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On Regular Square Prism Tilings in $\widetilde{\mathbf{SL_2R}}$ Space

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ABSTRACT

In [9] and [10] we have studied the regular prisms and prism tilings and their geodesic ball packings in $\widetilde{\mathbf{SL_2R}}$ space that is one among the eight Thurston geometries. This geometry can be derived from the 3-dimensional Lie group of all 2×2 real matrices with determinant one.

In this paper we consider the regular infinite and bounded square prism tilings whose existence was proved in [9]. We determine the data of the above tilings and visualize them in the hyperboloid model of $\widetilde{\mathbf{SL_2R}}$ space.

We use for the computations and visualization of the $\widetilde{\mathbf{SL_2R}}$ space its projective model introduced by E. Molnár.

Key words: Thurston geometries, $\widetilde{\mathbf{SL_2R}}$ geometry, tiling, prism tiling

MSC 2010: 52C17, 52C22, 53A35, 51M20

O popločavanju pravilnim kvadratskim prizmama u prostoru $\widetilde{\mathbf{SL_2R}}$

SAŽETAK

U [9] i [10] smo proučavali pravilne prizme, popločavanje prizmama te njihovo popunjavanje geodetskim kuglama u prostoru $\widetilde{\mathbf{SL_2R}}$, koji je jedan od osam Thurstonovih geometrija. Ova se geometrija može dobiti iz 3-dimenzionalne Lieve grupe svih 2×2 matrica s jediničnom determinantom.

U ovom članku promatramo popločavanje pravilnim beskonačnim i omeđenim kvadratskim prizmama čije je postojanje dokazano u [10]. Određujemo podatke gore spomenutog popločavanja i vizualiziramo ih u modelu hiperboloida u $\widetilde{\mathbf{SL_2R}}$ prostoru.

Za računanje i vizualizaciju $\widetilde{\mathbf{SL_2R}}$ prostora koristimo projektivni model koji je uveo E. Molnár.

Ključne riječi: Thurstonova geometrija, $\widetilde{\mathbf{SL_2R}}$ geometrija, popločavanje, popločavanje prizmama

1 The $\widetilde{\mathbf{SL_2R}}$ geometry

The $\mathbf{SL_2R}$ Lie-group consists of the real 2×2 matrices $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ with unit determinant $ad - cb = 1$. The $\widetilde{\mathbf{SL_2R}}$ geometry is the universal covering group of this group, and is a Lie-group itself. Because of the 3 independent coordinates, $\widetilde{\mathbf{SL_2R}}$ is a 3-dimensional manifold, with its usual neighbourhood topology. In order to model the above structure on the projective sphere $\mathcal{PS}^3$ and space $\mathcal{P}^3$ we introduce the new projective coordinates (x^0, x^1, x^2, x^3), where

$$a := x^0 + x^3, b := x^1 + x^2, c := -x^1 + x^2, d := x^0 - x^3, \quad (1)$$

with positive resp. non-zero multiplicative equivalence as projective freedom. Through the equivalence $\mathbf{SL_2R} \sim \mathbf{PSL_2R}$ it follows, that

$$0 > bc - ad = -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 \quad (2)$$

describes the interior af the above one-sheeted hyperboloid solid $\mathcal{H}$ in the usual Euclidean coordinate simplex with the origin $E_0(1;0;0;0)$ and the ideal points of the axes $E_1^\infty(1;1;0;0), E_2^\infty(1;0;1;0), E_3^\infty(1;0;0;1)$. We shall consider the collineation group $\mathbf{G}_*$, which acts on the projective space $\mathcal{P}^3$ and preserves the polarity, ie. a scalar product of signature $(--++)$, moreover certain additional fibering structure. This group leaves the one sheeted hypeboloid $\mathcal{H}$ invariant. Choosing an appropriate subgroup $\mathbf{G}$ of $\mathbf{G}_*$ as isometry group, the universal covering space $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ will be the hyperboloid model of $\widetilde{\mathbf{SL_2R}}$. (See fig. 1.)

