_\alpha}}.$$

Then from $\mu_\alpha(a+ib) \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ it follows that $\mu_\alpha a, \mu_\alpha b \in U_\alpha$ or $p_\alpha(\mu_\alpha a) < 1$ and $p_\alpha(\mu_\alpha b) < 1$. Therefore

$$\max\{p_\alpha(a), p_\alpha(b)\} < \mu_\alpha^{-k_\alpha} < q_\alpha(a+ib) + \varepsilon. \tag{1.3.3}$$

Since ε is arbitrary, then from (1.3.3) it follows that

$$\max\{p_\alpha(a), p_\alpha(b)\} \leqslant q_\alpha(a+ib)$$

for each $a, b \in A$. Taking this and the inequality (1.3.2) into account, it is clear that the statement b) holds.

c) Let $a \in A$, $\alpha \in \mathcal{A}$ and $\rho^{k_\alpha} > q_\alpha(a + i\theta_A)$. Then from

$$\left(\frac{a}{\rho} + i\theta_A\right) \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$$

it follows that $a \in \rho U_\alpha$ or $p_\alpha(a) < \rho^{k_\alpha}$. It means that the set of numbers ρ^{k_α} for which $\rho^{k_\alpha} > q_\alpha(a + i\theta_A)$ is bounded below by $p_\alpha(a)$. Therefore

$$p_\alpha(a) \leqslant q_\alpha(a + i\theta_A).$$

Let now $\rho^{k_\alpha} > p_\alpha(a)$. Then $a \in \rho U_\alpha$ and from

$$\left(\frac{a}{\rho} + i\theta_A\right) \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$$

it follows that $q_\alpha(a + i\theta_A) < \rho^{k_\alpha}$. Hence $q_\alpha(a + i\theta_A) \leqslant p_\alpha(a)$. Thus $q_\alpha(a + i\theta_A) = p_\alpha(a)$ for each $a \in A$ and $\alpha \in \mathcal{A}$.

d) It is clear that the set

$$\{a + ib \in \widetilde{A} : q_\alpha(a + ib) < 1\} \subset \Gamma_{k_\alpha}(U_\alpha + i\theta_A).$$

Now we show that $\Gamma_{k_\alpha}(U_\alpha + i\theta_A) \subset \{a + ib \in \widetilde{A} : q_\alpha(a + ib) < 1\}$. For it, let $a + ib \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$. Then

$$a + ib = \sum_{k=1}^{n} (\lambda_k + i\mu_k)(a_k + i\theta_A)$$

for some elements $a_1, \ldots, a_n \in U_\alpha$ and real numbers $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ such that

$$\sum_{k=1}^{n} |\lambda_k + i\mu_k|^{k_\alpha} \leqslant 1.$$

Since $p_\alpha(a_k) < 1$ for each $k = 1, \ldots, n$, we can choose $\varepsilon_\alpha > 0$ so that

$$\max\{p_\alpha(a_1), \ldots, p_\alpha(a_n)\} < \varepsilon_\alpha^{k_\alpha} < 1.$$

Then $a_k \in \varepsilon_\alpha U_\alpha$ for each $\alpha \in \mathcal{A}$ and each $k = 1, \ldots, n$. Therefore

$$\frac{a + ib}{\varepsilon_\alpha} \in \sum_{k=1}^{n} (\lambda_k + i\mu_k)(\frac{a_k}{\varepsilon_\alpha} + i\theta_A) \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A).$$

Hence,

$$(a + ib) \in \varepsilon_\alpha \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$$

or $q_\alpha(a + ib) \leqslant \varepsilon_\alpha^{k_\alpha} < 1$. It means that the statement d) holds. □

A topological algebra (A, τ) is called a *Fréchet algebra* if the underlying topological vector space of (A, τ) is a *Fréchet space* that is, (A, τ) is complete and metrizable. It means, that every Cauchy net[3] in (A, τ) converges in (A, τ) and there exists a metrics d on A such that the topology on A defined by d, consider with τ. It is well known that (A, τ) is metrizable if it has a countable base of neighbourhoods of zero.

Corollary 1.3.2. *If (A, τ) is a real locally pseudoconvex Fréchet algebra, then $(\widetilde{A}, \widetilde{\tau})$ is a complex locally pseudoconvex Fréchet algebra.*

Proof. Let (A, τ) be a real locally pseudoconvex Fréchet algebra and $\{p_n, n \in \mathbb{N}\}$ a countable family of k_n-homogeneuos seminorms (with $k_n \in (0, 1]$ for each $n \in \mathbb{N}$), which defines the topology τ on A. Then $\{q_n : n \in \mathbb{N}\}$ defines on $\widetilde{A}$ a metrizable locally pseudoconvex topology $\widetilde{\tau}$ by Theorem 1.3.1. If $(a_n + ib_n)$ is a Cauchy sequence in $\widetilde{A}$, then (a_n) and (b_n) are Cauchy sequences in A by the inequality b) of Theorem 1.3.1. Because A is complete, then (a_n) converges to $a_0 \in A$ and (b_n) converges to $b_0 \in A$. Hence $(a_n + ib_n)$ converges in $\widetilde{A}$ to $a_0 + ib_0 \in \widetilde{A}$ by the same inequality b). Thus, $\widetilde{A}$ is a complex locally pseudoconvex Fréchet algebra. □

Theorem 1.3.3. *Let A be a real locally A-pseudoconvex* (*locally m-pseudoconvex*) *algebra. Then $\widetilde{A}$ is a complex locally A-pseudoconvex* (*respectively, locally m-pseudoconvex*) *algebra.*

Proof. Let (A, τ) be a real locally A-pseudoconvex algebra and $\mathcal{P}_A = \{p_\alpha : \alpha \in \mathcal{A}\}$ a family of k_α-homogeneous A-multiplicative seminorms on A (with $k_\alpha \in (0, 1]$ for each $\alpha \in \mathcal{A}$), which defines the topology τ on A and $(\widetilde{A}, \widetilde{\tau})$ the complexification[4] of (A, τ). Then

[3]A net $(x_\lambda)_{\lambda \in \Lambda}$ of elements of topological vector space X is called a *Cauchy net* if for each neighbourhood U of zero of X there is an index $\lambda_0 \in \Lambda$ such that $x_\lambda - x_\mu \subset U$, whenever $\lambda > \mu > \lambda_0$.

[4]Here $\widetilde{\tau}$ denotes the topology on $\widetilde{A}$ defined by the system $\{q_\alpha : \alpha \in \mathcal{A}\}$ (see Theorem 1.3.1).

every fixed $a_0 \in A$ and $\alpha \in \mathcal{A}$ there are positive numbers $M_\alpha(a_0)$ and $N_\alpha(a_0)$ such that

$$p_\alpha(a_0 a) \leqslant M_\alpha(a_0) p_\alpha(a) \quad \text{and } p_\alpha(a a_0) \leqslant N_\alpha(a_0) p_\alpha(a)$$

for all $a \in A$. If $a_0 + ib_0$ is a fixed and $a + ib$ an arbitrary element in $\widetilde{A}$, then

$$\begin{aligned} q_\alpha((a_0 + ib_0)(a + ib)) = q_\alpha((a_0 a - b_0 b) + i(a_0 b + b_0 a)) \leqslant \\ \leqslant 2 \max\{p_\alpha(a_0 a - b_0 b), p_\alpha(a_0 b + b_0 a)\} \end{aligned}$$

for each $\alpha \in \mathcal{A}$ by the inequality b) of Theorem 1.3.1. If now

$$p_\alpha(a_0 a - b_0 b) \geqslant p_\alpha(a_0 b + b_0 a),$$

then

$$\begin{aligned} &\max\{p_\alpha(a_0 a - b_0 b), p_\alpha(a_0 b + b_0 a)\} = \\ &= p_\alpha(a_0 a - b_0 b) \leqslant M_\alpha(a_0) p_\alpha(a) + M_\alpha(b_0) p_\alpha(b) \leqslant \\ &\leqslant \max\{p_\alpha(a), p_\alpha(b)\}(M_\alpha(a_0) + M_\alpha(b_0)) \leqslant \\ &\leqslant \frac{1}{2} M_\alpha(a_0, b_0) q_\alpha(a + ib) \end{aligned}$$

by the same inequality b) of Theorem 1.3.1, where

$$M_\alpha(a_0, b_0) = 2(M_\alpha(a_0) + M_\alpha(b_0)).$$

Hence

$$q_\alpha((a_0 + ib_0)(a + ib)) \leqslant M_\alpha(a_0, b_0) q_\alpha(a + ib) \tag{1.3.4}$$

for each $a + ib \in \widetilde{A}$ and $\alpha \in \mathcal{A}$.

The proof for the case, when $p_\alpha(a_0 a - b_0 b) < p_\alpha(a_0 b + b_0 a)$, is similar. Thus, the inequality (1.3.4) holds for both cases. In the same way it is easy to show that the inequality

$$q_\alpha((a + ib)(a_0 + ib_0)) \leqslant N_\alpha(a_0, b_0) q_\alpha(a + ib)$$

holds for each $a + ib \in \widetilde{A}$ and $\alpha \in \mathcal{A}$. Consequently, $(\widetilde{A}, \widetilde{\tau})$ is a complex locally A-pseudoconvex algebra.

Let now (A, τ) be a real locally m-pseudoconvex algebra, then each $p_\alpha \in \mathcal{P}_A$ is a submultiplicative seminorm on A. If $a + ib$, $a' + ib' \in \widetilde{A}$, then

$$q_\alpha((a+ib)(a'+ib')) \leqslant 2\max\{p_\alpha(aa'-bb'), p_\alpha(ab'+ba')\}$$

by the inequality b) of Theorem 1.3.1. Again, if

$$p_\alpha(aa'-bb') \geqslant p_\alpha(ab'+ba'),$$

then

$$\begin{aligned}&\max\{p_\alpha(ab'-bb'), p_\alpha(ab'+bb')\} = p_\alpha(ab'-bb') \leqslant \\ &\leqslant p_\alpha(a)p_\alpha(a') + p_\alpha(b)p_\alpha(b') \leqslant \\ &\leqslant 2\max\{p_\alpha(a), p_\alpha(b)\}\max\{p_\alpha(a'), p_\alpha(b')\} \leqslant \\ &\leqslant 2q_\alpha(a+ib)q_\alpha(a'+ib')\end{aligned}$$

by the inequality b) of Theorem 1.3.1 Hence,

$$q_\alpha((a+ib)(a'+ib')) \leqslant 4q_\alpha(a+ib)q_\alpha(a'+ib').$$

Putting $r_\alpha = 4q_\alpha$ for each $\alpha \in \mathcal{A}$, we see that

$$r_\alpha((a+ib)(a'+ib')) \leqslant r_\alpha(a+ib)r_\alpha(a'+ib') \tag{1.3.5}$$

for each $a+ib, a'+ib' \in \widetilde{A}$ and $\alpha \in \mathcal{A}$.

The proof for the case, when $p_\alpha(ab'-bb') < p_\alpha(ab'+bb')$, is similar. Hence, the inequality (1.3.5) holds for both cases. Since the families $\{q_\alpha : \ \alpha \in \mathcal{A}\}$ and $\{r_\alpha : \ \alpha \in \mathcal{A}\}$ define on $\widetilde{A}$ the same topology $\widetilde{\tau}$, then $(\widetilde{A}, \widetilde{\tau})$ is a complex locally m-pseudoconvex algebra. □

1.4 Real galbed algebras and their complexifications

Let ℓ^0 be the set of all sequences $(\alpha_n) \in \mathbb{R}^{\mathbb{N}}$, in which only finite number of members α_n are different from zero. Moreover, let k be a positive real number, ℓ^k the set of all sequences $(\alpha_n) \in \mathbb{R}^{\mathbb{N}}$, for which the series

$$\sum_{k=0}^{\infty} |\alpha_n|^k$$

converges and let $\ell = \ell^1 \setminus \ell^0$.

A topological algebra A is called a *galbed algebra* if its underlying topological vector space is a *galbed space,* that is, there exist a sequence $(\alpha_n) \in \ell$ and for each neighbourhood O of zero in A another neighbourhood U of zero such that

$$\Big\{\sum_{k=0}^{n} \alpha_k a_k : a_0, ..., a_n \in U\Big\} \subset O \qquad (1.4.1)$$

for each $n \in \mathbb{N}$.

Now we give two particular cases of galbed algebras[5]. A topological algebra A is

- *strongly galbed* if its underlying topological vector space is a *strongly galbed space* that is, if there exists a sequence $(\alpha_n) \in \ell$ with $\alpha_0 \neq 0$ and
$$\alpha = \inf_{n>0} |\alpha_n|^{\frac{1}{n}} > 0$$
such that the condition (1.4.1) has been satisfied;

- *exponentially galbed* if A is a (2^{-n})-galbed algebra.

It is easy to see that every locally pseudoconvex algebra is an exponentially galbed algebra and every exponentially galbed algebra is an (α_n)-galbed algebra with $\alpha_n = 2^{-n}$ for each $n \in \mathbb{N}$. Hence, the class of galbed algebras is much larger than the class of exponentially galbed algebras. Herewith, there exists a metrizable algebra, which is not a galbed algebra (see [16], Proposition 5).

A topological algebra A is called a *topological algebra with bounded elements* if all elements of A are *bounded* that is, for each $a \in A$ there is $\lambda_a \in \mathbb{R} \setminus \{0\}$ such that the set

$$\Big\{\Big(\frac{a}{\lambda_a}\Big)^n : n \in \mathbb{N}\Big\}$$

is bounded in A.

Next we will find conditions for a real (strongly) galbed algebra A in order to the complexification $\widetilde{A}$ of A is a (strongly) galbed algebra.

[5]In case, when we have already specified the sequence $(\alpha_n) \in \ell$, then we talk about an (α_n)-*galbed algebra.*

Theorem 1.4.1. *Let A be a real galbed algebra (a commutative real strongly galbed algebra with jointly continuous multiplication and bounded elements). Then $\widetilde{A}$ is a complex galbed algebra (respectively, a commutative complex strongly galbed algebra with bounded elements).*

Proof. Let A be a real galbed algebra and $\widetilde{O}$ a neighbourhood of zero in $\widetilde{A}$. Then there are a sequense $(\alpha_n) \in \ell$, a neighbourhood O of zero in A such that $O + iO \subset \widetilde{O}$ and another neighbourhood U of zero in A such that

$$\Big\{\sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$. Since $U + iU$ is a neighbourhood of zero in A and

$$\Big\{\sum_{k=0}^{n} \alpha_k (a_k + ib_k) : a_0 + ib_0, \ldots, a_n + ib_n \in U + iU\Big\} \subset \\ \subset O + iO \subset \widetilde{O} \tag{1.4.2}$$

for each $n \in \mathbb{N}$, then $\widetilde{A}$ is a complex galbed algebra.

Let now A be a commutative real strongly galbed algebra with jointly continuous multiplication and bounded elements. It is clear, that $\widetilde{A}$ is a commutative complex strongly galbed algebra. We will show that every element in $\widetilde{A}$ is bounded. For it, let $\widetilde{O}$ be an arbitrary neighbourhood of zero in $\widetilde{A}$ and $a+ib \in \widetilde{A}$ an arbitrary element. Then there are a neighbourhood O of zero in A such that $O + iO \subset \widetilde{O}$ and $\lambda_a, \lambda_b \in \mathbb{C} \setminus \{0\}$ such that the sets

$$\Big\{\Big(\frac{a}{\lambda_a}\Big)^n : \ n \in \mathbb{N}\Big\} \quad \text{and} \quad \Big\{\Big(\frac{b}{\lambda_b}\Big)^n : \ n \in \mathbb{N}\Big\}$$

are bounded in A. The neighbourhood O defines now a balanced neighbourhood U of zero in A such that (1.4.2) holds and U defines a balanced neighbourhood V of zero in A such that $VV \subset U$ (because the multiplication in A is jointly continuous). Now there are numbers $\mu_a, \mu_b > 0$ such that

$$\Big(\frac{a}{|\lambda_a|}\Big)^n \in \mu_a V \quad \text{and} \quad \Big(\frac{b}{|\lambda_b|}\Big)^n \in \mu_b V$$

for each $n \in \mathbb{N}$. Let

$$\kappa = \frac{2(|\lambda_a| + |\lambda_b|)}{\alpha},$$

where $\alpha = \inf_{n>0} |\alpha_n|^{\frac{1}{n}} > 0$ because A is strongly galbed. Since $a + ib = (a + i\theta_A) + i(b + i\theta_A)$, then

$$\begin{aligned} \Big(\frac{a+ib}{\kappa}\Big)^n &= \sum_{k=0}^{n} \binom{n}{k} \Big(\Big(\frac{a}{\kappa}\Big)^k + i\theta_A\Big) i^{n-k} \Big(\Big(\frac{b}{\kappa}\Big)^{n-k} + i\theta_A\Big) = \\ &= \mu_a \mu_b \sum_{k=0}^{n} \alpha_k \widetilde{x}_k \end{aligned}$$

for each $n \in \mathbb{N}$, where

$$\widetilde{x}_k = \varrho_{nk} \frac{1}{\mu_a \mu_b} \Big(\Big(\frac{a}{|\lambda_a|}\Big)^k \Big(\frac{b}{|\lambda_b|}\Big)^{n-k} + i\theta_A\Big)$$

and

$$\varrho_{nk} = \frac{1}{\alpha_k} i^{n-k} \binom{n}{k} \Big(\frac{|\lambda_a|}{\kappa}\Big)^k \Big(\frac{|\lambda_b|}{\kappa}\Big)^{n-k}$$

for each $k \leqslant n$. As $|\alpha_k| \geqslant \alpha^k$, then

$$|\varrho_{nk}| = \frac{1}{|\alpha_k|\kappa^n} \binom{n}{k} |\lambda_a|^k |\lambda_b|^{n-k} \leqslant \frac{1}{\alpha^n \kappa^n} (|\lambda_a| + |\lambda_b|)^n \leqslant \Big(\frac{1}{2}\Big)^n < 1$$

and

$$\Big(\frac{a}{|\lambda_a|}\Big)^k \Big(\frac{b}{|\lambda_b|}\Big)^{n-k} + i\theta_A \in \mu_a \mu_b VV + i\theta_A \subset \mu_a \mu_b (U + iU).$$

As U is a balanced set, then $\widetilde{x}_k \in U + iU$ for each $k = 0, \ldots, n$. Hence,

$$\Big(\frac{a+ib}{\kappa}\Big)^n \in \mu_a \mu_b (O + iO) \subset \mu_a \mu_b \widetilde{O}$$

by (1.4.2) for each $n \in \mathbb{N}$. It means that $a + ib$ is bounded in $\widetilde{A}$. Consequently, $\widetilde{A}$ is a commutative complex strongly galbed algebra with bounded elements. □

Corollary 1.4.2. *If A is a real exponentially galbed algebra (a commutative real exponentially galbed algebra with jointly continuous multiplication and bounded elements). Then $\widetilde{A}$ is a complex exponentially galbed algebra (respectively, a commutative complex exponentially galbed algebra with bounded elements).*

Proof. According to the definition, every exponentially galbed algebra is a strongly galbed algebra with $\alpha = \frac{1}{2}$. □

1.5 Strictly and formally real algebras

1. Let A be a (not necessary topological) algebra over $\mathbb{C}$ with the unit element e_A and Inv$A$ the set of all invertible[6] elements in A. Then the *spectrum* of $a \in A$ is the set

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \notin \mathrm{Inv}A\}.$$

If A is a real algebra, then the *spectrum* of $a \in A$ is defined by

$$\mathrm{sp}_A(a) = \mathrm{sp}_{\widetilde{A}}(a + i\theta_A),$$

where $\widetilde{A}$ is the complexification of A. Real algebras have two main subclasses.

a) A real unital algebra A is *strictly real* if $a^2 + e_A \in \mathrm{Inv}A$ for each $a \in A$.