The specific isometries $\mathbf{S}(\phi)$ is a one parameter group given by the matrices:

$$\mathbf{S}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ d & b & \cos\phi & -\sin\phi \\ d & b & \sin\phi & \cos\phi \end{pmatrix} \quad (3)$$

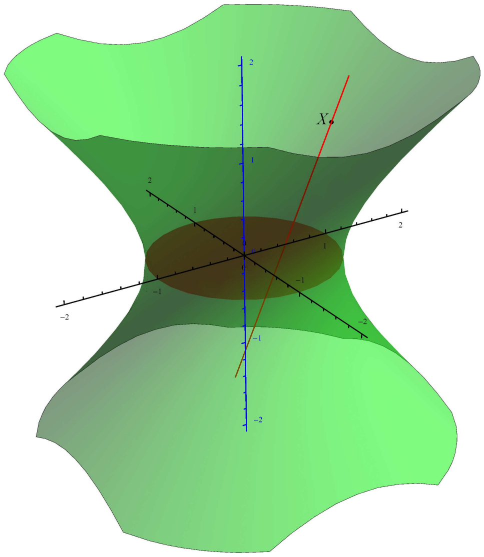

Figure 1: *The hypeboloid model of the* $\widetilde{\mathbf{SL_2R}}$ *space with the "base plane" and the fibre line e.g. through the point* $X(1;1;\frac{1}{2};\frac{1}{3})$

The specific isometries $\mathbf{S}(\phi)$ is a one parameter group given by the matrices:

The elements of $\mathbf{S}(\phi)$ are the so-called "fibre translations" for $\phi \in \mathbb{R}$. We obtain an unique fibre-line to each $X(x^0;x^1;x^2;x^3) \in \widetilde{\mathcal{H}}$ as the orbit by right action of $\mathbf{S}(\phi)$ on X. The coordinates of points lying on the fibre line through X can be expressed as the images of X by $\mathbf{S}(\phi)$:

$$(x^0;x^1;x^2;x^3) \stackrel{\mathbf{S}(\phi)}{\longrightarrow} (x^0\cos\phi - x^1\sin\phi; \\ x^0\sin\phi + x^1\cos\phi; x^2\cos\phi + x^3\sin\phi; -x^2\sin\phi + x^3\cos\phi) \tag{4}$$

The points of a fibre line through X by the usual inhomogeneous Euclidean coordinates $x = \frac{x^1}{x^0}, y = \frac{x^2}{x^0}, z = \frac{x^3}{x^0}$ are given by:

$$(1;x;y;z) \stackrel{\mathbf{S}(\phi)}{\longrightarrow} (1; \frac{x+\tan\phi}{1-x\tan\phi}; \frac{y+z\tan\phi}{1-x\tan\phi}; \frac{z-y\tan\phi}{1-x\tan\phi}). \tag{5}$$

From formulas (4) and (5) we can see the π periodicity of the above maps.

The elements of the isometry group of $\widetilde{\mathbf{SL_2R}}$ can be described in the above basis by the following matrix:

$$(a_i^j) = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & a_0^3 \\ \mp a_0^1 & \pm a_0^0 & \pm a_0^3 & \mp a_0^2 \\ a_2^0 & a_2^1 & a_2^2 & a_2^3 \\ \pm a_2^1 & \mp a_2^0 & \mp a_2^3 & \pm a_2^2 \end{pmatrix} \tag{6}$$

where

$$\begin{aligned} &-(a_0^0)^2-(a_0^1)^2+(a_0^2)^2+(a_0^3)^2=-1,\\ &-(a_2^0)^2-(a_2^1)^2+(a_2^2)^2+(a_2^3)^2=-1,\\ &-a_0^0a_2^0-a_0^1a_2^1+a_0^2a_2^2+a_0^3a_2^3=0\\ &-a_0^0a_2^1-a_0^1a_2^0+a_0^2a_2^3+a_0^3a_2^2=0. \end{aligned}$$

We define the translation group $\mathbf{G}_T$ as a subgroup of $\widetilde{\mathbf{SL_2R}}$ isometry group acting transitively on the points of $\widetilde{\mathcal{H}}$ and mapping the origin $E_0(1;0;0;0)$ onto $X(x^0;x^1;x^2;x^3)$. These isometries and their inverses (up to a positive determinant factor) can be given by the following matrices:

$$\mathbf{T}:\ (t_i^j) = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^0 & x^1 & x^2 & x^3 \\ x^1 & -x^0 & -x^3 & x^2 \end{pmatrix}$$

$$\mathbf{T^{-1}}:\ (T_j^k) = \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^0 & -x^1 & x^2 & -x^3 \\ -x^1 & x^0 & x^3 & x^2 \end{pmatrix} \tag{7}$$

The rotation about the fibre line through the origin $E_0(1;0;0;0)$ by angle ω can be expressed by the following matrix:

$$\mathbf{R}_{E_0}(\omega):(r_i^j(E_0,\omega)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\omega & \sin\omega \\ 0 & 0 & -\sin\omega & \cos\omega \end{pmatrix}, \tag{8}$$

while the rotation about the fibre line through point $X(x^0;x^1;x^2;x^3)$ by angle ω can be expressed by conjugation with the following formula: $(r_i^j(X,\omega)) = \mathbf{R}_X(\omega) = \mathbf{T}^{-1}\mathbf{R}_{E_0}(\omega)\mathbf{T}$.

We can introduce the so called hyperboloid parametrization as follows

$$\begin{aligned} &x^0 = \cosh r\cos\phi,\\ &x^1 = \cosh r\sin\phi,\\ &x^2 = \sinh r\cos(\theta-\phi),\\ &x^3 = \sinh r\sin(\theta-\phi), \end{aligned} \tag{9}$$

where (r,θ) are the polar coordinates of the base plane, and ϕ is the fibre coordinate. We note, that

$$-x^0x^0-x^1x^1+x^2x^2+x^3x^3 = -\cosh^2 r+\sinh^2 r = -1 < 0. \tag{10}$$

The inhomogeneous coordinates corresponding to (9), that play an important role in visualization, are given by

$$\begin{aligned} x &= \frac{x^1}{x^0} = \tan\phi, \\ y &= \frac{x^1}{x^0} = \tanh r \frac{\cos(\theta-\phi)}{\cos\phi}, \\ z &= \frac{x^1}{x^0} = \tanh r \frac{\sin(\theta-\phi)}{\cos\phi}. \end{aligned} \tag{11}$$

2 Geodesics and geodesic balls

In the following we are going to introduce the notion of the geodesic sphere and ball, using the concept of the metric tensor field and geodesic curve. After this we visualize the effects of the $\widetilde{\mathbf{SL_2R}}$ isometries using geodesic balls.

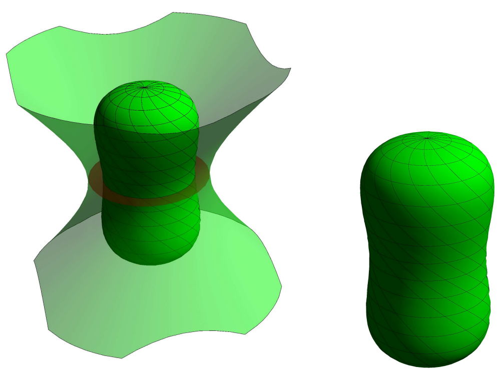

Figure 2: *Geodesic sphere of radius* 1 *centered at the origin*

The infinitesimal arc-length square can be derived by the standard method called pull back into the origin. By acting of (7) on the differentials $(dx^0; dx^1; dx^2; dx^3)$, we obtain by [2], [1] and [3] that in this parametrization the infinitesimal arc-length square at any point of $\widetilde{\mathbf{SL_2R}}$ is the following:

$$(ds)^2 = (dr)^2 + \cosh^2 r \sinh^2 r (d\theta)^2 + [(d\phi) + \sinh^2 r (d\theta)]^2. \tag{12}$$

Hence we get the symmetric metric tensor field g_{ij} on $\widetilde{\mathbf{SL_2R}}$ by components:

$$g_{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r(\sinh^2 r + \cosh^2 r) & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}, \tag{13}$$

The geodesic curves of $\widetilde{\mathbf{SL_2R}}$ are generally defined as having locally minimal arc length between any two of their (close enough) points.