Next result gives a sufficient condition for the strict reality of a real algebra.

Proposition 1.5.1. *Let A be a commutative real unital algebra. If $\mathrm{sp}_A(a) \subset \mathbb{R}$ for each $a \in A$, then A is strictly real.*

Proof. Let A be a commutative real unital algebra and $\widetilde{A}$ the complexification of A. Suppose that there is an element $a \in A$ such that $\alpha + i\beta \in \mathrm{sp}_A(a)$, with $\beta \neq 0$. Then

$$x = (a - \alpha e_A) - i\beta e_A = (a + i\theta_A) - (\alpha + i\beta)e_A \notin \mathrm{Inv}\widetilde{A}.$$

[6]An element $a \in A$ is *invertible* in A if there is an element $b \in A$ such that $ab = ba = e_A$.

Put $\overline{x} = a - (\alpha + i\beta)e_A$. If now $x\overline{x} \in \mathrm{Inv}A$, then there is an element $c \in A$ such that $(x\overline{x})c = c(x\overline{x}) = e_A$. Therefore,

$$(x\overline{x})(c + i\theta_A) = (x\overline{x})c + i\theta_A = e_A + i\theta_A.$$

Since A is a commutative algebra, then $x \in \mathrm{Inv}\widetilde{A}$, what is not possible. Hence,

$$x\overline{x} = (a - \alpha e_A)^2 + \beta^2 e_A \notin \mathrm{Inv}A$$

or

$$\Big(\frac{a - \alpha e_A}{\beta}\Big)^2 + e_A \notin \mathrm{Inv}A.$$

It means, that A is not strictly real. $\square$

Corollary 1.5.2. *Let A be a strictly real algebra, M a two-sided ideal in A and π_M the canonical homomorphism from A onto A/M. Then A/M is a strictly real algebra, too.*

Proof. If $x \in A/M$, then there is $a \in A$ such that $x = \pi_M(a)$. Since $\mathrm{sp}_{A/M}(\pi_M(a)) \subset \mathrm{sp}_A(a) \subset \mathbb{R}$, then A/M is strictly real algebra. $\square$

b) A real algebra A is *formally real* if

$$\text{from } a, b \in A \text{ and } a^2 + b^2 = \theta_A \quad \text{it follows} \quad \text{that } a = b = \theta_A. \quad (1.5.1)$$

The condition (1.5.1) shows that formally real algebras are "similar" to the field $\mathbb{R}$. It is known (see [20], Proposition 1.6.20) that the complexification $\widetilde{A}$ of a commutative real division algebra A is a division algebra if and only if A is formally real and every commutative real division algebra, which is not formally real, has the complex structure. Moreover (see [20], Proposition 1.9.14), a formally real division algebra is strictly real and a commutative strictly real division algebra is formally real.

Next result gives a necessary and sufficient condition for a quotient algebra A/I (over a two-sided ideal I) to be formally real.

Proposition 1.5.3. *Let A be a real algebra and I a two-sided ideal in A. Then the quotient algebra A/I is formally real if and only if A satisfies the condition*

$$\text{from } a, b \in A \text{ and } a^2 + b^2 \in I \quad \text{it follows} \quad \text{that} \quad a, b \in I. \quad (1.5.2)$$

Proof. Let A be a real algebra, I a two-sided ideal in A, π_I the quotient map of A onto A/I and let $a,\ b \in A$ be such that $a^2+b^2 \in I$. Then

$$\pi_I(a)^2 + \pi_I(b)^2 = \pi_I(a^2+b^2) = \theta_{A/I}.$$

If A/I is formally real, then $\pi_I(a) = \pi_I(b) = \theta_{A/I}$ or $a \in I$ and $b \in I$.

Let now algebra A satisfy the condition (1.5.2) and $x, y \in A/I$ be such that $x^2+y^2 = \theta_{A/I}$. Then there are $a,\ b \in A$ such that $x = \pi_I(a)$, $y = \pi_I(b)$ and

$$\pi_I(a^2+b^2) = x^2+y^2 = \theta_{A/I}.$$

Hence, from $a^2+b^2 \in I$ it follows that $x = y = \theta_{A/I}$ by the condition (1.5.2). □

2. Example. Let $C(X)$ be an algebra[7] of all continuous functions $f : X \to \mathbb{R}$, where X is a compact Hausdorff space. Then every maximal ideal M in $C(X)$ satisfies the condition (1.5.2), because every maximal ideal M defines $x \in X$ such that

$$M = M_x = \{f \in A : f(x) = 0\}.$$

More generally, if A is an subalgebra of $C(X)$, in which every maximal ideal M defines $x \in X$ such that $M = M_x$, then every maximal ideal M in A satisfies the condition (1.5.2) as well. Hence, the quotient algebra A/M is formally real algebra for each maximal ideal M in A.

1.6 Properties of the complexification of some real topological algebras

A unital topological algebra A is called a *Q-algebra* if the set Inv$A$ of all invertible elements of A is open in A and is called a *Waelbroeck algebra* if A is a Q-algebra in which the inverse $a \to a^{-1}$ is continuous in Inv$A$.

Let now A be a real unital topological algebra and $\widetilde{A}$ its complexification. Next we describe properties of $\widetilde{A}$ that we need later on.

[7] All algebraic operations in $C(X)$ are defined point-wisely.

Proposition 1.6.1. *If A is a commutative strictly real topological Hausdorff division algebra with continuous inversion. Then the complexification $\widetilde{A}$ of A is a commutative complex topological Hausdorff division algebra with continuous inversion.*

Proof. Let A be a commutative strictly real topological division algebra. Then $\widetilde{A}$ is a complex division algebra (see Propositions 1.6.20 and 1.9.14 from [20]). Since the underlying topological space of A is a Hausdorff space, then A is a Q-algebra. If the inversion in A is continuous, then A is a commutative real Waelbroeck algebra. Therefore, $\widetilde{A}$ is a commutative complex Hausdorff Waelbroeck algebra (see Proposition 3.6.31 from [20], or Proposition on the page 237 from [36]). Thus, $\widetilde{A}$ is a commutative complex Hausdorff division algebra with continuous inversion. □

Proposition 1.6.2. *Let A be a real topological algebra and $\widetilde{A}$ the complexification of A. If the topological dual A^* of A is not empty, then the topological dual $\widetilde{A}^*$ of $\widetilde{A}$ is also not empty.*

Proof. It is easy to see that if $\psi \in A^*$, then $\widetilde{\psi}$, defined by

$$\widetilde{\psi}(a+ib) = \psi(a) + i\psi(b)$$

for each $a + ib \in \widetilde{A}$, is an element of $\widetilde{A}^*$. □

Proposition 1.6.3. *Let A be a commutative strictly real division algebra and $\widetilde{A}$ the complexification of A. Then*

$$\mathrm{sp}_{\widetilde{A}}(a+ib) = \{\alpha + i\beta \in \mathbb{C} : \alpha \in \mathrm{sp}_A(a) \text{ and } \beta \in \mathrm{sp}_A(b)\}.$$

Proof. Let $\alpha + i\beta \in \mathrm{sp}_{\widetilde{A}}(a+ib)$. Since A is a commutative strictly real division algebra, then $\widetilde{A}$ is a commutative complex division algebra (see [20], Propositions 1.6.20 and 1.9.14). Therefore

$$a + ib - (\alpha + i\beta)(e_A + i\theta) = (a - \alpha e_A) + i(b - \beta e_A) = \theta_A + i\theta_A$$

if and only if $\alpha \in \mathrm{sp}_A(a)$ and $\beta \in \mathrm{sp}_A(b)$. □

1.7 Real Gelfand-Mazur division algebras

1. Let (A,τ) be a topological algebra, $M \in m(A)$ and π_M from A to A/M be the canonical homomorphism, then the *quotient topology* on A/M is defined by

$$\tau_{A/M} = \{U \subset A/M : \pi^{-1}(U) \in \tau\}.$$

A real topological algebra A is called a *real Gelfand-Mazur algebra* if the quotient algebra A/M (in the quotient topology) is topologically isomorphic[8] to $\mathbb{R}$ for each $M \in m(A)$. Complex Gelfand-Mazur algebras are defined similary (see [1], [2], [11]–[14]). Hence, a real Gelfand-Mazur algebra is a real topological algebra in which every $M \in m(A)$ defines a homomorhism $\psi_M \in \hom(A)$ such that $M = \ker\psi_M$. Here[9] and later on $\hom(A)$ denotes the set of all nontrivial continuous homomorphisms from A to $\mathbb{R}$.

The next result describes several classes[10] of real topological algebras which belong to the class of real Gelfand-Mazur division algebras.

Theorem 1.7.1. *Let A be a commutative strictly real topological division algebra. If there is a topology τ on A such that (A,τ) is one of the following algebras:*

a) *a locally pseudoconvex Hausdorff algebra with continuous inversion;*

b) *a Hausdorff algebra with continuous inversion for which the topological dual space A^* is not empty;*

c) *a strongly galbed* (*in particular, an exponentially galbed*) *Hausdorff algebra with jointly continuous multiplication and bounded elements;*

[8] Algebras A and B are topologically isomorphic if there is an isomorphism ρ from A onto B such that ρ and ρ^{-1} are continuous.

[9] The set $\hom(A)$ we endow, as usual, with the *Gelfand topology.* In this topology the sets $\{\psi \in \hom(A) : \quad |(\psi - \psi_o)(a)| < \varepsilon\}$ with $a \in A$ and $\varepsilon > 0$ form a subbase of neighbourhoos of $\psi_0 \in \hom(A)$.

[10] Several classes of complex Gelfand-Mazur algebras have been described in [11]–[13] and [17].

d) *a topological Hausdorff algebra for which the spectrum* $\mathrm{sp}_A(a)$ *is not empty for each* $a \in A$,

then A *and* $\mathbb{R}$ *are topologically isomorphic.*

Proof. If A is a commutative strictly real division algebra, then its complexification $\widetilde{A}$ is a commutative complex division algebra as above. Since (A, τ) satisfies

1) the condition a), then the complexification $(\widetilde{A}, \widetilde{\tau})$ of (A, τ) is a commutative complex locally pseudoconvex Hausdorff division algebra with continuous inversion (by Theorem 1.3.1 and Proposition 1.6.1);

2) the condition b), then the complexification $(\widetilde{A}, \widetilde{\tau})$ of (A, τ) is a commutative complex topological Hausdorff division algebra with continuous inversion for which the set $\widetilde{A}^*$ is not empty (by Propositions 1.6.1 and 1.6.2);

3) the condition c), then the complexification $(\widetilde{A}, \widetilde{\tau})$ of (A, τ) is a commutative complex strongly galbed Hausdorff division algebra with bounded elements (by Theorem 1.4.1) (in particular, a commutative complex exponentially galbed Hausdorff division algebra with bounded elements by Corollary 1.4.2);

4) the condition d), then the complexification $(\widetilde{A}, \widetilde{\tau})$ of (A, τ) is a commutative complex topological Hausdorff division algebra such that the spectrum $\mathrm{sp}_{\widetilde{A}}(a+ib)$ is not empty for each $a+ib \in \widetilde{A}$ (by Proposition 1.6.3).

Therefore $(\widetilde{A}, \widetilde{\tau})$ and $\mathbb{C}$ are topologically isomorphic (in cases a), b), and d) see [11], Theorem 1, and in case c) see [9], Proposition 4.1). Hence, every element $a + ib \in \widetilde{A}$ is representable in the form

$$a + ib = \lambda e_{\widetilde{A}}$$

for some $\lambda \in \mathbb{C}$. Now, for each $a \in A$ there is a real number μ_a such that $a = \mu_a e_A$, because A is strictly real. It is easy to see that ρ, defined by $\rho(a) = \mu_a$ for each $a \in A$, is an isomorphism from A to $\mathbb{R}$, whose inverse map is, continuous. To show the continuity of ρ, let O

be a neighbourhood of zero in $\mathbb{R}$. Then there is a number $\epsilon > 0$ such that

$$O_\epsilon = \{\alpha \in \mathbb{R} : \ |\alpha| < \epsilon\} \subset O.$$

If $\lambda_0 \in O_\epsilon \setminus \{0\}$, then there is a balanced neighbourhood V of zero of A such that $\lambda_0 e_A \notin V$ (because A is a Hausdorff space). If $|\mu_a| \geqslant |\lambda_0|$, then $|\mu_a^{-1}\lambda_0| \leqslant 1$. Therefore, $\lambda_0 e_A = (\lambda_0\mu_a^{-1})a \in V$ for each $a \in V$. As it is not possible, then $\mu_a \in O$ for each $a \in V$. Consequently, ρ is a continuous map. $\square$

Remark. In Theorem 1.7.1 the topology τ can be different from the preliminary topology of A.

Corollary 1.7.2. *Let A be a commutative strictly real division algebra. If there is a topology τ on A such that (A, τ) is one of the following algebras:*

a) *a locally pseudoconvex Hausdorff algebra with continuous inversion;*

b) *a locally A-pseudoconvex (in particular, a locally m-pseudoconvex) Hausdorff algebra;*

c) *a locally pseudoconvex Fréchet algebra;*

d) *a strongly galbed (in particular, an exponentially galbed) Hausdorff algebra with jointly continuous multiplication and bounded elements,*

e) *a topological Hausdorff algebra, for which the spectrum $\operatorname{sp}_A(a)$ is not empty for each $a \in A$,*

then A is a commutative real Gelfand-Mazur division algebra.

Proof. It is easy to see that A is a commutative real Gelfand-Mazur division algebra by Theorem 1.7.1 in cases a), d) and e). Since the inversion is continuous in every unital locally m-pseudoconvex Hausdorff algebra (see Lemma 2.2 in [17]) and every locally A-pseudoconvex Hausdorff algebra has a locally m-pseudoconvex topology (see Lemma 2.2 in [17]), then A is a commutative real Gelfand-Mazur division algebra by a).

Let now A be a commutative strictly real locally pseudoconvex Fréchet division algebra. Then A is a commutative strictly real locally pseudoconvex algebra with continuous inversion (see Corollary 7.6 from [35]). Hence, A is a commutative real Gelfand-Mazur division algebra by Theorem 1.7.1 a). □

Let now A be a real topological algebra with unit element e_A. An element $a \in A$ is *topologically invertible* in A if there is a net $(a_\lambda)_{\lambda\in\Lambda}$ of elements of A such that $(aa_\lambda)_{\lambda\in\Lambda}$ and $(a_\lambda a)_{\lambda\in\Lambda}$ converge in A to e_A (see, for example, [10], p. 14). The set of all topologically invertible elements of A we denote by $\mathrm{Tinv}A$.

Proposition 1.7.3. *Let A be a commutative real unital complete locally m-pseudoconvex Hausdorff algebra and B a strictly real subalgebra of A with the same unit e_A. If $m(B) \neq \emptyset$ and B satisfies the condition* (1.5.2) *for each $M \in m(B)$, then $\mathrm{cl}_A B$ is a commutative real unital locally m-pseudoconvex Hausdorff algebra, which satisfies the condition* (1.5.2) *for each $M \in m(\mathrm{cl}_A B)$.*

Proof. Let A be a commutative real unital complete locally m-pseudoconvex Hausdorff algebra and B a strictly real subalgebra of A which satisfies the condition (1.5.2) for each $M \in m(B)$. Then A is a real topological algebra with jointly continuous multiplication and B is a unital strictly real locally m-pseudoconvex Hausdorff algebra which satisfies the condition (1.5.2) for each $M \in m(B)$. Hence B is a real Gelfand-Mazur algebra by Corollary 1.7.2. Since $m(B) \neq \emptyset$, then $\hom(B) \neq \emptyset$. Therefore for each $\phi \in \hom(B)$ there is $\psi \in \hom(\mathrm{cl_A} B)$ such that $\phi = \psi|_B$ by Proposition 3 from [15] and the map $\phi \to \psi$ is bijection from $\hom(B)$ onto[11] $\hom(\mathrm{cl_A} B)$ by Theorem 4 from [15]. Now, let $b \in \mathrm{cl}_A B$ be an arbitrary element. Then there is in B a net $(a_\lambda)_{\lambda\in\Lambda}$ such that $(a_\lambda)_{\lambda\in\Lambda}$ converges to b in the topology of A and $e_A + a_\lambda^2 \in \mathrm{Inv}B$ for each $\lambda \in \Lambda$ because B is strictly real. Since

$$\psi(e_A + b^2) = \lim_\lambda \psi(e_A + a_\lambda^2) = 1 + \lim_\lambda \phi(a_\lambda)^2 = 1 + |\psi(b)|^2 \geq 1$$

for each $\psi \in \hom(\mathrm{cl}_A B)$, then $e_A + b^2 \in \mathrm{Tinv}(\mathrm{cl_A} B)$ by Proposition 11 e) from [10]. As $\mathrm{cl}_A B$ is complete and locally m-pseudoconvex, then

[11]Here and later on by $\mathrm{cl}_X(U)$ is denoted the closure of $U \subset X$ in the topology of the space X.