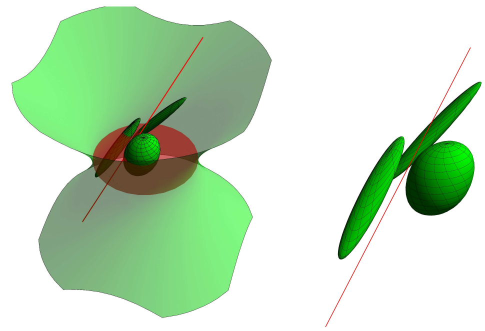

Figure 3: *Geodesic sphere rotated in 3rd order about a fibre line*

By (13) the second order differential equation system of the $\widetilde{\mathbf{SL_2R}}$ geodesic curve of form (11) is the following:

$$\begin{aligned} \ddot r &= \sinh(2r)\,\dot\theta\,\dot\phi + \frac{1}{2}\big(\sinh(4r) - \sinh(2r)\big)\dot\theta\,\dot\theta, \\ \ddot\phi &= 2\dot r \tanh(r)(2\sinh^2(r)\,\dot\theta + \dot\phi), \\ \ddot\theta &= \frac{2\dot r}{\sinh(2r)}\big((3\cosh(2r)-1)\dot\theta + 2\dot\phi\big). \end{aligned} \tag{14}$$

We can assume, that the starting point of a geodesic curve is $(1,0,0,0)$, because we can transform a curve into an arbitrary starting point. Moreover, $r(0) = 0$, $\phi(0) = 0$, $\theta(0) = 0$, $\dot r(0) = \cos\alpha$, $\dot\phi(0) = -\dot\theta(0) = \sin\alpha$ and so unit velocity can be assumed as follows in Table 1 from [1].

Table 1

Types	
$0 \le \alpha < \frac{\pi}{4}$ ($\mathbf{H}^2$ – like direction)	$r(s,\alpha) = \operatorname{arsinh}\Big(\frac{\cos\alpha}{\sqrt{\cos 2\alpha}}\sinh(\sqrt{\cos 2\alpha}\,s)\Big)$ $\theta(s,\alpha) = -\arctan\Big(\frac{\sin\alpha}{\sqrt{\cos 2\alpha}}\tanh(\sqrt{\cos 2\alpha}\,s)\Big)$ $\phi(s,\alpha) = 2\sin\alpha s + \theta(s,\alpha)$
$\alpha = \frac{\pi}{4}$ (light direction)	$r(s,\alpha) = \operatorname{arsinh}\Big(\frac{\sqrt{2}}{2}s\Big)$ $\theta(s,\alpha) = -\arctan\Big(\frac{\sqrt{2}}{2}s\Big)$ $\phi(s,\alpha) = \sqrt{2}s + \theta(s,\alpha)$
$\frac{\pi}{4} < \alpha \le \frac{\pi}{2}$ (fibre – like direction)	$r(s,\alpha) = \operatorname{arsinh}\Big(\frac{\cos\alpha}{\sqrt{-\cos 2\alpha}}\sin(\sqrt{-\cos 2\alpha}\,s)\Big)$ $\theta(s,\alpha) = -\arctan\Big(\frac{\sin\alpha}{\sqrt{-\cos 2\alpha}}\tan(\sqrt{-\cos 2\alpha}\,s)\Big)$ $\phi(s,\alpha) = 2\sin\alpha s + \theta(s,\alpha)$

The equation of the geodesic curve in the hyperboloid model – using the usual geographical coordiantes (λ, α), $(-\pi < \lambda \le \pi)$, as general longitude and altitude parameters for the later geodesic sphere $(-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2})$, and the arc-length parameter $0 \le s \in \mathbf{R}$ – are determined in [1]. The Euclidean coordinates $X(s,\lambda,\alpha)$, $Y(s,\lambda,\alpha)$, $Z(s,\lambda,\alpha)$ of the geodesic curves can be determined by substituting the results of Table 1 (see [1]) into the following equations by (11):

$$
\begin{aligned}
X(s,\lambda,\alpha) &= \tan\phi(s,\alpha), \\
Y(s,\lambda,\alpha) &= \tanh r(s,\alpha)\Big(\frac{\cos(\theta(s,\alpha)-\phi(s,\alpha))}{\cos\phi(s,\alpha)}\cos\lambda- \\
&\quad -\frac{\sin(\theta(s,\alpha)-\phi(s,\alpha))}{\cos\phi(s,\alpha)}\sin\lambda\Big) \\
&= \frac{\tanh r(s,\alpha)}{\cos\phi(s\alpha)}\cos[\theta(s,\alpha)-\phi(s,\alpha)+\lambda], \qquad (15) \\
Z(s,\lambda,\alpha) &= \tanh r(s,\alpha)\Big(\frac{\cos(\theta(s,\alpha)-\phi(s,\alpha))}{\cos\phi(s,\alpha)}\sin\lambda+ \\
&\quad \frac{\sin(\theta(s,\alpha)-\phi(s,\alpha))}{\cos\phi(s,\alpha)}\cos\lambda\Big) \\
&= \frac{\tanh r(s,\alpha)}{\cos\phi(s\alpha)}\sin[\theta(s,\alpha)-\phi(s,\alpha)+\lambda].
\end{aligned}
$$

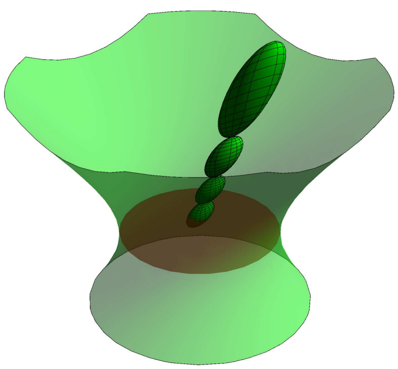

Figure 4: *Touching geodesic spheres of radius $\frac{1}{6}$ centered on a fibre line*

Definition 2.1 *The distance $d(P_1,P_2)$ between the points P_1 and P_2 is defined as the arc length of the geodesic curve from P_1 to P_2.*

In [10] the third author has investigated the notion of the geodesic spheres and balls, with the following definition:

Definition 2.2 *The geodesic sphere of radius ρ and center P is defined as the set of all points Q in the space with the additional condition $d(P,Q)=\rho\in\left[0,\frac{\pi}{2}\right)$.*

Remark 2.3 *The geodesic sphere above is a simply connected surface without self intersection in the space $\widetilde{\mathbf{SL_2R}}$.*

Figure 2 shows a sphere with radius $\rho=1$ and the origin as its center.

3 Regular prisms in $\widetilde{\mathbf{SL_2R}}$ space

In the paper [9] the third author has defined the prism and prism-like tilings in $\widetilde{\mathbf{SL_2R}}$ space, and also classified the infinite and bounded regular prism tilings. Now, we study the square prisms and prism tilings in $\widetilde{\mathbf{SL_2R}}$ space, review their most important properties and compute their metric data.

Definition 3.1 *Let $\mathcal{P}^i$ be a $\widetilde{\mathbf{SL_2R}}$ infinite solid that is bounded by one-sheeted hyperbolid surfaces of the model space, generated by neighbouring "side fibre lines" passing through the vertices of a p-gon ($\mathcal{P}^b$) lying in the "base plane". The images of solids $\mathcal{P}^i$ by $\widetilde{\mathbf{SL_2R}}$ isometry are called infinite p-sided $\widetilde{\mathbf{SL_2R}}$ prisms.*

The common part of $\mathcal{P}^i$ with the base plane is called the *base figure* of $\mathcal{P}^i$ and is denoted by $\mathcal{P}$. Its vertices coincide with the vertices of $\mathcal{P}^b$.