$\mathrm{Inv}(\mathrm{cl}_{\mathrm{A}}(B)) = \mathrm{Tinv}(\mathrm{cl}_A(B))$ by Corollary 2 from [10]. Hence

$$e_A + b^2 \in \mathrm{Inv}(\mathrm{cl}_{\mathrm{A}} B).$$

Thus, $\mathrm{cl}_A B$ is a commutative strictly real locally m-pseudoconvex algebra. Therefore $\mathrm{cl}_A B$ is a strictly real Gelfand-Mazur algebra by Proposition 3 from [15] and $m(\mathrm{cl}_A B) \neq \emptyset$.

Let now $M \in m(\mathrm{cl}_A B)$. Then there is $\phi \in \mathrm{hom}(\mathrm{cl}_A B)$ such that $M = \mathrm{ker}\phi$. If $a, b \in \mathrm{cl}_A B$ and $a^2 + b^2 \in M$, then from $\phi(a)^2 + \phi(b)^2 = 0$ it follws that $\phi(a) = \phi(b) = 0$ or $a, b \in M$. Consequently, $\mathrm{cl}_A B$ satisfies the condition (1.5.2) for each $M \in m(\mathrm{cl}_A B)$. $\square$

Chapter 2

Real Gelfand-Mazur algebras

In this Chapter we give a description of classes of real (commutative and noncommutative) Gelfand-Mazur algebras. We present conditions for a real topological algebra A, for which the center of A/P (the quotient algebra of A by a closed primitive ideal P) is homeomorphic to $\mathbb{R}$. Using this result, we give a description of closed maximal left (right or two-sided) ideals in real unital Gelfand-Mazur algebra.

Results of this Chapter are published in [28].

2.1 Properties of quotient algebras and of the center of a topological algebra

Let A be a real topological algebra, I a closed two-sided ideal in A, $\pi_I : A \to A/I$ the canonical homomorphism and

$$Z(A) = \{z \in A : za = az \text{ for each } a \in A\}$$

the center of A.

Proposition 2.1.1. *Let* (A, τ) *be a real locally pseudoconvex (in particular, a locally A-pseudoconvex or a locally m-pseudoconvex) algebra and* I *a closed two-sided ideal in* A*. Then* $(A/I, \tau_{A/I})$ *(in the quotient topology) and* $(Z(A/I), \tau_Z)$ *(in the subspace topology) are also* *real locally pseudoconvex (in particular, locally A-pseudoconvex or locally m-pseudoconvex) algebras.*

Proof. Let (A, τ) be a locally pseudoconvex algebra, then there is a base

$$\mathcal{B} = \{U_\alpha : \quad \alpha \in \mathcal{A}\}$$

of neighbourhoods of zero, consisting of balanced and pseudoconvex subsets of A. It is easy to see that

$$\mathcal{B}' = \pi_I(\mathcal{B}) = \{\pi_I(U_\alpha) : \quad U_\alpha \in \mathcal{B},\ \alpha \in \mathcal{A}\}$$

is a base of neighbourhoods of zero in $(A/I, \tau_{A/I})$, consisting of balanced and pseudoconvex subsets of A/I. Thus, $(A/I, \tau_{A/I})$ is a real locally pseudoconvex algebra.

Since $\tau_Z = \{O' \cap Z(A/I) : \ O' \in \tau_{A/I}\}$ is the subspace topology on $Z(A/I)$, generated by $\tau_{A/I}$, then it is clear that $(Z(A/I), \tau_Z)$ is a real locally pseudoconvex algebra.

Analogously, $(Z(A/I), \tau_Z)$ is a real locally A-pseudoconvex (respectively, locally m-pseudoconvex) algebra if A is a real locally A-pseudoconvex (respectively, locally m-pseudoconvex) algebra. □

Proposition 2.1.2. *Let (A, τ) be a real locally pseudoconvex Fréchet algebra and I a closed two-sided ideal in A. Then $(A/I, \tau_{A/I})$ (in the quotient topology) and $(Z(A/I), \tau_Z)$ (in the subspace topology) are also real locally pseudoconvex Fréchet algebras.*

Proof. By Theorem 2, p. 138, from [23] the quotient algebra $(A/I, \tau_{A/I})$ is a Fréchet algebra. Since $Z(A/I)$ is closed in A/I, then $(Z(A/I), \tau_Z)$ is complete and metrizable, hence $(Z(A/I), \tau_Z)$ is a Fréchet algebra. □

Proposition 2.1.3. *Let (A, τ) be a real topological algebra with bounded elements and I a closed two-sided ideal in A. Then $(A/I, \tau_{A/I})$ (in the quotient topology) and $(Z(A/I), \tau_Z)$ (in the subspace topology) are also real topological algebras with bounded elements.*

Proof. Let $x \in A/I$ be an arbitrary element and U an arbitrary neighbourhood of zero in A/I. Then there is an element $a \in A$ such that $x = \pi_I(a)$. Moreover, $\pi_I^{-1}(U)$ is a neigbourhood of zero in A.

Since every element in A is bounded, then there exist $\lambda \in \mathbb{R} \setminus \{0\}$ and a number $\mu > 0$ such that

$$\left(\frac{a}{\lambda}\right)^n \in \mu\pi_I^{-1}(U)$$

for each $n \in \mathbb{N}$. Now

$$\left(\frac{x}{\lambda}\right)^n = \left(\frac{\pi_I(a)}{\lambda}\right)^n = \pi_I\left(\left(\frac{a}{\lambda}\right)^n\right) \in \mu(\pi_I(\pi_I^{-1}(U))) = \mu U.$$

Hence, $(A/I, \tau_{A/I})$ is a real topological algebra with bounded element.

Let now $y \in Z(A/I)$ be an arbitrary element and W'' an arbitrary neighbourhood of zero in $Z(A/I)$. Then $y \in A/I$ and there exists a neighbourhood W' of zero in A/I such that $W'' = W' \cap Z(A/I)$. Since every element in A/I is bounded, then there exist $\lambda_y \in \mathbb{R} \setminus \{0\}$ and number $\mu'_W > 0$ such that

$$\left(\frac{y}{\lambda_y}\right)^n \in \mu'_W W'$$

for each $n \in \mathbb{N}$. As $y \in Z(A/I)$, then

$$\left(\frac{y}{\lambda_y}\right)^n \in Z(A/I)$$

for each $n \in \mathbb{N}$. Therefore,

$$\left(\frac{y}{\lambda_y}\right)^n \in \mu'_W W''$$

for each $n \in \mathbb{N}$. Thus, every element in $Z(A/I)$ is bounded. $\square$

Proposition 2.1.4. *Let (A, τ) be a real galbed (in particular, strongly or exponentially galbed) algebra and I a closed two-sided ideal in A. Then $(A/I, \tau_{A/I})$ (in the quotient topology) and $(Z(A/I), \tau_Z)$ (in the subspace topology) are also real galbed (in particular, strongly or exponentially galbed) algebras.*

Proof. Let (A, τ) be a real galbed algebra. Then there is a sequence $(\alpha_n) \in \ell$ such that (A, τ) is a (α_n)-galbed algebra. Moreover, let O be a neighbourhood of zero in $(A/I, \tau_{A/I})$. Then $U = \pi_I^{-1}(O)$ is a neighbourhood of zero in (A, τ) and there is in A a neighbourhood V of zero such that

$$\Big\{\sum_{k=0}^{n} \alpha_k x_k; \quad x_0, \ldots, x_n \in V\Big\} \subset U$$

for each $n \in \mathbb{N}$. Now $\pi_I(V)$ is a neighbourhood of zero in $(A/I, \tau_{A/I})$, because π_I is an open map. Let now $n \in \mathbb{N}$ and $x_0, \ldots, x_n \in \pi_I(V)$. Then there are elements $a_0, \ldots a_n \in V$ such that $x_k = \pi_I(a_k)$ for each $k = 0, ..., n$ and

$$\sum_{k=0}^{n} \alpha_k x_k = \pi_I\Big(\sum_{k=0}^{n} \alpha_k a_k\Big) \in \pi_I(\pi_I^{-1}(O)) = O$$

for each $n \in \mathbb{N}$. Hence, $(A/I, \tau_{A/I})$ is an (α_n)-galbed algebra.

Since τ_Z is the subspace topology on $Z(A/I)$ generated by $\tau_{A/I}$, then every neighbourhood O'' of zero in $Z(A/I)$ in this topology is representable in the form $O'' = O' \cap Z(A/I)$, where O' is a neighbourhood of zero in A/I. Now we find a neighbourhood V' of zero in A/I such that

$$\Big\{\sum_{k=0}^{n} \alpha_k a_k : \quad a_0, \ldots, a_n \in V'\Big\} \subset O'$$

for each $n \in \mathbb{N}$ and put $V'' = V' \cap Z(A/I)$. Since

$$\Big\{\sum_{k=0}^{n} \alpha_k x_k : \quad x_0, \ldots, x_n \in V''\Big\} \subset O''$$

for each $n \in \mathbb{N}$, then $(Z(A/I), \tau_Z)$ is a real (α_n)-galbed algebra. The proof for strongly and exponentially galbed algebras is similar. □

Proposition 2.1.5. *Let A be a real topological algebra and $M \in m(A)$. If for each $a \in A$ there exists $\lambda \in \mathbb{R} \setminus 0$ such that $a - \lambda e_A \in M$, then the spectrum $sp_{A/M}(x)$ is not empty for each element $x \in A/M$ and $\mathrm{sp}_{Z(A/M)}(b)$ is not empty for each $b \in Z(A/M)$.*

Proof. Let A be a real topological algebra, $M \in m(A)$, π_M be the canonical homomorphism from A onto A/M and $x \in A/M$ an arbitrary element. Then there exists an element $a \in A$ such that

$x = \pi_M(a)$. Let $\lambda_a \in \mathbb{R}$ be such that $a - \lambda_a e_A \in M$ by assumption. Then

$$\pi_M(a) - \lambda_a e_{A/M} = \pi_M(a - \lambda_a e_A) = \theta_{A/M}.$$

Since A/M is a division algebra (see [22], the proof of the Theorem 24.9.6, or [27], Theorem 2.4.12), then

$$\pi_M(a) - \lambda_a e_{A/M} \notin \mathrm{Inv} A/M.$$

Hence, $\lambda_a \in \mathrm{sp}_{A/M}(\pi(a))$.

Moreover, $\mathrm{sp}_{A/M}(b) \subset \mathrm{sp}_{Z(A/M)}(b)$ for each $b \in Z(A/M)$ (because $Z(A/M) \subset A/M$) and

$$\mathrm{Inv} Z(A/M) = \mathrm{Inv} A/M \cap Z(A/M).$$

Thus, $\mathrm{sp}_{Z(A/M)}(b)$ is not empty for each $b \in Z(A/M)$ and

$$\mathrm{sp}_{Z(A/M)}(b) = \mathrm{sp}_{A/M}(b).$$

□

2.2 Commutative real Gelfand-Mazur algebras

In this section we describe some real commutative Gelfand-Mazur algebras.

Theorem 2.2.1. *Let A be a commutative real topological algebra. If A satisfies the condition*

$$\text{from } a, b \in A \text{ and } a^2 + b^2 \in M \text{ it follows that } a, b \in M \tag{2.2.1}$$

for each $M \in m(A)$ and there is a topology τ on A such that (A, τ) is one of the following algebras:

a) *a locally pseudoconvex Waelbroeck algebra;*

b) *a locally A-pseudoconvex (in particular, a locally m-pseudoconvex) algebra;*

c) *a locally pseudoconvex Fréchet algebra;*

d) *a strongly galbed* (*in particular, an exponentially galbed*) *algebra with jointly continuous multiplication and bounded elements;*

e) *a topological algebra in which for each $a \in A$ and $M \in m(A)$ there is $\lambda \in \mathbb{R}$ such that $a - \lambda e_A \in M$,*

then A is a commutative real Gelfand-Mazur algebra.

Proof. Let (A, τ) be a commutative real topological algebra which satisfies the condition (2.2.1) for each $M \in m(A)$ and let M be a fixed element in $m(A)$. Then $(A/M, \tau_{A/M})$ is a topological division Hausdorff algebra. Moreover, $(A/M, \tau_{A/M})$ is a commutative formally real algebra by Proposition 1.5.3. Hence, it is a commutative strictly real algebra. Herewith, if (A, τ) satisfies

1) the condition a), then $(A/M, \tau_{A/M})$ is a locally pseudoconvex Waelbroeck algebra by Proposition 2.1.1 and Corollary 3.6.27 from [20];

2) the condition b), then $(A/M, \tau_{A/M})$ is a locally A-pseudoconvex (in particular, a locally m-pseudoconvex) algebra by Proposition 2.1.1;

3) the condition c), then $(A/M, \tau_{A/M})$ is a locally pseudoconvex Fréchet algebra by Proposition 2.1.2;

4) the condition d), then $(A/M, \tau_{A/M})$ is a strongly galbed (in particular, an exponentially galbed) algebra with jointly continuous multiplication and bounded elements by Propositions 2.1.3 and 2.1.4;

5) the condition e), then $(A/M, \tau_{A/M})$ is a topological algebra for which the spectrum $\mathrm{sp}_{A/M}(x)$ is not empty for each $x \in A/M$ by Proposition 2.1.5.

Hence, in all these cases A/M (in the quotient topology defined by the topology τ of A) is topologically isomorphic to $\mathbb{R}$ for each $M \in m(A)$ by Theorem 1.7.1 and Corollary 1.7.2. Therefore, A is a commutative real Gelfand-Mazur algebra. □

2.3 Some properties of ideals

Let A be an algebra over $\mathbb{R}$ and $M \in m_l(A)$ $(M \in m_r(A))$. Then

$$P_M = \{a \in A : aA \subset M\} \text{ (respectively, } P_M = \{a \in A : Aa \subset M\})$$

is the *primitive ideal* in A defined by M. Herewith, P_M is a closed two-sided ideal in A and if $M \in m_t(A)$, then $P_M = M$.

Let now A be a real algebra and $\widetilde{A}$ the complexification of A. Then the following results are true.

Proposition 2.3.1. *Let A be a real topological Hausdorff algebra, $\widetilde{A}$ the complexification of A and let $M \in m_l(A)$. Then*

a) *every ideal $\widetilde{M} \in m_l(\widetilde{A})$ is representable in the form*

$$\widetilde{M} = M + iM,$$

where $M \in m_l(A)$, and $M + iM \in m_l(\widetilde{A})$ for every $M \in m_l(A)$;

b) *the primitive ideal $\widetilde{P}_{\widetilde{M}}$ in $\widetilde{A}$, defined by $\widetilde{M}$=M+iM, is representable in the form $\widetilde{P}_{\widetilde{M}} = P_M + iP_M$, where P_M is the primitive ideal in A defined by $M \in m_l(A)$;*

c) $\widetilde{A}/\widetilde{P}_{\widetilde{M}} = A/P_M + iA/P_M$ *for each* $\widetilde{M} = M + iM \in m_l(\widetilde{A})$.

Similar results are true for ideals in $m_r(A)$ and $m_t(A)$.

Proof. a) Let A be a real topological algebra, $\widetilde{A}$ its complexification, $\widetilde{M} \in m_l(\widetilde{A})$,

$$M = \{a \in A : \ a + ib \in \widetilde{M} \quad \text{for some } b \in A\}$$

and

$$M' = \{b \in A : \ a + ib \in \widetilde{M} \quad \text{for some } a \in A\}.$$

Since

$$\begin{aligned} ca + icb &= (c + i\theta_A)(a + ib), \\ -b + ia &= i(a + ib), \\ b + i(-a) &= -i(a + ib) \end{aligned}$$

for each $a, b, c \in A$, then M and M' are left ideals in A, $M = M'$, $M + iM \neq \widetilde{A}$, $M + iM$ is a left ideal in $\widetilde{A}$ and

$$\widetilde{M} \subset M + iM' = M + iM.$$

Hence, $\widetilde{M} = M + iM$ (because $\widetilde{M}$ is a left maximal ideal in $\widetilde{A}$).

Now we show that M is a maximal ideal. For it, let I be an arbitrary ideal in A, such that $M \subset I$. Since $I + iI$ is an left ideal in $\widetilde{A}$, $\widetilde{M} \subset I + iI$ and $\widetilde{M}$ is maximal in $\widetilde{A}$, then $\widetilde{M} = I + iI$. Therefore, $M = I$ that is, M is maximal in A.

Next we show that M is a closed ideal in A. For it, let $a_0 \in \mathrm{cl}_A M$. Then there is a net $(m_\lambda)_{\lambda \in \Lambda}$ in A, which converges in A to a_0. Now $(m_\lambda + i\theta_A)$ is a net in $\widetilde{A}$, which converges in $\widetilde{A}$ to $a_0 + i\theta_A$ (because $\widetilde{A}$ is a Hausdorff space). Hence, $a_0 + i\theta_A \in \widetilde{M}$ (because $\widetilde{M}$ is closed in $\widetilde{A}$). Consequently, $a_0 \in M$. It means that $M \in m_l(A)$.