Definition 3.2 *A p-sided prism is an isometric image of a solid, which is bounded by the side surfaces of a p-sided infinite prism $\mathcal{P}^i$, its base figure $\mathcal{P}$ and the translated copy $\mathcal{P}^t$ of $\mathcal{P}$ by a fibre translation.*

The side faces $\mathcal{P}$ and $\mathcal{P}^t$ are called *"cover faces"*, and are related by fibre translation along fibre lines joining their points.

Definition 3.3 *An infinite prism in $\widetilde{\mathbf{SL_2R}}$ is regular if $\mathcal{P}^b$ is a regular p-gon with center at the origin in the "base plane" and the side surfaces are congruent to each other under an $\widetilde{\mathbf{SL_2R}}$ isometry.*

Definition 3.4 *The regular p-sided prism in $\widetilde{\mathbf{SL_2R}}$ space is a prism derived by Definition 3.2 from an infinite regular p-sided prism (see Definition 3.3).*

We consider a monohedral tessellation of the space $\widetilde{\mathbf{SL_2R}}$ with congruent regular infinite or bounded prisms. A tiling is called face-to-face, if the intersection of any two tiles is either empty or a common face, edge or vertex of both tiles, otherwise it is non-face-to-face.

A regular infinite tiling $\mathcal{T}^i_p(q)$ in the $\widetilde{\mathbf{SL_2R}}$ space is derived by a rotation subgroup $G^r_p(q)$ of the symmetry group $G_p(q)$ of $\mathcal{T}^i_p(q)$. $G^r_p(q)$ is generated by rotations $r_1,r_2,\dots,r_p$ with angles $\omega=\frac{2\pi}{q}$ about the fibre lines $f_1,\dots,f_p$ through the vertices of the given $\widetilde{\mathbf{SL_2R}}$ p-gon $\mathcal{P}^b$, and let $\mathcal{P}^i_p(q)$ be one of its tiles, where we can suppose without the loss of generality, that its p-gonal base figure is centered at the origin.

The vertices $A_1, A_2, \dots, A_p$ of the base figure $\mathcal{P}$ coincide with the vertices of a regular hyperbolic p-gon in the base plane with center at the origin, and we can introduce the following homogeneous coordinates to neighbouring vertices of the base figure of $\mathcal{P}^i_p(q)$ in the hyperboloid model of $\widetilde{\mathcal{H}} = \widetilde{\mathbf{SL_2R}}$.

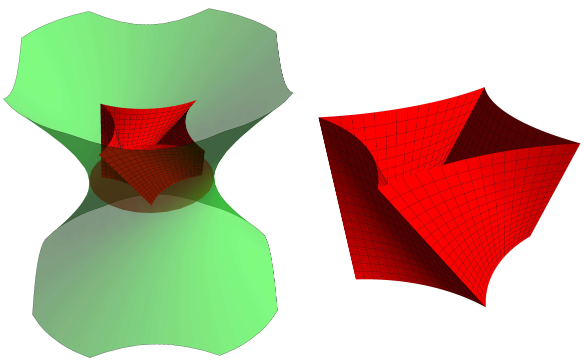

Figure 5: *The $\mathcal{P}^i_4(8)$ tile centered at the origin of the regular infinite tiling $\mathcal{T}^i_4(8)$*

$$A_1 = (1;0;0;x_3),$$
$$A_2 = (1;0;-x_3\sin(\frac{2\pi}{p});x_3\cos(\frac{2\pi}{p})),$$
$$A_3 = (1;0;-x_3\sin(\frac{4\pi}{p});x_3\cos(\frac{4\pi}{p})),$$
$$\dots,$$
$$A_p = (1;0;-x_3\sin((p-1)\frac{2\pi}{p});x_3\cos((p-1)\frac{2\pi}{p})) \qquad (16)$$

The side curves $c(A_iA_{i+1})(i = 1, \dots, p; A_{p+1} \equiv A_1)$ of the base figure are derived from each other by $\frac{2\pi}{p}$ rotation about the x-axis, so they are congruent in $\widetilde{\mathbf{SL_2R}}$ sense. The necessary requirement to the existence of $\mathcal{T}^i_p(q)$, that the surfaces of the neighbouring side faces of $\mathcal{P}^i_p(q)$ are derived from each other by $\frac{2\pi}{q}$ $(\frac{2p}{p-2} < q \in \mathbb{N})$ rotation about the common fibre line.