Now we show that $M + iM \in m_l(\widetilde{A})$ for $M \in m_l(A)$. It is easy to see that $M + iM$ is a left ideal in $\widetilde{A}$. If $\widetilde{J}$ is a left ideal in $\widetilde{A}$ such that $M + iM \subset \widetilde{J}$, then

$$M \subset J = \{a \in A : \ a + ib \in \widetilde{J} \quad \text{for some } b \in A\}.$$

As J is an left ideal in A, $J \neq A$ and M is maximal, then $M = J$. Hence, similarily as above $M + iM = \widetilde{J}$. Consequently, $M + iM$ is a maximal ideal in A.

Let now $a_0 + ib_0 \in \mathrm{cl}_{\widetilde{A}}(M + iM)$. Then there is a net $(m_\lambda + in_\lambda)_{\lambda \in \Lambda}$ in $M + iM$ such that $(m_\lambda + in_\lambda)_{\lambda \in \Lambda}$ converges to $a_0 + ib_0$. Therefore $(m_\lambda)_{\lambda \in \Lambda}$ converges in A to a_0 and $(n_\lambda)_{\lambda \in \Lambda}$ converges in A to b_0. Hence, $a_0 + ib_0 \in M + iM$, because m_λ, $n_\lambda \in M$ for each $\lambda \in \Lambda$ and M is closed. Consequently, $M + iM \in m_l(\widetilde{A})$.

The proof for ideals in $m_r(A)$ is similar. If $M \in m_t(A)$, then $M \in m_l(A)$ and $M \in m_r(A)$, therefore result is also true.

b) Let $\widetilde{P}_{\widetilde{M}}$ be the primitive ideal in $\widetilde{A}$, defined by $\widetilde{M} \in m_l(\widetilde{A})$. Then there is an ideal $M \in m_l(A)$ such that $\widetilde{M} = M + iM$. Let $a, b \in P_M$ and $v + iw \in \widetilde{A}$. Since

$$(a + ib)(v + iw) = av - bw + i(aw + bv) \in \widetilde{M},$$

then $P_M + iP_M \subset \widetilde{P}_{\widetilde{M}}$. Let now $a + ib \in \widetilde{P}_{\widetilde{M}}$ and $v + i\theta_A \in \widetilde{A}$. Then

$$(a + ib)(v + i\theta_A) = av + ibv \in \widetilde{M}$$

if and only if $av,\ bv \in M$ or $a,\ b \in P_M$. Thus $\widetilde{P}_{\widetilde{M}} \subset P_M + iP_M$. The proof for ideals in $m_r(A)$ is similar.

c) Let $\widetilde{M} \in m_l(\widetilde{A})$ and $a,\ b \in A$. Then again there is an ideal $M \in m_l(A)$ such that $\widetilde{M} = M + iM$ and

$$a + P_M + i(b + P_M) = (a + ib) + (P_M + iP_M) = (a + ib) + \widetilde{P}_{\widetilde{M}} \in \widetilde{A}/\widetilde{P}_{\widetilde{M}}.$$

Hence $A/P_M + iA/P_M \subset \widetilde{A}/\widetilde{P}_{\widetilde{M}}$ and similarly

$$\widetilde{A}/\widetilde{P}_{\widetilde{M}} \subset A/P_M + iA/P_M.$$

The proof for ideals in $m_r(A)$ is similar. □

Proposition 2.3.2. *Let A be a real topological algebra and $\widetilde{A}$ the complexification of A. Then*

$$Z(\widetilde{A}) = Z(A) + iZ(A).$$

Proof. It is clear that $Z(A) + iZ(A) \subset Z(\widetilde{A})$. Now we show that $Z(\widetilde{A}) \subset Z(A) + iZ(A)$. For it, let $a_0 + ib_0 \in Z(\widetilde{A})$. Since

$$aa_0 + iab_0 = (a + i\theta_A)(a_0 + ib_0) = (a_0 + ib_0)(a + i\theta_A) = a_0a + ib_0a$$

for each $a \in A$, then $a_0 \in Z(A)$ and $b_0 \in Z(A)$. □

A topological algebra A is called a *topologically primitive algebra* if there is an ideal $M \in m_l(A)$ ($M \in m_r(A)$) such that $P_M = \{\theta_A\}$.

Corollary 2.3.3. *If A is a real topologically primitive algebra, then the complexification $\widetilde{A}$ of A is a complex topologically primitive algebra.*

Proof. Let A be a real topologically primitive topological algebra. Then there is $M \in m_l(A)$ such that $P_M = \{\theta_A\}$. Since

$$\widetilde{P}_{\widetilde{M}} = P_M + iP_M = \theta_A + i\theta_A$$

for $\widetilde{M} = M + iM \in m_l(\widetilde{A})$, then $\widetilde{A}$ is a complex topologically primitive topological algebra. □

2.4 Noncommutative real Gelfand-Mazur algebras

Let A be a real topological algebra and $m_l'(A)$ the set of such $M \in m_l(A)$ for which the primitive ideal P_M satisfies the condition

$$\text{from } a, b \in A \text{ and } a^2 + b^2 \in P_M \text{ it follows that } a, b \in P_M. \quad (2.4.1)$$

The set $m_r'(A)$ we define simirarily. If $M \in m_t(A)$, then $P_M = M$. Therefore, by $m_t'(A)$ we define the set of such $M \in m_t(A)$ for which the condition (2.2.1) is true.

Theorem 2.4.1. *Let A be a real unital locally A-pseudoconvex algebra or a real unital locally pseudoconvex Fréchet algebra. Then $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$ (in the subset topology on $Z(A/P_M)$) for each $M \in m_l'(A)$ ($M \in m_r'(A)$). If $M \in m_t'(A)$, then $Z(A/M)$ is topologically isomorphic to $\mathbb{R}$.*

Proof. Let (A, τ) be a real unital locally A-pseudoconvex (or locally pseudoconvex Fréchet) algebra, $M \in m_l'(A)$, P_M the primitive ideal in A, defined by M, $\pi_M : A \to A/P_M$ the canonical homomorphism and τ_M the quotient topology on A/P_M, defined by τ and π_M. Then $(A/P_M, \tau_M)$ is a unital formally real locally A-pseudoconvex (respectively, locally pseudoconvex Fréchet) algebra by Propositions 1.5.3 and 2.1.1 (respectively, by Propositions 1.5.3 and 2.1.2).

Now, let $\widetilde{A}/\widetilde{P}_{\widetilde{M}}$ be the complexification of A/P_M (see Proposition 2.3.1), where $\widetilde{P}_{\widetilde{M}}$ is the primitive ideal in $\widetilde{A}$ defined by $\widetilde{M}$. Then $(\widetilde{A}/\widetilde{P}_{\widetilde{M}}, \widetilde{\tau}_{\widetilde{M}})$ is a unital complex locally A-pseudoconvex algebra by Theorem 1.3.3 or a unital complex locally pseudoconvex Fréchet algebra by Corollary 1.3.2. Therefore $Z(\widetilde{A}/\widetilde{P}_{\widetilde{M}})$ is topologically isomorphic to $\mathbb{C}$ by Theorem 1 from [1] or by Theorem 2.17 from [2]. Since A/P_M is formally real, then $Z(A/P_M)$ is formally real, too.

As

$$Z(\widetilde{A}/\widetilde{P}_{\widetilde{M}}) = Z(A/P_M) + iZ(A/P_M)$$

by Proposition 2.3.2 and $Z(A/P_M)$ is formally real (see [20], Proposition 1.6.20), then $Z(A/P_M)$ is isomorphic to $\mathbb{R}$. In the same way as in Theorem 1.7.1, it is easy to show that $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$, because $Z(A/P_M)$ is a Hausdorff space in the subset topology.

The proof for $M \in m'_r(A)$ and $M \in m'_t(A)$ is similar. $\square$

Corollary 2.4.2. *Let A be a real unital locally m-pseudoconvex topological algebra. Then $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$ for every $M \in m'_l(A)$ and $M \in m'_r(A)$. If $M \in m'_t(A)$, then $Z(A/M)$ is topologically isomorphic to $\mathbb{R}$.*

Proof. Since every locally m-pseudoconvex algebra is locally A-pseudoconvex, then $Z(A/P_M)$ is topologically isomorphic to $\mathbb{R}$ by Theorem 2.4.1. $\square$

Corollary 2.4.3. *Let A be a formally real unital topologically primitive locally A-pseudoconvex Hausdorff algebra or a formally real unital topologically primitive locally pseudoconvex Fréchet algebra. Then $Z(A)$ is topologically isomorphic to $\mathbb{R}$.*

Proof. Since A is a topologically primitive topological algebra, then there is a closed maximal left (right) ideal M of A such that $P_M = \{\theta_A\}$. Hence, $Z(A)$ is topologically isomorphic to $\mathbb{R}$ by Theorem 2.4.1. $\square$

2.5 Extendible ideals

Let A be a real topological algebra with a unit element e_A, B a closed subalgebra of $Z(A)$ containing e_A and $M \in m(B)$. If

$$I(M) = \mathrm{cl}_A\Big\{\sum_{k=1}^{n} a_k m_k;\ n \in \mathbb{N},\ a_1, ..., a_n \in A;\ m_1, ..., m_n \in M\Big\} \neq A,$$

then M is called an *extendible ideal* in A. We denote the set of all extendible ideals of B by $m_e(B)$.

Proposition 2.5.1. *Let*[1] *A be a real locally A-pseudoconvex (in particular, a real locally m-pseudoconvex) algebra with a unit element e_A or a real locally pseudoconvex Fréchet algebra with a unit element e_A. Let $M \in m'_l(A)$ ($M \in m'_r(A)$ or $M \in m'_t(A)$) and B be a closed subalgebra of $Z(A)$ containing e_A. Then*

1) *every $b \in B$ defines a number $\lambda \in \mathbb{R}$ such that $b - \lambda e_A \in M$;*

[1]For complex topological algebra this result has been proved in [2], p. 50−51.

2) $M \cap B \in m_e(B)$.

Proof. Let $M \in m'_l(A)$, P_M be the primitive ideal in A, defined by M, and $\pi_M : A \to A/P_M$ a canonical homomorphism. Then (by Theorem 2.4.1) there exists a topological isomorphism μ from $Z(A/P_M)$ onto $\mathbb{R}$. Since $\pi_M(b) \in Z(A/P_M)$ for every $b \in B$, then we can find a number $\lambda_b \in \mathbb{R}$ such that

$$\mu(\pi_M(b)) = \lambda_b = \mu(\pi_M(\lambda_b e_A)).$$

Therefore, from $\pi_M(b) = \pi_M(\lambda_b e_A)$ it follws that $b - \lambda_b e_A \in P_M \subset M$.

Let $M_B = M \cap B$. Then M_B is a closed ideal in B. Moreover, let I be an ideal in B such that $M_B \subset I$. If $M_B \neq I$, then there exists an element $b \in I \backslash M_B$ and by the statement 1) a number $\lambda_b \in \mathbb{R}$ such that $b - \lambda_b e_A \in M_B$. Since $b \notin M_B$, then $\lambda_b \neq 0$. Now, from $b - \lambda_b e_A \in I$ it follows that $e_A = \lambda_b^{-1}[b - (b - \lambda_b e_A)] \in I$. Therefore $I = B$, which is not possible. Hence, $M_B \in m(B)$. Since $M_B \subset M \neq A$, then $I(M_B) \subset M \neq A$. Thus, $M_B \in m_e(B)$.

The proof for closed maximal right ideals is similar. Consequently, the results are true for closed two-sided ideals, too. □

2.6 Description of closed maximal ideals

Let A be a real topological algebra and B a closed subalgebra of $Z(A)$. Here and later on we assume that $m_e(B)$ is not empty. Then for each $\mathcal{M} \in m_e(B)$ let $A_{\mathcal{M}} = A/I(\mathcal{M})$ and $\kappa_{\mathcal{M}} : A \to A_{\mathcal{M}}$ denote the canonical homomorphism. To describe the sets $m'_l(A)$ ($m'_r(A)$ and $m_t(A)$),we need the following results.

Lemma 2.6.1. *Let A be a real unital locally A-pseudoconvex (in particular, a real unital locally m-pseudoconvex) algebra or a unital real locally pseudoconvex Fréchet algebra and B a unital closed subalgebra of $Z(A)$. If $M \in m'_l(A)$ and $\mathcal{M} \in m_e(B)$ is such that*

$$\kappa_{\mathcal{M}}(M) = \{\kappa_{\mathcal{M}}(a) : a \in M\} \neq A_{\mathcal{M}},$$

then $\kappa_{\mathcal{M}}(M) \in m'_l(A_{\mathcal{M}})$.

Similar result holds in case, when $M \in m'_r(A)$ and $M \in m'_t(A)$.

Proof. Let $M \in m_l'(A)$ and $\mathcal{M} \in m_e(B)$ be such that $\kappa_{\mathcal{M}}(M) \neq A_{\mathcal{M}}$. First we show that $\kappa_{\mathcal{M}}(M)$ is a left ideal in $A_{\mathcal{M}}$ for each $M \in m_l'(A)$ and $\mathcal{M} \in m_e(B)$. For it, let $b_1, b_2 \in \kappa_{\mathcal{M}}(M)$, $\lambda \in \mathbb{R}$ and $d \in A_{\mathcal{M}}$. Then there are $a_1^M, a_2^M \in M$ and $c \in A$ such that $\kappa_{\mathcal{M}}(a_i^M) = b_i$ for $i = 1, 2$ and $\kappa_{\mathcal{M}}(c) = d$. Since M is a left ideal in A, then $a_1^M + a_2^M$, λa_1^M, $ca_1^M \in M$. Therefore,

$$\begin{aligned} b_1 + b_2 &= \kappa_{\mathcal{M}}(a_1^M) + \kappa_{\mathcal{M}}(a_2^M) = \kappa_{\mathcal{M}}(a_1^M + a_2^M) \in \kappa_{\mathcal{M}}(M), \\ \lambda b_1 &= \lambda \kappa_{\mathcal{M}}(a_1^M) = \kappa_{\mathcal{M}}(\lambda a_1^M) \in \kappa_{\mathcal{M}}(M) \end{aligned}$$

and

$$db_1 = \kappa_{\mathcal{M}}(c)\kappa_{\mathcal{M}}(a_1^M) = \kappa_{\mathcal{M}}(ca_1^M) \in \kappa_{\mathcal{M}}(M).$$

Hence, $\kappa_{\mathcal{M}}(M)$ is a left ideal in $A_{\mathcal{M}}$.

Now we show that $\kappa_{\mathcal{M}}(M)$ is maximal. For it, let W be a left ideal in $A_{\mathcal{M}}$ such that $\kappa_{\mathcal{M}}(M) \subset W$, then

$$M \subset \kappa_{\mathcal{M}}^{-1}(\kappa_{\mathcal{M}}(M)) \subset \kappa_{\mathcal{M}}^{-1}(W)$$

and there are two possibilities: $\kappa_{\mathcal{M}}^{-1}(W) = A$ (it gives us a contradiction $W = A_{\mathcal{M}}$) or $\kappa_{\mathcal{M}}^{-1}(W) \neq A$ (then $M = \kappa_{\mathcal{M}}^{-1}(W)$ or $\kappa_{\mathcal{M}}(M) = W$). Thus, $\kappa_{\mathcal{M}}(M)$ is a maximal left ideal in $A_{\mathcal{M}}$.

Next, we show that $\kappa_{\mathcal{M}}(M)$ is closed in $A_{\mathcal{M}}$. Again, we have two possibilities:

$$\mathrm{cl}_{A_{\mathcal{M}}}(\kappa_{\mathcal{M}}(M)) \neq A_{\mathcal{M}} \quad \text{or} \quad \mathrm{cl}_{A_{\mathcal{M}}}(\kappa_{\mathcal{M}}(M)) = A_{\mathcal{M}}.$$

If $\mathrm{cl}_{A_{\mathcal{M}}}(\kappa_{\mathcal{M}}(M)) \neq A_{\mathcal{M}}$, then $\mathrm{cl}_{A_{\mathcal{M}}}(\kappa_{\mathcal{M}}(M))$ is a closed left ideal in $A_{\mathcal{M}}$ (see [26], p.169). Since $\kappa_{\mathcal{M}}(M)$ is a maximal left ideal, then

$$\kappa_{\mathcal{M}}(M) = \mathrm{cl}_{A_{\mathcal{M}}}(\kappa_{\mathcal{M}}(M)).$$

Otherwise, $\kappa_{\mathcal{M}}(e_A) \in \mathrm{cl}_{A_{\mathcal{M}}}(\kappa_{\mathcal{M}}(M))$. Therefore, there is a net $(m_\lambda)_{\lambda \in \Lambda}$ in M such that $(\kappa_{\mathcal{M}}(m_\lambda))_{\lambda \in \Lambda}$ converges to $\kappa_{\mathcal{M}}(e_A)$ in $A_{\mathcal{M}}$.

Let O' be an arbitrary neighbourhood of zero in A, then $O = \kappa_{\mathcal{M}}(O')$ is a neighbourhood of zero in $A_{\mathcal{M}}$. Therefore we can find an index $\mu \in \Lambda$ such that $\kappa_{\mathcal{M}}(m_\lambda - e_A) \in O$ for each $\lambda > \mu$. Since

$$\kappa_{\mathcal{M}}(I(M)) = \theta_{A_{\mathcal{M}}} \in \kappa_{\mathcal{M}}(M),$$

then

$$I(M) \subset \kappa_{\mathcal{M}}^{-1}(\kappa_{\mathcal{M}}(M)) = M$$

by maximality of M (because $\kappa_{\mathcal{M}}^{-1}(\kappa_{\mathcal{M}}(M)) \neq A$). Let now $\lambda_0 > \mu$. Then

$$m_{\lambda_0} - e_A \in \kappa_{\mathcal{M}}^{-1}(\kappa_{\mathcal{M}}(O')) = I(M) + O' \subset M + O'.$$

Therefore

$$e_A = (e_A - m_{\lambda_0}) + m_{\lambda_0} \in M + O' + M \subset M + O'.$$

Hence

$$\begin{aligned} e_A &\in \bigcap\{M + O' : O' \text{ is a neighbourhood of zero in A}\} = \\ &= \mathrm{cl}_A(M) = M \end{aligned}$$

(see [31], p. 13). In this case $M = A$, which is not possible. Consequently, $\kappa_{\mathcal{M}}(M)$ is a closed maximal left ideal in $A_{\mathcal{M}}$.