We have the following theorem ([9]):

Theorem 3.5 *There exists regular infinite prism tiling $\mathcal{T}^i_p(q)$ for each $3 \le p \in \mathbb{N}$, where $\frac{2p}{p-2} < q$.*

The coordinates of the $A_1, A_2, \dots, A_p$ vertices of the base figure and thus the corresponding "fibre side lines" (the fibre lines through the vertices of the base figure) can be computed for any given (p,q) pair of parameters. Moreover the equation of the $c(A_2A_3)$ curve can be determined as follows.

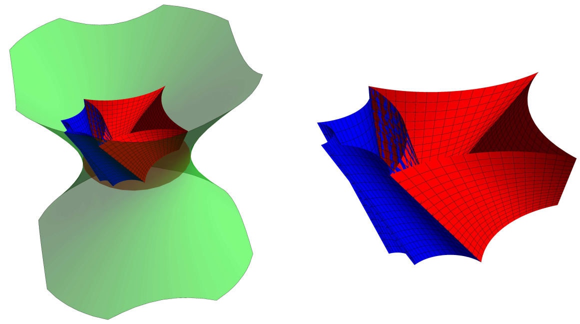

Figure 6: *Regular infinite prism tiling $\mathcal{T}^i_4(8)$*

Let $\mathcal{R}_{A_2}^{\frac{-2\pi}{q}}$ be the rotation matrix of the angle $\omega = -\frac{2\pi}{q}$ about the fibre line through A_2. Consider the half point F of the fibre line segment between the points A_3 and $A_1^{\mathcal{R}_{A_2}^{\frac{-2\pi}{q}}}$. The base curve $c(A_2A_3)$ will be the locus of common points of the fibre lines through the line segment A_2F with the "base plane" of the model. This also determines the side surfaces of $\mathcal{P}^i_p(q)$.

Using the above described method we can compute the x_3 parameter of the vertex coordinates, we obtain the following theorem (see [9]):

Theorem 3.6 *The vertices A_1, A_2, A_3 of the base figure $\mathcal{P}$ of $\mathcal{P}^i_3(q)$ are determined for parameters $p = 3$, and $7 \le q \in \mathbb{N}$ by coordinates in (16) where*

$$x_3 = \sqrt{\frac{\sqrt{3}\cos\frac{2\pi}{q} - \sin\frac{2\pi}{q}}{2\sin\frac{2\pi}{q} + \sqrt{3}}}. \qquad (17)$$

Therefore, the vertices of the prisms $\mathcal{P}^i_3(q)$ base figure $\mathcal{P}$ are the following:

$$A_1 = (1;0;0;\sqrt{\frac{\sqrt{3}\cos\frac{2\pi}{q} - \sin\frac{2\pi}{q}}{2\sin\frac{2\pi}{q} + \sqrt{3}}}),$$
$$A_2 = (1;0;-\frac{\sqrt{3}}{2}\sqrt{\frac{\sqrt{3}\cos\frac{2\pi}{q} - \sin\frac{2\pi}{q}}{2\sin\frac{2\pi}{q} + \sqrt{3}}};-\frac{1}{2}\sqrt{\frac{\sqrt{3}\cos\frac{2\pi}{q} - \sin\frac{2\pi}{q}}{2\sin\frac{2\pi}{q} + \sqrt{3}}}),$$
$$A_3 = (1;0;\frac{\sqrt{3}}{2}\sqrt{\frac{\sqrt{3}\cos\frac{2\pi}{q} - \sin\frac{2\pi}{q}}{2\sin\frac{2\pi}{q} + \sqrt{3}}};-\frac{1}{2}\sqrt{\frac{\sqrt{3}\cos\frac{2\pi}{q} - \sin\frac{2\pi}{q}}{2\sin\frac{2\pi}{q} + \sqrt{3}}}) \qquad (18)$$