Now we show that $\kappa_{\mathcal{M}}(M)$ satisfies the condition (2.4.1). For it, let $M_{\mathcal{M}} = \kappa_{\mathcal{M}}(M)$ and $u_1, u_2 \in A_{\mathcal{M}}$ be such that $u_1^2 + u_2^2 \in P_{M_{\mathcal{M}}}$.

Since

$$\kappa_{\mathcal{M}}(P_M) = \{\kappa_{\mathcal{M}}(a) \in A_{\mathcal{M}} : \kappa_{\mathcal{M}}(a)A_{\mathcal{M}} \subset M_{\mathcal{M}}\} = P_{M_{\mathcal{M}}},$$

then there are $x_0 \in P_M$, such that $u_1^2 + u_2^2 = \kappa_{\mathcal{M}}(x_0)$, and $a_1, a_2 \in A$ such that $u_i = \kappa_{\mathcal{M}}(a_i)$, $i = 1, 2$. As

$$\begin{aligned} \kappa_{\mathcal{M}}((a_1^2 + a_2^2)A) &= (u_1^2 + u_2^2)\kappa_{\mathcal{M}}(A) = \kappa_{\mathcal{M}}(x_0)\kappa_{\mathcal{M}}(A) = \\ &= \kappa_{\mathcal{M}}(x_0 A) \in \kappa_{\mathcal{M}}(M), \end{aligned}$$

then

$$(a_1^2 + a_2^2)A \in \kappa_{\mathcal{M}}^{-1}(\kappa_{\mathcal{M}}(M)) = M.$$

Hence, $a_1^2 + a_2^2 \in P_M$. Therefore, $a_1, a_2 \in P_M$, by the condition (2.4.1). Consequently, $u_1, u_2 \in P_{M_{\mathcal{M}}}$ and $M_{\mathcal{M}} \in m_l'(A_{\mathcal{M}})$.

The proof for $M \in m_r'(A)$ is similar. Consequently, the result is true for closed maximal two-sided ideals as well. $\square$

Lemma 2.6.2. *Let A be a real unital locally A-pseudoconvex* (*in particular, a unital real locally m-pseudoconvex*) *algebra or a real unital locally pseudoconvex Fréchet algebra and B a closed subalgebra of $Z(A)$. If $\mathcal{M} \in m_e(B)$ and $M_{\mathcal{M}} \in m'_l(A_{\mathcal{M}})$, then*

$$\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}}) \in m'_l(A).$$

Similar result holds in case, when $M \in m'_r(A)$ or $M \in m'_t(A)$.

Proof. First we show that $\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$ is a left ideal in A. For it, let $c_1, c_2 \in \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$ and $\lambda \in \mathbb{R}$. Then there are elements $b_1, b_2 \in M_{\mathcal{M}}$ such that $\kappa_{\mathcal{M}}(c_i) = b_i$ for $i = 1, 2$. Since $M_{\mathcal{M}}$ is a left ideal in $A_{\mathcal{M}}$, then

$$\kappa_{\mathcal{M}}(c_1 + c_2) = b_1 + b_2 \in M_{\mathcal{M}} \text{ and } \kappa_{\mathcal{M}}(\lambda c_1) = \lambda b_1 \in M_{\mathcal{M}}.$$

Therefore $c_1 + c_2, \lambda c_1 \in \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$. If $a \in A$ and $d = \kappa_{\mathcal{M}}(a)$, then $\kappa_{\mathcal{M}}(ac_1) = db_1 \in M_{\mathcal{M}}$ or $ac_1 \in \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$. Moreover, it is easy to see that $\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}}) \neq A$ and $\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$ is closed in A (because $M_{\mathcal{M}}$ is closed in $A_{\mathcal{M}}$). Thus, $\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$ is a closed left ideal in A.

Now we show that $\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$ is maximal in A. For it, let H be a left ideal in A such that $\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}}) \subset H$. Then

$$M_{\mathcal{M}} \subset \kappa_{\mathcal{M}}(\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})) \subset \kappa_{\mathcal{M}}(H).$$

If $\kappa_{\mathcal{M}}(H) = A_{\mathcal{M}}$, then there is an element $h \in H$ such that

$$\kappa_{\mathcal{M}}(h) = e_{A_{\mathcal{M}}} = \kappa_{\mathcal{M}}(e_A).$$

Since

$$h - e_A \in \kappa_{\mathcal{M}}^{-1}(\theta_{A_{\mathcal{M}}}) \in \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}}) \subset H,$$

then

$$e_A = h - (h - e_A) \in H$$

implies that $H = A$, which is a contradiction. Therefore, $\kappa_{\mathcal{M}}(H) \neq A$. Thus, $\kappa_{\mathcal{M}}(H)$ is a left ideal in $A_{\mathcal{M}}$ (see the proof of Lemma 2.6.1). Hence, $\kappa_{\mathcal{M}}(H) = M_{\mathcal{M}}$ (because $M_{\mathcal{M}}$ is maximal) and from

$$H \subset \kappa_{\mathcal{M}}^{-1}(\kappa_{\mathcal{M}}(H)) = \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}}) \subset H$$

it follows that $\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}}) = H$. Hence, $\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$ is a closed maximal left ideal in A.

Now we show that $M^* = \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$ satisfies the condition (2.4.1). For it, let $a_1, a_2 \in A$ be such that $a_1^2 + a_2^2 \in P_{M^*}$. Then

$$(a_1^2 + a_2^2)A \subset M^*.$$

Since

$$\begin{aligned}(\kappa_{\mathcal{M}}(a_1)^2 + \kappa_{\mathcal{M}}(a_2)^2)A_{\mathcal{M}} &= \kappa_{\mathcal{M}}((a_1^2 + a_2^2)A) \subset \\ \subset \kappa_{\mathcal{M}}(\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})) &= M_{\mathcal{M}},\end{aligned}$$

then $\kappa_{\mathcal{M}}(a_1)^2 + \kappa_{\mathcal{M}}(a_2)^2 \in P_{M_{\mathcal{M}}}$. Consequently, $\kappa_{\mathcal{M}}(a_i) \in P_{M_{\mathcal{M}}}$ for $i = 1, 2$, because $M_{\mathcal{M}} \in m_l'(A_{\mathcal{M}})$ (in this case from $x_1, x_2 \in A_{\mathcal{M}}$ and $x_1^2 + x_2^2 \in P_{M_{\mathcal{M}}}$ it follows that $x_1, x_2 \in P_{M_{\mathcal{M}}}$). Now

$$a_i A \subset \kappa_{\mathcal{M}}^{-1}[\kappa_{\mathcal{M}}(a_i)A_{\mathcal{M}}] \subset \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}}) = M^*.$$

Therefore, $a_i \in P_{M^*}$ for $i = 1, 2$. Thus, $M^* \in m_l'(A)$.

The proof for right ideals is similar. Consequently, the result is true for closed maximal two-sided ideals, too. □

Now we prove the main result of this chapter.

Theorem 2.6.3. *Let A be a real unital locally A-pseudoconvex (in particular, a locally m-pseudoconvex) algebra or a real unital locally pseudoconvex Fréchet algebra, B a closed subalgebra of $Z(A)$ with the same unit element as A and let $M \in m_l'(A)$. Then*

1) *$M = \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$ for some $M_{\mathcal{M}} \in m_l'(A_{\mathcal{M}})$ (here $\mathcal{M} = M \cap B$);*

2) *there exists a bijection*

$$\Lambda_l : \bigcup_{\mathcal{M} \in m_e(B)} \{\mathcal{M}\} \times m_l'(A_{\mathcal{M}}) \to m_l'(A).$$

Similar results hold in case, when $M \in m_r'(A)$ or $M \in m_t'(A)$.

Proof. Let $M \in m_l'(A)$. Then $\mathcal{M} = M \cap B \in m_e(B)$, by Proposition 2.5.1, and

$$M_{\mathcal{M}} = \kappa_{\mathcal{M}}(M) \in m_l'(A_{\mathcal{M}}),$$

by Lemma 2.6.1, because in this case $\kappa_{\mathcal{M}}(M) \neq A_{\mathcal{M}}$ (otherwise there is an element $m \in M$ such that $m - e_A \in I(\mathcal{M}) \subset M$, therefore

$e_A = m - (m - e_A) \in M$). As $M \subset \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$, then $M = \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}})$ because M is maximal. Moreover, if $\mathcal{M} \in m_e(B)$ and $M_{\mathcal{M}} \in m'_l(A_{\mathcal{M}})$, then $\kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}}) \in m'_l(A)$ by Lemma 2.6.2. Hence, for each $\mathcal{M} \in m_e(B)$ and $M_{\mathcal{M}} \in m'_l(A_{\mathcal{M}})$ the map Λ_l, defined by

$$\Lambda_l((\mathcal{M}, M_{\mathcal{M}})) = \kappa_{\mathcal{M}}^{-1}(M_{\mathcal{M}}),$$

is an onto map.

Now we show that Λ_l is one-to-one. For it, let

$$M = \kappa_{\mathcal{M}_1}^{-1}(M_{\mathcal{M}_1}) = \kappa_{\mathcal{M}_2}^{-1}(M_{\mathcal{M}_2})$$

for some $\mathcal{M}_i \in m_e(B)$ and $M_{\mathcal{M}_i} \in m'_l(A_{\mathcal{M}_i})$, where $i = 1, 2$. Then $M \in m'_l(A)$ by Lemma 2.6.2, and

$$\mathcal{M} = M \cap B \in m_e(B),$$

by Proposition 2.5.1. Since

$$\kappa_{\mathcal{M}_i}(I(\mathcal{M}_i)) = \theta_{A_{\mathcal{M}_i}} \in M_{\mathcal{M}_i}$$

for each $i = 1, 2$, then

$$\mathcal{M}_i \subset I(\mathcal{M}_i) \subset \kappa_{\mathcal{M}_i}^{-1}(M_{\mathcal{M}_i}) = M$$

Hence, $\mathcal{M}_i \subset \mathcal{M}$ for $i = 1, 2$. Since $\mathcal{M}_i$ is maximal in B for each $i = 1, 2$, then $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$ and

$$M_{\mathcal{M}_1} = \kappa_{\mathcal{M}_1}[\kappa_{\mathcal{M}_1}^{-1}(M_{\mathcal{M}_1})] = \kappa_{\mathcal{M}_2}[\kappa_{\mathcal{M}_2}^{-1}(M_{\mathcal{M}_2})] = M_{\mathcal{M}_2}.$$

Therefore, from $\Lambda_l((\mathcal{M}_1, M_{\mathcal{M}_1})) = \Lambda_l((\mathcal{M}_2, M_{\mathcal{M}_2}))$ it follows that

$$(\mathcal{M}_1, M_{\mathcal{M}_1}) = (\mathcal{M}_2, M_{\mathcal{M}_2}).$$

Hence, Λ_l is an one-to-one map. Consequently, Λ_l is a bijection.

The proof for right ideals is similar. Consequently, similar results are true for closed maximal two-sided ideals, too. □

Chapter 3

Description of ideals in subalgebras of $C(X, A, \sigma)$

We study in this Chapter properties of the topological algebra $C(X, A; \sigma)$, a description of closed maximal left (right or two-sided) ideals and of all nontrivial continuous linear multiplicative functionals in subalgebras $\mathfrak{A}(X, A; \sigma)$ of $C(X, A; \sigma)$.

Results of this Chapter are published in [29].

3.1 Properties of $C(X, A; \sigma)$ and of its subalgebras

Let A be a real topological algebra with jointly continuous multiplication, X a topological space, σ a cover of X and $C(X, A; \sigma)$ the set of all continuous functions $f : X \to A$ for which the closure of $f(S)$ (in the topology of A) is compact in A for each $S \in \sigma$. All algebraic operations on $C(X, A; \sigma)$ we define point-wisely and endow $C(X, A; \sigma)$ with the topology, whose subbase of neighbourhoods of zero is

$$\{T(S, O) : \quad S \in \sigma, \text{ O is a neighbourhood of zero in A}\},$$

where $T(S, O) = \{f \in C(X, A; \sigma) : \ f(S) \subset O\}$. Then $C(X, A, \sigma)$ is a real topological algebra. It is easy to see that $C(X, A; \sigma)$ is a Hausdorff space if A is a Hausdorff space. Now we describe these properties of $C(X, A; \sigma)$, which we need later on.

Lemma 3.1.1. *Let*[1] *X be a topological space, σ its cover and A a unital real locally m-pseudoconvex algebra. Then $C(X, A; \sigma)$ is also a unital real locally m-pseudoconvex algebra.*

Proof. Since A is a unital real locally m-pseudoconvex algebra, then A has a base $\mathcal{B}_A = \{U_\alpha; \alpha \in \mathcal{A}\}$ of neigbourhoods of zero, consisting of balanced, pseudoconvex and idempotent sets. Let O be a neigbourhood of zero in $C(X, A; \sigma)$. Then there are $n \in \mathbb{N}$, $S_1, \ldots, S_n \in \sigma$ and neigbourhoods $O_1, \ldots O_n$ of zero in A (by definition of topology of $C(X, A; \sigma)$) such that

$$\bigcap_{k=1}^{n} T(S_k, O_k) \subset O.$$

Now, for every k there is a neigbourhood $U_{\alpha_k} \in \mathcal{B}_A$ of zero such that $U_{\alpha_k} \subset O_k$. It is easy to see that

$$\left\{\bigcap_{k=1}^{n} T(S_k, U_{\alpha_k}) : n \in \mathbb{N}, S_1, \ldots, S_n \in \sigma, U_{\alpha_k} \in \mathcal{B}_A\right\}$$

is a base of neigbourhoods of zero in $C(X, A; \sigma)$, which consists of balanced, pseudoconvex and idempotent sets. Thus $C(X, A; \sigma)$ is a unital real locally m-pseudoconvex algebra. □

Lemma 3.1.2. *Let X be a completely regular Hausdorff space and σ a compact cover*[2] *of X, which is closed with respect to finite unions. Then*

1) *for every $\phi \in \hom(C(X, \mathbb{R}; \sigma))$ defines an element $x_\phi \in X$ such that $\phi = \phi_{x_\phi}$, where $\phi_{x_\phi}(\alpha) = \alpha(x_\phi)$ for each $\alpha \in C(X, \mathbb{R}; \sigma)$;*

2) *every $M \in m(C(X, \mathbb{R}; \sigma))$ defines an element $x_M \in X$ such that*

$$M = \{\alpha \in C(X, \mathbb{R}; \sigma) : \quad \alpha(x_M) = 0\}.$$

Proof. See [5], the proof of Theorem 2 b) and v) in case of compact cover. □

[1] For complex topological algebra A this result has been proved in [2], p. 67.
[2] That is, every $S \in \sigma$ is a compact subset of X.

Lemma 3.1.3. *Let*[3] *A be a real topological Hausdorff algebra and $a \in A \setminus \theta_A$. Then $\nu_a : \mathbb{R} \to \mathbb{R}a \subset A$, defined by $\nu_a(\lambda) = \lambda a$ for each $\lambda \in \mathbb{R}$, is a homeomorphism.*

Proof. Let A be a real topological Hausdorff algebra and $a \in A \setminus \theta_A$. It is clear that $\nu_a : \mathbb{R} \to A$, defined by $\nu_a(\lambda) = \lambda a$ for each $\lambda \in \mathbb{R}$, is a continuous bijection. We show that ν_a^{-1} is continuous. For it, let O be a neigbourhood of zero in $\mathbb{R}$. Then there is $\varepsilon > 0$ such that

$$O_\varepsilon = \{\lambda \in \mathbb{R} : \quad |\lambda| \leq \varepsilon\} \subset O.$$

If $\lambda_0 \in O_\varepsilon \setminus \{0\}$, then $\lambda_0 a \neq \theta_A$. Since A is a Hausdorff space, then there is a neigbourhood O_A of zero in A such that $\lambda_0 a \notin O_A$. Let V_A be a balanced neigbourhood of zero in A such that $V_A \subset O_A$. Now $O' = V_A \cap (\mathbb{R}a)$ is a neigbourhood of zero in $\mathbb{R}a$. If $\lambda a \in V_A$ and $|\lambda_0| \leq |\lambda|$, then $|\lambda_0 \lambda^{-1}| \leq 1$ and $\lambda_0 a = (\lambda_0 \lambda^{-1}) \lambda a \subset V_A$, which is a contradiction. Therefore, from $\lambda a \in O'$ it follows that $\lambda \in 0_\varepsilon \subset O$, which means that ν_a^{-1} is continuous. □

3.2 Description of ideals in subalgebras of $C(X, A, \sigma)$

1. Let $\mathfrak{A}(X, A; \sigma)$ be a subalgebra of $C(X, A; \sigma)$, endowed with the subset topology. The following results hold.

Lemma 3.2.1. *Let*[4] *X be a topological space, σ its cover, A a (not necessary real) topological algebra with jointly continuous multiplication and $\mathfrak{A}(X, A; \sigma)$ a subalgebra of $C(X, A; \sigma)$. If*

$$\{f(x) : f \in \mathfrak{A}(X, A; \sigma)\} = A$$

for each $x \in X$, then

$$Z(\mathfrak{A}(X, A; \sigma)) = \mathfrak{A}(X, A; \sigma) \cap C(X, Z(A); \sigma).$$

[3]For complex topological algebra A this result has been proved in [2], p. 70.
[4]For complex topological algebra this result has been proved in [2], p. 70−71.

Proof. Since $\mathfrak{A}(X,A;\sigma)$ is a subalgebra of $C(X,A;\sigma)$, then

$$\begin{aligned}&\mathfrak{A}(X,A;\sigma) \quad \cap \quad C(X,Z(A);\sigma) = \\ &\mathfrak{A}(X,A;\sigma) \quad \cap \quad Z(C(X,A;\sigma)) \subset Z(\mathfrak{A}(X,A;\sigma)).\end{aligned}$$

Now we show that $Z(\mathfrak{A}(X,A;\sigma)) \subset C(X,Z(A);\sigma)$. Let $x \in X$ and $g \in Z(\mathfrak{A}(X,A;\sigma))$. By assumption, every $a \in A$ defines a function $f_a \in \mathfrak{A}(X,A;\sigma)$ such that $f_a(x) = a$. Since $f_a g = g f_a$ for each $a \in A$, then

$$f_a g(x) = g f_a(x) \quad \text{or} \quad a g(x) = g(x) a$$

for each $x \in X$ and each $a \in A$. Thus $g(x) \in Z(A)$, which implies that $Z(\mathfrak{A}(X,A;\sigma)) \subset C(X,Z(A);\sigma)$. □

Next let $\varepsilon_x : \mathfrak{A}(X,A;\sigma) \to A$ be the homomorphism, defined by $\varepsilon_x(f) = f(x)$ for each $f \in \mathfrak{A}(X,A;\sigma)$.

Lemma 3.2.2. *Let X be a topological space, σ its cover, A a unital topological algebra with jointly continuous multiplication and $\mathfrak{A}(X,A;\sigma)$ a subalgebra of $C(X,A;\sigma)$. If*

$$\{f_a : a \in A, f_a(x) = a \text{ for each } x \in X\} \subset \mathfrak{A}(X,A;\sigma), \qquad (3.2.1)$$

then

$$\mathfrak{M}_{x,M} = \{f \in \mathfrak{A}(X,A;\sigma) : f(x) \in M\} \in m'_l(\mathfrak{A}(X,A;\sigma))$$

for each $x \in X$ and $M \in m'_l(A)$.

Similar result holds for the pairs $m'_r(\mathfrak{A}(X,A;\sigma))$, $m'_r(A)$ and $m'_t(\mathfrak{A}(X,A;\sigma))$, $m'_t(A)$.

Proof. Let $x \in X$ and $M \in m'_l(A)$. It is clear that $\mathfrak{M}_{x,M}$ is a left ideal in $\mathfrak{A}(X,A;\sigma)$. We show that $\mathfrak{M}_{x,M}$ is closed. For it, let $f_0 \in \mathrm{cl}_{\mathfrak{A}(X,A;\sigma)}(\mathfrak{M}_{x,M})$. Then there is a net $(f_\lambda)_{\lambda\in\Lambda}$ of elements of $\mathfrak{M}_{x,M}$, which converges to f_0 in $\mathfrak{A}(X,A;\sigma)$. Since $\varepsilon_x(T(S,O)) \subset O$ for each neighbourhood O of zero in A and a set $S \in \sigma$ such that $x \in S$, then ε_x is continuous. Therefore $(\varepsilon_x(f_\lambda))_{\lambda\in\Lambda}$ converges to $\varepsilon_x(f_0)$ and $\varepsilon_x(f_0) \in M$, because M is closed. Thus, $f_0 \in \mathfrak{M}_{x,M}$ which means that $\mathfrak{M}_{x,M}$ is a closed ideal.

Next, we show that $\mathfrak{M}_{x,M}$ is maximal. For it, let I be a left ideal of $\mathfrak{A}(X,A;\sigma)$ such that $\mathfrak{M}_{x,M} \subset I$. Then $\varepsilon_x(\mathfrak{M}_{x,M}) \subset \varepsilon_x(I)$. Suppose that $\varepsilon_x(I) = A$. Then there is an element $g \in I$ such that $e_A = \varepsilon_x(g)$. Therefore

$$\varepsilon_x(f_{e_A} - g) = f_{e_A}(x) - \varepsilon_x(g) = \theta_A \in M,$$

which means that $f_{e_A} - g \subset \mathfrak{M}_{x,M} \subset I$. Hence $f_{e_A} = (f_{e_A} - g) + g \in I$, which is not possible, because f_{e_A} is the unit element in $\mathfrak{A}(X, A; \sigma)$. Thus, $\varepsilon_x(I) \neq A$, $\varepsilon_x(I)$ is a left ideal in A and $M \subset \varepsilon_x(\mathfrak{M}_{x,M}) \subset \varepsilon_x(I)$. Since M is a maximal ideal in A, then $M = \varepsilon_x(I)$. Taking this into account, from

$$I \subset \varepsilon_x^{-1}(\varepsilon_x(I)) = \varepsilon_x^{-1}(M) = \mathfrak{M}_{x,M} \subset I$$

it follows that $I = \mathfrak{M}_{x,M}$. Consequently, $\mathfrak{M}_{x,M} \in m_l(\mathfrak{A}(X, A; \sigma))$.

Now we show that $P_{\mathfrak{M}_{x,M}}$ satisfies the condition (2.4.1). For it let $f, g \in \mathfrak{A}(X, A; \sigma)$ be such that $f^2 + g^2 \in P_{\mathfrak{M}_{x,M}}$. Then from

$$(f^2 + g^2)\mathfrak{A}(X, A; \sigma) \subset \mathfrak{M}_{x,M}$$

it follows that

$$(f^2 + g^2) f_a \in \mathfrak{M}_{x,M}$$

for each $a \in A$ by the assumption (3.2.1). It means that

$$(f^2(x) + g^2(x))a \in M$$

for each $a \in A$. Hence $f^2(x) + g^2(x) \in P_M$. As $M \in m_l'(A)$, then P_M satisfies the condition (2.4.1). Hence, $f(x), g(x) \in P_M$ or $f(x)A \subset M$ and $g(x)A \subset M$. Therefore,

$$f(x)h(x), g(x)h(x) \subset M \quad \text{or} \quad fh, gh \in \mathfrak{M}_{x,M}$$

for each $h \in \mathfrak{A}(X, A; \sigma)$. Consequently, $f, g \in P_{\mathfrak{M}_{x,M}}$. Thus, for each $x \in X$ and $M \in m_l'(A)$ holds $\mathfrak{M}_{x,M} \in m_l'(\mathfrak{A}(X, A; \sigma))$.

The proof for right and two-sided ideals is similar. □

2. Let again X be a topological space, σ its cover and A a real topological algebra. For each[5] $\alpha \in C(X, \mathbb{R})$ and $a \in A$ let αa denote the map defined by $(\alpha a)(x) = \alpha(x)a$ for each $x \in X$. Moreover, let

$$\mathfrak{A}_a(X, \mathbb{R}; \sigma) \;=\; \{\alpha \in C(X, \mathbb{R}) : \quad \alpha a \in \mathfrak{A}(X, A; \sigma)\}$$

and

$$\mathfrak{A}(X, \mathbb{R}; \sigma) \;=\; \mathfrak{A}_{e_A}(X, \mathbb{R}; \sigma).$$

[5]The set of all continuous functions $f : X \to A$ we denote by $C(X, A)$. It is clear that $C(X, A; \sigma) \subset C(X, A)$.

Lemma 3.2.3. *Let*[6] *X be a topological space, σ its cover, which is closed with respect to finite unions, A a real topological Hausdorff algebra, $a \in A \setminus \{\theta_A\}$ and $\mathfrak{A}_a = \mathfrak{A}_a(X, \mathbb{R}; \sigma)a$. Then the map μ_a, defined by $\mu_a(\alpha) = \alpha a$ for each $\alpha \in \mathfrak{A}_a(X, \mathbb{R}; \sigma)$, is a topological isomorphism between $\mathfrak{A}_a(X, \mathbb{R}; \sigma)$ and $\mathfrak{A}_a$.*

Proof. It is clear that μ_a is a bijection between the sets $\mathfrak{A}_a(X, \mathbb{R}; \sigma)$ and $\mathfrak{A}_a$. We show that μ_a is continuous. For it, let O be a neighbourhood of zero in $\mathfrak{A}_a$. Then $O = O' \cap \mathfrak{A}_a$, where O' is a neighbourhood of zero in $\mathfrak{A}(X, A; \sigma)$. Now there is $S \in \sigma$, neighbourhood O_A of zero in A and neighbourhood $O_\varepsilon = \{\lambda \in \mathbb{K} : |\lambda| \le \varepsilon\}$ of zero in $\mathbb{R}$ such that $T(S, O_A) \cap \mathfrak{A}(X, A; \sigma) \subset O'$ and $O_\varepsilon a \subset O_A$. Since $T(S, O_\varepsilon) \cap \mathfrak{A}_a(X, \mathbb{R}; \sigma)$ is a neighbourhood of zero in $\mathfrak{A}_a(X, \mathbb{R}; \sigma)$ and

$$\begin{aligned}&\mu_a(T(S, O_\varepsilon) \cap \mathfrak{A}_a(X, \mathbb{R}; \sigma)) \subset T(S, O_\varepsilon a) \cap \mathfrak{A}_a \subset \\ &\subset (T(S, O_\varepsilon a) \cap \mathfrak{A}(X, A; \sigma)) \cap \mathfrak{A}_a \subset O' \cap \mathfrak{A}_a = O,\end{aligned}$$

then μ_a is continuous.

Next we show, that μ_a^{-1} is continuous. For it, let U be a neighbourhood of zero in $\mathfrak{A}_a(X, \mathbb{R}; \sigma)$. Then there is $S \in \sigma$ and $\varepsilon > 0$ such that $T(S, O_\varepsilon) \cap \mathfrak{A}_a(X, \mathbb{R}; \sigma) \subset U$. Because $O_\varepsilon a = \nu_a(O_\varepsilon)$ and ν_a is a homeomorphism, by Lemma 3.1.3, then $O_\varepsilon a$ is a neighbourhood of zero in $\nu_a(\mathbb{R})$. Hence, there is a neighbourhood U_A of zero in A such that $\nu_a(O_\varepsilon) = U_A \cap \nu_a(\mathbb{R})$. Therefore, $U' = T(S, U_A) \cap \mathfrak{A}_a$ is a neighbourhood of zero in $\mathfrak{A}_a$ and $\mu_a^{-1}(U') \subset T(S, O_\varepsilon) \cap \mathfrak{A}_a(X, \mathbb{R}; \sigma) \subset U$ which means that μ_a^{-1} is continuous. □

3.3 Description of closed maximal ideals in subalgebras of $C(X, A, \sigma)$

To describe closed maximal ideals in subalgebras of $C(X, A, \sigma)$, we need the following result.

Lemma 3.3.1. *Let X be a topological space, σ a cover of X, which is closed with respect to finite unions, A a real unital locally m-pseudoconvex Hausdorff algebra and $\mathfrak{A}(X, A; \sigma)$ a complete subalgebra of $C(X, A; \sigma)$, which contains the unit element of $C(X, A; \sigma)$. If*

[6]For complex topological algebra A this result has been proved in [2], p. 72−73.

$\mathfrak{A}(X,\mathbb{R};\sigma)$ *is strictly real and every* $M \in m(\mathfrak{A}(X,\mathbb{R};\sigma)e_A)$ *defines a point* $x \in X$ *such that*

$$M = M_x = \{\alpha e_A \in \mathfrak{A}(X,\mathbb{R};\sigma)e_A : \ \alpha(x) = 0\},$$

then

$$B = \mathrm{cl}_{Z(\mathfrak{A}(X,A;\sigma))}(\mathfrak{A}(X,\mathbb{R};\sigma)e_A)$$

is a commutative real unital complete Gelfand-Mazur Hausdorff algebra.

Proof. Let X, σ, A and B be such as in the formulation of Lemma 3.3.1. Then B is a real closed Hausdorff subalgebra of the center $Z(\mathfrak{A}(X,A;\sigma))$ of $\mathfrak{A}(X,A;\sigma)$. The map μ_{e_A}, defined by

$$\mu_{e_A}(\alpha) = \alpha e_A$$

for each $\alpha \in \mathfrak{A}(X,\mathbb{R};\sigma)$, is a topological isomorphism of $\mathfrak{A}(X,\mathbb{R};\sigma)$ into B, by Lemma 3.2.3. Let τ be the topology on $\mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma))$, induced by the topology of $Z(\mathfrak{A}(X,A;\sigma))$. Since σ is closed with respect to finite unions, then every element of a base of neighbourhoods of zero in $\mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma))$ has the form

$$\mathfrak{B} = \{\alpha e_A : \quad \alpha \in \mathfrak{A}(X,\mathbb{R};\sigma),\ \alpha(S)e_A \subset O_{Z(A)} \cap \mathbb{R}e_A\}$$

for some $S \in \sigma$ and neighbourhood $O_{Z(A)}$ of zero in $Z(A)$, by Lemma 3.2.1. Since the map $\lambda \to \lambda e_A$ is continuous (by Lemma 3.1.3), then there exists a number $\varepsilon \in (0,1)$ such that $O_\varepsilon e_A \subset O_{Z(A)} \cap \mathbb{R}e_A$, where $O_\varepsilon = \{\lambda \in \mathbb{R} : |\lambda| \leq \varepsilon\}$. It is easy to see that

$$\{T(S, O_\varepsilon e_A) \cap \mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma)) : \quad S \in \sigma, \varepsilon > 0\}$$

is also a base of neighbourhoods of zero in $\mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma))$ in the topology τ. Because every set $T(S, O_\varepsilon e_A) \cap \mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma))$ is idempotent and absolutely convex, then $(\mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma)), \tau)$ is a commutative real locally m-convex algebra.

Now we show that $\mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma))$ is strictly real. By assumption the unit element f_{e_A} of $C(X,A;\sigma)$ belongs to $\mathfrak{A}(X,\mathbb{R};\sigma)$. Since $f_{e_A} = ee_A$ (here $e(x) \equiv 1$), then $e \in \mathfrak{A}(X,\mathbb{R};\sigma)$ and ee_A is the unit element in $\mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma))$. Now it is easy to show that

$$\mathrm{sp}_{\mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma))}(\alpha e_A) \subset \mathrm{sp}_{\mathfrak{A}(X,\mathbb{R};\sigma)}(\alpha)$$

for each $\alpha \in \mathfrak{A}(X, \mathbb{R}; \sigma)$. As

$$\mathrm{sp}_{\mathfrak{A}(X,\mathbb{R};\sigma)}(\alpha) \subset \mathbb{R}$$

for each $\alpha \in \mathfrak{A}(X, \mathbb{R}; \sigma)$ by Proposition 1.5.1 (because $\mathfrak{A}(X, \mathbb{R}; \sigma)$ is strictly real), then

$$\mathrm{sp}_{\mu_{e_A}(\mathfrak{A}(X,\mathbb{R};\sigma))}(f) \subset \mathbb{R}$$

for each $f \in \mu_{e_A}(\mathfrak{A}(X, \mathbb{R}; \sigma))$. Hence, $\mu_{e_A}(\mathfrak{A}(X, \mathbb{R}; \sigma))$ is also strictly real by Proposition 1.5.1. If p is a homogeneous submultiplicative seminorm on $\mu_{e_A}(\mathfrak{A}(X, \mathbb{R}; \sigma))$, then (similarily as in [21], Proposition 4, p. 129) its extension p' onto B is a homogeneuos submultiplicative seminorm on B. Hence B is a commutative real unital locally m-pseudoconvex Hausdorff algebra. By assumption of Lemma 3.3.1, every $M \in m(\mathfrak{A}(X, \mathbb{R}; \sigma)e_A)$ defines a point $x \in X$ such that $M = M_x$. Therefore from $\alpha, \beta \in \mathfrak{A}(X, \mathbb{R}; \sigma)e_A$ and $\alpha^2 + \beta^2 \in M$ it follows that $\alpha, \beta \in M$. Thus $\mathfrak{A}(X, \mathbb{R}; \sigma)e_A$ satisfies the condition (2.2.1) for each $M \in m(\mathfrak{A}(X, \mathbb{R}; \sigma)e_A)$. Since $\mathfrak{A}(X, A; \sigma)$ is complete, then the center $Z(\mathfrak{A}(X, A; \sigma))$ (as a closed subspace) is complete. Thus B is also complete and, by Lemma 1.7.3, satisfies the condition (2.2.1) for each $M \in m(B)$. Consequently, B is a real unital commutative Gelfand-Mazur Hausdorff algebra, by Corollary 1.7.2. □

Now we prove the main result of Chapter 3, which describes closed maximal ideals in subalgebras of $C(X, A; \sigma)$.

Theorem 3.3.2. *Let X be a completely regular Hausdorff space, σ a cover of X, which is closed with respect to finite unions, A a real unital locally m-pseudoconvex Hausdorff algebra and $\mathfrak{A}(X, A; \sigma)$ a complete subalgebra of $C(X, A; \sigma)$ (with the same unit as $C(X, A, \sigma)$). If*

1) $\{f_a : a \in A, f_a(x) = a$ *for each* $x \in X\} \subset \mathfrak{A}(X, A; \sigma\}$;

2) *every* $M \in m(\mathfrak{A}(X, \mathbb{R}; \sigma))$ *defines an element* $x \in X$ *such that*

$$M = M_x = \{\alpha \in \mathfrak{A}(X, \mathbb{R}; \sigma) : \quad \alpha(x) = 0\};$$

3) $\alpha f \in \mathfrak{A}(X, A; \sigma)$ *for each* $\alpha \in C(X, \mathbb{R})$ *and* $f \in \mathfrak{A}(X, A; \sigma)$;

4) $\mathfrak{A}(X, \mathbb{R}; \sigma)$ *is strictly real,*

then every $\mathfrak{M} \in m'_l(\mathfrak{A}(X, A; \sigma))$ *is representable in the form*

$$\mathfrak{M} = \mathfrak{M}_{x,M} = \{f \in \mathfrak{A}(X, A; \sigma) : \quad f(x) \in M\}$$

for some $x \in X$ *and* $M \in m'_l(A)$.

Similar result is true for ideals in $m'_r(\mathfrak{A}(X, A; \sigma))$ *and for ideals in* $m'_t(\mathfrak{A}(X, A; \sigma))$.