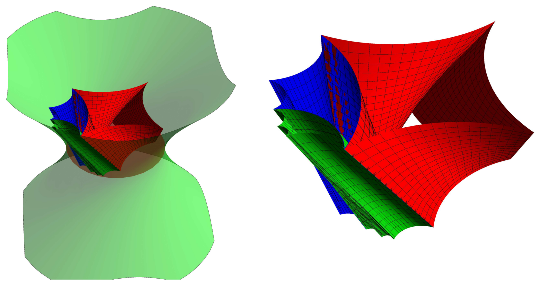

Figure 7: *Regular infinite prism tiling* $\mathcal{T}^i_4(8)$

With an analogous argument we also proved the following theorem, which seems to be a new, important result:

Theorem 3.7 *The vertices* A_1, A_2, A_3, A_4 *of the base figure* $\mathcal{P}$ *of* $\mathcal{P}^i_4(q)$ *are determined for parameters* $p = 4$, *and* $5 \leq q \in \mathbb{N}$ *by coordinates in (16) where*

$$x_3 = \sqrt{\frac{\cos\frac{\pi}{q} - \sin\frac{\pi}{q}}{\cos\frac{\pi}{q} + \sin\frac{\pi}{q}}}, \tag{19}$$

Using this, the vertices of the prisms $\mathcal{P}$ *base figure are:*

$$\begin{aligned} A_1 &= (1; 0; 0; \sqrt{\frac{\cos\frac{\pi}{q} - \sin\frac{\pi}{q}}{\cos\frac{\pi}{q} + \sin\frac{\pi}{q}}}), \\ A_2 &= (1; 0; -\sqrt{\frac{\cos\frac{\pi}{q} - \sin\frac{\pi}{q}}{\cos\frac{\pi}{q} + \sin\frac{\pi}{q}}}; 0), \\ A_3 &= (1; 0; 0; -\sqrt{\frac{\cos\frac{\pi}{q} - \sin\frac{\pi}{q}}{\cos\frac{\pi}{q} + \sin\frac{\pi}{q}}}), \\ A_4 &= (1; 0; \sqrt{\frac{\cos\frac{\pi}{q} - \sin\frac{\pi}{q}}{\cos\frac{\pi}{q} + \sin\frac{\pi}{q}}}; 0). \end{aligned} \tag{20}$$

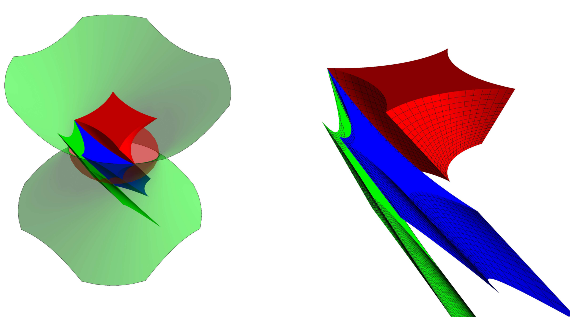

Figure 8: *Regular bounded prism tiling* $\mathcal{T}_4(8)$

Similarly to the regular infinite prism tilings we get the types of the regular bounded prism tilings which are classified in [9] where the third author has proved, that a regular bounded prism tiling are non-face-to-face one. In this paper we visualize in Fig. 8 only some neighbouring prisms of a bounded regular prism tiling $\mathcal{T}_4(8)$ where the height of the prisms are $\frac{3}{4}$. When visualizing prism tilings we use different colors to note the neighbourhooding prisms.

In this paper we have mentioned only some problems in discrete geometry of the $\widetilde{\mathbf{SL_2R}}$ space, but we hope that from these it can be seen that our projective method suits to study and solve similar problems (see [4], [7], [8], [10]).

References

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