Proof. We give the proof only for left ideals (the proof for right and two-sided ideals is similar). Let X, σ and A be as in the formulation of Theorem 3.3.2,

$$B = \mathrm{cl}_{Z(\mathfrak{A}(X,A;\sigma))}(\mathfrak{A}(X, \mathbb{R}; \sigma)e_A)$$

and $M \in m(\mathfrak{A}(X, \mathbb{R}; \sigma)e_A)$. Then $\mu_{e_A}^{-1}(M) \in m(\mathfrak{A}(X, \mathbb{R}; \sigma))$ by Lemma 3.2.3. Now by the condition 2) there is a point $x \in X$ such that

$$\mu_{e_A}^{-1}(M) = \{\alpha \in C(X, \mathbb{R}; \sigma) : \quad \alpha(x) = 0\}.$$

Hence,

$$M = \{\alpha e_A \in \mathfrak{A}(X, \mathbb{R}; \sigma)e_A : \quad \alpha(x) = 0\}.$$

Consequently, B is a commutative real unital complete Gelfand-Mazur Hausdorff algebra by Lemma 3.3.1 and the condition 4). In this case the set $m(B)$ is not empty, because every $\psi \in \hom(\mathfrak{A}(X, \mathbb{R}; \sigma)e_A)$ has the extension $\overline{\psi} \in \hom(B)$ by Proposition 3 from [15]. Therefore, every $M \in m(B)$ defines a map $\psi_M \in \hom(B)$ such that $M = \ker\, \psi_M$. Since μ_{e_A} is a topological isomorphism from $\mathfrak{A}(X, \mathbb{R}; \sigma)$ into B, by Lemma 3.2.3, and $\mu_{e_A}(\mathfrak{A}(X, \mathbb{R}; \sigma))$ is dense in B, then

$$\psi_M \circ \mu_{e_A} \in \hom(\mathfrak{A}(X, \mathbb{R}; \sigma)).$$

By the condition 2), there is a unique element $x_0 \in X$ such that

$$\ker(\psi_M \circ \mu_{e_A}) = \{\alpha \in C(X, \mathbb{R}; \sigma) : \quad \alpha(x_0) = 0\}.$$

Since $\xi_{x_0} : \mathfrak{A}(X, \mathbb{R}; \sigma) \to \mathbb{R}$, defined by $\xi_{x_0}(\alpha) = \alpha(x_0)$ for each $\alpha \in \mathfrak{A}(X, \mathbb{R}; \sigma)$, is a homomorphism and $\ker \xi_{x_0} = \ker(\psi_M \circ \mu_{e_A})$, then $\psi_M \circ \mu_{e_A} = \xi_{x_0}$. Now

$$\begin{aligned}\mu_{e_A}(\ker\xi_{x_0}) &= \mu_{e_A}(\mu_{e_A}^{-1}(\ker\psi_M)) = \ker\psi_M \cap \mu_{e_A}(\mathfrak{A}(X, \mathbb{R}; \sigma)) = \\ &= M \cap \mu_{e_A}(\mathfrak{A}(X, \mathbb{R}; \sigma)).\end{aligned}$$

Since B is a commutative real unital Gelfand-Mazur algebra, then

$$\mathrm{cl}_B(\mu_{e_A}(\ker\xi_{x_0})) = \mathrm{cl}_B(M \cap \mu_{e_A}(\mathfrak{A}(X, \mathbb{R}; \sigma))) = M,$$

by Corollary 1 from [15]. Hence, every M defines an element $x \in X$ such that

$$M = \overline{M}_x = \mathrm{cl}_B(\{\alpha e_A : \quad \alpha \in (\mathfrak{A}(X, \mathbb{R}; \sigma)),\ \alpha(x) = 0\}).$$

Let $f \in \mathfrak{A}(X, A; \sigma)$,

$$\kappa_{\overline{M}_x} : \mathfrak{A}(X, A; \sigma) \to Y = \mathfrak{A}(X, A; \sigma)/I(\overline{M}_x)$$

be the quotient map[7] and $\delta_x : Y \to A$ the map, defined by

$$\delta_x(\kappa_{\overline{M}_x}(f)) = \varepsilon_x(f)$$

for each $x \in X$ and $f \in \mathfrak{A}(X, A; \sigma)$. To show that δ_x is well defined, we show that $\ker\varepsilon_x = I(\overline{M}_x)$ for each $x \in X$. For it, let $f \in I(\overline{M}_x)$. If we define the multiplication over A in $\mathfrak{A}(X, A; \sigma)$ by $(af)(x) = af(x)$ for each $x \in X$ and $a \in A$, then $\overline{M}_x \subset \ker\varepsilon_x$ (because ε_x is continuous and A is a Hausdorff spase) and

$$\varepsilon_x\Big(\sum_{k=1}^{n} f_k m_k\Big) = \theta_A$$

for each $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathfrak{A}(X, A; \sigma)$ and $m_1, \ldots, m_n \in \overline{M}_x$. Hence,

$$\varepsilon_x(I(\overline{M}_x)) = \theta_A \quad \text{or} \quad I(\overline{M}_x) \subset \ker\varepsilon_x$$

for each $x \in X$.

Next, we show that $\ker\varepsilon_x \subset I(\overline{M}_x)$ for each $x \in X$. For it, let $x_0 \in X$, $f \in \ker\varepsilon_{x_0}$ and $O(f)$ be any neighbourhood of f in $\mathfrak{A}(X, A; \sigma)$. Since σ is closed with respect to finite unions, then

$$f + (T(S_0, O_0) \cap \mathfrak{A}(X, A; \sigma)) \subset O(f)$$

for some $S_0 \in \sigma$ and balanced neighbourhood O_0 of zero in A. Now there exists an open neighbourhood O' of zero in A such that $O' \subset O_0$.

[7] Here $I(\overline{M}_x)$ is extendible ideal in $\mathfrak{A}(X, A; \sigma)$, defined by $\overline{M}_x$.

Since $f(x_0) \in O'$, then $X_{O'} = X \setminus f^{-1}(O')$ is closed in X and $x_0 \notin X_{O'}$. By assumption, X is a completely regular space. Therefore, there is $\alpha \in C(X, [0, 1])$ such that $\alpha(x_0) = 0$ and $\alpha(X_{O'}) = \{1\}$.

Let now $x \in S_0$. If $x \in X_{O'}$, then

$$(\alpha f - f)(x) = (\alpha(x) - 1)f(x) \in O_0.$$

If $x \notin X_{O'}$, then $x \in f^{-1}(O')$ and

$$(\alpha f - f)(x) = (\alpha(x) - 1)f(x) \in (\alpha(x) - 1)O_0 \in O_0,$$

because $|\alpha(x) - 1| \leq 1$ and O_0 is balanced set. Therefore,

$$\alpha f - f \in T(S_0, O_0) \cap \mathfrak{A}(X, A; \sigma),$$

by the condition 3) and[8] $\alpha f = f\alpha e_A \in I(\overline{M}_{x_0})$, then

$$I(\overline{M}_{x_0}) \cap O(f) \neq \emptyset.$$

Consequently, $f \in I(\overline{M}_{x_0})$, which implies that $\ker\varepsilon_{x_0} \subset I(\overline{M}_{x_0})$. Therefore, $I(\overline{M}_{x_0}) = \ker\varepsilon_{x_0}$. Since every $M \in m(B)$ defines an element $x \in X$ such that $M = \overline{M}_x$ and $\ker\varepsilon_x \neq \mathfrak{A}(X, A; \sigma)$, because $\mathfrak{A}(X, A; \sigma)$ has the unit element, then every closed maximal ideal in B is extendible.

It is easy to see that $C(X, A; \sigma)$ is a Hausdorff algebra, if A is a Hausdorff algebra. By the assumption of Theorem 3.3.2 and by Lemma 3.1.1 we see that $C(X, A; \sigma)$ is locally m-pseudoconvex. Therefore $\mathfrak{A}(X, A; \sigma)$, as a subalgebra of $C(X, A; \sigma)$, is a real unital locally m-pseudoconvex Hausdorff algebra. It is shown in the proof of Theorem 2.6.3 that every ideal $\mathfrak{M} \in m_l'(\mathfrak{A}(X, A; \sigma))$ has the form

$$\mathfrak{M} = \kappa_{\overline{M}_x}^{-1}(\kappa_{\overline{M}_x}(\mathfrak{M}))$$

for some $x \in X$ and $\overline{M}_x = \mathfrak{M} \cap B \in m(B)$ (because every ideal of $m(B)$ is extendible). Hence,

$$\mathfrak{M} = \{f \in \mathfrak{A}(X, A; \sigma) : \quad f(x) \in \varepsilon_x(\mathfrak{M})\}.$$

[8] It is clear that $\alpha e_A = \alpha f_{e_A} \in \mathfrak{A}(X, A; \sigma)$ by the conditions 1) and 3). Therefore $\alpha \in \mathfrak{A}(X, A; \sigma)$ and $\alpha e_A \in \overline{M}_{x_0}$.

Next we show that $\varepsilon_x(\mathfrak{M})$ is a closed maximal left ideal in A. If $\varepsilon_x(\mathfrak{M}) = A$, then there is an element $g \in \mathfrak{M}$ such that $\varepsilon_x(g) = e_A$. Since $hg \in \mathfrak{M}$ for each $h \in \mathfrak{A}(X, A; \sigma)$, then from

$$\varepsilon_x(h) = h(x)e_A = (hg)(x) = \varepsilon_x(hg)$$

it follows that $h - hg \in \ker\varepsilon_x = I(\overline{M}_x)$. Since $\overline{M}_x = \mathfrak{M} \cap B$ (that is, $\kappa_{\overline{M}_x}(\mathfrak{M})$ is an ideal in Y by Lemma 2.6.1) and

$$\kappa_{\overline{M}_x}(I(\overline{M}_x)) = \theta_Y \in \kappa_{\overline{M}_x}(\mathfrak{M}),$$

then

$$I(\overline{M}_x) \subset \kappa_{\overline{M}_x}^{-1}[\kappa_{\overline{M}_x}(\mathfrak{M})] = \mathfrak{M}.$$

Thus $\mathfrak{A}(X, A; \sigma) = \mathfrak{M}$, which gives us a contradiction. Hence $\varepsilon_x(\mathfrak{M}) \neq A$ and $\varepsilon_x(\mathfrak{M})$ is a left ideal in A. Let now I be a left ideal in A such that $\varepsilon_x(\mathfrak{M}) \subset I$. Then

$$\mathfrak{M} \subset \varepsilon_x^{-1}(\varepsilon_x(\mathfrak{M})) \subset \varepsilon_x^{-1}(I) \neq \mathfrak{A}(X, A; \sigma)$$

because of which $\varepsilon_x^{-1}(I)$ is a left ideal in $\mathfrak{A}(X, A; \sigma)$. Since $\mathfrak{M}$ is a maximal left ideal in $\mathfrak{A}(X, A; \sigma)$, then $\mathfrak{M} = \varepsilon_x^{-1}(I)$ or $\varepsilon_x(\mathfrak{M}) = I$, because ε_x is an onto map by the condition 1). Consequently, $\varepsilon_x(\mathfrak{M})$ is a maximal left ideal in A.

Next, we show that $\varepsilon_x(\mathfrak{M})$ is closed. For it, let a_0 be an arbitrary element of $\mathrm{cl}_A(\varepsilon_x(\mathfrak{M}))$. Then there is a net $(m_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{M}$ such that $\varepsilon_x(m_\lambda)$ converges to a_0. Let $\rho : A \to C(X, A; \sigma)$ be a map, defined by $(\rho(a))(x) = a$ for each $x \in X$ and $a \in A$. Then ρ is continuous (because ρ is linear and $\rho(O) \subset T(S, O)$ for each neighbourhood O of zero in A and $S \in \sigma$). Therefore, $\rho(\varepsilon_x(m_\lambda))$ converges to $\rho(a_0)$. Since

$$\varepsilon_x[\rho(\varepsilon_x(m_\lambda))] = (\rho(\varepsilon_x(m_\lambda)))(x) = \varepsilon_x(m_\lambda)$$

and δ_x is an one-to-one map, then $\kappa_{\overline{M}_x}[\rho(\varepsilon_x(m_\lambda))] = \kappa_{\overline{M}_x}(m_\lambda)$ for each $\lambda \in \Lambda$. Thus

$$\rho(\varepsilon_x(m_\lambda)) \in \kappa_{\overline{M}_x}^{-1}[\kappa_{\overline{M}_x}(\rho(\varepsilon_x(m_\lambda)))] = \kappa_{\overline{M}_x}^{-1}[\kappa_{\overline{M}_x}(m_\lambda)] \in \mathfrak{M}$$

for each $\lambda \in \Lambda$. Hence $\rho(a_0) \in \mathfrak{M}$, because $\mathfrak{M}$ is closed in $\mathfrak{A}(X, A; \sigma)$. Therefore, $a_0 = \varepsilon_x((\rho(a_0)) \in \varepsilon_x(\mathfrak{M})$. So we have proved that

$\mathrm{cl}_A(\varepsilon_x(\mathfrak{M})) = \varepsilon_x(\mathfrak{M})$, which means that $M_{\mathfrak{M}} = \varepsilon_x(\mathfrak{M})$ is a closed maximal left ideal in A. Consequently,

$$\mathfrak{M} = \mathfrak{M}_{x,M_{\mathfrak{M}}}$$

for some $x \in X$ and $M_{\mathfrak{M}} \in m_l(A)$.

Next we show, that $\varepsilon_x(\mathfrak{M}) \in m_l'(A)$. For it, let $a, b \in A$ be such that $a^2 + b^2 \in \varepsilon_x(\mathfrak{M})$. Then from

$$\varepsilon_x(f_a^2 + f_b^2) = (f_a^2 + f_b^2)(x) = a^2 + b^2 \in \varepsilon_x(\mathfrak{M})$$

it follows that $\kappa_{\overline{M}_x}(f_a^2 + f_b^2) \in \kappa_{\overline{M}_x}(\mathfrak{M})$ because σ_x is a one-to-one map. Hence,

$$f_a^2 + f_b^2 \in \kappa_{\overline{M}_x}^{-1}[\kappa_{\overline{M}_x}(f_a^2 + f_b^2)] \subset \kappa_{\overline{M}_x}^{-1}[\kappa_{\overline{M}_x}(\mathfrak{M})] = \mathfrak{M}.$$

Since $\mathfrak{M} \in m_l'(\mathfrak{A}(X, A; \sigma))$, and $f_a, f_b \in \mathfrak{A}(X, A; \sigma)$ by condition 1), then $f_a, f_b \in \mathfrak{M}$ or $a, b \in \varepsilon_x(\mathfrak{M})$. Consequently, $\varepsilon_x(\mathfrak{M}) \in m_l'(A)$. $\square$

Corollary 3.3.3. *Let X be a completely regular Hausdorff k-space*[9]*, σ a compact cover of X, which is closed with respect to finite unions, A a real unital complete locally m-pseudoconvex Hausdorff algebra and $\mathfrak{A}(X, A; \sigma)$ a closed subalgebra of $C(X, A; \sigma)$ (with the same unit as $C(X, A, \sigma)$). If all conditions* 1) – 4) *of Theorem* 3.3.2 *have been satisfied, then every $\mathfrak{M} \in m_l'(\mathfrak{A}(X, A; \sigma))$ is representable in the form $\mathfrak{M} = \mathfrak{M}_{x,M}$ for some $x \in X$ and $M \in m_l'(A)$.*
Similar result is true for ideals in $m_r'(\mathfrak{A}(X, A; \sigma))$ and $m_t'(\mathfrak{A}(X, A; \sigma))$.

Proof. Since X is a k-space, σ is a compact cover and A is complete, then $C(X, A; \sigma)$ is complete by Theorem 43.11 from [34]. Therefore $\mathfrak{A}(X, A; \sigma)$ is also complete as a closed subset. Taking this into account, Corollary 3.3.3 is true by Theorem 3.3.2. $\square$

Corollary 3.3.4. *Let all assumtions and conditions of Theorem* 3.3.2 *be fullfilled. Then the map $\Omega : m_l'(\mathfrak{A}(X, A; \sigma)) \to X \times m_l'(A)$, defined by*

$$\Omega(\mathfrak{M}_{x,M}) = (x, M)$$

[9] A topological space is a k-*space* or a *compactly generated space* if the following condition holds: $A \subset X$ is open if and only if $A \cap K$ is open in K for each compact set K in X (see [25]). The collection of k-spaces contains a considerably wide class of topological spaces. It is known (see [25], p. 172, or [34], p. 285) that every locally compact Hausdorff space and every Hausdorff space, satisfying the first axiom of countability, are k-spaces.

for each $x \in X$ and $M \in m'_l(A)$, is a bijection.
Similar result is true for ideals in $m'_r(\mathfrak{A}(X, A; \sigma))$ and $m'_t(\mathfrak{A}(X, A; \sigma))$.

Proof. It is clear that Ω maps $m'_k(\mathfrak{A}(X, A; \sigma))$ onto $X \times m'_k(A)$ by Lemma 3.2.2 and Theorem 3.3.2. If now $\Omega(\mathfrak{M}_{x_1,M_1}) = \Omega(\mathfrak{M}_{x_2,M_2})$, then from $(x_1, M_1) = (x_2, M_2)$ it follows that $x_1 = x_2$ and $M_1 = M_2$. Hence $\mathfrak{M}_{x_1,M_1} = \mathfrak{M}_{x_2,M_2}$ and thus Ω is a bijection. $\square$

Remark. In case, when A is a complex unital locally m-pseudoconvex Hausdorff algebra, similar results to Theorem 3.3.2 and Corollary 3.3.4 have been proved in [2] (see also [3, 4]).

3.4 Description of homomorphisms in subalgebras of $C(X, A, \sigma)$

In this section we give some results, which follow from Theorem 3.3.2, and describe homomorphisms from subalgebras of $C(X, A, \sigma)$ onto $\mathbb{R}$.

Proposition 3.4.1. *Let all assumptions and conditions of Theorem* 3.3.2 *be fullfilled. If, in addition, A is a commutative algebra, then every homomorphism $\Phi \in \hom(\mathfrak{A}(X, A; \sigma))$ defines $x \in X$ and $\phi \in \hom(A)$ such that $\Phi = \phi \circ \varepsilon_x$.*

Proof. If $\Phi \in \hom(\mathfrak{A}(X, A; \sigma))$, then $\ker\Phi$ is a closed maximal two-sided ideal in $\mathfrak{A}(X, A; \sigma)$ (see [24], p. 68). Let now $f, g \in \mathfrak{A}(X, A; \sigma)$ be such that $f^2 + g^2 \in \ker\Phi$. Then

$$\Phi(f^2 + g^2) = (\Phi(f))^2 + (\Phi(g))^2 = 0.$$

Since $\Phi(f), \Phi(g) \in \mathbb{R}$, then $\Phi(f) = \Phi(g) = 0$. Thus, the condition (2.2.1) is true in the present case. Hence, $\ker\Phi = \mathfrak{M}_{x,M}$ for some $x \in X$ and $M \in m(A)$ by Theorem 3.3.2.

Since A is a real Gelfand-Mazur algebra by Corollary 1.7.2 case b), then there is $\phi \in \hom(A)$ such that $M = \ker\phi$. Now from $f \in \ker\phi$ it follws that $\varepsilon_x(f) \in \ker\phi$ or $f \in \ker(\phi \circ \varepsilon_x)$. Therefore $\ker\Phi \subset \ker(\phi \circ \varepsilon_x)$. Since $\ker(\phi \circ \varepsilon_x)$ is a two-sided ideal in $\mathfrak{A}(X, A; \sigma)$ and $\ker\Phi$ is a maximal two-sided ideal, then $\ker\Phi = \ker(\phi \circ \varepsilon_x)$ and therefore $\Phi = \phi \circ \varepsilon_x$ by Lemma 7.2 from [24]. $\square$

A subset $H \subset \hom(A)$ is *equicontinuous* at $a_0 \in A$ if for every $\varepsilon > 0$ there is a neighbourhood O_{a_0} such that $|\phi(a) - \phi(a_0)| < \varepsilon$ for all $\phi \in H$ and $a \in O_{a_0}$ and $H \subset \hom(A)$ is *equicontinuous* if it is equicontinuous at every $a \in A$. The set $\hom(A)$ (in the Gelfand topology) is *locally equicontinuous* if every $\phi_0 \in \hom(A)$ has an equicontinuous neighbourhood O_{ϕ_0} of ϕ_0 in $\hom(A)$ that is, for each $\varepsilon > 0$ and $a_0 \in A$ there is a neighbourhood O_{a_0} of a_0 in A such that $|\phi(a) - \phi_0(a_0)| < \varepsilon$ for all $a \in O_{a_0}$ and $\phi \in O_{\phi_0}$.

Theorem 3.4.2. *Let all assumptions and conditions of Theorem* 3.3.2 *be fullfilled. Let*

$$\hom(\mathfrak{A}(X, A; \sigma)) = \{\xi_x : \quad x \in X\},$$

where $\xi_x(\alpha) = \alpha(x)$ *for each* $\alpha \in \mathfrak{A}(X, \mathbb{R}; \sigma)$, *and the map* $\xi_x \to x$ *from* $\hom(\mathfrak{A}(X, \mathbb{R}; \sigma))$ *onto* X *be continuous. If, in addition,* A *is a commutative algebra, for which* $\hom(A)$ *is locally equicontinuous, then* $\hom(\mathfrak{A}(X, A; \sigma))$ *and* $X \times \hom(A)$ *are homeomorphic.*

Proof. By Proposition 3.4.1, every $\Phi \in \hom(\mathfrak{A}(X, A; \sigma))$ is representable in the form

$$\Phi = \Phi_{(x,\phi)} = \phi \circ \varepsilon_x$$

for some $x \in X$ and $\phi \in \hom(A)$. Since ε_x is a continuous homomorphism from $\mathfrak{A}(X, A; \sigma)$ onto A for each $x \in X$, then $\phi \circ \varepsilon_x \in \hom(\mathfrak{A}(X, \mathbb{R}; \sigma))$ for each $x \in X$ and $\phi \in \hom(A)$. Therefore Ω, defined by

$$\Omega(\Phi) = \Omega(\Phi_{(x,\phi)}) = (x, \phi)$$

for each $x \in X$ and $\phi \in \hom(A)$ maps $\hom(\mathfrak{A}(X, A; \sigma))$ onto $X \times \hom(A)$.

We show that Ω is one-to-one. For it, let $\Omega(\Phi_{(x,\phi)}) = \Omega(\Phi_{(x_1,\phi_1)})$. Then $(x, \phi) = (x_1, \phi_1)$. Hence, $\Phi_{(x,\phi)} = \Phi_{(x_1,\phi_1)}$, which means that Ω is a bijection.

Next we show that Ω is continuous. For it, let

$$(\Phi_i)_{i \in I} = (\Phi_{(x_i,\phi_i)})_{i \in I}$$

be a net in $\hom(\mathfrak{A}(X, A; \sigma))$, which converges to $\Phi_0 = \Phi_{(x_0,\phi_0)}$ in the Gelfand topology on $\hom(\mathfrak{A}(X, A; \sigma))$. Then the net $(\Phi_i(f))_{i \in I}$ converges to $\Phi_0(f)$ for each $f \in \hom(\mathfrak{A}(X, A; \sigma))$. Since (by the condition

2) of Theorem 3.3.2) $f_a \in \mathfrak{A}(X, A; \sigma)$ for each $a \in A$, and

$$\Phi_i(f_a) = \phi_i \circ \varepsilon_{x_i}(f_a) = \phi_i(\varepsilon_{x_i}(f_a)) = \phi_i(f_a(x_i)) = \phi_i(a)$$

for each $i \in I \cup \{0\}$, then $(\phi_i(a))_{i\in I}$ converges to $\phi_0(a)$ for each $a \in A$. Hence $(\phi_i)_{i\in I}$ converges to ϕ_0 in the Gelfand topology on $\hom(A)$.

Moreover, $\alpha e_A = \alpha f_{e_A} \in \mathfrak{A}(X, A; \sigma)$ for each $\alpha \in \mathfrak{A}(X, A; \sigma)$ by condition 1) and 3) of Theorem 3.3.2. Therefore the net $(\Phi_i(\alpha e_A))_{i\in I}$ converges to $\Phi_0(\alpha e_A)$ for each $\alpha \in \mathfrak{A}(X, \mathbb{R}; \sigma)$. Since

$$\Phi_{(x_i,\phi)}(\alpha e_A) = \alpha(x_i)\phi(e_A) = \alpha(x_i) = \xi_{x_i}(\alpha)$$

for each $\phi \in \hom(A)$ and $\alpha \in \mathfrak{A}(X, \mathbb{R}; \sigma)$, then $(\xi_{x_i}(\alpha))_{i\in I}$ converges to $\xi_{x_0}(\alpha)$ for each $\alpha \in \mathfrak{A}(X, \mathbb{R}; \sigma)$. Hence $(\xi_{x_i})_{i\in I}$ converges to ξ_{x_0} in the Gelfand topology on $\hom(\mathfrak{A}(X, \mathbb{R}; \sigma))$. By assumption, $\xi_x \to x$ is continuous. Thus $(x_i)_{i\in I}$ converges to x_0 in the topology of X and therefore $(x_i, \phi_i)_{i\in I}$ converges to (x_0, ϕ_0) in the product topology on $X \times \hom A$. It means that Ω is continuous.

Now we show that Ω^{-1} is continuous. First, we show that for each $x_0 \in X$, $\phi_0 \in \hom(A)$ and neighbourhood $O_{\Phi_{(x_0,\phi_0)}}$ of $\Phi_{(x_0,\phi_0)}$ in the Gelfand topology on $\hom(\mathfrak{A}(X, A; \sigma))$ there is a neighbourhood $O_{(x_0,\phi_0)}$ of (x_0, ϕ_0) such that

$$\Omega^{-1}(O_{(x_0,\phi_0)}) \subset O_{\Phi_{(x_0,\phi_0)}}.$$

For that, it is enogh to show that for each $\varepsilon > 0$, $x_0 \in X$, $\phi_o \in \hom(A)$ and $f \in \mathfrak{A}(X, A; \sigma)$ there is a neighbourhood $O_{(x_0,\phi_0)}$ such that

$$|\Phi_{(x,\phi)}(f) - \Phi_{(x_0,\phi_0)}(f)| < \varepsilon,$$

whenever $(x, \phi) \in O_{(x_0,\phi_0)}$.

Since $\hom(A)$ is locally equicontinuous, then every $\phi_0 \in \hom(A)$ has an equicontinuous neighbourhood O_{ϕ_0} of ϕ_0 in the Gelfand topology on $\hom(A)$. Therefore, for every $x_0 \in X$, $f \in \mathfrak{A}(X, A; \sigma)$ and $\varepsilon > 0$ there is a neighbourhood $O_{f(x_0)}$ of $f(x_0)$ such that

$$|\phi(a) - \phi(f(x_0))| = |\phi(a - f(x_0))| < \frac{\varepsilon}{2}$$

for each $a \in O_{f(x_0)}$ and $\phi \in O_{\phi_0}$. Since f is continuous, then there is a neighbourhood O_{x_0} of x_0 such that $f(x) \in O_{f(x_0)}$, whenever $x \in O_{x_0}$. Hence

$$|\phi(f(x) - f(x_0))| < \frac{\varepsilon}{2},$$

whenever $\phi \in O_{\phi_0}$ and $x \in O_{x_0}$. Let now

$$O(\phi_0) = O_{\phi_0} \bigcap \Big\{ \phi \in \hom(A) : \quad |(\phi - \phi_0)(f(x_0))| < \frac{\varepsilon}{2} \Big\}.$$

Since $O_{x_0} \times O_{\phi_0}$ is a neighbourhood of (x_0, ϕ_0) in the product topology of $X \times \hom(A)$ and

$$\begin{aligned} &|\Phi_{(x,\phi)}(f) - \Phi_{(x_0,\phi_0)}(f)| = |\phi(f(x)) - \phi_0(f(x_0))| \leq \\ &\leq |\phi(f(x) - f(x_0))| + |(\phi - \phi_0)(f(x_0))| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for each $f \in (\mathfrak{A}(X, A; \sigma)$, whenever $(x, \phi) \in O_{x_0} \times O_{\phi_0}$, then Ω^{-1} is continuous. Hence, Ω is a homeomorphism. □

3.5 Some results for $C(X, A; \sigma)$

Next we describe ideals in $C(X, A; \sigma)$.

Proposition 3.5.1. *Let X be a completely regular Hausdorff k-space, σ a compact cover of X, which is closed with respect to finite unions, and A a real unital complete locally m-pseudoconvex Hausdorff algebra. Then*

a) *every $\mathfrak{M} \in m'_l(C(X, A; \sigma))$ is representable in the form*

$$\mathfrak{M} = \mathfrak{M}_{x,M} = \{f \in C(X, A; \sigma) : f(x) \in M\}.$$

for some $x \in X$ and $M \in m'_l(A)$;

b) *$\mathfrak{M}_{x,M} \in m'_l(C(X, A; \sigma))$ for each $x \in X$ and each $M \in m'_l(A)$;*

c) *the map $\Omega : m'_l(C(X, A; \sigma)) \to X \times m'_l(A)$, defined by*

$$\Omega(\mathfrak{M}_{x,M}) = (x, M)$$

for each $x \in X$ and $M \in m'_l(A)$, is a bijection.

Similar results are true for ideals in $m'_r(\mathfrak{A}(X, A; \sigma))$ and in $m'_t(\mathfrak{A}(X, A; \sigma))$.

Proof. It is easy to see that $C(X, A; \sigma)$ is strictly real and the conditions of Lemma 3.2.1 and Theorem 3.3.2 are fullfilled by Lemma 3.1.1 and Lemma 3.1.2. Thus, the statements a), b) and c) are true by Lemma 3.2.2 and Corollaries 3.3.3 and 3.3.4. $\square$

Proposition 3.5.2. *Let X be a completely regular Hausdorff k-space, σ a compact cover of X, which is closed with respect to finite unions, A a commutative real unital complete locally m-pseudoconvex Hausdorff algebra.*

Then every $\Phi \in \hom(C(X, A; \sigma))$ defines $x \in X$ and $\phi \in \hom(A)$ such that $\Phi = \phi \circ \varepsilon_x$. If, in addition, $\hom(A)$ is locally equicontinuous, then $\hom(C(X, A; \sigma))$ and $X \times \hom(A)$ are homeomorphic.

Proof. All assumptions and conditions of Corollary 3.3.3 and Theorem 3.4.2 have been fullfilled by Lemmas 3.1.1 and 3.1.2. Therefore, every $\Phi \in \hom(C(X, A; \sigma)$ defines $x \in X$ and $\phi \in \hom(A)$ such that $\phi = \phi \circ \varepsilon_x$ by Proposition 3.4.1. It is shown in [5], Theorem 2, that $\hom(C(X, \mathbb{R}; \sigma)) = \{\xi_x : \ x \in X\}$ and the map $\xi_X \to x$ is a homeomorphism from $C(X, \mathbb{R}; \sigma)$ onto X. Consequently, $\hom(C(X, A; \sigma)$ and $X \times \hom(A)$ are homeomorphic by Theorem 3.4.2, because $\hom(A)$ is locally equicontinuous. $\square$

Kokkuvõte

1. Olgu A topoloogiline algebra üle reaalarvude korpuse $\mathbb{R}$, s.t. selline topoloogiline vektorruum üle korpuse $\mathbb{R}$, milles on defineeritud assotsiatiivne eraldi pidev korrutamine, ning olgu $m(A)$ kõigi selliste kinniste kahepoolsete ideaalide hulk algebras A, mis on maksimaalsed kui vasakpoolsed ideaalid või kui parempoolsed ideaalid. Topoloogilist algebrat üle $\mathbb{R}$ nimetatakse *reaalseks Gelfand-Mazuri algebraks*, kui faktoralgebra A/M (faktortopoloogias) on topoloogiliselt isomorfne korpusega $\mathbb{R}$. Analoogiliselt defineeritakse kompleksne Gelfand-Mazuri algebra.

Komplekse Gelfand-Mazuri algebra mõiste võtsid teineteisest sõltumatult kasutusele Mati Abel (vt. [6]–[8]) ja Anastasios Mallios (vt. [24]). Komplekssete Gelfand-Mazuri algebrate struktuur on seni küllaltki hästi uuritud. Reaalse Gelfand-Mazuri algebra mõiste on kasutusele võetud töös [18].

Reaalse Gelfand-Mazuri algebra A omaduste uurimiseks sisestatakse A komplekssesse Gelfand-Mazuri algebrasse $\widetilde{A}$. Kasutades komplekse Gelfand-Mazuri algebra korral teada olevaid tulemusi algebra $\widetilde{A}$ korral, saame kirjeldada algebra A omadusi. Seda meetodit on edukalt kasutatatud reaalsete Banachi algebrate uurimisel (vt. [21] ja [30]).

2. Käesolev väitekiri koosneb kolmest peatükist. Esimeses vaadeldakse topoloogiliste (s.o. lokaalselt pseudokumerate ja gälb) algebrate kompleksifitseerimist ja selle omadusi. Antakse põhiliste reaalsete Gelfand-Mazuri jagamisega algebrate kirjeldus (Järeldus 1.7.2).

Teises peatükis näidatakse (Teoreem 2.2.1), et juhul kui kommutatiivne reaalne topoloogiline algebra A rahuldab tingimust

$$\text{kui } a, b \in A \text{ ning } a^2 + b^2 \in M, \text{ siis } a, b \in M$$

iga $M \in m(A)$ korral, ning algebral A leidub selline topoloogia τ, et (A, τ) on üks järgmistest topoloogilistest algebratest:

a) lokaalselt pseudokumer Waelbroecki algebra;

b) lokaalselt A-pseudokumer (erijuhul lokaalselt m-pseudokumer) algebra;

c) lokaalselt pseudokumer Fréchet algebra;

d) pideva korrutamisega ja tõkestatud elementidega tugevalt gälb (erijuhul eksponentsiaalselt gälb) Hausdorffi algebra;

e) topoloogiline Hausdorffi algebra, milles iga $a \in A$ ja $M \in m(A)$ korral leidub selline $\lambda \in \mathbb{R}$, et $a - \lambda e_A \in M$,

siis A on kommutatiivne reaalne Gelfand-Mazuri algebra.

Kasutades saadud tulemusi, leitakse selliste topoloogiliste algebrate A kirjeldus, mille korral faktoralgebra A/P (üle kinnise primitiivse ideaali P) tsenter $Z(A/P)$ on topoloogiliselt isomorfne korpusega $\mathbb{R}$.

Kasutades G. Allani ja L. Waelbroecki ideid (vt. [19, 32]) ja Mart Abeli poolt saadud tulemusi (vt. [2] ja [4]) kinniste maksimaalsete ideaalide kirjeldamiseks komplekssetes Gelfand-Mazuri algebrates, antakse ühe- ja kahepoolsete ideaalide kirjeldus lokaalselt A-pseudokumerates (erijuhul lokaalselt m-pseudokumerates) algebrates ning lokaalselt pseudokumerates Fréchet algebrates.

Kolmandas peatükis kirjeldatakse topoloogilise algebra $C(X, A; \sigma)$ omadusi juhul, kui A on reaalne ühikuga lokaalselt m-pseudokumer algebra (vt. Lemma 3.1.1). Leitakse piisavad tingimused (Teoreemid 3.3.2 ja 3.4.1) selleks, et algebra $C(X, A; \sigma)$ alamalgebrates oleks võimalik kirjeldada kõiki kinniseid maksimaalseid ühe- ja kahepoolseid ideaale (samuti kõiki pidevaid lineaarseid multiplikatiivseid funktsionaale) ruumide X ja $\hom(A)$ punktide abil.

3. Käesoleva väitekirja tulemused on publitseeritud töödes [18] ja [28] (töö [29] ilmumisel). Oma tulemusi on tutvustatud järgmistel rahvusvahelistel konverentsidel: "International Conference on Topological Algebras and its Applications" (Oulu, 2001), "Topological algebras, their applications and related results" (Będlewo, 2003), "International conference dedicated to 125-th anniversary of Hans Hahn"

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